WIENER ESTIMATES FOR PARABOLIC OBSTACLE PROBLEMS*

by

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Introduction

We study the pointwise regularity of local weak solutions of parabolic obstacle problems of the following type

$$u \geqslant \psi \text{ q.e.}$$

$$\frac{\partial u}{\partial t} + Lu \geqslant 0 \qquad \text{in } Q$$

$$\left\langle \frac{\partial u}{\partial t} + Lu,\ u - \psi \right\rangle = 0$$

where ψ is a given arbitrary Borel function in Q, $Q = \Omega \times (0,T)$, and L is an operator of the form

$$L = -\sum_{i,j=1}^{N} D_{x_i}(a_{ij} D_{x_j}) \quad .$$

By $u \geqslant \psi$ q.e. we mean that $u \geqslant \psi$ except sets of capacity zero with respect to the capacity introduced in Sec. 1.

The continuity of u at an arbitrarily given point $z_0 = (x_0, t_0) \in Q$ is obtained as a consequence of an estimate of the quantity

$$V(r) = \operatorname{osc}_{Q(z_0, r)} u + \left(\int_{Q(z_0, r)} |D_x u|^2 G^{z_0} dxdt \right)^{1/2}$$

(G^{z_0} is the Green's function of the parabolic operator with singularity at z_0) which also provides an estimate of the modulus of continuity of u at z_0.

These estimates, given in Theorems 4.1 and 4.2, are expressed in terms of a function

$$\omega_\sigma(r, R)$$

which we call the Wiener modulus of ψ at z_0 and is defined in Sec. 3.

In Sec. 2 we also show that the solutions of parabolic obstacle problems, whose existence has been proved by F. Mignot and J.P. Puel [10] for a.e. obstacles and by M. Pierre [14] for q.e. obstacles under suitable boundary conditions, both are local solutions in Q in the sense of the present paper. The result of this paper extends an estimate given in the elliptic case by [8, 12, 13].

For general parabolic equations a Wiener condition for the regularity of weak solutions at boundary points of the domain was given by W. P. Ziemer [16], and estimates, similar to those obtained in the present paper, were obtained in [5]. We note, however, that the capacity considered in [5, 16] is stronger than the one used in the present paper, where we have to deal with supersolutions.

Let us finally remark that our results provide a pointwise condition for the Hölder continuity of u at z_0 which holds, in particular, if ψ is Hölder continuous at z_0.

1. Notation and Preliminaries

By Ω we denote a (bounded) open set in R^N, $N \geq 3$, and by $B(x,r)$, $x \in R^N$, $r > 0$, we denote the open ball

$$B(x,r) = \left\{ y \in R^N \,\Big|\, |y-x| < r \right\}.$$

For a given $T > 0$ we put

$$Q = \Omega \times (0,T)$$

and for every $z = (x,t) \in R^{N+1}$ and $r > 0$ we define

$$Q(z,r) = B(x,r) \times (t-r^2, t+r^2)$$

$$Q^-(z,r) = B(x,r) \times (t-r^2, t)$$

$$Q^-_\theta(z,r) = B(x,r) \times (t-r^2, t-6\theta r^2)$$

where $0 < \theta < 1/6$.

Let E be a compact subset of Q; we define the capacity of E relative to Q by setting

$$\mathrm{cap}_Q E = \inf\left\{ \int_Q |D_x \phi|^2 \, dxdt \,\Big|\, \phi \in C_0^\infty(Q), \ \phi \geq 1 \text{ on a neighborhood of } E \right\} .$$

It turns out that $\mathrm{cap}_Q E$ is a Choquet capacity, see Appendix, and it can be extended by the standard procedure to define the (external) capacity $\mathrm{cap}_Q E$ for an arbitrary subset $E \subset Q$.

For the properties of this capacity, which are relevant in what follows, we refer to the Appendix.

By $H^{1,p}(\Omega)$, $1 \leq p < +\infty$, we denote as usual the Sobolev space of all functions $w \in L^p(\Omega)$ with distribution derivatives $D_{x_i} w \in L^p(\Omega)$, normed by

$$\|w\|_{H^{1,p}} = \left(\|w\|^p_{L^p} + \sum_{i=1}^N \|D_{x_i} w\|^p_{L^p} \right)^{1/p}$$

and by $H_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega)$. We then define $H^1(\Omega) = H^{1,2}(\Omega)$, $H_0^1(\Omega) = H_0^{1,2}(\Omega)$.

By $L^2(0,T; H^1(\Omega))$ $(L^2(0,T; H_0^1(\Omega)))$ we denote the space of all functions $w(x,t)$ such that $w(\cdot,t) \in H^1(\Omega)$ $(H_0^1(\Omega))$ for a.e. $t \in (0,T)$ and $t \to w(\cdot,t)$ is square-integrable in $(0,T)$ with values in $H^1(\Omega)$ $(H_0^1(\Omega))$.

By $L^\infty(0,T; L^2(\Omega))$ we denote the space of all functions $w(x,t)$ on Q such that $w(\cdot,t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ and $\|w(\cdot,t)\|_{L^2}$ is (essentially) bounded on $(0,T)$.

For given functions $a_{ij} \in L^\infty(Q)$, $i,j = 1,\dots,N$, satsifying the conditions

$$(1.1) \qquad |a_{jj}(x,t)| \leqslant \Lambda \qquad \forall (x,t) \text{ a.e. } \in Q .$$

$$(1.2) \qquad \sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j \geqslant \lambda|\xi|^2 \qquad \forall (x,t) \text{ a.e. } \in Q$$

for some constants $\Lambda \geqslant \lambda > 0$, we consider the (formal) operator

$$P = \frac{\partial}{\partial t} - \sum_{i,j=1}^N D_{x_i}\left(a_{ij}(x,t) D_{x_j}\right) .$$

We define in $H^1(\Omega)$ the bilinear forms

$$a(t; u, v) = \sum_{i,j=1}^N \int_\Omega a_{ij}(\cdot,t) D_{x_j} u \, D_{x_i} v \, dx ,$$

and for $u \in L^2(0,T; H^1(\Omega))$ we define Pu in the distribution sense in Q by setting

$$(1.3) \qquad \langle Pu, \phi\rangle = -\int_Q u \frac{\partial\phi}{\partial t} dxdt + \sum_{i,j=1}^N \int_Q a_{ij} D_{x_j} u \, D_{x_i} \phi \, dxdt \qquad \forall \phi \in C_0^\infty(Q) .$$

Given $z = (x,t)$ we denote by $G^z = G(z,w)$ the Green's function for the operator P in a large cylinder $Q_0 = Q_0 \times (-T_0, T_0)$, with homogeneous Cauchy-Dirichlet boundary conditions, [1]. In [1] it is shown that $G^z(w)$, $w = (y,s)$, can be estimated as follows

$$\gamma_1 \frac{1}{|t-s|} \exp\left(-\gamma_1' \frac{|x-y|^2}{|t-s|}\right) \leqslant G^z(w) \leqslant \gamma_2 \frac{1}{|t-s|} \exp\left(-\gamma_2' \frac{|x-y|^2}{|t-s|}\right)$$

for arbitrary $y \in \Omega$ and $s < t$, $G^z(w) = 0$, $s > t$, γ_1, γ_1' and γ_2, γ_2' being suitable positive constants which depend only on N, λ, Λ.

By G^z_ρ, $\rho > 0$, we denote the "regularized" Green's function, which is the (unique) solution in $L^2(-T_0, T_0; H^1_0(\Omega_0))$ of the problem

$$\int_{Q_0} G^z_\rho D_t \phi + \sum_{i,j=1}^{N} \int_{Q_0} a_{ij} D_{x_i} G^z_\rho D_{x_j} \phi \, dxdt = \fint_{Q(z,\rho)} \phi \, dxdt$$

for every $\phi \in C^\infty_0(Q_0)$, with the "initial" condition $G^z_\rho(x, T_0) = 0$ for a.e. $x \in \Omega_0$, where we denote

$$\fint_Q v \, dxdt = \frac{1}{|Q|} \int_Q v \, dxdt \,, \qquad |Q| = \text{meas.}(Q) \,.$$

By Moser's theorem G^z_ρ is Hölder continuous in Q_0 and as $\rho \to 0+$ we have $G^z_\rho(w) \to G^z(w)$ for every $w \neq z$ and uniformly on every compact set of $Q_0 - \{z\}$; Moreover, $G^z_\rho \to G^z$ weakly in $H^1(Q - \{z_0\})$ and in $L^1(-T_0, T_0; W^{1,1}(\Omega_0))$.

2. Local Weak Solutions

Let $\psi : R^{n+1} \to [-\infty, +\infty)$ be a given Borel function, defined up to sets of capacity zero, which we assume to be essentially bounded from above in the capacity sense.

With notation from Sec. 1, we now consider a function u satisfying the following conditions:

(2.1) $$u \geq \psi \text{ q.e.}, \qquad u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

(2.2) u is a supersolution for P, that is,

$$\langle Pu, \phi \rangle \geq 0 \qquad \forall \phi \in C_0^\infty(Q) \qquad \phi \geq 0$$

(2.3) For every $\Phi \in C_0^\infty(Q)$, $\Phi \geq 0$ and every constant d such that $d \geq \psi$ q.e. in $\{z | \Phi(z) > 0\}$, we have

$$\frac{1}{2} \|(u-d)^+ \Phi\|^2_{L^2(\Omega)}(t) - \int_0^t \int_\Omega D_t\Phi \, \Phi \, |(u-d)^+|^2 \, dxds + \int_0^t a(s, u(s), (u-d)^+ \Phi^2(s)) dx = 0 \; .$$

We then say that u is a local (weak) solution of the obstacle problem formally stated in the introduction.

Let us now state an "energy inequality" statisfied by every supersolution u.

Lemma 2.1. *Let $u \in L^2(0,T; H^1(\Omega))$ be such that (2.2) holds; then for every $\Phi \in C_0^\infty(Q)$, $\Phi \geq 0$ in Q and every constant $d \in R$ we have*

(2.4) $$\frac{1}{2} \|(u-d)^- \Phi\|^2_{L^2(\Omega)}(t) - \int_0^t \int_\Omega D_t\Phi \, \Phi \, |(u-d)^-|^2 \, dxds + \int_0^t a(s, u(s), (u-d)^- \Phi^2(s)) \, ds \leq 0 \; .$$

The proof is based on the approximation of u in $Q' \subset\subset Q$, $Q' = \Omega' \times (T_1, T_2)$, $0 < T_1 < T_2 < T$, by a sequence of supersolutions u_n of P in Q', $u_n \in L^2(T_1, T_2; H^1(\Omega')) \cap H^1(T_1, T_2; H^{-1}(\Omega'))$ which converges to u in $L^2(T_1, T_2; H^1(\Omega'))$ as $n \to +\infty$, see lemma 2.2 below. The inequality (2.4) for every u_n and Φ with $\mathrm{supp}(\Phi) \subset Q'$ is proved in a standard way, by choosing in (2.2) $\phi = (u-d)^{-}\Phi^2$, which is admissible as it is seen by a simple density argument; then the inequality (2.4) for the given u is obtained in the limit as $n \to +\infty$.

Lemma 2.2. Let u be a supersolution of P in Q. Then for every $Q' \subset\subset Q$, $Q' = \Omega' \times (T_1, T_2)$, there exists a sequence $u_n \in L^2(T_1, T_2; H^1(\Omega')) \cap H^1(T_1, T_2; H^{-1}(\Omega'))$, u_n supersolution of P in Q', such that $u_n \to u$ in $L^2(T_1, T_2; H^1(\Omega'))$.

Let $\chi \in C_0^\infty(\Omega)$ with $\chi = 1$ in Ω' and $\tau \in C_0^\infty(0, T)$ with $\tau = 1$ in (T_1, T_2).

Let u be a supersolution of P in Q; for every $n = 1, 2, \ldots$ let us define u_n to be the solution of the problem

$$n^{-1} P u_n + u_n = \chi\tau u + n^{-1} D_t \tau\chi u - 2n^{-1}\tau \sum_{i,j=1}^{N} (a_{ij} + a_{ji}) D_{x_i}\chi D_{x_j} u -$$

$$- n^{-1}\tau u \sum_{i,j=1}^{N} D_{x_i}(a_{ij} D_{x_j}\chi)$$

$$u_n(\cdot, 0) = 0, \quad u_n|_{\partial\Omega} = 0 .$$

We can easily prove that $u_n \to \chi\tau u$ strongly in $L^2(Q)$ and weakly in $L^2(0, T; H_0^1(\Omega))$; moreover

$$\langle P u_n, \phi \rangle \geqslant 0$$

for every $\phi \in C_0^\infty(Q')$, $\phi \geqslant 0$.

Let now v_n be defined by the problem

$$n^{-1} D_t v_n(\cdot, x) + v_n(\cdot, x) = \chi\tau u(\cdot, x) , \qquad v_n(0, x) = 0 \text{ a.e. in } \Omega ;$$

we have $v_n \to \chi\tau u$ in $L^2(0, T; H_0^1(\Omega))$.

Define $w_n = u_n - v_n$ for every n. We have

$$n^{-1}\left(D_t w_n - \sum_{i,j=1}^{N} D_{x_i}(a_{ij} D_{x_j} w_n)\right) + w_n = n^{-1} g$$

where

$$g = D_t \tau \chi u - 2\tau \sum_{i,j=1}^{N} (a_{ij} + a_{ji}) D_{x_i} \chi D_{x_j} u - \tau u \sum_{i,j=1}^{N} D_{x_i}(a_{ij} D_{x_j} \chi) \in L^2(Q) \ ;$$

therefore

(2.5) $$\frac{1}{2}\frac{d}{dt}\|w_n\|^2_{L^2(\Omega)} + a\big(t; u_n(t), w_n(t)\big) \leqslant \int_\Omega g w_n \, dx \, .$$

We observe that

$$\int_Q g w_n \, dxdt \to 0$$

and $$a\big(t; v_n(t), w_n(t)\big) \to 0 \quad \text{in } L^2(0,T) \ .$$

From (2.5) we have $w_n \to 0$ in $L^2(0,T; H^1_0(\Omega))$, then $u_n \to \chi\tau u$ $L^2(0,T; H^1_0(\Omega))$ and the result is proved. ∎

From (2.3) and lemma 2.1 we obtain the following "energy inequality" for local solutions of our problem:

Corollary 2.1. Let u satisfy (2.1), (2.2), (2.3) and let Φ, d be as in (2.3). Then we have

$$\frac{1}{2}\|(u-d)\Phi\|^2_{L^2(\Omega)}(t) - \int_0^t \int_\Omega D_t\Phi\,\Phi\,|u-d|^2 \, dxdt + \int_0^t a\big(s; u(s), (u-d)\Phi^2(s)\big)\, ds \leqslant 0 \ .$$

The remainder of this section, which is not essential for what follows, is devoted to showing that the weak solutions of an a.e. obstacle problem in the sense of Mignot-Puel [10], as well as the weak solution of a q.e. obstacle problem in the sense of Pierre [14], are both local weak solutions according to the definition above.

(a) The a.e. obstacle

Let ψ be q.l.s.c. and such that there exists $v_0 \in L^2(0,T;V) \cap H^1(0,T;V')$, $v_0 \geq \psi$ q.e. in Q, where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ is as in [10]. A function u is a weak solution of the obstacle problem in the sense of Mignot-Puel, if u is the minimum solution of the problem

$$\int_0^T \left[\langle D_t v, v-u\rangle_{V',V} + a(\cdot, u, v-u)\right] dt \geq \frac{1}{2} \|v(0)-u_0\|^2_{L^2(\Omega)} \tag{2.6}$$

$$\forall v \in L^2(0,T;V) \cap H^1(0,T;V'), \quad v \geq \psi \text{ a.e. in } Q$$

$$u \in L^2(0,T;V), \qquad u \geq \psi \quad \text{a.e. in } Q.$$

Under the above assumptions on ψ there exists a Mignot-Puel solution of the a.e. obstacle problem and this is obtained as the limit in $L^2(Q)$ strong and in $L^2(0,T;V)$ weak of the solutions u_ε of the following penalized problems:

$$\langle D_t u_\varepsilon, v\rangle_{V',V} + a(\cdot, u_\varepsilon, v) + \frac{1}{\varepsilon}\int_\Omega \beta(u_\varepsilon) v\, dx = 0 \qquad \forall v \in V$$

(2.6$_\varepsilon$)

$$u_\varepsilon(0) = u_0$$

where $\beta(u_\varepsilon) = -(u_\varepsilon - \psi)^-$.

We now prove that u satisfies (2.1), (2.2), (2.3).

By standard methods we can prove that the solutions u_ε of the penalized problems (2.6$_\varepsilon$) are uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, therefore $u \in L^\infty(0,T;L^2(\Omega))$.

The function u being quasi-continuous, see Appendix, and ψ being q.l.s.c., from $u \geq \psi$ a.e. in Q we obtain $u \geq \psi$ q.e. in Q, hence (2.1) holds.

We observe that u_ε is a supersolution for P in Q, therefore u is also a supersolution of P in Q and (2.2) holds.

Using the approximation lemma 2.2, we obtain

$$(2.7) \qquad \frac{1}{2}\left\|(u-d)^+\Phi\right\|^2_{L^2(\Omega)}(t) - \int_0^t\int_\Omega D_t\Phi\,\Phi\,\left|(u-d)^+\right|^2 dxdt + \int_0^t a\left(s; u(s), (u-d)^+\Phi^2(s)\right) dx \geqslant 0$$

$\forall \Phi \in C_0^\infty(Q)$, $\Phi \geqslant 0$, $d \in R$.

Moreover, by using the approximation by penalization, we have also

$$(2.8) \qquad \frac{1}{2}\left\|(u-d)^+\bar{\Phi}\right\|^2_{L^2(\Omega)}(t) - \int_0^t D_t\bar{\Phi}\,\bar{\Phi}\,\left|(u-d)^+\right|^2 dxdt + \int_0^t a\left(s; u(s); (u-d)^+\bar{\Phi}^2(s)\right) ds \leqslant 0$$

where $\bar{\Phi}$, d are as in (2.3).

From (2.7), (2.8) we have that (2.3) holds for u.

(b) <u>The q.e. obstacle</u>

Here we use the notation of [14].

Let $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ be as in [14]. By $\tau_{u_0} \in L^2(0,T;V) \cap H^1(0,T;V')$ we denote the solution of the Cauchy problem with initial data $u_0 \in L^2(\Omega)$ and by $\mathcal{P}_a$ the class of parabolic potentials relative to $a(t; u, v)$ (for the definition of parabolic potentials, see [14]).

For $u \in \mathcal{P}_a$ we define

$$\hat{u} = \inf q.ess.(P)\left\{v \,\middle|\, v \text{ q.cont.}(P),\ v \geqslant u \text{ a.e.}\right\}$$

where q.ess.(P) (q.cont.(P), q.e.(P)) means q.ess. (q.cont., q.e.) with respect to the capacity introduced by M. Pierre, [14], denoted below by (P)-cap. By $\xi(u)$ we denote the measure associated with the potential u as in Prop. I-1 of [14]. Finally we denote by $\hat{\psi}$ the q.u.s.c. (P) "regularization" of ψ as defined in Prop. IV-2 of [14].

By assuming $u_0 \geqslant \hat{\psi}(0)$ and that there exists $v_0 \in L^2(0,T; V) \cap H^1(0,T; V')$ with $v_0 \geqslant \psi$ q.e., it is proved in [14] that there exists a function $u \in \mathcal{P}_a + \tau_{u_0}$ such that

$$\hat{u} \geqslant \hat{\psi} \text{ q.e. (P)} , \qquad \hat{u}(0) = u_0 , \qquad \hat{u} = \widehat{u - \tau_{u_0}} + \tau_{u_0}$$

$$\int_Q (\hat{u} - \hat{\psi})\, d\xi(u - \tau_{u_0}) = 0$$

and that such a u is unique. We say that u is a weak Pierre solution.

Now let u be a weak Pierre solution of the obstacle problem. Since $u \in \mathcal{P}_a + \tau_{u_0}$, we have $u \in L^\infty(0,T; L^2(\Omega))$. Moreover, since $\hat{u} \geqslant \hat{\psi}$ q.e.(P), it follows from Prop. IV-2 [14] that $u \geqslant \psi$ q.e.. Thus (2.1) holds.

Furthermore, $u \in \mathcal{P}_a + \tau_{u_0}$ is a supersolution, hence (2.2) holds.

Let now Φ be in $C_0^\infty(Q)$ with $\Phi \geqslant 0$ and $d \geqslant \psi$ q.e. on the set $\{z \mid \Phi(z) > 0\}$. We will prove that $d \geqslant \hat{\psi}$ q.e. on $\{z \mid \Phi(z) > 0\}$. The set $\{z \mid \Phi(z) > 0\}$ is open, therefore we can consider a sequence A_n of compact sets such that (P)-cap$\{\{z \mid \Phi(z) > 0\} - A_n\} \to 0$, $A_n \subset Q$. We observe that v_0 is q.cont.(P), hence, from Prop. IV-2 of [14], $v_0 \geqslant \hat{\psi}$ q.e. (P).

Consider now $\eta_n \in C_0^\infty(\{\Phi > 0\})$ with $\eta_n = 1$ in A_n. We have $v_{0,n} = (1-\eta_n)v_0 + \eta_n d \geqslant \psi$ q.e.. Since v_0 is q.cont.(P), then $v_{0,n}$ is also q.cont.(P), therefore by Prop. IV-2 [14], we have $v_{0,n} \geqslant \hat{\psi}$ q.e. (P). Thus $d \geqslant \psi$ q.e. (P) in A_n. It follows that $d \geqslant \psi$ q.e. (P) on $\{z \mid \Phi(z) > 0\}$.

We remark that $\xi(u - \tau_{u_0})$ is a positive measure, then, by taking into account Coroll. I-2 of [14], we obtain

$$\int_Q (\hat{u}-d)^+\phi^2 d\xi(u-\tau_{u_0}) \geqslant 0 .$$

Moreover,

$$\int_Q (\hat{u}-d)^+\phi^2 d\varepsilon(u-\tau_{u_0}) \leqslant \int_Q (\hat{u}-\hat{\psi})\phi^2 d\xi(u-\tau_{u_0}) \leqslant C\int_Q (\hat{u}-\hat{\psi})\, d\xi(u-\tau_{u_0}) = 0 ,$$

therefore

$$\int_Q (\hat{u}-d)^+\phi^2 d\xi(u-\tau_{u_0}) = 0 .$$

We consider now $w \in \mathcal{P}_a + \tau_{u_0}$ with $D_t w \in L^2(0,T,V')$. From Prop. II-1 of [14] we have

$$(2.9) \qquad \int_0^t \Big(\big\langle D_t w, (w_n-d)^+\phi^2 \big\rangle_{V',V} + a\big(s; w(s), (w_n-d)^+\phi^2(s)\big)\Big)\, ds$$

$$= \int_0^t \int_\Omega (w_n-d)^+\phi^2 d\xi(w-\tau_{u_0})$$

where $w_n \in L^2(0,T;V) \cap H^1(0,T;L^2(\Omega))$, $w_n \to w$ in $L^2(0,T;V) \cap H^1(0,T;V')$. Going to the limit as $n \to +\infty$ in (2.9), after some easy computations we obtain

$$(2.10) \qquad \frac{1}{2}\left\|(w-d)^+\phi\right\|^2_{L^2(\Omega)}(t) - \int_0^t\int_\Omega D_t\phi\,\phi\left|(w-d)^+\right|^2 dxdt +$$

$$+ \int_0^t a\big(s, w(s), (w-d)^+\phi^2(s)\big)\, ds = \int_0^t\int_\Omega (w-d)^+\phi^2 d\xi(w-\tau_{u_0})$$

Using now the approximation result of Prop. II-7 [14], and the lemma II-5 [14] we can prove that (2.10) holds also for w q.cont.(P).

From the Remark of p. 1192 [14], we can now consider a sequence $w_n \in \mathcal{P}_a + \tau_{u_0}$ of quasi-continuous functions, such that $\hat{w}_n \downarrow \hat{u}$ q.e.(P), where $\hat{w}_n = \widehat{(w_n - \tau_0)} + \tau_0$. For every fixed p, we have

$$0 \leqslant \limsup_{n \to +\infty} \int_0^t \int_\Omega (\hat{w}_n - d)^+ \phi^2 \, d\xi (w_n - \tau_{u_0}) \leqslant \limsup_{n \to +\infty} \int_0^t \int_\Omega (\hat{w}_p - d)^+ \phi^2 \, d\xi (w_n - \tau_{u_0}) .$$

Therefore from lemma II-5, [14], we obtain

$$\int_0^t \int_\Omega (\hat{w}_p - d)^+ \phi^2 \, d\xi (u - \tau_{u_0}) \geqslant \limsup_{n \to +\infty} \int_0^t \int_\Omega (\hat{w}_n - d)^+ \phi^2 \, d\xi (w_n - \tau_{u_0}) \quad .$$

From the monotone convergence theorem, we have

$$0 \leqslant \liminf_{n \to +\infty} \int_0^t \int_\Omega (\hat{w}_n - d)^+ \phi^2 d\xi (w_n - \tau_{u_0}) \leqslant \limsup_{n \to +\infty} \int_0^t \int_\Omega (\hat{w}_n - d)^+ d\xi (w_n - \tau_{u_0}) \leqslant$$

$$\leqslant \lim_{p \to +\infty} \int_0^t \int_\Omega (\hat{w}_p - d)^+ \phi^2 \, d\xi (u - \tau_{u_0}) = \int_0^t \int_\Omega (\hat{u} - d)^+ \phi^2 \, d\xi (u - \tau_{u_0}) = 0 \;,$$

then

$$\lim_{n \to +\infty} \int_0^t \int_\Omega (\hat{w}_n - d)^+ \phi^2 \, d\xi (w_n - \tau_{u_0}) = 0 \;.$$

We observe that from Th. III-2 [14], we can suppose $w_n \downarrow u$ in $L^2(Q)$ and in $L^2(0, T; V)$ weakly. It follows then from (2.7) and (2.10) that (2.3) holds for u, and the result is proved. ∎

3. A Wiener Modulus for ψ

Let a function $\psi : R^{N+1} \to [-\infty, +\infty)$ be given, defined up to sets of capacity zero (q.e.) in R^{N+1}.

Let $z_0 = (x_0, t_0)$ be a fixed point in R^{N+1}.

Let $\theta \in (0, 1/6)$ be given; for arbitrary $\varepsilon > 0$ and $r > 0$ we consider the level sets

$$(3.1) \qquad E(\varepsilon, r) = E_\theta(\psi, z_0, \varepsilon, r) = \Big\{ z = (x,t) \in Q_\theta^-(z_0, r) \,\Big|\, \psi(x,t) \geqslant \geqslant \sup_{B(x_0, r/2) \times (t,\, t_0 + r^2/4)} \psi - \varepsilon \Big\},$$

where the supremum is taken in the q.e. sense, and the relative capacities

$$(3.2) \qquad \delta(\varepsilon r) = \delta_\theta(\psi, z_0, \varepsilon, r) = \frac{\Delta(\varepsilon, r)}{\sigma_N R^N}$$

where

$$(3.3) \qquad \Delta(\varepsilon, r) = \Delta_\theta(\psi, z_0, \varepsilon, r) = \operatorname{cap}_{Q(z_0, 2r)} E(\varepsilon, r)$$

and $\sigma_N = \operatorname{cap}_{Q(0,2)} Q(0,1)$.

We then define the function

$$\omega_\sigma(r, R) = \omega_\sigma(\psi, z_0; r, R)$$

for arbitrary $\sigma > 0$ and $0 < r \leqslant R$ by setting

$$(3.4) \qquad \omega_\sigma(r, R) = \inf\left\{ \omega > 0 \,\middle|\, \omega \exp \int_r^R \delta(\sigma\omega, \rho) \frac{d\rho}{\rho} \geqslant 1 \right\}.$$

This is well defined, since $\delta(\varepsilon, \rho)$ is not decreasing in ε, and we have

$$r/R \leqslant \omega_\sigma(r, R) \leqslant 1$$

We call $\omega_\sigma(r, R)$ the Wiener modulus of ψ at the point z_0.

Remark 3.1. In some applications, as for instance to monotone obstacles ψ, i.e., $\psi(x,t) \geqslant \psi(x',t')$ whenever $x \leqslant x'$ in R^N and $t \leqslant t'$, it is convenient to introduce an additional scaling factor $m \geqslant 1$ in the above definition of the Wiener modulus. This is done by replacing the ball $B(x_0, r/2)$ in the definition (3.1) of the level sets $E(\varepsilon, r)$ by a ball $B(x_0, r/m)$. This factor m is fixed together with ψ and z_0 and is independent of $\theta, \varepsilon, r, R$. ∎

Let $F = \{\psi > -\infty\}$. With notation

$$Q^F(z_0,\rho) = Q_\theta^-(z_0,\rho) \cap F \quad \text{if} \quad \mathrm{cap}_{Q(z_0,2\rho)}\left(Q(z_0,\rho) \cap F\right) > 0$$

$$Q^F(z_0,\rho) = Q_\theta^-(z_0,\rho) \qquad \text{if} \quad \mathrm{cap}_{Q(z_0,2\rho)}\left(Q(x_0,\rho) \cap F\right) = 0$$

$\rho > 0$, we define for arbitrary $0 < r \leqslant R$

$$W(r,R) = W_F(r,R) = \exp\left(-\int_r^R \frac{\mathrm{cap}_{Q(z_0,2\rho)}\left(Q^F(z_0,\rho)\right)}{\sigma_N \rho^N} \frac{d\rho}{\rho}\right) .$$

Then the following estimate of $\omega_\sigma(r,R)$ holds, which can be proved as in the elliptic case, see [13].

Proposition 3.1. <u>The Wiener modulus</u> $\omega_\sigma(r,R)$ <u>of</u> ψ <u>at</u> z_0 <u>satisfies the estimate</u>

$$(3.5) \qquad r/R \leqslant \omega_\sigma(r,R) \leqslant \min\left\{1, \max\left[W(r,R), \sigma^{-1} \mathrm{osc}_{Q(z_0,R) \cap F}\, \psi\right]\right\} .$$

Let us consider some important special cases.

Let $\psi = -\infty$ on some neighborhood of z_0 in R^{N+1}; then

$$\omega_\sigma(r,R) = r/R .$$

Moreover, if ψ is continuous at z_0 by choosing

$$\sigma = \frac{R}{r} \mathrm{osc}_{Q(z_0,R) \cap F}\, \psi$$

we obtain again

$$\omega_\sigma(r,R) = r/R .$$

In the case $\delta(\varepsilon, \rho) = \delta(\rho)$ we have easily from the definition of the Wiener modulus

$$\omega_\sigma(r,R) = \exp\left(-\int_r^R \delta(\rho)\, \frac{d\rho}{\rho}\right) .$$

Finally, if ψ is continuous on F, as in the case of "thin obstacles," and z_0 is such that $\operatorname{cap}_{Q(z_0, 2R)}\left(Q(z_0,\rho) \cap F\right) > 0$ for every $\rho > 0$, then by choosing

(3.6) $$\sigma = W_F(r,R)^{-1} \operatorname{osc}_{Q(z_0,R) \cap F} \psi$$

we obtain from (3.5)

(3.7) $$\omega_\sigma(r,R) \leqslant W_F(r,R) .$$

4. The Wiener Estimates and the Results

In this section we consider a given "obstacle" ψ, that is, a function

$$\psi : R^{N+1} \longrightarrow [-\infty, +\infty) \ , \tag{4.1}$$

defined up to sets of capacity zero in R^{N+1}, and we fix arbitrarily a point

$$z_0 = (x_0, t_0) \in R^{N+1} \ .$$

We then consider an arbitrary local solution u in a cylinder $Q = Q(z_0, R_0)$ as defined by (2.1), (2.2), (2.3).

Our goal is to estimate the quantity

$$V(r) = V(u, z_0; r) = \operatorname{osc}_{Q(z_0, r)} u + \left(\int_{Q(z_0, r)} |Du|^2 G^{z_0} \, dxdt \right)^{1/2}$$

as $r \longrightarrow 0$ in terms of the Wiener modulus of ψ at a given z_0. We shall prove the following results.

Theorem 4.1. _There exists_ $\theta_0 \in (0, 1/6)$ _such that for every_ $\theta \in (0, \theta_0)$ _and every_ $\sigma > 0$

$$V(r) \leq CV(R)\omega_\sigma(r,R)^\beta + k\sigma\omega_\sigma(r,R), \quad 0 < r \leq \theta R, \quad R < R_0,$$

where C _is an absolute constant,_ θ_0 _and_ k _are constants depending only on_ N, λ, Λ _and_ $\beta > 0$ _is a constant depending only on_ N, λ, Λ _and_ θ.

In particular we remark that the constants in Th. 1 do not depend on ψ, z_0, Q, u and they depend on the coefficients a_{ij} only by the ellipticity constant and $\sup_Q |a_{ij}|$. Let us point out that $\omega_\sigma(r,R)$ depends on θ, _via_ the sets (3.1).

The proof of Th. 1 is given in Sec. 7.

Let $Z(R)$ be defined by

$$Z(R) = \inf\left\{ \left(R^{-(N+2)} \int_{Q(z_0;R)} |u - d|^2 \, dxdt \right)^{1/2} ; \ d \geq \psi \text{ in } Q(z_0;R) \right\}.$$

Theorem 4.2. There exist constants c and K such that

$$V(cR) \leqslant K\,Z(R) \;,$$

in particular

$$V(R) \leqslant K\Big\{R_0^{-(N+2)/2}\|u\|_{L^2(Q(z_0,R_0))} + \max\big(\operatorname*{supess}_{Q(z_0,R_0)} \psi, 0\big)\Big\}$$

$R \leqslant cR_0$, where the constants c and K depend only on N, λ, Λ.

The proof of Th. 4.2 is given in Sec. 5.

Let F be the set $\{\psi > -\infty\}$ and $W(r,R) = W_F(r,R)$ as in Sec. 3; the following result is an easy consequence of Th. 4.1 and (3.6), (3.7) of Sec. 3.

Corollary 4.1. We have, with the same constants C, θ_0, k and β as in Th. 4.1,

$$V(r) \leqslant CV(R)W(r,R)^{\beta} + k\operatorname{osc}_{Q(z_0,R)\cap F}\psi \;, \qquad 0 < r \leqslant \theta R \;, \quad R \leqslant R_0 \;,$$

for every $0 < \theta < \theta_0$.

Remark 4.1. The Coroll. 4.1 can be applied in particular to the case of the equations ($\psi = -\infty$), of parabolic potentials of a given set F ($\psi = 1$ in F and $\psi = -\infty$ in $R^{N+1} \setminus F$) and of obstacles which are continuous at z_0 on F.

The condition in the following corollary extends an analogue condition considered in [7] and can be applied in particular to the case of obstacles which are decreasing in t, and monotone in x as in [7].

Corollary 4.2. Suppose there exist constants $\theta \in (0,\theta_0)$, $\alpha > 0$ such that

$$\operatorname{cap}_{Q(z_0,2\rho)}\big(E(\omega,\rho)\big) \geqslant \alpha \operatorname{cap}_{Q(z_0,2\rho)}\big(Q(z_0,\rho)\big)$$

for every $\rho, \omega > 0$; then there exists a constant $\gamma = \gamma(N,\lambda,\Lambda,\theta,\alpha) > 0$ such that

$$V(r) \leqslant CV(R_0)(r/R_0)^{\gamma}$$

for every $r \leqslant \theta R_0$; in particular u is Hölder continuous at z_0.

Finally from Theorems 4.1 and 4.2 we obtain the following result for local solutions of parabolic equations, where we define

$$A_q(z_0) = \{(x,t) \mid t_0 - t \geqslant q|x-x_0|^2\}, \quad q > 0 ;$$

Corollary 4.3. Let u be a local solution of the parabolic equation $Pu = 0$ relative to the operator P in $Q(z_0, R_0)$, then we have

$$\left(\int_{Q(z_0,r)\cap A_q(z_0)} |D_x u|^2 \, dxdt\right)\left(\int_{Q(z_0,R)} |D_x u|^2 \, dxdt\right)^{-1} \leqslant c(r/R)^{N+\beta} ,$$

$$0 < r < 2\theta^2 R$$

where θ and β are as in Th. 4.1 and c is a constant depending on $q, N, \lambda, \Lambda, \theta$.

We will prove this result in Sec. 8.

5. Estimates of Caccioppoli - De Giorgi Type

In this section we consider an arbitrary local solution, that is, a function satisfying (2.1), (2.2), (2.3) of Sec. 2; moreover, we fix a point $z_0 = (x_0, t_0) \in Q$. Since u is a (local) supersolution, u is locally bounded from below in Q. Moreover, since ψ is assumed to be q.e. bounded from above in Q, u is also locally bounded in Q, as can be shown by a simple adaptation of the truncation method for equations, see, for instance, [9]. Therefore u is locally bounded in Q.

We shall prove now the following:

Lemma 5.1. Let $d \geqslant \psi$ in $B(x_0, R/2) \times (t_0 - 6\theta R^2, t_0 + R^2/4)$ q.e. Then for $\bar{z} \in Q(z_0, R/4)$, $R \leqslant R_0$, the following estimate holds for a suitable $\sigma > 0$:

$$\int_{\bar{t}-3\theta R^2}^{\bar{t}} \int_{B(\bar{x}, R/8)} |D_x u|^2 G^{\bar{z}} \,dxdt + |u-d|^2(\bar{z}) \leqslant$$

$$\leqslant CR^{-2} \int_{\bar{t}-3\theta R^2}^{\bar{t}} \int_{B(\bar{x}, R/3) \setminus B(\bar{x}; R/16)} |u-d|^2 \cdot \Big(G^{\bar{z}} + \sigma R^{-N/2} (G^{\bar{z}})^{1/2} + \sigma^{-1} R^{N/2} (G^{\bar{z}})^{3/2} + \sigma R^{-N}\Big) dxdt$$

$$+ \theta^{-1} CR^{-2} \int_{\bar{t}-5\theta R^2}^{\bar{t}-3\theta R^2} \int_{B(R/4; \bar{x})} |u-d|^2 G^{\bar{z}} \,dxdt$$

where $\theta \in (0, 1/6)$ and the constant C depends only on N, λ, Λ.

Let $\eta = \eta(x)$ and $\tau = \tau(t)$ be such that

$$\eta = 1 \qquad x \in B(\bar{x}; R/8)$$

$$\eta = 0 \qquad x \notin B(\bar{x}; R/4)$$

$$0 \leqslant \eta \leqslant 1 \qquad x \in B(\bar{x}; R/4)$$

$$\eta \in C_0^{\infty}(R^N)$$

$$|D_x \eta| \leqslant C/R$$

$$\tau = 1 \qquad t \geqslant \bar{t} - 3\theta R^2$$

$$\tau = 0 \qquad t \leqslant \bar{t} - 5\theta R^2$$

$$0 \leqslant \tau \leqslant 1 \qquad t - 5\theta R^2 \leqslant t \leqslant t - 3\theta R^2$$

$$\tau \in C_0^{\infty}(R)$$

$$|D_t \tau| \leqslant C\theta R^{-2}$$

By replacing $\phi = \eta^2 \tau^2 G_\rho^{\bar{z}}$ with $\rho < \theta R^2$, and d as above into (2.3), by Corollary 2.1 we have

(5.1)
$$\int_{Q(z,R)} |D_x u|^2 \eta^2 \tau^2 G_\rho^{\bar{z}}\, dxdt + \frac{1}{|Q(\bar{z},\rho)|} \int_{Q(\bar{z},\rho)} |u-d|^2 \eta^2 \tau^2\, dxdt$$

$$\leqslant C \left\{ \int_{\bar{t}-5\theta R^2}^{\bar{t}+\rho^2} \int_{B(\bar{x},R/4)} |u-d|^2 |D_x \eta| G_\rho^{\bar{z}}\, dxdt + \right.$$

$$+ \int_{\bar{t}-R^2}^{\bar{t}+\rho^2} \int_{B(\bar{x},R/4)} |u-d|^2 \eta \tau^2 |D_x \eta| \left| D_x G_\rho^{\bar{z}} \right| dxdt +$$

$$\left. + \int_{\bar{t}-5\theta R^2}^{\bar{t}-3\theta R^2} \int_{B(\bar{x},R/4)} |u-d|^2 |D_t \tau| G_\rho^{\bar{z}}\, dxdt \right\} ,$$

where by C we denote constants which depend only on N, λ, Λ. Passing to the limit as $\rho \to 0$ we have for q.e. $\bar{z}$

$$(5.2)\qquad \int_{Q(\bar z,R)} |D_x u|^2 \eta^2 \tau^2 G^{\bar z} + |u-d|^2(\bar z) \leqslant$$

$$\leqslant C\Big\{R^{-2}\int_{\bar t-5\theta R^2}^{\bar t}\int_{B(\bar x,R/4)-B(\bar x,R/8)} |u-d|^2 G^{\bar z}\,dxdt +$$

$$+R^{-2}\theta^{-1}\int_{\bar t-5\theta R^2}^{\bar t-3\theta R^2}\int_{B(\bar x,R/4)} |u-d|^2 G^{\bar z}\,dxdt +$$

$$+\int_{\bar t-R^2}^{\bar t}\int_{B(\bar x,R/4)} |u-d|^2 \eta\tau^2 |D_x\eta|\,|D_x G^{\bar z}|\,dxdt\Big\}.$$

Let us consider now the last term on the right hand side of (5.2); we have

$$(5.3)\qquad \int_{\bar t-R^2}^{\bar t}\int_{B(\bar x,R/4)} |u-d|^2 \eta\tau^2 |D_x\eta|\,|D_x G^{\bar z}|\,dxdt \leqslant$$

$$\leqslant \sigma R^{-N/2}\int_{\bar t-R^2}^{\bar t}\int_{B(\bar x,R/4)\setminus B(\bar x,R/8)} \eta^2\tau^2 |u-d|^2 |D_x G^{\bar z}|^2 (G^{\bar z})^{-3/2}\,dxdt$$

$$+ 4\sigma^{-1}R^{+\frac{N}{2}-2}\int_{\bar t-5\theta R^2}^{\bar t}\int_{B(R/4;\,\bar x)-B(R/8;\,\bar x)} |u-d|^2 (G^{\bar z})^{3/2}\,dxdt$$

where $\sigma > 0$.

The term containing $|D_x G^{\bar z}|^2$ can be estimated by Lemma 5.2 below; we then obtain

$$(5.4)\qquad \int_{\bar t-R^2}^{\bar t}\int_{B(\bar x,R/4)-B(\bar x,R/8)} |u-d|^2 \eta^2\tau^2 |D_x G^{\bar z}|^2 (G^{\bar z})^{-3/2}\,dxdt \leqslant$$

$$\leqslant C\Big[R^{-2}\int_{\bar t-R^2}^{\bar t}\int_{B(\bar x,R/3)-B(\bar x,R/16)} |u-d|^2 (G^{\bar z})^{1/2}\,dxdt +$$

$$+ R^{N/2} \int_{\bar t - R^2}^{\bar t} \int_{B(\bar x, R)} |D_x u|^2 \eta^2 \tau^2 G^{\bar z}\, dxdt +$$

$$+ R^{-(N/2+2)} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(\bar x, R/3) - B(\bar x, R/16)} |u - d|^2\, dxdt \quad .$$

From (5.2), (5.3), (5.4), by choosing $\sigma > 0$ suitably we have

$$(5.5) \qquad \int_{\bar t - R^2}^{\bar t} \int_{B(\bar x, R)} |D_x u|^2 \eta^2 \tau^2 G^{\bar z}\, dxdt + |u-d|^2(\bar z) \leqslant$$

$$\leqslant C \Bigg[R^{-2} \int_{\bar t - 3\theta R^2}^{\bar t} \int_{B(\bar x, R/3) - B(\bar x, R/16)} |u-d|^2 \Big(G^{\bar z} + \sigma R^{-N/2} (G^{\bar z})^{1/2} +$$

$$+ \sigma^{-1} R^{N/2} (G^{\bar z})^{3/2} + \sigma R^{-N} \Big) dxdt +$$

$$+ \theta^{-1} R^{-2} \int_{\bar t - 5\theta R^2}^{\bar t - 3\theta R^2} \int_{B(\bar x, R/4)} |u-d|^2 \, G^{\bar z}\, dxdt \quad .$$

The result of Lemma 5.1 can be easily obtained from (5.5). ∎

Let now ω be such that

$$\omega = \tilde\omega \eta$$

where $\tilde\omega \in C_0^\infty(B(\bar x, R))$

$\tilde\omega = 0$ in $B(\bar x, R/16)$ and for $x \notin B(\bar x, 5R/16)$

$\tilde\omega = 1$ in $B(\bar x, R/4) - B(\bar x, R/8)$

$0 \leqslant \tilde\omega \leqslant 1.$

We complete the proof of Lemma 5.1 by proving the following

Lemma 5.2. Let the conditions of Lemma 5.1 be satisfied. Then the following estimate holds:

$$\int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},R)} \omega^2 \tau^2 |u-d|^2 |D_x G^{\bar{z}}|^2 (G^{\bar{z}})^{-3/2} dxdt$$

$$\leq C \left[R^{-2} \int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},R/4) - B(\bar{x},R/16)} \tau^2 [u-d]^2 (G^{\bar{z}})^{1/2} dxdt \right.$$

$$+ R^{N/2} \int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},R)} |D_x u|^2 G^{\bar{z}} \eta^2 \tau^2 dxdt +$$

$$+ R^{-(N/2+2)} \int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},5R/16) - B(\bar{x},R/16)} |u-d|^2 \tau^2 dxdt \quad .$$

From the definition of the regularized Green's function we have easily, for $\rho < 1/16$,

$$\left\langle -D_t G_\rho^{\bar{z}} - \sum_{i,j=1}^{N} D_{x_j}(a_{ij} D_{x_i} G_\rho^{\bar{z}}),\ \omega^2 \tau^2 ({}_\delta G_\rho^{\bar{z}})^{1/2} |u-d|^2 \right\rangle = 0$$

where ${}_\delta G_\rho^{\bar{z}} = G_\rho^{\bar{z}} + \delta,\ \ \delta > 0.$

Then, u being a local solution of our obstacle problem, we have, after some computations,

$$(5.6)\quad \int_{\bar{t}-R}^{\bar{t}-\delta} \int_{B(\bar{x},R)} ({}_\delta G_\rho^{\bar{z}})^{1/2} D_t \tau \pi \omega^2 |u-d|^2 dxdt - \left\| ({}_\delta G_\rho^{\bar{z}})^{1/2} \omega^2 |u-d|^2 \right\|_{L^2(B(\bar{x},R))} (\bar{t}-\delta) -$$

$$- \frac{1}{2} \sum_{ij=1}^{N} \int_{\bar{t}-R^2}^{\bar{t}-\delta} \int_{B(\bar{x},R)} a_{ij} D_{x_j} G_\rho^{\bar{z}} D_{x_i} G_\rho^{\bar{z}} ({}_\delta G_\rho^{\bar{z}})^{-3/2} \omega^2 \tau^2 |u-d|^2 dxdt +$$

$$+\sum_{i,j=1}^{N}\int_{\bar t-R^2}^{\bar t-\delta}\int_{B(\bar x,R)}\Big\{2a_{ij}D_{x_i}G_\rho^{\bar z}D_{x_i}\omega\omega\tau^2|u-d|^2({}_\delta G_\rho^{\bar z})^{-1/2}-$$

$$-2a_{ij}({}_\delta G_\rho^{\bar z})^{1/2}D_{x_i}\omega D_{x_j}(u-d)\omega(u-d)\tau\Big\}dxdt\geqslant 0\quad .$$

From (5.6), by similar estimates as in the Appendix of [5], the result follows. ■

We now prove the following

Lemma 5.3. Let the conditions of Lemma 5.1 hold and $\theta\in(0,\theta_0]$; then we have

$$\int_{t_0-\theta R^2}^{t_0}\int_{B(x_0,\theta^{1/2}R)}|D_x u|^2 G^{z_0}dxdt+\sup_{Q(z_0,\theta^{1/2}R)}|u-d|^2\leqslant$$

$$\leqslant C\exp(C\theta^{-1})\,\theta^{-3N/4}R^{-(N+2)}\int_{Q(z_0,R)}|u-d|^2dxdt+$$

$$+C\theta^{-(3N/4+1)}R^{-(N+2)}\int_{t_0-6\theta R^2}^{t_0-2\theta R^2}\int_{B(x_0,3/8R)}|u-d|^2dxdt$$

where C, θ_0 are constants dependent on N, λ, Λ.

We obtain the result by estimating $G^{\bar z}$ as in Sec. 1 and taking the supremum for $\bar z\in Q(z_0,\theta^{1/2}R)$.

From Lemma 5.3 the inequality of Th. 2 follows by choosing $\theta=\theta_0$.

From Lemma 5.3 we also obtain

$$(5.7)\quad\int_{Q(z_0,\theta^{1/2}R)}|D_x u|^2G^{z_0}dxdt+\sup_{Q(z_0,\theta^{1/2}R)}|u-d|^2\leqslant$$

$$\leqslant C\exp(C\theta^{-1})\theta^{-3N/4}\sup_{Q(z_0,R)}|u-d|^2+$$

$$+ C\theta^{-(\frac{3N}{4}+1)} R^{-(N+2)} \int_{\bar{t}-6\theta R^2}^{\bar{t}-2\theta R^2} \int_{B(x_0, 3/8R)} |u-d|^2 \, dxdt \quad .$$

We choose now $\hat{d}$ such that $\inf_{Q(z_0,R)} u \leqslant \hat{d} \leqslant \sup_{Q(z_0,R)} \hat{u}$ and $d = \hat{d}+\varepsilon$, $\varepsilon > 0$ ($\hat{d}$ can be chosen as in Sec. 6 below). We observe that

$$(5.8) \qquad \sup_{Q(z_0,\theta^{1/2}R)} |u-d|^2 \geqslant \frac{1}{4}\Big(\operatorname*{osc}_{Q(z_0,\theta^{1/2}R)} (u-\hat{d})\Big)^2 - C\varepsilon^2 \geqslant$$

$$\geqslant \frac{1}{4}\Big(\operatorname*{osc}_{Q(z_0,\theta^{1/2}R)} u\Big)^2 - C\varepsilon^2$$

$$(5.9) \qquad \sup_{Q(z_0,R)} |u-d|^2 \leqslant 2\Big(\operatorname*{osc}_{Q(R,z_0)} (u-d)\Big)^2 + C\varepsilon^2 \ .$$

Then from (5.7) we obtain

Lemma 5.4. <u>Let us consider a constant</u> $d = \hat{d}+\varepsilon$ <u>with</u> $\inf_{Q(z_0,R)} u \leqslant \hat{d} \leqslant \sup_{Q(z_0,R)} u$, $\varepsilon > 0$ <u>and let us suppose that</u> d <u>satisfies in addition the assumptions of</u> <u>Lemma</u> 5.1. <u>Then if</u> $\theta \in (0,\theta_0)$, <u>we have</u>

$$(5.10) \qquad \int_{Q(z_0,\theta^{1/2}R)} |D_x u|^2 G^{z_0} \, dxdt + \Big(\operatorname*{osc}_{Q(z_0,\theta^{1/2}R)} u\Big)^2 \leqslant$$

$$\leqslant C\exp(C\theta^{-1})\,\theta^{-3N/4}\Big(\operatorname*{osc}_{Q(z_0,R)} u\Big)^2 +$$

$$+ C\theta^{-(\frac{3N}{4}+1)} R^{-(N+2)} \int_{t_0-6\theta R^2}^{t_0-2\theta R^2} \int_{B(x_0, 3/8R)} |u-\hat{d}|^2 \, dxdt + K_1(\theta)\varepsilon^2$$

<u>where</u> C <u>are constants depending only on</u> N, λ, Λ <u>and</u> $K_1(\theta)$ <u>depends also on</u> <u>on</u> θ.

6. A Poincaré's Inequality for Local Solutions

In this section we will prove a Poincaré's inequality for a local solution, involving only the spatial gradient: this inequality will be used in Sec. 7 in the proof of Th. 4.1.

Let u be a local solution, satisfying (2.1), (2.2), (2.3) of Sec. 2 in $B(x_0,R)\times(t_0-R^2, t_0+R^2)$ and $\eta\in C_0^\infty(B(x_0,R/2))$, $0\leq\eta\leq 1$, $\eta=1$ in $B(x_0,3/8\,R)$, $|D_x\eta|\leq CR^{-1}$ and denote

$$E(\psi, z_0, \mathcal{E}, R) = E$$

$$E_t = E\cap\{t\}$$

(The set E_t is defined a.e. in t and, E being a Borel set, E_t is a Borel set in R^N.)

We choose now $K(t)\in\left[\inf_{E_t} u, \sup_{E_t} u\right]$ to be a measurable function such that if $E_t^1=\left\{(x,t)\,|\,(u(x,t)-K(t))^+=0\right\}$ and $E_t^2=\left\{(x,t)\,|\,(u(x,t)-K(t))^-=0\right\}$, we have

$$\text{(6.1)}\qquad N\text{-}\mathrm{cap}\,E_t^1\geq\frac{1}{4}N\text{-}\mathrm{cap}\,E_t\qquad\qquad N\text{-}\mathrm{cap}\,E_t^2\geq\frac{1}{4}N\text{-}\mathrm{cap}\,E_t$$

a.e. in t, where N-cap is the Newtonian capacity in R^N (the proof of the existence of such a function can be obtained by the same methods used in [8] for the elliptic case, taking into account that u is quasi-continuous). From (2.3) we have

$$\text{(6.2)}\qquad \sup_{s\in(t_0-6\theta R^2,\, t_0-\theta R^2)}\left\|(u(s)-K(t))^+\eta\right\|^2_{L^2}\leq$$

$$\leq C\left[\left\|(u(t)-K(t))^+\right\|^2_{L^2(B(x_0,R/2))}+\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)}|D_x u|^2\,dx\,ds+\varepsilon^2\right]$$

where C is a constant depending only on N, λ, Λ and $t\in(t_0-R^2, t_0-6\theta R^2)$.

By Lemma 2.1 we have also

$$(6.3)\qquad \sup_{s\in(t_0-6\theta R^2,\ t_0-\theta R^2)} \left\|(u(s)-K(t))^-\eta\right\|^2_{L^2} \leqslant$$

$$\leqslant C\left[\left\|(u(t)-K(t))^-\right\|^2_{L^2(B(x_0,R/2))} + \int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxds + \varepsilon^2\right].$$

We apply now in (6.2) and (6.3) the elliptic Poincaré's inequality, [7], and we obtain

$$(6.4_1)\qquad \sup_{s\in(t_0-6\theta R^2,\ t_0-\theta R^2)} \left\|(u(s)-K(t))^+\right\|^2_{L^2(B(x_0,3/8R))} \leqslant$$

$$\leqslant C\left[\frac{R^{2-N}}{N\text{-cap}\,E^1_t}\left\|D_x u\right\|^2_{L^2(B(x_0,R))}(t) + \int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxds + \varepsilon^2\right]$$

$$(6.4_2)\qquad \sup_{s\in(t_0-6\theta R^2,\ t_0-\theta R^2)} \left\|(u(s)-K(t))^-\right\|^2_{L^2(B(x_0,R/8))} \leqslant$$

$$\leqslant C\left[\frac{R^{2-N}}{N\text{-cap}\,E^2_t}\left\|D_x u\right\|^2_{L^2(B(R;\,x_0))}(t) + \int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxds + \varepsilon^2\right]$$

where C is a constant depending only on N, λ, Λ.

We consider now $\hat{d}$ defined by

$$\hat{d} = \sup\left\{K \;\middle|\; \int_{\{t\in(t_0-R^2,\ t_0-6\theta R^2),\ K(t)\leqslant K\}} N\text{-cap}\,E_t\,dt \geqslant \frac{1}{4}\,\text{cap}\,E\right\}.$$

By the definition of $\hat{d}$ and by remarking that $K(t)$ and $N\text{-cap}\,E_t$ are measurable in t (see Appendix), we have

$$\int_{\{t\in(t_0-R^2,\,t_0-6\,\theta R^2),\ K(t)\leqslant\hat{d}\}} N\text{-cap}\,E_t \geqslant \frac{1}{4}\,\text{cap}\,E$$

$$\int_{\{t\in(t_0-6\theta R^2,\ t_0-\theta R^2),\ K(t)\geqslant \hat d\}} N\text{-cap}\,E_t \geqslant \frac{1}{4}\ \text{cap}\,E\ .$$

We integrate now (6.4_1) on the set $\{t\in(t_0-R^2,\ t_0-6\theta R^2),\ K(t)\leqslant \hat d\}$; we find

$$(6.5)\qquad \sup_{s\in(t_0-6\theta R^2,\ t_0-\theta R^2)} \|(u(s)-\hat d)^+\|^2_{L^2(B(x_0,\ 3/8R))} \leqslant$$

$$\leqslant C\left\{\delta_\theta(\xi,R)^{-1}\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxds + \xi^2\right\}.$$

Analogously from (6.4_2) we obtain

$$(6.6)\qquad \sup_{s\in(t_0-6\theta R^2,\ t_0-\theta R^2)} \|(u(s)-d)^-\|^2_{L^2(B(x_0,3/8R))} \leqslant$$

$$\leqslant C\left[\delta_\theta(\xi,R)^{-1}\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxds + \xi^2\right\}.$$

From (6.5) and (6.6) we then have:

Proposition 6.1. Let $\varepsilon > 0$ be arbitrary. Then there exists a constant $\hat d$ such that

$$\hat d \geqslant \psi(x,t) - \varepsilon \quad \text{in}\ B(x_0,R/2)\times(t_0-6\theta R^2,\ t_0+R^2/4)$$

and

$$\int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,\ 3/8R)} (u-\hat d)^2\,dxdt \leqslant$$

$$\leqslant C\left[R^2\delta_\theta(\varepsilon,R)^{-1}\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxdt + \varepsilon^2\right]$$

where C is a constant depending only on $N,\ \lambda,\ \Lambda$.

7. Proof of Theorem 4.1

We now give the proof of Theorem 4.1.

By Lemma 5.4, we know that u satisfies the inequality (5.10). We now apply the Poincaré's inequality, as given in Prop. 6.1 and we obtain

$$(7.1)\qquad \int_{Q(z_0,\theta^{1/2}R)} |D_x u|^2 G^{z_0}\,dxdt + \Big(\operatorname*{osc}_{Q(z_0,\theta^{1/2}R)} u\Big)^2 \leqslant$$

$$\leqslant C\exp(-C\theta^{-1})\,\theta^{-3N/4}\Big(\operatorname*{osc}_{Q(z_0,R)} u\Big)^2 +$$

$$+ C\theta^{-(\frac{3N}{4}+1)} \frac{1}{R^N \delta_\theta(\xi,R)} \int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dxdt + K_1(\theta)\,\xi^2$$

where C are constants depending only on N, λ, Λ.

Now, by using the estimates on the Green's function [1], we obtain the following estimate:

$$(7.2)\qquad \int_{Q(z_0,\theta^{1/2}R)} |D_x u|^2 G^{z_0}\,dxdt + \Big(\operatorname*{osc}_{Q(z_0,\theta^{1/2}R)} u\Big)^2 \leqslant$$

$$\leqslant C_1 K_2(\theta)\Big(\operatorname*{osc}_{Q(z_0,R)} u\Big)^2 + \frac{1}{C_2 K_3(\theta)\,\delta_\theta(\xi,R)} \int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2 G^{z_0}\,dxdt +$$

$$+ C_3 K_4(\theta)\,\xi^2$$

where C_1, C_2, C_3 are always constants depending only on N, λ, Λ, and

$$K_1(\theta) = \exp(-C\theta^{-1})\,\theta^{-3N/4}$$

$$K_2(\theta) = \exp(-C\theta^{-1})\,\theta^{-(\frac{N}{4}+1)}$$

$$K_3(\theta) = \theta^{-3N/4}\ .$$

We now prove the following estimate:

Lemma 7.1. There exists $\theta_1 > 0$ depending only on N, λ, Λ, such that for $\theta \in (0, \theta_1)$ if

$$V^2(\theta^{1/2}R) \geqslant 2C_3K_4(\theta)\varepsilon^2 \tag{7.3}$$

then we have

$$V^2(\theta^{1/2}R) \leqslant \frac{1}{1+K_5(\theta)\delta_\theta(\xi, R)} V^2(R)$$

where V(R) is as in Sec. 4.

Let

$$\Phi(r) = \left(\underset{Q(z_0, r)}{\operatorname{osc}} u\right)^2$$

$$\mu(r) = \int_{Q(z_0,R)} |D_x u|^2 G^{z_0} \,dxdt \quad .$$

Let (7.3) hold. From (7.2) we have

$$\left(1+2^{-1}C_2K_3(\theta)\delta_\theta(\xi,R)\right)\left(\Phi(\theta^{1/2}R)+\mu(\theta^{1/2}R)\right) \leqslant$$

$$\leqslant \left(1+C_1C_2K_2(\theta)K_3(\theta)\delta_\theta(\varepsilon, R)\right)\Phi(R) + \mu(R)$$

Then

$$\Phi(\theta^{1/2}R) + \mu(\theta^{1/2}R) \leqslant$$

$$\leqslant \frac{\left(1+C_1C_2K_2(\theta)K_3(\theta)\delta_\theta(\xi, R)\right)}{\left(1+2^{-1}C_2K_3(\theta)\delta_\theta(\xi, R)\right)}\Phi(R) + \frac{1}{1+2^{-1}C_2K_3(\theta)\delta_\theta(\xi, R)}\mu(R) \quad .$$

We observe that

$$\frac{1+C\sigma x}{1+Cx} \leqslant \frac{1}{1+\frac{C}{2}x} \qquad 0 \leqslant x \leqslant 1$$

if $0 < C < 1$ and $0 < \sigma < 1/3$; then if we choose θ such that $0 < 2^{-1}C_2K_3(\theta) < 1$ and $0 < 2C_1K_2(\theta) < 1/3$ we have the result.

By similar arguments as in the elliptic case, [8], we obtain from Lemma 7.1 the following:

Lemma 7.2. Let $\theta \in (0, \theta_1)$, then for $r \leqslant \theta^{1/2}R$ we have

$$V^2(r) \leqslant C^2 \exp\left(-2\beta \int_r^R \delta_\theta(\varepsilon, \rho) \frac{d\rho}{\rho}\right) V^2(R) + 2C_3K_4(\theta)\,\xi^2$$

where

$$\beta = \frac{1}{4} \frac{C_2K_3(\theta)}{4 + C_2K_3(\theta)} \left|\ln \theta\right|^{-1}$$

and $C > 0$ is an absolute constant.

We now conclude the proof of Th. 4.1.

From Lemma 7.2 we have

$$V(r) \leqslant C \exp\left(-\beta \int_r^R \delta_\theta(\varepsilon, \rho) \frac{d\rho}{\rho}\right) V(R) + K_5(\theta)\xi \ .$$

Choosing now $\xi = \sigma\omega_\sigma(r, R) + \eta$, $\eta > 0$, we obtain

$$V(r) \leqslant C \exp\left(-\beta \int_r^R \delta_\theta\big(\sigma\omega_\sigma(r, R) + \eta, \rho\big) \frac{d\rho}{\rho}\right) V(R) + K_5(\theta)\big(\sigma\omega_\sigma(r, R) + \eta\big) \ ;$$

then the result of Th. 4.1 follows by letting $\eta \to 0$.

8. Proof of Corollary 4.3

We observe that from the definition of $A_q(z_0)$ we have

$$\frac{1}{r^N}\int_{Q(z_0,r)\cap A_q(z_0)} |D_x u|^2 \,dxdt \leqslant c_q \int_{Q(z_0,r)} |D_x u|^2 G^{z_0} \,dxdt \, . \tag{8.1}$$

From Corollary 4.1 we have

$$\int_{Q(z_0,r)} |D_x u|^2 G^{z_0} \,dxdt \leqslant c\,(r/R)^{\beta} V(\theta R) \qquad 0 < r < \theta^2 R \, . \tag{8.2}$$

From (8.1), (8.2) and Th. 4.2 we obtain

$$\frac{1}{r^N}\int_{Q(z_0,r)\cap A_q(z_0)} |D_x u|^2 \,dxdt \leqslant (r/R)^{\beta} \frac{c_q^1}{R^{N+2}} \int_{Q(z_0,R)} |u-u_R|^2 \,dxdt \, ,$$

where u_R denotes the average of u on $Q(z_0,R)$.

By applying now Poincaré's inequality for the local solution u of the parabolic equation as proved in [5], we obtain

$$\int_{Q(z_0,r)\cap A_q(z_0)} |D_x u|^2 \,dxdt \leqslant c\,(r/R)^{N+\beta} \int_{Q(z_0,2R)} |D_x u|^2 \,dxdt$$

where c depends on q, N, λ, Λ, θ.

Appendix

We will study here some properties of the capacity defined in Sec. 1.

From the definition we have, if $P_1 \subset P_2$ (P_1, P_2 parabolic open cylinders) and $E \subset \bar{E} \subset P_1 \subset P_2$,

$$\text{(A.1)} \qquad \mathrm{cap}_{P_1} E \geqslant \mathrm{cap}_{P_2} E \quad .$$

Proposition A.1. Let $E \subset \bar{E} \subset P$. If $\mathrm{cap}_{R^{N+1}} E = 0$, then $\mathrm{cap}_P E = 0$.

Suppose at first E compact. Then there exists a sequence $\{w_s\} \subset C_0^\infty(R^{N+1})$, with $w_s = 1$ on a neighborhood of E, such that

$$\lim_{s \to \infty} w_s = 0 \quad \text{in } L^2(R, H^1(R^N)) \;.$$

Now let $\phi \in C_0^\infty(P)$ with $\phi = 1$ on a neighborhood of E. We have

$$\lim_{s \to +\infty} w_s \phi = 0 \quad \text{in } L^2(\tau_1, \tau_2; H_0^1(B)) \;;$$

then $\mathrm{cap}_P E = 0$.

Let now E be an arbitrary set with $\mathrm{cap}_{R^{N+1}} E = 0$ and $\bar{E} \subset P$. Then there exists a sequence of open sets $A_s \subset \bar{A}_s \subset A \subset \bar{A} \subset P$ (A fixed open set in P) such that

$$\lim_{s \to +\infty} \mathrm{cap}_{R^{N+1}} A_s = 0.$$

If $\mathrm{cap}_P A_s \geqslant \delta > 0$ there exists a sequence of compact sets $\{K_s\}$ such that

$$\text{(a)} \quad \lim_{s \to +\infty} \mathrm{cap}_{R^{N+1}} K_s = 0 \qquad\qquad \text{(b)} \quad \mathrm{cap}_P K_s \geqslant \delta/2 \;.$$

From (a) there exists a sequence $\{w_s\} \subset C_0^\infty(R^{N+1})$ with $w_s = 1$ in a neighborhood of K_s such that

(A.2) $$\lim_{s \to +\infty} w_s = 0 \quad \text{in } L^2(R; H^1(R^N)) \ .$$

Let now $\phi \in C_0^\infty(P)$ with $\phi = 1$ in R. We then have

(A.3) $$\lim_{s \to +\infty} \operatorname{cap}_P K_s = 0 \ .$$

From (b) and (A.2) we have a contradiction, and the Proposition is proved.

Let now $D \subset \bar{D} \subset B$. By $N\text{-cap}_B D$ we denote the usual Newtonian capacity and we put $E_\tau = E \cap \{t = \tau\}$. We then have the following result:

Proposition A.2. Let $E \subset P$ compact; then

$$\operatorname{cap}_P E = \int_{\tau_1}^{\tau_2} N\text{-cap}_B(E_\tau)\, d\tau \ .$$

From the definition of $\operatorname{cap}_P E$ we have

(A.4) $$\operatorname{cap}_P E \geqslant \int_{\tau_1}^{\tau_2} N\text{-cap}_P(E_\tau)\, d\tau \ .$$

If $E = D \times [s_1, s_2]$, $D \subset B$ compact, $\forall \eta > 0$ there exists $w \in C_0^\infty(B)$ such that

$$\int_B |D_x w|^2 dx - \eta \leqslant N\text{-cap}_B D \leqslant \int_B |D_x w|^2 dx$$

$w = 1$ in a neighborhood of D.

Let $\phi(t)$ be such that $\phi \in C_0(s_1-\eta, s_2-\eta)$, $[s_1-\eta, s_2+\eta] \subset]\tau_1, \tau_2[$, $\phi = 1$ in $[s_1-\eta/2, s_2+\eta/2]$.

Let us consider the function

$$g(x, t) = w(x)\phi(t) \ .$$

We have $g(x, t) = 1$ in a neighborhood of E and $g \in C_0(P)$. Therefore

$$\mathrm{cap}_P E \leqslant \int_{\tau_1}^{\tau_2} |D_x g|^2 \, dxdt \leqslant (\tau_1 - \tau_2)(N\text{-}\mathrm{cap}_B D + \eta) + 2\eta \int_B |D_x w|^2 \, dxdt.$$

Thus we have

$$\mathrm{cap}_P E \leqslant (\tau_1 - \tau_2) N\text{-}\mathrm{cap}_B D \ . \tag{A.5}$$

From (A.4), (A.5) we obtain

$$\mathrm{cap}_P E = (\tau_1 - \tau_2) N\text{-}\mathrm{cap}_B D \ .$$

We come back now to the general case.

We consider the union V_r of the cubs $j_k/r \leqslant x_k \leqslant (j_k+1)/r$, $j_0/r \leqslant t \leqslant (j_0+1)/r$, $j_k, j_0 = 1, 2, \ldots,$ which have in the interior a point of E; we have

$$V_r \subset P \qquad r \geqslant \bar{r}$$

$$E = \bigcap_r V_r \ , \quad V_r \text{ decreasing in } r$$

$$E_\tau = \bigcap_r V_{r,\tau} \ , \quad V_{r,\tau} \text{ decreasing in } r \ .$$

From the right continuity and the subadditivity of cap_P and $N\text{-}\mathrm{cap}_B$ we have

$$\mathrm{cap}_P E = \mathrm{Inf}_r \, \mathrm{cap}_P V_r \leqslant \mathrm{Inf}_r \int_{\tau_1}^{\tau_2} N\text{-}\mathrm{cap}_B V_{r,\tau} \, d\tau = \int_{\tau_1}^{\tau_2} \mathrm{Inf}_r \, N\text{-}\mathrm{cap}_B V_{r,\tau} \, d\tau = \tag{A.6}$$

$$= \int_{\tau_1}^{\tau_2} N\text{-}\mathrm{cap}_B E_\tau \, d\tau$$

From (A.4), (A.6) we have the result.

Proposition A.3. Let $E \subset \bar{E} \subset P$ be an open set, then

$$\mathrm{cap}\, E = \int_{\tau_1}^{\tau_2} N\text{-}\mathrm{cap}\, E_\tau \, d\tau$$

The proof is analogous to the one of Prop. A.4 and it will only be sketched below.

We prove first the result in the case of open cylinders. Then we approximate E by a union of open cylinders containing the set E with the exception of a subset of E of capacity zero, which corresponds to a countable sequence of value of t.

Proposition A.4. Let $E \subset \bar{E} \subset P$ be an arbitrary Borel set. Then

$$\operatorname{cap} E = \int_{\tau_1}^{\tau_2} N\text{-}\operatorname{cap} E_\tau \, d\tau \quad .$$

Let us consider an increasing sequence of compact sets $K_s \subset E$. We have

(A.7) $$\sup_s \operatorname{cap} K_s = \operatorname{cap} E$$

Let $\tilde{\delta}(t) = \sup N\text{-}\operatorname{cap} K_{st}$; we easily obtain

(A.8) $$\tilde{\delta}(t) \leqslant N\text{-}\operatorname{cap} E_t = \delta(t) \quad .$$

Consider now a decreasing class of open sets $A_s \supset E$. We have:

(A.9) $$\inf_s \operatorname{cap} A_s = \operatorname{cap} E$$

Let $\underset{\sim}{\delta}(t) = \inf N\text{-}\operatorname{cap} A_{s,t}$. We easily obtain

(A.10) $$\underset{\sim}{\delta}(t) \geqslant N\text{-}\operatorname{cap} E_t = \delta(t) \quad .$$

From (A.7), (A.9) we have $\tilde{\delta}(t) = \underset{\sim}{\delta}(t)$ a.e. in $]\tau_1, \tau_2[$ therefore

$$\delta(t) = \tilde{\delta}(t) = \underset{\sim}{\delta}(t) \text{ a.e. in }]\tau_1, \tau_2[$$

and the result follows.

Let now $P' = B' \times]\tau'_1, \tau'_2[\subset \bar{P}' \subset P$.

Proposition A.5. Let v, $D_x v \in L^2(P')$; then v is quasi-continuous in P' (for cap_P); moreover, if $\{v_n\}$ is a sequence such that

$$\lim_{n\to\infty} v_n = v , \qquad \lim_{n\to\infty} D_x v_n = D_x v$$

then $v_n \to v$ quasi-uniformly in P'.

We prove the first part of Prop. A.5; the proof of the second part is analogous.

There exist $\forall\, \varepsilon > 0$ an open set V in $]\tau_1', \tau_2'[$ such that mes $V \leqslant \varepsilon$ and $v(t, \cdot)$ is bounded in $L^2(B')$ for $t \in]\tau_1, \tau_2[- V$. From the results of [16] on the Γ-capacity, v is quasi-continuous for the Γ-capacity, therefore it is also q.c. for the cap_P. Since

$$\text{(A.11)} \qquad cap(V \times B') = \int_V N\text{-}cap_B B' \, d\tau \leqslant \varepsilon N\text{-}cap_B B'$$

the quasi-continuity of v follows.

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