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Stationary Non-Newtonian Fluid Flows in Channel-like and Pipe-like Domains

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Abstract

This paper is concerned with stationary non-Newtonian fluid in an unbounded domain which geometrically is channel-like in 2-d or axisymmetric pipe-like in 3-d. The flow satisfies no-slip boundary conditions, and behaves as Poiseuille flow at infinity. Existence and uniqueness are proved under the assumption of a large kinematic viscosity. The results apply to a family of models including second-order, Maxwell and Oldroyd-B fluids. For second-order fluids, the existence and uniqueness results are extended to the case when the boundary has corner points.

Keywords. Non-Newtonian fluid, second-order fluid, Navier-Stokes equation, Poiseuille flow.

AMS subject classification. 76A05, 76DOS, 35M20, 35Q30.

1 Introduction

The present paper is concerned with non-Newtonian stationary flow in a domain R with non-slip boundary conditions on the boundary ∂R. The case of a second-order fluid in bounded domains with smooth boundary was considered in [7] and [3] using, as an essential tool, the Helmholtz decomposition of the velocity vector field (See also [6] which also applies to domains with corner points but assumes, however, that the fluid satisfies an additional, rather unnatural, condition).

We shall use an entirely different approach which allows us to deal with unbounded domains; furthermore, we can relax the smoothness condition on the boundary, allowing a finite number of corner points. In Sections 2-6 we deal with second-order fluid in 2-d channel-like domain with smooth boundary; we prove existence and uniqueness of a flow with no-slip boundary conditions and with Poiseuille flow conditions at infinity. In Section 7 we extend these results to more general non-Newtonian fluids (including Maxwell and Oldroyd-B models). In Sections 8-10 we extend the results of Sections 2-6 to domains

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with corner points. Finally, Section 11 extends the results of the paper to 3-d axially symmetric flows.

We briefly describe our approach in the 2-d case for second-order fluids. The differential equation representing the flow can be written in the form:

$$\Delta^2\psi + \varepsilon(\overrightarrow{v} \cdot \overrightarrow{\nabla})\Delta^2\psi = \varepsilon F(D^\alpha\psi, |\alpha| \leq 4) , \tag{1.1}$$

where $\overrightarrow{v} = (\psi_y, -\psi_x)$ is the velocity vector, ψ is the stream function and $F(\cdot)$ is a nonlinear function in its arguments. As in [7] we make a "smallness" assumption which, in the present context, means that ε is a small parameter. We shall view (1.1) as a hyperbolic equation for $\Delta^2\psi$, and integrate it along characteristics, i.e., along curves $(x(s), y(s))$ which satisfy

$$\frac{dx}{ds} = \psi_y(x(s), y(s)) , \tag{1.2}$$

$$\frac{dy}{ds} = -\psi_x(x(s), y(s)) . \tag{1.3}$$

In this way we obtain an equation of the form

$$\Delta^2\psi = G(\psi) , \tag{1.4}$$

where G is a nonlinear functional of ψ, with non-slip boundary conditions. We then proceed to prove existence and uniqueness for (1.4), by a fixed point theorem.

When the domain has a corner, the characteristic curves develop a singularity near the corner and the situation becomes more delicate. However, due to the fact that the highest derivatives of ψ in $F(D^\alpha\psi)$ are of order ≤ 4, we are able to extend the analysis of the case of smooth domains and establish existence and uniqueness for second-order fluids in domains with corners.

2 The model

Second-order non-Newtonian fluid is characterized by a constitutive relation of the form (cf. [13]):

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} ,$$

where $\mathbf{T}$ is the stress tensor, p the pressure,

$$\mathbf{S} = 2\mu\mathbf{D} + 2\alpha_1\mathbf{A} + 4\alpha_2\mathbf{D}^2 ,$$

$\mathbf{D}$ the rate of deformation tensor, $\mathbf{A}$ is a tensor given by

$$\mathbf{A} = \frac{d\mathbf{D}}{dt} + \mathbf{D}\overrightarrow{\nabla}\,\overrightarrow{v} + (\overrightarrow{\nabla}\,\overrightarrow{v})^T\mathbf{D} ,$$

μ is the kinematic viscosity coefficient and α_1, α_2 are normal stress moduli. When $\alpha_1 = \alpha_2 = 0$ we recover the Newtonian constitutive relation. In a stationary situation we can write

$$\mathbf{A} = (\overrightarrow{v} \cdot \overrightarrow{\nabla})\mathbf{D} + \mathbf{D}\overrightarrow{\nabla}\,\overrightarrow{v} + (\overrightarrow{\nabla}\,\overrightarrow{v})^T\mathbf{D} .$$

The stationary conservation of momentum and mass equations are:

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla} \cdot \mathbf{T} , \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{v} = 0 , \tag{2.2}$$

where $\mathbf{T}$ is defined as above, with stationary $\mathbf{A}$.

Restricting ourselves to a two dimensional situation, we can write, in Cartesian coordinates,

$$\mathbf{D} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \frac{\partial v_y}{\partial y} \end{pmatrix} , \tag{2.3}$$

$$\vec{\nabla}\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix} , \tag{2.4}$$

with $\vec{v} = (v_x, v_y)$.

From (2.2) it follows that $\vec{v}$ is the curl of a function ψ, i.e.,

$$v_x = \psi_y ,$$
$$v_y = -\psi_x ;$$

ψ is called the *stream function.*

Applying the curl operator $(\partial_y, -\partial_x)$ to the vector equation (2.1), we eliminate the pressure p thereby obtaining the equation:

$$\frac{\partial}{\partial y}\left(\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} - v_x\frac{\partial v_x}{\partial x} - v_y\frac{\partial v_x}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial S_{21}}{\partial x} + \frac{\partial S_{22}}{\partial y} - v_x\frac{\partial v_y}{\partial x} - v_y\frac{\partial v_y}{\partial y}\right) .$$

This equation can be written in terms of ψ:

$$\frac{\partial^2 S_{11}}{\partial x \partial y} + \frac{\partial^2 S_{12}}{\partial y^2} - \frac{\partial^2 S_{21}}{\partial x^2} - \frac{\partial^2 S_{22}}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\psi_y\psi_{yx} - \psi_x\psi_{yy}\right) + \frac{\partial}{\partial x}\left(\psi_y\psi_{xx} - \psi_x\psi_{xy}\right)$$

$$\equiv \sum_{\substack{i+j=4 \\ k+l=i\geq 1 \\ m+n=j\geq 1}} \tilde{b}_{klmn}\frac{\partial^i\psi}{\partial x^k\partial y^l}\frac{\partial^j\psi}{\partial x^m\partial y^n} . \tag{2.5}$$

On the other hand, from the definition of S,

$$\frac{\partial^2 S_{11}}{\partial x \partial y} + \frac{\partial^2 S_{12}}{\partial y^2} - \frac{\partial^2 S_{21}}{\partial x^2} - \frac{\partial^2 S_{22}}{\partial x \partial y} = 2\mu\left(\frac{\partial^2 D_{11}}{\partial x \partial y} + \frac{\partial^2 D_{12}}{\partial y^2} - \frac{\partial^2 D_{21}}{\partial x^2} - \frac{\partial^2 D_{22}}{\partial x \partial y}\right)$$

$$+2\alpha_1\left(\frac{\partial^2 A_{11}}{\partial x \partial y} + \frac{\partial^2 A_{12}}{\partial y^2} - \frac{\partial^2 A_{21}}{\partial x^2} - \frac{\partial^2 A_{22}}{\partial x \partial y}\right) + 4\alpha_2\left(\frac{\partial^2 D_{11}^2}{\partial x \partial y} + \frac{\partial^2 D_{12}^2}{\partial y^2} - \frac{\partial^2 D_{21}^2}{\partial x^2} - \frac{\partial^2 D_{22}^2}{\partial x \partial y}\right) . \tag{2.6}$$

and, by direct computation,

$$\frac{\partial^2 D_{11}}{\partial x \partial y} + \frac{\partial^2 D_{12}}{\partial y^2} - \frac{\partial^2 D_{21}}{\partial x^2} - \frac{\partial^2 D_{22}}{\partial x \partial y} = \frac{1}{2}\Delta^2\psi ,$$

$$\frac{\partial^2 A_{11}}{\partial x \partial y} + \frac{\partial^2 A_{12}}{\partial y^2} - \frac{\partial^2 A_{21}}{\partial x^2} - \frac{\partial^2 A_{22}}{\partial x \partial y} = \frac{1}{2}(\overrightarrow{v} \cdot \overrightarrow{\nabla})\Delta^2 \psi \,, \tag{2.7}$$

$$\frac{\partial^2 D_{11}^2}{\partial x \partial y} + \frac{\partial^2 D_{12}^2}{\partial y^2} - \frac{\partial^2 D_{21}^2}{\partial x^2} - \frac{\partial^2 D_{22}^2}{\partial x \partial y} = 0 \,,$$

(see also [14]). Hence equation (2.5) can be rewritten in the form

$$\Delta^2 \psi + \varepsilon(\overrightarrow{v} \cdot \overrightarrow{\nabla})\Delta^2 \psi = \sum_{\substack{i+j=4 \\ k+l=i\geq 1 \\ m+n=j\geq 1}} b_{klmn} \frac{\partial^i \psi}{\partial x^k \partial y^l} \frac{\partial^j \psi}{\partial x^m \partial y^n} \,, \tag{2.8}$$

with $\varepsilon = \frac{\alpha_1}{\mu}$ and $b_{klmn} = \frac{\widetilde{b_{klmn}}}{\mu}$. To fix ideas, we shall consider $\alpha_1 > 0$. The case $\alpha_1 < 0$ is also covered by the theory and we discuss it at the end of section 6. We shall also assume that μ is large enough so that

$$|b_{klmn}| < C\varepsilon \,. \tag{2.9}$$

where ε is sufficiently small. Note that when $\varepsilon = 0$ we recover the biharmonic equation

$$\Delta^2 \psi = 0 \,,$$

which describes the stationary Stokes flow. The limit $\varepsilon \to 0$ is singular in the sense that highest order derivatives of ψ disappear. This makes, in principle, the solution of the biharmonic equation inaccurate for describing the non-Newtonian flow.

Throughout most of this paper we consider the non-Newtonian flow contained in channel-like domains R, i.e., in domains which are a perturbation of a rectangular channel $\mathbb{R} \times [-1, 1]$. For simplicity we shall take R to be symmetric about the line $y = 0$; all our results however extend to no-symmetric domains. More precisely, we take

$$\partial R = \pm h(x) = \pm(1 + \varepsilon \xi(x)) \,, \tag{2.10}$$

with $\xi(x)$ a bounded continuous function such that

$$\begin{aligned} \lim_{x \to -\infty} \xi(x) &= 0 \,, \\ \lim_{x \to +\infty} \xi(x) &= 1 \,, \end{aligned} \tag{2.11}$$

(the analysis for the more general case $\lim_{x \to +\infty} \xi(x) = b$ with b arbitrary is similar). We shall specify the regularity assumptions on $\xi(x)$ later on.

The boundary conditions will be

$$\overrightarrow{v} = 0 \quad \text{on } \partial R \tag{2.12}$$

and we shall also impose a Poiseuille type behavior on the flow at $\pm\infty$:

$$\begin{aligned} \psi &= y\left(1 - \frac{y^2}{3}\right) + o(1) \quad \text{as } x \to -\infty \,, \\ \psi &= \frac{y}{1+\varepsilon}\left(1 - \frac{y^2}{3(1+\varepsilon)^2}\right) + o(1) \quad \text{as } x \to +\infty \,. \end{aligned} \tag{2.13}$$

3 A lemma on characteristics

Throughout Sections 3-6 we assume that ∂R is smooth in the sense that

$$\begin{cases} |\xi|_{C^{5+\alpha}(-\infty,+\infty)} < C \\ \left|\xi^{(j)}(x)\right| \le C(1+|x|)^{-\beta-j} \quad 1 \le j \le 4 \, , \end{cases} \tag{3.1}$$

for some $0 < \alpha < 1$, $\beta > \frac{1}{2}$.

We associate to the Poiseuille flow in the channel $-1 < y < 1$,

$$P(x,y) = y\left(1 - \frac{y^2}{3}\right) \, , \tag{3.2}$$

the flow

$$G(x,y) = P(x, \frac{y}{h(x)}) \, . \tag{3.3}$$

Then

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0 \quad \text{at } \partial R \, ,$$

and

$$\Delta^2 G = 0 \quad \text{if } \varepsilon = 0 \, .$$

In the next lemma we consider characteristic curves given by a velocity field $\overrightarrow{v} = (\psi_y, -\psi_x)$:

$$\begin{aligned} \frac{dx}{ds} &= \psi_y(x(s), y(s)) \, , \\ \frac{dy}{ds} &= -\psi_x(x(s), y(s)) \, , \\ x(0) &= x, \; y(0) = y \, . \end{aligned} \tag{3.4}$$

Lemma 3.1 *Let*

$$\psi(x,y) = G(x,y) + \varepsilon^{\frac{1}{2}} \widetilde{\psi}(x,y) \, ,$$

satisfy (2.8), (2.12), (2.13) and assume that $\left\|\widetilde{\psi}\right\|_{C^2(R)} < 1$. *Then, for any* ε *small enough:*

1.- There does not exist any stagnation point for ψ, *i.e.,* $(\psi_y, -\psi_x) \neq 0$ *everywhere in the interior of* R;

2.- The characteristic curves (3.4) do not intersect each other and cover the whole region R, *and*

$$\left|\frac{dy}{dx}\right| \le C \quad \textit{everywhere in } R \, . \tag{3.5}$$

Proof. It is easy to verify that for ε small enough

$$\left|\frac{\partial G}{\partial y}\right| \ge f(x,y) \, ,$$

where

$$f(x,y) = \begin{cases} M(h(x) - |y|) \text{ if } |y| \ge h(x) - \delta \\ M\delta \text{ if } |y| < \delta \end{cases}$$

and M, δ are suitable positive numbers. Let us write $(v_x, v_y) = (\psi_y, -\psi_x)$ and $(\widetilde{v_x}, \widetilde{v_y}) = (\widetilde{\psi}_y, -\widetilde{\psi}_x)$. The function $\widetilde{\psi}$ satisfies the boundary conditions

$$\frac{\partial \widetilde{\psi}}{\partial x} = \frac{\partial \widetilde{\psi}}{\partial y} = 0 \quad \text{at } \partial R \ ,$$

as both ψ and G satisfy them. Clearly

$$\widetilde{v}_x(x,y) = \widetilde{v}_x(x, -h(x)) + \int_{-h(x)}^{y} \frac{\partial \widetilde{v}_x}{\partial y}(x,\xi)d\xi = \int_{-h(x)}^{y} \frac{\partial^2 \widetilde{\psi}}{\partial y^2}(x,\xi)d\xi \ ,$$

so that

$$|\widetilde{v}_x| \leq \left(\sum_{i=0}^{2} \sup \left| \frac{\partial^2 \widetilde{\psi}}{\partial x^i \partial y^{2-i}} \right|^2 \right)^{\frac{1}{2}} (h(x) - |y|) \leq \left\| \widetilde{\psi} \right\|_{C^2(R)} (h(x) - |y|) \ ,$$

and therefore

$$|v_x| = \left| \frac{\partial G}{\partial y} + \varepsilon^{\frac{1}{2}} \widetilde{v}_x \right| \geq \left| \frac{\partial G}{\partial y} \right| - \varepsilon^{\frac{1}{2}} |\widetilde{v}_x| \geq f(x,y) - \varepsilon^{\frac{1}{2}} \left\| \widetilde{\psi} \right\|_{C^2(R)} (h(x) - |y|) \ .$$

Since the right-hand side is strictly positive inside R (and vanishes at ∂R), no stagnation points exist inside R. We can also write

$$v_x = \left(1 - \frac{y^2}{h^2(x)} \right) \left(1 + \varepsilon^{\frac{1}{2}} g_1(x,y) \right) \ , \tag{3.6}$$

where $|g_1| < C$ for some constant C independent of ε. From the structure of $G(x,y)$, the hypothesis $\left\| \widetilde{\psi} \right\|_{C^2(R)} < 1$, and the boundary conditions, it also follows

$$v_y = \varepsilon^{\frac{1}{2}} \left(1 - \frac{y^2}{h^2(x)} \right) g_2(x,y) \ , \tag{3.7}$$

with $|g_2| < C$. Therefore we can write the characteristics equations in the form

$$\begin{aligned} \frac{dx}{ds} &= v_x = \left(1 - \frac{y^2}{h^2(x)} \right) \left(1 + \varepsilon^{\frac{1}{2}} g_1(x,y) \right) \ , \\ \frac{dy}{ds} &= v_y = \varepsilon^{\frac{1}{2}} \left(1 - \frac{y^2}{h^2(x)} \right) g_2(x,y) \ . \end{aligned} \tag{3.8}$$

The characteristic curves do not intersect, by uniqueness of the solutions of the system (3.8). A curve passing through any point (x_0, y_0) can never cross the lines $y = \pm h(x)$, which are themselves characteristics, and (since $\frac{dx}{ds} > 0$) can be continued up to $x = +\infty$ and down to $x = -\infty$. Moreover, the values of $y(s)$ as $s \to +\infty$ $(-\infty)$ cover the entire interval $-1 - \varepsilon < y < 1 + \varepsilon$ $(-1 < y < 1)$. Therefore the characteristics cover the whole region R. Finally, from (3.8) we immediately infer that (3.5) holds. □

4 Generalized Poiseuille flow

Consider the system

$$
\begin{aligned}
\Delta^2\psi_0 &= 0 \quad \text{in } R=\mathbb{R}\times[-1,1] \ , & (4.1)\\
\frac{\partial\psi_0}{\partial x} &= \frac{\partial\psi_0}{\partial y}=0 \quad \text{at } \partial R \ , & (4.2)\\
\psi_0 &= y\left(1-\frac{y^2}{3}\right)+o(1) \quad \text{as } x\to-\infty \ , & (4.3)\\
\psi_0 &= \frac{y}{1+\varepsilon}\left(1-\frac{y^2}{3(1+\varepsilon)^2}\right)+o(1) \quad \text{as } x\to+\infty \ , & (4.4)
\end{aligned}
$$

where $h(x)=1+\varepsilon\xi(x)$. We may view ψ_0 as a generalized Poiseuille flow. We shall refer to

$$\overrightarrow{v_0}=\left(\frac{\partial\psi_0}{\partial y},-\frac{\partial\psi_0}{\partial x}\right) \ ,$$

as the *Stokes velocity field.*

Lemma 4.1 *For any ε small enough there exists a unique solution of (4.1)-(4.4) of the form*

$$\psi_0=G(x,y)+\varepsilon\widetilde{\psi}(x,y) \ , \quad \left\|\widetilde{\psi}\right\|_{H^2(R)}<\infty \ ,$$

and $\left\|\widetilde{\psi}\right\|_{C^{5+\alpha}(R)}\le C(\|\xi\|_{C^{5+\alpha}(R)})$, *$C$ independent of ε.*

Proof. From (3.2), (3.3), the definition of $h(x)$ and the fact that $P(x,y)$ is biharmonic, we deduce that

$$\Delta^2 G=-\varepsilon H(x,y;\varepsilon) \ ,$$

where H and its derivatives up to any order are uniformly bounded, and

$$|H(x,y;\varepsilon)|\le C\,(1+|x|)^{-\beta} \ , \ \|H\|_{C^{1+\alpha}(R)}<\infty \ .$$

Consider the following problem for $\widetilde{\psi}$:

$$
\begin{aligned}
\Delta^2\widetilde{\psi} &= H(x,y;\varepsilon) \ ,\\
\frac{\partial\widetilde{\psi}}{\partial x} &= \frac{\partial\widetilde{\psi}}{\partial y}=0 \quad \text{at } \partial R \ ,\\
\widetilde{\psi} &\to 0 \text{ as } |x|\to 0 \ .
\end{aligned}
$$

Existence and uniqueness of a weak solution in $H^2(R)$ follows from the Lax-Milgram theorem and standard elliptic estimates (note that the function H is in $L^2(R)$ since $\beta>\frac{1}{2}$). Regularity of the weak solution as well as the estimate

$$\left\|\widetilde{\psi}\right\|_{C^{5+\alpha}(R)}\le C\,\|H\|_{C^{1+\alpha}(R)}\le C \ ,$$

follow from [1]. The function $\psi_0=G+\varepsilon\widetilde{\psi}$ then clearly satisfies all the assertions of the lemma.□

5 A fixed point framework

Let us introduce the characteristic equations:

$$\begin{aligned}\frac{dx}{ds} &= v_x(x(s),y(s)) = \psi_y(x(s),y(s)) \, , \\ \frac{dy}{ds} &= v_y(x(s),y(s)) = -\psi_x(x(s),y(s)) \, .\end{aligned}$$

Then we can write (2.8) in the form:

$$\Delta^2\psi + \varepsilon \frac{d\Delta^2\psi}{ds} = \varepsilon F(x,y) \, , \tag{5.1}$$

where

$$\varepsilon F(x,y) = B[\psi,\psi] := \sum_{\substack{i+j=4 \\ k+l=i\geq 1 \\ m+n=j\geq 1}} b_{klmn} \frac{\partial^i \psi}{\partial x^k \partial y^l} \frac{\partial^j \psi}{\partial x^m \partial y^n} \, , \tag{5.2}$$

and $\frac{d}{ds}$ is the derivative along the characteristic curve at the point (x,y). Thus the flow problem we wish to solve consists of (5.1) and the boundary conditions:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \quad \text{at } \partial R \, , \tag{5.3}$$

$$\psi = y\left(1 - \frac{y^2}{3}\right) + o(1) \quad \text{as } x \to -\infty \, , \tag{5.4}$$

$$\psi = \frac{y}{1+\varepsilon}\left(1 - \frac{y^2}{3(1+\varepsilon)^2}\right) + o(1) \quad \text{as } x \to +\infty \, . \tag{5.5}$$

We shall adopt the following scheme:
Given a stream function $\widetilde{\psi}$ which is a small perturbation of the Stokes flow,

$$\widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}} \widetilde{\psi_1} \, , \tag{5.6}$$

we seek a solution ψ of

$$\Delta^2\psi + \varepsilon \frac{\widetilde{d}}{\widetilde{ds}} \Delta^2\psi = \varepsilon F(x,y) = B\left[\widetilde{\psi}, \widetilde{\psi}\right] \, ,$$

and the boundary conditions (5.3)-(5.5), where $\frac{\widetilde{d}}{\widetilde{ds}}$ is the derivative along the characteristic curves of (5.6). If we write ψ in the form

$$\psi = \psi_0 + \varepsilon^{\frac{1}{2}} \psi_1 \, ,$$

where ψ_0 is the stream function corresponding to the Stokes flow, then ψ_1 is a solution of

$$\Delta^2\psi_1 + \varepsilon \frac{\widetilde{d}\Delta^2\psi_1}{\widetilde{ds}} = \varepsilon^{\frac{1}{2}} F(x,y) \, , \tag{5.7}$$

$$\frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_1}{\partial y} = 0 \quad \text{at } \partial R \, , \tag{5.8}$$

$$|\psi_1| \to 0 \quad \text{as } |x| \to \infty \, . \tag{5.9}$$

We define a mapping T by:

$$T\widetilde{\psi_1} = \psi_1 \ . \tag{5.10}$$

and shall later on prove that T has a unique fixed point. This establishes the solution to (5.1)-(5.5).

The system (5.7)-(5.9) can be written in the following form

$$\begin{aligned} \Delta^2\psi_1 &= W(x,y)\ , & (5.11)\\ W(x,y) &= \varepsilon^{-\frac{1}{2}}\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}}F(x(\sigma),y(\sigma))d\sigma\ , & (5.12)\\ \frac{\partial\psi_1}{\partial x} &= \frac{\partial\psi_1}{\partial y} = 0 \quad \text{at } \partial R\ , & (5.13)\\ |\psi_1| &\to 0 \quad \text{as } |x|\to\infty\ , & (5.14)\end{aligned}$$

where $(x(\sigma),y(\sigma))$ is the characteristic curve with $(x(0),y(0))=(x,y)$ associated to $\widetilde{\psi}$.

In what follows we shall need the following version of Gronwall's inequality:

Lemma 5.1 *Suppose that* $u(s):\mathbb{R}^+\to\mathbb{R}^+$ *is such that*

$$u(s)\le g(s)+K\int_0^s u(\sigma)d\sigma\ . \tag{5.15}$$

Then

$$u(s)\le g(s)+K\int_0^s e^{K(s-\sigma)}g(\sigma)d\sigma\ . \tag{5.16}$$

The proof is standard and will be omitted.

We shall also need the following lemma concerning the distance between two characteristics:

Lemma 5.2 *Let* $\overrightarrow{v_1}$ *be a velocity field such that* $|\overrightarrow{v_1}|_{C^1(R)}<1$, *and set*

$$\overrightarrow{v} = \overrightarrow{v_0}+\varepsilon^{\frac{1}{2}}\overrightarrow{v_1}\ .$$

Let $\overrightarrow{x}(s)$ *and* $\overrightarrow{\widetilde{x}}(s)$ *be characteristics associated to the velocity field* $\overrightarrow{v}$ *and starting from different points* $\overrightarrow{x}(0)$ *and* $\overrightarrow{\widetilde{x}}(0)$. *Then*

$$\left|\overrightarrow{x}-\overrightarrow{\widetilde{x}}\right|(s)\le e^{c(\varepsilon)|s|}\left|\overrightarrow{x}-\overrightarrow{\widetilde{x}}\right|(0)\ , \tag{5.17}$$

where

$$c(\varepsilon)=\|\overrightarrow{v_0}\|_{C^1(R)}+\varepsilon^{\frac{1}{2}}\ .$$

Proof. By definition of characteristics

$$\frac{d(\overrightarrow{x}-\overrightarrow{\widetilde{x}})}{ds}=(\overrightarrow{v_0}(\overrightarrow{x}(s))-\overrightarrow{v_0}(\overrightarrow{\widetilde{x}}(s)))+\varepsilon^{\frac{1}{2}}(\overrightarrow{v_1}(\overrightarrow{x}(s))-\overrightarrow{v_1}(\overrightarrow{\widetilde{x}}(s))) \tag{5.18}$$

and, by integration,

$$(\overrightarrow{x}-\overrightarrow{\widetilde{x}})(s)=(\overrightarrow{x}-\overrightarrow{\widetilde{x}})(0)$$

$$+\int_0^s (\overrightarrow{v_0}(\overrightarrow{x}(\sigma)) - \overrightarrow{v_0}(\overrightarrow{\widetilde{x}}(\sigma)))d\sigma + \varepsilon^{\frac{1}{2}} \int_0^s (\overrightarrow{v_1}(\overrightarrow{x}(\sigma)) - \overrightarrow{v_1}(\overrightarrow{\widetilde{x}}(\sigma)))d\sigma \ ,$$

so that

$$\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|(s) \leq \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|(0)$$

$$+\left|\int_0^s \left|\overrightarrow{v_0}(\overrightarrow{x}(\sigma)) - \overrightarrow{v_0}(\overrightarrow{\widetilde{x}}(\sigma))\right| d\sigma\right| + \varepsilon^{\frac{1}{2}} \left|\int_0^s \left|\overrightarrow{v_1}(\overrightarrow{x}(\sigma)) - \overrightarrow{v_1}(\overrightarrow{\widetilde{x}}(\sigma))\right| d\sigma\right|$$

$$\leq \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|(0) + \left(|D\overrightarrow{v_0}|_0 + \varepsilon^{\frac{1}{2}} |D\overrightarrow{v_1}|_0\right) \left|\int_0^s \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|(\sigma)\right| \ .$$

Applying Gronwall's inequality and the fact that $|D\overrightarrow{v_1}|_0 < 1$, the assertion (5.17) follows.□

The following lemma shows that the mapping T given in (5.10) is well defined on the unit ball in $C^{5+\alpha}(R) \cap H^4(R)$:

Lemma 5.3 *There exists an ε_0 positive and small enough such that, for any $0 < \varepsilon < \varepsilon_0$, $F \in C^{1+\alpha}(R) \cap L^2(R)$ and $\left\|\widetilde{\psi_1}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$, there exists a unique solution ψ_1 of (5.7)-(5.9) in $H^2(R)$, and*

$$\|\psi_1\|_{C^{5+\alpha}(R)\cap H^4(R)} < C\varepsilon^{\frac{1}{2}} \|F\|_{C^{1+\alpha}(R)\cap L^2(R)} \ ,$$

where C is a constant independent of ε.

Proof. The problem (5.11), (5.13) with $W(x,y) \in L^2(R)$ admits a unique weak solution $\psi_1 \in H_0^2(R)$ (by the Lax-Milgram theorem). Since ψ_1 is in $H_0^2(R)$, it also satisfies (5.14). By regularity results of [1] it follows that whenever $W \in C^{1+\alpha}(R) \cap L^2(R)$ the unique weak solution of (5.11), (5.13), (5.14) satisfies the estimate :

$$\|\psi_1\|_{C^{5+\alpha}(R)\cap H^4(R)} \leq C \|W\|_{C^{1+\alpha}(R)\cap L^2(R)} \ .$$

Therefore the proof of the lemma will be completed if we can establish the estimates

$$\|W\|_{C^{1+\alpha}(R)} \leq C\varepsilon^{\frac{1}{2}} \|F\|_{C^{1+\alpha}(R)} \ , \tag{5.19}$$

$$\|W\|_{L^2(R)} \leq C\varepsilon^{\frac{1}{2}} \|F\|_{C^{1+\alpha}(R)\cap L^2(R)} \ . \tag{5.20}$$

Clearly,

$$\sup_R |W| \leq \varepsilon^{-\frac{1}{2}} \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} \sup_R |F| \, d\sigma = \varepsilon^{\frac{1}{2}} |F|_0 \ . \tag{5.21}$$

In order to estimate the first derivatives of W, we consider two points $\overrightarrow{x_0}$ and $\overrightarrow{x_1}$ such that $|\overrightarrow{x_1} - \overrightarrow{x_0}| = 1$ and the line segment joining them:

$$\overrightarrow{x_\lambda} = \overrightarrow{x_0} + \lambda(\overrightarrow{x_1} - \overrightarrow{x_0}), \quad \lambda \in (0,1) \ .$$

We also introduce the characteristic curve $\overrightarrow{x_\lambda}(s) = (x_\lambda(s), y_\lambda(s))$ with $\overrightarrow{x_\lambda}(0) = \overrightarrow{x_\lambda}$, associated with $\widetilde{\psi}$. Then,

$$D_{x_0x_1}W \equiv \lim_{\lambda\to 0} \frac{W(\overrightarrow{x_\lambda}) - W(\overrightarrow{x_0})}{\lambda} = \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} \left(F(x_\lambda(\sigma), y_\lambda(\sigma)) - F(x_0(\sigma), y_0(\sigma))\right) d\sigma}{\lambda}$$

$$= \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} D_{x_0x_\lambda} F(\overrightarrow{x_{\lambda 0}}(\sigma_0)) \cdot (\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x_0}(\sigma))\, d\sigma}{\lambda} \ , \tag{5.22}$$

where $0 \le \lambda 0 = \lambda_0(\sigma,\lambda) \le \lambda < 1$ and $\sigma_0 = \sigma_0(\sigma,\lambda)$; here $D_{x_0x_\lambda}$ denotes a directional derivative along $\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x_0}(\sigma)$ and $\overrightarrow{x_{\lambda_0}}(\sigma_0)$ is a point in the interval between $\overrightarrow{x_\lambda}(\sigma)$ and $\overrightarrow{x_0}(\sigma)$ belonging to the characteristic through $\overrightarrow{x_{\lambda_0}}$ and with parameter σ_0.

By Lemma 5.2,

$$|D_{x_0x_1} W| \le \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} \, |DF|_0 \, |\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x_0}(\sigma)|\, d\sigma}{\lambda}$$

$$\le \varepsilon^{-\frac{1}{2}} \, |DF|_0 \lim_{\lambda\to 0} \frac{\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} e^{-c(\varepsilon)\sigma} \, |\overrightarrow{x_\lambda} - \overrightarrow{x_0}|(0) d\sigma}{\lambda} = \frac{\varepsilon^{\frac{1}{2}}}{1-\varepsilon c(\varepsilon)} \, |DF|_0 \ , \tag{5.23}$$

provided ε is small enough so that $\varepsilon c(\varepsilon) < 1$.

Next we estimate the Hölder coefficient of any partial derivative DW of W. For any two points $\overrightarrow{x} = (x,y)$ and $\overrightarrow{\widetilde{x}} = (\widetilde{x},\widetilde{y})$ in R, we define the intervals

$$\overrightarrow{x_\lambda} = \overrightarrow{x} + \lambda(\overrightarrow{x_1} - \overrightarrow{x}), \quad \lambda \in (0,1) \ ,$$

$$\overrightarrow{\widetilde{x}_\lambda} = \overrightarrow{\widetilde{x}} + \lambda(\overrightarrow{\widetilde{x}_1} - \overrightarrow{\widetilde{x}}), \quad \lambda \in (0,1) \ ,$$

with $(\overrightarrow{x_1} - \overrightarrow{x}) = (\overrightarrow{\widetilde{x}_1} - \overrightarrow{\widetilde{x}})$ and $|\overrightarrow{x_1} - \overrightarrow{x}| = \left|\overrightarrow{\widetilde{x}_1} - \overrightarrow{\widetilde{x}}\right| = 1$. We have, by (5.22)

$$D_{xx_1} W(x,y) - D_{\widetilde{x}\widetilde{x}_1} W(\widetilde{x},\widetilde{y}) =$$

$$= \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{1}{\lambda} \left[\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} \left(D_{xx_\lambda} F(\overrightarrow{x_{\lambda_0}}(\sigma_0)) \cdot (\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma)) \right) \right.$$
$$\left. - D_{\widetilde{x}\widetilde{x}_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) \cdot \left(\overrightarrow{\widetilde{x}_\lambda}(\sigma) - \overrightarrow{\widetilde{x}}(\sigma) \right) \right) d\sigma \Big]$$

$$= \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{1}{\lambda} \left[\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} \left(D_{xx_\lambda} F(\overrightarrow{x_0}(\sigma_0)) - D_{xx_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) \right) (\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma))\, d\sigma \right]$$

$$+ \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{1}{\lambda} \left[\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} \left(D_{xx_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) - D_{\widetilde{x}\widetilde{x}_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) \right) (\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma))\, d\sigma \right]$$

$$+ \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{1}{\lambda} \left[\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} \left(D_{\widetilde{x}\widetilde{x}_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}) \cdot \left((\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma)) - \left(\overrightarrow{\widetilde{x}_\lambda}(\sigma) - \overrightarrow{\widetilde{x}}(\sigma) \right) \right) \right) d\sigma \right] .$$

If we now take into account that

$$\lim_{\lambda\to 0} \left| D_{xx_\lambda} F(\overrightarrow{x_{\lambda_0}}(\sigma_0)) - D_{xx_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) \right| \le \lim_{\lambda\to 0} |DF|_\alpha \left| \overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0) - \overrightarrow{x_{\lambda_0}}(\sigma_0) \right|^\alpha =$$

$$= |DF|_\alpha \left| \overrightarrow{\widetilde{x}}(\sigma) - \overrightarrow{x}(\sigma) \right|^\alpha \ ,$$

and

$$\lim_{\lambda\to 0} \left| D_{xx_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) - D_{\widetilde{x}\widetilde{x}_\lambda} F(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) \right|$$

$$= \lim_{\lambda\to 0} \left| (\overrightarrow{\nabla} F)(\overrightarrow{\widetilde{x}_{\widetilde{\lambda}_0}}(\widetilde{\sigma}_0)) \cdot \frac{(\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma)) - \left(\overrightarrow{\widetilde{x}_\lambda}(\sigma) + \overrightarrow{\widetilde{x}}(\sigma) \right)}{\lambda} \right|$$

$$\leq |DF|_0 \lim_{\lambda\to 0} \frac{\left|\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{\widetilde{x}}_\lambda(\sigma) - \overrightarrow{x}(\sigma) + \overrightarrow{\widetilde{x}}(\sigma)\right|}{\lambda}$$

we then we clearly have

$$|DW(x,y) - DW(\widetilde{x},\widetilde{y})| \leq \varepsilon^{-\frac{1}{2}} |DF|_\alpha \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} \left|\overrightarrow{\widetilde{x}}(\sigma) - \overrightarrow{x}(\sigma)\right|^\alpha \left(\lim_{\lambda\to 0} \frac{|\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma)|}{\lambda}\right)$$

$$+\varepsilon^{-\frac{1}{2}} |DF|_0 \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} \left(\lim_{\lambda\to 0} \frac{\left|\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{\widetilde{x}}_\lambda(\sigma) - \overrightarrow{x}(\sigma) + \overrightarrow{\widetilde{x}}(\sigma)\right|}{\lambda}\right) \left(\lim_{\lambda\to 0} \frac{|\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x}(\sigma)|}{\lambda}\right)$$

$$+\varepsilon^{-\frac{1}{2}} |DF|_0 \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} \left(\lim_{\lambda\to 0} \frac{\left|\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{\widetilde{x}}_\lambda(\sigma) - \overrightarrow{x}(\sigma) + \overrightarrow{\widetilde{x}}(\sigma)\right|}{\lambda}\right) \equiv J_1 + J_2 + J_3 \,. \tag{5.24}$$

In order to estimate J_1, consider

$$\frac{d}{ds}\left(\frac{\overrightarrow{x_\lambda} - \overrightarrow{x}}{\lambda}\right) = \frac{1}{\lambda}(\overrightarrow{v}(\overrightarrow{x_\lambda}) - \overrightarrow{v}(\overrightarrow{x}))\,. \tag{5.25}$$

By integration

$$\frac{|\overrightarrow{x_\lambda} - \overrightarrow{x}|}{\lambda}(s) \leq \frac{|\overrightarrow{x_\lambda} - \overrightarrow{x}|}{\lambda}(0) + \int_0^{|s|} \left|\frac{1}{\lambda}(\overrightarrow{v}(\overrightarrow{x_\lambda}) - \overrightarrow{v}(\overrightarrow{x}))\right| d\sigma$$

$$\leq 1 + \frac{1}{\lambda}\int_0^{|s|} |D\widetilde{v}|_0 \, |\overrightarrow{x_\lambda} - \overrightarrow{x}|(\sigma)\cdot d\sigma\,.$$

so that by Gronwall's inequality

$$\frac{|\overrightarrow{x_\lambda}(s) - \overrightarrow{x}(s)|}{\lambda} \leq e^{|D\widetilde{v}|_0 |s|}\,. \tag{5.26}$$

Using Lemma 5.2 and (5.26) we find that

$$\begin{aligned} |J_1| &\leq \varepsilon^{-\frac{1}{2}} |DF|_\alpha \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} e^{|D\widetilde{v}|_0|\sigma|} e^{\alpha c(\varepsilon)|\sigma|} \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha (0)\cdot d\sigma \\ &= \frac{\varepsilon^{\frac{1}{2}}}{1 + (\alpha c(\varepsilon) + |D\widetilde{v}|_0)\,\varepsilon} \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha (0) \leq C\varepsilon^{\frac{1}{2}} |DF|_\alpha \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha (0)\,, \end{aligned} \tag{5.27}$$

if ε is small enough.

To estimate J_2 and J_3, consider the function

$$\overrightarrow{A}(s) = \left(\frac{(\overrightarrow{x_\lambda} - \overrightarrow{x}) - \left(\overrightarrow{\widetilde{x}}_\lambda - \overrightarrow{\widetilde{x}}\right)}{\lambda}\right)(s)\,.$$

It satisfies:

$$\frac{d}{ds}\overrightarrow{A}(s) = \frac{1}{\lambda}(\overrightarrow{v}(\overrightarrow{x_\lambda}) - \overrightarrow{v}(\overrightarrow{\widetilde{x}}_\lambda) - \overrightarrow{v}(\overrightarrow{x}) + \overrightarrow{v}(\overrightarrow{\widetilde{x}}))$$

$$= \frac{1}{\lambda}\left(\left(D\overrightarrow{\widetilde{v}}(\overrightarrow{x_{\lambda_0}})\right)\cdot(\overrightarrow{x_\lambda} - \overrightarrow{x}) - \left(D\overrightarrow{\widetilde{v}}(\overrightarrow{\widetilde{x}_{\widetilde{\lambda_0}}})\right)\cdot(\overrightarrow{\widetilde{x}_\lambda} - \overrightarrow{\widetilde{x}})\right)$$

$$= \left(D\overrightarrow{\widetilde{v}}(\overrightarrow{x_{\lambda_0}}) - D\overrightarrow{\widetilde{v}}(\overrightarrow{\widetilde{x}_{\lambda_0}})\right)\cdot\frac{(\overrightarrow{x_\lambda} - \overrightarrow{x})}{\lambda} + \left(D\overrightarrow{\widetilde{v}}(\overrightarrow{\widetilde{x}_{\widetilde{\lambda_0}}})\right)\cdot\overrightarrow{A}(s)\,.$$

By integration

$$\overrightarrow{A} = \int_0^s \left(D\overrightarrow{\widetilde{v}}(\overrightarrow{x_{\lambda_0}}) - D\overrightarrow{\widetilde{v}}(\overrightarrow{\widetilde{x}_{\lambda_0}})\right)\cdot\frac{(\overrightarrow{x_\lambda} - \overrightarrow{x})}{\lambda}d\sigma + \int_0^s \left(D\overrightarrow{\widetilde{v}}(\overrightarrow{\widetilde{x}_{\widetilde{\lambda_0}}})\right)\cdot\overrightarrow{A}\,d\sigma\,,$$

which implies, using (5.26), that

$$\left|\overrightarrow{A}\right| \leq |D\widetilde{v}|_\alpha \int_0^{|s|}\left|\overrightarrow{x_{\lambda_0}} - \overrightarrow{\widetilde{x}_{\lambda_0}}\right|^\alpha(\sigma)\frac{|\overrightarrow{x_\lambda} - \overrightarrow{x}|}{\lambda}(\sigma)\cdot d\sigma + |D\widetilde{v}|_0\int_0^s\left|\overrightarrow{A}\right|(\sigma)\cdot d\sigma$$

$$\leq |D\widetilde{v}|_\alpha \int_0^{|s|}\left|\overrightarrow{x_{\lambda_0}} - \overrightarrow{\widetilde{x}_{\lambda_0}}\right|^\alpha(\sigma)\cdot e^{|D\widetilde{v}|_0|\sigma|}d\sigma + |D\widetilde{v}|_0\int_0^{|s|}\left|\overrightarrow{A}\right|(\sigma)\cdot d\sigma$$

or, in the limit $\lambda \to 0$ (which implies that also $\lambda_0 \to 0$),

$$\left|\overrightarrow{A}\right| \leq |D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)\int_0^{|s|}e^{(\alpha c(\varepsilon)+|D\widetilde{v}|_0)|\sigma|}d\sigma + |D\widetilde{v}|_0\int_0^{|s|}\left|\overrightarrow{A}\right|(\sigma)\cdot d\sigma$$

$$= \frac{e^{(\alpha c(\varepsilon)+|D\widetilde{v}|_0)|s|} - 1}{\alpha c(\varepsilon) + |D\widetilde{v}|_0}|D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0) + |D\widetilde{v}|_0\int_0^{|s|}\left|\overrightarrow{A}\right|(\sigma)\cdot d\sigma$$

Using Gronwall's inequality and defining $a(\varepsilon) = \alpha c(\varepsilon) + |D\widetilde{v}|_0$, we get

$$\left|\overrightarrow{A}\right| \leq \left(\frac{e^{a(\varepsilon)|s|} - 1}{a(\varepsilon)} + |D\widetilde{v}|_0\int_0^{|s|}\frac{e^{a(\varepsilon)|\sigma|} - 1}{a(\varepsilon)}e^{|s|-|\sigma|}d\sigma\right)|D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)$$

$$= \left(\frac{e^{a(\varepsilon)|s|} - 1}{a(\varepsilon)} + |D\widetilde{v}|_0\frac{-e^{a(\varepsilon)|s|} - a(\varepsilon) + 1 + e^{|s|}a(\varepsilon)}{a(\varepsilon)\,(1 - a(\varepsilon))}\right)|D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)$$

$$\leq Ce^{b(\varepsilon)|s|}\,|D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)\,,$$

where $b(\varepsilon) = \max(a(\varepsilon), 1)$ and C is a constant depending on $a(\varepsilon)$ and $|D\widetilde{v}|_0$. Recalling (5.26) we get

$$|J_2| \leq C\varepsilon^{-\frac{1}{2}}\left(\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}}e^{\alpha|D\widetilde{v}|_0|\sigma|}e^{b(\varepsilon)|\sigma|}d\sigma\right)|DF|_0\,|D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)$$

$$\leq \frac{C\varepsilon^{\frac{1}{2}}}{1 - \varepsilon\,(b(\varepsilon) + \alpha\,|D\widetilde{v}|_0)}|DF|_0\left|D\overrightarrow{\widetilde{v}}\right|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)$$

$$\leq 2C\varepsilon^{\frac{1}{2}}\,|DF|_0\left|D\overrightarrow{\widetilde{v}}\right|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)\,, \tag{5.28}$$

provided $1 - \varepsilon\,(b(\varepsilon) + \alpha\,|D\widetilde{v}|_0) > \frac{1}{2}$. Similarly,

$$|J_3| \leq C\varepsilon^{-\frac{1}{2}}\left(\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}}e^{b(\varepsilon)|\sigma|}d\sigma\right)|DF|_0\,|D\widetilde{v}|_\alpha\left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha(0)$$

$$\leq \frac{C\varepsilon^{\frac{1}{2}}}{1-\varepsilon b(\varepsilon)} |DF|_0 \left|D\overrightarrow{\widetilde{v}}\right|_\alpha \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha (0) \leq 2C\varepsilon^{\frac{1}{2}} |DF|_0 \left|D\overrightarrow{\widetilde{v}}\right|_\alpha \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha (0) , \tag{5.29}$$

Combining (5.28), (5.28) and (5.29) we conclude from (5.24) that

$$|DW(x,y) - DW(\widetilde{x},\widetilde{y})| \leq C\varepsilon^{\frac{1}{2}} \left(|DF|_\alpha + 4\left|D\overrightarrow{\widetilde{v}}\right|_\alpha |DF|_0\right) \left|\overrightarrow{x} - \overrightarrow{\widetilde{x}}\right|^\alpha (0) ,$$

so that

$$|DW|_\alpha \leq \varepsilon^{\frac{1}{2}} C(\left\|\overrightarrow{\widetilde{v}}\right\|_{C^{1+\alpha}(R)}) \|F\|_{C^{1+\alpha}(R)} \ . \tag{5.30}$$

Combining the inequalities (5.21), (5.23), (5.30), the assertion (5.19) follows.

In order to prove (5.20) we write the differential equation satisfied by W:

$$W + \varepsilon \left(\overrightarrow{\widetilde{v}} \cdot \overrightarrow{\nabla}\right) W = \varepsilon^{\frac{1}{2}} F , \tag{5.31}$$

where

$$\overrightarrow{\widetilde{v}} = (\widetilde{\psi}_y, -\widetilde{\psi}_x) .$$

We multiply equation (5.31) by W and integrate in R to obtain

$$\int_R W^2 dxdy + \frac{\varepsilon}{2}\int_R \left(\widetilde{\psi}_y \frac{\partial W^2}{\partial x} - \widetilde{\psi}_x \frac{\partial W^2}{\partial y}\right) dxdy = \varepsilon^{\frac{1}{2}} \int_R FW dxdy \tag{5.32}$$

or, integrating by parts the second integral at the left-hand side of (5.32) and using the fact that $(\widetilde{\psi}_y, -\widetilde{\psi}_x) = (0,0)$ at the boundary of R,

$$\int_R W^2 dxdy + \frac{\varepsilon}{2}\left(\lim_{x\to\infty}\int_{-h(x)}^{h(x)} \widetilde{\psi}_y W^2 - \lim_{x\to-\infty}\int_{-h(x)}^{h(x)} \widetilde{\psi}_y W^2\right) = \varepsilon^{\frac{1}{2}} \int_R FW dxdy . \tag{5.33}$$

By the hypothesis on $\widetilde{\psi}$ and inequality (5.19) we have

$$\left|\widetilde{\psi}_y W^2\right| \leq C\varepsilon^{\frac{1}{2}} \left(\|\overrightarrow{v_0}\|_{C^1(R)} + \varepsilon^{\frac{1}{2}}\right) \|F\|_{C^{1+\alpha}(R)}$$

and therefore, from equation (5.33) we can write

$$\|W\|^2_{L^2(R)} \leq C\varepsilon^{\frac{3}{2}} \left(\|\overrightarrow{v_0}\|_{C^1(R)} + \varepsilon^{\frac{1}{2}}\right) \|F\|^2_{C^{1+\alpha}(R)} + \varepsilon^{\frac{1}{2}} \int_R FW dxdy$$

$$\leq C\varepsilon^{\frac{3}{2}} \left(\|\overrightarrow{v_0}\|_{C^1(R)} + \varepsilon^{\frac{1}{2}}\right) \|F\|^2_{C^{1+\alpha}(R)} + \frac{1}{2}\left(\varepsilon \|F\|^2_{L^2(R)} + \|W\|^2_{L^2(R)}\right) ,$$

which implies

$$\|W\|_{L^2(R)} \leq C\varepsilon^{\frac{1}{2}} \|F\|_{C^{1+\alpha}(R)\cap L^2(R)} ,$$

and this completes the proof of the lemma.□

6 Existence and uniqueness

Lemma 6.1 *Denote by* ψ_1, $\overline{\psi_1} \in C^{5+\alpha}(R) \cap H^4(R)$ *the solutions of (5.7)-(5.9) corresponding to* $F(x,y) \in C^{1+\alpha}(R) \cap L^2(R)$ *and* $\overline{F}(x,y) \in C^{1+\alpha}(R) \cap L^2(R)$, *respectively, with characteristics corresponding to* $\widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\psi_1}$ *and* $\widetilde{\overline{\psi}} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\overline{\psi_1}}$, *respectively. Then*

$$\left\|\psi_1 - \overline{\psi_1}\right\|_{C^{4+\alpha}(R)} \leq C\left(\varepsilon^{\frac{3}{2}}\left\|\widetilde{\psi_1} - \widetilde{\overline{\psi_1}}\right\|_{C^{4+\alpha}(R)} + \varepsilon^{\frac{1}{2}}\left\|F - \overline{F}\right\|_{C^{\alpha}(R)}\right), \quad (6.1)$$

provided ε *is small enough, independently of* F *and* $\overline{F}$.

Proof. We have to estimate the difference of the solutions of system (5.7)-(5.9) corresponding to $F(x,y)$ and $\overline{F}(x,y)$ and with characteristics corresponding to $\widetilde{\psi}$ and $\widetilde{\overline{\psi}}$. The differential equation for the difference is:

$$\Delta^2(\psi_1 - \overline{\psi_1}) + \varepsilon((\overrightarrow{v_0} + \varepsilon^{\frac{1}{2}}\overrightarrow{\overline{v_1}}) \cdot \overrightarrow{\nabla})\Delta^2(\psi_1 - \overline{\psi_1}) = -\varepsilon^{\frac{3}{2}}((\overrightarrow{\widetilde{v_1}} - \overrightarrow{\widetilde{\overline{v_1}}}) \cdot \overrightarrow{\nabla})\Delta^2\overline{\psi_1}$$
$$+\varepsilon^{\frac{1}{2}}\left(F(x,y) - \overline{F}(x,y)\right).$$

where $\overrightarrow{\widetilde{v}} = \overrightarrow{v_0} + \varepsilon^{\frac{1}{2}}\overrightarrow{\widetilde{v_1}}$ and $\overrightarrow{\widetilde{\overline{v}}} = \overrightarrow{v_0} + \varepsilon^{\frac{1}{2}}\overrightarrow{\widetilde{\overline{v_1}}}$ are the velocity fields corresponding to $\widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\psi_1}$ and $\widetilde{\overline{\psi}} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\overline{\psi_1}}$. Proceeding as in Lemma 5.3, we obtain

$$\left\|\psi_1 - \overline{\psi_1}\right\|_{C^{4+\alpha}(R)} \leq C\left(\varepsilon^{\frac{3}{2}}\left\|((\overrightarrow{\widetilde{v_1}} - \overrightarrow{\widetilde{\overline{v_1}}}) \cdot \overrightarrow{\nabla})\Delta^2\overline{\psi_1}\right\|_{C^{\alpha}(R)} + \varepsilon^{\frac{1}{2}}\left\|F - \overline{F}\right\|_{C^{\alpha}(R)}\right)$$

and, since

$$\left\|((\overrightarrow{\widetilde{v_1}} - \overrightarrow{\widetilde{\overline{v_1}}}) \cdot \overrightarrow{\nabla})\Delta^2\overline{\psi_1}\right\|_{C^{\alpha}(R)} \leq C\left\|\overline{\psi_1}\right\|_{C^{5+\alpha}(R)}\left\|\widetilde{\psi_1} - \widetilde{\overline{\psi_1}}\right\|_{C^{5+\alpha}(R)} \leq C\left\|\widetilde{\psi_1} - \widetilde{\overline{\psi_1}}\right\|_{C^{4+\alpha}(R)},$$

the proof is complete.□

We wish to prove that the mapping T defined in (5.10) maps the unit ball in $C^{5+\alpha}(R) \cap H^4(R)$ into itself and is a contraction. To do that we need to estimate the explicit expressions of both F and $\overline{F}$ as given by (11.2) when $\psi = \widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\psi_1}$ and $\psi = \widetilde{\overline{\psi}} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\overline{\psi_1}}$, respectively, as well as the difference $F - \overline{F}$. Notice that

$$\begin{aligned} F(x,y) &= \varepsilon^{-1}B\left[\widetilde{\psi},\widetilde{\psi}\right] \\ &= \varepsilon^{-1}\left(B[\psi_0,\psi_0] + \varepsilon^{\frac{1}{2}}B\left[\psi_0,\widetilde{\psi_1}\right] + \varepsilon^{\frac{1}{2}}B\left[\widetilde{\psi_1},\psi_0\right] + \varepsilon B\left[\widetilde{\psi_1},\widetilde{\psi_1}\right]\right), \end{aligned} \quad (6.2)$$

$$\begin{aligned} \overline{F}(x,y) &= \varepsilon^{-1}B\left[\widetilde{\overline{\psi}},\widetilde{\overline{\psi}}\right] \\ &= \varepsilon^{-1}\left(B[\psi_0,\psi_0] + \varepsilon^{\frac{1}{2}}B\left[\psi_0,\widetilde{\overline{\psi_1}}\right] + \varepsilon^{\frac{1}{2}}B\left[\widetilde{\overline{\psi_1}},\psi_0\right] + \varepsilon B\left[\widetilde{\overline{\psi_1}},\widetilde{\overline{\psi_1}}\right]\right) \end{aligned} \quad (6.3)$$

and recall that the coefficients of the bilinear form B are bounded by $C\varepsilon$.

Lemma 6.2 *Let $\widetilde{\psi_1}$ and $\widetilde{\widetilde{\psi_1}}$ be such that $\left\|\widetilde{\psi_1}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$ and $\left\|\widetilde{\widetilde{\psi_1}}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$. Then*

$$\|F\|_{C^{1+\alpha}(R)\cap L^2(R)} \leq$$

$$\leq C\left(\|\psi_0\|^2_{C^{5+\alpha}(R)\cap H^4(R)} + \varepsilon^{\frac{1}{2}}\|\psi_0\|_{C^{5+\alpha}(R)\cap H^4(R)}\left\|\widetilde{\psi_1}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} + \varepsilon\left\|\widetilde{\psi_1}\right\|^2_{C^{5+\alpha}(R)\cap H^4(R)}\right), \tag{6.4}$$

$$\left\|\overline{F}\right\|_{C^{1+\alpha}(R)\cap L^2(R)} \leq$$

$$\leq C\left(\|\psi_0\|^2_{C^{5+\alpha}(R)\cap H^4(R)} + \varepsilon^{\frac{1}{2}}\|\psi_0\|_{C^{5+\alpha}(R)\cap H^4(R)}\left\|\widetilde{\widetilde{\psi_1}}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} + \varepsilon\left\|\widetilde{\widetilde{\psi_1}}\right\|^2_{C^{5+\alpha}(R)\cap H^4(R)}\right), \tag{6.5}$$

$$\left\|F - \overline{F}\right\|_{C^{\alpha}(R)} \leq C\varepsilon^{\frac{1}{2}}\|\psi_0\|_{C^{4+\alpha}(R)}\left\|\widetilde{\psi_1} - \widetilde{\widetilde{\psi_1}}\right\|_{C^{4+\alpha}(R)}. \tag{6.6}$$

Proof. The estimates (6.4), (6.5) follow from the form of F and $\overline{F}$ and the inequality:

$$\|fg\|_{C^{5+\alpha}(R)\cap H^4(R)} \leq C\|f\|_{C^{5+\alpha}(R)\cap H^4(R)}\|g\|_{C^{5+\alpha}(R)\cap H^4(R)}. \tag{6.7}$$

The inequality (6.6) follows from an inequality similar to (6.7) for $C^{4+\alpha}(R)$ instead of $C^{5+\alpha}(R)$ together with the assumptions that $\left\|\widetilde{\psi_1}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$, $\left\|\widetilde{\widetilde{\psi_1}}\right\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$.□

We are now in a position to prove the existence and uniqueness:

Theorem 6.3 *If ε is small enough then there exists a unique solution ψ of system (5.1)-(5.5) of the form*

$$\psi = \psi_0 + \varepsilon^{\frac{1}{2}}\psi_1,$$

with $\|\psi_1\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$.

Proof. If ε is small enough then, by Lemmas 5.3 and 6.2, the mapping T defined by (5.10) maps the unit ball in $C^{5+\alpha}(R)\cap H^4(R)$ into itself. Writing

$$\varepsilon F = B\left[\widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\psi_1}, \widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\psi_1}\right],$$

Lemmas 6.1 and 6.2 show that for ε small enough T is a contraction in $C^{4+\alpha}(R)$. Hence, the proof of the Banach fixed point theorem can be applied in $C^{4+\alpha}(R)$ to conclude that there exists a unique solution of the system (5.1)-(5.5) in $C^{5+\alpha}(R)\cap H^4(R)$.□

Remark 6.1. The proof of Theorem 6.3 can actually be carried out with $C^{4+\alpha}$ estimates rather than $C^{5+\alpha}$ estimates (i.e., showing that T maps a unit ball in $C^{4+\alpha}(R)\cap H^4(R)$ into itself and is a contraction). This is so because the nonlinear perturbation F involves only third order derivatives of ψ. However, the results of Section 7 as well as Section 11 do require $C^{5+\alpha}$ estimates since in these models derivatives of ψ up to order four occur in the perturbation terms.

Remark 6.2. When the normal stress modulus α_1 is negative, we can formulate a system identical to (5.11)-(5.14) except for the term at the right-hand side of (5.12) which now is

$$\varepsilon^{-\frac{1}{2}} \int_0^\infty e^{\frac{\sigma}{\varepsilon}} F(x(\sigma), y(\sigma)) d\sigma ,$$

with the characteristics $(x(s), y(s))$ defined as before. All the results of sections 5 and 6 may be proved analogously and, in particular, Theorem 6.3 holds.

7 Existence and uniqueness for more general non-Newtonian fluids

In this section we consider several non-Newtonian models grouped in the so called Oldroyd 8-Constant Model (see for example [2]). The stress tensor is

$$\mathbf{T} = -pI + \mathbf{S} ,$$

and $\mathbf{S}$ satisfies the following constitutive equation:

$$\mathbf{S} + \lambda_1 \mathbf{S}_{(1)} + \lambda_3 \{\mathbf{DS} + \mathbf{SD}\} + \lambda_5 (tr\mathbf{S})\mathbf{D} + \lambda_6 (\mathbf{S} : \mathbf{D})\mathbf{I} =$$

$$= 2\mu\mathbf{D} + \lambda_2 \mathbf{D}_{(1)} + \lambda_4 \mathbf{D}^2 + \lambda_7 (\mathbf{D} : \mathbf{D})\mathbf{I} , \qquad (7.1)$$

where we have used the notation

$$\mathbf{Q}_{(1)} = \left(\overrightarrow{v} \cdot \overrightarrow{\nabla}\right) \mathbf{Q} + \mathbf{Q} \overrightarrow{\nabla} \overrightarrow{v} + \left(\overrightarrow{\nabla} \overrightarrow{v}\right)^T \mathbf{Q} , \qquad (7.2)$$

and

$$(A : B) = \sum_{ij} A_{ij} B_{ij} .$$

We note that (7.2) corresponds to the so called "lower convected" derivative. There are also models which use "upper convected" derivative of $\mathbf{Q}$, namely,

$$\left(\overrightarrow{v} \cdot \overrightarrow{\nabla}\right) \mathbf{Q} - \overrightarrow{\nabla} \overrightarrow{v} \mathbf{Q} - \mathbf{Q} \left(\overrightarrow{\nabla} \overrightarrow{v}\right)^T ;$$

the results of this section apply to such models as well. We shall make the rather natural assumption that λ_1, λ_2 have the same sign and $|\lambda_2| \le C |\lambda_1|$ for some positive constant C. We concentrate on the case λ_1, λ_2 positive but, as in the case of second order fluid, the results can be easily extended to the case of λ_1, λ_2 negative (see Remark 6.2).

The model (7.1) includes as special cases:

(a) the Newtonian model (λ_1 to λ_7 are all zero),

(b) the second order fluid model ($\lambda_1 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 0$),

(c) the Maxwell model (λ_2 to λ_7 are all zero),

(d) the Oldroyd-B model (λ_3 to λ_7 are all zero),

(e) the Gordon-Schowalter or Johnson-Segalman model ($\lambda_2 = 2\mu_s\lambda_1$, $\lambda_3 = \xi\lambda_1$, $\lambda_4 = \xi\lambda_2$, $\lambda_5 = \lambda_6 = \lambda_7 = 0$).

Applying the operator D_{ij}, defined by

$$D_{ij}\mathbf{Q} = \frac{\partial^{i+j}\mathbf{Q}}{\partial x^i \partial y^j} ,$$

to the constitutive equation (7.1) we obtain the following equation:

$$D_{ij}\mathbf{S} + \lambda_1(\overrightarrow{v} \cdot \overrightarrow{\nabla})D_{ij}\mathbf{S} = 2\mu D_{ij}\mathbf{D} + \lambda_2(\overrightarrow{v} \cdot \overrightarrow{\nabla})D_{ij}\mathbf{D} + B_{ij}(\overrightarrow{v}, \mathbf{S}) , \qquad (7.3)$$

where

$$B_{ij}(\overrightarrow{v}, \mathbf{S}) = \sum_{\substack{k+l+m+n=i+j+2 \\ k+l\geq 2}} a_{klmn} D_{kl}\overrightarrow{v} D_{mn}\overrightarrow{v}$$

$$+ \sum_{\substack{k+l+m+n=i+j+1 \\ k+l\geq 2}} b_{klmn} D_{kl}\overrightarrow{v} D_{mn}\mathbf{S} + \sum_{\substack{k+l+m+n=i+j \\ k+l\geq 0}} c_{klmn} D_{kl}\mathbf{S} D_{mn}\mathbf{S} ,$$

with $D_{kl}\overrightarrow{v}$ or $D_{kl}\mathbf{S}$ meaning the derivative D_{kl} of any of the 2 or 2×2 components of $\overrightarrow{v}$ and $\mathbf{S}$ respectively.

Given the hyperbolic structure of the constitutive equation (7.3) (which is analogous to equation (1.1)), we can integrate along the characteristics of $\overrightarrow{v}$ to obtain the following implicit expression for $D_{ij}\mathbf{S}$:

$$D_{ij}\mathbf{S} = \frac{1}{\lambda_1}\int_{-\infty}^{0} e^{\frac{\sigma}{\lambda_1}} \left(2\mu D_{ij}\mathbf{D} + \lambda_2(\overrightarrow{v} \cdot \overrightarrow{\nabla})D_{ij}\mathbf{D} + B_{ij}(\overrightarrow{v}, \mathbf{S})\right) d\sigma$$

$$= \frac{\lambda_2}{\lambda_1} D_{ij}\mathbf{D} + \frac{1}{\lambda_1}\int_{-\infty}^{0} e^{\frac{\sigma}{\lambda_1}} \left(\left(2\mu - \frac{\lambda_2}{\lambda_1}\right) D_{ij}\mathbf{D} + B_{ij}(\overrightarrow{v}, \mathbf{S})\right) d\sigma . \qquad (7.4)$$

Lemma 7.1 *(i) Assume that*

$$\overrightarrow{v} = \overrightarrow{v_0} + \varepsilon^{\frac{1}{2}}\overrightarrow{v_1} ,$$

where $\overrightarrow{v_0}$ *is the Stokes flow in* R *and* $\|\overrightarrow{v_1}\|_{C^{4+\alpha}(R)\cap H^3(R)} < 1$. *Then, for* ε, λ_i $(1 \leq i \leq 7)$ *small enough and* $\mathbf{D}$ *the rate of deformation tensor based on* $\overrightarrow{v}$ *(see formula (2.3)), there exists a unique solution* $\mathbf{S}$ *of equation (7.3) in* $C^{3+\alpha}(R) \cap H^2(R)$ *and it satisfies*

$$\|\mathbf{S}\|_{C^{3+\alpha}(R)\cap H^2(R)} \leq \frac{C}{\lambda_1}\left(\left|2\mu - \frac{\lambda_2}{\lambda_1}\right| + \lambda_2\right) \|\overrightarrow{v}\|_{C^{4+\alpha}(R)\cap H^3(R)} .$$

(ii) Consider two velocity fields

$$\overrightarrow{v} = \overrightarrow{v_0} + \varepsilon^{\frac{1}{2}}\overrightarrow{v_1} ,$$
$$\overrightarrow{\tilde{v}} = \overrightarrow{v_0} + \varepsilon^{\frac{1}{2}}\overrightarrow{\tilde{v}_1} ,$$

and the corresponding solutions $\mathbf{S}$,$\tilde{\mathbf{S}}$ *of equation (7.3). Then*

$$\left\|\mathbf{S} - \tilde{\mathbf{S}}\right\|_{C^{2+\alpha}(R)} \leq C\varepsilon^{\frac{1}{2}}\left(\left|2\mu - \frac{\lambda_2}{\lambda_1}\right| + \lambda_2\right) \left\|\overrightarrow{v_1} - \overrightarrow{\tilde{v}_1}\right\|_{C^{3+\alpha}(R)} .$$

Proof. The proof is a simple fixed point argument for $\mathbf{S}$ and its derivatives in equation (7.4) along the same lines of Lemmas 5.3, 6.1, 6.2 and Theorem 6.3.□

If we apply curl$(\cdot)$ to the conservation of momentum equation (2.1) we get

$$\mathrm{curl}((\overrightarrow{v}\cdot\overrightarrow{\nabla})\overrightarrow{v}) = \mathrm{curl}(\mathrm{div}(\mathbf{T})) = -\,\mathrm{curl}(\mathrm{grad}(p)) + \mathrm{curl}(\mathrm{div}(\mathbf{S})) = \mathrm{curl}(\mathrm{div}(\mathbf{S}))\,. \tag{7.5}$$

Note that

$$\mathrm{curl}(\mathrm{div}(\mathbf{Q}_{(1)})) = \left(\overrightarrow{v}\cdot\overrightarrow{\nabla}\right)\mathrm{curl}(\mathrm{div}\,\mathbf{Q})$$
$$+\,\mathrm{curl}\left(\mathrm{div}\left(\mathbf{Q}\overrightarrow{\nabla}\,\overrightarrow{v} + \left(\overrightarrow{\nabla}\,\overrightarrow{v}\right)^T\mathbf{Q}\right)\right) + \left[\mathrm{curl}(\mathrm{div}(\cdot)), \overrightarrow{v}\cdot\overrightarrow{\nabla}\right]\mathbf{Q} \tag{7.6}$$

where $[A,B]\,f \equiv (AB - BA)\,f$.

An elementary but lengthy computation shows that

$$\mathrm{curl}(\mathrm{div}(\mathbf{D}_{(1)})) = (\overrightarrow{v}\cdot\overrightarrow{\nabla})\,\mathrm{curl}(\mathrm{div}(\mathbf{D}))\,; \tag{7.7}$$

this is in fact (2.7) written in terms of ψ instead of $\mathbf{D}$ on the right-hand side.

If we apply curl(div$(\cdot)$) to equation (7.1) and use (7.6), (7.7) we obtain

$$\mathrm{curl}(\mathrm{div}(\mathbf{S})) + \lambda_1\left(\overrightarrow{v}\cdot\overrightarrow{\nabla}\right)\mathrm{curl}(\mathrm{div}(\mathbf{S})) = 2\mu\,\mathrm{curl}(\mathrm{div}(\mathbf{D}))$$
$$+\lambda_2\left(\overrightarrow{v}\cdot\overrightarrow{\nabla}\right)\mathrm{curl}(\mathrm{div}(\mathbf{D})) + B^*(\overrightarrow{v},\mathbf{S})\,, \tag{7.8}$$

where

$$B^*(\overrightarrow{v},\mathbf{S}) = \mathrm{curl}(\mathrm{div}(\lambda_4\mathbf{D}^2 + \lambda_7\,(\mathbf{D}:\mathbf{D})\,\mathbf{I} - \lambda_3\,\{\mathbf{DS}+\mathbf{SD}\} - \lambda_5(tr\mathbf{S})\mathbf{D} - \lambda_6\,(\mathbf{S}:\mathbf{D})\,\mathbf{I}))$$
$$-\lambda_1\left[\mathrm{curl}(\mathrm{div}(\cdot)), \overrightarrow{v}\cdot\overrightarrow{\nabla}\right]\mathbf{S} - \lambda_1\,\mathrm{curl}(\mathrm{div}(\mathbf{S}\overrightarrow{\nabla}\,\overrightarrow{v} + \left(\overrightarrow{\nabla}\,\overrightarrow{v}\right)^T\mathbf{S}))\,.$$

Note that (7.5) comes from the conservation of momentum, whereas (7.8) comes from the constitutive law. We wish to combine both equations in order to obtain a simple reformulation of the fluid flow problem.

We define the stream function ψ such that

$$\overrightarrow{v} = (\psi_y, -\psi_x)\ ,$$

substitute the explicit expression for $\mathbf{D}$ given by (2.3) into the right-hand side of (7.8) and then insert curl(div$(\mathbf{S})$) from (7.5) into the first two terms on the left-hand side of (7.8) to deduce the following equation for ψ:

$$\Delta^2\psi + \varepsilon\left(\overrightarrow{v}\cdot\overrightarrow{\nabla}\right)\Delta^2\psi = -\frac{1}{\mu}B^*(\overrightarrow{v},\mathbf{S}) + \frac{1}{\mu}C\left[\overrightarrow{v},\overrightarrow{v}\right]\ , \tag{7.9}$$

where $\varepsilon = \frac{\lambda_2}{2\mu}$, and

$$C\left[\overrightarrow{v},\overrightarrow{v}\right] = \mathrm{curl}((\overrightarrow{v}\cdot\overrightarrow{\nabla})\overrightarrow{v}) + \lambda_1\left(\overrightarrow{v}\cdot\overrightarrow{\nabla}\right)\mathrm{curl}((\overrightarrow{v}\cdot\overrightarrow{\nabla})\overrightarrow{v})\,.$$

Note that the right-hand side of (7.9) contains derivatives of ψ and $\mathbf{S}$ of, at most, fourth order and second order respectively.

The formulation (7.9) enables us to use Lemmas 5.3, 6.1 and 6.2 together with Lemma 7.1 in order to proceed as in the proof of Theorem 6.3. In this way we can establish the following existence and uniqueness result.

Theorem 7.2 *If $|\lambda_i|$ $(1 \le i \le 7)$ are small enough and μ is large enough, then there exists a unique solution ψ of equation (7.9) with* **S** *given by equation (7.1) and the boundary conditions (5.3)-(5.5), which has the form*

$$\psi = \psi_0 + \varepsilon^{\frac{1}{2}} \psi_1 ,$$

where ψ_0 is the stream function associated to Stokes flow and $\|\psi_1\|_{C^{5+\alpha}(R)\cap H^4(R)} < 1$.

Remark 7.1. In the limiting case $\lambda_2 = 0$ (which corresponds to Maxwell model if λ_3 to λ_7 are all zero) we arrive to equation (7.9) with $\varepsilon = 0$ and Theorem 7.2 would follow by simply using the theory for the biharmonic equation.

8 Introduction to the problem with corners

In this section we study the case in which the boundary is regular everywhere except for a finite number of corners. For simplicity we shall deal with the case where the domain is symmetric with respect to the x-axis and has just one corner at each component of its boundary.

We recall that existence of solutions inside an infinite wedge is well known (cf. [5]). In particular we have:

Lemma 8.1 *Consider the wedge $S = \{r \in [0, \infty), \theta \in [0, \gamma]\}$, where $\pi \le \gamma \le \pi + \varepsilon$. If ε is small enough then there exist at least two solutions ψ_1, ψ_2 of the problem*

$$\Delta^2 \psi = 0 \quad in\ S , \tag{8.1}$$

$$\psi = \psi_\theta = 0 \quad in\ \partial S , \tag{8.2}$$

having the form

$$\psi_1(r, \theta) = r^{l_1(\gamma)+1} f(\theta; l_1(\gamma)) ,$$
$$\psi_2(r, \theta) = r^{l_2(\gamma)+1} f(\theta; l_2(\gamma)) ,$$

where $l_1(\gamma)$ and $l_2(\gamma)$ are roots of the trascendental equation $l^2 \sin^2 \gamma = \sin^2(l\gamma)$ near $l = 1$ and $l = 2$ respectively, and $0 < l_1(\gamma) < 1,\ 1 < l_2(\gamma) < 2$ if $\gamma > \pi$.

Proof. If we substitute $\psi(r, \theta) = r^{l+1} f(\theta)$ and introduce it into equation (8.1), we find that $f(\theta)$ must satisfy the equation:

$$f^{(iv)} + ((l+1)^2 + (l-1)^2) f'' + (l+1)^2 (l-1)^2 f = 0 ,$$

so that

$$f(\theta) = A \sin((l-1)\theta) + B \cos((l-1)\theta) + C \sin((l+1)\theta) + C \cos((l+1)\theta) .$$

Imposing the boundary conditions (8.2), we get the eigenvalue condition

$$\left|\begin{array}{cccc} 0 & 1 & 0 & 1 \\ (l-1) & 0 & l+1 & 0 \\ \sin((l-1)\gamma) & \cos((l-1)\gamma) & \sin((l+1)\gamma) & \cos((l+1)\gamma) \\ (l-1)\cos((l-1)\gamma) & -(l-1)\sin((l-1)\gamma) & (l+1)\cos((l+1)\gamma) & -(l+1)\sin((l+1)\gamma) \end{array}\right| = 0 \, ,$$

which reduces to the equation:

$$F(l,\gamma) \equiv l \sin\gamma \pm \sin(l\gamma) = 0 \, . \qquad (8.3)$$

Defining $f(\gamma) = \frac{\sin\gamma}{\gamma}$ we can rewrite (8.3) in the form

$$f(l\gamma) = \pm f(\gamma) \, .$$

For $\gamma = \pi$ there exists an infinite family of roots $l_m = m$, $m = 1, 2, \ldots$. The implicit function theorem shows that there exist continuous branches $l(\gamma)$ of $F(l,\gamma) = 0$ in $m - \varepsilon \le l \le m + \varepsilon$ with ε small enough, and that

$$\left.\frac{dl}{d\gamma}\right|_{l=m,\gamma=\pi} = -\left.\frac{F_\gamma}{F_l}\right|_{l=m,\gamma=\pi} = -\left.\frac{l\cos\gamma \pm l\cos(l\gamma)}{\sin\gamma \pm \gamma\cos(l\gamma)}\right|_{l=m,\gamma=\pi} = -\frac{-m \pm (-1)^m i}{\pm\pi(-1)^m} = \frac{-m \pm (-1)^m m}{\pi} \, ,$$

an expression which is always negative if we pick the "$-$" sign for m even or the "$+$" sign for m odd. Such branches for $m = 1$ and $m = 2$ yield the asserted solutions.□

In this and in the next two sections we shall consider domains R bounded by curves $y = \pm h(x)$ where:

$$h(x) = \begin{cases} 1 \quad \text{if } x \le 0 \, , \\ 1 + x\tan\gamma \quad \text{if } 0 < x \le 2\delta\,|\cos\gamma| \, , \\ f(x) \quad \text{if } 2\delta\,|\cos\gamma| < x \le 3\delta\,|\cos\gamma| \, , \\ 1 + 3\delta\sin\gamma \quad \text{if } x > 3\delta\,|\cos\gamma| \, , \end{cases} \qquad (8.4)$$

δ is a small positive number, $\gamma = \pi + \varepsilon$ and $f(x)$ is a nondecreasing function chosen in such a way that $h(x) \in C^\infty(\mathbb{R}^+)$ (Our results extend to more general domains; see Remark 10.2). Note that R has corner points at $(0, \pm 1)$. We are interested in proving the existence of solutions of the non-Newtonian system when ε is small enough. In order to do that we split the domain R in five regions:

$$\begin{aligned} A_1 &\equiv B_\delta(0,1) \cap R \, , \\ A_2 &\equiv B_\delta(0,-1) \cap R \, , \\ A_3 &= (B_{2\delta}(0,1) \backslash B_\delta(0,1)) \cap R \, , \\ A_4 &= (B_{2\delta}(0,-1) \backslash B_\delta(0,-1)) \cap R \, , \\ A_5 &= R \backslash (A_1 \cup A_2 \cup A_3 \cup A_4) \, ; \end{aligned}$$

note that A_2 and A_4 are the reflections about the $x-$axes of A_1 and A_3, respectively.

We introduce a function $\overline{\psi}$ defined as

$$\overline{\psi} = \eta_1 \psi_1 + \eta_2 \psi_2 + (1 - \eta_1 - \eta_2)\, \psi_3 \ ,$$

where η_1 and η_2 are $C^\infty(R)$ functions such that:

$$\eta_1 = \begin{cases} 1 \ \text{ if } (x,y) \in A_1 \ , \\ 0 \ \text{ if } (x,y) \in R \backslash (A_1 \cup A_3) \ , \end{cases}$$

$$\eta_2 = \begin{cases} 1 \ \text{ if } (x,y) \in A_2 \ , \\ 0 \ \text{ if } (x,y) \in R \backslash (A_2 \cup A_4) \ ; \end{cases}$$

the function ψ_1 is defined as

$$\psi_1 = \frac{2}{3} - r^{l_1(\gamma)+1} f(\theta; l_1(\gamma)) + \frac{1}{3} r^{l_2(\gamma)+1} f(\theta; l_2(\gamma)) \ , \tag{8.5}$$

with

$$r = \sqrt{x^2 + (y-1)^2} \ , \ \ \theta = -\arctan \frac{1-y}{x} \ ; \tag{8.6}$$

the function ψ_2 is defined as

$$\psi_2 = -\frac{2}{3} + r^{l_1(\gamma)+1} f(\theta; l_1(\gamma)) - \frac{1}{3} r^{l_2(\gamma)+1} f(\theta; l_2(\gamma)) \ , \tag{8.7}$$

with

$$r = \sqrt{x^2 + (y+1)^2} \ , \ \ \theta = \arctan \frac{-1+y}{x} \ , \tag{8.8}$$

and the function ψ_3 is defined as

$$\psi_3(x,y) = \frac{y}{\widetilde{h}(x)} \left(1 - \frac{y^2}{3\widetilde{h}^2(x)} \right)$$

where $\widetilde{h}(x)$ is a smooth function everywhere, which coincides with $h(x)$ for $|x| < \delta\, |\cos \gamma|$.

It is easy to verify that

$$\frac{\partial \overline{\psi}}{\partial x} = \frac{\partial \overline{\psi}}{\partial y} = 0 \quad \text{at } \partial R \ .$$

It is important to observe, using (8.6), (8.8) that, when $\gamma = \pi$, $\overline{\psi}$ coincides with the Poiseuille solution

$$\psi = y \left(1 - \frac{y^2}{3} \right) = \frac{2}{3} - (1-y)^2 + \frac{1}{3}(1-y)^3 = \frac{2}{3} - r^2 \sin^2 \theta + \frac{1}{3} r^3 \sin^3 \theta \ ,$$

of the biharmonic equation.

The main idea upon which the subsequent analysis is based is the proximity of the solutions of the non-Newtonian system, when ε is small, to the function $\overline{\psi}$ that we have just constructed.

9 The Stokes flow in a domain with corners

Let R be the domain bounded by $\pm h(x)$, h as in (8.4). We write the stream function ψ_0, corresponding to the Stokes flow, in the form

$$\psi_0 = \overline{\psi} + \varepsilon \widetilde{\psi} \ .$$

Since

$$\begin{aligned} \Delta^2 \psi_0 &= 0 \quad \text{in } R \ , \\ \frac{\partial \psi_0}{\partial x} &= \frac{\partial \psi_0}{\partial y} = 0 \quad \text{in } \partial R \ , \end{aligned}$$

we have

$$\begin{aligned} \Delta^2 \widetilde{\psi} &= -\varepsilon^{-1} \Delta^2 \overline{\psi} \quad \text{in } R \ , \\ \frac{\partial \widetilde{\psi}}{\partial x} &= -\varepsilon^{-1} \frac{\partial \overline{\psi}}{\partial x} = 0 \quad \text{in } \partial R \ , \\ \frac{\partial \widetilde{\psi}}{\partial y} &= -\varepsilon^{-1} \frac{\partial \overline{\psi}}{\partial y} = 0 \quad \text{in } \partial R \ . \end{aligned}$$

The function $\widetilde{\psi}$ then satisfies the equation:

$$\Delta^2 \widetilde{\psi} = H(x, y; \varepsilon) \quad \text{in } R \ , \tag{9.1}$$

where $H(x, y) = -\varepsilon^{-1} \Delta^2 \overline{\psi}$, with the boundary conditions:

$$\frac{\partial \widetilde{\psi}}{\partial x} = \frac{\partial \widetilde{\psi}}{\partial y} = 0 \quad \text{in } \partial R \ , \tag{9.2}$$

$$\left| \widetilde{\psi} \right| \to 0 \quad \text{as } |x| \to \infty \ . \tag{9.3}$$

Lemma 9.1 *If $\varepsilon_0 > 0$ is sufficiently small then, for any $0 < \varepsilon < \varepsilon_0$, the function $H(x, y; \varepsilon)$ satisfies the following estimate:*

$$\|H\|_{C^{1+\alpha}(R)} \leq C \ , \tag{9.4}$$

where C is a constant independent of ε.

Proof. Clearly

$$H(x, y) \equiv -\varepsilon^{-1} \Delta^2 \overline{\psi} = 0 \quad \text{in } S \equiv A_1 \cup A_2 \cup \{(x, y) / \, |x| > 3\delta\} \ .$$

Outside S we have

$$\Delta^2 \overline{\psi} = \Delta^2 \psi_3 + \Delta^2 \left(\eta_1 (\psi_1 - \psi_3) \right) + \Delta^2 \left(\eta_2 (\psi_2 - \psi_3) \right) \equiv J_1 + J_2 + J_3 \ ,$$

and we need to estimate the $C^{1+\alpha}$ norm of the J_i. We begin with

$$J_1 = \Delta^2 \psi_3 = \Delta^2 P \left(x, \frac{y}{\widetilde{h}(x)} \right) \ .$$

Reasoning as in Lemma 4.1, we get

$$J_1 = \varepsilon H_1(x, y; \varepsilon) ,$$

where $\|H_1\|_{C^{1+\alpha}(R)} \leq C$ uniformly in ε.

On the other hand,

$$J_2 = \Delta^2 \left(\eta_1(\psi_1 - \psi_3)\right) = \sum_{\alpha=0}^{4} \eta_1^{(\alpha)} D^{4-\alpha}(\psi_1 - \psi_3) ,$$

so that, in order to obtain the bound

$$\|J_2\|_{C^{1+\alpha}(A_1)} \leq C\varepsilon , \tag{9.5}$$

we only have to prove that

$$\|\psi_1 - \psi_3\|_{C^{5+\alpha}(A_1)} \leq C\varepsilon , \tag{9.6}$$

We write

$$\|\psi_1 - \psi_3\|_{C^{5+\alpha}(A_1)} \leq \|\psi_1 - \varphi\|_{C^{5+\alpha}(A_1)} + \|\psi_3 - \varphi\|_{C^{5+\alpha}(A_1)} ,$$

where

$$\varphi = \frac{2}{3} - r^2 \sin^2\left(\frac{\pi}{\gamma}\theta\right) + \frac{1}{3} r^3 \sin^3\left(\frac{\pi}{\gamma}\theta\right) .$$

and then, by simple calculation, find that

$$\begin{aligned} \|\psi_1 - \varphi\|_{C^{5+\alpha}(A_1)} &\leq C\varepsilon , \\ \|\psi_3 - \varphi\|_{C^{5+\alpha}(A_1)} &\leq C\varepsilon . \end{aligned} \tag{9.7}$$

Thus (9.6) follows.

Since the estimate for J_3 is identical to that for J_2, the proof of (9.4) is complete.□

The existence of a unique weak solution in $H_0^2(R)$ of (9.1)-(9.3) is can be established using the Lax-Milgram theorem (the fact that R has corners does not affect the proof). But we shall need further estimates on the asymptotic behavior of ψ near the corners.

The asymptotic behavior of the solutions of elliptic equations in non-smooth domains has been the subject of intense research since the first work of Kondrati'ev (cf. [9]). Subsequent work is reported in the books [11], [10], [12] and the references cited there.

In the sequel we shall work with functions f that are $O(r^\beta)$ near corner points $(0,1)$, $(0,-1)$, whose m^{th}−order derivatives are bounded by $O(r^{\beta-m})$, and their Hölder coefficients have a corresponding growth. It is useful to introduce the weight functions $\rho_s(X;\beta)$ defined by

$$\rho_s(X;\beta) = \begin{cases} \delta^{\beta-s}\,|X-(0,1)|^{s-\beta} & \text{if } X \in A_1 , \\ \delta^{\beta-s}\,|X-(0,-1)|^{s-\beta} & \text{if } X \in A_2 , \\ 1 & \text{if } R \setminus (A_1 \cup A_2) . \end{cases}$$

where $X = (x, y)$, and the norms

$$\|f\|_{C^n_\beta(R)} = \sum_{|m|\leq n} \sup \left(\rho_{|m|}(X;\beta) \, |D^m f| \right) \ ,$$

$$\|f\|_{C^{n+\alpha}_\beta(R)} = \|f\|_{C^n_\beta(R)} + \sum_{|m|=n} \sup_{X,X'} \frac{|\rho_{n+\alpha}(X;\beta) D^n_X f(X) - \rho_{n+\alpha}(X';\beta) D^n_{X'} f(X')|}{|X - X'|^\alpha} \ ,$$

$$\|f\|^2_{H^n_\beta(R)} = \sum_{|m|\leq n} \int_R \rho^2_{|m|}(X;\beta) \, |D^m_X f(X)|^2 \, dX \ .$$

One can immediately verify the following continuous embedding properties:

$$C^n_\alpha(R) \subset C^m_\beta(R), \ H^n_\alpha(R) \subset H^m_\beta(R) \quad \text{if } \beta < \alpha \text{ and } m < n \ . \tag{9.8}$$

It is also easy to prove the following inequality:

$$\|ab\|_{C^m_{2s_1}(R)\cap H^n_{2s_2}(R)} \leq C \, \|a\|_{C^m_{s_1}(R)\cap H^n_{s_2}(R)} \, \|b\|_{C^m_{s_1}(R)\cap H^m_{s_2}(R)} \ . \tag{9.9}$$

Lemma 9.2 *The unique solution of (9.1)-(9.3) in $H^2_0(R)$ satisfies the estimate*

$$\left\| \widetilde{\psi} \right\|_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} \leq C \, \|H(x, y; \varepsilon)\|_{C^\alpha(R)\cap L^2(R)} \ , \tag{9.10}$$

where C is independent of ε.

Proof. From the work of Maz'ya and Plamenevskij (see [10] and [12]) it follows that:

$$\left\| \widetilde{\psi} \right\|_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} \leq C \, \|H(x, y; \varepsilon)\|_{C^\alpha_{l_1-3}(R)\cap H^0_2(R)} \ .$$

and, since the support of $H(x, y; \varepsilon)$ lies outside $A_1 \cup A_2$, the right-hand side is bounded by the right-hand side of (9.10).□

Lemma 9.3 *Let*

$$\psi(x, y) = \overline{\psi}(x, y) + \varepsilon^{\frac{1}{2}} \widetilde{\psi}(x, y) \ ,$$

satisfy (2.8), (2.12), (2.13), and assume that $\left\| \widetilde{\psi} \right\|_{C^2_{l_1+1}(R)} < 1$. Then for any ε small enough:

1.- There does not exist any stagnation point for ψ, i.e., $(\psi_y, -\psi_x) \neq 0$ everywhere in the interior of R.

2.- The characteristic curves (3.4) do not intersect, they cover the entire region R, and

$$\left| \frac{dy}{dx} \right| \leq C \quad \textit{everywhere in } R. \tag{9.11}$$

Proof. The proof is similar to the proof of Lemma 3.1 with the only difference being that now, near the corners, the equations for the characteristics are:

$$
\begin{aligned}
\frac{dr}{ds} &= v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left(-r^{l_1+1}f'(\theta;l_1) + \frac{1}{3}r^{l_2+1}f'(\theta;l_2)\right)\left(1+\varepsilon^{\frac{1}{2}}g_1(r,\theta)\right) , \\
r\frac{d\theta}{ds} &= v_\theta = -\frac{\partial\psi}{\partial r} = \left((l_1+1)r^{l_1}f(\theta;l_1) - \frac{(l_2+1)}{3}r^{l_2}f(\theta;l_2)\right)\left(1+\varepsilon^{\frac{1}{2}}g_2(r,\theta)\right) ,
\end{aligned}
\tag{9.12}
$$

where r and θ are polar coordinates around the corners and, by the hypothesis on $\widetilde{\psi}$, the functions g_1 and g_2 are bounded (uniformly in ε). For angles γ close to π and r small enough (independent of ε), the function

$$
(l_1+1)r^{l_1}f(\theta;l_1) - \frac{(l_2+1)}{3}r^{l_2}f(\theta;l_2) , \tag{9.13}
$$

has zeros only at $\theta = 0$ and $\theta = \gamma$, i.e., the angular velocity component v_θ does not vanish for $0 < \theta < \gamma$. In order to see this we note that both $f(\theta;l_1)$ and $f(\theta;l_2)$ do not vanish if $0 < \theta < \gamma$, and vanish just linearly at $\theta = 0$, $\theta = \pi$; hence near the origin the leading term in (9.13) is $(l_1+1)r^{l_1}f(\theta;l_1)$, from which the above assertion follows. This fact enables us to proceed as in Lemma 3.1 and show that ψ has no stagnation points, its characteristics do not intersect, and they fill the entire domain R.

Recall that inequality (9.11) was proved in Lemma 3.1 for smooth boundaries. In the present situation we have, near a corner,

$$
\frac{dy}{dx} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta} \sim \frac{-\frac{1}{(l_1+1)}r\frac{f'(\theta;l_1)}{f(\theta;l_1)}\sin\theta + r\cos\theta}{-\frac{1}{(l_1+1)}r\frac{f'(\theta;l_1)}{f(\theta;l_1)}\cos\theta - r\sin\theta} = \frac{\frac{1}{(l_1+1)}f'(\theta;l_1)\sin\theta - f(\theta;l_1)\cos\theta}{\frac{1}{(l_1+1)}f'(\theta;l_1)\cos\theta + f(\theta;l_1)\sin\theta} .
$$

But

$$
\begin{aligned}
f(\theta;l_1) &= \sin^2\frac{\pi}{\gamma}\theta + \varepsilon g(\theta;l_1,\varepsilon) , \\
f'(\theta;l_1) &= 2\frac{\pi}{\gamma}\sin\frac{\pi}{\gamma}\theta\cos\frac{\pi}{\gamma}\theta + \varepsilon g'(\theta;l_1,\varepsilon)
\end{aligned}
$$

where g and g' are uniformly bounded in ε, and therefore

$$
\frac{dy}{dx} = \frac{2\frac{\pi}{\gamma}\frac{1}{(l_1+1)}\cos\frac{\pi}{\gamma}\theta\sin\theta - \sin\frac{\pi}{\gamma}\theta\cos\theta}{2\frac{\pi}{\gamma}\frac{1}{(l_1+1)}\cos\frac{\pi}{\gamma}\theta\cos\theta + \sin\frac{\pi}{\gamma}\theta\sin\theta} + \varepsilon G(\theta;l_1,\varepsilon) , \tag{9.14}
$$

with G uniformly bounded. The estimate (9.11) now follows by noting that the first term on the right-hand side of (9.14) is bounded as its denominator is strictly uniformly positive for $0 < \theta < \gamma$.□

10 Non-Newtonian flow in a domain with corners

Proceeding as in the case of domains with smooth boundaries, we shall establish existence and uniqueness of the non-Newtonian flow problem in domains with corners as defined in Section 8. Although there is a substantial difference due to singularities near the corner, nevertheless, the integration along the characteristics leads to an extra regularization which will allow us to consider the non-Newtonian flow as an small perturbation of a Newtonian flow. Again, if we write

$$\psi = \psi_0 + \varepsilon^{\frac{1}{2}}\psi_1 \ ,$$

we have to deal with the following system:

$$\Delta^2\psi_1 = W(x,y) \ , \tag{10.1}$$

$$W(x,y) = \varepsilon^{-\frac{1}{2}}\int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} F(x(\sigma),y(\sigma))d\sigma \ , \tag{10.2}$$

$$\frac{\partial\psi_1}{\partial x} = \frac{\partial\psi_1}{\partial y} = 0 \quad \text{at } \partial R \ , \tag{10.3}$$

$$|\psi_1| \to 0 \quad \text{as } |x| \to \infty \ , \tag{10.4}$$

with $F(x,y) = \varepsilon^{-1}B[\psi,\psi]$ and $(x(s),y(s))$ the characteristic curve initiating at (x,y) and associated with ψ.

We define a mapping T as follows:

Given a stream function $\widetilde{\psi}$ of the form

$$\widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}}\widetilde{\psi_1} \ , \ \left\|\widetilde{\psi_1}\right\|_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} < 1 \ ,$$

define a function W by (10.2) where $(x(s),y(s))$ are the characteristic curves associated with $\widetilde{\psi}$, and

$$F(x,y) = \varepsilon^{-1}B\left[\widetilde{\psi},\widetilde{\psi}\right] = \varepsilon^{-1}B\left[\psi_0+\varepsilon^{\frac{1}{2}}\widetilde{\psi_1},\psi_0+\varepsilon^{\frac{1}{2}}\widetilde{\psi_1}\right] \ . \tag{10.5}$$

We shall prove, later on, that there exists a unique solution of (10.1), (10.3), (10.4), (10.5) in $C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)$, and set

$$T\widetilde{\psi_1} = \psi_1 \ .$$

Note that a fixed point of T is a solution to (5.1)-(5.5), and vice-versa.

Theorem 10.1 *If ε is small enough then there exists a unique solution ψ of system (5.1)-(5.5) in $C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)$ of the form*

$$\psi = \psi_0 + \varepsilon^{\frac{1}{2}}\psi_1 \ ,$$

with $\|\psi_1\|_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} < 1$.

Proof of Theorem 10.1:

Observe that $F(x,y)$ contains a singular term near a corner point, namely,

$$B[\psi_0,\psi_0] = \varepsilon r^{2l_1-2}g(\theta) + \varepsilon O(r^{2l_1-2+\kappa}) \ , \text{ for some } \kappa > 0 \ .$$

It is for this reason that we shall be working with the norm $H^0_{-2}(R)$ of F rather than $L^2(R)$. In particular, we shall use the estimate

$$\|\psi_1\|_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} \le C\,\|W\|_{C^{\alpha}_{l_1-3}(R)\cap H^0_{-2}(R)} \tag{10.6}$$

for solutions of (10.1), (10.3), (10.4); this estimate, with a somewhat different notation, was established by Maz'ya and Plamenevskij (see [11], [10] and [12]) for domains with corners.

Lemma 10.2 *Let*

$$W(x,y) = \varepsilon^{-\frac{1}{2}} \int_{-\infty}^{0} e^{\frac{\sigma}{\varepsilon}} F(x(\sigma), y(\sigma)) d\sigma \,,$$

where $(x(s), y(s))$ *are the characteristics associated to* $\widetilde{\psi} = \psi_0 + \varepsilon^{\frac{1}{2}} \widetilde{\psi_1}$ *with* $\left\| \widetilde{\psi_1} \right\|_{C^{4+\alpha}_{l_1+1}(R) \cap H^4_2(R)} < 1$*, and assume* $F(x,y) \in C^1_\beta(R) \cap H^1_{-2}(R)$ *with* $\beta < 0$*. Then*

$$\|W\|_{C^1_\beta(R) \cap H^1_{-2}(R)} \leq M \varepsilon^{\frac{1}{2}} \|F\|_{C^1_\beta(R) \cap H^1_{-2}(R)} \tag{10.7}$$

if ε *is sufficiently small, where* M *is a constant independent of* ε*.*

Proof. The proof is similar to the proof of Lemma 5.3 except for the regions around the corner points. The characteristics near the corners are, to a leading order, given by

$$\begin{aligned} \frac{dr}{ds} &= v_r = \frac{1}{r} \frac{\partial \widetilde{\psi}}{\partial \theta} = -r^{l_1} f'(\theta; l_1) \,, \\ r \frac{d\theta}{ds} &= v_\theta = -\frac{\partial \widetilde{\psi}}{\partial r} = (l_1+1) r^{l_1} f(\theta; l_1) \,. \end{aligned} \tag{10.8}$$

Near the corners, these curves can be written as

$$\widetilde{\psi} = r^{l_1+1} f(\theta; l_1) = C \,. \tag{10.9}$$

We now fix our attention on characteristic curves close to the corner, i.e., such that $r(0) = r_0$ with r_0 positive and sufficiently small. If we substitute $f(\theta; l_1)$ by $\sin^2(\frac{\pi}{\gamma}\theta)$, we introduce only errors of order ε. By doing so we can write the equation for the characteristic curves (not only for their leading order terms) in the form

$$\begin{aligned} \frac{dr}{ds} &= \left(\pm \frac{2\pi}{\gamma} r^{l_1} g(r,C)^{\frac{1}{2}} \sqrt{1 - g(r,C)} \right) (1 + O(\varepsilon)) \,, \\ r(0) &= r_0 \,, \end{aligned} \tag{10.10}$$

where

$$g(r,C) = \frac{C}{r^{l_1+1}} \,;$$

the sign "+" corresponds to $r(s)$ before its minimum $r_m = C^{\frac{1}{l_1+1}})$ at say $s = s_{\min}$, and the sign "−" corresponds to $r(s)$ after it has reached its minimum. For clarity, we first drop out the $O(\varepsilon)$-term in (10.10). We can then write

$$s(r) = \begin{cases} -\frac{\gamma}{2\pi} \int_r^{r_0} \frac{dr}{r^{l_1} \sqrt{g(r,C)} \sqrt{1-g(r,C)}} & \text{if } s > s_{\min} \,, \\ -\frac{\gamma}{2\pi} \int_{r_m}^{r_0} \frac{dr}{r^{l_1} \sqrt{g(r,C)} \sqrt{1-g(r,C)}} - \frac{\gamma}{2\pi} \int_{r_m}^{r} \frac{dr}{r^{l_1} \sqrt{g(r,C)} \sqrt{1-g(r,C)}} & \text{if } s < s_{\min} \,. \end{cases}$$

Note that by the change of variables $\tilde{r} = C^{-\frac{1}{l_1+1}} r$,

$$\int_r^{r_0} \frac{dr}{r^{l_1}\sqrt{g(r,C)}\sqrt{1-g(r,C)}} = C^{\frac{1-l_1}{1+l_1}} \int_{C^{-\frac{1}{l_1+1}} r}^{C^{-\frac{1}{l_1+1}} r_0} \frac{\tilde{r}d\tilde{r}}{\sqrt{\tilde{r}^{l_1+1}-1}} \ ,$$

and, therefore, for $s > s_{\min}$,

$$s(r) = -C^{\frac{1-l_1}{1+l_1}} \frac{\gamma}{2\pi} \int_{C^{-\frac{1}{l_1+1}} r}^{C^{-\frac{1}{l_1+1}} r_0} \frac{\tilde{r}d\tilde{r}}{\sqrt{\tilde{r}^{l_1+1}-1}} = -C^{\frac{1-l_1}{1+l_1}} \left(G(C^{-\frac{1}{l_1+1}} r_0) - G(C^{-\frac{1}{l_1+1}} r) \right) \ , \tag{10.11}$$

where

$$G(x) = \frac{\gamma}{2\pi} \int_1^x \frac{\tilde{r}d\tilde{r}}{\sqrt{\tilde{r}^{l_1+1}-1}} \ ,$$

whereas

$$s(r) = -C^{\frac{1-l_1}{1+l_1}} \left(G(C^{-\frac{1}{l_1+1}} r_0) + G(C^{-\frac{1}{l_1+1}} r) \right)$$

for $s < s_{\min}$.

The function $G(x)$ is strictly monotone increasing and so is $G^{-1}(x)$. Since

$$G(x) \sim \frac{\gamma}{\pi(3-l_1)} x^{\frac{3-l_1}{2}} + o(x^{\frac{3-l_1}{2}}) \quad \text{as } x \to \infty \ , \tag{10.12}$$

we have

$$G^{-1}(x) \sim \left(\frac{\pi(3-l_1)}{\gamma} x \right)^{\frac{2}{3-l_1}} + o(x^{\frac{2}{3-l_1}}) \quad \text{as } x \to \infty \ . \tag{10.13}$$

By inverting the function $s = s(r)$ in (10.11), we obtain

$$r(s) = C^{\frac{1}{l_1+1}} G^{-1} \left(C^{\frac{l_1-1}{1+l_1}} s + G(C^{-\frac{1}{l_1+1}} r_0) \right) \quad \text{if } s > s_{\min} \ , \tag{10.14}$$

$$r(s) = C^{\frac{1}{l_1+1}} G^{-1} \left(-C^{\frac{l_1-1}{1+l_1}} s - G(C^{-\frac{1}{l_1+1}} r_0) \right) \quad \text{if } s < s_{\min} \ . \tag{10.15}$$

This formulas will now be used to estimate W.

By assumption,

$$|F| \le M r^{\beta} \ , \ |DF| \le M r^{\beta-1},$$

where $M = \|F\|_{C^1_\beta(R) \cap H^1_{-2}(R)}$. Therefore

$$|W(r_0,\theta_0)| \le M\varepsilon^{-\frac{1}{2}} \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} r(\sigma)^{\beta} d\sigma =$$

$$= M\varepsilon^{-\frac{1}{2}} \int_{s_{\min}}^0 e^{\frac{\sigma}{\varepsilon}} r(\sigma)^{\beta} d\sigma + M\varepsilon^{-\frac{1}{2}} \int_{-\infty}^{s_{\min}} e^{\frac{\sigma}{\varepsilon}} r(\sigma)^{\beta} d\sigma \equiv M\varepsilon^{-\frac{1}{2}} J_1 + M\varepsilon^{-\frac{1}{2}} J_2 \ . \tag{10.16}$$

Recalling (10.14) and making a change of variables

$$\sigma' = C^{\frac{l_1-1}{1+l_1}} \sigma + G(C^{-\frac{1}{l_1+1}} r_0) \ , \tag{10.17}$$

we obtain

$$J_1 \equiv \int_{s_{\min}}^{0} e^{\frac{\sigma}{\varepsilon}} r(\sigma)^\beta d\sigma = \int_{s_{\min}}^{0} e^{\frac{\sigma}{\varepsilon}} \left(C^{\frac{1}{l_1+1}} G^{-1} \left(C^{\frac{l_1-1}{1+l_1}} \sigma + G(C^{-\frac{1}{l_1+1}} r_0) \right) \right)^\beta d\sigma =$$

$$= C^{\frac{1-l_1}{1+l_1}} e^{-\frac{1}{\varepsilon} C^{\frac{1-l_1}{1+l_1}} G(C^{-\frac{1}{l_1+1}} r_0)} \int_0^{G(C^{-\frac{1}{l_1+1}} r_0)} e^{C^{\frac{1-l_1}{1+l_1}} \frac{\sigma'}{\varepsilon}} \left(C^{\frac{1}{l_1+1}} G^{-1}(\sigma') \right)^\beta d\sigma' \tag{10.18}$$

$$= C^{\frac{1-l_1+\beta}{1+l_1}} e^{-\frac{1}{\varepsilon} C^{\frac{1-l_1}{1+l_1}} G(C^{-\frac{1}{l_1+1}} r_0)} \int_0^{G(C^{-\frac{1}{l_1+1}} r_0)} e^{C^{\frac{1-l_1}{1+l_1}} \frac{\sigma'}{\varepsilon}} \left(G^{-1}(\sigma') \right)^\beta d\sigma' .$$

Here we have used the fact that s reaches its minimum along a stream line whenever r does, and that happens when G^{-1} has its minimum value, i.e., when its argument is zero.

Recalling (10.13) we get

$$J_1 \sim D C^{\frac{1-l_1+\beta}{1+l_1}} e^{-Ka} \int_1^a e^{K\sigma'} \sigma'^\nu d\sigma' , \tag{10.19}$$

where

$$\begin{aligned} a &= \left(\frac{\gamma}{\pi(3-l_1)} \right)^{\frac{3-l_1}{2}} C^{-\frac{3-l_1}{2(l_1+1)}} r_0^{\frac{3-l_1}{2}} , \\ K &= \frac{C^{\frac{1-l_1}{1+l_1}}}{\varepsilon} , \ \nu = \frac{2\beta}{3-l_1} , \\ D &= \left(\frac{\pi(3-l_1)}{\gamma} \right)^{\frac{2\beta}{3-l_1}} . \end{aligned}$$

Making use of the asymptotic expansion:

$$\int_1^a e^{Kx} x^\nu dx \sim K^{-1} e^{Ka} (a^\nu + O(a^{\nu-1})) \quad \text{as } a \to \infty ,$$

we conclude that

$$J_1 \sim D\varepsilon r_0^\beta \quad \text{as } C^{-\frac{1}{l_1+1}} r_0 \to \infty, \ i.e., \text{ as } \theta_0 \to \gamma$$

(note that $C \sim r_0^{l_1+1}(\theta_0-\gamma)^2$). If $C^{-\frac{1}{l_1+1}} r_0 \to \eta < \infty$, then it is immediate from the expression for J_1 that

$$J_1 \le M\varepsilon C^{\frac{1-l_1+\beta}{1+l_1}} \le M\varepsilon \eta^{1-l_1+\beta} r_0^{1-l_1+\beta} \le M\varepsilon \left(\frac{r_0}{\delta} \right)^{1-l_1+\beta} ,$$

where M is a generic constant, and therefore, we can always write

$$J_1 \le M\varepsilon \left(\frac{r_0}{\delta} \right)^\beta .$$

To estimate J_2 we proceed as in (10.18),

$$J_2 = \int_{-\infty}^{s_{\min}} e^{\frac{\sigma}{\varepsilon}} r(\sigma)^\beta d\sigma = C^{\frac{1-l_1+\beta}{1+l_1}} e^{-\frac{1}{\varepsilon} C^{\frac{1-l_1}{1+l_1}} G(C^{-\frac{1}{l_1+1}} r_0)} \int_{-\infty}^{0} e^{C^{\frac{1-l_1}{1+l_1}} \frac{\sigma'}{\varepsilon}} \left(G^{-1}(-\sigma') \right)^\beta d\sigma' ,$$

where we have performed the change of variables (10.17). Using (10.13) and a change of variables $x = \varepsilon^{-1} C^{\frac{1-l_1}{1+l_1}} \sigma'$, we find that the asymptotic behavior of the last integral as $C \to 0$ is

$$\left(\frac{\pi(3-l_1)}{\gamma}\right)^{\frac{2\beta}{3-l_1}} \Gamma\left(\frac{2\beta}{3-l_1}+1\right) \varepsilon^{\frac{2\beta}{3-l_1}+1} C^{-\frac{1-l_1}{1+l_1}\left(\frac{2\beta}{3-l_1}+1\right)} ,$$

which leads to:

$$J_2 \sim M \varepsilon^{\frac{2\beta}{3-l_1}+1} C^{\frac{\beta}{3-l_1}} e^{-\frac{1}{\varepsilon} C^{\frac{1-l_1}{1+l_1}} G(C^{-\frac{1}{l_1+1}} r_0)} .$$

If we introduce $\lambda = C^{-\frac{1}{l_1+1}} r_0$ (note that $0 < \lambda_0 < \lambda < \infty$ for some λ_0), we can the write

$$J_2 \sim M \varepsilon^{\frac{2\beta}{3-l_1}+1} r_0^{\frac{\beta(l_1+1)}{3-l_1}} \lambda^{-\frac{\beta(l_1+1)}{3-l_1}} e^{-\frac{1}{\varepsilon} r_0^{1-l_1} \lambda^{l_1-1} G(\lambda)} ,$$

or more compactly

$$J_2 \sim M \varepsilon r_0^{\beta} F(x,\lambda) ,$$

where

$$F(x,\lambda) = x^{-\frac{2\beta}{3-l_1}} \lambda^{-\beta} e^{-xG(\lambda)}$$

and

$$x = \lambda^{l_1-1} \left(\frac{r_0}{\varepsilon^{\frac{1}{1-l_1}}}\right)^{1-l_1} .$$

The function $F(x,\lambda)$ is bounded for all values of $x \in \mathbb{R}^+$ and $\lambda \in (\lambda_0, +\infty)$. To prove it, notice that for every λ finite, $F(x,\lambda)$ is bounded. When λ goes to infinity, $G(\lambda) \sim C\lambda^{\frac{3-l_1}{2}}$ (cf. equation (10.12)) and therefore

$$F(x,\lambda) \sim y^{-\frac{2\beta}{3-l_1}} e^{-Cy} ,$$

with $y = x\lambda^{\frac{3-l_1}{2}}$, which is clearly a bounded function. Therefore

$$J_2 \le M \varepsilon r_0^{\beta} .$$

Thus in all cases

$$J_2 \le M \varepsilon \left(\frac{r_0}{\delta}\right)^{\beta} .$$

So far we have assumed points (r_0, θ_0) past the corners and belonging to stream lines close to the boundary. Outside these regions J_1 and J_2 are bounded functions. Thus, altogether, by (10.16),

$$\begin{aligned} W(r,\theta) &\le M \varepsilon^{\frac{1}{2}} \left(\frac{r}{\delta}\right)^{\beta} \text{ in } A_1 , \\ W(r,\theta) &\le M \varepsilon^{\frac{1}{2}} \left(\frac{r}{\delta}\right)^{\beta} \text{ in } A_2 , \end{aligned} \qquad (10.20)$$

where (r,θ) denotes polar coordinates around each corner. It is now evident that

$$\sup |\rho_0(X;\beta) W(X)| \le M \varepsilon^{\frac{1}{2}} .$$

and

$$\int_R |\rho_0(X;\beta)W(X)|^2\, dX \le M^2\varepsilon$$

Next we shall estimate for the first derivative of W. Take two points $\overrightarrow{x_0}$ and $\overrightarrow{x_1}$ in R such that $|\overrightarrow{x_1} - \overrightarrow{x_0}| = 1$ and introduce the line segment joining them:

$$\overrightarrow{x_\lambda} = \overrightarrow{x_0} + \lambda(\overrightarrow{x_1} - \overrightarrow{x_0}), \quad \lambda \in (0,1) \ ,$$

Then

$$(D_{x_0x_1}W)(x_0) = \lim_{\lambda\to 0} \frac{W(\overrightarrow{x_\lambda}) - W(\overrightarrow{x_0})}{\lambda}$$

$$= \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} \left(F(x_\lambda(\sigma), y_\lambda(\sigma)) - F(x_0(\sigma), y_0(\sigma))\right) d\sigma}{\lambda}$$

$$= \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} D_{x_0x_\lambda} F(\overrightarrow{x_{\lambda 0}}(\sigma_0)) \cdot (\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x_0}(\sigma))\, d\sigma}{\lambda} \ ,$$

for some $\lambda_0 \in (0,1)$. By Lemma 5.2,

$$|D_{x_0x_1}W| \le \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} |D_{x_0x_\lambda} F(\overrightarrow{x_{\lambda 0}}(\sigma_0))|\, |\overrightarrow{x_\lambda}(\sigma) - \overrightarrow{x_0}(\sigma)|\, d\sigma}{\lambda}$$

$$\le \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \frac{\int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} e^{-c(\varepsilon)\sigma} |D_{x_0x_\lambda} F(\overrightarrow{x_{\lambda 0}}(\sigma_0))|\, |\overrightarrow{x_\lambda} - \overrightarrow{x_0}|\,(0) d\sigma}{\lambda}$$

$$= \varepsilon^{-\frac{1}{2}} \lim_{\lambda\to 0} \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} e^{-c(\varepsilon)\sigma}\, |D_{x_0x_\lambda} F(\overrightarrow{x_{\lambda 0}}(\sigma_0))|\, d\sigma$$

$$= \varepsilon^{-\frac{1}{2}} \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} e^{-c(\varepsilon)\sigma} \left| D_{\overrightarrow{e}(\sigma)} F(\overrightarrow{x_0}(\sigma)) \right| d\sigma \le \varepsilon^{-\frac{1}{2}} \int_{-\infty}^0 e^{\frac{\sigma}{\varepsilon}} e^{-c(\varepsilon)\sigma} \left| (\overrightarrow{\nabla} F)(\overrightarrow{x_0}(\sigma)) \right| d\sigma \ ,$$

where $D_{\overrightarrow{e}(\sigma)}$ denotes the directional derivative along a direction $\overrightarrow{e}(\sigma)$ (which varies with σ).

Taking into account that near the corners,

$$\left| \overrightarrow{\nabla} F(\overrightarrow{x_0}(\sigma)) \right| \le M r(\sigma)^{\beta-1} \ , \tag{10.21}$$

where $M = \|F\|_{C^1_\beta(R)\cap H^1_{-2}(R)}$, and proceeding as in the proof of the estimate for W (but changing the exponent β into $\beta - 1$) we obtain, near the corners

$$|D_{x_0x_1}W| \le M\varepsilon^{\frac{1}{2}} r^{\beta-1} \ .$$

Since away from the corners the estimates are the same as for the case of smooth boundary, the proof of the lemma is complete.

We have now completed the proof of the lemma in case the $O(\varepsilon)$-term in (10.10) is omitted. By introducing this term however, the analysis changes just slightly, and does not affect the final estimate.□

From the previous lemma and the embeddings (9.8) we can deduce the following inequality:

$$\|W\|_{C^\alpha_{l_1-3}(R)\cap H^0_{-2}(R)} \le M\varepsilon^{-\frac{1}{2}} \|F\|_{C^1_{l_1-3}(R)\cap H^1_{-2}(R)} \le M'\varepsilon^{-\frac{1}{2}} \|F\|_{C^1_{2l_1-2}(R)\cap H^1_0(R)} \ .$$

Using (9.9), the function $F(x,y) = \varepsilon^{-1} B\left[\widetilde{\psi}, \widetilde{\psi}\right]$ can be estimated as follows:

$$\|F\|_{C^1_{2l_1-2}(R)\cap H^1_0(R)} \leq$$

$$\leq C\varepsilon\left(\|\nabla v_0\|^2_{C^1_{l_1-1}(R)\cap H^1_0(R)} + \varepsilon^{\frac{1}{2}}\|\nabla v_0\|_{C^1_{l_1-1}(R)\cap H^1_0(R)}\|\nabla \widetilde{v}_1\|_{C^1_{l_1-1}(R)\cap H^1_0(R)} + \varepsilon\|\nabla \widetilde{v}_1\|^2_{C^1_{l_1-1}(R)\cap H^1_0(R)}\right)$$

$$\begin{aligned} &\leq C\varepsilon\left(\|\psi_0\|^2_{C^3_{l_1+1}(R)\cap H^3_2(R)} + \varepsilon^{\frac{1}{2}}\|\psi_0\|^2_{C^3_{l_1+1}(R)\cap H^3_2(R)}\left\|\widetilde{\psi}_1\right\|^2_{C^3_{l_1+1}(R)\cap H^3_2(R)} + \varepsilon\left\|\widetilde{\psi}_1\right\|^2_{C^3_{l_1+1}(R)\cap H^3_2(R)}\right) \\ &\leq C\varepsilon\left(\|\psi_0\|^2_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} + \varepsilon^{\frac{1}{2}}\|\psi_0\|_{C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)} + \varepsilon\right) . \end{aligned}$$

We can now proceed as in Lemmas 6.1 and 6.2 to show that T is a contraction (in the $C^{4+\alpha}_{l_1+1}(R)$ norm) in the unit ball of $C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)$ and, consequently, by the proof of Banach fixed point theorem, the existence of a unique solution of the problem in $C^{4+\alpha}_{l_1+1}(R)\cap H^4_2(R)$ follows.□

Remark 10.1. One could also establish Theorem 10.1 by working with $C^{5+\alpha}$ norm (cf. Remark 6.1), but this will make the analysis much more complicated due to the singularity at the corner.

Remark 10.2. The proof of Theorem 10.1 extends to more general domains when $h(x)$ in (8.4) satisfies for $x > 3\delta|\cos\gamma|$ and for $x < 0$ the same conditions as $h(x)$ in (2.10), (2.11), (3.1); furthermore, one can take $\gamma = \pi + C\varepsilon$ or $\gamma = \pi + C\varepsilon^\lambda$ (where $C > 0t, \lambda > 0$) instead of $\gamma = \pi + \varepsilon$.

11 Axially symmetric flows

The results of the preceding sections can easily be extended to 3-dimensional axially symmetric flows in pipe-like domains which are either smooth or have corners. If we introduce cylindrical coordinates (r,z), and a stream function ψ such that

$$\begin{aligned} v_z &= \frac{1}{r}\psi_r , \\ v_r &= -\frac{1}{r}\psi_z , \end{aligned}$$

then analogously to (5.1)-(5.5) the flow problem can be formulated as follows:

$$\frac{\Delta^2\psi}{r} + \varepsilon\frac{d}{ds}\frac{\Delta^2\psi}{r} = \varepsilon F(r,z) , \tag{11.1}$$

where

$$\begin{aligned} \varepsilon F(r,z) &= B[\psi,\psi] := \frac{1}{r}\sum_{\substack{i+j=6\\ k+l=i\geq 2\\ m+n=j\geq 2}} a_{klmn}\left(D^k_r\frac{\partial^l\psi}{\partial z^l}\right)\left(D^n_r\frac{\partial^m\psi}{\partial z^m}\right) \\ &+\frac{1}{r}\sum_{\substack{i+j=4\\ k+l=i\geq 1\\ m+n=j\geq 1}} b_{klmn}\left(D^k_r\frac{\partial^l\psi}{\partial z^l}\right)\left(D^n_r\frac{\partial^m\psi}{\partial z^m}\right) , \end{aligned} \tag{11.2}$$

(here $\frac{d}{ds}$ is the derivative along the characteristic curve at the point (r, z) and D_r means either differentiation with respect to r or multiplication by $1/r$), with the boundary conditions

$$\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial z} = 0 \quad \text{at } \partial R \,, \tag{11.3}$$

$$\psi = \frac{r^2}{2}\left(1 - \frac{r^2}{2}\right) + o(1) \quad \text{as } z \to -\infty \,, \tag{11.4}$$

$$\psi = \frac{r^2}{2(1+\varepsilon)^2}\left(1 - \frac{r^2}{2(1+\varepsilon)^2}\right) + o(1) \quad \text{as } z \to +\infty \,. \tag{11.5}$$

Although the terms on the right-hand side of (11.2) contain fourth order derivatives, Lemmas 5.3, 6.1 and 6.2 are still valid and so is therefore also Theorem 6.3. The analysis for more general constitutive equations as in Section 7 also extends to the present case of axially symmetric flows in domains with smooth boundaries.

In the case of a domain with corners, one need to compute the expression $B[\psi, \psi]$ more carefully than in (11.2). Following the calculations as in (2.8) we find that

$$B[\psi,\psi] = \frac{1}{r}\left(-\psi_r\psi_{rrrz} + \psi_z\psi_{rrrr} + \frac{3}{2}\psi_r\psi_{zzzz} - \frac{3}{2}\psi_r\psi_{rrzz} + \psi_r\psi_{rrrr}\right) + Q(D^\alpha\psi,\ |\alpha| \le 3)\,, \tag{11.6}$$

where $Q(D^\alpha\psi,\ |\alpha| \le 3)$ is the sum of products of at most third order derivatives (in r or z) of ψ by expressions $D_r^n\frac{\partial^m\psi}{\partial z^m}$ with $n + m \le 2$; thus the main difference between the 2-dimensional and the axially symmetric cases is in the occurrence of fourth order derivatives of ψ on the right-hand side of (11.6).

Near each corner, any H^2 solution ψ for the wedge problem satisfies:

$$\psi_r, \psi_z \sim \rho^{1-\kappa} \,,$$

where ρ is the distance of a point to the corner and κ is some small positive number. Therefore,

$$B[\psi,\psi] \sim \rho^{-2-2\kappa}$$

which is of lower order than the singularity of $\Delta^2\psi$. Thus the function $B[\psi, \psi]$ constitutes a regular perturbation of $\Delta^2\psi$ and therefore the analysis of Sections 8-10 extends (with minor differences) to the present case.

Remark 11.1. The behavior of non-Newtonian flow near a corner is generally an open problem (see [14] for a discussion on second order creeping flows, and [4], [8] for discussions on the Oldroyd-B model). Theorem 10.1 and the previous discussion show that for second-order non-Newtonian fluid (either planar or axisymmetric) the singularity near the corner is the same as for Newtonian fluid, i.e.,

$$\psi \sim \rho^{1+l_1(\gamma)} f(\theta; l_1)\,.$$

However, our analysis does not include the more general fluid described in Section 7, since for such fluids some of the terms (which appear in $B[\psi, \psi]$) have higher order singularity than $\Delta^2\psi$ near the corner.

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