Saturated Linear Recurrent Neural Networks with Maximum Capacity

*Fernanda Botelho**
botelhof@msci.memphis.edu
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152

Abstract

We consider discrete and recurrent neural network models with saturated linear neurons. A definition of capacity is discussed and conditions that assure infinite capacity are established. The aim of this paper is to study networks with maximum capacity and to what extend maximal capacity relates to the network connecting weights.

1 Introduction

Neural networks are mathematical models inspired by the parallelism of our brain. Using neural networks we can build machines that reproduce some functions easily performed by our neural system but for which sequential algorithms either do not exist or are so hard to implement that become unpractical. The architecture of an artificial neural network consists on a finite number of cells or neurons with connections between them.

Pattern recognition and associative memories are examples of neural networks whose function consists in retrieving stored information. Information, represented by sequences of real numbers in the unit interval, is equally distributed among the cells composing the net, each such number is called a state of the cell and a whole sequence, a configuration of the network. As signals pass across the net connections from one cell to another, they are changed via a multiplicative factor, denoted connecting weight. Each neuron acts on the sum of all received signals, in a certain instant, via a transfer function of saturated linear type. We assume that the state update is done simultaneously at all the cells at discrete intervals of time. The dynamical system capturing the activity of the net is a continuous multivariable and multivalued real function.

Stationary configurations of a network model are those that remain unchanged after a network update.In dynamical systems standard language, these configurations correspond to fixed points

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of the function representing the network dynamics. Stationary configurations, especially stable ones, are good storage places for pattern recognition problems. In this case, networks are expected to recover the real information from a noisy one, such as retrieving a clear picture from a fuzzy or incomplete one. Eventually fixed or periodic orbits are used to mimick associative memories, where a network is expected to associate by relating information. The capacity of a network is related to the number of storage places the network accepts.

In this paper we discuss the existence of stationary configurations for saturated linear recurrent neural networks and how this set changes with the connecting weights. First, we establish conditions for the existence of infinite capacity. Next, we determine a sharp upper bound of the network capacity, and determine necessary and sufficient conditions under which the network presents maximum capacity. We also identify those configurations that are stable.

2 Basic Definitions and Notation

An n-dimensional recurrent neural network is a system with n cells and connecting channels between them. Fig.1 represents one such network. The change of a signal passing across a channel from cell i to cell j is characterized by a multiplicative factor, called connecting weight, and denoted by ω_{ji}. In Fig. 1, the path that a signal follows from cell i to cell j is represented by an oriented segment. Each cell may also have a self-connecting weight measuring a feedback connection to itself.

A configuration is an assignment of real numbers to the cells composing the net, each such number quantifies some measurable physical quantity associated to the cell, such as, the average difference potential across its membrane. An initial configuration, also called input, is a point in the unit hypercube, $\mathcal{H} = [0,1]^n$. The dynamics of the net is considered to be the discrete time evolution of an arbitrary input and is given by the following rule:

$$T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sigma\left(\sum_{i=1}^{n} \omega_{1i} x_i\right) \\ \vdots \\ \sigma\left(\sum_{i=1}^{n} \omega_{ni} x_i\right) \end{bmatrix}. \tag{1}$$

The sum $S_i = \sum_{i=1}^{n} \omega_{ij} x_j$ represents the incoming signal into cell i whose reaction is represented by the saturated linear transfer function, σ:

$$\sigma(x) = \begin{cases} 0 & if \quad x \leq 0 \\ x & if \quad 0 < x < 1 \\ 1 & if \quad x \geq 1. \end{cases}$$

Neural networks whose cells act on incoming signals via saturated linear functions are called neural networks of saturated linear type. A saturated linear neuron simulates the indifference of a real cell outside some preassigned threshold value. In fact, no cell activity is spontaneously generated, the cell saturates after a certain value of the signal's intensity and leaves it unchanged otherwise.

We set notation to be followed throughout the entire paper. Boldface lower case letters represent configurations of the network which, in fact, are points in $\mathcal{H}$. A point $\mathbf{x}$ is an n-tuple $\{x_i\}_{i=1}^{n}$, with $0 \leq x_i \leq 1$, and when convenient is also represented as a column vector, $[x_i]_{i=1}^{n}$, where

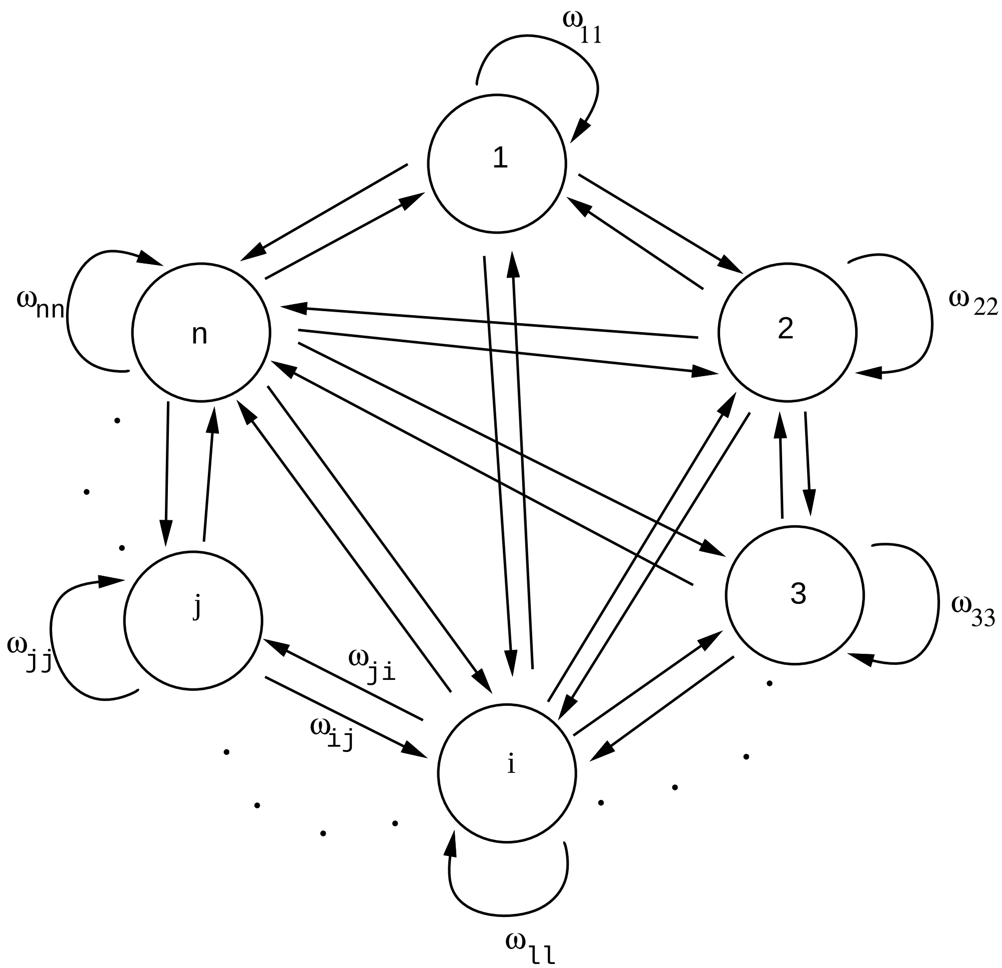

Figure 1: A Recurrent Neural Network with n Cells

x_i is the state of cell i. We denote by $\mathbf{O}$ the origin or the point in $\mathcal{H}$ with all its coordinates equal to 0. We define $\mathbf{N}$ to be the set $\{1, 2, \cdots, n\}$. Matrices are generically represented by $\mathbf{M}$ and the weight matrix, $[\omega_{ij}]_{i,j=1,\ldots,n}$, by $\mathbf{W}$. The action of $\mathbf{W}$ on some point $\mathbf{x}$ is given by the standard matrix multiplication and denoted by $\mathbf{W}(\mathbf{x})$ or just $\mathbf{Wx}$. For simplicity of notation we define $\sigma(\mathbf{x})$ to be the column vector $[\sigma(x_i)]_i$. The map π_j represents the standard projection onto the j-th component.

Definition 2.1 *Given a subset $\mathbf{A} = \{i_1, i_2, \cdots, i_k\}$ of $\mathbf{N}$, a k-simplex of $\mathcal{H}$ associated with $\mathbf{A}$ and denoted by $\mathcal{S}^k_{\mathbf{A}}$ is the set*

$$\{(x_i)_i \in \mathcal{H} \ : \ x_i = 0 \text{ or } 1 \quad \text{if } i \notin \mathbf{A}\}.$$

The interior of $\mathcal{S}^k_{\mathbf{A}}$, int($\mathcal{S}^k_{\mathbf{A}}$)= $\Big\{(x_i)_i \in \mathcal{S}^k_{\mathbf{A}} \ : \ 0 < x_i < 1, \text{ for } i \in \mathbf{A}\Big\}$. The set of all k-simpleces of $\mathcal{H}$ is denoted by $\mathcal{S}^k$ and $\mathcal{S}$ is the set of all possible simpleces in $\mathcal{H}$.

In particular, we notice that the cardinality of $\mathcal{S}^0$ is 2^n. Each 0-simplex is a vertex of the hypercube $\mathcal{H}$. There is only one n-simplex, the hypercube itself.

Definition 2.2 *We say that a k-simplex associated with $\mathbf{A}$, $\mathcal{S}^k_{\mathbf{A}}$, is adjacent to $\mathbf{O}$ if $\mathbf{O} \in \mathcal{S}^k_{\mathbf{A}}$. We denote the set of all k-simpleces adjacent to $\mathbf{O}$ by $\mathbf{Adj}_k$ and its complement by $\mathbf{NAdj}_k$.*

Given a k-simplex, Δ, associated with a set $\mathbf{A}$ we consider its two corresponding sets

$$\mathbf{A}_0 = \{\, i \in \mathbf{N} : \, \pi_i(\mathrm{int}(\Delta)) = 0 \,\} \text{ and } \mathbf{A}_1 = \{\, i \in \mathbf{N} : \, \pi_i(\mathrm{int}(\Delta)) = 1 \,\}.$$

The pair $\{\mathbf{A}_0, \mathbf{A}_1\}$ define a partition of the complement of $\mathbf{A}$.

Next, we establish necessary and sufficient conditions on the weights under which every vertex (or 0-simplex), $\mathbf{x} = \{x_i\}_i \in \mathcal{H}$ (with $x_i = 0$ or 1) is a stationary configuration i.e. $T(\mathbf{x}) = \mathbf{x}$.

Definition 2.3 *A point $\mathbf{x}$ in $\mathcal{H}$ represents a stationary configuration of the network if it is fixed under T, ie. $T(\mathbf{x}) = \mathbf{x}$. The number of stationary configurations, if finitely many, is called the* **capacity** *of the network. If this number is infinite we simply say that the network has* **infinite capacity**.

Lemma 2.1 *All 0-simpleces are stationary if and only if for every subset $\mathbf{A}_1$ of $\mathbf{N}$ the 0-simplex, $\mathbf{x} = \{x_i\}_i$ such that $_i = 1$ if and only if $i \in \mathbf{A}_1$, satisfies the condition:*

$$\pi_i(\mathbf{W}\mathbf{x}) \begin{cases} \geq 1 & \text{if} \quad i \in \mathbf{A} \\ \leq 0 & \quad\text{otherwise.} \end{cases}$$

Proof: Let $\mathbf{x}$ be a vertex of $\mathcal{H}$, such that $x_i = 1$ if and only if $i \in \mathbf{A}_1$. We have that:

$$\sigma(\mathbf{W}\mathbf{x})_i = \sigma\left(\sum_{j \in \mathbf{A}_1} \omega_{ij}\right) = \begin{cases} 1 & \text{if} \quad i \in \mathbf{A}_1 \iff \pi_i(\mathbf{W}\mathbf{x}) = \sum_{j \in \mathbf{A}_1} \omega_{ij} \geq 1 \\ 0 & \text{if} \quad i \notin \mathbf{A}_1 \iff \pi_i(\mathbf{W}\mathbf{x}) = \sum_{j \in \mathbf{A}_1} \omega_{ij} \leq 0. \end{cases} \diamondsuit$$

3 Networks with Infinite Capacity

We establish conditions that guarantee the existence of a whole segment pointwise fixed. This situation creates serious difficulties since longterm iterations may accumulate stepwise errors that grow exponentially fast. Computer generated orbits might be misleading by presenting a behavior not encountered in the real system. The next two Lemmas establish sufficient conditions for the network to have infinite capacity.

Lemma 3.1 *If $\Delta \in \mathbf{Adj}_k$, $\mathbf{x} \in \mathit{int}(\Delta)$, and $T(\mathbf{x}) = \mathbf{x}$ then T has infinitely many fixed points.*

Proof: Let $\mathbf{A}$ be the set associated with the simplex Δ and $\mathbf{x} = (x_i)_i$, then

$$\sum_{j \in \mathbf{A}} \omega_{ij} x_j \begin{cases} = x_i & \text{if} \quad i \in \mathbf{A} \\ \leq 0 & \quad \text{otherwise.} \end{cases}$$

(Δ is adjacent to $\mathbf{O}$ and therefore $\mathbf{A}_1 = \{ i \in \mathbf{N} \,|\, \pi_i(\mathrm{int}(\Delta)) = 1\} = \emptyset$.) Now, we check that the segment connecting $\mathbf{O}$ to $\mathbf{x}$ is pointwise fixed. Let $\mathbf{y} = \{\lambda x_i\}_i$, with $0 < \lambda < 1$, then

$$\pi_i\left(T(\mathbf{y})\right) = \sigma\left(\sum_{j\in\mathbf{A}} \omega_{ij}\lambda x_j\right) = \sigma\left(\lambda\sum_{j\in\mathbf{A}} \omega_{ij} x_j\right) = \left\{ \begin{array}{ccc} \sigma(\lambda x_i) = \lambda x_i & \Longleftrightarrow & i \in \mathbf{A} \\ \leq 0 & \Longleftrightarrow & i \notin \mathbf{A}. \end{array} \right. \diamondsuit$$

Lemma 3.2 *If Δ is a k-simplex non-adjacent to $\mathbf{O}$ and $\mathbf{x}$, $\mathbf{y}$ are two distinct fixed points in int(Δ) then T has infinitely many fixed points.*

Proof: Let $\mathbf{A}$ be the set associated with λ and $\mathbf{A}_0$ and $\mathbf{A}_1$ its two corresponding sets. Since Δ is not adjacent to $\mathbf{O}$ we conclude that $\mathbf{A}_1$ is nonempty. Now, we show that the segment connecting $\mathbf{x}$ to $\mathbf{y}$ is pointwise fixed. For $0 < \lambda < 1$, we have that

$$\begin{aligned} \pi_i\left(T\left(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\right)\right) &= \sigma\mathbf{W}\left(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\right)_i = \\ = \sigma\left(\lambda\mathbf{W}\mathbf{x} + (1-\lambda)\mathbf{W}\mathbf{y}\right)_i &= \left\{ \begin{array}{ll} \lambda x_i + (1-\lambda) y_i & \text{if} \quad i \in \mathbf{A} \\ 1 & \text{if} \quad i \in \mathbf{A}_1 \\ 0 & \text{if} \quad i \in \mathbf{A}_0. \end{array} \right. \diamondsuit \end{aligned}$$

The statements of these two Lemmas imply the following Theorem. The set of all stationary configurations of the network or all the fixed points of T is denoted by Fix(T).

Theorem 3.1 *If T has finitely many fixed points then Fix(T)$\subset \mathcal{S}^0 \cup \mathbf{NAdj}$ and for every $\Delta \in \mathbf{NAdj}_k$ the intersection Fix(T) $\cap$ int(Δ) has at most one point.*

4 Networks with Maximum Capacity

Throughout this section we assume that T has finitely many fixed points. The capacity of the network associated with T is the number of elements in Fix(T). Theorem 3.1 allows us to find a sharp upper bound of the number of fixed points. We also determine conditions on the weights that guarantee maximum capacity and we show that the set of stable configurations grows exponentially fast with the number of cells composing the network.

Theorem 4.1 *The capacity of a recurrent saturated linear network is at most $3^n - 2^n + 1$.*

Proof: We denote the number of elements in a set $\mathbf{A}$ by $\#\mathbf{A}$. We denote by T the dynamics associated with a recurrent saturated linear network. T has at most one fixed point in the interior of each non-adjacent (to $\mathbf{O}$) k-simplex and $\#\mathbf{NAdj}_k = \binom{n}{k}(2^{n-k} - 1)$. Therefore we have that,

$$\#\mathrm{Fix}(T) \leq \sum_{k=0}^{n} \binom{n}{k} (2^{n-k} - 1) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} - \sum_{k=1}^{n} \binom{n}{k} = 3^n - 2^n + 1. \ \diamondsuit$$

Example:

We provide an example of a 2-dimensional network with exactly six fixed points (cf. Fig. 2), weight matrix $\mathbf{W} = \begin{bmatrix} 4 & -1 \\ -1 & 5 \end{bmatrix}$, and saturated linear neurons.

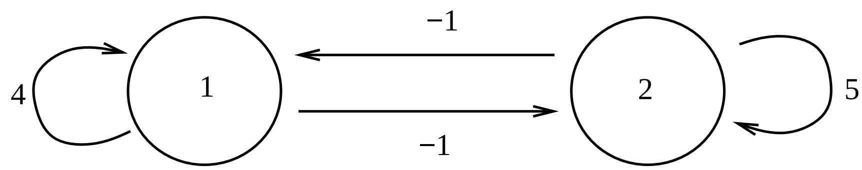

Figure 2: A two-dimensional recurrent network with six fixed points.

It is easy to verify that Fix(T)= $\left\{\mathbf{O}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\right\}$.

Next, we establish conditions on the weights of an n-cell network that guarantee its dynamics to have exactly $3^n - 2^n + 1$ fixed points. We represent by $\widetilde{\mathbf{W}} = [\omega_{ij}]$ the matrix $\mathbf{W} - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Given a subset of $\mathbf{N}$, $\mathbf{A}$, and $p \in \mathbf{N}$ we define the operator $\mathcal{L}^p_{\mathbf{A}}$, on the set of all $n \times n$ matrices, such that to each matrix $\mathbf{M} = [m_{ij}]$, associates an $n \times n$ matrix $\mathcal{L}^p_{\mathbf{A}}(\mathbf{M}) = [a_{ij}]_{i,j=1,\dots,n}$, where

$$a_{ij} = \begin{cases} m_{ij} & \text{if} \quad j \neq p \\ \sum_{j \in \mathbf{A}} m_{ij} & \text{if} \quad j = p. \end{cases}$$

Given two subsets of $\mathbf{N}$, $\mathbf{A}_0$ and $\mathbf{A}_1$ with cardinality p and q, respectively, we define the operator $\mathcal{L}_{\{\mathbf{A}_0, \mathbf{A}_1\}}$ that associates to each $n \times n$ matrix, $\mathbf{M}$, an $(n-p) \times (n-q)$ matrix obtained from $\mathbf{M}$ by deleting those rows specified in $\mathbf{A}_0$ and those columns specified in $\mathbf{A}_1$. When $\mathbf{A}_0 = \mathbf{A}_1$ we simply denote this operator by $\mathcal{L}_{\mathbf{A}_0}$. Given an $n \times n$ matrix $\mathbf{M}$, det$\mathbf{M}$ represents the determinant of $\mathbf{M}$. We denote by $\circ$ the usual composition between maps.

Theorem 4.2 *A network has maximum capacity if and only if det*$\widetilde{\mathbf{W}} \geq 0$, *and for every pair of disjoint subsets of* $\mathbf{N}$, $\{\mathbf{A}_0,\, \mathbf{A}_1\}$ *with* $\mathbf{A}_1 \neq \emptyset$, *and* $\{i\} \subset \mathbf{A}_0 \cup \mathbf{A}_1$,

$$det\ \left(\mathcal{L}_{\mathbf{A}_0 \cup \mathbf{A}_1 \setminus \{i\}} \circ \mathcal{L}^i_{\mathbf{A}_1}(\widetilde{\mathbf{W}}\right) \begin{cases} > 0 & if \quad i \in \mathbf{A}_1 \\ < 0 & if \quad i \in \mathbf{A}_0. \end{cases}$$

Proof: We show that the condition stated in the Theorem implies the existence of $3^n - 2^n + 1$ stationary configurations. It is sufficient to prove that every 0-simplex is stationary, and every non-adjacent simplex has a fixed point in its interior (cf. Theorem 4.1).

Let $\mathbf{B}_0$ and $\mathbf{B}_1 (\neq \emptyset)$ be subsets of $\mathbf{N}$. We denote by $\mathbf{B}_2 = \mathbf{N} \setminus (\mathbf{B}_0 \cup \mathbf{B}_1)$ and by Δ the simplex associated with $\mathbf{B}_2$ such that

$$\mathbf{x} \in \Delta \text{ if and only if } \pi_i(\mathbf{x}) = 1 \text{ if } i \in \mathbf{B}_1 \text{ and } \pi_i(\mathbf{x}) = 0, \text{ if } i \in \mathbf{B}_0 \setminus \mathbf{B}_1.$$

Case 1. $\mathbf{B}_2 = \emptyset$.
In this case Δ is a 0-simplex then

$$\pi_i\,(T(\mathbf{x})) = \textstyle\sum_{j\in\mathbf{B}_1}\omega_{ij} = \det\,\left(\mathcal{L}_{\mathbf{N}\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{B}_1}\right)(\mathbf{W})\begin{cases} >1 & \text{if} \quad i\in\mathbf{B}_1 \\ <0 & \text{if} \quad i\in\mathbf{B}_0. \end{cases}$$

Consequently, $\mathbf{x}$ is a fixed point.

Case 2. $\mathbf{B}_2 \neq \emptyset$.
Let $\mathbf{B}_2 = \{i_1, i_2, \ldots, i_p\}$ and, for simplicity of notation, assume that $\mathbf{B}_1 \cap \mathbf{B}_2 = \emptyset$. If T has a fixed point, $\mathbf{x} = (x_i)_i \in \text{int}(\Delta)$, then it satisfies

$$\sum_{j\in\mathbf{B}_1}\omega_{kj} + \sum_{t=1}^{p}\omega_{ki_t}x_{i_t}\begin{cases} = x_k & \text{if} \quad k\in\mathbf{B}_2 \qquad (1) \\ \geq 1 & \text{if} \quad k\in\mathbf{B}_1 \qquad (2) \\ \leq 0 & \text{if} \quad k\in\mathbf{B}_0. \qquad (3) \end{cases}$$

System (1) is equivalent to

$$\sum_{t=1}^{p}\widetilde{\omega}_{ki_t}x_{i_t} = -\sum_{j\in\mathbf{B}_1}\widetilde{\omega}_{kj} \quad (\,k\in\mathbf{B}_2\,),$$

and it has a unique solution if the determinant of the coefficient matrix, $\widetilde{\mathbf{W}}_c$, is not equal to 0. Since $\det\,\widetilde{\mathbf{W}}_c = \det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{i_t\}}\circ\mathcal{L}^{i_t}_{\{i_t\}}\big)\left(\widetilde{\mathbf{W}}\right)$, the condition stated in the Theorem (with $\mathbf{A}_0 = \mathbf{B}_0\cup\mathbf{B}_1$ and $\mathbf{A}_1 = \{i_t\}$) implies that $det\widetilde{\mathbf{W}}_c > 0$. Therefore system (1) has a unique solution given by

$$x_{i_t} = -\frac{\det\left(\widetilde{\mathbf{W}}^t_\Delta\right)}{\det\left(\widetilde{\mathbf{W}}_c\right)},$$

with

$$\det\,\widetilde{\mathbf{W}}^t_\Delta = \det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\circ\mathcal{L}^{i_t}_{\mathbf{B}_1}\big)(\widetilde{\mathbf{W}}).$$

Since $i_t \notin \mathbf{B}_1$ we have that $\det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\circ\mathcal{L}^{i_t}_{\mathbf{B}_1}\big)\left(\widetilde{\mathbf{W}}\right) < 0$ and therefore $x_{i_t} > 0$.

On the other hand, the matrices $\widetilde{\mathbf{W}}^t_\Delta$ and $\widetilde{\mathbf{W}}_c$ are identical except at the t-column. We define a new matrix $\widetilde{\mathbf{W}}^t_*$ obtained from $\widetilde{\mathbf{W}}_c$ by adding its t-column with the t-column of $\widetilde{\mathbf{W}}^t_\Delta$. We have that

$$\det\left(\widetilde{\mathbf{W}}^t_\Delta\right) + \det\left(\widetilde{\mathbf{W}}_c\right) = \det\left(\widetilde{\mathbf{W}}^t_*\right).$$

This shows that

$$\det\,\widetilde{\mathbf{W}}^t_* = \det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\circ\mathcal{L}^{i_t}_{\mathbf{B}_1\cup\{i_t\}}\big)(\widetilde{\mathbf{W}}) > 0,$$

which implies $x_{i_t} < 1$. Moreover, inequalities (2) and (3) follow from the well-known formula that computes the determinant of a matrix along the entries of a given row using (cf. [6], ch.8). This is stated in the following Lemma whose proof is omitted.

Lemma 4.1 *If* $k \in \mathbf{B}_0 \cup \mathbf{B}_1$ *then*

$$det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{k\}}\circ\mathcal{L}^k_{\mathbf{B}_1}\big)(\widetilde{\mathbf{W}}) = \sum_{j\in\mathbf{B}_1}\widetilde{\omega}_{kj}\, det\left(\widetilde{\mathbf{W}}_c\right) - \sum_{t=1}^{p}\widetilde{\omega}_{ki_t}\, det\left(\widetilde{\mathbf{W}}^t_\Delta\right).$$

Conversely, if T has $3^n - 2^n + 1$ fixed points we first prove that for every pair of disjoint sets $\mathbf{A}_0$ and $\mathbf{A}_1$ such that $\mathbf{A}_1 \neq \emptyset$ and $\mathbf{A}_0 \cup \mathbf{A}_1 = \mathbf{N}$ we have

$$\det\ (\mathrm{L}_{\mathbf{N}\setminus\{i\}}\circ\mathcal{L}^i_{\mathbf{A}_1})\left(\widetilde{\mathbf{W}}\right)\begin{cases} > 0 & \text{if} \quad i\in\mathbf{A}_1\\ < 0 & \text{if} \quad i\in\mathbf{A}_0.\end{cases}$$

The point $\mathbf{x} = (x_i)_i$ such that $x_i = 1$ if $i \in \mathbf{A}_1$ and $x_i = 0$ if $i \in \mathbf{A}_0$ is a fixed point then

$$\det\big(\mathcal{L}_{\mathbf{N}\setminus\{i\}}\circ\mathcal{L}^i_{\mathbf{A}_1}\big)\left(\widetilde{\mathbf{W}}\right)\begin{cases} \geq 0 & \text{if} \quad i\in\mathbf{A}_1\\ \leq 0 & \text{if} \quad i\in\mathbf{A}_0.\end{cases}$$

We consider Δ, a 1-simplex associated with $\mathbf{A} = \{i\}$ where $\pi_j(\mathrm{int}(\Delta)) = 1$ if and only if $j \in \mathbf{A}_1 \setminus \{i\}$. Since T has a unique fixed point $\mathbf{x}$ in $\mathrm{int}(\Delta)$ then x_i satisfies the following equation

$$\omega_{ii}x_i + \sum_{s\in\mathbf{A}_1\setminus\{i\}}\omega_{is} = x_i.$$

This implies that $\omega_{ii} > 1$ and $\sum_{s\in\mathbf{A}_1}\omega_{is} < 0$, if $i \notin \mathbf{A}_1$. If $\mathbf{A}_1 = \{i\}$ then Δ is adjacent to $\mathbf{O}$ which leads to a contradiction (cf Theorem 4.1). If $i \in \mathbf{A}_1$ and since $x_i < 1$ we have that $\sum_{s\in\mathbf{A}_1}\omega_{is} > 1$. This completes the proof of the statement above.

Now, we proceed by induction, let us assume that for every $\mathbf{B}_0$ and $\mathbf{B}_1$, disjoint subsets of $\mathbf{N}$, such that $\mathbf{B}_1 \neq \emptyset$ and $\#(\mathbf{N} \setminus \mathbf{B}_0 \cup \mathbf{B}_1) \leq k - 1$ we have

$$\det\ \big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{i\}}\circ\mathcal{L}^i_{\mathbf{B}_1}\big)(\widetilde{\mathbf{W}})\begin{cases} > 0 & \text{if} \quad i\in\mathbf{B}_1\\ < 0 & \text{if} \quad i\in\mathbf{B}_0.\end{cases}$$

We want to show that given $\mathbf{A}_0$ and $\mathbf{A}_1$, disjoint subsets of $\mathcal{N}$, such that $\mathbf{A}_1 \neq \emptyset$ and $\mathbf{A}_2 = (\mathbf{N} \setminus \mathbf{A}_0 \cup \mathbf{A}_1) = \{j_1, \ldots, j_k\}$ with $k < n - 1$ we have

$$\det\ \big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}}\circ\mathcal{L}^i_{\mathbf{A}_1}\big)(\widetilde{\mathbf{W}})\begin{cases} > 0 & \text{if} \quad i\in\mathbf{A}_1\\ < 0 & \text{if} \quad i\in\mathbf{A}_0.\end{cases}$$

Let Δ be the k-simplex associated with $\mathbf{A}_2$ where $\pi_i(\mathrm{int}(\Delta)) = 1$ if and only if $i \in \mathbf{A}_1$. T has a unique fixed point, $\mathbf{x} \in \mathrm{int}\Delta$, which satisfies the relations

$$\sum_{j\in\mathbf{A}_1}\omega_{ij} + \sum_{t=1}^{p}\omega_{ij_t}x_{j_t}\begin{cases} = x_i & \text{if} \quad i\in\mathbf{A}_2 & \text{(a)}\\ \geq 1 & \text{if} \quad i\in\mathbf{A}_1 & \text{(b)}\\ \leq 0 & \text{if} \quad i\in\mathbf{A}_0. & \text{(c)}\end{cases}$$

System (a) is equivalent to $\sum_{t=1}^{k}\widetilde{\omega}_{ij_t}x_{i_t} = -\sum_{j\in\mathbf{A}_1}\widetilde{\omega}_{ij}$ (for $i \in \mathbf{A}_2$) which has a unique solution if and only if

$$\det\,(\widetilde{\mathbf{W}}_c) = \det\,\big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{j_t\}}\circ\mathcal{L}^{j_t}_{\{j_t\}}\big)(\widetilde{\mathbf{W}}) \neq 0.$$

Let $\mathbf{B}_0 = \mathbf{A}_0 \cup \mathbf{A}_1,$ and $\mathbf{B}_1 = \{j_t\}$, (therefore $\#\,(\mathbf{N} \setminus (\mathbf{B}_0 \cup \mathbf{B}_1)) = k-1$) then the hypothesis implies that $\det\,\big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{j_t\}}\circ\mathcal{L}^{j_t}_{\{j_t\}}\big)(\widetilde{\mathbf{W}}) > 0$. The solution of system (a) is given by

$$x_{j_t} = -\frac{\det(\widetilde{\mathbf{W}^t_\Delta})}{\det(\widetilde{\mathbf{W}_c})},\ \ t = 1,\,\ldots,\,k.$$

Since x_{j_t} we have that $\det\,(\widetilde{\mathbf{W}}^t_\Delta) = \det\,\big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1}\circ\mathcal{L}^{j_t}_{\mathbf{A}_1}\big)(\widetilde{\mathbf{W}}) < 0$. Lemma 4.1 allows us to write inequalities (b) and (c) as follows

$$\det\,\big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{A}_1}\big)(\widetilde{\mathbf{W}}) = \textstyle\sum_{j\in\mathbf{A}_1}\widetilde{\omega}_{ij}\ \det(\widetilde{\mathbf{W}}_c) - \sum_{s=1}^{k}\widetilde{\omega}_{ij_s}\det(\widetilde{\mathbf{W}^t_\Delta}) \begin{cases} \geq 0 & \text{if} \quad i \in \mathbf{A}_1 \\ \leq 0 & \text{if} \quad i \in \mathbf{A}_0. \end{cases}$$

It remains to show that these inequalities are strict. We consider $i \in \mathbf{A}_0 \cup \mathbf{A}_1$ and define a $k+1$-simplex, Δ, associated with the set $\{j_1,\,j_2,\,\ldots,\,j_k,\,i\}$, where $\pi_k(\text{int}(\Delta)) = 1$ if $k \in \mathbf{A}_1 \setminus \{i\}$ and $\pi_k(\text{int}(\Delta)) = 0$ if $k \in \mathbf{A}_0 \setminus \{i\}$. Similar arguments show that

$$0 \,<\, x_i = -\frac{\det(\widetilde{\mathbf{W}^i_\Delta})}{\det(\widetilde{\mathbf{W}_c}}) \,<\, 1.$$

The induction hypothesis implies that $\det\big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{A}_1}\big)(\widetilde{\mathbf{W}}) = \det(\widetilde{\mathbf{W}^i_\Delta}) < 0$ if $i \in \mathbf{A}_0$ and $\det\,\big(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{A}_1}\big)(\widetilde{\mathbf{W}}) = \det(\widetilde{\mathbf{W}^i_\Delta}) > 0$ if $i \in \mathbf{A}_1$.

It remains to prove that $\det(\widetilde{\mathbf{W}}) \geq 0$. If Δ is the $(n-1)-$simplex associated with $\{1,\,2,\,\ldots,\,n-1\}$ then the only fixed point of T in the $\text{int}(\Delta)$ satisfies the system

$$\sum_{j=1}^{n-1}\widetilde{\omega}_{ij}x_j = -\widetilde{\omega}_{in},\ \ i = 1,\,\cdots,\,n-1$$

and the inequality

$$\sum_{j=1}^{n-1}\widetilde{\omega}_{nj}x_j + \widetilde{\omega}_{nn} \geq 0.$$

This implies that $\det\,(\widetilde{\mathbf{W}}) \geq 0.$ $\diamondsuit$

4.1 Overall Dynamics

As we mentioned in the introduction, information stored in stable fixed points can be retrieved from corrupted inputs. Under the assumption of finitely many fixed points, it is important to determine which fixed points are good storage places. We start by reviewing the definition of stable fixed point.

Definition 4.1 *A fixed point* $\mathbf{x}$ *is called* <u>*stable*</u> *if and only if there exists a neighborhood of* $\mathbf{x}$, $\mathbf{N_x}$, *such that for every* $\mathbf{y} \in \mathbf{N_x}$, $T^n(\mathbf{y})$ *converges to* $\mathbf{x}$ *as* n *approaches infinity.*
The basin of attraction of a stable fixed point $\mathbf{x}$ *is defined to be the set*

$$\{\, \mathbf{y} \in \mathcal{H} \,|\, \{T^n(\mathbf{y})\}_n \, \textit{converges to} \;\; \mathbf{x} \,\}.$$

Lemma 4.2 *Every* 0*-simplex* Δ*, different from* $\mathbf{O}$*, is stable.*

Proof: This result follows from Lemma 2.1 and the continuity the map

$$\phi \;:\; \mathbf{x} \;\rightarrow \sum_{\{j:\,\pi_j(\Delta)=1\}} \omega_{ij} x_j. \quad \diamondsuit$$

A k-simplex with a fixed point in its interior is expanded under the action of T. This way we may conclude that the only stable fixed points are the 0-simpleces different from $\mathbf{O}$. We illustrate in Fig. 3 a typical dynamical phase portrait encountered in both adjacent and non-adjacent 2-simpleces, similar situation occurs for higher dimension simpleces. We remark that the basin of attraction of a stable fixed point is bounded by a hyperpolygon.

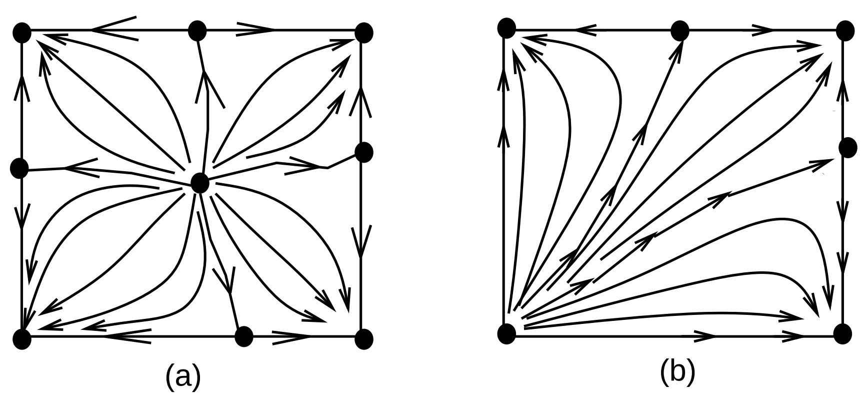

Figure 3: Typical phase portraits for: (a)non-adjacent simpleces, (b)adjacent simpleces

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