Minimizing the Elastic Energy of Knots

Heiko von der Mosel
Mathematisches Institut, Universität Bonn,
Beringstraße 4, 53115 Bonn
e mail: heiko@johnson.iam.uni-bonn.de

Mar 20, 1998

Abstract

We consider the problem of minimizing the bending energy $E_b = \int \kappa^2\, ds$ on isotopy classes of closed curves in $I\!R^3$ to model the elastic behaviour of knotted loops of springy wire. A potential of Coulomb type with a small factor θ as a measure for the thickness of the wire is added to the elastic energy in order to preserve the isotopy class. With a direct method we show existence of minimizers $\mathbf{x}^\theta$ under a given topological knot type for each $\theta > 0$. Moreover, allowing smaller and smaller thickness ($\theta \searrow 0$) and looking at a subsequence of the corresponding minimizers $\mathbf{x}^\theta$, we obtain a generalized minimizer $\mathbf{x}$ of the bending energy E_b as a limit. It turns out that $\mathbf{x}$ is the once covered circle, if one considers the class of unknotted loops in $I\!R^3$. In nontrivial knot classes, however, $\mathbf{x}$ must have double points, whose multiplicity and position on the curve is controlled by the value of the bending energy $E_b(\mathbf{x})$.

AMS Classification: 49Q10, 53A04, 57R52

1 Introduction

Physical experiments with knotted loops of elastic wire yield nicely symmetric knot configurations with several points of selfcontact. In [16] and [17] we have modelled this behaviour by minimizing the classical bending energy

$$E_b = \int \kappa^2\, ds \tag{1}$$

on isotopy classes of closed curves in $I\!R^3$. In order to preserve the given isotopy class, i.e. knot type, we imposed a certain selfobstacle condition on the set of competing curves. The present paper will give an alternative approach to this variational problem. Instead of the obstacle condition we consider a linear combination of the bending energy E_b and a potential energy E_c^α as total elastic energy:

$$E = E_b + \theta E_c^\alpha , \tag{2}$$

where $\theta > 0$ is a small factor and the potential E_c^α for $\alpha \in [2,3)$ is given by

$$E_c^\alpha(\mathbf{y}) := \int_{S^1}\int_{S^1} \left\{ \frac{1}{|\mathbf{y}(s)-\mathbf{y}(t)|^\alpha} - \frac{1}{D(\mathbf{y}(s),\mathbf{y}(t))^\alpha} \right\} |\dot{\mathbf{y}}(t)||\dot{\mathbf{y}}(s)|\, dsdt\ . \tag{3}$$

Here, $D(\mathbf{y}(s),\mathbf{y}(t))$ denotes the distance between the points $\mathbf{y}(s)$ and $\mathbf{y}(t)$ on the curve.

Thus we combine two different types of energies. The bending energy E_b — suggested as early as 1738 by D. Bernoulli and studied intensively in the case of planar curves by L. Euler ([2]) —

is a model for the elastic energy of springy wire. There are quite a few publications regarding existence and shape of critical points of E_b — so called elastica — such as the contributions by J. Radon and R. Irrgang ([14],[4]), J. Langer and D.A. Singer as well as R. Bryant and P. Griffiths ([7]–[9],[1]). There is also some work on the gradient flow and on the evolution problem for E_b, see [10] and [13].

The potential energy E_c^α on the other hand, first introduced by J. O'Hara for $\alpha=2$ and thoroughly investigated by M. Freedman, Z. He and Z. Wang turned out to be a valuable tool to distinguish different knot types ([12],[3]). It was also used to produce nicely embedded minimal knot configurations as standard representatives of certain irreducible knot classes and essential links ([3],[5]). The inverse power of the distance of different curve points in the integrand of E_c^α resembles a repulsive Coulomb type potential suitable for describing nonelastic but electrically charged wires. The complicated interplay between the globally effective bending forces, that move the elastic wire in the physical experiment on one hand, and the locally repulsive forces, that prevent the solid wire from passing through itself on the other hand, is reflected in an idealized way in expression (2), the total elastic energy E. Here, the small factor θ monitoring the influence of the repulsive potential E_c^α on the total elastic energy E corresponds to the thickness of the wire, if E is bounded. In fact, the value $E_c^\alpha \le E/\theta$ determines explicitely a lower bound on the euclidean distance of two different points on the curve, see Lemma 2.3 in section 2.

We are going to show that for any $\theta>0$ there exists a minimizer of (2) in any given knot class. More precisely, modelling knotted wires as regular, closed space curves in the Sobolev class

$$H^{2,2}(S^1,\mathbb{R}^3) := \{\, \mathbf{y}\in H^{2,2}((0,2\pi),\mathbb{R}^3) \mid \mathbf{y}(0)=\mathbf{y}(2\pi)\,,\ \dot{\mathbf{y}}(0)=\dot{\mathbf{y}}(2\pi)\,,\ \dot{\mathbf{y}}(s)\neq 0 \text{ for all } s\in[0,2\pi]\,\}$$

we can determine the knot or isotopy type of any loop $\mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3)$ by deforming it continuously and without selfintersections into a representative $\mathbf{z}$ of a standard knot in $\mathbb{R}^3$. Such a deformation is a continuous mapping $\Phi : S^1\times[0,1]\longrightarrow \mathbb{R}^3$ with $\Phi(.,0)=\mathbf{y}(.)$, $\Phi(.,1)=\mathbf{z}(.)$, and $\Phi(.,\sigma)$ closed and 1–1 for all $\sigma\in[0,1]$. Thus, for a given list $\omega_0,\omega_1,\omega_2,\ldots$ of equivalence classes of topologically different standard knots in $\mathbb{R}^3$, starting with $\omega_0=[S^1]$, we can define the subclass $C(n)\subset H^{2,2}(S^1,\mathbb{R}^3)$ as

$$C(n) := \{\, \mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3) \mid L(\mathbf{y})=l \text{ and } \exists \text{ a parametrization } \mathbf{z}\in\omega_n \text{ isotopic to } \mathbf{y}\,\}\,,$$

where $L(\mathbf{y}) := \int_0^{2\pi} |\dot{\mathbf{y}}(\tau)|\,d\tau$ denotes the length of the curve $\mathbf{y}$ and $l>0$ is fixed.

We look at the following variational problem:

Minimize the total elastic energy $E = E_b + \theta E_c^\alpha$ *in* $C(n)$.

Remark. We have fixed the length l, since E is not scale invariant. Consequently, without any normalization one cannot expect to find a minimizer $\mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3)$ with finite total energy, since rescaling $\mathbf{y}\mapsto R\mathbf{y}$ with $R>1$ yields a curve with smaller energy (without changing the knot type):

$$E(R\mathbf{y}) = E_b(R\mathbf{y}) + \theta E_c^\alpha(R\mathbf{y}) = E_b(\mathbf{y})/R + \theta E_c^\alpha(\mathbf{y})/R^{\alpha-2} < E(\mathbf{y}) \text{ for } R>1\,, \text{ since } \alpha\in[2,3)\,.$$

With a direct method we will prove the above mentioned existence result (section 3):

Theorem 3.1 (Existence of minimizers) *For every* $\theta>0$ *and* $\alpha\in[2,3)$ *there exists* $\mathbf{x}^\theta\in C(n)$ *such that* $|\dot{\mathbf{x}}^\theta|\equiv l/2\pi$ *on* S^1 *and*

$$E(\mathbf{x}^\theta) = \inf\{\, E(\mathbf{y}) \mid \mathbf{y}\in C(n)\,\}\,.$$

The properties of E_c^α developed in section 2 guarantee that the topological side condition — the prescribed isotopy class — carries over to limit curves of appropriately regularized minimal sequences in $C(n)$.

Letting θ tend to zero, which corresponds to allowing smaller and smaller thickness of the elastic wires in the physical experiments, we obtain a limit curve that can be considered as a generalized minimizer of the bending energy E_b in the closure of the given knot class:

Theorem 3.2 (Existence of a limit curve) *There is a sequence* $\theta_j \searrow 0$ *and a curve* $\mathbf{x} \in H^{2,2}(S^1, I\!R^3) \cap \overline{C(n)}$ *such that* $\mathbf{x}^{\theta_j} \rightharpoonup \mathbf{x}$ *weakly in* $H^{2,2}((0,2\pi), I\!R^3)$*, where the curves* $\mathbf{x}^{\theta_j} \in C(n)$ *are minimizers of* $E = E_b + \theta_j E_c^\alpha$ *obtained by Theorem 3.1. Here* $\overline{C(n)}$ *denotes the closure of* $C(n)$ *with respect to the* C^1*–norm. In addition,*

(i) $|\dot{\mathbf{x}}| \equiv l/2\pi$,

(ii) $E_b(\mathbf{x}) \le E_b(\mathbf{y})$ *for all* $\mathbf{y} \in C(n)$.

The limit curve $\mathbf{x}$ is not embedded in general; actually, it turns out that for every nontrivial knot class, i.e. for $C(n)$ with $n \ge 1$, the generalized minimizer $\mathbf{x} \in \overline{C(n)}$ cannot be simple (Corollary 4.3). If, however, $\mathbf{x}$ is embedded, it must be the once covered circle, which is, on the other hand, the unique generalized minimizer in the case of the trivial knot class $C(0)$ of unknotted loops, see Proposition 4.1. This result confirms what one observes in the physical experiment: unknotted closed elastic wires spring into a circular stable configuration no matter how entangled they had been before, if one can sufficiently reduce the effects of friction.

Finally we are going to prove an inequality that demonstrates how the bending energy controls not only the multiplicity of double points but also the lengths of the connecting arcs that start and end at a double point:

Theorem 4.3 *Let* $\mathbf{y} \in H^{2,2}(S^1, I\!R^3)$ *be a curve of length* l *with a double point of multiplicity* m *dividing* $\mathbf{y}$ *into* m *subarcs* $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m$ *with lengths* $l_i := L(\mathbf{Y}_i), i = 1, \ldots, m,$ *and* $\sum_{i=1}^m l_i = l$. *Then*

$$\sum_{i=1}^{m} 1/l_i \le E_b(\mathbf{y})/16\,, \quad \textit{in particular,} \quad m \le \sqrt{l E_b(\mathbf{y})/16}\,. \tag{4}$$

As an application let us mention that there are infinitely many 2–bridge knots of length l representable by curves with bending energy E_b less than $(4\pi)^2/l + \epsilon$ for any given $\epsilon > 0$, thus coming arbitrarily close to the lower bound on E_b implied by the classical Fàry–Milnor theorem on the total curvature of knots. Hence, for the multiplicity m of double points of the respective minimizer $\mathbf{x}$ in the closure of such 2–bridge knot classes, the second inequality in (4) implies $m < 4$ for ϵ sufficiently small. In other words, the minimizing curve $\mathbf{x}$ can pass at most three times through one point in space. Also by (4) with $\mathbf{y}$ and $\mathbf{Y}_i$ replaced by $\mathbf{x}$ and $\mathbf{X}_i$,

$$\sum_{i=1}^{m} L(\mathbf{X}_i)^{-1} \quad \le \quad \pi^2/l + O(\epsilon) \quad < \quad 10/l \qquad \text{for } \epsilon \ll 1\,,$$

which implies that the connecting arcs $\mathbf{X}_i$ have lengths $L(\mathbf{X}_i) > l/10$ for all $i = 1, \ldots, m$, and that there must be at least one $i \in \{1, \ldots, m\}$ with $L(\mathbf{X}_i) > ml/10$ for $m = 2, 3$. That means, the double points of the generalized minimizer $\mathbf{x}$ cannot produce loops of small length compared to the given total length l.

Remark. In this paper we do not address the question of regularity of the generalized minimizer $\mathbf{x}$. Let us mention, however, that the regularity results of [17] carry over to $\mathbf{x}$, if one makes certain assumptions on the geometry of the touching situation. In particular, it is easy to check that $\mathbf{x}$ is C^∞–smooth away from its double points, since there $\mathbf{x}$ satisfies the Euler equation of E_b, compare [17, p.6].

Acknowledgements. Part of this research was done during the author's stay at the Institute for Mathematics and its Applications (IMA) at the University of Minnesota in Minneapolis. I would like to express my gratitude to the organizers of the IMA for their hospitality and support — especially to R. Gulliver and to J. Sullivan for many interesting discussions. I am also grateful to the Sonderforschungsbereich 256 in Bonn for its generous support.

2 Properties of the potential E_c^α

In the following we list some properties of the potential E_c^α for $\alpha\in[2,3)$. For $\alpha=2$ most of the proofs can be found in [3].

For $\mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3)$ we define $D(\mathbf{y}(s),\mathbf{y}(t))$ as the distance between the points $\mathbf{y}(s)$ and $\mathbf{y}(t)$ for $s,t\in S^1\cong \mathbb{R}/2\pi Z$ on the curve, i.e.

$$D(\mathbf{y}(s),\mathbf{y}(t)) := \min\{ \int_s^t |\dot{\mathbf{y}}(\tau)|\,d\tau, \int_t^s |\dot{\mathbf{y}}(\tau)|\,d\tau \}\ ,$$

where $\int_t^s |\dot{\mathbf{y}}(\tau)|\,d\tau := \int_t^{2\pi} |\dot{\mathbf{y}}(\tau)|\,d\tau + \int_0^s |\dot{\mathbf{y}}(\tau)|\,d\tau$, if $s<t$.

The potential E_c^α on $H^{2,2}(S^1,\mathbb{R}^3)$ is defined as the Cauchy principal value

$$E_c^\alpha(\mathbf{y}) := \lim_{\epsilon\searrow 0} \iint\limits_{|s-t|\geq\epsilon} \{\frac{1}{|\mathbf{y}(s)-\mathbf{y}(t)|^\alpha} - \frac{1}{D(\mathbf{y}(s),\mathbf{y}(t))^\alpha}\}\, |\dot{\mathbf{y}}(t)||\dot{\mathbf{y}}(s)|\,dsdt\,,$$

if this limit exists, otherwise we set $E_c^\alpha(\mathbf{y}) := \infty$. Note, that the rectifiability of $\mathbf{y}$ is sufficient to define this energy, compare [3]. The definition of E_c^α is parameter invariant, i.e. we have $E_c^\alpha(\mathbf{y}\circ\tau) = E_c^\alpha(\mathbf{y})$ for all C^1–diffeomorphisms $\tau : S^1 \longrightarrow S^1$. If we consider an affine linear transformation $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ of the form $T(\mathbf{z}) = A\circ r\mathbf{z}+\mathbf{b}$ for some $r\in\mathbb{R}\backslash\{0\}, \mathbf{b}\in\mathbb{R}^3$ and $A\in O(3)$, then we obtain the scaling property $E_c^\alpha(T\circ\mathbf{y}) = r^{2-\alpha}E_c^\alpha(\mathbf{y})$. In particular, E_c^2 is scale invariant. The exceptional role of the power $\alpha=2$ becomes even more evident, if one considers general Möbiustransformations in $\mathbb{R}^3$. The following result is due to M. Freedman, Z. He and Z. Wang ([3, Thm 2.1]):

Theorem 2.1 *Let* $\mathbf{y}$ *be a rectifiable simple closed curve in* $\mathbb{R}^3$ *and let* $M: S^3\longrightarrow S^3$ *be a Möbiustransformation. Then*

(i) $E_c^2(M\circ\mathbf{y}) = E_c^2(\mathbf{y})$, *if* $M\circ\mathbf{y}\subset\mathbb{R}^3$ *and*

(ii) $E_c^2((M\circ\mathbf{y})\cap\mathbb{R}^3) = E_c^2(\mathbf{y}) - 4$, *if* $M\circ\mathbf{y}\not\subset\mathbb{R}^3$.

In fact, this Möbiusinvariance of E_c^2 was a valuable tool for Freedman, He and Wang to prove existence and $C^{1,1}$–regularity of minimizers of E_c^2 in each isotopy class of irreducible knots.

The following Lemma asserts that curves $\mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3)$ are locally bi–Lipschitz, which implies that possible points of selfcontact $\mathbf{y}(s)=\mathbf{y}(t)$ for $s\neq t$ occur only for parameters $s,t\in S^1$ that are sufficiently far apart:

Lemma 2.2 *For any* $\mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3)$ *there exist constants* $d>0$ *and* $0<L_1\leq L_2<\infty$, *such that for all* $(s,t)\in S^1\times S^1$ *with* $|s-t|_{S^1}\leq d$

$$L_1|s-t|_{S^1} \leq |\mathbf{y}(s)-\mathbf{y}(t)| \leq L_2|s-t|_{S^1}\ , \qquad (5)$$

where $|s-t|_{S^1} := \min\{|s-t|, 2\pi-|s-t|\}$ *denotes the distance between* s *and* t *on* S^1.

Proof. We estimate

$$|\mathbf{y}(s)-\mathbf{y}(t)| \leq \int\limits_{[s,t]} |\dot{\mathbf{y}}(\tau)|\,d\tau \leq C_S\|\mathbf{y}\|_{H^{2,2}}|s-t|_{S^1} =: L_2|s-t|_{S^1}$$

for all $(s,t)\in S^1\times S^1$, where C_S denotes the constant of the Sobolev embedding

$$H^{2,2}((0,2\pi)) \hookrightarrow C^{1,1/2}([0,2\pi])\,. \tag{6}$$

On the other hand, we obtain (with $v := \min_{\sigma\in S^1}|\dot{\mathbf{y}}(\sigma)|$)

$$\begin{aligned}
|\mathbf{y}(s)-\mathbf{y}(t)| &\geq \Big|\int\limits_{[s,t]} \dot{\mathbf{y}}(s)\,d\tau\,\Big| - \Big|\int\limits_{[s,t]} (\dot{\mathbf{y}}(\tau)-\dot{\mathbf{y}}(s))\,d\tau\,\Big| \\
&\geq v|s-t|_{S^1} - \Big|\int\limits_{[s,t]}\int\limits_{[s,\tau]} \ddot{\mathbf{y}}(\sigma)\,d\sigma d\tau\,\Big| \\
&\geq \{\, v - \|\ddot{\mathbf{y}}\|_{L^2}|s-t|_{S^1}^{1/2}\,\}\,|s-t|_{S^1} \\
&\geq v|s-t|_{S^1}/2 \qquad \text{for} \quad |s-t|_{S^1} \leq d := [\,v/(2\|\ddot{\mathbf{y}}\|_{L^2})\,]^2 \\
&=: L_1|s-t|_{S^1}\,.
\end{aligned}$$

□

Remark. This proof also works for $\mathbf{y}\in C^{1,\mu}$ for some $\mu\in(0,1]$. Then d depends on the $C^{1,\mu}$–norm of $\mathbf{y}$ instead of the L^2–norm of $\ddot{\mathbf{y}}$.

Curves with finite potential E_c^α are in fact embedded, which was first observed by J. O'Hara ([12]) for $\alpha=2$.

Lemma 2.3 *For $\mathbf{y}\in H^{2,2}(S^1,\mathbb{R}^3)$ with $E_c^\alpha(\mathbf{y})<\infty$ there is a constant $L_0>0$, such that for all $(s,t)\in S^1\times S^1$*

$$L_0 D(\mathbf{y}(s),\mathbf{y}(t)) \leq |\mathbf{y}(s)-\mathbf{y}(t)| \leq D(\mathbf{y}(s),\mathbf{y}(t))\,, \tag{7}$$

where L_0 depends on $E_c^\alpha(\mathbf{y})$, on the length of $\mathbf{y}$ and on $\|\ddot{\mathbf{y}}\|_{L^2}$.

Proof. We only have to prove the lower bound, and we may assume that $|\dot{\mathbf{y}}|\equiv const =: v$, since E_c^α and relation (7) are parameter invariant. Then (7) is valid for $|s-t|_{S^1}\leq d=[v/(2\|\ddot{\mathbf{y}}\|_{L^2})]^2$, if we take $L_0 := L_1 = v/2$, see the proof of Lemma 2.2.

For $s,t\in S^1$ with $|s-t|_{S^1}>d$ define the neighbourhoods $B(t) := [t-d/3, t+d/3]$ and $B(s) := [s-d/3, s+d/3]$ with $B(t)\cap B(s)=\emptyset$. For $\tau\in B(t), \sigma\in B(s)$ we obtain expanding about t and s

$$\begin{aligned}
|\mathbf{y}(\tau)-\mathbf{y}(\sigma)|^\alpha &= \Big|\mathbf{y}(t)-\mathbf{y}(s)+\int_t^\tau \dot{\mathbf{y}}(r)\,dr - \int_s^\sigma \dot{\mathbf{y}}(r)\,dr\,\Big|^\alpha \\
&\leq 2^{2\alpha-2}\{\,|\mathbf{y}(s)-\mathbf{y}(t)|^\alpha + v^\alpha[\,|\tau-t|^\alpha + |\sigma-s|^\alpha\,]\,\}\,.
\end{aligned}$$

Consequently, using $D(\mathbf{y}(\sigma),\mathbf{y}(\tau)) = v|\sigma-\tau|_{S^1}\geq vd/3$ for $\tau\in B(t),\sigma\in B(s)$ one estimates for $\alpha\in[2,3)$, $e := |\mathbf{y}(s)-\mathbf{y}(t)|$

$$E_c^\alpha(\mathbf{y}) \geq v^2\int_{B(t)}\int_{B(s)}\left\{\frac{1}{2^{2\alpha-2}[e^\alpha + v^\alpha(|\tau-t|^\alpha+|\sigma-s|^\alpha)]} - \left(\frac{3}{vd}\right)^\alpha\right\}d\sigma d\tau\,.$$

Denoting $c_0(E_c^\alpha(\mathbf{y}),d) := 2^{2\alpha}[v^{\alpha-2}E_c^\alpha(\mathbf{y})/4 + (d/3)^{2-\alpha}]$ we arrive at

$$
\begin{aligned}
c_0(E_c^\alpha(\mathbf{y}),d) &\geq \int_{B(t)}\int_{B(s)} \frac{d\sigma\, d\tau}{(e/v)^\alpha + |\tau - t|^\alpha + |\sigma - s|^\alpha} = 2\int_{B(t)}\int_0^{d/3} \frac{dz\, d\tau}{(e/v)^\alpha + |\tau - t|^\alpha + z^\alpha} \\
&\geq 2\int_{B(t)}\int_0^{e} \frac{dz\, d\tau}{(e/v)^\alpha + |\tau - t|^\alpha + z^\alpha} \quad \geq 2\int_{B(t)}\int_0^{e} \frac{dz\, d\tau}{e^\alpha(1+v^{-\alpha}) + |\tau - t|^\alpha}
\end{aligned} \tag{8}
$$

assuming that $e \leq d/3$ (otherwise $e > d/3 \geq d|s-t|_{S^1}/3\pi$, which would finish the proof). The remaining integrand does not depend on z, hence

$$
(1+v^{-\alpha})c_0(E_c^\alpha(\mathbf{y}),d)/2 \geq e\int_{B(t)} \frac{d\tau}{e^\alpha + |\tau - t|^\alpha} \geq e\int_0^e \frac{dr}{e^\alpha} = e^{2-\alpha},
$$

which implies for $\alpha \in (2,3)$

$$
e \geq \big[\,(1+v^{-\alpha})c_0(E_c^\alpha(\mathbf{y}),d)/2\,\big]^{1/(2-\alpha)} =: c_1(E_c^\alpha(\mathbf{y}),d) \geq c_1(E_c^\alpha(\mathbf{y}),d)|s-t|_{S^1}/\pi\,.
$$

For $\alpha = 2$ one integrates in (8) with respect to z and obtains

$$
\begin{aligned}
c_0(E_c^\alpha(\mathbf{y}),d) &\geq 2\int_{B(t)} [(e/v)^2 + |\tau - t|^2]^{-1/2} \arctan([(e/v)^2 + |\tau - t|^2]^{-1/2} d/3)\, d\tau \\
&\geq 2c(d)\int_{B(t)} [(e/v)^2 + |\tau - t|^2]^{-1/2}\, d\tau\,, \qquad \text{since} \quad e < v\pi \text{ and } |\tau - t| < d/3\,, \\
&\geq 4c(d)\int_0^{d/3} \frac{dy}{(e/v) + y} = 4c(d)\ln\big[\frac{(e/v) + d/3}{e/v}\big]
\end{aligned} \tag{9}
$$

with some constant $c(d)$ as a lower bound for the arctan–term depending on d. Consequently $\exp[c_0(E_c^\alpha(\mathbf{y}),d)/4c(d)] - 1 \geq dv/(3e)$ or $e \geq dv(\exp[c_0(E_c^\alpha(\mathbf{y}),d)/4c(d)] - 1)^{-1}|s-t|_{S^1}/(3\pi)$. If we set

$$
L_0 := \begin{cases} \min\{\, v/2, c_1(E_c^\alpha(\mathbf{y}),d)/\pi \,\} & \text{for } \alpha \in (2,3)\,, \\ \min\{\, v/2, dv(\exp[c_0(E_c^\alpha(\mathbf{y}),d)/4c(d)] - 1)^{-1}/(3\pi) \,\} & \text{for } \alpha = 2\,, \end{cases}
$$

we arrive at inequality (7). □

Remarks. 1. The previous calculation demonstrates that the potential $E_c^\alpha(\mathbf{y})$ is not finite, if $\mathbf{y} \in H^{2,2}(S^1, I\!R^3)$ has a double point. In fact, assuming that there is a pair $(s_0,t_0) \in S^1 \times S^1$, $s_0 \neq t_0$ with $\mathbf{y}(s_0) = \mathbf{y}(t_0)$, we have by Lemma 2.2 that $\pi \geq |s_0 - t_0|_{S^1} > d$. Starting from (8) for any $\alpha \in [2,3)$ with $e = 0$ for $(t,s) = (t_0,s_0)$ we conclude

$$
\begin{aligned}
2^{2\alpha}[v^{\alpha-2}E_c^\alpha(\mathbf{y})/4 + (d/3)^{2-\alpha}] \quad &\geq \quad 2\int_{B(t)}\int_0^{d/6} \frac{dz\, d\tau}{|\tau - t|^\alpha + z^\alpha} \\
&\underset{(d/6)<1}{\geq} \quad 2\int_{B(t)}\int_0^{d/6} \frac{dz\, d\tau}{|\tau - t|^\alpha + z^2} \\
&\underset{(9)}{\geq} \quad 2c(d)\int_{B(t)} |\tau - t|^{-\alpha/2}\, d\tau \quad \geq 4c(d)\int_0^{d/6} y^{-1}\, dy\,.
\end{aligned}
$$

This last integral is divergent, hence $E_c^\alpha(\mathbf{y})$ is not finite.

2. The potential E_c^2 of the once covered circle is 4 and $E_c^2(\mathbf{y}) > 4$ for all other rectifiable loops $\mathbf{y} : S^1 \longrightarrow I\!R^3$, see [3, Cor.2.2]. Using residue calculus R. Kusner and G. Stengle ([6]) were able to calculate explicitely the potential (for $\alpha = 2$) for certain torus knots. For $\alpha \neq 2$, however, it

still seems unproven that the once covered circle actually minimizes E_c^α in the class of rectifiable closed curves, as one would expect.

Simple loops that are sufficiently smooth have finite potential, which is shown in the following proposition, a refinement of Prop.1.5 and Prop. 8.2 in [3].

Proposition 2.4 *$E_c^\alpha(\mathbf{y}) < \infty$ for any simple $\mathbf{y}\in H^{2,p}(S^1,\mathbb{R}^3)$ with $p>2/(3-\alpha), \alpha\in[2,3)$.*

Proof. We may assume $|\dot{\mathbf{y}}| \equiv v$ and use the expansion

$$\mathbf{y}(t) - \mathbf{y}(s) = \dot{\mathbf{y}}(s)(t-s) + \int_0^1 \ddot{\mathbf{y}}(s+\sigma(t-s))(1-\sigma)\, d\sigma\, (t-s)^2$$

to obtain

$$\begin{aligned}|\mathbf{y}(t)-\mathbf{y}(s)|^\alpha &= (t-s)^\alpha \Big[v^2 + 2(t-s)\int_0^1 (1-\sigma)\langle \dot{\mathbf{y}}(s), \ddot{\mathbf{y}}(s+\sigma(t-s))\rangle\, d\sigma \\ &\qquad + (t-s)^2 \Big| \int_0^1 (1-\sigma)\ddot{\mathbf{y}}(s+\sigma(t-s))\, d\sigma\Big|^2\Big]^{\alpha/2}.\end{aligned}$$

By Hölder's inequality we have $\int_0^1 |\ddot{\mathbf{y}}(s+\sigma(t-s))|\, d\sigma = O(|t-s|^{-1/p})$, which allows us to expand the integrand of $E_c^\alpha(\mathbf{y})$ for $|t-s| \ll v^2$:

$$\frac{1}{|\mathbf{y}(t)-\mathbf{y}(s)|^\alpha} = \frac{1}{(v(t-s))^\alpha}\Big[1 - \frac{\alpha(t-s)}{v^2}\int_0^1 (1-\sigma)\langle \dot{\mathbf{y}}(s), \ddot{\mathbf{y}}(s+\sigma(t-s))\rangle\, d\sigma + O(|t-s|^{2-(2/p)})\Big].$$

Hence, observing that $D(\mathbf{y}(s),\mathbf{y}(t)) = v|s-t|_{S^1}$ one calculates for
$\Sigma(\epsilon,\epsilon_0) := \{(s,t)\in S^1\times S^1 \mid 0<\epsilon\le |s-t|_{S^1} < \epsilon_0\}, \epsilon_0 \ll v^2$:

$$\begin{aligned}
(10)\quad & \int\!\!\int_{\Sigma(\epsilon,\epsilon_0)} \Big\{ \frac{1}{|\mathbf{y}(s)-\mathbf{y}(t)|^\alpha} - \frac{1}{D(\mathbf{y}(s),\mathbf{y}(t))^\alpha}\Big\}\, |\dot{\mathbf{y}}(s)||\dot{\mathbf{y}}(t)|\, ds dt \\
&= -\frac{\alpha}{v^\alpha}\int\!\!\int_{\Sigma(\epsilon,\epsilon_0)} (t-s)^{1-\alpha}\int_0^1 (1-\sigma)\langle \dot{\mathbf{y}}(s), \ddot{\mathbf{y}}(s+\sigma(t-s))\rangle\, d\sigma\, ds dt + \int\!\!\int_{\Sigma(\epsilon,\epsilon_0)} O(|t-s|^{2-(2/p)-\alpha})\, ds dt \\
&\underset{(w:=t-s)}{=} -\frac{\alpha}{v^\alpha}\int_{S^1}\int_{\epsilon\le|w|\le\epsilon_0} \Big\{ w^{1-\alpha}\int_0^1 (1-\sigma)\langle \dot{\mathbf{y}}(s), \ddot{\mathbf{y}}(s+\sigma w)\rangle\, d\sigma + O(|w|^{2-(2/p)-\alpha})\Big\}\, dw dt \\
&\le -\frac{\alpha}{v^\alpha}\int_{S^1}\int_{\epsilon\le|w|\le\epsilon_0} O(|w|^{2-(2/p)-\alpha})\, dw dt,
\end{aligned}$$

which has a bound independent of ϵ, if $2-(2/p)-\alpha > -1$, i.e. for $p > 2/(3-\alpha)$. For the last inequality in (10) we have used the fact that $\langle \dot{\mathbf{y}}(s+\sigma w), \ddot{\mathbf{y}}(s+\sigma w)\rangle = 0$, since $|\dot{\mathbf{y}}| \equiv v$ by assumption, and in addition $|\dot{\mathbf{y}}(s) - \dot{\mathbf{y}}(s+\sigma w)| \le \|\ddot{\mathbf{y}}\|_{L^p}|w|^{1-(1/p)}$ by Hölder's inequality. On $\{(s,t)\in S^1\times S^1 \mid |s-t|_{S^1}\ge\epsilon_0\}$ the integrand is bounded independent of ϵ, since $\mathbf{y}$ is simple by assumption, which conludes the proof. □

The following simple compactness result will be sufficient for the proof of the existence theorems in the following section. For the corresponding result in the context of rectifiable curves we refer to Proposition 8.3 in [3].

Lemma 2.5 (Compactness) *Let $A \subset H^{2,2}(S^1,\mathbb{R}^3)$ be a bounded set of loops with uniformly bounded potential E_c^α for some $\alpha\in[2,3)$:*

$$(11)\qquad \|\mathbf{y}\|_{H^{2,2}} + E_c^\alpha(\mathbf{y}) \le K < \infty \text{ for all } \mathbf{y}\in A.$$

In addition, assume that $|\dot{\mathbf{y}}| \equiv v$ for all $\mathbf{y}\in A$.
Then there is a sequence $\{\mathbf{y}_j\} \subset A$ and $\mathbf{x}\in H^{2,2}(S^1,\mathbb{R}^3)$ with

(i) $\mathbf{y}_j \rightharpoonup \mathbf{x}$ *weakly in* $H^{2,2}$ *and* $\|\mathbf{y}_j - \mathbf{x}\|_{C^1} \longrightarrow 0$ *for* $j \nearrow \infty$ *and*

(ii) $\lim_{j\nearrow\infty} E_c^\alpha(\mathbf{y}_j) = E_c^\alpha(\mathbf{x}) \le K$.

Proof. Using the weak compactness of bounded sets in $H^{2,2}((0,2\pi),I\!R^3)$ together with the embedding (6) and applying the theorem by Arzelà and Ascoli we obtain (i). Note, that the uniform convergence in C^1 guarantees that $\mathbf{x}$ satisfies the pointwise conditions in the definition of $H^{2,2}(S^1,I\!R^3)$. The uniform bound (11) and the constant velocity of the $\mathbf{y}_j$ independent of j imply the uniform bi–Lipschitz property by Lemma 2.3, (7):

$$L_0|s-t|_{S^1} \le |\mathbf{y}_j(s) - \mathbf{y}_j(t)| \le v|s-t|_{S^1} \text{ for all } j\in I\!N\,,$$

which according to (i) carries over to the limit curve $\mathbf{x}$. Consequently, the integrands $G_j(s,t) := v^2[\,(\mathbf{y}_j(s)-\mathbf{y}_j(t))^{-\alpha} - D(\mathbf{y}_j(s),\mathbf{y}_j(t))^{-\alpha}\,]$ are bounded independent of j: $G_j(s,t)\le v^2/(L_0\epsilon)^\alpha$ for $|s-t|_{S^1}\ge\epsilon$.

Moreover, by (i), $G_j(s,t) \longrightarrow G(s,t) := v^2[\,(\mathbf{x}(s)-\mathbf{x}(t))^{-\alpha} - D(\mathbf{x}(s),\mathbf{x}(t))^{-\alpha}\,]$. Hence,

$$\int\int_{|s-t|_{S^1}\ge\epsilon} G(s,t)\,dsdt = \lim_{j\nearrow\infty}\int\int_{|s-t|_{S^1}\ge\epsilon} G_j(s,t)\,dsdt \text{ for all } \epsilon>0.$$

Passing to another subsequence we can assume that $\lim_{j\nearrow\infty} E_c^\alpha(\mathbf{y}_j)$ exists and we obtain

$$\begin{aligned} E_c^\alpha(\mathbf{x}) &= \lim_{\epsilon\searrow 0}\int\int_{|s-t|_{S^1}\ge\epsilon} G(s,t)\,dsdt = \lim_{\epsilon\searrow 0}\lim_{j\nearrow\infty}\int\int_{|s-t|_{S^1}\ge\epsilon} G_j(s,t)\,dsdt \\ &= \lim_{j\nearrow\infty}\lim_{\epsilon\searrow 0}\int\int_{|s-t|_{S^1}\ge\epsilon} G_j(s,t)\,dsdt = \lim_{j\nearrow\infty} E_c^\alpha(\mathbf{y}_j) \le K\,. \end{aligned}$$

□

3 Existence results

Theorem 3.1 (Existence of minimizers) *For every* $\theta>0$ *and* $\alpha\in[2,3)$ *there exists* $\mathbf{x}^\theta\in C(n)$ *such that* $|\dot{\mathbf{x}}^\theta|\equiv l/2\pi, \mathbf{x}^\theta(0)=0$ *and*

$$E(\mathbf{x}^\theta) = \inf\{\,E(\mathbf{y})\,|\,\mathbf{y}\in C(n)\,\} =: e_\theta\,.$$

Proof. The class $C(n)$ is not empty and e_θ is finite, since any smooth parametrization $\mathbf{z}\in C(n)\cap C^\infty([0,2\pi],I\!R^3)$ of a standard knot ω_n has finite energy according to Proposition 2.4. We may choose a minimal sequence $\{\mathbf{y}_j\}\subset C(n)$ with $\mathbf{y}_j(0)=0$, since the total energy E is translationally invariant. In addition we may assume that $|\dot{\mathbf{y}}_j|\equiv l/2\pi$ for all $j\in I\!N$ after a suitable reparametrization as carried out in [17, p.3]. Since e_θ is finite, there is a constant $K<\infty$, such that $E(\mathbf{y}_j)\le K$ for all $j\in I\!N$, whence $E_b(\mathbf{y}_j)\le K$. This implies $\|\mathbf{y}_j\|_{H^{2,2}}\le K'$ for some constant $K'<\infty$, because, for curves with constant velocity $E_b(.)$ reduces to a constant multiple of the L^2–norm squared of the second derivative. By Lemma 2.5 (i) we obtain a subsequence $\{\mathbf{y}_{j'}\}\subset C(n)$ and a limit curve $\mathbf{x}^\theta\in H^{2,2}(S^1,I\!R^3)$ with

$$\begin{cases} \mathbf{y}_{j'} \rightharpoonup \mathbf{x}^\theta & \text{weakly in } H^{2,2} \text{ and} \\ \mathbf{y}_{j'} \longrightarrow \mathbf{x}^\theta & \text{uniformly in } C^1\,. \end{cases} \tag{12}$$

As an immediate consequence of (12) one obtains $L(\mathbf{x}^\theta)=l, \mathbf{x}^\theta(0)=0$ and $|\dot{\mathbf{x}}^\theta|\equiv l/2\pi$. In addition, $E_c^\alpha(\mathbf{x}^\theta)\le K/\theta<\infty$, hence $\mathbf{x}^\theta$ is embedded. Use Lemma A.1 of the appendix to conclude that $\mathbf{y}_{j'}$ and $\mathbf{x}^\theta$ are isotopic for j' sufficiently large, i.e. $\mathbf{x}^\theta\in[\mathbf{y}_{j'}]$. But then $\mathbf{x}^\theta\in C(n)$, for isotopy is an equivalence relation, see the simple argument in [16, p.28].

Finally, since the bending energy $E_b(.)$ is a bounded nonnegative quadratic form, hence lower semicontinuous on any set of curves in $H^{2,2}(S^1,I\!R^3)$ with uniformly constant velocity, and because of (ii) in Lemma 2.5, we conclude

$$\begin{aligned} e_\theta \le E(\mathbf{x}^\theta) &= E_b(\mathbf{x}^\theta) + \theta E_c^\alpha(\mathbf{x}^\theta) \\ &\le \liminf_{j'\nearrow\infty} E_b(\mathbf{y}_{j'}) + \theta \lim_{j'\nearrow\infty} E_c^\alpha(\mathbf{y}_{j'}) \\ &= \liminf_{j'\nearrow\infty} E(\mathbf{y}_{j'}) = e_\theta \,. \end{aligned}$$

□

Theorem 3.2 (Existence of a limit curve) *There is a sequence* $\theta_j \searrow 0$ *and a curve* $\mathbf{x} \in H^{2,2}(S^1,I\!R^3)\cap\overline{C(n)}$ *such that* $\mathbf{x}^{\theta_j} \rightharpoonup \mathbf{x}$ *weakly in* $H^{2,2}((0,2\pi),I\!R^3)$ *, where the curves* $\mathbf{x}^{\theta_j}\in C(n)$ *are minimizers of* $E = E_b + \theta_j E_c^\alpha$ *obtained by Theorem 3.1. Here* $\overline{C(n)}$ *denotes the closure of* $C(n)$ *with respect to the* C^1*–norm. In addition,*

(i) $|\dot{\mathbf{x}}| \equiv l/2\pi$ *,*

(ii) $E_b(\mathbf{x}) \le E_b(\mathbf{y})$ *for all* $\mathbf{y} \in C(n)$ *.*

Proof. First observe that

$$\begin{aligned} \left(\frac{2\pi}{l}\right)^3 \int_{S^1} |\ddot{\mathbf{x}}^\theta|^2 = E_b(\mathbf{x}^\theta) &\le E_b(\mathbf{x}^\theta) + \theta E_c^\alpha(\mathbf{x}^\theta) \\ &\le E_b(\mathbf{y}) + \theta E_c^\alpha(\mathbf{y}) \\ &\le E_b(\mathbf{y}) + \theta_1 E_c^\alpha(\mathbf{y}) \ < \ \infty \end{aligned} \tag{13}$$

for all $\mathbf{y}\in C(n)\cap C^\infty([0,2\pi],I\!R^3)$, $\alpha\in[2,3)$ and $\theta\le\theta_1$. (Recall that $C(n)\cap C^\infty([0,2\pi],I\!R^3)$ is not empty.) Also, by Theorem 3.1 $|\dot{\mathbf{x}}^\theta| \equiv l/2\pi$ and $\mathbf{x}^\theta(0)=0$. This implies weak compactness of the set of minimizers $\{\mathbf{x}^\theta \,|\, \theta>0\}$ in $H^{2,2}$ and compactness in C^1. Consequently, we may extract a sequence $\theta_j \searrow 0$ such that $\mathbf{x}_{\theta_j} \rightharpoonup \mathbf{x}$ weakly in $H^{2,2}((0,2\pi),I\!R^3)$ and $\mathbf{x}_{\theta_j} \longrightarrow \mathbf{x}$ uniformly in $C^1([0,2\pi],I\!R^3)$, hence $|\dot{\mathbf{x}}| \equiv l/2\pi$ and
$\mathbf{x}\in H^{2,2}(S^1,I\!R^3)$.

In order to show (ii) we use the lower semicontinuity of $E_b(.)$ and the fact that $\mathbf{x}^\theta$ is the E–minimizer in $C(n)$ to estimate

$$\begin{aligned} E_b(\mathbf{x}) &\le \liminf_{j\nearrow\infty} E_b(\mathbf{x}_{\theta_j}) \le \liminf_{j\nearrow\infty}[\, E_b(\mathbf{x}_{\theta_j}) + \theta_j E_c^\alpha(\mathbf{x}_{\theta_j})\,] \\ &\le \liminf_{j\nearrow\infty}[\, E_b(\mathbf{z}) + \theta_j E_c^\alpha(\mathbf{z})\,] = E_b(\mathbf{z}) \end{aligned} \tag{14}$$

for all $\mathbf{z}\in C(n)\cap C^\infty([0,2\pi],I\!R^3)$, since $E_c^\alpha(\mathbf{z})<\infty$ for such $\mathbf{z}$ by Proposition 2.4.

Arbitrary $\mathbf{y}\in C(n)$ can be approximated by smooth members of $C(n)$, see Lemma A.2 in the appendix. More precisely, for any given $\epsilon>0$ there exists a curve $\mathbf{y}_\epsilon \in C(n)\cap C^\infty([0,2\pi],I\!R^3)$, such that $\|\mathbf{y}-\mathbf{y}_\epsilon\|_{H^{2,2}}\le\epsilon$. Then we arrive at

$$E_b(\mathbf{x}) \underset{(14)}{\le} E_b(\mathbf{y}_\epsilon) \le E_b(\mathbf{y}) + O(\epsilon) \qquad \text{for } \epsilon \ll 1\,,$$

where we have used that the bending energy $E_b(.)$ is continuous on $H^{2,2}(S^1,I\!R^3)$ with respect to the $H^{2,2}$–norm, see Lemma A.3 in the appendix. Letting ϵ tend to zero we obtain statement (ii) for all $\mathbf{y}\in C(n)$ and hence, again by the continuity of $E_b(.)$ for all $\mathbf{y}\in\overline{C(n)}$, which concludes the proof. □

Remark. Using the same argument one can show that (ii) holds even for all $\mathbf{y}$ in the closure of $C(n)$ with respect to the $H^{2,2}$–norm.

4 The generalized minimizer

In view of the previous theorem the limit curve $\mathbf{x} \in \overline{C(n)}$ may be regarded as a generalized minimizer of the bending energy in the class $C(n)$. In the following we are going to prove some properties concerning the shape of $\mathbf{x}$.

Proposition 4.1 *Let* $\mathbf{x}$ *be the limit curve of Theorem 3.2. Then* $\mathbf{x}$ *is the once covered circle of length* l*, if the potential* $E_c^\alpha(\mathbf{x})$ *is finite or if* $n=0$*.*

Proof. Assume that $E_c^\alpha(\mathbf{x})<\infty$. By Lemma 2.3 $\mathbf{x}$ is embedded and property (i) of Theorem 3.2 tells us that $\mathbf{x}$ is also regular. Hence, we can apply Lemma A.1 of the appendix to obtain $\mathbf{x}\in C(n)$, since $\mathbf{x}\in\overline{C(n)}$. But then, again applying Lemma A.1 we find that all $\mathbf{y}\in H^{2,2}(S^1,I\!R^3)$ with length $L(\mathbf{y})=l$ and $\|\mathbf{y}-\mathbf{x}\|_{H^{2,2}}\leq\epsilon\ll 1$ are in fact members of the same isotopy class, i.e. $\mathbf{y}\in C(n)$. Consequently, by Theorem 3.2, (ii) we have $E_b(\mathbf{x})\leq E_b(\mathbf{y})$ for all such $\mathbf{y}$, which means that $\mathbf{x}$ is a relative minimizer of E_b in $H^{2,2}(S^1,I\!R^3)$ under the constraint of fixed length. In particular, $\mathbf{x}$ is a stable closed elastica in $I\!R^3$, which according to a result of J. Langer and D.A. Singer ([10]) must be the once covered circle.

Let $\mathbf{c}$ be a smooth parametrization of the once covered circle of length l. Now, if $n=0$ then we have for the generalized minimizer $\mathbf{x}\in\overline{C(0)}$ using the classical estimate of W. Fenchel for the total curvature of closed space curves together with Hölder's inequality:

$$E_b(\mathbf{x})\leq E_b(\mathbf{c})=4\pi^2/l\leq E_b(\mathbf{y}) \text{ for all } \mathbf{y}\in H^{2,2}(S^1,I\!R^3) \text{ with } L(\mathbf{y})=l\,.$$

But then again by the stability result [10] $\mathbf{x}$ must be a parametrization of the once covered circle thus finishing the proof of the second claim. □

It is an easy consequence of this result that we cannot expect to obtain a priori bounds on the potential, if we work in any nontrivial knot class $C(n)$ with $n\geq 1$. In fact, we have

Corollary 4.2 *If* $n\geq 1$*, then* $E_c^\alpha(\mathbf{x})$ *is not finite.*

Proof. If $E_c^\alpha(\mathbf{x})$ were finite, then by Proposition 4.1 $\mathbf{x}$ would be the once covered circle, hence embedded and $\mathbf{x}\in C(0)$. On the other hand $\mathbf{x}\in\overline{C(n)}, n\geq 1$ and even $\mathbf{x}\in C(n)$ by Lemma A.1, since $\mathbf{x}$ having finite potential is embedded. This gives a contradiction, because $C(n)\cap C(m)=\emptyset$ for $n\neq m$. □

Of geometric significance is the following result that confirms what physical experiments with knotted elastic wires suggest: There are points of selfcontact for any generalized minimizer of the bending energy in nontrivial knot classes.

Corollary 4.3 (Double points) *If* $n\geq 1$*, then* $\mathbf{x}$ *has at least one double point (of possibly higher multiplicity).*

Proof. Assuming on the contrary that $\mathbf{x}$ is embedded we can apply the same arguments as in the proof of Proposition 4.1 to show that $\mathbf{x}$ must be the once covered circle contradicting the fact that $n\geq 1$. □

Remark. One might ask if one can control the multiplicity of double points of the generalized minimizer $\mathbf{x}$. A preliminary answer is given by Lemma 2.2, which implies that the maximal multiplicity of double points is estimated from above by $2\pi/d$, since any two parameters $s_1\neq s_2$ in S^1 with $\mathbf{x}(s_1)=\mathbf{x}(s_2)$ must satisfy $|s_1-s_2|_{S^1}>d$, where $d=[l/(4\pi\|\ddot{\mathbf{x}}\|_{L^2})]^2=\pi/(2lE_b(\mathbf{x}))$ is controlled by the bending energy $E_b(\mathbf{x})$.
With the following one dimensional analogue of the Li–Yau inequality ([11, Theorem 6]) we give a more satisfactory estimate on the maximal multiplicity and even on the location of the corresponding double point on the curve:

Theorem 4.4 *Let* $\mathbf{y}\in H^{2,2}(S^1,I\!R^3)$ *be a curve of length* l *with a double point of multiplicity* m *dividing* $\mathbf{y}$ *into* m *subarcs* $\mathbf{Y}^1,\mathbf{Y}^2,\dots,\mathbf{Y}^n$ *with lengths* $l_i := L(\mathbf{Y}^i), i=1,\dots,m,$ *and* $\sum_{i=1}^m l_i = l$ *. Then*

$$\sum_{i=1}^m 1/l_i \le E_b(\mathbf{y})/16\,, \qquad \textit{in particular,} \quad m\le\sqrt{lE_b(\mathbf{y})/16}\,. \tag{15}$$

Proof. We are going to use an idea of L. Simon ([15, Lemma 1.4]), see also [13, p.60]. Without loss of generality we may assume that the origin in $I\!R^3$ is the double point under consideration and that $|\dot{\mathbf{y}}|\equiv const.$

Testing the identity

$$\int_0^{2\pi}\langle\dot{\mathbf{y}}(\tau), D\phi(\mathbf{y}(\tau))\circ\dot{\mathbf{y}}(\tau)\rangle\,d\tau = -\int_0^{2\pi}\langle\ddot{\mathbf{y}}(\tau),\phi(\mathbf{y}(\tau))\rangle\,d\tau\,, \tag{16}$$

valid for all $\phi\in C^{0,1}(I\!R^3, I\!R^3)$, with

$$\phi(\mathbf{z}) := \begin{cases}\mathbf{z}/\sigma & \text{for } |\mathbf{z}|\le\sigma\,,\\ \mathbf{z}/|\mathbf{z}| & \text{for } |\mathbf{z}|>\sigma\,, \qquad \sigma>0\end{cases}$$

we arrive at

$$\frac{1}{\sigma}\int_\Sigma|\dot{\mathbf{y}}|^2 + \int_{S^1\setminus\Sigma}\Big[\frac{|\dot{\mathbf{y}}|^2}{|\mathbf{y}|} - \frac{\langle\dot{\mathbf{y}},\mathbf{y}\rangle^2}{|\mathbf{y}|^3}\Big] = -\frac{1}{\sigma}\int_\Sigma\langle\ddot{\mathbf{y}},\mathbf{y}\rangle - \int_{S^1\setminus\Sigma}\frac{\langle\ddot{\mathbf{y}},\mathbf{y}\rangle}{|\mathbf{y}|}\,, \tag{17}$$

where $\Sigma := \{\,\tau\in S^1\,|\,|\mathbf{y}(\tau)|\le\sigma\,\}$.

Now, let $\mathbf{y}^\perp := \mathbf{y}-\langle\mathbf{y},\dot{\mathbf{y}}\rangle\dot{\mathbf{y}}/|\dot{\mathbf{y}}|^2$ be the normal component of the position vector, then $|\mathbf{y}^\perp-\mathbf{y}|^2 = \langle\dot{\mathbf{y}},\mathbf{y}\rangle^2/|\dot{\mathbf{y}}|^2$, and we may write

$$\begin{aligned}\int_{S^1\setminus\Sigma}\frac{\langle\dot{\mathbf{y}},\mathbf{y}\rangle^2}{|\mathbf{y}|^3} &= \int_{S^1\setminus\Sigma}[\,|\mathbf{y}^\perp|^2 - 2\langle\mathbf{y}^\perp,\mathbf{y}\rangle + |\mathbf{y}|^2\,]\cdot|\dot{\mathbf{y}}|^2/|\mathbf{y}|^3\\ &= \int_{S^1\setminus\Sigma}\Big[\,\frac{1}{|\mathbf{y}|} - \frac{|\mathbf{y}^\perp|^2}{|\mathbf{y}|^3}\,\Big]\cdot|\dot{\mathbf{y}}|^2\,.\end{aligned}$$

Substituting this into (17) one obtains

$$\frac{1}{\sigma}\int_\Sigma|\dot{\mathbf{y}}|^2 + \int_{S^1\setminus\Sigma}\frac{|\mathbf{y}^\perp|^2|\dot{\mathbf{y}}|^2}{|\mathbf{y}|^3} = -\frac{1}{\sigma}\int_\Sigma\langle\ddot{\mathbf{y}},\mathbf{y}\rangle - \int_{S^1\setminus\Sigma}\frac{\langle\ddot{\mathbf{y}},\mathbf{y}\rangle}{|\mathbf{y}|}\,. \tag{18}$$

Note, that

$$\frac{|\mathbf{y}^\perp|^2}{|\mathbf{y}|^3} = |\mathbf{y}|\Big|\,\frac{\mathbf{y}^\perp}{|\mathbf{y}|^2} + \frac{\ddot{\mathbf{y}}}{2|\dot{\mathbf{y}}|^2}\,\Big|^2 - \frac{\langle\mathbf{y}^\perp,\ddot{\mathbf{y}}\rangle}{|\mathbf{y}||\dot{\mathbf{y}}|^2} - \frac{|\ddot{\mathbf{y}}|^2|\mathbf{y}|}{4|\dot{\mathbf{y}}|^4} \qquad \text{a.e. on} \quad S^1$$

and $\langle\mathbf{y}^\perp,\ddot{\mathbf{y}}\rangle = \langle\mathbf{y},\ddot{\mathbf{y}}\rangle$, since $\ddot{\mathbf{y}}\perp\dot{\mathbf{y}}$ a.e. on S^1. Hence from (18) we infer

$$\frac{1}{\sigma}\int_\Sigma|\dot{\mathbf{y}}|^2 + \int_{S^1\setminus\Sigma}|\dot{\mathbf{y}}|^2|\mathbf{y}|\cdot\Big|\,\frac{\mathbf{y}^\perp}{|\mathbf{y}|^2} + \frac{\ddot{\mathbf{y}}}{2|\dot{\mathbf{y}}|^2}\,\Big|^2 = -\frac{1}{\sigma}\int_\Sigma\langle\ddot{\mathbf{y}},\mathbf{y}\rangle + \frac{1}{4}\int_{S^1\setminus\Sigma}\frac{|\ddot{\mathbf{y}}|^2|\mathbf{y}|}{|\dot{\mathbf{y}}|^2}\,. \tag{19}$$

Observe that, since $\mathbf{y}\in H^{2,2}(S^1,I\!R^3)$,

$$\lim_{\sigma\searrow0}\frac{1}{\sigma}\int_\Sigma|\dot{\mathbf{y}}|^2 = 2m|\dot{\mathbf{y}}|\,,$$

$$\lim_{\sigma\searrow0}\frac{1}{\sigma}\int_\Sigma\langle\ddot{\mathbf{y}},\mathbf{y}\rangle = 0 \qquad \text{and}$$

$$\lim_{\sigma\searrow0}\int_{S^1\setminus\Sigma}\frac{|\ddot{\mathbf{y}}|^2|\mathbf{y}|}{|\dot{\mathbf{y}}|^2} = \int_{S^1}\frac{|\ddot{\mathbf{y}}|^2|\mathbf{y}|}{|\dot{\mathbf{y}}|^2}\,.$$

By taking the limit $\sigma \searrow 0$ and omitting the second nonnegative integral on the left in (19) we get the estimate

$$8m \leq \int_{S^1} \frac{|\ddot{\mathbf{y}}|^2|\mathbf{y}|}{|\dot{\mathbf{y}}|^3} = \int_{\mathbf{y}} \kappa^2|\mathbf{y}|\,ds\,, \tag{20}$$

where the last equality sign is valid, since $\mathbf{y}$ has constant velocity.

If the double point is given by $\mathbf{y}(s_1)=\mathbf{y}(s_2)=\ldots=\mathbf{y}(s_m)=0$ for $0\leq s_1<s_2<\ldots<s_m<2\pi$, we define the connecting arcs by

$$\begin{aligned}\mathbf{Y}^i &:= \mathbf{y}([s_i, s_{i+1}]) \text{ for } i=1,\ldots,m-1 \text{ and}\\ \mathbf{Y}^m &:= \mathbf{y}([s_m, 2\pi]\cup(0,s_1])\end{aligned}$$

with their respective lengths $l_i := L(\mathbf{Y}^i)$, $\sum_{i=1}^m = l$.
Reparametrizing the arcs $\mathbf{Y}^i$ to constant velocity curves $\mathbf{y}_i$ on $[0,\pi]$ we can define the reflection at the origin by

$$\boldsymbol{\eta}_i(t) := \begin{cases}\mathbf{y}_i(t) & \text{for } t\in[0,\pi]\,,\\ -\mathbf{y}_i(2\pi-t) & \text{for } t\in(\pi,2\pi)\,.\end{cases}$$

Note, that since $\mathbf{y}_i(0)=\mathbf{y}_i(\pi)=0$ and

$$\dot{\boldsymbol{\eta}}_i(t) = \begin{cases}\dot{\mathbf{y}}_i(t) & \text{for } t\in[0,\pi]\,,\\ \dot{\mathbf{y}}_i(2\pi-t) & \text{for } t\in(\pi,2\pi)\end{cases}$$

one obtains $\boldsymbol{\eta}_i \in H^{2,2}(S^1,\mathbb{R}^3)$ with the origin $0=\boldsymbol{\eta}_i(0)=\boldsymbol{\eta}_i(\pi)$ as a double point with mutliplicity two. In addition, $L(\boldsymbol{\eta}_i)=2l_i$ by construction for each $i\in\{1,\ldots,m\}$. Applying (20) to $\boldsymbol{\eta}_i$ with $m=2$, one derives

$$16 \leq \int_{\boldsymbol{\eta}_i} \kappa_i^2|\boldsymbol{\eta}_i|\,ds \leq \frac{l_i}{2}\int_{\boldsymbol{\eta}_i} \kappa_i^2\,ds = l_i\int_{\mathbf{Y}^i}\kappa_i^2\,ds\,, \tag{21}$$

since $|\boldsymbol{\eta}_i|\leq l_i/2$ and $\int_{\boldsymbol{\eta}_i}\kappa_i^2\,ds=2\int_{\mathbf{Y}^i}\kappa_i^2\,ds$ by symmetry. Adding up (21) over i from 1 to m we arrive at (15) keeping in mind that $m^2\leq\sum_{i=1}^m n_i^{-1}$ for $n_i\in\mathbb{R}, n_i>0$ with $\sum_{i=1}^m n_i=1$. □

A Appendix

Lemma A.1 (Isotopy) *Let $\boldsymbol{\eta}\in C^1(S^1,\mathbb{R}^3)$ be a regular simple closed curve in $\mathbb{R}^3$. Then there exists a constant $\epsilon^*>0$ depending on $\boldsymbol{\eta}$, such that all $\boldsymbol{\xi}\in C^1(S^1,\mathbb{R}^3)$ with $\|\dot{\boldsymbol{\xi}}-\dot{\boldsymbol{\eta}}\|_{C^0}\leq\epsilon^*$ are isotopic to $\boldsymbol{\eta}$.*

Proof. Since $\boldsymbol{\eta}$ is regular, there is a constant $v>0$ such that $|\dot{\boldsymbol{\eta}}(t)|\geq v$ for all $t\in S^1$. Hence

$$\begin{aligned}|\boldsymbol{\eta}(t)-\boldsymbol{\eta}(t')| &= \Big|\int_t^{t'}\dot{\boldsymbol{\eta}}(\tau)\,d\tau\Big| \geq \Big|\int_t^{t'}\dot{\boldsymbol{\eta}}(t)\,d\tau\Big| - \Big|\int_t^{t'}|\dot{\boldsymbol{\eta}}(t)-\dot{\boldsymbol{\eta}}(\tau)|\,d\tau\Big|\\ &\geq v|t-t'|_{S^1} - \sup_{\tau\in[t',t]}|\dot{\boldsymbol{\eta}}(t)-\dot{\boldsymbol{\eta}}(\tau)|\cdot|t-t'|_{S^1}\\ &\geq \frac{v}{2}|t-t'|_{S^1} \text{ for } |t-t'|_{S^1}\leq v/(4\|\dot{\boldsymbol{\eta}}\|_{C^0})\,.\end{aligned}$$

For $\boldsymbol{\eta}$ is simple, we get

$$|\boldsymbol{\eta}(t)-\boldsymbol{\eta}(t')| \geq \min\{\,\frac{v}{2}|t-t'|_{S^1},\theta^*\,\}\,, \tag{22}$$

where $\theta^* := \min\{ |\boldsymbol{\eta}(\tau) - \boldsymbol{\eta}(\tau')| \,|\, (\tau,\tau') \in S^1 \times S^1, |\tau - \tau'|_{S^1} \geq v/(8\|\dot{\boldsymbol{\eta}}\|_{C^0}) \} > 0$.
The homotopy $\Phi : S^1 \times [0,1] \longrightarrow I\!R^3$ defined by

$$\Phi(t,\sigma) := (1-\sigma)\boldsymbol{\eta}(t) + \sigma \boldsymbol{\xi}(t)$$

satisfies $\Phi(t,0) = \boldsymbol{\eta}(t)$, $\Phi(t,1) = \boldsymbol{\xi}(t)$, and the curves $\Phi(.,\sigma)$ are closed for all $\sigma \in [0,1]$.
In addition, $\Phi(.,\sigma)$ is injective for $\epsilon > 0$ sufficiently small, since by (22)

$$\begin{aligned}
|\Phi(t,\sigma) - \Phi(t',\sigma)| &= |\,(1-\sigma)(\boldsymbol{\eta}(t) - \boldsymbol{\eta}(t')) + \sigma \int_{[t,t']} \dot{\boldsymbol{\eta}}(\tau)\, d\tau \\
&\quad - \sigma \int_{[t,t']} [\, \dot{\boldsymbol{\eta}}(\tau) - \dot{\boldsymbol{\xi}}(\tau)\,]\, d\tau \,| \\
&\geq |\boldsymbol{\eta}(t) - \boldsymbol{\eta}(t')| - \sigma \|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \cdot |t - t'|_{S^1} \\
&\geq \min\{ \frac{v}{2} |t - t'|_{S^1}, \theta^* \} - \|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \cdot |t - t'|_{S^1} \\
&\geq \frac{1}{2} \min\{ \frac{v}{2} |t - t'|_{S^1}, \theta^* \} > 0 \qquad \text{for all } \sigma \in [0,1]\,, (t,t') \in S^1 \times S^1\,, t \neq t',
\end{aligned}$$

if $\epsilon^* := \min\{\theta^*/2\pi, v/4\}$. (Obviously this proof also works for vector functions $\boldsymbol{\eta}, \boldsymbol{\xi}$ that are merely Lipschitz continuous.) □

Lemma A.2 (Approximation in $C(n)$**)** *For any* $\mathbf{y} \in C(n)$ *and any given* $\epsilon > 0$ *there exists* $\mathbf{y}_\epsilon \in C(n) \cap C^\infty([0,2\pi], I\!R^3)$ *with*

$$\|\mathbf{y}_\epsilon - \mathbf{y}\|_{H^{2,2}} < \epsilon\,.$$

Proof. Since $\mathbf{y} \in C(n)$ is a simple, regular and closed curve with continuous first derivatives, we may apply Lemma A.1 to find a constant ϵ^* depending on $\mathbf{y}$, such that all $\mathbf{z} \in C^1(S^1, I\!R^3)$ with $\|\dot{\mathbf{y}} - \dot{\mathbf{z}}\|_{C^0} \leq \epsilon^*$ are isotopic to $\mathbf{y}$.
Now, given an arbitrary $\epsilon > 0$, one can show (see e.g. [16, p.35]) that there is a curve $\tilde{\mathbf{y}}_\epsilon \in H^{2,2}(S^1, I\!R^3) \cap C^\infty([0,2\pi], I\!R^3)$, such that

$$\text{(23)} \qquad \|\mathbf{y} - \tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}} < \min\{l/(4\pi C_S), \epsilon/[1 + \|\mathbf{y}\|_{H^{2,2}} + l/(4\pi C_S)], \epsilon^*/C_S\},$$

where C_S denotes the constant of the Sobolev embedding

$$\text{(24)} \qquad H^{2,2}((0,2\pi)) \hookrightarrow C^{1,1/2}([0,2\pi])\,.$$

For the length $l_\epsilon := L(\tilde{\mathbf{y}}_\epsilon)$ we find

$$\text{(25)} \qquad |l_\epsilon - l| \leq 2\pi C_S \|\mathbf{y} - \tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}} < l/2 \quad \text{by (23)}\,.$$

Now define $\mathbf{y}_\epsilon := (l/l_\epsilon) \cdot \tilde{\mathbf{y}}_\epsilon \in H^{2,2}(S^1, I\!R^3) \cap C^\infty([0,2\pi], I\!R^3)$ with length $L(\mathbf{y}_\epsilon) = l$.
By (23) and (25)

$$\begin{aligned}
\|\mathbf{y} - \mathbf{y}_\epsilon\|_{H^{2,2}} &\leq \|\mathbf{y} - \tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}} + \|\tilde{\mathbf{y}}_\epsilon - \mathbf{y}_\epsilon\|_{H^{2,2}} \\
&\leq \|\mathbf{y} - \tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}} + (l_\epsilon - l)\|\tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}}/l_\epsilon \\
&\leq (1 + \|\mathbf{y} - \tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}} + \|\mathbf{y}\|_{H^{2,2}}) \cdot \|\mathbf{y} - \tilde{\mathbf{y}}_\epsilon\|_{H^{2,2}} < \epsilon \qquad \text{and}
\end{aligned}$$

$$\|\mathbf{y} - \mathbf{y}_\epsilon\|_{C^1} \leq C_S \|\mathbf{y} - \mathbf{y}_\epsilon\|_{H^{2,2}} \leq \epsilon^*\,,$$

which allows us to conclude $\mathbf{y}_\epsilon \in C(n) \cap C^\infty([0,2\pi], I\!R^3)$. □

Lemma A.3 (Continuity of $E_b(.)$) *The bending energy $E_b(.)$ is continuous on the space $H := \{\mathbf{z}\in H^{2,2}((0,2\pi),\mathbb{R}^3)\,|\,\dot{\mathbf{z}}(s)\neq 0$ for all $s\in[0,2\pi]\}$ with respect to the $H^{2,2}$–norm.*

Proof. Note, that it makes sense to prescribe a pointwise condition on the first derivatives of $\mathbf{z}\in H$ because of the Sobolev embedding (24). For $\mathbf{z}\in H$ we have $|\dot{\mathbf{z}}|\geq v$ for all $s\in S^1$ for some $v>0$ depending on $\mathbf{z}$, hence $|\dot{\mathbf{y}}(s)|\geq v-O(\epsilon)$ for all $\mathbf{y}\in H$ with $||\mathbf{y}-\mathbf{z}||_{H^{2,2}}\leq\epsilon$. Now we estimate

$$\begin{aligned}
|E_b(\mathbf{z})-E_b(\mathbf{y})| &\leq \int_0^{2\pi}\Big|\frac{|\dot{\mathbf{z}}\wedge\ddot{\mathbf{z}}|^2}{|\dot{\mathbf{z}}|^5}-\frac{|\dot{\mathbf{y}}\wedge\ddot{\mathbf{y}}|^2}{|\dot{\mathbf{y}}|^5}\Big| \\
&\leq \frac{1}{v^5}\int_0^{2\pi}[\,|\dot{\mathbf{z}}\wedge\ddot{\mathbf{z}}|+|\dot{\mathbf{y}}\wedge\ddot{\mathbf{y}}|\,]\cdot[\,|\dot{\mathbf{z}}\wedge(\ddot{\mathbf{z}}-\ddot{\mathbf{y}})|+|(\dot{\mathbf{z}}-\dot{\mathbf{y}})\wedge\ddot{\mathbf{y}}|\,] \\
&\quad+\int_0^{2\pi}\Big|\frac{1}{|\dot{\mathbf{z}}|^5}-\frac{1}{|\dot{\mathbf{y}}|^5}\Big|\cdot|\dot{\mathbf{y}}\wedge\ddot{\mathbf{y}}|\cdot[|\dot{\mathbf{z}}\wedge\ddot{\mathbf{z}}|+|\dot{\mathbf{y}}\wedge\ddot{\mathbf{y}}|] \\
&\leq \frac{1}{v^5}\cdot[\,||\mathbf{z}||^2_{H^{2,2}}+O(\epsilon)\,]\Big\{||\mathbf{z}-\mathbf{y}||_{H^{2,2}}+\frac{C||\mathbf{z}||^4_{C^1}||\mathbf{z}-\mathbf{y}||_{C^1}}{v^{10}-O(\epsilon)}\Big\} \\
&\leq C(||\mathbf{z}||_{H^{2,2}},v)||\mathbf{z}-\mathbf{y}||_{H^{2,2}} \\
&= O(\epsilon)\quad \text{for } \epsilon\searrow 0\,.
\end{aligned}$$

□

References

[1] R. BRYANT, P. GRIFFITHS, Reduction of Order for the Constrained Variational Problem and $\int k^2/2\,ds$, *Amer. J. Math.* **108**, 1986, pp. 525–570 .

[2] L. EULER, Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattisimo sensu accepti, *Bousquet, Lausannae et Genevae, 1744, E65A. O.O.* Ser. I, vol. 24.

[3] M.H. FREEDMAN, ZHENG-XU HE, ZHENGHAN WANG, On the Möbius Energy of Knots and Unknots, *Annals of Math.* **139** no. 1, 1994, pp. 1–50.

[4] R. IRRGANG, Ein singuläres bewegungsinvariantes Variationsproblem, *Math. Z.* **37**, 1933, pp. 381–401.

[5] R. KUSNER, J. SULLIVAN, Möbius Energies for Knots and Links, Surfaces and Submanifolds, *Geometric Topology*, Proc. Georgia International Topology Conference, 1993, (ed. W.H. Kazez),*AMS/IP Studies in Advanced Math.* vol. 2, 1996, pp. 570–604.

[6] D. KIM, R. KUSNER, Torus Knots Extremizing the Möbius Energy, *Experimental Math.* **2** 1993, pp. 1–9.

[7] J. LANGER, D.A. SINGER, Knotted Elastic Curves in $\mathbb{R}^3$, *J. London Math. Soc.* (2), **30**, 1984, pp. 512–520.

[8] J. LANGER, D.A. SINGER, Curves in the Hyperbolic Plane and Mean Curvature of Tori in 3–Space, *Bull. London Math. Soc.* **16**, 1984, pp. 531–534.

[9] J. LANGER, D.A. SINGER, The Total Squared Curvature of Closed Curves, *J. Differential Geometry* **20**, 1984, pp. 1–22.

[10] J. LANGER, D.A. SINGER, Curve Straightening and a Minimax Argument for Closed Elastic Curves, *Topology* **24**, no. 1, 1985, pp. 75–88.

[11] P. LI, S.T. YAU, A New Conformal Invariant and Its Applications to the Willmore Conjecture and the First Eigenvalue on Compact Surfaces, *Invent. Math.* **69**, 1982, 269–291.

[12] J. O'HARA, Energy of a Knot, *Topology* **30**, 1991, pp. 241–247.

[13] A. POLDEN, Curves and Surfaces Of Least Total Curvature and Fourth–Order Flows, *Dissertation Univ. Tübingen* 1996.

[14] J. RADON, Über das Minimum des Integrals $\int_{s_1}^{s_2} F(x, y, \vartheta, \kappa)\, ds$, *Sitzungsber. Kaiserliche Akad. Wiss., Wien. Math.-nat. Kl.*, **69**, 1910, pp. 1257–1326.

[15] L. SIMON, Existence of Surfaces Minimising the Willmore Functional, *Comm. Analysis and Geometry* **1**, 251–326 (1993).

[16] H. VON DER MOSEL, Geometrische Variationsprobleme höherer Ordnung, *Bonner Math. Schriften* 293, 1996.

[17] H. VON DER MOSEL, Elastic knots in Euclidean 3–space, to appear in: *Ann. de l'I.H.P. – Analyse Non Linéaire.*