CHEMICALLY REACTING FLUID FLOWS: WEAK SOLUTIONS AND GLOBAL ATTRACTORS

DAVID E. NORMAN

ABSTRACT. In this paper we present a model for incompressible chemically reacting flows where reactants enter the domain, react and then leave the domain. In the interior of the domain we have the Navier-Stokes equations for the fluid flow coupled with reaction diffusion equations for the chemistry and temperature. On the boundary we use mixed Neumann and inhomogeneous Robin boundary conditions for the chemistry and temperature equations which fix the amount of chemicals and heat flowing into the system. For three dimensional domains we then show the existence of weak solutions which are physically reasonable, i.e., which satisfy certain maximum principle properties. These solutions are sufficient to provide the basis for a dynamical systems study of these reacting flows. We do not show that the solutions are unique, however we are able to show the existence of global attractors for the system.

1. INTRODUCTION

A chemically reacting flow is a fluid flow in which a chemical reaction is also occurring. Such flows occur in a wide range of fields including combustion, chemical engineering, biology and pollution abatement. In this paper we present a general model for chemically reacting flows in which chemicals enter a domain, react and leave the domain. For example, this could be used to model a chemical reactor in which raw materials enter the system, react to produce a product, which in turn leaves the system.

In this paper we will work with a bounded three dimensional domain Ω. In the interior of the domain we model the reacting flow by coupling the Navier-Stokes equations for the fluid flow with reaction-diffusion equations for the chemical reactions, a model also considered in two dimensions by Manley, Marion and Temam [20]. Except for the chemistry, this is essentially the same as the Bénard problem, as studied in Foias, Manley and Temam [7]. While these earlier papers dealt with the long time dynamics of the 2D problem, our focus in this work is on the 3D problem. Our goal is to lay the foundation for the dynamical systems study for the 3D problem, which can then be applied to the analysis of any chemically reacting plant.

A novel feature of our model is found in the boundary conditions we use for the chemistry and temperature equations. In many situations, important quantities in the reacting flow system are the amounts of the chemicals and heat entering the system. We show that in

Date: March 12, 1998.

1991 *Mathematics Subject Classification.* Primary 80A25; Secondary 35Q99, 35B40, 58F39.

Key words and phrases. Chemically Reacting Flow, Combustion, Arrhenius Kinetics, Incompressible Navier-Stokes Equations, Reaction Diffusion Equations, Boussinesq Models, Mixed Neumann and Robin Boundary Conditions, Weak Solution, Global Attractor, Nonuniqueness, Equations with Constraints.

This research was supported in part by the Army Research Office and the AHPCRC and IMA at the University of Minnesota.

order to have these quantities specified as parameters of the system, the boundary conditions for the chemistry and temperature equations are required to be inhomogeneous mixed Robin and Neumann boundary conditions depending in a specific way on the fluid flow at the boundary and the diffusion coefficients.

We also show the existence of weak solutions to our set of equations and, in particular, give estimates showing that these solutions dissipate energy. These estimates depend on the precise way that we handle the inhomogeneous boundary data. For instance, for the fluid flow u, we write

$$u(x,t) = v(x,t) + w(x),$$

where v satisfies homogeneous boundary conditions and w satisfies inhomogeneous boundary conditions and does not depend on time. We also do a similar procedure for the temperature and chemistry equations. We call this process homogenizing the boundary conditions and call w the homogenizing term. We will see that picking the correct homogenizing terms for the fluid and temperature equations is critical for the *a priori* estimates, in particular for showing that the system dissipates energy. Our work here is a significant generalization of Manley, Marion and Temam [20], since they consider only premixed reactions where the inflow concentrations are constant in space, which allows them to pick the homogenizing terms to be constant in space. An additional difficulty comes from the fact that the classical Poincaré inequality is not valid for either the temperature or chemistry equations, due to the Robin-Neumann boundary conditions. Instead, we prove a generalized Poincaré inequality, lemma 6, which is based on the consideration of a boundary integral which arises naturally in the analysis of the problem.

It is important that the weak solutions be physically reasonable, by which we mean that that the concentrations remain nonnegative, mass is conserved and the temperature remains bounded below. Although these results follow formally from the maximum principle, our solutions are not smooth enough to use this approach. Furthermore, since we consider the 3D problem, our solutions have less smoothness than the 2D solutions considered in Manley, Marion and Temam [20], hence we cannot use their arguments either. Instead, we prove physical reasonableness as part of the Bubnov-Galerkin method of showing existence of solutions. In particular, we take the limit of the approximate solutions in two stages, first for the chemistry and temperature equations and then for the velocity equation. This has the advantage that when we work with the chemistry and temperature equations we have a smooth approximate velocity.

In summary, we are able to show that there exist sufficient physically reasonable weak solutions in order to study the system from a dynamical systems viewpoint. For instance, in Norman [23], these solutions are used in comparing the dynamics of the reacting flow model to the dynamics of a continuous flow stirred tank reactor (CSTR). A CSTR is an ODE approximation to a reacting flow based on the assumption that the chemical concentrations and temperature are spatially homogeneous. In Norman [23] it is shown that, for large chemical and thermal diffusivities, the CSTR ODE is a good approximation to the full reacting flow PDE, and in particular that the global attractor for the reacting flow converges to the global attractor for the CSTR in a suitable sense.

We are unable to show the existence of global strong solutions or uniqueness of solutions for much the same reason that no such proofs exist for 3D Navier-Stokes equations alone. However, based on the work of Sell [24], we are able to show the existence of global attractors for our system. We do this by proving a general global attractor existence theorem for systems of equations without unique solutions. One of the interesting features of our approach is that we need exactly the same properties of the equations to show both the existence of solutions and the existence of the global attractor.

An outline of the paper is as follows. In section 2 we give a general overview of reacting flows and give a short discussion of what the chemistry implies about the equations. We also present the model we will be working with and justify our boundary conditions. In section 3 we present the mathematical background and notation for the problem and state our main theorems. We then present the method for homogenizing the boundary conditions and state and prove the basic properties of the equations that will be used in both the existence of weak, physically reasonable solutions and the existence of the global attractor. Then in section 4 we show the necessary *a priori* estimates and exhibit the the existence of weak solutions. In section 5 we state and prove our general global attractor existence theorem and apply it to the chemically reacting flow. Finally in section 6 we summarize our results and compare them to the previous work of Manley, Marion and Temam [20], see also Manley and Marion [19], Marion [21] and Marion and Temam [22].

2. Background for Reacting Flows

There are three quantities arising in the study of chemically reacting flows, the fluid flow, the chemical concentrations and the temperature. In particular, all three of these quantities are coupled. The fluid flow moves around the chemicals and the heat, the chemicals react which produces heat and affects the temperature, and finally the temperature affects the fluid flow through buoyancy effects.

We will use the following notation for the various quantities:

- u Velocity field, a vector
- p Pressure, a scalar
- T Temperature, a scalar
- Y_i, $i = 1, \dots N$ Mass fraction of the chemical species i, a scalar

Note that the mass fractions and concentrations C_i are related by $Y_i = \frac{\rho}{m_i} C_i$, where m_i is the molecular weight of chemical species i and ρ is the density, which we assume to be constant. We choose to work with mass fractions instead of concentrations because we then have the properties

$$\sum_{i=1}^{N} Y_i(x,t) = 1 \qquad \text{for all } x \in \Omega \text{ and } t \geq 0,$$

and

$$0 \leq Y_i(x,t) \leq 1, \qquad \text{for all } x \in \Omega \text{ and } t \geq 0.$$

We assume that the equations hold in a C^2 bounded domain $\Omega \subset \mathbb{R}^3$. Using the above notation our model for incompressible reacting flows takes the form

$$\partial_t u - Pr\Delta u + (u\cdot\nabla)u + \nabla p = f_0(T), \tag{2.1a}$$

$$\nabla\cdot u = 0, \tag{2.1b}$$

$$\partial_t T - \Delta T + (u\cdot\nabla)T = -\sum_{i=1}^{N} h_i W_i(Y_1,\dots,Y_N,T), \tag{2.1c}$$

$$\partial_t Y_i - \frac{1}{Le}\Delta Y_i + (u\cdot\nabla)Y_i = W_i(Y_1,\dots,Y_N,T), \tag{2.1d}$$

where Pr is the Prandl number, Le is the Lewis number, the terms with $(u\cdot\nabla)$ are fluid transport terms, $f_0(T)$ is the forcing term from buoyancy, $W_i(Y_1,\dots,Y_N,T)$ describes the change in mass fractions due to the reaction, and h_i is the enthalpy of species i divided by its molecular weight, i.e., a measure of the amount of heat contained in species i. The first two equations are the usual Navier-Stokes equations, and the second two are reaction-diffusion equations with a transport term added. The central assumption made in deriving these equations is that the fluid has constant density. For further physical background on these equations, see Buckmaster and Ludford [3] and Williams [27].

The most common model of chemical kinetics is the so called Arrhenius model in which the W_i take the form

$$W_i(Y_1,\dots,Y_N,T) = \sum_{j=1}^{m_i} A_j e^{-E_j/R_0T}\prod_{k=1}^{N} C_k^{\nu_{j,k}}$$

where A_j are the frequency factors, E_j are the activation energies, R_0 is the universal gas constant, C_i are the concentrations, i.e., the mass fractions Y_i divided by the molecular weight, and the $\nu_{j,k}$ are nonnegative integers, where at least one of $\nu_{j,k}$, for $k=1,\dots N$, is nonzero for each j.

Although we will not explicitly assume the Arrhenius model, it does provide motivation for the more general assumptions which we make. In particular, we assume that each W_i is continuous and Lipschitz for $(Y_1,\dots,Y_N,T)\in[0,1]^N\times[T_l,\infty)$, and that mass is conserved, i.e.,

$$\sum_{i=1}^{N} W_i(Y_1,\dots,Y_N,T) = 0 \qquad \text{for all } (Y_1,\dots,Y_N,T)\in[0,1]^N\times[T_l,\infty). \tag{2.2}$$

We also assume that

$$W_i(Y_1,\dots,Y_N,T)\geq 0 \qquad \text{whenever } Y_i = 0, \tag{2.3}$$

which implies that in order for a species to be consumed in the reaction, i.e., for $W_i<0$, its mass fraction Y_i must be positive.

An important assumption for the mathematical analysis is that each W_i is everywhere bounded, namely that,

$$|W_i(Y_1,\dots,Y_N,T)|\leq B \qquad \text{for all } (Y_1,\dots,Y_N,T)\in[0,1]^N\times[T_l,\infty), \tag{2.4}$$

where $T_l>0$ is some minimum temperature which we fix later. The boundedness in each Y_i follows from the continuity of W_i, however the boundedness as $T\to\infty$ is more physical. It can be justified intuitively based on the idea that the reaction rate will always be constrained by the quantities of the chemical species, regardless of the temperature. Note that if we assume the Arrhenius model, then these assumptions hold directly.

We also assume that that at some minimum temperature T_l the reaction always produces heat:

$$-\sum_{i=1}^{N} h_i W_i(Y_1, \ldots, Y_N, T_l) \geq 0 \qquad \text{for all } (Y_1, \ldots, Y_N) \in [0,1]^N, \tag{2.5}$$

One uses this assumption to show that the temperature never falls below T_l.

We of course assume that the initial condition satisfies constraints compatible with the above, namely, that the sum of the mass fractions is identically one, that the mass fractions are everywhere nonnegative, and that the temperature is everywhere greater than or equal to T_l. From the above assumptions we show that for all time these properties are preserved, i.e., if $(u, T, Y_i)(x,t)$ is a solution, then for all x, t one has

$$\sum_{i=1}^{N} Y_i(x,t) \equiv 1, \tag{2.6a}$$

$$1 \geq Y_i(x,t) \geq 0, \tag{2.6b}$$

$$T(x,t) \geq T_l, \tag{2.6c}$$

We say that a solution satisfying these properties is physically reasonable.

The origins of the buoyant forcing term $f_0(T)$ are unfortunately less physical than the above. The problem is that buoyant forces originate from spatial variation in density, e.g., where the temperature is high, the fluid is less dense, hence it has an upward force acting on it. However we want to work with incompressible fluids and assume that the density is constant. The usual way one gets around this difficulty is to assume that the change in density is so small that entire effect can be contained in the Navier-Stokes forcing term. This is the so called Boussinesq model. Since the density change is small, one further assumes that $f_0(T)$ is linear. For instance we will assume that

$$f_0(T) = -c_0 \vec{g}(T - T_0) \tag{2.7}$$

where $T_0, c_0 > 0$ are constants and $\vec{g}$ is a unit vector pointing in the direction of gravity.

2.1. **Boundary Conditions.** We now want to discuss the boundary conditions for the reacting flow system. It is, of course, easier to study things with homogeneous boundary data, but in this case that is not physically realistic. For instance, one of the main things one would like to model with the system (2.1) is combustion in some sort of a burner. In this case fuel is being injected into the system and the combustion products and heat are being removed from the system. This clearly requires inhomogeneous boundary data. In fact many chemical systems are of this form, since rarely does one want to just study a closed system. Frequently one wants to study open systems where chemicals enter the system, react, and the products leave.

An example of an Ω that one might consider is schematically pictured in figure 2.1. In this case one set of reactants enter on one portion of the boundary, another set enter on another portion, they mix and react in the middle, then the products leave. We note that the picture is somewhat misleading in that we do require that the domain to be at least C^2.

Clearly, since we have chemicals and heat flowing into and out of the system, an important quantity to consider is how much of each flows in and out of the system. We can formally

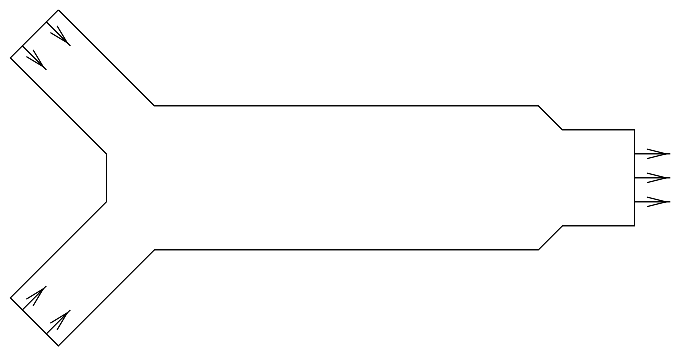

Figure 1. A prototypical domain Ω

derive this quantity from the equations in (2.1) in the following way. We consider first the rate of change of the total amount of Y_i in the domain.

$$\begin{aligned}\partial_t \int_\Omega Y_i\, dx &= \int_\Omega \partial_t Y_i\, dx \\ &= \int_\Omega \frac{1}{Le}\Delta Y_i - (u\cdot\nabla)Y_i + W_i\, dx.\end{aligned}$$

Now using the the identity $\nabla\cdot(uY_i) = Y_i\nabla\cdot u + (u\cdot\nabla)Y_i$, the divergence free property of u and the divergence theorem we get

$$\begin{aligned}\partial_t \int_\Omega Y_i\, dx &= \int_\Omega \frac{1}{Le}\nabla\cdot\nabla Y_i - \nabla\cdot(uY_i) + W_i\, dx \\ &= \int_{\partial\Omega} \frac{1}{Le}\frac{\partial Y_i}{\partial n} - Y_i(u\cdot n)\, dS + \int_\Omega W_i\, dx,\end{aligned}$$

where n is the outward unit normal to $\partial\Omega$. This implies that the rate of change of the total amount of Y_i in the domain is due to three terms which we can physically interpret as diffusion across the boundary, fluid transport across the boundary and changes due to reactions in the domain. In particular, the above implies that the flux of Y_i across the boundary at any point is given by

$$\text{Flux}_{Y_i} = \frac{1}{Le}\frac{\partial Y_i}{\partial n} - Y_i(u\cdot n). \tag{2.8}$$

Similarly the flux for T is given by

$$\text{Flux}_T = \frac{\partial T}{\partial n} - T(u\cdot n). \tag{2.9}$$

We now want to explain the physical basis for the boundary conditions we will be using. We will assume the existence of a partition $\partial\Omega = \Gamma_I \cup \Gamma_O \cup \Gamma_W$ corresponding to the portions of the boundary where fluid flows into the domain, where fluid flows out and the walls of the container, respectively. On all of $\partial\Omega$, the fluid flow u will be specified by Dirichlet boundary data, however the boundary conditions for T and Y_i will vary across the partition.

We first consider Γ_I. Here the fluid is flowing into the domain, so one has

$$u(x,t) = \phi(x) \qquad \text{in } \Gamma_I, \tag{2.10a}$$
$$\phi(x)\cdot n < 0 \qquad \text{in } \Gamma_I,$$

where $\phi(x)$ is a prescribed function. For T and Y_i, we use so called Robin boundary data, i.e.,

$$\frac{\partial T}{\partial n} - (\phi(x)\cdot n)T = T_f(x) \qquad \text{in } \Gamma_I, \tag{2.10b}$$
$$\frac{1}{Le}\frac{\partial Y_i}{\partial n} - (\phi(x)\cdot n)Y_i = Y_{i,f}(x) \qquad \text{in } \Gamma_I, \tag{2.10c}$$

where $T_f(x)$ and $Y_{i,f}(x)$ are prescribed nonnegative functions on Γ_I. These boundary conditions imply that we are fixing the flux of temperature and chemical species at each point of Γ_I, which implies that we are fixing the total amount of heat and chemicals flowing into the system.

Another possible option for Γ_I would be Dirichlet boundary data, as used in Manley, Marion and Temam [20]. This would be used if one wants to specify the composition of the incoming chemicals. However, in this case the flux would be unspecified, implying that, although one knows the composition of the inflowing fluid, one does not know how much of each chemical is actually entering the domain.

On Γ_O where the fluid flows out of the domain, we fix boundary conditions of the form

$$u(x,t) = \phi(x) \qquad \text{in } \Gamma_O, \tag{2.10d}$$
$$\phi(x)\cdot n > 0 \qquad \text{in } \Gamma_O,$$
$$\frac{\partial T}{\partial n}(x,t) = \frac{\partial Y_i}{\partial n}(x,t) = 0 \qquad \text{in } \Gamma_O. \tag{2.10e}$$

This is equivalent to specifying that the only flux out of the system is due to the fluid flow, with none due to diffusion. We do not want to specify the flux out of the system, since that is often the principal quantity one wants to study.

The boundary conditions on Γ_W are now relatively simple. We simply want there to be no flux across the walls. This implies

$$u(x,t) = 0 \qquad \text{in } \Gamma_W \text{ and} \tag{2.10f}$$
$$\frac{\partial T}{\partial n}(x,t) = \frac{\partial Y_i}{\partial n}(x,t) = 0 \qquad \text{in } \Gamma_W. \tag{2.10g}$$

Later in the paper we will frequently want to make statements about the temperature and chemistry equations simultaneously. To make this easier we will use the following notation for the homogeneous version of the boundary conditions

$$\kappa\frac{\partial \psi}{\partial n} - (\phi\cdot n)\psi = 0 \qquad \text{on } \Gamma_I, \tag{2.11a}$$
$$\frac{\partial \psi}{\partial n} = 0 \qquad \text{on } \partial\Omega - \Gamma_I. \tag{2.11b}$$

where $\kappa = 1$ for temperature and $\kappa = \frac{1}{Le}$ for the chemistry equations, and ϕ denotes the boundary values for u.

We also note that the divergence free condition and the divergence theorem imply that

$$\int_{\partial\Omega} \phi \cdot n \, dS = 0, \tag{2.12}$$

which in turn implies that

$$\int_{\Gamma_I} \phi \cdot n \, dS + \int_{\Gamma_O} \phi \cdot n \, dS = 0. \tag{2.13}$$

There is also a further physical constraint on the $Y_{i,f}(x)$. Recall that by the definition of mass fraction one has

$$Y(x,t) = \sum_{i=1}^{N} Y_i(x,t) = 1 \qquad \text{for } (x,t) \in \Omega \times \mathbb{R}^+.$$

Summing the boundary condition (2.10c) over i gives

$$\frac{1}{Le}\frac{\partial Y}{\partial n} - (\phi(x) \cdot n)Y = \sum_{i=1}^{N} Y_{i,f}(x) \qquad \text{for } x \in \Gamma_I.$$

Since $Y(x,t) \equiv 1$ should satisfy this equation, we must have that

$$\sum_{i=1}^{N} Y_{i,f}(x) = -(\phi(x) \cdot n), \qquad \text{for } x \in \Gamma_I. \tag{2.14}$$

Therefore, since $Y_{i,f}(x) \geq 0$, one has for each i that

$$Y_{i,f}(x) = -Y_{i,mf}(x)(\phi(x) \cdot n),$$

for some function $Y_{i,mf}(x)$, which can by physically interpreted as the mass fraction of chemical species i in the incoming fluid.

In order that a generalized Poincaré inequality will hold for the Laplacian terms in the temperature and chemistry equations, we will assume that Γ_I is nonempty. We also assume that $\phi \in H^{3/2}(\partial\Omega)$, so that it can be extended into the interior of Ω to give a function in $H^2(\Omega)$. Similarly, we assume that $T_f, Y_{i,f} \in H^{3/2}(\partial\Omega)$ so that there exist functions in $H^2(\Omega)$ which satisfy the temperature and chemistry parts of the boundary conditions (2.10). See Lions and Magenes [18] or Adams [1] for the necessary trace theorems. We note that these assumptions imply that ϕ, T_f and $Y_{i,f}$ are bounded functions.

3. Preliminaries

In this section we want to discuss certain preliminary material which will be used later in the paper. We will give definitions and state certain theorems which will be used later. In addition we will reformulate the equations in terms of new variables which satisfy homogeneous boundary conditions. Finally we will show that the equations satisfy certain boundedness and continuity properties which, in turn, are used to show both the existence of solutions and global attractors.

3.1. **Definitions.** We define the usual spaces for the velocity

$$H = \text{Closure}_{L^2(\Omega)^3}\left\{v \in C_0^\infty(\Omega)^3 : \nabla\cdot v = 0\right\},$$
$$V = \text{Closure}_{H^1(\Omega)^3}\left\{v \in C_0^\infty(\Omega)^3 : \nabla\cdot v = 0\right\},$$

i.e., H and V are the spaces of divergence free vector fields with zero boundary conditions in L^2 and H^1 respectively. We will also denote the dual of V by V^{-1}. We then define the Leray projection $\mathbb{P}$ to be the orthogonal projection of $L^2(\Omega)$ onto H. By using the divergence theorem one can show that any gradient is orthogonal to H, hence if we apply $\mathbb{P}$ to (2.1a) the pressure term will disappear, leaving us with a parabolic evolutionary equation. The linear part of this equation is the Stokes operator $A = -\mathbb{P}\Delta$ and $B(u,u) = \mathbb{P}(u\cdot\nabla)u$ is the nonlinearity. We also define the trilinear form $b(u,v,w) = \langle (u\cdot\nabla)v, w\rangle$. Note that $b(u,v,w) = \langle B(u,v), w\rangle$ if $w \in H$.

We define similar concepts for the chemistry and temperature equations. For the temperature equation we define

$$\mathcal{D}_1 = \left\{\psi \in C^\infty(\Omega) : \frac{\partial\psi}{\partial n} - (\phi\cdot n)\psi = 0 \text{ on } \Gamma_I \text{ and } \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega - \Gamma_I\right\},$$
$$H_1 = \text{Closure}_{L^2(\Omega)}\,\mathcal{D}_1,$$
$$V_1 = \text{Closure}_{H^1(\Omega)}\,\mathcal{D}_1,$$

and for the chemistry equations we have

$$\mathcal{D}_2 = \left\{\psi \in C^\infty(\Omega) : \frac{1}{Le}\frac{\partial\psi}{\partial n} - (\phi\cdot n)\psi = 0 \text{ on } \Gamma_I \text{ and } \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega - \Gamma_I\right\},$$
$$H_2 = \text{Closure}_{L^2(\Omega)}\,\mathcal{D}_2,$$
$$V_2 = \text{Closure}_{H^1(\Omega)}\,\mathcal{D}_2.$$

For $\phi \in V_i$ and $\psi \in H_i$, $i = 1,2$, we also define the bilinear form $B_1(u,\phi) = (u\cdot\nabla)\phi$ and the trilinear form $b_1(u,\phi,\psi) = \langle (u\cdot\nabla)\phi, \psi\rangle$.

We will also use the product spaces

$$\overline{H} = H \times H_1 \times \prod_{i=1}^N H_2,$$
$$\overline{V} = V \times V_1 \times \prod_{i=1}^N V_2$$

One can show that the Stokes operator A is a positive self-adjoint operator with compact resolvent, which implies that A has a complete set of eigenvectors. In particular, we let $\lambda_1 > 0$ be the smallest eigenvalue of A, so that

$$\lambda_1\|v\|^2 \le \|A^{1/2}v\|^2 \qquad \text{for all } v \in V \tag{3.1}$$

It can also be shown, see Constantin and Foias [4], that since Ω is C^2, the norm defined by the operator A on V is equivalent to the Sobolev norms, i.e., for all $v \in V$

$$C_e^1\|v\|^2_{H^1(\Omega)} \le \|A^{1/2}v\|^2 \le C_e^2\|v\|^2_{H^1(\Omega)} \qquad \text{for all } v \in V, \tag{3.2}$$

for some positive constants C_e^i depending only on Ω. One can also show that $V = \mathcal{D}(A^{1/2})$ and $V^{-1} = \mathcal{D}(A^{-1/2})$.

For symmetry in notation we let $A_1 = -\Delta$ with the boundary conditions (2.11) with $\kappa = 1$ and $A_2 = -\Delta$ with $\kappa = 1/Le$. Then one can show that A_1 and A_2 also have compact resolvent, since their inverse is compact, which follows from the fact that solutions to the associated elliptic problem are smooth, see Lions and Magenes [18].

We will also use the following equivalent norms for $\psi \in V_i$.

$$C_e^1\|\psi\|^2_{H^1(\Omega)} \leq \|A_i^{1/2}\psi\|^2 \leq C_e^2\|\psi\|^2_{H^1(\Omega)} \qquad \text{for all } \psi \in V_i, \tag{3.3}$$

for some constants C_e^i depending only on Ω. We will give a proof of this fact below in section 3.3

We will also use the following notation introduced in Sell [24]. Let X be any Banach space with norm $\|\cdot\|_X$. Then for $1 \leq p < \infty$ let $L^p_{loc}(0,\infty;H)$ be the set of all functions ϕ from $(0,\infty)$ to X such that for each t_0 and t with $0 < t_0 \leq t$, one has

$$\int_{t_0}^t \|\phi\|_X^p \, ds < \infty.$$

We also let $L^p_{loc}[0,\infty;X)$ be the set of all functions $\phi \in L^p_{loc}(0,\infty;X)$ such that for all $0 < t < \infty$ we have

$$\int_0^t \|\phi\|_X^p \, ds < \infty.$$

We also make the analogous definitions for $p = \infty$.

We will make heavy use of the fact that both $L^p_{loc}[0,\infty;X)$ and $L^p_{loc}(0,\infty;X)$ are Fréchet spaces, meaning that they are complete metrizable locally convex topological vector spaces. The topologies of both spaces can be generated by a countable family of pseudonorms, for example we can use

$$N_n(\phi) = \left(\int_{1/n}^n \|\phi\|_X^p \, ds\right)^{1/p} \quad n = 2, 3, 4, \ldots$$

for $L^p_{loc}(0,\infty;X)$ and

$$N_n(\phi) = \left(\int_{n-1}^n \|\phi\|_X^p \, ds\right)^{1/p} \quad n = 1, 2, 3, \ldots$$

for $L^p_{loc}[0,\infty;X)$. From these seminorms one can construct a translation invariant metric, for instance,

$$d(\phi,\psi) = d(\phi-\psi,0) = \sum_{i=1}^N 2^{-n}\min\{1, N_n(\phi-\psi)\}.$$

We also use the fact that a set U of a Fréchet space is bounded if for each seminorm N_n we have

$$\sup_{\phi\in U} N_n(\phi) < \infty.$$

3.2. **Statements of Theorems.** In this section we want to state the main theorems of the paper. The first of these theorems gives the existence and properties of weak solutions to the reacting flow system, and the second describes the existence of global attractors.

The first step in the mathematical analysis of the reacting flow system is to homogenize the boundary conditions, i.e., we want to rewrite the system so that the new variables satisfy homogeneous boundary conditions. In particular we will let

$$\begin{aligned} u(x,t) &= v(x,t) + w(x), \\ T(x,t) &= \theta(x,t) + \theta_0(x), \\ Y_i(x,t) &= \eta_i(x,t) + \eta_{i,0}(x), \end{aligned}$$

where v, θ and η satisfy homogeneous boundary conditions, and w, θ_0 and $\eta_{i,0}$ satisfy the inhomogeneous boundary conditions and are constant in time with $\nabla\cdot w = 0$ in Ω. In section 4.2 we show that the solutions we obtain are independent of the choice of w, θ_0 and $\eta_{i,0}$, however our existence proof itself depends heavily on exactly how w and θ_0 are chosen, as discussed in section 3.4.

If we make these substitutions, and at the same time apply the Leray projection $\mathbb{P}$ to the v equation, we get the system:

$$\partial_t v + PrAv + B(v+w, v+w) = f(\theta), \tag{3.4a}$$

$$\partial_t \theta + A_1\theta + B_1(v+w, \theta+\theta_0) = g(\eta_1, \ldots, \eta_N, \theta), \tag{3.4b}$$

$$\partial_t \eta_i + \frac{1}{Le}A_2\eta_i + B_1(v+w, \eta_i+\eta_{i,0}) = \omega_i(\eta_1, \ldots, \eta_N, \theta) \tag{3.4c}$$

where

$$f(\theta) = \mathbb{P}f_0(\theta+\theta_0) + Pr\mathbb{P}\Delta w, \tag{3.5a}$$

$$g(\eta_1, \ldots, \eta_N, \theta) = -\sum_{i=1}^{N} h_i W_i(\eta_1+\eta_{1,0}, \ldots, \eta_N+\eta_{N,0}, \theta+\theta_0) + \Delta\theta_0, \tag{3.5b}$$

$$\omega_i(\eta_1, \ldots, \eta_N, \theta) = W_i(\eta_1+\eta_{1,0}, \ldots, \eta_N+\eta_{N,0}, \theta+\theta_0) + \frac{1}{Le}\Delta\eta_{i,0}. \tag{3.5c}$$

Note that because of the smoothness of the boundary conditions, we can assume that w, θ_0 and $\eta_{i,0}$ are all in $H^2(\Omega)$, so that their Laplacians are in L^2.

The theorem requires the following condition on ϕ, the implications of which are discussed in section 3.4.

Assumption 1. *Let Γ'_i, $i = 1, \ldots, N_c$ be the different components of $\partial\Omega$. Then assume that the boundary condition ϕ for the velocity u satisfies*

$$\int_{\Gamma'_i} \phi \cdot n \, dS = 0$$

for each $i = 1, \ldots, N_c$.

We note that if Ω is simply connected then $\partial\Omega$ has only one component, so that (2.13) implies that assumption 1 is satisfied.

For conciseness and consistency with the general attractor theorem given below, we introduce the following notation. Let

$$F_v(v,\theta,\eta_i) = -PrAv - B(v+w,v+w) + f(\theta),$$
$$F_\theta(v,\theta,\eta_i) = -A_1\theta - B_1(v+w,\theta+\theta_0) + g(\eta_1,\dots,\eta_N,\theta)$$
$$F_{\eta_i}(v,\theta,\eta_i) = -\frac{1}{Le}A_2\eta_i - B_1(v+w,\eta_i+\eta_{i,0}) + \omega_i(\eta_1,\dots,\eta_N,\theta),$$

where f, g and ω_i are given by (3.5). Then the reacting flow system (3.4) becomes simply

$$\partial_t v = F_v(v,\theta,\eta_i),$$
$$\partial_t \theta = F_\theta(v,\theta,\eta_i),$$
$$\partial_t \eta_i = F_{\eta_i}(v,\theta,\eta_i).$$

In the following theorem the constants C_i depend only on the data for the problem, and in particular are independent of the initial conditions of the various solutions.

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded C^2 domain. Assume further that W_i, h_i, ϕ and $Y_{i,f}$ satisfy the conditions discussed in section 2 and also that ϕ satisfies assumption 1. For every initial condition $(v(0),\theta(0),\eta_i(0)) \in \overline{H}$ that is physically reasonable in the sense of equations (2.6), the reacting flow system has a weak solution satisfying the following properties.*

1. *The solutions themselves satisfy*

$$v \in L^\infty(0,\infty;H) \cap L^2_{loc}[0,\infty;V), \tag{3.6a}$$
$$\theta \in L^\infty(0,\infty;H_1) \cap L^2_{loc}[0,\infty;V_1), \tag{3.6b}$$
$$\eta_i \in L^\infty(0,\infty;H_2) \cap L^2_{loc}[0,\infty;V_2). \tag{3.6c}$$

2. *The derivatives of the solutions satisfy*

$$\partial_t v \in L^p_{loc}[0,\infty;V^{-1}), \tag{3.7a}$$
$$\partial_t \theta \in L^p_{loc}[0,\infty;V_1^{-1}), \tag{3.7b}$$
$$\partial_t \eta_i \in L^p_{loc}[0,\infty;V_2^{-1}) \tag{3.7c}$$

   with $p = 4/3$.

3. *Let σ, M_1, M_2 and C_1 be positive constants, depending only on the parameters of the problem and not the initial conditions, given by (4.15), (4.13a), (4.13b) and (4.17), respectively. Then for almost all t and t_0 with $t > t_0$ we have*

$$\|v(t)\|^2 + M_1\|\theta(t)\|^2 + M_2\sum_{i=1}^N \|\eta_i(t)\|^2 \le C_1 + e^{-\sigma(t-t_0)}\left(\|v(t_0)\|^2 + M_1\|\theta(t_0)\|^2 + M_2\sum_{i=1}^N\|\eta_i(t_0)\|^2\right). \tag{3.8}$$

   and there exist functions $\overline{L}'_v, \overline{L}'_\theta$ and $\overline{L}'_\eta$ given by (4.20), (4.22) and (4.25) which are continuous increasing functions of each argument such that for almost all t_0 and all

$t > t_0$ one has

$$Pr \int_{t_0}^{t} \|A^{1/2}v\|^2 \, ds \le \overline{L}'_v(\|v(t_0)\|, \|\theta(t_0)\|, \|\eta_i(t_0)\|, t - t_0) \tag{3.9a}$$

$$C_e^1 \int_{t_0}^{t} \|A_1^{1/2}\theta\|^2 \, ds \le \overline{L}'_\theta(\|v(t_0)\|, \|\theta(t_0)\|, \|\eta_i(t_0)\|, t - t_0) \tag{3.9b}$$

$$\frac{C_e^1}{Le} \int_{t_0}^{t} \|A_2^{1/2}\eta_i\|^2 \, ds \le \overline{L}'_\eta(\|v(t_0)\|, \|\theta(t_0)\|, \|\eta_i(t_0)\|, t - t_0) \tag{3.9c}$$

4. *For all $0 \le t_0 \le t$, $\overline{v} \in V$ and $\psi_i \in V_i, i = 1, 2$, we have*

$$\langle v(t) - v(t_0), \overline{v} \rangle = \int_{t_0}^{t} \langle F_v(v, \theta, \eta_i), \overline{v} \rangle \, ds, \tag{3.10a}$$

$$\langle \theta(t) - \theta(t_0), \psi_1 \rangle = \int_{t_0}^{t} \langle F_\theta(v, \theta, \eta_i), \psi_1 \rangle \, ds, \tag{3.10b}$$

$$\langle \eta_i(t) - \eta_i(t_0), \psi_2 \rangle = \int_{t_0}^{t} \langle F_{\eta_i}(v, \theta, \eta_i), \psi_2 \rangle \, ds. \tag{3.10c}$$

5. *For almost every x and t we have that the solutions are physically reasonable, i.e.,*

$$\sum_{i=1}^{N} Y_i(x, t) = 1, \tag{3.11a}$$

$$0 \le Y_i(x, t) \le 1, \tag{3.11b}$$

$$T_l \le T(x, t) \tag{3.11c}$$

Theorem 2. *Assuming the same hypotheses as in the existence theorem, theorem 1, then there exists a phase space W containing the solutions described in the existence theorem such that the reacting flow system has a global attractor.*

We will give a proof of this theorem and a description of the system's phase space in section 5.

3.3. **Preliminary Lemmas.** In this section we want to give some general lemmas which will be used throughout the rest of the paper.

One of the very important properties of the trilinear forms b and b_1 in the homogeneous boundary case is given in the following lemma.

Lemma 3. *Let $\nabla \cdot u = 0$ in Ω. Suppose that at each point in $\partial\Omega$ one of the following three conditions holds:*

- $u \cdot n = 0$, *or*
- $v = 0$, *or*
- $w = 0$,

where n is the unit outward normal to $\partial\Omega$. Then

$$b(u, v, w) = -b(u, w, v). \tag{3.12}$$

Similarly, if at each point in $\partial\Omega$ one of the following

- $u \cdot n = 0$, *or*
- $\psi_1 = 0$, *or*

- $\psi_2 = 0$,

holds on $\partial\Omega$, *then*

$$b_1(u, \psi_1, \psi_2) = -b_1(u, \psi_2, \psi_1). \tag{3.13}$$

This lemma is shown using integration by parts then noting that the condition on the boundary is sufficient for the boundary integral to be zero. For instance, in the second case we have

$$\begin{aligned} b_1(u, \psi_1, \psi_2) + b_1(u, \psi_2, \psi_1) &= \int_\Omega u \cdot \nabla(\psi_1 \psi_2)\, dx \\ &= \int_\Omega (\nabla \cdot u)\psi_1\psi_2\, dx + \int_{\partial\Omega} \psi_1\psi_2(u \cdot n)\, dS \end{aligned}$$

As a corollary we have under the above conditions that

$$b(u, v, v) = 0 \qquad \text{for all } u \in H^1(\Omega) \text{ and } v \in V \tag{3.14}$$

and

$$b_1(v, \psi, \psi) = 0 \qquad \text{for all } v \in V \text{ and } \psi \in H^1(\Omega). \tag{3.15}$$

For the lemmas below we will need the following theorem of Deny and Lions [5].

Theorem 4. *If* Ω *is a bounded, Lipschitz domain, then there exists a constant* $C_2 = C_2(\Omega)$ *such that for each* $\psi_0 \in H^1(\Omega)$ *satisfying*

$$\int_\Omega \psi_0\, dx = 0$$

we have

$$\|\psi_0\|_{L^2(\Omega)} \leq C_2(\Omega)\|\nabla\psi_0\|_{L^2(\Omega)^n}.$$

As a corollary we have

Corollary 5. *Suppose* Ω *is bounded and Lipschitz, and let* $\xi \in L^\infty(\partial\Omega)$ *be a nonnegative function which is not identically zero. Then there exists a constant* C_3 *depending on* ξ *and* Ω *such that for each* $\psi \in H^1(\Omega)$ *we have*

$$\|\psi\|^2_{L^2(\Omega)} \leq C_3 \left(\|\nabla\psi\|^2_{L^2(\Omega)^n} + \int_{\partial\Omega} \xi\psi^2\, dS \right).$$

One proves the corollary first by decomposing $\psi = \psi_0 + \int_\Omega \psi\, dx$ where $\int_\Omega \psi\, ds = 0$. Then one notes that constant functions are orthogonal to ψ_0.

Using this corollary we can prove the following generalized Poincaré inequality for the chemistry and temperature equations.

Lemma 6. *Fix* κ *a positive real number and assume that* u *is a divergence free vector field with boundary values* ϕ *and let* $\Gamma_I \subset \partial\Omega$ *be that portion of the boundary where* $\phi \cdot n < 0$ *and* Γ_O *be that portion where* $\phi \cdot n > 0$. *Suppose further that* ψ *is a scalar field satisfying the boundary conditions* (2.11). *Then, if* $u \in H^1(\Omega)$ *and* $\psi \in H^2(\Omega)$, *one has*

$$-\kappa\langle\Delta\psi, \psi\rangle + b_1(u, \psi, \psi) = \kappa\|\nabla\psi\|^2 + \frac{1}{2}\int_{\partial\Omega} \psi^2\, |\phi \cdot n|\, dS \tag{3.16a}$$

$$\geq \mu(\kappa)\|\psi\|^2. \tag{3.16b}$$

where $\mu > 0$ *is a constant which depends on* Ω, κ *and* ϕ, *but not on* u.

When we apply this lemma we will use it in the cases where $\kappa = 1$ and $\kappa = \frac{1}{Le}$, hence we let $\mu_1 = \min\{\mu(1), \mu(\kappa)\}$.

Proof. The equality (3.16a) follows directly from integration by parts. In particular, by the divergence theorem and the boundary condition (2.11) we have

$$\begin{aligned} -\kappa \int_\Omega \psi \Delta \psi \, dx &= \kappa \int_\Omega |\nabla \psi|^2 \, dx - \kappa \int_{\partial\Omega} \psi \frac{\partial \psi}{\partial n} \, dS \\ &= \kappa \int_{\Gamma_I} |\nabla \psi|^2 \, dx - \int_{\partial\Omega} \psi^2 (\phi \cdot n) \, dS \end{aligned}$$

Further, since $\psi(u \cdot \nabla)\psi = \frac{1}{2} u \cdot \nabla \psi^2$ we have

$$\begin{aligned} b_1(u, \psi, \psi) &= \frac{1}{2} \int_\Omega u \cdot \nabla \psi^2 \, dx \\ &= \frac{1}{2} \int_\Omega (\nabla \cdot u) \psi^2 \, dx + \frac{1}{2} \int_{\partial\Omega} \psi^2 (\phi \cdot n) \, dS. \end{aligned} \tag{3.17}$$

Therefore, combining these two equalities gives

$$\begin{aligned} -\kappa \langle \Delta \psi, \psi \rangle + b_1(u, \psi, \psi) =& \kappa \|\nabla \psi\|^2 + \frac{1}{2} \int_{\Gamma_O} \psi^2 (\phi \cdot n) \, dS - \frac{1}{2} \int_{\Gamma_I} \psi^2 (\phi \cdot n) \, dS \\ =& \kappa \|\nabla \psi\|^2 + \frac{1}{2} \int_{\partial\Omega} \psi^2 \, |\phi \cdot n| \, dS \end{aligned}$$

which gives (3.16a). The inequality (3.16b) is a direct consequence of corollary 5. □

We can also use corollary 5 to show (3.3), the equivalent norms for V_i. We prove the norms first by considering a smooth $\psi \in V_i$, then completing in V_i. We begin by showing the first inequality in (3.3). To prove this, we use the self-adjointness of A_i, integration by parts and the boundary condition (2.11) to note

$$\begin{aligned} \|A_i^{1/2} \psi\|^2 &= \langle A_i \psi, \psi \rangle \\ &= -\langle \Delta \psi, \psi \rangle \\ &= \|\nabla \psi\|^2 - \int_{\partial\Omega} \psi \frac{\partial \psi}{\partial n} \, dS \\ &= \|\nabla \psi\|^2 - \frac{1}{\kappa} \int_{\Gamma_I} (\phi \cdot n) \psi^2 \, dS. \end{aligned} \tag{3.18}$$

Now since $\phi \cdot n < 0$ on Γ_I one can apply corollary 5 to get

$$\|\psi\|_{L^2}^2 \leq C_4 \|A_i^{1/2} \psi\|^2.$$

Also, in (3.18) the boundary integral is positive, so we have

$$\|\nabla \psi\|^2 \leq \|A_i^{1/2} \psi\|^2.$$

Adding these two equations gives the first inequality in (3.3), i.e.,

$$\|\psi\|_{H^1}^2 \leq (C_e^1)^{-1} \|A_i^{1/2} \psi\|^2.$$

One obtains the other inequality in (3.3) by observing that

$$\|\nabla\psi\|^2 \le \|\psi\|_{H^1}^2$$

by the definition of the Sobolev norm, and also that

$$0 \le -\frac{1}{\kappa}\int_{\Gamma_I}(\phi\cdot n)\psi^2\,dS \le C_5\int_{\Gamma_I}\psi^2\,dS \le C_6\|\psi\|_{H^1}^2,$$

since ϕ is bounded and the trace operator is bounded from $H^1(\Omega)$ into $L^2(\partial\Omega)$. Substituting these two inequalities into (3.18) gives the other inequality in (3.3).

We will also use the usual Sobolev estimates for the bilinear and trilinear forms, see Sell and You [25] or Constantin and Foias [4], namely there is a constant C_s depending only on Ω such that

(3.19a) $$\|B(u,v)\|_{H^{-1}} \le C_s\|u\|^{1/4}\|u\|_{H^1}^{3/4}\|v\|^{1/4}\|v\|_{H^1}^{3/4}$$

(3.19b) $$\|B_1(u,\psi)\|_{H^{-1}} \le C_s\|u\|^{1/4}\|u\|_{H^1}^{3/4}\|\psi\|^{1/4}\|\psi\|_{H^1}^{3/4}$$

(3.19c) $$|b(u,v,w)| \le C_s\|u\|^{1/4}\|u\|_{H^1}^{3/4}\|v\|_{H^1}\|w\|^{1/4}\|w\|_{H^1}^{3/4},$$

(3.19d) $$|b_1(u,\phi,\psi)| \le C_s\|u\|^{1/4}\|u\|_{H^1}^{3/4}\|\phi\|_{H^1}\|\psi\|^{1/4}\|\psi\|_{H^1}^{3/4},$$

(3.19e) $$|b_1(u,\phi,\psi)| \le C_s\|u\|_{H^1}\|\phi\|_{H^1}\|\psi\|^{1/2}\|\psi\|_{H^1}^{1/2},$$

(3.19f) $$|b_1(u,\phi,\psi)| \le C_s\|u\|_{H^2}\|\phi\|_{H^1}\|\psi\|,$$

(3.19g) $$|b_1(u,\phi,\psi)| \le C_s\|u\|\|\phi\|_{H^2}\|\psi\|_{H^1},$$

for $u, v, w \in V$ and $\phi, \psi \in V_i$. We will also use the estimates

(3.20a) $$\|b(u,v,w)\| \le C_s\|u\|_{H^1}\|v\|_{H^1}\|w\|_{H^1},$$

(3.20b) $$\|b_1(u,\psi_1,\psi_2)\| \le C_s\|u\|_{H^1}\|\psi_1\|_{H^1}\|\psi_2\|_{H^1},$$

which follow from (3.19c) and (3.19d). We will also freely use the equivalent norms (3.2) and (3.3) in the above estimates by replacing the constant C_s with a larger constant, if necessary. For example, instead of the estimate (3.20b) we will use the estimate

$$\|b_1(u,\psi_1,\psi_2)\| \le C_s\|u\|_{H^1}\|A_i^{1/2}\psi_1\|\|A_i^{1/2}\psi_2\|.$$

3.4. **The Choice of Homogenizing Functions.** In this section we want to describe how we choose w, θ_0 and $\eta_{i,0}$. At first glance, it would seem simplest to simply pick θ_0 and $\eta_{i,0}$ to be harmonic and w to be the solution to a Stokes problem. However, we do not take this approach because the proper choice of w and θ_0 is critical to obtaining our *a priori* estimates. More precisely, in order to show dissipativity for the v equation we will need a good w and in order to show dissipativity for the coupled v and θ equations, we will need to a good θ_0 as well. The theorems given below will allow us to pick w and θ_0 satisfying

(3.21a) $$|b(v,w,v)| \le \frac{Pr}{8}\|A^{1/2}v\|^2,$$

(3.21b) $$|b_1(v,\theta_0,\theta)| \le \epsilon\|A^{1/2}v\|\|A_1^{1/2}\theta\|,$$

for all $v \in V$ and $\theta \in V_1$ and where $\epsilon > 0$ satisfies (4.6), (4.7) and (4.14), and in particular ϵ depends only on Pr, Le, c_0 and Ω. The only assumptions we will make on $\eta_{i,0}$ is that it satisfies the boundary conditions (2.10) and that it is in $H^2(\Omega)$. In section 4.2 we will show that the dynamics of the equations (2.1) do not change with different choices of w, θ_0 and $\eta_{i,0}$.

The following two lemmas show when we can find appropriate w and θ_0.

Lemma 7. *Suppose assumption 1 is satisfied and let $\phi \in H^{3/2}(\partial\Omega)$. Then for any $\epsilon > 0$ there exists $w(x) \in H^2(\Omega)$ which agrees with ϕ on the boundary and satisfies*

$$|b(v, w, v)| \leq \epsilon \|A^{1/2} v\|^2 \tag{3.22}$$

for all $v \in V$.

Lemma 8. *For any function $\psi_1 \in H^2(\Omega)$ and for any $\epsilon > 0$ there exists a $\delta > 0$ and $\psi_0(x) \in H^2(\Omega)$ which agrees with ψ_1 in a δ neighborhood of the boundary and satisfies*

$$|b_1(v, \psi_0, \psi)| \leq \epsilon \|A^{1/2} v\| \|\psi\|_{H^1} \tag{3.23}$$

for all $v \in V$ and $\psi \in H^1(\Omega)$.

The first proof of lemma 7 was given by Leray in his 1933 thesis, see also Leray [16], Hopf [14], Fujita [9] and Temam [26], in the context of showing the existence of steady state solutions to the inhomogeneous Navier-Stokes equations. Our proof of lemma 8, which is also based on Hopf's ideas, is given below. We then give a short discussion of the added complexities of lemma 7.

Proof of lemma 8. Throughout the proof, C will denote a constant depending only on Ω. Let $\xi : \mathbb{R} \to [0, 1]$ be a smooth function which satisfies

$$\begin{aligned} \xi(x) &= 1 \qquad && \text{for } x < 1/2, \\ \xi(x) &= 0 && \text{for } x > 1, \end{aligned}$$

Let $\rho(x)$ be the distance from x to $\partial\Omega$ and define

$$\psi_\delta(x) = \xi\left(\frac{\rho(x)}{\delta}\right) \psi_1.$$

We note that ψ_δ agrees with ϕ in a neighborhood of the boundary. We want to show that for small δ, we can pick $\psi_0 = \psi_\delta$.

Denote the components of v by (v^1, v^2, v^3). Then by using (3.13) and the Cauchy and Hölder inequalities, we get:

$$\begin{aligned} |b_1(v, \psi_\delta, \psi)| &= |b_1(v, \psi, \psi_\delta)| \\ &\leq \sum_{i=1}^{3} \int_\Omega \left|\frac{v_i}{\rho}\right| |D_i \psi| \, |\rho \psi_\delta| \; dx \\ &\leq \sum_{i=1}^{3} \|\rho \psi_\delta\|_\infty \|\frac{v_i}{\rho}\| \|D_i \psi\| \end{aligned}$$

It is known, see for example Temam [26], that there exists a constant C_7 depending only on Ω such that

$$\|\frac{v_i}{\rho}\| \leq C_7 \|v_i\|_{H_0^1}, \qquad \text{for every } v_i \in H_0^1(\Omega).$$

Also, since the support of ψ_δ is within δ of the boundary one has

$$\|\rho\psi_\delta\|_\infty \le \delta\|\psi_\delta\|_\infty,$$

and by the definition of ψ_δ, with $0 < \delta \ll 1$,

$$\|\rho\psi_\delta\|_\infty \le \delta\|\psi_1\|_\infty \le C\delta\|\psi_1\|_{H^2},$$

since H^2 embeds in L^∞. Therefore,

$$|b_1(v, \psi_\delta, \psi)| \le C\delta\|\psi_0\|_{H^2}\|v_i\|_{H_0^1}\|\psi\|_{H^1}.$$

So, for δ sufficiently small one obtains (3.23) □

We now want to say a few words about the proof of lemma 7. The key problem is in finding a w with support near the boundary. The solvability of the Stokes problem, see Galdi [10] or Temam [26], guarantees the existence of a w' which is divergence free and agrees with ϕ on the boundary. However we cannot just cut the function off by multiplying by a function with support near the boundary, since the resulting function need not be divergence free. Hopf's approach is to find another function ζ such that $w' = \operatorname{curl}\zeta$. Then if ξ can be used as a cutoff function, by setting

$$w = \operatorname{curl}(\xi\zeta),$$

one obtains

$$\nabla\cdot w = \nabla\cdot \operatorname{curl}(\xi\zeta) = 0.$$

However, such a ζ does not exist in general. Foias and Temam [8], see the appendix in Temam [26], show that the assumption 1 is sufficient for the existence of such a ζ.

We should point out that lemma 7 has been used previously to get estimates analogous to ours for the Navier-Stokes equations alone, see Ghidaglia [11] and Lafon [15]. In particular, for the purpose of showing the existence of global attractors, they use lemma 7 to show that the 2D Navier-Stokes equations are dissipative in L^2. It is an open problem, see Heywood [13], to show that the 2D Navier-Stokes equations are dissipative for general inhomogeneous boundary conditions on multiply connected domains.

3.5. **Extension of W_i.** All of the discussion of W_i in section 2 is concerned only with the domain $(T, Y_i) \in [0,1]^N \times [T_l, \infty)$, i.e., with physically reasonable values. However, *a priori* we do not know that our solutions will remain physically reasonable, so we need to extend the W_i to all of $\mathbb{R}^{N+1}$. To do this, we follow Manley, Marion and Temam [20] and let

$$\alpha(s) = \begin{cases} T_l & \text{if } s \le T_l, \\ s & \text{if } s \ge T_l, \end{cases}$$

$$\beta(s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } 0 \le s \le 1, \\ 1 & \text{if } 1 \le s. \end{cases}$$

Then we can then extend W_i to all of $\mathbb{R}^{N+1}$ by setting

$$W_i(Y_1, \ldots, Y_N, T) = W_i(\beta(Y_1), \ldots, \beta(Y_n), \alpha(T)), \quad \text{for } (Y_i, T) \in \mathbb{R}^{N+1} - [0,1]^N \times [T_l, \infty).$$

With this extension we have that on $\mathbb{R}^{N+1}$, W_i is bounded, globally Lipschitz, satisfies the mass conservation property (2.2), and an extended version of (2.3)

$$W_i(Y_1, \dots, Y_N, T) \geq 0 \qquad \text{whenever } Y_i \leq 0. \tag{3.24}$$

In section 4.3 we will show the existence of physically reasonable solutions, so that this extension has no effect on these solutions.

3.6. **Analysis of the Equations.** One of the noteworthy features of our approach is that we use essentially the same two properties of F_v, F_θ and F_{η_i} for both the existence of solutions and the existence of the global attractor. In this section we state the two properties and prove that they are satisfied.

The first property involves a real parameter p. For our reacting flow system, as for the 3D Navier-Stokes, we will require that $p = 4/3$, however the attractor existence theorem only requires $p > 1$.

Property 1 (Boundedness Property). *We say that a function $F(\phi_1, \dots \phi_N)$ satisfies the boundedness property for a given $p \in (1, \infty]$ if it maps sets which are bounded in $L^2(t_0, t; \overline{V}) \cap L^\infty(t_0, t; \overline{H})$ into sets which are uniformly bounded in $L^p(t_0, t; \overline{V}^{-1})$. In particular, we assume that for some function B_F we have for $t > t_0$*

$$\int_{t_0}^{t} \|F(\phi_1(\tau), \dots, \phi_n(\tau))\|_{V^{-1}}^p \, d\tau \leq \\ \leq B_F \left((t - t_0) + \max_i \int_{t_0}^{t} \|\phi_i(\tau)\|_{V_2} \, d\tau + \max_i \operatorname{ess\,sup}_{t_0 \leq \tau \leq t} \|\phi_i(\tau)\| \right), \tag{3.25}$$

for all $(\phi_1, \dots, \phi_N)$.

We note that functions satisfying the boundedness property map sets which are bounded in $L^2_{loc}(0, \infty; \overline{V}) \cap L^\infty_{loc}(0, \infty; \overline{H})$ into sets which are uniformly bounded in $L^p_{loc}(0, \infty; \overline{V}^{-1})$.

The other property is a continuity property, which we formulate by using the conclusions of the following compactness lemma.

Lemma 9 (Compactness Lemma). *Assume that, for $i = 1, \dots, N$ we have two Hilbert spaces V_i and H_i, such that $V_i \hookrightarrow H_i$, where the embedding is dense and compact. Let ϕ^n be a sequence in $L^2_{loc}(0, \infty; V_i)$ such that ϕ^n is bounded in $L^2_{loc}(0, \infty; V_i)$ and the sequence $\partial_t \phi^n$ is bounded in $L^p_{loc}(0, \infty; V_i^{-1})$, where $1 < p \leq \infty$. Then there exists a subsequence, which we also denote by ϕ^n, and a function $\phi \in L^2_{loc}(0, \infty; V_i)$, with $\partial_t \phi \in L^p_{loc}(0, \infty; V_i^{-1})$, such that the following properties hold:*

1. *One has $\phi^n \overset{w}{\to} \phi$ weakly in $L^2_{loc}(0, \infty; V_i)$*
2. *One has $\partial_t \phi^n \overset{w}{\to} \partial_t \phi$ weakly in $L^p_{loc}(0, \infty; V_i^{-1})$*
3. *One has $\phi^n \to \phi$ strongly in $L^2_{loc}(0, \infty; H_i)$*
4. *For each $t \in (0, \infty)$, one has $\phi^n(t) \to \phi(t)$ strongly in V_i^{-1}.*
5. *There is a set E in $(0, \infty)$ having measure zero, such that for $t \in \mathbb{R}^+ - E$, one has $\phi^n(t) \to \phi(t)$ strongly in H_i.*

This lemma has been used many times before, see for example Constantin and Foias [4] and Lions [17]. In particular Sell [24] gives an outline of the proof.

Property 2 (Continuity Property). *A function $F(\phi_1,\ldots,\phi_N)$ is said to satisfy the continuity property if, for each sequence $\phi^j=(\phi_1^j,\ldots,\phi_N^j)$ in $L^2_{loc}(0,\infty;\overline{V})$ and limit function $\phi^0=(\phi_1^0,\ldots,\phi_N^0)$ in $L^2_{loc}(0,\infty;V)$ satisfying the conclusions of the compactness lemma 9 for each i, $1\le i\le N$, we have*

$$\lim_{j\to\infty} F(\phi_i^j)\overset{w}{\to}F(\phi_i^0)\ \text{ weakly in } L^1_{loc}(0,\infty;V^{-1}) \text{ for } 1\le i\le N$$

We will use these two properties in the following way. We begin with a sequence of weak solutions which we know to be bounded in $L^2_{loc}(0,\infty;\overline{V})\cap L^\infty_{loc}(0,\infty;\overline{H})$. The boundedness property then implies that the derivatives of the solutions are bounded in $L^p_{loc}(0,\infty;\overline{V}^{-1})$. This means that we can apply the compactness lemma and extract a convergent subsequence. The continuity property then implies that the limit function will satisfy the equation weakly in $\overline{V}^{-1}$, which is exactly the sense in which our weak solutions will satisfy the equation.

We now want to show that F_v, F_θ and F_{η_i} satisfy both of the properties. We first have a series of lemmas examining each of the various terms in the equations, then we look at their sum.

Lemma 10. *The maps $(u,v)\mapsto B(u,v)$ and $(u,\psi)\mapsto B_1(u,\psi)$ satisfy the continuity property.*

Proof. We first show the property for $B(u,v)$. Pick $u_i\to u_0$ and $v_i\to v_0$ satisfying the conditions in the continuity property. Then

$$B(u_i,v_i)-B(u_0,v_0)=B(u_i-u_0,v_i)-B(u_0,v_0-v_i).$$

Fixing $t>t_0$ and using the Hölder inequality and (3.19a), we get

$$\begin{aligned}\int_{t_0}^t \|A^{-1/2}B(u_i-u_0,v_i)\|\,ds \le & C_8\int_{t_0}^t \|u_i-u_0\|^{1/4}\|A^{1/2}(u_i-u_0)\|^{3/4}\|v_i\|^{1/4}\|A^{1/2}v_i\|^{3/4}\,ds\\ \le & C_9\left(\int_{t_0}^t\|u_i-u_0\|^2\right)^{1/8}\left(\int_{t_0}^t\|A^{1/2}(u_i-u_0)\|^2\right)^{3/8}\times\\ & \times\left(\int_{t_0}^t\|v_i\|^2\right)^{1/8}\left(\int_{t_0}^t\|A^{1/2}v_i\|^2\right)^{3/8}.\end{aligned}$$

The first term goes to zero by property 3 of the compactness lemma, while the other terms are bounded. The analysis of the term $B(u_0,u_0-u_i)$ is identical.

The same proof using (3.19b) instead of (3.19a) works for $B_1(u,\psi)$. □

Lemma 11. *The maps $(u,v)\mapsto B(u,v)$ and $(u,\psi)\mapsto B_1(u,\psi)$ satisfy the boundedness property where $p=4/3$.*

Proof. Fix u, v satisfying the hypotheses of property 1. For all $t > t_0$ we get, using that $p = 4/3$ and the Hölder inequality and (3.19a) we get,

$$\begin{aligned}\int_{t_0}^{t} \|A^{-1/2}B(u,v)\|^p \, ds \leq & C_{10} \int_{t_0}^{t} \|u\|^{p/4}\|A^{1/2}u\|^{3p/4}\|v\|^{p/4}\|A^{1/2}v\|^{3p/4} \, ds \\ \leq & C_{11}\|u\|_{L^\infty(0,\infty;H)}^{1/3}\|v\|_{L^\infty(0,\infty;H)}^{1/3} \int_{t_0}^{t} \|A^{1/2}u\|\|A^{1/2}v\| \, ds \\ \leq & C_{12}\|u\|_{L^\infty(0,\infty;H)}^{1/3}\|v\|_{L^\infty(0,\infty;H)})^{1/3} \\ & \left(\int_{t_0}^{t} \|A^{1/2}u\|^2 \, ds\right)^{1/2} \left(\int_{t_0}^{t} \|A^{1/2}v\|^2 \, ds\right)^{1/2},\end{aligned}$$

which is bounded by the assumptions of the boundedness property.

The identical proof works for $B_1(u, \psi)$.

□

Lemma 12. *The maps $u \mapsto Au$, $\psi \mapsto A_i\psi$, for $i = 1, 2$, satisfy the continuity property.*

Proof. We first consider Au. Let u^n be a sequence satisfying the hypotheses of the continuity property. Fix $t > t_0$. Then

$$\lim_{n\to\infty} \int_{t_0}^{t} \langle Au^n - Au, v\rangle = \lim_{n\to\infty} \int_{t_0}^{t} \langle u^n - u, Av\rangle$$

Which goes to zero, since $Av \in V^{-1}$ and $u^n \stackrel{w}{\to} u$ weakly in $L^2(0, T; V)$.

The same proof works for the $A_i\psi$. □

Lemma 13. *The maps $u \mapsto Au$, $\psi \mapsto A_i\psi, i = 1, 2$ satisfy the boundedness property for $p = 4/3$.*

Proof. First we look at Au. Fix $t > t_0$. Then we need to consider

$$\int_{t_0}^{t} \|A^{-1/2}Au\|^p \, ds = \int_{t_0}^{t} \|A^{1/2}u\|^p \, ds,$$

which is bounded since u is bounded in $L^2_{loc}(0, \infty; V)$ and $p < 2$.

The same proof works for the $A_i\psi$. □

Lemma 14. *The functions $\omega_k(\phi_1, \ldots, \phi_N)$ satisfy the continuity property.*

Proof. Pick $\phi_i^j \to \phi_i$ satisfying the conditions in the continuity property for each i. We will actually show that

$$\omega_k(\phi_1^j, \ldots, \phi_N^j) \to \omega_k(\phi_1, \ldots, \phi_N)$$

strongly in $L^2_{loc}(0, \infty; H)$. We note that since ω_k is Lipschitz in each coordinate, we have

$$\left|\omega_k(\phi_1^j, \ldots, \phi_N^j) - \omega_k(\phi_1, \ldots, \phi_N)\right| \leq C_{13} \sum_{i=1}^{N} \left|\phi_i^j - \phi_i\right|$$

for some constant C_{13}. This, with property 3 of the compactness lemma, gives the result after integrating in time and space. □

Lemma 15. *The functions $\omega_k(\eta_1, \ldots, \eta_N, \theta)$ satisfy the boundedness property for any $p > 1$.*

Proof. This is a direct consequence of the assumption that the ω_k are everywhere bounded. □

Corollary 16. *The maps F_v, F_θ and F_{η_i} all satisfy the boundedness and continuity properties.*

Proof. The statement follows directly from the above lemmas and the fact that the sets of functions satisfying the two properties are closed under scalar multiplication and addition. □

4. Existence and Properties of Weak Solutions

We now want to show the existence of physically reasonable weak solutions to our system. We use the Bubnov-Galerkin method of approximating the equations by projecting them onto finite dimensional subspaces. In order to show that the resulting solutions are physically reasonable we need to project the velocity equation onto an m dimensional subspace and the temperature and chemistry equations onto an n dimensional subspace, then take the limit first in n and then in m. Our motivation for doing this, as well as the proof of physical reasonableness, is found in section 4.3. In this section we will use the continuity and boundedness properties to show the existence of solutions to the equations.

We derive the Bubnov-Galerkin approximate system as follows. Let P_m be the orthogonal projections onto the first m eigenvalues of the Stokes operator A and let Q_n and R_n be the orthogonal projections onto the first n eigenvalues of the Laplacians A_1 and A_2, respectively. Since P_m, Q_n and R_n are spectral projections, they commute with A, A_1 and A_2, respectively. Applying the projections to the system (3.4) gives the equations for the m, n^{th} order Bubnov-Galerkin approximate solution:

$$\partial_t v^{m,n} + PrAv^{m,n} + P_m B(v^{m,n} + w, v^{m,n} + w) = P_m f(\theta^{m,n}), \tag{4.1a}$$

$$\partial_t \theta^{m,n} + A_1\theta^{m,n} + Q_n B_1(v^{m,n} + w, \theta^{m,n} + \theta_0) = Q_n g(\eta_1^{m,n}, \dots, \eta_N^{m,n}, \theta^{m,n}) \tag{4.1b}$$

$$\partial_t \eta_i^{m,n} + \frac{1}{Le} A_2 \eta_i^{m,n} + R_n B_1(v^{m,n} + w, \eta_i^{m,n} + \eta_{i,0}) = R_n \omega_i(\eta_1^{m,n}, \dots, \eta_N^{m,n}, \theta^{m,n}) \tag{4.1c}$$

$$v^{m,n}(0) = P_m v(0), \tag{4.1d}$$

$$\theta^{m,n}(0) = Q_n \theta(0), \tag{4.1e}$$

$$\eta_i^{m,n}(0) = R_n \eta_i(0), \tag{4.1f}$$

where $v(0)$, $\theta(0)$ and $\eta_i(0)$ are the initial conditions for the solution we are approximating.

4.1. Approximate Solution Estimates.

In this section we want to prove certain estimates of the Bubnov-Galerkin approximate solutions. It is here that we will use the properties of the homogenizing terms w and θ_0 given in (3.21). We note that in this section all constants C_i are independent of the orders of the approximation n, m and the initial conditions.

We begin by deriving energy estimates for the $v^{m,n}$ equation. The methods we use are basically the same as for the Navier-Stokes equations alone. We first take the inner product of the $v^{m,n}$ equation (4.1a) with $v^{m,n}$ to get

$$\frac{1}{2}\partial_t \|v^{m,n}\|^2 + Pr\|A^{1/2}v^{m,n}\|^2 + b(v^{m,n} + w, v^{m,n} + w, v^{m,n}) = \langle f(\theta^{m,n}), v^{m,n}\rangle$$

We can estimate the left hand side using the definitions of f and f_0 given by (3.5a) and (2.7), and the Hölder and Young inequalities, and the Poincaré inequality (3.1) to get

$$\begin{aligned}\langle f(\theta^{m,n}), v^{m,n}\rangle &= \langle -c_0\vec{g}(\theta+\theta_0-T_0)+\Delta w, v^{m,n}\rangle \\ &\leq c_0\langle \vec{g}\theta, v^{m,n}\rangle + C_{14}\|v^{m,n}\| \\ &\leq \frac{Pr}{4}\|A^{1/2}v^{m,n}\|^2 + \frac{2c_0^2\|\theta\|^2}{\lambda_1 Pr} + \frac{2C_{14}^2}{\lambda_1 Pr}.\end{aligned}$$

where C_{14} depends on Ω, T_0, Pr, θ_0 and Δw. Also from the Sobolev estimate (3.20a) and the Young inequality, we get

$$\begin{aligned}|b(w,w,v^{m,n})| &\leq C_s\|w\|_{H^1}^2\|A^{1/2}v^{m,n}\| \\ &\leq \frac{Pr}{8}\|A^{1/2}v^{m,n}\|^2 + \frac{2C_s^2}{Pr}\|w\|_{H^1}^4.\end{aligned}$$

Using these estimates with the condition (3.21a) on w and that $b(v+w,v,v)=0$ from lemma 3 we get:

$$\partial_t\|v^{m,n}\|^2 + Pr\|A^{1/2}v^{m,n}\|^2 - \frac{4c_0^2}{\lambda_1 Pr}\|\theta^{m,n}\|^2 \leq C_{15}, \tag{4.2}$$

where

$$C_{15} = \frac{4C_{14}^2}{\lambda_1 Pr} + \frac{4C_s^2\|w\|_{H^1}^4}{Pr}.$$

If we now multiply the $\theta^{m,n}$ equation (4.1b) by $\theta^{m,n}$ and use the integration by parts formula (3.16a) we get

$$\frac{1}{2}\partial_t\|\theta^{m,n}\|^2 - \langle \Delta\theta^{m,n}, \theta^{m,n}\rangle + b_1(v^{m,n}+w, \theta^{m,n}+\theta_0, \theta^{m,n}) = \langle g, \theta^{m,n}\rangle,$$

$$\frac{1}{2}\partial_t\|\theta^{m,n}\|^2 + \|\nabla\theta^{m,n}\|^2 + \frac{1}{2}\int_{\partial\Omega}(\theta^{m,n})^2\,|\phi\cdot n|\;dS + b_1(v^{m,n}+w,\theta_0,\theta^{m,n}) = \langle g, \theta^{m.n}\rangle, \tag{4.3}$$

Using the condition on θ_0 (3.21b), the Sobolev estimate (3.19f) and the Cauchy and Young inequalities we get

$$\begin{aligned}|b_1(w,\theta_0,\theta^{m,n})| &\leq C_s\|\theta^{m,n}\|\|\theta_0\|_{H^1}\|w\|_{H^2} \\ &\leq \frac{\mu_1}{6}\|\theta^{m,n}\|^2 + \frac{3C_s^2}{2\mu_1}\|\theta_0\|_{H^1}^2\|w\|_{H^2}^2 \\ |b_1(v^{m,n},\theta_0,\theta^{m,n})| &\leq 2\epsilon\|A^{1/2}v^{m,n}\|\|A_1^{1/2}\theta^{m,n}\| \\ &\leq \epsilon\|A_1^{1/2}\theta^{m,n}\|^2 + \epsilon\|A^{1/2}v^{m,n}\|^2 \\ |\langle g,\theta^{m,n}\rangle| &\leq \|g\|\|\theta^{m,n}\| \leq \frac{\mu_1}{6}\|\theta^{m,n}\|^2 + \frac{3C_{16}}{2\mu_1}\end{aligned}$$

where ϵ satisfies (4.6), (4.7) and (4.14) and where C_{16} is a constant depending on the bound g, which in turn depends on ω_i, θ_0 and Ω. These estimates give

$$\frac{1}{2}\partial_t\|\theta^{m,n}\| + \|\nabla\theta^{m,n}\|^2 + \frac{1}{2}\int_{\partial\Omega}(\theta^{m,n})^2\,|\phi\cdot n|\ dS - \\ - \frac{\mu_1}{3}\|\theta^{m,n}\|^2 - \epsilon\|A_1^{1/2}\theta^{m,n}\|^2 - \epsilon\|A^{1/2}v^{m,n}\|^2 \le \frac{C_{17}}{2},$$

where

$$C_{17} = \frac{3}{\mu_1}(C_{16}^2 + C_s^2\|\theta_0\|_{H^1}^2\|w\|_{H^2}^2)$$

The difficulty in this equation is how to absorb the negative $-\epsilon\|A_1^{1/2}\theta^{m,n}\|^2$ term into the positive $\|\nabla\theta^{m,n}\|^2$ term. The key property that allows our argument to work is that we can pick ϵ small, which we can do because of our freedom to choose θ_0. More precisely, we begin by using the equivalent norms (3.3) to get

$$\|A_1^{1/2}\theta^{m,n}\|^2 \le C_e^2\|\theta^{m,n}\|_{H^1}^2 = C_e^2(\|\nabla\theta^{m,n}\|^2 + \|\theta^{m,n}\|^2), \tag{4.5}$$

which gives

$$\frac{1}{2}\partial_t\|\theta^{m,n}\| + (1-\epsilon C_e^2)\|\nabla\theta^{m,n}\|^2 + \frac{1}{2}\int_{\partial\Omega}(\theta^{m,n})^2\,|\phi\cdot n|\ dS - \\ - (\frac{\mu_1}{3} + \epsilon C_e^2)\|\theta^{m,n}\|^2 - \epsilon\|A^{1/2}v^{m,n}\|^2 \le \frac{C_{17}}{2}.$$

Now if we assume that

$$\epsilon C_e^2 < 1 \tag{4.6}$$

we can apply the generalized Poincaré inequality in lemma 6 and use the positivity of the boundary integral to get

$$\frac{1}{2}\partial_t\|\theta^{m,n}\| + (\frac{2\mu_1}{3} - \epsilon C_e^2 - \epsilon\mu_1 C_e^2)\|\theta^{m,n}\|^2 - \epsilon\|A^{1/2}v^{m,n}\|^2 \le \frac{C_{17}}{2}.$$

Now if we assume that

$$\epsilon \le \frac{\mu_1}{6C_e^2(1+\mu_1)} \tag{4.7}$$

we get

$$\partial_t\|\theta^{m,n}\| + \mu_1\|\theta^{m,n}\|^2 - 2\epsilon\|A^{1/2}v^{m,n}\|^2 \le C_{17}. \tag{4.8}$$

The analysis of the $\eta^{m,n}$ equation is similar to the analysis of the $\theta^{m,n}$ equation. Again we take the inner product of the equation for $\eta_i^{m,n}$ with $\eta_i^{m,n}$ and apply the integration by parts formula (3.16a) to get

$$\frac{1}{2}\partial_t\|\eta_i^{m,n}\|^2 - \frac{1}{Le}\langle\Delta\eta_i^{m,n}, \eta_i^{m,n}\rangle + b_1(v^{m,n}+w, \eta_i + \eta_{i,0}, \eta_i^{m,n}) = \langle\omega_i, \eta_i^{m,n}\rangle,$$

$$\frac{1}{2}\partial_t\|\eta_i^{m,n}\|^2 + \frac{1}{Le}\|\nabla\eta_i^{m,n}\|^2 + \frac{1}{2}\int_{\partial\Omega}(\eta_i^{m,n})^2\,|\phi\cdot n|\ dS + b_1(v^{m,n}+w, \eta_{i,0}, \eta_i^{m,n}) = \langle\omega_i, \eta_i^{m,n}\rangle,$$

Let $\delta > 0$ be a constant satisfying (4.10), (4.11) and (4.23), which in particular only depends on Le, Pr and Ω. Then using the Sobolev estimates (3.19f) and (3.19g) and the Hölder and Young inequalities we get

$$\begin{aligned} |b_1(w, \eta_{i,0}, \eta_i^{m,n})| &\leq \frac{\mu_1}{6}\|\eta_i^{m,n}\|^2 + \frac{3C_s^2}{2\mu_1}\|\eta_{i,0}\|_{H^1}^2\|w\|_{H^2}^2, \\ |b_1(v, \eta_{i,0}, \eta_i^{m,n})| &\leq \delta\|A_2^{1/2}\eta_i^{m,n}\|^2 + \frac{C_s^2}{4\delta}\|\eta_{i,0}\|_{H^2}^2\|v^{m,n}\|^2, \\ |\langle\omega_i, \eta_i^{m,n}\rangle| &\leq \frac{\mu_1}{6}\|\eta_i^{m,n}\|^2 + C_{18}, \end{aligned}$$

where C_{18} is a constant depending on the bound for ω_i, Ω, μ_1, C_s, Le and $\Delta\eta_{i,0}$. Using these estimates and $b(v, \eta_i^{m,n}, \eta_i^{m,n}) = 0$ gives

$$\begin{aligned} \frac{1}{2}\partial_t\|\eta_i^{m,n}\|^2 + \frac{1}{Le}\|\nabla\eta_i^{m,n}\|^2 + \frac{1}{2}\int_{\partial\Omega}(\eta_i^{m,n})^2\,|\phi\cdot n|\;dS - \\ - \delta\|A_2^{1/2}\eta_i^{m,n}\|^2 - \frac{\mu_1}{3}\|\eta_i^{m,n}\|^2 - \frac{C_s^2}{4\delta}\|v\|^2\|\eta_{i,0}\|_{H^2}^2 \leq \frac{C_{19}}{2} \end{aligned} \tag{4.9}$$

where

$$C_{19} = 2C_{18} + \frac{3C_s^2}{\mu_1}\|\eta_{i,0}\|_{H^1}^2\|w\|_{H^2}^2.$$

We now proceed, as in the analysis for the θ estimate, to absorb the term $-\delta\|A_2^{1/2}\eta_i^{m,n}\|^2$ into the $\|\nabla\eta_i^{m,n}\|^2$ term. In this case, however, we use the fact that we can pick δ small, which came from an application of the Young inequality. The equivalent norms (3.3) gives the estimate

$$\|A_1^{1/2}\eta_i^{m,n}\|^2 \leq C_e^2\|\eta_i^{m,n}\|_{H^1}^2 = C_e^2(\|\nabla\eta_i^{m,n}\|^2 + \|\eta_i^{m,n}\|^2).$$

As a result we get

$$\begin{aligned} \frac{1}{2}\partial_t\|\eta_i^{m,n}\|^2 + (\frac{1}{Le} - \delta C_e^2)\|\nabla\eta_i^{m,n}\|^2 + \frac{1}{2}\int_{\partial\Omega}(\eta_i^{m,n})^2\,|\phi\cdot n|\;dS - \\ - \left(\frac{\mu_1}{3} + \delta C_e^2\right)\|\eta_i^{m,n}\|^2 - \frac{C_s^2}{4\delta}\|v\|^2\|\eta_{i,0}\|_{H^2}^2 \leq \frac{C_{19}}{2}. \end{aligned}$$

If we assume that

$$\delta C_e^2 Le < 1/2 \tag{4.10}$$

we can apply the generalized Poincaré inequality in lemma 6 and use the positivity of the boundary integral to get

$$\frac{1}{2}\partial_t\|\eta_i^{m,n}\|^2 + \left(\frac{2\mu_1}{3} - \delta C_e^2\mu_1 Le - C_e^2\delta\right)\|\eta_i^{m,n}\|^2 - \frac{C_s^2}{4\delta}\|v\|^2\|\eta_{i,0}\|_{H^2}^2 \leq \frac{C_{19}}{2}.$$

Now if we assume that

$$\delta \leq \frac{5\mu_1}{12C_e^2 Le(1+\mu_1)} \tag{4.11}$$

and multiply by 2 we get

$$\partial_t \|\eta_i^{m,n}\|^2 + \frac{\mu_1}{2}\|\eta_i^{m,n}\|^2 - \frac{C_s^2}{2\delta}\|v\|^2\|\eta_{i,0}\|_{H^2}^2 \le C_{19}. \tag{4.12}$$

Now fix

$$M_1 = \frac{8c_0^2}{\mu_1\lambda_1 Pr}, \tag{4.13a}$$

$$M_2 = \frac{\delta\lambda_1 Pr}{2C_s^2\|\eta_{i,0}\|_{H^2}^2}. \tag{4.13b}$$

If we add (4.2) to M_1 times (4.8) and M_2 times (4.12) and sum over i we get

$$\begin{aligned}\partial_t(\|v^{m,n}\|^2 + M_1\|\theta^{m,n}\|^2 + M_2\sum_{i=1}^N \|\eta_i^{m,n}\|^2) +& \\ (Pr - 2\epsilon M_1)\,\|v^{m,n}\| - \frac{C_s^2 M_2}{2\delta}\|\eta_i^0\|_{H^2}^2\|A^{1/2}v^{m,n}\|^2 +& \\ + \left(\mu_1 M_1 - \frac{4c_0^2}{\lambda_1 Pr}\right)\|\theta^{m,n}\|^2 + \frac{\mu_1 M_2}{2}\sum_{i=1}^N \|\eta_i^{m,n}\|^2& \\ \le C_{15} + M_1 C_{17} + M_2 N C_{19}.&\end{aligned}$$

Now if we pick

$$\epsilon < Pr/4M_1 \tag{4.14}$$

and use the definitions of M_1 and M_2 and the Poincaré inequality for v we get

$$\begin{aligned}\partial_t(\|v^{m,n}\|^2 + M_1\|\theta^{m,n}\|^2 + M_2\sum_{i=1}^N \|\eta_i^{m,n}\|^2) +& \\ + \frac{\lambda_1 Pr}{2}\|v^{m,n}\| + \frac{\mu_1 M_1}{2}\|\theta^{m,n}\|^2 + \frac{\mu_1 M_2}{2}\sum_{i=1}^N \|\eta_i^{m,n}\|^2& \\ \le C_{15} + M_1 C_{17} + M_2 N C_{19}.&\end{aligned}$$

If we let

$$\sigma = \min\{\lambda_1 Pr/2, \mu_1/2\} \tag{4.15}$$

then we have

$$\begin{aligned}\partial_t(\|v^{m,n}\|^2 + M_1\|\theta^{m,n}\|^2 + M_2\sum_{i=1}^N \|\eta_i^{m,n}\|^2) +& \\ + \sigma\left(\|v^{m,n}\| + M_1\|\theta^{m,n}\|^2 + M_2\sum_{i=1}^N \|\eta_i^{m,n}\|^2\right) \le& \\ \le C_{15} + M_1 C_{17} + M_2 N C_{19}.&\end{aligned}$$

The uniform Gronwall inequality then gives for any $t > t_0$

$$\|v^{m,n}(t)\|^2 + M_1\|\theta^{m,n}(t)\|^2 + M_2\sum_{i=1}^{N}\|\eta_i^{m,n}(t)\|^2 \leq C_1 + e^{-\sigma(t-t_0)}\left(\|v^{m,n}(t_0)\|^2 + M_1\|\theta^{m,n}(t_0)\|^2 + M_2\sum_{i=1}^{N}\|\eta_i^{m,n}(t_0)\|^2\right) \tag{4.16}$$

where

$$C_1 = \frac{C_{15} + C_{17}M_1 + C_{19}NM_2}{\sigma}, \tag{4.17}$$

or

$$\|v^{m,n}(t)\|^2 + M_1\|\theta^{m,n}(t)\|^2 + M_2\sum_{i=1}^{N}\|\eta_i^{m,n}(t)\|^2 \leq C_1 + \|v^{m,n}(t_0)\|^2 + M_1\|\theta^{m,n}(t_0)\|^2 + M_2\sum_{i=1}^{N}\|\eta_i^{m,n}(t_0)\|^2. \tag{4.18}$$

We should point out here the role that our choice of w and θ_0 play in the above analysis. For example, in the v equation we have the term which dissipates energy $Pr\|A^{1/2}v\|^2$ and also other terms from the nonlinearity which compete with the dissipation term. The choices of w and θ_0 allow us to make the coefficients of the competing terms small enough that they can be absorbed into the dissipation term. If we had not chosen w and θ_0 as carefully, then these other terms would dominate for small Pr. This would have led to estimates similar to the above, but with one crucial difference: σ would be negative, implying that the energy in the system could become unbounded in time. Although such estimates are much weaker, they would be enough to show the existence of solutions, but not enough to show the existence of global attractors.

We now want to get estimates for $v^{m,n}$, $\theta^{m,n}$ and $\eta_i^{m,n}$ in $L^2(t_0, t; V)$, $L^2(t_0, t; V_1)$ and $L^2(t_0, t; V_2)$, respectively. To get the estimate for $v^{m,n}$ we integrate (4.2) from t_0 to t, use (4.18) to eliminate $\|\theta^{m,n}\|$ to get

$$Pr\int_{t_0}^{t}\|A^{1/2}v^{m,n}(s)\|\,ds + \|v^{m,n}(t)\|^2 \leq \overline{L}'_v(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t-t_0), \tag{4.19}$$

where

$$\overline{L}'_v(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t-t_0) = \|v^{m,n}(t_0)\|^2 + C_{19}(t-t_0) + {} + \frac{4C_{14}(t-t_0)}{\lambda_1 Pr}\left(\frac{1}{M_1}\|v^{m,n}(t_0)\|^2 + \|\theta^{m,n}(t_0)\|^2 + \frac{1}{M_1}\sum_{i=1}^{N}\|\eta_i^{m,n}(t_0)\|^2\right) \tag{4.20}$$

and

$$C_{19} = C_{15} + \frac{4c_0^2}{\lambda_1\sigma Pr}\left(C_{17} + \frac{C_{15}}{M_1} + \frac{M_2C_{19}}{M_1}\right)$$

To get an estimate for $\theta^{m,n}$, we begin with (4.3) and use the positivity of the boundary integral, the estimates (4.4) and the estimate

$$|b_1(v^{m,n}, \theta_0, \theta^{m,n})| \leq C_e^1 \|A_1^{1/2}\theta^{m,n}\|^2 + \frac{1}{2C_e^1}\|v^{m,n}\|^2 \|\theta_0\|_{H^2}^2$$

we get

$$\frac{1}{2}\partial_t\|\theta^{m,n}\|^2 + \|\nabla\theta^{m,n}\|^2 - \frac{2\mu_1}{3}\|\theta^{m,n}\|^2 - C_e^1\|A_1^{1/2}\theta^{m,n}\|^2 \leq C_{17} + \frac{1}{2C_e^1}\|v^{m,n}\|^2\|\theta_0\|_{H^2}^2.$$

Now the equivalent norms (3.3) gives the estimate

$$C_e^1\|A_1^{1/2}\theta^{m,n}\| - \|\theta^{m,n}\|^2 \leq \|\nabla\theta^{m,n}\|^2$$

which we can use to get

$$\frac{1}{2}\partial_t\|\theta^{m,n}\|^2 + C_e^1\|A_1^{1/2}\theta^{m,n}\|^2 \leq C_{17} + \left(\frac{2\mu_1}{3} + 2\right)\|\theta^{m,n}\|^2 + \frac{1}{2C_e^1}\|v^{m,n}\|^2\|\theta_0\|_{H^2}^2.$$

Now if we use (4.18) to eliminate the norms of $v^{m,n}$ and $\theta^{m,n}$ from the right hand sides and integrate from t_0 to t we get

(4.21)

$$C_e^1\int_{t_0}^t \|A_1^{1/2}\theta^{m,n}(s)\|^2\, ds + \|\theta^{m,n}(t)\|^2 \leq \overline{L}'_\theta(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t-t_0),$$

where

$$\begin{aligned}(4.22)\quad \overline{L}'_\theta(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t-t_0) &= C_{20}(t-t_0) + \\ &+ (t-t_0)\left(\frac{2}{M_1} + \frac{2\mu_1}{3M_1} + \frac{1}{C_e^1}\|\theta_0\|_{H^2}^2\right)\|v^{m,n}(t_0)\|^2 + \\ &+ (t-t_0)\left(2 + \frac{2\mu_1}{3} + \frac{M_1}{C_e^1}\|\theta_0\|_{H^2}^2\right)\|\theta^{m,n}(t_0)\|^2 + \\ &+ M_2(t-t_0)\left(\frac{2}{M_1} + \frac{2\mu_1}{3M_1} + \frac{1}{C_e^1}\|\theta_0\|_{H^2}^2\right)\sum_{i=1}^N \|\eta_i^{m,n}(t_0)\|^2,\end{aligned}$$

and where C_{20} is a constant depending on C_{15}, C_{17}, C_{19}, M_1, M_2, σ, C_e^1 and $\|\theta_0\|_{H^2}$.

The estimate for the $\eta_i^{m,n}$ is similar to the estimate for $\theta^{m,n}$. From the equivalent norms (3.3) we get

$$C_e^1\|A_2^{1/2}\eta_i^{m,n}\|^2 - \|\eta_i^{m,n}\|^2 \leq \|\nabla\eta_i^{m,n}\|.$$

If we use this estimate, the positivity of the boundary integral and assume that

$$(4.23)\qquad \delta \leq \frac{C_e^1}{2Le}$$

we get from (4.9) that

$$\partial_t\|\eta_i^{m,n}\|^2 + \frac{C_e^1}{Le}\|A_2^{1/2}\eta_i^{m,n}\|^2 \leq C_{19} + \left(\frac{2\mu_1}{3} + \frac{2}{Le}\right)\|\eta_i^{m,n}\|^2 + \frac{C_s^2}{2\delta}\|v\|^2\|\eta_{i,0}\|_{H^2}^2.$$

Integrating from t_0 to t and using (4.18) to eliminate norms of $\eta_i^{m,n}$ and v from the right hand side gives

(4.24)
$$\frac{C_e^1}{Le}\int_{t_0}^t \|A_1^{1/2}\eta^{m,n}(s)\|^2\,ds + \|\eta^{m,n}(t)\|^2 \le \overline{L}_\eta'(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t-t_0),$$

where

$$\begin{aligned}(4.25)\quad \overline{L}_\eta'(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t-t_0) = {}&\\ (t-t_0)C_{21} + {}&\\ + (t-t_0)\left(\frac{2}{M_2 Le} + \frac{2\mu_1}{3M_2} + \frac{C_s^2}{2\delta}\sum_{i=1}^N \|\eta_{i,0}\|_{H^2}^2\right)\|v^{m,n}(t_0)\|^2 + {}&\\ + (t-t_0)M_1\left(\frac{2}{LeM_2} + \frac{2\mu_1}{3M_2} + \frac{C_s^2}{2\delta}\sum_{i=1}^N \|\eta_{i,0}\|_{H^2}^2\right)\|\theta^{m,n}(t_0)\|^2 + {}&\\ + (t-t_0)\sum_{i=1}^N\left(\frac{2}{Le} + \frac{2\mu_1}{3} + \frac{C_s^2 M_2}{2\delta}\|\eta_{i,0}\|_{H^2}^2\right)\|\eta_i^{m,n}(t_0)\|^2,&\end{aligned}$$

where C_{21} is a constant depending on C_{15}, C_{17}, C_{19}, M_1, M_2, σ, Le, C_e^1 and $\|\eta_{i,0}\|_{H^2}$.

4.2. **Existence of Solutions.** We are now in a position to prove our main existence theorem. In this section we will prove the existence of solutions and estimates on their norms, and in section 4.3 we will show that the solutions we get here are physically reasonable.

Proof of Parts 1 - 4 of Theorem 1. We begin by taking the limit as $n \to \infty$. In the discussion in the previous section showed that the Bubnov-Galerkin approximate solutions are uniformly bounded in various spaces. For instance, if we take $t_0 = 0$ in (4.18) and use the definition of the initial conditions (4.1) and that the norm of the projections is one we get:

$$\begin{aligned}(4.26)\quad \|v^{m,n}(t)\|^2 + M_1\|\theta^{m,n}(t)\|^2 + M_2\sum_{i=1}^N \|\eta_i^{m,n}(t)\|^2 \le {}&\\ \le C_1 + \|P_m v(0)\|^2 + M_1\|Q_n\theta(0)\|^2 + M_2\sum_{i=1}^N\|Q_n\eta_i(0)\|^2&\\ \le C_1 + \|v(0)\|^2 + M_1\|\theta(0)\|^2 + M_2\sum_{i=1}^N\|\eta_i(0)\|^2,&\end{aligned}$$

which shows that v, θ and η are bounded, uniformly in m and n, in the spaces $L^\infty(0,\infty;H)$, $L^\infty(0,\infty;H_1)$ and $L^\infty(0,\infty;H_2)$, respectively. Similarly, if we look at (4.19) with $t_0 = 0$ and use the monotonicity of $\overline{L}_v'$ we get

$$(4.27)\qquad Pr\int_{t_0}^t \|A^{1/2}v^{m,n}(s)\|\,ds + \|v^{m,n}(t)\|^2 \le \overline{L}_v'(\|v(0)\|, \|\theta(0)\|, \|\eta_i(0)\|, t-t_0),$$

which shows that $v^{m,n}$ is uniformly bounded in $L^2_{loc}[0,\infty;V)$. Similarly (4.21) and (4.24) can be used to show that $\theta^{m,n}$ and $\eta_i^{m,n}$ are uniformly bounded in $L^2_{loc}[0,\infty;V)$ and $L^2_{loc}[0,\infty;V)$.

Further we note that

$$\begin{aligned}\|A^{-1/2}\partial_t v^{m,n}\| &= \|A^{-1/2}P_m F_v(v^{m,n},\theta^{m,n},\eta_i^{m,n})\|\\ &= \|P_m A^{-1/2}F_v(v^{m,n},\theta^{m,n},\eta_i^{m,n})\|\\ &\le \|A^{-1/2}F_v(v^{m,n},\theta^{m,n},\eta_i^{m,n})\|,\end{aligned}$$

so that the boundedness property implies that the $\partial_t v^{m,n}$ are bounded in $L^p_{loc}[0,\infty;V^{-1})$ for $p = 4/3$. Similarly we get boundedness for $\partial_t\theta^{m,n}$ and $\partial_t\eta_i^{m,n}$ in $L^p_{loc}[0,\infty;V_1^{-1})$ and $L^p_{loc}[0,\infty;V_2^{-1})$.

Therefore, for fixed m, we can apply the compactness lemma $N+2$ times to the sequences $\{v^{m,n}\}$, $\{\theta^{m,n}\}$ and $\{\eta_i^{m,n}\}$ as $n\to\infty$ to get limit functions v^m, θ^m and η_i^m.

We can now take the limit as $m\to\infty$ to get the solutions we are looking for. In order to apply the compactness lemma again we need to know that (v^m,θ^m,η_i^m) is uniformly bounded in $L^\infty_{loc}(0,\infty;\overline{H})$ and $L^2_{loc}(0,\infty;\overline{V})$. However from the estimates in the previous section, we know that $(v^{m,n},\theta^{m,n},\eta_i^{m,n})$ are bounded in $L^\infty_{loc}(0,\infty;\overline{H})$ and $L^2_{loc}(0,\infty;\overline{V})$ uniformly in m and n, and the limiting process in the compactness lemma preserves these bounds, in particular using properties 1 and 2 of the compactness lemma. Therefore we can find a convergent subsequence, which we continue to label with m and limiting functions (v,θ,η_i) such that

$$(v^m,\theta^m,\eta_i^m)\to(v,\theta,\eta_i)\text{ as } m\to\infty,$$

where the convergence is in the sense of the compactness lemma.

Property 4, which gives the sense in which the equations are satisfied by the solution, is a direct consequence of the continuity properties of F_v, F_θ and F_{η_i} that were shown in section 3.6. In particular for the v equation we want to show:

$$\langle v(t)-v(t_0),\overline{v}\rangle = \int_{t_0}^t \langle F_v(v,\theta,\eta_i),\overline{v}\rangle\,ds,$$

for each $\overline{v}\in V$. Because the Bubnov-Galerkin solutions satisfy the ODE (4.1), we have for each n, m

$$\begin{aligned}\langle v^{m,n}(t)-v^{m,n}(t_0),\overline{v}\rangle &= \int_{t_0}^t \langle P_m F_v(v^{m,n},\theta^{m,n},\eta_i^{m,n}),\overline{v}\rangle\,ds\\ &= \int_{t_0}^t \langle F_v(v^{m,n},\theta^{m,n},\eta_i^{m,n}),P_m\overline{v}\rangle\,ds.\end{aligned}$$

Now taking the limit as $n\to\infty$ and using the continuity property and property 5 of the compactness lemma gives

$$\begin{aligned}\langle v^m(t)-v^m(t_0),\overline{v}\rangle &= \int_{t_0}^t \langle F_v(v^m,\theta^m,\eta_i^m),P_m\overline{v}\rangle\,ds\\ &= \int_{t_0}^t \langle P_m F_v(v^m,\theta^m,\eta_i^m),\overline{v}\rangle\,ds \qquad (4.28)\end{aligned}$$

for almost every t and t_0.

If we take the limit as $m \to \infty$ in (4.28), the left hand side converges for almost all t and t_0 by property 5 of the compactness lemma. For the right side, we note:

$$\begin{aligned}
\int_{t_0}^{t} |\langle P_m F_v(v^m, \theta^m, \eta_i^m) - F_v(v, \theta, \eta_i), \overline{v}\rangle| \; ds \\
&\leq \int_{t_0}^{t} |\langle P_m F_v(v^m, \theta^m, \eta_i^m) - F_v(v^m, \theta^m, \eta_i^m), \overline{v}\rangle| \; ds \\
&\quad + \int_{t_0}^{t} |\langle F_v(v^m, \theta^m, \eta_i^m) - F_v(v, \theta, \eta_i), \overline{v}\rangle| \; ds \\
&\leq \int_{t_0}^{t} |\langle F_v(v^m, \theta^m, \eta_i^m), (I - P_m)\overline{v}\rangle| \; ds \\
&\quad + \int_{t_0}^{t} |\langle F_v(v^m, \theta^m, \eta_i^m) - F_v(v, \theta, \eta_i), \overline{v}\rangle| \; ds \\
&\leq \int_{t_0}^{t} \left|\langle A^{-1/2} F_v(v^m, \theta^m, \eta_i^m), (I - P_m) A^{1/2}\overline{v}\rangle\right| \; ds \\
&\quad + \int_{t_0}^{t} |\langle F_v(v^m, \theta^m, \eta_i^m) - F_v(v, \theta, \eta_i), \overline{v}\rangle| \; ds
\end{aligned}$$

The first term goes to zero, since the $A^{-1/2} F_v(v^m, \theta^m, \eta_i^m)$ are bounded in $L^1_{loc}(0, \infty; H)$ and $(I - P_m) A^{1/2}\overline{v}$ goes to zero in H uniformly in t. The second term also goes to zero because of the continuity property of F_v. We now have that

$$\langle v(t) - v(t_0), \overline{v}\rangle = \int_{t_0}^{t} \langle F_v(v, \theta, \eta_i), \overline{v}\rangle \; ds$$

for almost all t_0 and t. We can now change v on a set of measure zero, which only changes the left hand side, so that the last equation holds for all t_0 and t with $t \geq t_0$. This is equation (3.10a) of property 4 in the existence theorem.

Identical arguments, with the exception that the two steps must be reversed, show that the other equations in property 4 are satisfied. We omit the details.

The estimates in property 3 of the solutions follow from taking the limit of the corresponding estimate for the Bubnov-Galerkin solutions and using properties 1 and 5 of the compactness lemma, the semicontinuity of the norm under weak convergence and the continuity of the functions $\overline{L}'_v, \overline{L}'_\theta$ and $\overline{L}'_\eta$ to show that the estimates hold in the limit. For example, consider (4.19). Observe that

$$\int_{t_0}^{t} \|A^{1/2} v(s)\| \; ds \leq \liminf_{m,n \to \infty} \int_{t_0}^{t} \|A^{1/2} v^{m,n}(s)\| \; ds,$$

since norms decrease under weak convergence. We also have that

$$\lim_{m,n \to \infty} \overline{L}'_v(\|v^{m,n}(t_0)\|, \|\theta^{m,n}(t_0)\|, \|\eta_i^{m,n}(t_0)\|, t - t_0) = \overline{L}'_v(\|v(t_0)\|, \|\theta(t_0)\|, \|\eta_i(t_0)\|, t - t_0)$$

for almost every t_0, by property 5 of the compactness lemma and the continuity of $\overline{L}'_v$. Therefore, we can take the limit of (4.19) to get

$$Pr\int_{t_0}^{t}\|A^{1/2}v(s)\|\,ds \le \overline{L}'_v(\|v(t_0)\|,\|\theta(t_0)\|,\|\eta_i(t_0)\|,t-t_0)$$

for almost all t_0. Finally we have that the last inequality holds for all $t > t_0$, since both sides are continuous in t. This is then corresponding property of the solution (3.9a).

□

We are now in a position to show that the dynamics of the system (3.4) do not depend on the choices of the homogenizing functions w, θ_0 and $\eta_{i,0}$. For simplicity, we will focus on the v equation, since the arguments for the other equations are similar. Suppose that we have two sets of homogenizing functions $(w_1, \theta_{0,1}, \eta_{i,0,1})$ and $(w_2, \theta_{0,2}, \eta_{i,0,2})$ with associated right hand sides $F_{v,1}$ and $F_{v,2}$. Then

$$F_{v,1}(u-w_1, T-\theta_{0,1}, Y_i-\eta_{i,0,1}) = F_{v,2}(u-w_2, T-\theta_{0,2}, Y_i-\eta_{i,0,2}),$$

for all (u, T, Y_i). This implies

$$F_{v,1}(v_1, \theta_1, \eta_{i,1}) = F_{v,2}(v_1+w_1-w_2, \theta_1+\theta_{0.1}-\theta_{0,2}, Y_i+\eta_{i,0,1}-\eta_{i,0,2}),$$

which implies that $v_2 = v_1 + w_1 - w_2$ is a solution to the second homogenized equation.

It is important to note that at this time we cannot prove the uniqueness of solutions to our reacting flow system, for much the same reason that there is no uniqueness proof for weak solutions of the 3D Navier-Stokes equations. In particular, just as the nonlinearity $B(u,u)$ causes problems for the usual Navier-Stokes equations, the terms $B_1(u,\theta)$ and $B_1(u,\eta_i)$ cause identical problems in our reacting flow system, which prevent us from even proving uniqueness for the θ and η_i components alone.

4.3. **Physical Reasonableness of the Solution.** In this section we want to return to the Bubnov-Galerkin limiting process from section 4.2 and show that, when the initial conditions are physically reasonable, then the resulting solutions are physically reasonable, i.e., that they satisfy equations (3.11) in the solution existence theorem 1. It is here that we will use the particular structure of the limiting process.

In this section we will revert back to the T and Y_i variables rather than using the θ and η_i variables.

It is worth noting here that if we have smooth solutions, then one can show that the solutions are physically reasonable based on classical maximum principle arguments. However, because our solutions are weak, we will use another method which is motivated by the method used in Manley, Marion and Temam [20] for the 2D problem.

We would like to give a short summary of how Manley, Marion and Temam show that the chemical concentrations remain nonnegative for the 2D problem as an introduction to our proof below. They begin by defining

$$Y_i^-(x,t) = \min\{Y_i(x,t), 0\}.$$

Then taking the inner product of the equation for Y_i with Y_i^- and using that $Y_i^- = 0$ whenever $Y_i^- \neq Y_i$ they get

$$\langle \partial_t Y_i^-, Y_i^-\rangle + \frac{1}{Le}\|\nabla Y_i^-\|^2 + b_1(u, Y_i^-, Y_i^-) = \langle W_i(Y_1,\dots,Y_N,T), Y_i^-\rangle.$$

Now, their choice of mixed Dirichlet-Neumann boundary conditions for Y_i implies that $b_1(u, Y_i^-, Y_i^-) = 0$, and also (3.24) implies that the right hand side is zero. Furthermore, since they are working in two dimensions, they have that $\partial_t Y_i \in L^2_{loc}[0,\infty; V_2^{-1})$, and hence

$$\langle \partial_t Y_i^-, Y_i^- \rangle = \frac{1}{2}\partial_t \|Y_i^-\|^2. \tag{4.29}$$

Combining all this gives that

$$\partial_t \|Y_i^-(t)\|^2 \leq 0,$$

which implies that $\|Y_i^-(t)\| = 0$ for almost all t, since $\|Y_i^-(0)\| = 0$.

The key difficulty in generalizing this to the 3D problem is the equality (4.29). In particular, in our case $\partial_t Y_i \in L^p_{loc}[0,\infty; V_2^{-1})$, and hence $\partial_t Y_i^- \in L^p_{loc}[0,\infty; V_2^{-1})$, with $p = 4/3$, while $Y_i^- \in L^2_{loc}[0,\infty; V_2)$. As a result, the left hand side of (4.29) need not even be locally integrable.

The reason that $\partial_t Y_i \in L^p_{loc}[0,\infty; V_2^{-1})$ can be traced to the nonlinearity $(u\cdot\nabla)Y_i$ and in particular to the lack of smoothness in u. If u were smooth then we could use the Sobolev estimate (3.19f) rather than (3.19d), which would allow us to get additional smoothness for $\partial_t Y_i$, in which case, an equality like (4.29) would apply. This is essentially the 2D argument.

It is for this reason in the 3D argument that we choose to take the limits as we have done in the previous section. In particular, for each fixed m, the approximate solution $u^{m,n}$ is smooth in both space and time uniformly in n. This allows us to use different Sobolev estimates to get the analogue of (4.29) for fixed m, and hence to show that the approximate solutions u^m, T^m and Y_i^m are physically reasonable. Then showing that the final solutions u, T and Y_i are physically reasonable is a direct consequence of the compactness lemma and the following lemma.

Lemma 17. *Let (u^m, T^m, Y_i^m) be a sequence of functions which converge to (u, T, Y_i) in the sense of the compactness lemma. Suppose further that they are physically reasonable, i.e.,*

$$\sum_{i=1}^{N} Y_i^m(x,t) = 1, \tag{4.30a}$$

$$0 \leq Y_i^m(x,t) \leq 1, \tag{4.30b}$$

$$T_l \leq T^m(x,t), \tag{4.30c}$$

for almost every x and t. Then the limit functions (u, T, Y_i) are also physically reasonable.

Proof. Property 5 of lemma 9 implies that for almost all x and t

$$\begin{aligned} T^m(x,t) &\to T(x,t) \\ Y_i^m(x,t) &\to Y_i(x,t) \end{aligned} \qquad \text{as } m \to \infty,$$

which implies the stated result. □

Proof of physical reasonableness. To show that the solutions obtained in the previous section are physically reasonable, we begin by showing that, for each m, the approximate solutions θ^m and η_i^m are physically reasonable, i.e., show that (4.30) are valid. In particular, we will use the notation

$$\begin{aligned} T^m(x,t) &= \theta^m(x,t) + \theta_0(x,t), \\ Y_i^m(x,t) &= \eta_i^m(x,t) + \eta_{i,0}(x,t). \end{aligned}$$

We begin by showing the first inequality in (4.30b), i.e., that the Y_i^m remain nonnegative. We define

$$Y_i^{m-}(x,t) = \min\{0, Y_i^m(x,t)\} \qquad Y_i^{m,n-}(x,t) = \min\{0, Y_i^{m,n}(x,t)\}$$

By proposition 2.6 of Edmunds and Evans [6].

$$\partial_{x_k} Y_i^{m,n-}(x) = \begin{cases} \partial_{x_k} Y_i^{m,n}(x) & \text{if } Y_i^{m,n}(x) < 0, \\ 0 & \text{if } Y_i^{m,n}(x) \geq 0, \end{cases} \tag{4.31}$$

for $k = 1, 2, 3$, with similar relationships valid for Y^{m-}. This implies that $Y_i^{m,n-}$ and Y_i^{m-} are both in $H^1(\Omega)$ whenever $Y_i^{m,n}$ and Y_i^m are.

We also have that $Y_i^{m,n-}(t) \to Y_i^{m-}(t)$ strongly in H_2 for almost every t by property 5 of the compactness lemma. This, plus the uniform boundedness in $L^\infty_{loc}[0,\infty; H_2)$ and the Lebesgue Dominated Convergence theorem imply that $Y_i^{m,n-} \to Y_i^{m-}$ strongly in $L^2_{loc}[0,\infty; H_2)$.

Now the equation for $Y_i^{m,n}$ can be obtained by substituting into (4.1), which gives

$$\partial_t Y_i^{m,n} - \frac{1}{Le}\Delta Y_i^{m,n} - \frac{1}{Le}\Delta\eta_{i,0} + R_n B_1(v^{m,n} + w, Y_i^{m,n}) = \\ = R_n W_i(Y_1^{m,n}, \ldots, Y_N^{m,n}, T) + R_n \frac{1}{Le}\Delta\eta_{i,0},$$

or

$$\partial_t Y_i^{m,n} - \frac{1}{Le}\Delta Y_i^{m,n} + R_n B_1(v^{m,n} + w, Y_i^{m,n}) = \\ = R_n W_i(Y_1^{m,n}, \ldots, Y_N^{m,n}, T) + (R_n - I)\frac{1}{Le}\Delta\eta_{i,0}.$$

Now if we take the inner product of this equation with $Y_i^{m,n-}$ we get

$$\langle \partial_t Y_i^{m,n}, Y_i^{m,n-}\rangle - \frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle + b_1(v^{m,n} + w, Y_i^{m,n}, R_n Y_i^{m,n-}) = \\ \langle R_n W_i(Y_1^{m,n}, \ldots, Y_N^{m,n}, T), Y_i^{m,n-}\rangle + \langle (R_n - I)\frac{1}{Le}\Delta\eta_{i,0}, Y_i^{m,n-}\rangle.$$

or

$$\langle \partial_t Y_i^{m,n}, Y_i^{m,n-}\rangle - \frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle \\ + b_1(v^{m,n} + w, Y_i^{m,n}, Y_i^{m,n-}) + b_1(v^{m,n} + w, Y_i^{m,n}, (R_n - I)Y_i^{m,n-}) = \\ \langle R_n W_i(Y_1^{m,n}, \ldots, Y_N^{m,n}, T), Y_i^{m,n-}\rangle + \langle (R_n - I)\frac{1}{Le}\Delta\eta_{i,0}, Y_i^{m,n-}\rangle. \tag{4.32}$$

Now integration by parts gives

$$-\frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle = \frac{1}{Le}\langle \nabla Y_i^{m,n}, \nabla Y_i^{m,n-}\rangle - \frac{1}{Le}\int_{\partial\Omega} Y_i^{m,n-}\frac{\partial}{\partial n} Y_i^{m,n}\, dS.$$

Using the definition of $Y_i^{m,n-}$ and the properties of its derivatives given by (4.31) one gets

$$-\frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle = \frac{1}{Le}\|\nabla Y_i^{m,n-}\|^2 - \frac{1}{Le}\int_{\partial\Omega} Y_i^{m,n-}\frac{\partial}{\partial n} Y_i^{m,n-}\, dS. \tag{4.33}$$

Similarly, the derivative relationships (4.31) give

$$b_1(v^{m,n}+w, Y_i^{m,n}, Y_i^{m,n-}) = b_1(v^{m,n}+w, Y_i^{m,n-}, Y_i^{m,n-}) \tag{4.34}$$

and

$$\frac{1}{2}\partial_t \|Y_i^{m,n-}\|^2 = \langle \partial_t Y_i^{m,n}, Y_i^{m,n-}\rangle. \tag{4.35}$$

Next we need to consider the analogue to the generalized Poincaré inequality in lemma 6 for this situation. We have using (4.33), (4.34) and a calculation identical to that of (3.17) that

$$\begin{aligned}
-\frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle &+ b_1(v^{m,n}+w, Y_i^{m,n}, Y_i^{m,n-}) = \\
&= -\frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle + b_1(v^{m,n}+w, Y_i^{m,n-}, Y_i^{m,n-}) \\
&= \frac{1}{Le}\|\nabla Y_i^{m,n-}\|^2 - \frac{1}{Le}\int_{\partial\Omega} Y_i^{m,n-}\frac{\partial}{\partial n} Y_i^{m,n-}\, dS. \\
&\quad + \frac{1}{2}\int_{\partial\Omega} (Y_i^{m,n-})^2(\phi\cdot n)\, dS
\end{aligned}$$

Now we claim that the boundary integrand $I(x)$ given by

$$I(x) = -\frac{1}{Le} Y_i^{m,n-}(x)\frac{\partial}{\partial n} Y_i^{m,n-}(x) + \frac{1}{2}(Y_i^{m,n-}(x))^2(\phi(x)\cdot n)$$

is pointwise nonnegative. There are two cases to consider. First, if $Y^{m,n}(x) \geq 0$, then the integrand is itself zero. Second, if $Y^{m,n}(x) < 0$, then by the above derivative properties $Y^{m,n-}$ satisfies the same inhomogeneous boundary conditions as $Y^{m,n}$ at x. So we proceed as in the proof of the lemma 6, but use inhomogeneous rather than homogeneous boundary conditions. If $x \in \Gamma_W$, then $\phi(x) = 0$ and the normal derivative of $Y^{m,n-}(x)$ are both zero, hence $I(x) = 0$. If $x \in \Gamma_O$, then the normal derivative is again zero, hence

$$I(x) = \frac{1}{2}(Y_i^{m,n-}(x))^2(\phi(x)\cdot n) \qquad \text{for } x \in \Gamma_O$$

which is positive. Finally, for $x \in \Gamma_I$ we have the boundary condition

$$\frac{1}{Le}\frac{\partial Y_i^{m,n-}}{\partial n} - (\phi(x)\cdot n)Y_i^{m,n-} = Y_{i,f}(x) \qquad \text{in } \Gamma_I,$$

where $Y_{i,f}(x) \geq 0$. So in this case we have

$$\begin{aligned}
I(x) &= -\frac{1}{Le} Y_i^{m,n-}(x)\frac{\partial}{\partial n} Y_i^{m,n-}(x) + \frac{1}{2}(Y_i^{m,n-}(x))^2(\phi(x)\cdot n) \\
&= -\frac{1}{2}\frac{1}{Le} Y_i^{m,n-}(x)\frac{\partial}{\partial n} Y_i^{m,n-}(x) + \frac{1}{2} Y_i^{m,n-}(x)\left(Y_i^{m,n-}(x)(\phi(x)\cdot n) - \frac{1}{Le}\frac{\partial}{\partial n} Y_i^{m,n-}(x)\right) \\
&= -\frac{1}{2} Y_i^{m,n-}(x)\left(Y_{i,f}(x) + (\phi(x)\cdot n)Y_i^{m,n-}(x)\right) - \frac{1}{2} Y_i^{m,n-}(x) Y_{i,f}(x),
\end{aligned}$$

all the terms of which are positive, since $Y_i^{m,n-}(x) < 0$ and $\phi \cdot n < 0$. Hence we have that

$$-\frac{1}{Le}\langle \Delta Y_i^{m,n}, Y_i^{m,n-}\rangle + b_1(v^{m,n}+w, Y_i^{m,n}, Y_i^{m,n-}) \geq 0.$$

Substituting into (4.32) gives

$$\frac{1}{2}\partial_t\|Y_i^{m,n-}\|^2 + b_1(v^{m,n}+w, Y_i^{m,n}, (R_n - I)Y_i^{m,n-}) \le$$
$$\le \langle R_n W_i(Y_1^{m,n},\dots,Y_N^{m,n},T), Y_i^{m,n-}\rangle + \langle (R_n - I)\frac{1}{Le}\Delta\eta_{i,0}, Y_i^{m,n-}\rangle.$$

Integrating from 0 to t gives

$$Y_i^{m,n-}(t) + \int_0^t b_1(v^{m,n}+w, Y_i^{m,n}, (R_n - I)Y_i^{m,n-})\, d\tau \le \int_0^t \langle W_i(Y_1^{m,n},\dots,Y_N^{m,n},T^{m,n}), R_n Y_i^{m,n-}\rangle\, d\tau + \int_0^t \langle (R_n - I)\frac{1}{Le}\Delta\eta_{i,0}, Y_i^{m,n-}\rangle\, d\tau, \tag{4.36}$$

since $Y_i^{m,n-}(0) = 0$. We now want to pass to the limit as $n \to \infty$. Note that by the Sobolev estimate (3.19e) and the Hölder inequality one gets

$$\begin{aligned}
\int_0^t b_1(v^{m,n}+w, Y_i^{m,n}, (R_n - I)Y_i^{m,n-})\, d\tau &\le C_s \int_0^t \|v^{m,n}+w\|_{H^1}\|Y_i^{m,n}\|_{H^1} \times \\
&\qquad \times \|(R_n - I)Y_i^{m,n-}\|^{1/2} \\
&\qquad \times \|(R_n - I)Y_i^{m,n-}\|_{H^1}^{1/2}\, d\tau \\
&\le \sup_{\tau\in(0,t)} \|v^{m,n}+w\|_{H^1} \times \left(\int_0^t \|Y_i^{m,n}\|_{H^1}^2\, d\tau\right)^{1/2} \times \\
&\qquad \times \left(\int_0^t \|(R_n - I)Y_i^{m,n-}\|_{H^1}^2\, d\tau\right)^{1/4} \times \\
&\qquad \times \left(\int_0^t \|(R_n - I)Y_i^{m,n-}\|^2\, d\tau\right)^{1/4}.
\end{aligned} \tag{4.37}$$

The key reason for taking the limits the way we have is that now $\sup_{\tau\in(0,t)}\|v^{m,n}+w\|_{H^1}$ is finite and uniformly bounded for fixed m. To see this, it suffices to consider $\sup_{\tau\in(0,t)}\|v^{m,n}\|_{H^1}$. Note that $v^{m,n}$ is smooth and contained in the finite dimensional space spanned by the first m eigenvectors of the Stokes operator. On this finite dimensional space all norms are equivalent, hence

$$\|v^{m,n}(t)\|_{H^1} \le C_{22}(m)\|v^{m,n}(t)\|.$$

But we know from (4.26) that $\|v^{m,n}(t)\|$ is uniformly bounded in n, m and t, hence, for fixed m, $\sup_{\tau\in(0,t)}\|v^{m,n}\|_{H^1}$ is uniformly bounded in n. Of course $C_{22}(m)$ becomes unbounded as $m \to \infty$, but this will cause no trouble.

Now returning to (4.37) we note that the first two integrals in the estimate are bounded by (4.24). Further we have that

$$\int_0^t \|(R_n - I)Y_i^{m,n-}\|^2\, d\tau \le \int_0^t \|R_n(Y_i^{m,n-} - Y_i^{m-})\|^2\, d\tau + $$
$$+ \int_0^t \|R_n Y_i^{m-} - Y_i^{m-}\|^2\, d\tau + \int_0^t \|Y_i^{m-} - Y_i^{m,n-}\|^2\, d\tau.$$

The first and third integrals go to zero as $n \to \infty$ because $Y_i^{m,n-} \to Y_i^{m-}$ as $n \to \infty$ in $L^2_{loc}(0,\infty;L^2)$. The second goes to zero because of the Lebesgue Dominated Convergence theorem and the strong convergence of R_n to the identity. Therefore, as $n \to \infty$, the b_1 integral in (4.36) goes to zero.

We also have that

$$\int_0^t \langle W_i(Y_1^{m,n},\dots,Y_N^{m,n},T^n), R_n Y_i^{m,n-}\rangle \, d\tau \to \int_0^t \langle W_i(Y_1^m,\dots,Y_N^m,T), Y_i^{m-}\rangle \, d\tau,$$

as $n \to \infty$, by the Lebesgue Dominated Convergence theorem, since both terms of the inner product are bounded in $L^\infty_{loc}(0,\infty;L^2)$ and since $Y_i^{m,n}(\tau) \to Y_i(\tau)$ and $Y_i^{m,n-}(\tau) \to Y_i^{m-}(\tau)$ strongly in $L^2(\Omega)$ for almost every τ. Now by the definition of Y_i^{m-} and using (3.24) we get

$$\langle W_i(Y_1^m,\dots,Y_N^m,T^m), Y_i^{m-}\rangle = 0.$$

Therefore, the W_i integral in (4.36) goes to zero.

The last term to consider is

$$\int_0^t \langle (R_n - I)\frac{1}{Le}\Delta\eta_{i,0}, Y_i^{m,n-}\rangle \, d\tau,$$

which again goes to zero by the Lebesgue Dominated Convergence theorem and the strong convergence of R_n to the identity.

Therefore, (4.36) becomes in the limit

$$\|Y_i^{m-}(t)\|^2 \le 0 \qquad \text{for } t \ge 0,$$

which implies that $Y_i^m(x,t) \ge 0$ for almost every x. This shows the first inequality in (4.30b).

Essentially the same proof can be used to show (4.30c), i.e., that $T^m(x,t) > T_l$ for almost every t and x, in particular using the assumption (2.5).

We now want to show that (4.30a) holds for each m. Let

$$Y^m(x,t) = \sum_{i=1}^N Y_i^m(x,t) \qquad Y^{m,n}(x,t) = \sum_{i=1}^N Y_i^{m,n}(x,t)$$

Note that $Y_i^{m,n}(t) \to Y_i^m(t)$ strongly in $L^2(\Omega)$ for almost every t. We can find an equation and boundary conditions for $Y^{m,n}$ by summing (2.1d) and (2.10) over i to get

$$\begin{aligned}\frac{\partial Y^{m,n}}{\partial t} - \frac{1}{Le}\Delta Y^{m,n} + R_n(u^{m,n}\cdot\nabla)Y^{m,n} &= R_n \sum_{i=1}^N W_i(Y_1^{m,n},\dots,Y_N^{m,n},T)\\ &= 0 \qquad \text{in } \Omega.\end{aligned}$$

On the boundary one has

$$\frac{1}{Le}\frac{\partial Y^{m,n}}{\partial n} - (\phi\cdot n)Y^{m,n} = -\phi\cdot n \qquad \text{on } \partial\Omega.$$

where we have used the mass conservation property (2.2) and the requirement on the boundary conditions (2.14). Note also that

$$Y^{m,n}(0) = \sum_{i=1}^N R_n Y_i(0) = R_n 1,$$

since the initial condition is physically reasonable. If we homogenize the boundary conditions by writing

$$Y^{m,n}(x,t) = \eta^{m,n}(x,t) - 1$$

Then $\eta^{m,n}$ satisfies

$$\begin{aligned} \frac{\partial \eta^{m,n}}{\partial t} - \frac{1}{Le}\Delta \eta^{m,n} + R_n(u^n \cdot \nabla)\eta^{m,n} &= 0, && \text{in } \Omega \\ \frac{1}{Le}\frac{\partial \eta^{m,n}}{\partial n} - (\phi \cdot n)\eta^{m,n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.38}$$

We note that, by uniqueness for ODE's, any solution of the Y^n equation necessarily corresponds to a sum of solutions to the $\eta^{m,n}$ equations. If we now take the inner product of (4.38) with $\eta^{m,n}$ and use the Poincaré inequality from lemma 6, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|\eta^{m,n}\|^2 \leq 0.$$

Integrating in time gives

$$\|\eta^{m,n}(t)\|^2 \leq \|\eta^{m,n}(0)\|^2.$$

Now we have

$$\begin{aligned} \eta^{m,n}(x,0) &= -1 + \sum_{i=1}^{N} R_n Y_i(x,0) \\ &= -1 + R_n 1. \end{aligned}$$

So as $n \to \infty$, $\|\eta^{m,n}\| \to 0$, hence $\eta^{m,n}(t) \to 0$ in $L^2(\Omega)$ as $n \to \infty$. Therefore, for almost every x and t, we have (4.30a). This result combined with the positivity of the Y_i implies the other half of (4.30b).

Therefore, we have shown (4.30), i.e., that Y_i^m and T^m, or η_i^m and θ^m, are physically reasonable. Now lemma 17 implies that when we take the limit in m the resulting solution is itself physically reasonable, since physical reasonableness is preserved under the convergence described in the compactness lemma. □

It is worth pointing out that we have not shown that all weak solutions to the reacting flow system are physically reasonable, for instance one could obtain solutions by letting $m, n \to \infty$ simultaneously. We have only shown the existence of a solution which is physically reasonable, assuming that the data are physically reasonable.

5. Existence of Global Attractors

We now want to discuss the existence of global attractors for the physically reasonable weak solutions to the reacting flow system. The central problem we face is that we are unable to prove uniqueness of weak solutions. Sell [24] shows the existence of global attractors for weak solutions to the 3D Navier-Stokes equations, for which uniqueness is also not known, and it is his work which we want to generalize to our situation. His approach is to use a phase space in which each point is a solution to the equation, hence two different solutions with the same initial condition correspond to different points in the phase space. A key part of the theory is to describe exactly what is meant by a solution, then to analyze the resulting phase space.

The sections below are a direct generalization of Sell's work. Our main contribution is to present the work as a general attractor existence theorem for systems of equations with constraints, in particular using the boundedness and continuity properties above.

5.1. **Notation.** We now want to present the notation for our general attractor theorem. We will assume that we have a system of N_1 equations

$$\partial_t \phi_i = F_i(\phi_1, \dots, \phi_{N_1}). \tag{5.1}$$

Associated with each equation i are two spaces $V_i \hookrightarrow H_i$, where the embedding is dense and compact.

We will also use the notation $\overline{H} = \prod H_i$ and $\overline{V} = \prod V_i$ and say that

$$\phi(t) = (\phi_1(t), \dots, \phi_{N_1}(t)) \in \overline{H}.$$

We also denote $\|\phi_i(t)\| = \|\phi_i(t)\|_{H_i}$ and $\|\phi(t)\| = \|\phi(t)\|_{\overline{H}}$.

We also want to consider the possibility of constraints on our solutions. In the case of the reacting flow system these constraints would correspond to the physical reasonableness conditions (3.11). We will denote these constraints by

$$G_k(\phi(t)) = 0, \qquad \text{for almost every } t \tag{5.2a}$$

$$G_l^*(\phi(t)) \geq 0 \qquad \text{for almost every } t \tag{5.2b}$$

for $k = 1, 2, \dots N_2$ and $l = 1, 2, \dots N_3$. Our only assumption on the constraints will be that they are closed under the type of convergence in the compactness lemma 9, i.e., if ϕ^j satisfy the constraints and $\phi_i^j \to \phi_i$, as in the compactness lemma for each i, then ϕ satisfies the constraints. For example, in the case of the reacting flow the positivity of chemical species i could be written as

$$G_1(u, T, Y_i) = \int_\Omega \min\{0, Y_i\}\, dx = 0.$$

Lemma 17 shows that the physical reasonableness constraints are closed under the convergence in the compactness lemma, and in fact using similar arguments one can show that any constraint which should hold pointwise almost everywhere, is closed under the convergence in the compactness lemma.

5.2. **Spaces of Weak Solutions.** The first step in the theory is to describe precisely what we mean by a solution in order be able to describe the phase space of our solutions. In particular, the definition we give is a direct generalization to systems of equations with constraints of Sell's notion of weak solutions of Leray-Hopf class.

Assume that there exist constants $p > 1$ and $\sigma_i, K_i > 0$ and nondecreasing continuous functions of N_1 real variables $L_i(a_1, \dots, a_{N_1})$ and $L_i'(a_1, \dots, a_{N_1})$. We say that ϕ is a weak solution of Leray-Hopf class of the system (5.1) with the constraints (5.2) and write $\phi \in W_{LH}$ if, for each i, the following five properties hold.

1. $\phi_i \in L^\infty(0, \infty; H_i) \cap L^2_{loc}[0, \infty; V_i)$ for $1 \leq i \leq N_1$.
2. $\partial_t \phi_i \in L^p_{loc}[0, \infty; V_i^{-1})$ for $1 \leq i \leq N_1$.
3. For almost all t and almost all t_0 with $t > t_0 > 0$ one has

$$\|\phi_i(t)\|^2 \leq e^{-\sigma_i(t-t_0)} L_i(\|\phi_1(t_0)\|, \dots, \|\phi_{N_1}(t_0)\|) + K_i \tag{5.3}$$

and for almost all $t_0 > 0$ and all $t \geq t_0$ one has

$$\int_{t_0}^{t} \|\phi_i\|_{V^i}^2 \, ds \leq L_i'(\|\phi_1(t_0)\|, \ldots, \|\phi_{N_1}(t_0)\|, t - t_0), \tag{5.4}$$

for $1 \leq i \leq N_1$.

4. For all $t \geq t_0 \geq 0$ one has

$$\langle \phi_i(t) - \phi_i(t_0), v \rangle = \int_{t_0}^{t} \langle F_i(\phi_1, \ldots, \phi_{N_1}), v \rangle ds \tag{5.5}$$

for all $v \in V_i$ and $1 \leq i \leq N_1$, where the integrand is in the sense of the dual product between V_i and V_i^{-1}.

5. The solution ϕ satisfies the constraints (5.2) for almost every t.

We would like to use W_{LH} with the $L^2_{loc}[0, \infty; \overline{H})$ topology as the phase space for our system, but unfortunately we cannot do this directly. In particular, the general attractor existence theorems we want to use below require that the phase space be complete, which in our case is equivalent to saying that W_{LH} is closed in $L^2_{loc}[0, \infty; \overline{H})$. However we are unable to prove this. The same problem arises in the study of the 3D Navier-Stokes equations.

Sell's solution to this problem is to widen the definition of a solution to include the possibility of a singularity at the origin. In particular, we say that ϕ is a generalized weak solution of the system (5.1) and write $\phi \in W \subset L^2_{loc}(0, \infty; \overline{H})$ if

1. Each ϕ_i satisfies

$$\phi_i \in L^2_{loc}[0, \infty; H_i) \cap L^\infty_{loc}(0, \infty; H_i) \cap L^\infty[1, \infty; H_i) \cap L^2_{loc}(0, \infty; V_i),$$

for $1 \leq i \leq N_1$.

2. $\partial_t \phi_i \in L^p_{loc}(0, \infty; V_i^{-1})$, for $1 \leq i \leq N_1$.
3. For almost all t and almost all t_0 with $t > t_0$ inequality (5.3) and for all $t > t_0$ inequality (5.4) holds.
4. For all $t > t_0 > 0$ equation (5.5) holds.
5. The solution ϕ satisfies the constraints (5.2) for almost every t.

It is this space W with the $L^2_{loc}[0, \infty; \overline{H})$ topology that we will use as our phase space. The remainder of this section will deals with analyzing the properties of W and its solutions.

Define the set

$$B(M_0) = \left\{ \phi \in W : \int_0^1 \|\phi\|^2 < M_0^2 \right\}. \tag{5.6}$$

Note that, if $B \subset W$ is a bounded set, then by the definition of boundedness,

$$\sup_{\phi \in B} \int_0^1 \|\phi\|^2 < \infty,$$

so $B \subset B(M_0)$ for some M_0. An important part of the next lemma that the converse is also true, i.e., that $B(M_0)$ is bounded in $L^2_{loc}[0, \infty; \overline{H})$.

Lemma 18. *The following statements about $B(M_0)$ are true.*

1. *If $\phi \in B(M_0)$, then for each i, with $1 \leq i \leq N_1$, one has*

$$\|\phi_i(\tau)\|^2 \leq \tau^{-1} M_0^2 + K_i \quad 0 < \tau \leq 1 \tag{5.7}$$

and

$$\|\phi_i(\tau)\|^2 \le e^{-\sigma_i(\tau-1)}(M_0^2 + K_i) + K_i \quad \tau \ge 1 \tag{5.8}$$

2. *If $\phi \in B(M_0)$, then for each i, with $1 \le i \le N_1$, one has*

$$\int_T^{T+1} \|\phi_i\|^2 \, ds \le e^{-\sigma_i(T-1)}(M_0^2 + K_i) + K_i \tag{5.9}$$

3. *The set $B(M_0)$ is bounded in W.*
4. *For each τ and T with $0 < \tau \le T < \infty$, there exist constants M_1 and M_2 which depend on τ and T such that for each $\phi \in B(M_0)$ and each i, with $1 \le i \le N_1$, one has*

$$\int_\tau^T \|\phi_i\|_{V_i}^2 \, ds \le M_1(\tau, T)^2 \tag{5.10}$$

and, assuming that the boundedness property is satisfied by F_i,

$$\int_\tau^T \|\partial_t \phi_i\|_{V_i^{-1}}^p \, ds \le M_2(\tau, T)^p \tag{5.11}$$

Proof. To show (5.7) it is important to work in the product space $\overline{H}$. We note that (5.3) implies that

$$\|\phi(t)\|^2 \le e^{-\sigma(t-t_0)} L(\|\phi(t_0)\|) + K$$

for almost all $t > t_0 > 0$, where $\sigma = \min_i \sigma_i$ and L is a nondecreasing function of its argument, K a positive constant and $\|\phi\|^2 = \|\phi\|_{\overline{H}}^2$. This implies that for $0 < \tau \le 1$ we have $\int_0^\tau \|\phi\|^2 \, ds \le \int_0^1 \|\phi\|^2 \, ds \le M_0^2$ for each i. Hence for each such τ the set

$$\left\{ t \in (0, \tau) : \|\phi(t)\|^2 \le \tau^{-1} M_0^2 \right\}$$

must have positive measure. So for each τ and there exists a $0 < t_\phi < \tau$ such that $\|\phi(t_\phi)\|^2 < \tau^{-1} M_0^2$. Hence, for each i we have $\|\phi_i(t_\phi)\|^2 < \tau^{-1} M_0^2$ This gives using (5.3),

$$\|\phi_i\|^2 \le e^{-\sigma_i(t-t_\phi)} L_i(\tau^{-1} M_0, \ldots, \tau^{-1} M_0) + K_i, \qquad \text{for } 1 \le i \le N_1,$$

which gives (5.7) and (5.8). Integrating then gives (5.9). The fact that $B(M_0)$ is bounded in W follows then from the definition of boundedness.

To derive (5.10) we can assume that $\tau < 1 < T$. Choose t_ϕ as above, then (5.4) implies

$$\int_{t_\phi}^T \|\phi_i\|_{V_i} \le \int_\tau^T \|\phi_i\|_{V_i} \le L'(\tau^{-1} M_0^2, \ldots, \tau^{-1} M_0^2, t - t_0) \qquad \text{for } 1 \le i \le N_1,$$

which gives (5.10).

To derive (5.11), we note that $\partial_t \phi_i = F_i(\phi)$, for $1 \le i \le N_1$, so the result follows from (5.7), (5.8) and the boundedness property. □

We can now show that W is closed in $L^2_{loc}[0, \infty; H)$, so that it is a suitable phase space.

Lemma 19. *Assume that the F_i, for $1 \le i \le N_1$ satisfy the continuity and boundedness properties and assume that the constraints G_k and G_l^* are closed under the convergence in the compactness lemma. Let ϕ^n be a sequence in W which converges to ϕ^0 in $L^2_{loc}[0, \infty; H)$. Then there exists a $\phi \in W$ such that $\phi \overset{a.e.}{=} \phi^0$. In particular, the set W is closed in $L^2_{loc}[0, \infty; H)$.*

Proof. We first note that since $\phi_i^n \to \phi_i^0$ strongly in $L^2_{loc}[0,\infty;H_i)$ we have that

$$\lim_{n\to\infty} \phi_i^n(t) = \phi_i^0(t) \qquad \text{for almost every } t \in (0,\infty) \text{ and } 1 \le i \le N_1 \ . \tag{5.12}$$

Further, since ϕ_i^n is convergent in $L^2_{loc}[0,\infty;H_i)$, it must be bounded, hence there is a positive number M_0 such that $\phi_i^n \in B(M_0)$. Now lemma 18 implies that the compactness lemma holds for each i, so we can apply it N_1 times, and continue to denote the subsequences by ϕ^n. Let the limit functions be ϕ_i and let $\phi = (\phi_i)$. Property 5 of the compactness lemma together with (5.12) implies that $\phi_i \overset{a.e.}{=} \phi_i^0$ for each i.

We now want to show that $\phi \in W$. Because of property 5 of the compactness lemma, we see that for each i, with $1 \le i \le N_1$, $\phi_i \in L^\infty_{loc}(0,2;H_i) \cap L^\infty(1,\infty;H_i)$. Also $\partial_t\phi_i \in L^p_{loc}(0,\infty;V_i^{-1})$, by property 2 of the compactness lemma. Also the estimate (5.3) holds by property 5. Hence we just need to show that (5.4) and (5.5) are satisfied.

To show (5.5), first note that the continuity property of the F_i implies that

$$\lim_{n\to\infty} F_i(\phi_1^n,\ldots,\phi_N^n) \overset{w}{\to} F_i(\phi_1,\ldots,\phi_N) \qquad \text{weakly in } L^1_{loc}(0,\infty;V_i^{-1})$$

for each i, with $1 \le i \le N_1$. Hence for all $t > t_0 > 0$ we have,

$$\begin{aligned}\int_{t_0}^t \langle F_i(\phi_1,\ldots,\phi_N), v\rangle \, ds &= \lim_{t\to\infty} \int_{t_0}^t \langle F_i(\phi_1^n,\ldots,\phi_N^n), v\rangle \, ds \\ &= \lim_{t\to\infty} \langle \phi^n(t) - \phi^n(t_0), v\rangle\end{aligned}$$

So by property 5 of the compactness lemma we have for almost every t and t_0 (5.5) holds. So by changing ϕ_i on a set of measure zero, we can arrange for (5.5) to hold for all t and t_0. Call the set where we changed ϕ_i E_i^c.

To show (5.4), we first discuss the t_0 for which the estimate will hold. Let E_i^n be the set of $t_0 \in (0,\infty)$ such that (5.4) does not hold for ϕ_i^n. We claim that the estimate holds for

$$t_0 \notin E \cup E_i^c \cup \bigcup_{n,i} E_i^n$$

Note that the set on the right hand side is of measure zero, since it is a countable union of sets of measure zero. Fix such a t_0. By property 1, the lower semicontinuity property for weak convergence, (5.4) applied to the ϕ_i^n and property 5, we have for each i, with $1 \le i \le N_1$ and for all $t \ge t_0$, one has

$$\begin{aligned}\int_{t_0}^t \|\phi_i\|_{V_i} \, ds &\le \liminf_{n\to\infty} \int_{t_0}^t \|\phi_i^n\|_{V_i} \, ds \\ &\le \liminf_{n\to\infty} L'(\|\phi_1^n(t_0)\|,\ldots,\|\phi_N^n(t_0)\|, t-t_0) \\ &\le L'(\|\phi_1(t_0)\|,\ldots,\|\phi_N(t_0)\|, t-t_0)\end{aligned}$$

Which implies (5.4) for ϕ_i.

We also note that ϕ satisfies the constraints (5.2), since it is almost everywhere equal to the limit of functions satisfying the constraints, with convergence in the sense of the compactness lemma.

□

Later in the paper we will need the following lemma which states that, when a generalized weak solution is appropriately bounded near $t = 0$, it is in fact a solution of Leray-Hopf class.

Lemma 20. *Suppose the F_i, for $1 \leq i \leq N_1$ satisfy the boundedness property and let $\phi \in W$ such that $\phi \in L^2_{loc}[0, \infty; \overline{V}) \cap L^\infty(0, \infty; H)$. Then $\phi \in W_{LH}$.*

Proof. We just need to show that

1. $\partial_t \phi \in L^p_{loc}[0, \infty; \overline{V})$ for $p = 4/3$ and
2. Equation (5.5) is satisfied for $t_0 = 0$.

Item (1) follows directly from the boundedness property of the F_i. Further, because of this, $\langle F_i, \overline{v} \rangle \in L^1_{loc}[0, \infty; \mathbb{R})$, hence item (2) is satisfied. ☐

5.3. **The Semiflow.** Now that we have discussed the phase space for our system, we need to discuss the dynamics on that space. To do this we need the notion of a semiflow. A semiflow $S(t)$ is a map from $\mathbb{R}^+ \times W$ to W, which we denote by $S : (t, \phi) \mapsto S(t)\phi$, which satisfies the following properties.

1. For all t, s and ϕ we have
$$S(t + s)\phi = S(t)S(s)\phi$$
2. For all ϕ
$$S(0)\phi = \phi$$
3. $S(t)\phi$ is continuous on $(0, \infty) \times W$.

In our case, the semiflow is particularly simple and is given by time translation. In particular we define

$$(S(t)\phi)(\tau) = \phi(t + \tau).$$

It is clear from this definition that properties 1 and 2 of a semiflow are satisfied by this definition, and property 3 follows from general principles, see lemma 7 in Sell [24]. It is also straightforward to check that W and W_{LH} are invariant under $S(t)$.

One particularly nice property we have is that $S(t)W \in W_{LH}$ for all $t > 0$. This means that after any strictly positive time we only need to be concerned with weak Leray-Hopf solutions.

Lemma 21. *Let $t > 0$. Then $S(t)W \subset W_{LH}$.*

Proof. Let $\phi_t \in S(t)W$. Then from the definition,

$$\phi_t(t + \tau) = \phi(\tau) \qquad \text{for } \tau \geq 0.$$

But this implies from (5.3) that $\phi_t \in L^\infty(0, \infty; H)$ and from (5.4) that $\phi_t \in L^2_{loc}[0, \infty; V)$. Therefore, by lemma 20, $\phi \in W_{LH}$. ☐

The general global attractor theorem we want to use depends on two definitions. We say that a semiflow $S(t)$ is **point dissipative** if there exists a bounded set $U \subset W$ such that for all $\phi \in W$ there exists a time T such that for $t > T$, $S(t) \in U$. We also say that U is an absorbing set. We also say that the semiflow is **compact** if for each bounded set $U \subset W$ there exists a time τ such that $S(\tau)U$ has compact closure. It follows then that $S(\tau + t)U$ has compact closure for $t \geq 0$. We then have the following theorem.

Theorem 22 (Billotti-LaSalle)**.** *Let $S(t)$ be a semiflow which is compact and point dissipative. Then $S(t)$ has a global attractor $\mathbb{A}$ which attracts all bounded sets.*

For a proof, see for example Billotti and LaSalle [2], Sell and You [25] or Hale [12]. One can further show that the attractor satisfies the following properties:

1. $\mathbb{A}$ is maximal in the sense that every compact invariant set in W lies in $\mathbb{A}$.
2. $\mathbb{A}$ is minimal in the sense that if B is a closed set which attracts each compact set in W, then $\mathbb{A} \subset B$.
3. For each bounded set $B \subset W$, the omega limit set $\omega(B)$ lies in $\mathbb{A}$.
4. $\mathbb{A}$ is connected.
5. $\mathbb{A}$ is Lyapunov stable, i.e., for every neighborhood V of $\mathbb{A}$ and every $\tau > 0$, there is a neighborhood U of $\mathbb{A}$ such that $S(t)U \subset V$ for all $t > \tau$.

Given our previous work, is is actually fairly straightforward to show that the semiflow on W is both compact and point dissipative.

Lemma 23. *Assume that the F_i, for $1 \le i \le N_1$ satisfy the boundedness and continuity properties and that the constraints are closed under the convergence of the compactness lemma. Then the semiflow $S(t)$ restricted to the set W is compact for $t > 0$, i.e., for each bounded set B in W and each $t > 0$, $S(t)B$ has compact closure.*

Proof. Fix $0 < \tau < 1$ and a bounded set $B \subset W$. We claim that $S(\tau)B$ has compact closure. We note that in order to show compactness, we only need to check sequential compactness, since W is a metric space. Since B is bounded, we have $B \subset B(M_0)$ for some $M_0 > 0$. Therefore, by lemma 18, for any $\phi \in B$ and any i, we have from (5.7) and (5.8)

$$\|S(\tau)\phi_i\|_\infty^2 \le \tau^{-1} M_0^2 + K_i$$

From (5.10) we get that for any m,

$$\int_m^{m+1} \|S(\tau)\phi_i\|_V^2 \le M_1(\tau + m, \tau + m + 1)^2$$

and using (5.11)

$$\int_m^{m+1} \|\partial_t S(\tau)\phi_i\|_{V^{-1}}^2 \le M_2(\tau + m, \tau + m + 1)^2.$$

Therefore we get that B is bounded in $L^2_{loc}(0, \infty; V)$ and $\partial_t B$ is bounded in $L^2_{loc}(0, \infty; V^{-1})$.

This implies that we can apply the compactness lemma to any sequence in $S(\tau)B$. After relabeling, we let $S(\tau)\phi^n$ denote the sequence and $\phi_\tau \in L^2_{loc}(0, \infty; \overline{V})$ the limit function. But property 3 of the compactness lemma implies that the $S(\tau)\phi^n$ converges in $L^2_{loc}(0, \infty; \overline{H})$, hence ϕ_τ is in the closure of $S(\tau)B$, which implies that its closure is compact. □

Lemma 24. *The restriction of the semiflow $S(t)$ to W is point dissipative.*

Proof. Let $U \subset W$ be defined by

$$U = \left\{ \phi : \int_m^{m+1} \|\phi_i\|^2 \, ds < 2K_i \text{ for } 1 \le i \le N_1 \text{ and } m = 0, 1, \dots \right\}.$$

Clearly U is bounded in W. We claim that U is an absorbing set. If we fix $\phi \in W$, then for some M_0, $\phi \in B(M_0)$. Then (5.10) implies that for all $m = 0, 1, \dots$ and i we have

$$\int_m^m +1\|S(\tau)\phi_i\|\, ds = \int_{m+\tau}^{m+1+\tau} \|\phi_i\|\, ds$$
$$\leq e^{-\sigma_i(m+\tau-1)}(M_0^2 + K_i) + K_i$$

Now fix τ_0 large, so that

$$e^{-\sigma_i(\tau-1)}(M_0^2 + K_i) < K_i.$$

Then for $\tau > \tau_0$ we have that $S(\tau)\phi \in U$. Hence $S(t)$ is point dissipative. □

Theorem 25 (Main Theorem). *Let $\partial_t\phi_i = F_i(\phi_1, \dots, \phi_N)$, $i = 1, \dots, N$ be a system of equations where the F_i satisfy the boundedness and continuity properties and suppose that the constraints G_k and G_l^* are closed under the convergence in the compactness lemma. Then the set of generalized weak solutions for the system has a global attractor in the set of weak solutions of Leray-Hopf class.*

Proof. The existence of the global attractor follows from theorem 22 and lemmas 23 and 24.

The fact that the attractor lies in W_{LH} follows from the fact that the attractor is invariant and lemma 21. □

We can then directly apply this theorem to the reacting flow system.

Corollary 26. *Assume the hypotheses of the existence theorem 1. The solutions to the chemically reacting flow system* (3.4) *described in that theorem posses a global attractor.*

Proof. Corollary 16 implies that the reacting flow system satisfies the continuity and boundedness properties, while lemma 17 implies that the physical reasonableness constraints are closed under the convergence in the compactness lemma. Furthermore, the existence theorem 1 describes solutions of weak Leray-Hopf class, in particular because of the estimates given in the theorem. Therefore theorem 25 holds and there exists a global attractor. □

6. Conclusion

It is appropriate at this point to compare our results to the previous work of Manley, Marion and Temam [20], see also Manley and Marion [19], Marion [21] and Marion and Temam [22]. While we have focused on the 3D problem, they consider the equations (3.4) with similar assumptions, but work on a two dimensional rectangular domain with specific boundary conditions, in particular the chemistry and temperature satisfy constant Dirichlet conditions on the inflow region. This in turn allows them to pick their homogenizing functions $w(x)$, $\theta_0(x)$, and $\eta_{i,0}(x)$ to be constant in space, which allows them to avoid most of the difficulties with the *a priori* estimates that we encounter in section 4.1. Additionally, since they are working in two dimensions, the time derivatives of their solutions are in $L^2_{loc}[0, \infty; V_i^{-1})$, as opposed to $L^{4/3}_{loc}[0, \infty; V_i^{-1})$. As a result, they are able to give a simpler proof of the existence of physically reasonable solutions for the 2D problem. They also show the existence of compact global attractors in the Hilbert space $L^2(\Omega)$ by showing the existence of an absorbing set in $H^1(\Omega)$. Furthermore, for the 2D problem they derive estimates for its fractal dimension. For the 3D problem, the phase space is no longer a Hilbert space and the issue of the attractor's dimension for the 3D problem is an open question.

7. Acknowledgments

The author wishes to thank Professor George Sell for many helpful discussions during the preparation of this paper.

References

[1] R A Adams, *Sobolev spaces*, Academic Press, Inc., San Diego, Calif., 1978.

[2] J E Billotti and J P LaSalle, *Dissipative periodic processes*, Bull Amer Math Soc **77** (1971), 1082–1088.

[3] J D Buckmaster and G S S Ludford, *Lectures on mathematical combustion*, CBMS-NSF regional conference series in applied mathematics, no. 43, SIAM, Philadelphia, PA, 1983.

[4] P Constantin and C Foias, *Navier-Stokes equations*, Univ. Chicago Press, Chicago, 1988.

[5] J Deny and J L Lions, *Les espaces du type de beppo-levi*, Amm. Inst. Fourier (Grenoble) **5** (1954), 305–370.

[6] D E Edmunds and W D Evans, *Spectral theory and differential operators*, Clarendon Press, Oxford, 1987.

[7] C Foias, O Manley, and R Temam, *Attractors for the Bénard problem: Existence and physical bounds on their fractal dimension*, Nonlinear analysis, Theory, Methods and Applications **11** (1987), no. 8, 939–967.

[8] C Foias and R Temam, *Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation*, Annali Scuola Norm. Sup. di Pisa **5** (1978), no. 1, 28–63.

[9] Hiroshi Fujita, *On the existence and regularity of the steady-state solutions of the Navier-Stokes theorem*, J. Fac. Sci. Univ. Tokyo Sect. I **9** (1961), 59–102.

[10] G P Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, vol. I, Springer-Verlag, New York, 1994.

[11] J-M Ghidaglia, *On the fractal dimension of attractors for viscous incompressible fluid flows*, SIAM J. Math. Anal. **17** (1986), no. 5, 1139–1157.

[12] J Hale, *Asymptotic behavior of dissipative systems*, vol. 25, American Math Soc, Providence R.I., 1988.

[13] J G Heywood, *Open problems in the theory of the Navier-Stokes equations for viscous incompressible flow*, The Navier-Stokes Equations (Oberwolfach, 1988), Lecture Notes in Math., vol. 1431, Springer, Berlin, 1990, pp. 1–22.

[14] E Hopf, *On nonlinear partial differential equations*, Lecture Series of the Symposium on Partial Differential Equations, Berkeley, 1955.

[15] A Lafon, *Borne sur la dimension de Hausdorff de l'attracteur pour les equations de Navier-Stokes a duex dimensions.*, C.-R.-Acad.-Sci.-Paris-Ser.-IMath **298** (1984), no. 18, 453–456.

[16] J Leray, *Études de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynmaique*, J. Math. Pures Appl **12** (1933), 1–82.

[17] J L Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Gauthier-Villar, Paris, 1969.

[18] J L Lions and E Magenes, *Non-homogeneous boundary value problems and applications*, vol. 1, Springer-Verlag, Berlin, 1973.

[19] O Manley and M Marion, *Attractor dimension for a simple premixed flame model*, Combust. Sci. and Tech. **88** (1992), 15–32.

[20] O Manley, M Marion, and R Temam, *Equations of combustion in the presence of complex chemistry*, Indiana University Mathematics Journal **42** (1993), no. 3, 941–967.

[21] M Marion, *Attractors and turbulence for some combustion models*, IMA Volume in Maths and its Applications, vol. 35, Springer, New York, 1991, pp. 213–228.

[22] M Marion and R Temam, *Some remarks on turbulent combustion from the attractor point of view*, Mathematical Modeling in Combustion and Related Topics, vol. 35, Martinus Nyhoff, Dordrecht, 1988, pp. 155–172.

[23] D E Norman, *Chemically reacting flows and CSTRs: The large diffusion limit*, submitted for publication, 1997.

[24] G R Sell, *Global attractors for the three-dimensional Navier-Stokes equations*, Journal of Dynamics and Differential Equations **8** (1996), no. 1, 1–33.

[25] G R Sell and Y You, *Dynamics of evolutionary equations*, Springer-Verlag, New York, (to appear).
[26] R Temam, *Navier-Stokes equations*, 3d rev. ed., North Holland, Amsterdam, 1984.
[27] F A Williams, *Combustion theory: the fundamental theory of chemically reacting flow systems*, 2nd ed., Benjamin/Cummings Pub. Co., Menlo Park, Calif., 1985.

Department of Mathematics, 127 Vincent Hall, University of Minnesota, Minneapolis MN 55455
E-mail address: norman@math.umn.edu