A Non–overlapping Additive Schwarz Method for the Biharmonic Equation

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Abstract

We present a non–overlapping additive Schwarz method for the Morley and the incomplete biquadratic nonconforming finite elements for the biharmonic equation. It is shown that the condition number of the preconditioned finite element system has an upper bound of the form $O(\max_{1\leq j\leq J}(H_j/h_j)^3)$, where H_j is the size of the jth subdomain Ω_j and h_j is the mesh size of the "triangulation" of Ω_j. Numerical results are also presented to validate the theory and to illustrate the efficiency of the method.

1 Introduction

The additive Schwarz method (ASM) refers to a general methodology, based on the idea of domain decomposition or divide–and–conquer, for solving system of linear algebraic equations resulting from finite element and other type discretizations of partial differential equations. A general abstract framework for the additive Schwarz method had been developed by Dryja, Widlund, Matsokin, Nepomnyaschikh and others during the late 80's and the early 90's. For the past several years the method has been successfully used to solve various application problems from science and engineering, see [6] and the references therein for detailed exposition.

In this paper we shall develop an additive Schwarz method for solving systems of equations resulting from nonconforming finite element discretizations of the biharmonic equation. The method is based on the idea of using non–overlapping subdomains and the fact that the interactions between the subdomains will take place only through a coarse space. The coarse space is constructed without explicitly introducing any coarse mesh, instead, it is defined as the range of some interpolation–like operator which is defined as a composition of two simple averaging operators. Consequently, one is free

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to use arbitrary, irregular subdomains. We like to emphasize that the action of the composite operator is local, so it is very cheap to compute the action. It is shown that the non–overlapping additive Schwarz method has a rate of convergence of the order $O((H/h)^{\frac{3}{2}})$ when combined with the Conjugate Gradient method.

The work of this paper is inspired by and closely related to the paper [1], where two non–overlapping additive Schwarz methods were developed for second order elliptic problems with discontinuous coefficients. For earlier works on two–level overlapping additive Schwarz methods using nonconforming plate elements, for the biharmonic equation, we refer to [2, 4, 7] and the references therein.

The paper is organized as follows. In Section 2, the biharmonic problem and its nonconforming Morley finite element approximation are recalled. In Section 3, the non–overlapping Schwarz method is proposed and analyzed for the Morley finite element. The construction of the coarse space as the range of an interpolation–like operator and the establishment of the approximation property of this operator are the two main components of this section. In Section 4, the non–overlapping additive Schwarz method is extended to use the incomplete biquadratic nonconforming finite element (cf. [5]). In Section 5, we present some numerical results to verify our theory and to illustrate the efficiency of the method.

2 Preliminaries

We consider the biharmonic problem

$$\Delta^2 u = f \qquad \text{in } \Omega, \tag{2.1}$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{2.2}$$

Where $\Omega \subset \mathbf{R}^2$ is a bounded domain and $f \in L^2(\Omega)$. This boundary value problem describes the bending of a clamped plate Ω under the distributed force f (cf. [3]).

The weak formulation of (2.1)–(2.2) is defined by seeking $u \in H_0^2(\Omega)$ such that

$$a_\Omega(u,v) = F(v) \qquad \forall v \in H_0^2(\Omega), \tag{2.3}$$

where

$$a_\Omega(u,v) = \int_\Omega [\ \Delta u \Delta v + (1-\sigma)\left(2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right)]\, dx, \tag{2.4}$$

$$F(v) = \int_\Omega f v \, dx, \tag{2.5}$$

and $0 < \sigma < \frac{1}{2}$ is the Poisson's coefficient of the plate.

Let $\mathcal{T}_h$ be a quasi–uniform triangulation of Ω and let V^h denote the Morley finite element space associated with $\mathcal{T}_h$. Then $v \in V^h$ if the following properties are satisfied:

(i) $v|_K \in P_2(K)$ for each triangle $K \in \mathcal{T}_h$.

(ii) v is continuous at the vertices of $\mathcal{T}_h$.

(iii) $\frac{\partial v}{\partial n}$ is continuous at the midpoints of the sides of $\mathcal{T}_h$.

(iv) v vanishes at the vertices along $\partial\Omega$, and $\frac{\partial v}{\partial n}$ vanishes at the midpoints of the sides along $\partial\Omega$.

Clearly, $V^h \not\subset C^0(\overline{\Omega})$, so the Morley element is a strongly nonconforming plate element. The finite element solution $u \in V^h$ is defined as the solution of

$$a^h(u,v) = F(v) \qquad \forall v \in V^h, \tag{2.6}$$

where

$$a^h(u,v) = \sum_{K\in\mathcal{T}_h} a_K(u,v) \tag{2.7}$$

and $a_K(u,v)$ is defined by (2.4).

Define for any $v \in V^h$

$$||v||_{h,m,\Omega} = \left(\sum_{K\in\mathcal{T}_h, K\subset\Omega} |v|^2_{H^m(K)} \right)^{\frac{1}{2}}. \tag{2.8}$$

It is well–known (see [3]) that $||v||_{h,2,\Omega}$ is a norm on V^h and that there exists two positive constants C_1 and C_2 such that

$$C_1 ||v||^2_{h,2,\Omega} \le a^h(v,v) \le C_2 ||v||^2_{h,2,\Omega}, \quad \forall v \in V^h. \tag{2.9}$$

Let $N = N(h)$ denote the dimension of the Morley finite element space V^h, clearly $N = O(h^{-2})$. Using the basis functions of the space V^h the finite element problem (2.6) reduces to an $N \times N$ linear system

$$A\mathbf{x} = \mathbf{b}, \tag{2.10}$$

where the coefficient matrix $A \in \mathbf{R}^{N\times N}$, also called the stiffness matrix, is symmetric and positive definite.

It is well–known that the (2–norm) condition number of A (cond(A)) is of the order $O(h^{-4})$ (cf. [2, 4] and the references therein). So the finite element system becomes ill–conditioned for small mesh size h. Consequently, it is not efficient to solve the linear system (2.10) directly using the classical iterative methods. On the other hand, if one can find a good preconditioner B, a symmetric positive definite $N\times N$ matrix, for the stiffness matrix A such that BA is well–conditioned, then any of the classical iterative methods (in particular, the Conjugate Gradient method) works effectively on the preconditioned system

$$BA\mathbf{x} = B\mathbf{b}. \tag{2.11}$$

The goal of this paper is to develop an additive Schwarz preconditioner, based on non–overlapping domain decomposition, for the finite element problem (2.6) and to solve the preconditioned system using the Conjugate Gradient method. For background knowledge and a general theory on the additive Schwarz method, we refer to the recent book by Smith, Bjørstad and Gropp [6].

3 The non–overlapping additive Schwarz method

3.1 A space decomposition

Let $\{\Omega_j\}_{j=1}^J$ be a non–overlapping partion of Ω such that Ω_j consists of a cluster of finite elements (triangles) of $\mathcal{T}_h$. Introduce the notation

$$\begin{aligned}
\mathcal{N}^h &= \{\text{The set of all vertices of } \mathcal{T}_h\}. \\
\mathcal{M}^h &= \{\text{The set of midpoints of all edges of } \mathcal{T}_h\}. \\
\Omega_h &= \Omega \cap \mathcal{N}^h, \qquad \partial\Omega_h = \partial\Omega \cap \mathcal{N}^h. \\
\Omega_h^* &= \Omega \cap \mathcal{M}^h, \qquad \partial\Omega_h^* = \partial\Omega \cap \mathcal{M}^h. \\
\Omega_{jh} &= \Omega_j \cap \mathcal{N}^h, \qquad \partial\Omega_{jh} = \partial\Omega_j \cap \mathcal{N}^h. \\
\Omega_{jh}^* &= \Omega_j \cap \mathcal{M}^h, \qquad \partial\Omega_{jh}^* = \partial\Omega_j \cap \mathcal{M}^h. \\
\overline{v}_j &= \{\text{The arithmetic average of } v \text{ values over all vertices on } \partial\Omega_j\}. \\
h_j &= \max_{K\subset\Omega_j} \operatorname{diam}(K), \qquad H_j = \operatorname{diam}(\Omega_j).
\end{aligned}$$

Let us define the subspace V_j^h corresponding to the subdomain Ω_j by

$$V_j^h = \{v \in L^2(\Omega);\ v|_{\Omega_j} \in \tilde{V}_j^h,\ v = 0 \text{ for } x \notin \Omega_j\}, \tag{3.1}$$

with $\tilde{V}_j^h$ defined as

$$\tilde{V}_j^h = V^h|_{\Omega_j} \cap \{v;\ v(x) = 0 \text{ for } x \in \partial\Omega_{jh},\ \frac{\partial v}{\partial n_j}(x) = 0 \text{ for } x \in \partial\Omega_{jh}^*\}. \tag{3.2}$$

Clearly,

$$V_j^h \subset V^h, \quad j = 1, 2, \cdots, J. \tag{3.3}$$

To construct a valid decomposition of V^h, we need to find another subspace V_0^h, called the “coarse space”, such that

$$V^h = V_0^h + V_1^h + \cdots + V_J^h \tag{3.4}$$

holds. In the following, we shall define our coarse space V_0^h using the idea given in [1], that is, to define V_0^h as the range of some averaging operator I_A. This averaging operator, which we now present here, is constructed as a composition of two simple interpolation–like operators. To give the definitions of the operators, we need to introduce a few more notations.

$$\begin{aligned}
\mathcal{S}_1(\Omega_j) &= \{K;\ K \subset \Omega_j,\ \text{exactly one edge of } K \text{ lies on } \partial\Omega_j\}.\\
\mathcal{S}_2(\Omega_j) &= \{K;\ K \subset \Omega_j,\ \text{exactly two edges of } K \text{ lie on } \partial\Omega_j\}.\\
\mathcal{G}_0(\Omega_j) &= \{K;\ K \subset \Omega_j,\ \partial K \cap \partial\Omega_j = \emptyset\}.\\
\mathcal{G}_1(\Omega_j) &= \{K;\ K \subset \Omega_j,\ \text{exactly one vertex of } K \text{ lies on } \partial\Omega_j\}.\\
\mathcal{G}_2(\Omega_j) &= \{K;\ K \subset \Omega_j,\ \text{exactly two vertices of } K \text{ lie on } \partial\Omega_j,\\
&\qquad \text{meas}(\partial K \cap \partial\Omega_j) = 0\}.
\end{aligned}$$

We call a triangle a "boundary" element if it has at least one vertex on a subdomain boundary, all other elements are termed as the "interior" elements. Accordingly, the set $\mathcal{G}_0(\Omega_j)$ above contains interior elements, whereas, all other sets contain elements that are boundary elements. Let

$$W(\Omega) = \{v;\ v|_K \in P_2(K)\ \ \forall K \in \mathcal{T}_h,\ v(x) = 0\ \ \forall x \in \partial\Omega_h,\ \frac{\partial v}{\partial n}(x) = 0\ \ \forall x \in \partial\Omega_h^*\}.$$

The first operator, $\widehat{I}_A$: $V^h \to W(\Omega)$, is defined as follows in two steps. For any $v \in V^h$, $\widehat{I}_A v$ on each Ω_j $(j = 1, 2, \cdots, J)$ is defined by

(i).

$$(\widehat{I}_A v)(x) = \begin{cases} v(x), & \text{if } x \in \partial\Omega_{jh}, \\ \overline{v}_j, & \text{if } x \in \Omega_{jh}. \end{cases} \tag{3.5}$$

(ii). $\widehat{I}_A v$ is extended to all of Ω_j by interpolating its nodal values. Here a linear interpolation is used in each element $K \in \mathcal{G}_0(\Omega_j) \cup \mathcal{G}_2(\Omega_j)$, a bilinear interpolation in each boundary element $K \in \mathcal{G}_1(\Omega_j) \cup \mathcal{S}_1(\Omega_j)$, and a Morley quadratic interpolation in each boundary "corner" element $K \in \mathcal{S}_2(\Omega_j)$. More precisely,

$$(\widehat{I}_A v)(x) = \begin{cases} \overline{v}_j, & x \in K,\ K \in \mathcal{G}_0(\Omega_j), \\ \ell_K(x), & x \in K,\ K \in \mathcal{G}_2(\Omega_j), \\ q_K(x), & x \in K,\ K \in \mathcal{G}_1(\Omega_j) \cup \mathcal{S}_1(\Omega_j), \\ p_K(x), & x \in K,\ K \in \mathcal{S}_2(\Omega_j), \end{cases} \tag{3.6}$$

where ℓ_K denotes the P_1 polynomial which interpolates $\widehat{I}_A v$ at the three vertices of K. q_K denotes the bilinear function satisfying

$$q_K(x) = (\hat{I}_A v)(x), \quad x \in \partial K \cap \mathcal{N}^h,\ K \in \mathcal{G}_1(\Omega_j) \cup \mathcal{S}_1(\Omega_j), \tag{3.7}$$

$$\frac{\partial q_K}{\partial n_K}(x) = \begin{cases} \frac{\partial v}{\partial n_j}(x), & x \in \partial K \cap \partial\Omega^*_{jh},\ K \in \mathcal{S}_1(\Omega_j), \\ 0, & x \in e_K \cap \Omega^*_{jh},\ K \in \mathcal{G}_1(\Omega_j), \end{cases} \tag{3.8}$$

where e_K denotes the edge of $K \in \mathcal{G}_1(\Omega_j)$ that does not intersect $\partial\Omega_j$. p_K is the P_2 function which satisfies the following conditions.

$$p_K(x) = (\hat{I}_A v)(x), \quad x \in \partial K \cap \mathcal{N}^h,\ K \in \mathcal{S}_2(\Omega_j), \tag{3.9}$$

$$\frac{\partial p_K}{\partial n_K}(x) = \begin{cases} \frac{\partial v}{\partial n_j}(x), & x \in \partial K \cap \partial\Omega^*_{jh},\ K \in \mathcal{S}_2(\Omega_j), \\ \frac{\partial r_{K'}}{\partial n_K}(x), & x \in \partial K \cap \partial K' \cap \Omega^*_{jh},\ K \in \mathcal{S}_2(\Omega_j), \end{cases} \tag{3.10}$$

where K' denotes the triangle in $\mathcal{G}_2(\Omega_j) \cup \mathcal{S}_1(\Omega_j)$, which shares an edge with $K \in \mathcal{S}_2(\Omega_j)$, and $r_{K'}$ represents either $\ell_{K'}$ or $q_{K'}$ depending on whether K' belongs to $\mathcal{G}_2(\Omega_j)$ or $\mathcal{S}_1(\Omega_j)$ respectively. Step (ii) of the definition of $\hat{I}_A$ ends here.

Since the normal derivative of the function $\hat{I}_A v$ may not be continuous at the midpoints of those edges of boundary triangles that intersect $\partial\Omega_j$, $\hat{I}_A v$ does not belong to V^h in general. Hence, we need an extra step to approximate $\hat{I}_A v$ by a function in V^h. This will be done by using an "averaging operator" for the Morley element (cf. [2, 4]).

Define the second operator $\hat{J}_A : W(\Omega) \to V^h$ such that for any $w \in W(\Omega)$, $\hat{J}_A w$ satisfies

$$(\hat{J}_A w)(x) = \begin{cases} w(x), & \forall x \in \Omega_h, \\ 0, & \forall x \in \partial\Omega_h, \end{cases} \tag{3.11}$$

$$\frac{\partial(\hat{J}_A w)}{\partial n_K}(x) = \begin{cases} R\left[\frac{\partial w}{\partial n_j}(x)\right], & \forall x \in \partial K \cap \Omega^*_h,\ K \in \mathcal{T}_h, \\ 0, & \forall x \in \partial K \cap \partial\Omega^*_h,\ K \in \mathcal{T}_h. \end{cases} \tag{3.12}$$

Where $R[\cdot]$ denote the "averaging operator" or the "random choice operator" (cf. [4]).

Now we define the operator I_A as the composite operator of $\hat{J}_A$ and $\hat{I}_A$, that is,

$$I_A = \hat{J}_A \circ \hat{I}_A. \tag{3.13}$$

Let

$$V^h_0 = \{I_A v;\ \forall v \in V^h\}. \tag{3.14}$$

Clearly, V^h_0 is a subspace of V^h.

Remark 3.1 *Although the operator $\widehat{J}_A$ is defined globally over Ω, the action $I_Av = \widehat{J}_A(\widehat{I}_Av)$ can be computed independently in each subdomain Ω_j. In fact, the action of $\widehat{J}_A$ on $\widehat{I}_Av$ is computed only in the boundary finite elements (i.e. triangles having at least one vertex on $\partial\Omega_j$) of each Ω_j, since $\widehat{I}_Av$ is constant in the interior finite elements (triangles) of Ω_j.*

Lemma 3.1 *The following space decomposition of V^h exists*

$$V^h = V_0^h + V_1^h + \cdots + V_J^h. \tag{3.15}$$

Proof: For any $v \in V^h$, let $w = v - I_Av$. Then

$$w \in V_1^h + V_2^h \cdots + V_J^h.$$

Hence, there exists $v_j \in V_j^h$ for $j = 1, 2, \cdots, J$ such that

$$w = v_1^h + v_2^h \cdots + v_J^h.$$

Therefore,

$$v = v_0 + v_1 + \cdots + v_J, \qquad v_0 = I_Av \in V_0^h.$$

Lemma 3.2 *There is a positive constant C, which only depends on the quasi–uniformity constant (the minimal angle) of the triangulation $\mathcal{T}_h$, such that*

$$||\widehat{J}_Aw||_{h,2,\Omega} \le C||w||_{h,2,\Omega} \qquad \forall w \in W(\Omega). \tag{3.16}$$

Proof: The proof is a straightforward application of the scaling argument. See §3 of [4] for a detailed proof of similar type.

Lemma 3.3 *There exists a constant $C > 0$ such that for any $v \in V^h$,*

$$||v - \widehat{I}_Av||_{0,K}^2 \le Ch_j^2||v||_{h,1,\Omega_j}^2 \qquad \forall K \subset \Omega_j. \tag{3.17}$$

Proof: For each fixed $K \subset \Omega_j$, by the definition of $\widehat{I}_A$, we know that $P_0(\Omega_j)$ is contained in the kernel of the linear operator $(\mathcal{I} - \widehat{I}_A)|_{\Omega_j}$ which maps $V^h|_{\Omega_j}$ into $P_2(K)$, here $\mathcal{I}$ denotes the identity operator. Clearly, $||v||_{h,1,\Omega_j}$ is a norm in the quotient space $(V^h|_{\Omega_j})/P_0(\Omega_j)$. The proof is completed by using the Bramble–Hilbert Lemma (cf. [3]), and the following scaling argument,

$$\widetilde{K} = \{\tilde{x} := x/h_j;\ x \in K\},\ \widetilde{\Omega}_j = \{\tilde{x} := x/h_j;\ x \in \Omega_j\},\ \ \tilde{v}(\tilde{x}) := v(h_j\tilde{x}),$$

as well as the facts

$$||\widetilde{(\mathcal{I} - \widehat{I}_A)\tilde{v}}||_{0,\widetilde{K}} = h_j^{-1}||(\mathcal{I} - \widehat{I}_A)v||_{0,K}, \qquad ||\tilde{v}||_{h,1,\widetilde{\Omega}_j} = ||v||_{h,1,\Omega_j}.$$

Corollary 3.1 *Summing (3.17) over all triangles in Ω_j gives*

$$||v-\widehat{I}_A v||^2_{0,\Omega_j} \le CH_j^2||v||^2_{h,1,\Omega_j} \qquad \forall v\in V^h. \tag{3.18}$$

Here we have used the fact that $\sum_{K\subset\Omega_j} = O(\frac{H_j^2}{h_j^2})$.

From Lemmas 3.2 and 3.3 we have the following lemma.

Lemma 3.4 *There exists a constant $C>0$ such that*

$$||I_A v||^2_{h,2,\Omega} \le C\sum_{j=1}^{J}\left(\frac{H_j}{h_j}\right)^3 ||v||^2_{h,2,\Omega_j} \qquad \forall v\in V^h. \tag{3.19}$$

Proof: For any $v\in V^h$, let v^I denote the continuous piecewise linear interpolation of v, which interpolates v at all vertices of $\mathcal{T}_h$, and let Ω_j' denote the union of all boundary triangles in Ω_j. From Lemmas 3.2 and 3.3, the inverse inequality and the Poincaré inequality we have

$$||I_A v||^2_{h,2,\Omega} \le C||\widehat{I}_A v||^2_{h,2,\Omega} = C\sum_{j=1}^{J}||\widehat{I}_A v||^2_{h,2,\Omega_j}. \tag{3.20}$$

$$\begin{aligned} ||\widehat{I}_A v||^2_{h,2,\Omega_j} &= ||\widehat{I}_A v||^2_{h,2,\Omega_j'} \\ &= ||\widehat{I}_A v - v^I||^2_{h,2,\Omega_j'} \\ &\le Ch_j^{-4}\sum_{K\subset\Omega_j'}||\widehat{I}_A v - v^I||^2_{0,K} \\ &\le Ch_j^{-4}\sum_{K\subset\Omega_j'}[||\widehat{I}_A v - v||^2_{0,K} + ||v-v^I||^2_{0,K}] \\ &\le Ch_j^{-4}\sum_{K\subset\Omega_j'}[h_j^2||v||^2_{h,1,\Omega_j} + h_j^4|v|^2_{H^2(K)}] \\ &\le C\left(\frac{H_j}{h_j}\right)^3||v||^2_{h,2,\Omega_j}, \end{aligned} \tag{3.21}$$

where we have used that facts that $\sum_{K\subset\Omega_j'} = O(\frac{H_j}{h_j})$ and $H_j/h_j>1$. The proof is completed by using the inequality (3.21) in (3.20).

3.2 The additive Schwarz preconditioner

To formulate our additive Schwarz preconditioner, we introduce bilinear forms $a_j(\cdot,\cdot)$ on $V_j^h\times V_j^h$ of the form

$$a_j(u,v) = a^h(u,v), \quad j=0,1,2,\cdots,J. \tag{3.22}$$

Remark 3.2 *The integration in $a_0(u,v)$ is computed only on the finite elements which touch the subdomain boundaries $\bigcap_{j=1}^{J} \partial\Omega_j$. Hence, $a_0(u,v)$ is cheap to compute.*

Define the additive operator

$$T = T_0 + T_1 + \cdots + T_J, \tag{3.23}$$

where

$$a_j(T_j u, v) = a^h(u,v) \qquad \forall v \in V_j^h,\ j = 0, 1, \cdots, J. \tag{3.24}$$

Following the framework given in [6], the additive Schwarz method is to replace the discrete problem (2.6) by the equation

$$Tu = g, \tag{3.25}$$

where $g = \sum_{j=0}^{J} g_j$, and $g_j = T_j u$ is defined as the solution of

$$a_j(g_j, v) = F(v) \qquad \forall v \in V_j^h,\ j = 0, 1, \cdots, J. \tag{3.26}$$

We are now ready to state the main result of this paper.

Theorem 3.1 *The additive operator T, defined by (3.23), is self-adjoint and positive definite in V^h with the inner product $a^h(\cdot,\cdot)$. Moreover, there exists a positive constant C, which is independent of both H_j and h_j, such that the following estimates hold*

$$\lambda_{\max}(T) \le 2, \qquad \lambda_{\min}(T) \ge \frac{1}{C} \max_{1 \le j \le J} \left(\frac{h_j}{H_j}\right)^3, \tag{3.27}$$

and hence

$$\mathrm{cond}\ (\mathrm{T}) = \frac{\lambda_{\max}}{\lambda_{\min}} \le 2\mathrm{C} \max_{1 \le \mathrm{j} \le \mathrm{J}} \left(\frac{\mathrm{H_j}}{\mathrm{h_j}}\right)^3. \tag{3.28}$$

Proof: Using the general convergence theory of Additive Schwarz methods given in [6], we need to check three key assumptions. Due to the choices of the space decomposition and the exact subdomain solvers $a_j(\cdot,\cdot)$, it is easy to see that Assumption 2 holds with $\mathcal{E}$ being the $J \times J$ identity matrix, hence, $\rho(\mathcal{E}) = 1$; and Assumption 3 holds with $\omega = 1$ (see [6] for the notations). Finally, to check Assumption 1, for any $u \in V^h$, let $u_0 = I_A u$ and $u_j = (u - u_0)|_{V_j^h}$, then $u_j \in V_j^h$ and

$$u = u_0 + u_1 + u_2 + \cdots + u_J.$$

Now using this decomposition and Lemma 3.4 we get

$$\begin{aligned}\sum_{j=0}^{J} a_j(u_j,u_j) &= \sum_{j=1}^{J} a^h(u-u_0,u-u_0)|_{\Omega_j} + a_0(u_0,u_0)\\ &= a^h(u-u_0,u-u_0)+a^h(u_0,u_0)\\ &\le 2a^h(u,u)+3a^h(u_0,u_0)\\ &\le C[||u||^2_{h,2,\Omega}+||I_Au||^2_{h,2,\Omega}]\\ &\le C\max_{1\le j\le J}\left(\frac{H_j}{h_j}\right)^3 a^h(u,u).\end{aligned}$$

Hence, Assumption 1 holds with

$$C_0^2 = C\max_{1\le j\le J}\left(\frac{H_j}{h_j}\right)^3. \tag{3.29}$$

The proof is completed by applying Lemma 3 of §5.2 of [6].

4 The non–overlapping additive Schwarz method for the incomplete biquadratic element

In this section, we shall briefly explain how to extend the additive Schwarz method of the previous sections to use, instead of the Morley element, its rectangular version, the incomplete biquadratic plate element (cf. [5]).

Let $\mathcal{T}_h$ now denote a quasiuniform rectangular partition of Ω and V^h denote the incomplete biquadratic finite element space associated with $\mathcal{T}_h$. We recall that a function $v\in V^h$ if it satisfies the following conditions:

(i) $v|_K \in P(K) \equiv P_2(K)\cup\{x_1^2x_2, x_1x_2^2\} \quad \forall K\in\mathcal{T}_h$.

(ii) v is continuous at the vertices of $\mathcal{T}_h$.

(iii) $\frac{\partial v}{\partial n}$ is continuous at the midpoints of the sides of $\mathcal{T}_h$ or the mean value of $\frac{\partial v}{\partial n}$ is continuous across each side of $\mathcal{T}_h$.

(iv) v vanishes at the vertices of the sides along $\partial\Omega$; $\frac{\partial v}{\partial n}$ vanishes at the midpoints of the sides along $\partial\Omega$ or the mean value of $\frac{\partial v}{\partial n}$ on each side along $\partial\Omega$ vanishes.

It is easy to check that $V^h \not\subset C^0(\overline{\Omega})$. Hence, like the Morley element, the incomplete biquadratic element is also a strongly nonconforming plate element.

Due to the similarity between the Morley element and the incomplete biquadratic element, the non–overlapping additive Schwarz method for the incomplete biquadratic element becomes quite similar to that for the Morley element developed in Section 2. Only a small modification in the construction of $\widehat{I}_A$ is needed. Note that we have here elements of type $\mathcal{G}_0, \mathcal{S}_1$ and $\mathcal{S}_2$ only. In order to define $\widehat{I}_A$ appropriately, we use the same definition as is given in section 3, but by replacing step (ii) completely with the following. $\widehat{I}_A v$ is now extended to the whole Ω by using, in each Ω_j, piecewise bilinear interpolation in the interior rectangles and piecewise quadratic interpolation in the boundary rectangles. That is,

$$(\widehat{I}_A v)(x) = \begin{cases} \overline{v}_j, & x \in K,\ K \in \mathcal{G}_0(\Omega_j), \\ p_K(x), & x \in K,\ K \in \mathcal{S}_1(\Omega_j) \cup \mathcal{S}_2(\Omega_j), \end{cases} \tag{4.1}$$

where p_K is a P_2 function satisfying

$$p_K(x) = (\widehat{I}_A v)(x), \quad x \in \partial K \cap \mathcal{N}^h,\ K \in \mathcal{S}_1(\Omega_j) \cup \mathcal{S}_2(\Omega_j), \tag{4.2}$$

$$\frac{\partial p_K}{\partial n_K}(x) = \begin{cases} \frac{\partial v}{\partial n_j}(x), & x \in \partial K \cap \partial\Omega^*_{jh},\ K \in \mathcal{S}_1(\Omega_j) \cup \mathcal{S}_2(\Omega_j), \\ 0, & x \in e_K \cap \Omega^*_{jh},\ K \in \mathcal{S}_1(\Omega_j), \end{cases} \tag{4.3}$$

where e_K denotes the edge of $K \in \mathcal{S}_1(\Omega_j)$ that does not intersect $\partial\Omega_j$.

The operator $\widehat{J}_A$: $W(\Omega) \to V^h$, is kept unchanged and so is I_A and V_0^h, see (3.11)–(3.14) for their definitions.

The main result of this section is given by the following theorem.

Theorem 4.1 *The conclusions of all lemmas and remarks in the previous section still hold for the new operators $\widehat{I}_A$, $\widehat{J}_A$ and I_A for the incomplete biquadratic element. Therefore, the conclusion of Theorem 3.1 is also valid for the incomplete biquadratic element.*

5 Numerical experiments

In this section, we shall present some numerical results showing the performance of our new non–overlapping additive Schwarz preconditioner using the Morley finite element. The objectives here are to validate our theory and to demonstrate the effectiveness of this new preconditioner for the biharmonic problem by comparing the performances of the Conjugate Gradient method with and without the preconditioner, in solving the Morley finite element equations.

We consider the biharmonic problem of (2.1) with Ω defined as the square domain $[-1,1]^2$. As for the boundary conditions we choose

$$u = \sin(\frac{\pi}{2}x)\ \sin(\frac{\pi}{2}y), \qquad \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega. \tag{5.1}$$

This is different from (2.2) in the sense that we are now letting u to be nonzero on the boundary. Non-homogeneous boundary conditions will make no difference in our theory.

The force function is so chosen that the solution of the problem will exactly equal the expression for u in (5.1). We shall be mentioning another problem later in connection with Table 2, and hence for the sake of clarity we call this one as our model problem A.

Triangulation of the domain Ω was done by first generating a rectangular grid using equidistant grid lines parallel to the sides of the domain. For the purpose of our tests we let equal number of intervals along each side of the domain and denote that number by n. A regular triangulation is then created by dividing each rectangular block into a pair of triangles. The overall domain is partitioned into d^2 nonoverlapping rectangular subdomains each containing equal number of elements. Note that the mesh sizes h and H are now inversely proportional to the numbers n and d, respectively. Consequently, keeping $\frac{n}{d}$ constant will simply give a constant H/h.

At this stage of building a non-overlapping preconditioner for the biharmonic problem, we are only interested in the numerical property of the preconditioner, the algorithm has therefore been implemented in the Matlab environment. So, much of the features like parallelism, which is inherent in the method, and optimal computational speed etc. were not considered when implementing. In all experiments presented here, we use exact solver for both the subdomain problem and the coarse problem, for convenience. The CG iteration stops whenever $\| r \|_2 \; / \; \| b \|_2 < \; 10^{-6}$ is reached. In all tables of this section both n and d are used to represent respectively the number of intervals and the number of subdomains along any side of the rectangular domain.

# of intervals (n) along each side	Morley FEM			
	No preconditioning		Non-overlap ASP	
12	138	(4053.7)	31	(107.6)
24	520	(64145.3)	60	(972.9)
36	1114	(303254.0)	90	(3393.3)
48	1832	(541962.0)	122	(8179.9)

Table 1: *Iteration counts and condition number estimates (in bracket) using Conjugate Gradient method on model problem A. Precondtioning is done with our Non–overlapping Additive Schwarz preconditioner using* 3^2 *subdomains.*

Table 1 shows the performance of the CG method, on problem A, using the new non-overlapping preconditioner compared to that of using no preconditioner. The condition number estimate for $n = 12$ is already big for no preconditioning and it grows bigger as the problem size n increases. Very soon, it becomes too big for any computer to handle, using only single precision floating point operations. By using the new preconditioner we have managed to obtain a better result. For this case the condition number estimates are also increasing with n but with a smaller pace. Of course, we have used only a fixed number of subdomains (3^2 subdomains) for all values of n in the table, these estimates can in fact be made much smaller if we increase the number of subdomains. For instance, consider $n = 36$, increasing the number of subdomains from 3^2 to 9^2 reduces the condition number estimate from 3393.3 to 798.3.

One feature of the method, that the theory predicts in (3.28), is that if the subdomain size H is kept fixed then the condition number of the preconditioned system should change with a rate of $O((1/h)^3)$. The behavior of the condition number estimates from Table 1 is reasonably close enough to what we may use as an argument to conclude that the results do comply with the above statement.

# of interv. (n) along each side	# of subd. (d) along each side	Non-overlap ASP			
		Problem A		Problem B	
4	2	9	(7.5)	9	(7.5)
6	3	18	(10.5)	19	(10.5)
12	6	25	(13.3)	29	(13.4)
24	12	25	(15.0)	33	(15.4)
48	24	26	(15.3)	37	(16.4)
60	30	26	(15.3)	38	(16.5)

Table 2: *Iteration count and condition number estimate (in bracket) using Conjugate Gradient method with Non–overlapping Additive Schwarz preconditioner using d^2 subdomains. The results here represent a constant H/h corresponding to $\frac{n}{d} = 2$.*

Another feature, that we wish to examine here, is the behavior of the condition number of the preconditioned system as the ratio H/h is kept constant, which according to (3.28) should remain constant. This is true for $\frac{n}{d} = 2$ as seen from Table 2. Beside our model problem A, the table now includes another test problem, we call it model problem B. This time, we consider (2.1) and the homogeneous boundary conditions (2.2) with $\Omega = [0, 1]^2$, and we choose the force function so that the exact solution becomes

$$u = x^2 y^2 (x-1)^2 (y-1)^2. \tag{5.2}$$

For both model problems A and B the results from Table 2 indicate that the condition number estimates tend to stabilize near a constant number as the problem size n increases.

The situation is not so clear for other values of $\frac{n}{d}$ as it is for the value 2. Table 3 shows results from two different values of H/h corresponding to $\frac{n}{d} = 3$ and 4 respectively, using model problem A. Here the condition number estimates don't seem to keep constant for constant H/h, we haven't been able to understand this discrepancy properly.

Yet, if we look carefully at the ratios of the consecutive estimates for any constant H/h in the table we find that the ratios are getting smaller as n is increasing, in other words the estimates are coming closer. Recall that this was also the case for $\frac{n}{d} = 2$ but there the effect was more significant, and visible even for small n. By looking at the tables and after running several other tests, we believe that the results here simply reflect the asymptotic behavior of the finite element theory. However, it is difficult to

# of subd. (d) along each side	$n = 3 \times d$			$n = 4 \times d$		
			ratio			ratio
8	66	(193.5)		109	(635.9)	
10	77	(290.2)	1.4997	130	(979.7)	1.5407
12	89	(408.2)	1.4066	152	(1399.7)	1.4287
14	100	(547.6)	1.3415	172	(1895.8)	1.3544
16	110	(708.2)	1.2933	191	(2468.1)	1.3019
18	121	(890.3)	1.2571	210	(3116.7)	1.2628
20	132	(1093.8)	1.2286	229	(3841.5)	1.2326
22	142	(1318.6)	1.2055	248	(4642.5)	1.2085

Table 3: *Iteration count and condition number estimate (in bracket) using Conjugate Gradient method with Non–overlapping Additive Schwarz preconditioner using d^2 subdomains. Ratios of consecutive condition number estimates for constant H/h (corresponding to $\frac{n}{d}$ = 3 and 4) are shown in the third column of each $n = 3 \times d$ and $4 \times d$.*

tell, just from Table 3, whether we shall see any convergence of those numbers if we let n grow indefinitely. Also, there is no reason to believe that the condition numbers will actually converge monotonously. So far, we haven't been able to run large enough problems to see the situation. It is however impossible to increase the size of a problem indefinitely on any present day computer as they are limited by their memory and speed.

6 Conclusion

An additive Schwarz method with no subdomain overlap, for solving the biharmonic equation, is proposed and analyzed in this paper. The method is easy to implement and flexible to use, because it requires no overlap among the subdomains and imposes no regularity condition on the subdomains. We have shown that the condition number of our preconditioned finite element system has an upper bound of the form $O(\max_{1\leq j\leq J}(H_j/h_j)^3)$. An immediate consequence of this result is that if the Conjugate Gradient method is used to solve the preconditioned system (3.25), then the method will converge with a rate proportional to $\max_{1\leq j\leq J}(H_j/h_j)^{\frac{3}{2}}$. We also remark that the upper bound on the condition number of our method is one order smaller than those of the *overlapping* additive Schwarz methods with minimal overlap, developed in [2] and [4]. Due to the non–overlapping feature, the new method is both cheaper to use and more convenient for parallel implementation than its overlapping counterparts. Moreover, the new method can easily be generalized to solve other fourth order problems with variable, discontinuous coefficients.

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