# EIGENVALUES OF ELLIPTIC BOUNDARY VALUE PROBLEMS WITH AN INDEFINITE WEIGHT FUNCTION

By

Jacqueline FLECKINGER[+,*] and Michel L. LAPIDUS[++,**]

[+] Université Paul Sabatier
Département de Mathématiques 1R2
118, route de Narbonne
31062, Toulouse, Cédex, France

[++] University of Southern California
Department of Mathematics DRB 306
University Park - MC-1113
Los Angeles, CA 90089-1113, USA

**1980 Mathematics Subject Classification.** Primary 35P20, 35J20, 35J25, 35J35, 35J40, 35P15; Secondary 34B25, 35J10, 35P30, 47A70, 49G05.

**Key words and phrases.** Elliptic boundary value problems, indefinite weight function, asymptotic behavior of eigenvalues, lower bounds of eigenvalues, spectral theory, variational methods, operators of Schrödinger type.

---

* Research partially supported by the CNRS and Argonne National Laboratory.

** Work supported in part by a Grant from the USC Faculty Research and Innovation Fund and Argonne National Laboratory. The second author would like to dedicate his research in this work to Odile.

# EIGENVALUES OF ELLIPTIC BOUNDARY VALUE PROBLEMS WITH AN INDEFINITE WEIGHT FUNCTION

By

Jacqueline FLECKINGER* and Michel L. LAPIDUS**

**ABSTRACT.** We consider general self-adjoint elliptic eigenvalue problems (P) $Au = \lambda r(x)u$, in an open set $\Omega \subset \mathbb{R}^k$. Here, the "weight" r is a function which changes sign in $\Omega$ and is allowed to be discontinuous. A scalar $\lambda$ is said to be an eigenvalue of (P) if $Au = \lambda ru$ - in the variational sense - for some non zero u satisfying the appropriate growth and boundary conditions. In the case when $\Omega$ is bounded, we assume Dirichlet or Neumann boundary conditions and, when $\Omega = \mathbb{R}^k$, $A$ is an operator of "Schrödinger type". In both situations, we determine the asymptotic behavior of the eigenvalues of (P), under suitable assumptions. When $\Omega$ is bounded, we also give lower bounds for the eigenvalues of the Dirichlet problem.

**1. Introduction.** The purpose of this paper is to provide estimates on the eigenvalues of "right non definite" elliptic boundary value problems: $Au = \lambda ru$ in $\Omega \subset \mathbb{R}^k$ (with Dirichlet or Neumann boundary conditions) when the weight function r is "indefinite", (i.e., changes sign in $\Omega$).

The main results concern the asymptotics of the eigenvalues; for self-adjoint elliptic operators of order 2m, when the open set $\Omega$ is bounded, we extend the classical Weyl's formula; when $\Omega = R^k$, we then generalize its analogue - also called de Wet Mandl formula - for operators of Schrödinger type.
We also obtain lower bounds for the eigenvalues of the Dirichlet problem. These results extend the earlier ones obtained independently by Lapidus [La 1, 2] and Fleckinger-El Fetnassi [FlF 1, 2].

Many nonlinear problems lead, after linearization, to elliptic eigenvalue problems with an indefinite weight function. (See, e.g., the survey paper by de Figueiredo [dF] and the work of Hess-Kato [HK].) A vast literature in engineering, physics and applied mathematics, deals with such problems arising, for instance, in the study of transport theory, reaction-diffusion equations, fluid dynamics ... (See, e.g., [DGHa, GN, LuR, LuW and KKLeZ].)

Most of the literature on this subject - studied since the end of the last century (Richardson [Ri], Bôcher [Bo], Hilbert [Hi],...) - is concerned with the one dimensional case, which is still the object of active research. (See, for instance, Mingarelli's monograph [Mi, Chapter IV].) To our knowledge, the first result in the multidimensional case is due to Holmgren (1904) [Ho]; he considers the Dirichlet problem $\Delta u + \lambda r(x,y)u = 0$, on a bounded open set $\Omega \subset \mathbf{R}^2$, when r is continuous and changes sign; he proves in this case the existence of an infinite number of positive (and negative) eigenvalues which can be characterized by the "min-max principle". The asymptotic distribution of these eigenvalues has been established by Pleijel (1942) [Pl] for the Dirichlet and Neumann problems.

The spectral theory of similar abstract problems has been studied more recently by Weinberger [Wn] and its application to the Dirichlet problem for an elliptic operator of order 2, by Manes and Micheletti [MM].

Recently, unaware of Pleijel's work, Lapidus [La 1] determined the asymptotic behavior of the eigenvalues of the Laplacian with Dirichlet or Neumann boundary conditions on a bounded open set of $\mathbf{R}^k$, $k \geq 1$, in the case when the weight function r is not necessarily continous; he also gave lower bounds having the correct "coupling constant behavior" for the eigenvalues of the Dirichlet problem. Together with some complementary work on remainder estimates, these results were presented - in a slightly extended form - in [La 2].

At about the same time, Fleckinger and El Fetnassi [FlF 1, 2] proved an analogous asymptotic estimate for the Dirichlet problem corresponding to a general self-adjoint elliptic operator of order 2m with a continuous weight function; asymptotics were also obtained for operators of "Schrödinger-type" on unbounded domains for a sufficiently smooth weight function.

The present paper extends both results: for general elliptic problems, we do not assume that the weight function is continous; this is of interest in view of possible applications to nonlinear problems.

We now indicate how this work is organized. After having stated our main results in the next section, we present our methods in §3 by treating the prototypal example of the Laplacian on a bounded open set of $\mathbb{R}^k$. We then extend this study in §4 to general elliptic boundary value problems on a bounded open set. Finally, in §5, we determine the asymptotic distribution of the eigenvalues of certain Schrödinger operators with an indefinite weight function.

**2. Notations and main results.** We introduce some notations which will be used throughout the article:

Let m and k be integers $\geq 1$.

Let $\Omega$ be an open set in $\mathbb{R}^k$ with boundary $\partial\Omega$.

The interior (resp., the closure) of $Q \subset \mathbb{R}^k$ is denoted by $Q^o$ (resp., $\overline{Q}$); if, in addition, Q is Lebesgue measurable, $|Q|$ stands for its Lebesgue measure in $\mathbb{R}^k$.

If $R \subset Q$, the notation $Q \setminus R$ indicates the complement of R in Q.

For $j \in \mathbb{N}$, $C^j(\Omega)$ is the space of j times continuously differentiable functions in $\Omega$.

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, $D^\alpha$ is the derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_k$: $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}$; if, in addition,

$\xi = (\xi_1,\dots,\xi_k) \in \mathbb{R}^k$, then $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_k^{\alpha_k}$.

$H^m(\Omega)$ denotes the Sobolev space of functions $u \in L^2(\Omega)$ such that the distributional derivatives $D^\alpha u$ are in $L^2(\Omega)$ for $|\alpha| \leq m$ (see, e.g., [Ad]). Recall that $H^m(\Omega)$ is a Hilbert space for the norm

$$\|u\|_{H^m(\Omega)} = (\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u(x)|^2\, dx)^{1/2}.$$

$H_0^m(\Omega)$ is the subspace of $H^m(\Omega)$ obtained by completing $C_0^\infty(\Omega)$ with respect to the norm $H^m(\Omega)$; here, $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support contained in $\Omega$.

Finally, if f is a real-valued function defined on $\Omega$, we set $f_+ = \max(f,0)$ and $f_- = \max(-f,0)$.

We can now state our hypotheses and present our main results:

(2.1) Let A be a formally self-adjoint uniformly elliptic operator of order 2m defined on $\Omega$:

$$A = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq m}} D^\alpha(a_{\alpha\beta}(x)\, D^\beta),$$

where $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$ for $|\alpha| \leq m$ and $|\beta| \leq m$; we assume that $a_0 \geq 0$ and – when $\Omega$ is bounded (resp., $\Omega = \mathbb{R}^k$) – $a_0 \in L^{k/2m}(\Omega)$ if $k > 2m$ and $a_0 \in L^1(\Omega)$ if $k < 2m$ [resp., $a_0 \in L^\infty(\Omega)$]; moreover, $a_{\alpha\beta} \in L^\infty(\Omega)$ if $0 < |\alpha| + |\beta| \leq 2m$ and $a_{\alpha\beta} \in C^0(\overline{\Omega})$ if $|\alpha| + |\beta| = 2m$.

(2.2) Let r be a measurable real-valued function on $\Omega$; we suppose that $\Omega_+ = \{x \in \Omega: r(x) > 0\}$ and $\Omega_- = \{x \in \Omega: r(x) < 0\}$ are of positive Lebesgue measure. We also assume that $r \in L^p(\Omega)$ [resp., $r \in L^p_{loc}(\Omega)$] when $\Omega$ is bounded [resp., $\Omega = R^k$], with $p > k/2m$ if $k \geq 2m$ and $p = 1$ if $k < 2m$.

We study the spectrum (i.e., the set of eigenvalues $\lambda$) of the homogeneous Dirichlet or Neumann boundary value problem (in the variational sense):

(P) $$Au = \lambda ru, \quad \text{in } \Omega.$$

When $\Omega$ is bounded, this spectrum is discrete and consists of a double sequence of real eigenvalues of finite multiplicity (one nonnegative and one negative):

$$\cdots \leq \lambda^-_{n+1} \leq \lambda^-_n \leq \cdots \leq \lambda^-_2 \leq \lambda^-_1 < (0) \leq \lambda^+_1 \leq \lambda^+_2 \leq \cdots \leq \lambda^+_n \leq \lambda^+_{n+1} \leq \cdots,$$

with $|\lambda^\pm_n|$ tending to $+\infty$ as n tends to $\infty$ (each eigenvalue is repeated according to multiplicity).
The same result holds, under appropriate assumptions, for an operator of "Schrödinger type" on $\mathbb{R}^k$.

Let $N^+(\lambda)$ [resp., $N^-(\lambda)$] be the number of nonnegative [resp., negative] eigenvalues $\lambda^+_n$ less than or equal to $\lambda > 0$ [resp., $\lambda^-_n$ greater than or equal to $\lambda < 0$].

Assume that $\Omega$ is bounded. If $|\Omega_+ \setminus \Omega_+^o| = 0$ (this is the case, for example, if r is continuous), we show for the Dirichlet problem that (see Theorem 4.1):

$$(2.3) \qquad N^+(\lambda) \sim \int_{\Omega_+} (\lambda r(x))^{k/2m} \mu_A'(x)\, dx, \quad \text{as } \lambda \to +\infty,$$

where $\mu_A'(x)$ is the "Browder–Gårding density" on $\Omega$:

$$\mu_A'(x) = (2\pi)^{-k} \int_{\{\xi \in \mathbb{R}^k:\ A'(x,\xi) < 1\}} d\xi\ , \quad x \in \Omega,$$

and $A'(x,\xi)$ is the symbol of the principal part of $A$:

$$A'(x,\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\, \xi^{\alpha+\beta}, \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}^k.$$

Similarly, if $|\Omega_- \setminus \Omega_-^o| = 0$, then

$$(2.4) \qquad N^-(\lambda) \sim \int_{\Omega_-} (\lambda r(x))^{k/2m} \mu_A'(x)\, dx, \quad \text{as } \lambda \to -\infty.$$

In particular, when $A = -\Delta$, we have $\mu_A'(x) = (2\pi)^{-k} B_k$, where $B_k = \pi^{k/2} / \Gamma(1 + k/2)$ denotes the volume of the unit ball in $\mathbb{R}^k$; since $N^+(\lambda_n^+) = n$, we then obtain an extension of Hermann Weyl's famous asymptotic formula (see Theorems 3.1 and 3.2):

$$(2.5) \qquad |\lambda_n^{\pm}| \sim C_k\ n^{2/k}\ \|r_\pm\|^{-1}_{L^{k/2}(\Omega)}\ , \quad \text{as } n \to \infty,$$

where $C_k$ denotes Weyl's constant: $C_k = (2\pi)^2\ (B_k)^{-2/k}$.

**REMARKS.** 1. Actually, (2.5) contains two equations: one involving the minus signs alone and a second one for the plus signs; we shall use this convenient sign convention throughout the paper.

2. Note that the results for negative eigenvalues can be deduced directly from those for positive eigenvalues by changing r into -r.

3. For simplicity, we write $\Omega^{o}_{\pm}$ instead of $(\Omega_{\pm})^{o}$.

If, in addition, $\Omega$ satisfies the segment property (for example, if $\partial\Omega$ is locally Lipschitz) and if $|\partial\Omega| = 0$, the formulae (2.3) and (2.4) still hold for the Neumann problem (see Theorem 4.2).

When $r(x) \equiv 1$, $N^{-}(\lambda) = 0$ and (2.3) reduces to the usual formula (see, e.g., [CH, Chapter VI, §4, pp. 429-445; Br; Ga; Kc; FlMe; Me; RS, Theorem XIII.78, p. 271; ...]).

When $\Omega = \mathbb{R}^k$, we assume that $A$ is an operator of "Schrödinger type", i.e., $A = L + q$, where $L$ satisfies (2.1).

(2.6) Here, the "potential" q is a nonnegative function defined on $\mathbb{R}^k$; moreover, $q(x) \to +\infty$ and $q^{-1}(x)r(x) \to 0$, as $|x| \to +\infty$.

Under other suitable hypotheses, we show that the spectrum of (P) is discrete and that (see Theorem 5.1):

$$(2.7) \qquad N^{\pm}(\lambda) \sim \int_{\Omega_{\pm} \cap \Omega_{\lambda}} (\lambda r(x) - q(x))^{k/2m} \mu'_{L}(x)\, dx, \quad \text{as } \lambda \to \pm\infty,$$

where $\Omega_{\lambda} = \{x \in \Omega : \lambda r(x) > q(x)\}$ and $\mu'_{L} = \mu'_{A}$, by definition.

This estimate is well known when $r(x) \equiv 1$ (see, e.g., [T, Chapter 17, §§8-14, pp. 174-185; Ro 1; Me; Fl; RS, Theorem XIII.81, p. 275; ...]

We also establish lower bounds for the eigenvalues of the Dirichlet problem on a bounded domain $\Omega$; for instance, for the Laplacian and for $k \geq 3$, we have (see Theorem 3.3):

$$(2.8) \qquad |\lambda_n^{\pm}| \geq \Theta_k \, n^{2/k} \, \|r_{\pm}\|^{-1}_{L^{k/2}(\Omega)} ,$$

where $\Theta_k$ is a constant depending only on k which is explicitly known. In view of the extended Weyl's formula (2.5), these lower bounds exhibit the correct "coupling constant behavior".

We refer the reader to Courant-Hilbert [CH, Chapter VI], Reed-Simon [RS, Chapter XIII, §1 and §15] or Weinberger [We, Chapter 3] for the Courant-Weyl method of proof of Weyl's formula. For the variational theory of elliptic boundary value problems, we point out Lions-Magenes [LM] and Agmon [Ag]. Finally, the basic properties of Sobolev spaces used in this paper can be found in Adams [Ad].

**3. An example: the Laplacian on a bounded open set.** We now illustrate our methods for finding the asymptotic behavior of the eigenvalues, in the case of the Laplacian with Dirichlet or Neumann boundary conditions on a bounded open set (parts A and B). In part C, we obtain lower bounds for the eigenvalues of the corresponding Dirichlet problem.

**A. The Dirichlet problem.** We consider the eigenvalue problem

(E) $$-\Delta u = \lambda r u, \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

(3.1) $\Omega$ is a bounded open set of $\mathbb{R}^k$ such that $|\Omega_+ \setminus \Omega_+^o| = 0$ and $|\Omega_- \setminus \Omega_-^o| = 0$.

The eigenvalues of the variational problem associated with (E) are the complex numbers $\lambda$ such that there exists a non zero $u \in H_0^1(\Omega)$ satisfying $-\Delta u = \lambda r u$ (in the distributional sense).

It is well known (see, e.g., [Ho; Pl; Wn, Chapter 3; MM; dF, Chapter I, §§1-2; ...] that the eigenvalues $\lambda_n^{\pm}$ (when they exist) are characterized by the "max-min principle":

(3.2) $$\frac{1}{\lambda_n^+} = \max_{F_n \in \mathcal{F}_n} \min_{u \in F_n} \{\textstyle\int_\Omega r|u|^2 : \|\text{grad } u\|^2_{L^2(\Omega)} = 1\},$$

where $\mathcal{F}_n$ denotes the set of n-dimensional subspaces of $H_0^1(\Omega)$; an analogous formula holds for $\lambda_n^-$.

The hypothesis $|\Omega_{\pm}| > 0$ implies that $\lambda_n^{\pm}$ exists for all $n \geq 1$ and $\lambda_n^{\pm} \to \pm\infty$ as $n \to \infty$. The precise asymptotic behavior is given by

**THEOREM 3.1.** Under the assumptions (2.2) and (3.1), the estimates (2.3), (2.4) and (2.5) hold.

If $r$ is positive in $\Omega$, we have $N^-(\lambda) = 0$; (actually, this happens if and only if $|\Omega_+| = 0$). The formulae (2.3) and (2.5) - which are clearly equivalent - are classical when $r$ is positive and continuous (see, e.g., [CH, Theorem 14, p. 435; RS, Theorem XIII.78, p. 271; Me, Theorem 5.12, p. 188; ...]).

We shall need two lemma in order to derive Theorem 3.1. The following result is essentially known and for $k \geq 3$, for instance, can be obtained by combining [RS, Theorem XIII.80, p. 274] and [LiY, Step (iv), pp. 317-318].

**LEMMA 3.1.** _When_ $r$ _is positive in_ $L^p(\Omega)$ _with_ $p > k/2$ _if_ $k \geq 3$ _and_ $p = 1$ _if_ $k < 3$, (2.3) _and_ (2.5) _hold._

In the following, we shall denote by $\lambda_n^+(r,\Omega)$ the $n^{th}$ nonnegative eigenvalue of (E) to indicate the dependence on $r$ and $\Omega$.

**LEMMA 3.2.** (Monotonicity principles). $\lambda_n^+(r,\Omega)$ _does not increase when_ $r$ _or_ $\Omega$ _increases._

**REMARKS.** 1. Here and thereafter, we adopt the following convention: whenever we make use of an eigenvalue $\lambda_n^+$, we implicitly assume that it exists (equivalently, that the right-hand side of (3.2) - or of its counterpart - is finite).

2. The monotonicity with respect to $\Omega$ is only valid for Dirichlet problems.

Lemma 3.2 is a simple consequence of (3.2) and is well known when r is positive (see, e.g., [CH, Theorem 3, p. 409 and Theorem 7, p. 411], [RS, Proposition 4, p. 270] and [We, Theorem 7.1, p. 58, Example 7.2, p. 60 and Theorem 8.1, p. 62].

**PROOF OF THEOREM 3.1.** We follow the method used in [La 1, 2]: the idea consists in finding lower and upper bounds for $\lambda_n^+$ having the same asymptotics. With this aim and the help of the monotonicity principles we reduce the problem to the study of positive weights.

It clearly suffices to establish (2.5) for $\lambda_n^+$ since $\lambda_n^-(r,\Omega) = -\lambda_n^+(-r,\Omega)$ and $r_- = (-r)_+$.

Fix $\varepsilon > 0$. By Lemma 3.2, we have:

$$\lambda_n^+(r_+ + \varepsilon, \Omega) \le \lambda_n^+(r, \Omega) \le \lambda_n^+(r, \Omega_+^o); \tag{3.5}$$

this holds since $r \le r_+ + \varepsilon$ and $\Omega_+^o$ is an open subset of $\Omega$ of positive measure, by (2.2) and (3.1).

By Lemma 3.1 applied to the positive weights $r_+ + \varepsilon$ in $\Omega$ and $r$ in $\Omega_+^o$, we see that:

$$\lim_{n\to\infty} n^{-2/k} \lambda_n^+(r_+ + \varepsilon, \Omega) = C_k \, \| r_+ + \varepsilon \|^{-1}_{L^{k/2}(\Omega)} \tag{3.6}$$

and

$$\lim_{n\to\infty} n^{-2/k} \lambda_n^+(r, \Omega_+^o) = C_k \, \| r \|^{-1}_{L^{k/2}(\Omega_+^o)} . \tag{3.7}$$

We infer from Equations (3.5) to (3.7) that:

$$\begin{aligned} C_k \, \| r_+ + \varepsilon \|^{-1}_{L^{k/2}(\Omega)} &\le \underline{\lim}_{n\to\infty} n^{-2/k} \lambda_n^+(r, \Omega) \\ &\le \overline{\lim}_{n\to\infty} n^{-2/k} \lambda_n^+(r, \Omega) \le C_k \, \| r \|^{-1}_{L^{k/2}(\Omega_+^o)} . \end{aligned} \tag{3.8}$$

Now, it follows from the Lebesgue's dominated convergence theorem that $\| r_+ + \varepsilon \|_{L^{k/2}(\Omega)}$ tends to $\| r \|_{L^{k/2}(\Omega_+^o)}$ as $\varepsilon$ tends to $0$;

in fact, $\Omega$ is bounded and, by (2.1), $|r|^{k/2} \in L^1(\Omega)$ for $k > 1$; moreover, by (3.1),

$$\int_{\Omega_+^o} r^{k/2} = \int_{\Omega_+} r^{k/2} = \int_\Omega (r_+)^{k/2} .$$

[Note that by omitting the zero set of r, one does not alter the value of the latter integral.]

Consequently, we obtain (2.5) by letting $\varepsilon$ tend to zero in (3.8). □

**REMARKS. 1.** When r is continuous, $\Omega_+$ is open and Theorem 3.1 holds for an arbitrary bounded open set $\Omega \subset \mathbb{R}^k$.

2. When $\Omega_+$ is a "Jordan contented set" (roughly speaking, if $\Omega_+$ is well approximated from within and without by a finite union of cubes - see e.g., [LoS, Chapter 8, §§6-7; RS, p. 271; Pe, Chapter 2, §§11-12]), we have $|\Omega_+ \setminus \Omega_+^o| = 0$ since then, by [LoS, Proposition 6.1, p. 332], $|\partial(\Omega_+)| = 0$; this was the assumption of [La 1, Theorem 1, p. 266].

**B. The Neumann problem.** We suppose that

(3.9) $\Omega$ is a bounded open set satisfying the segment property (see [Ag, Definition 2.1, p. 11]) such that $|\partial\Omega| = 0$, $|\Omega_+ \setminus \Omega_+^o| = 0$ and $|\Omega_- \setminus \Omega_-^o| = 0$.

We consider the eigenvalue problem

(E') $$- \Delta u = \lambda r u \text{ in } \Omega; \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega;$$

where $\partial/\partial n$ denotes the normal derivative.

The scalar $\lambda$ is said to be an eigenvalue of the variational problem associated with (E') if $-\Delta u = \lambda r u$ for some non zero $u \in H^1(\Omega)$.

The corresponding nonnegative eigenvalues $\Lambda_n^+(r,\Omega)$ are, in this case, given by (3.2), where $H_0^1(\Omega)$ is replaced by $H^1(\Omega)$.

Since $H_0^1(\Omega) \subset H^1(\Omega)$, it follows from the "max-min principle" as in [CH, Theorem 5, p. 410], [RS, Proposition 4, p. 270] or [We, Theorem 7.1, p. 58 and Example 7.4, pp. 61-62] that:

(3.10) $$\Lambda_n^+(r,\Omega) \le \lambda_n^+(r,\Omega);$$

and, as above, the eigenvalues are monotone with respect to the weight; therefore, for $\varepsilon > 0$, we have:

(3.11) $$\Lambda_n^+(r_+ + \varepsilon,\Omega) \le \Lambda_n^+(r,\Omega) \le \lambda_n^+(r,\Omega).$$

As for the Dirichlet problem, using an extension of (2.3) or (2.5) when $\Omega$ satisfies (3.9) and the weight is positive [FlMe; Me, Theorem 5.12, p. 188], we obtain:

**THEOREM 3.2.** Under the assumptions (2.1), (2.2) and (3.9), the estimates (2.3), (2.4) and (2.5) hold.

Next, we comment on possible extensions or interpretations of the above results:

**REMARKS.** 1. The hypothesis concerning the segment property in Theorem 3.2 can be weakened as in [Me, Condition (C'), p. 156].

2. It is noteworthy that no assumption has been made about $\Omega_0 = \{x \in \Omega : r(x) = 0\}$ in order to obtain Theorems 3.1 and 3.2.

3. It is easy to derive from Theorems 3.1 and 3.2 the corresponding result for mixed Dirichlet-Neumann boundary conditions; naturally, the asymptotic distribution of the eigenvalues remains unchanged in this case.

4. The conclusion of Theorem 3.1 or 3.2 can be interpreted as follows: the positive (resp., negative) eigenvalues of (E) or (E') have the same asymptotic behavior as the eigenvalues - in the usual sense - of the elliptic operator $-\Delta/r(x)$ [resp., $+\Delta/r(x)$], defined in $\Omega_+^o$ (resp., $\Omega_-^o$) for the corresponding boundary conditions. Here, we use implicitly the fact that $\|r_\pm\|_{L^{k/2}(\Omega)} = \|r\|_{L^{k/2}(\Omega_\pm^o)}$.

[The preceding remarks apply, with the obvious changes, to the general elliptic boundary value problems studied in §4.B.]

5. Physically, when r is positive, the eigenvalues of (E) represent the natural frequencies of a vibrating membrame fixed along its boundary $\partial\Omega$ and of mass density equal to r - as well as of tension unity. (See, e.g., [CH, Chapters V and VI] or [Kc].) Therefore, in the case when r changes sign, we may interpret Theorem 3.1 by saying that the large values of $\lambda_n^+$ (resp., $\lambda_n^-$) are determined by the "positive mass" (resp., "negative mass") distribution of the "membrane".

**EXAMPLE 3.1.** We now consider a simple example of discontinuous weight function. Naturally, our assumptions would allow us to treat much more singular weights.

Let $\Omega$ be a bounded open set in $\mathbb{R}^k$. Let $\Omega_+$ and $\Omega_-$ be two disjoint measurable subsets of $\Omega$ of positive measure such that $|\Omega_+ \setminus \Omega_+^o| = 0$ and $|\Omega_- \setminus \Omega_-^o| = 0$. We define:

$$r(x) = \begin{cases} -1 & \text{if } x \in \Omega_- , \\ 0 & \text{if } x \in \Omega \setminus (\Omega_- \cup \Omega_+), \\ +1 & \text{if } x \in \Omega_+. \end{cases}$$

We then deduce from Theorem 3.1, that, for the Dirichlet problem, we have:

**COROLLARY 3.1.** $$\lambda_n^\pm \sim C_k \left(\frac{n}{|\Omega_\pm|}\right)^{2/k}, \quad \underline{\text{as}}\ n \to \infty.$$

The same estimate holds for the Neumann problem under the additional assumptions that $|\partial\Omega| = 0$ and $\Omega$ satisfies the segment property.

**REMARK.** The geometric features of the above example suggest the following natural extension of the well known isospectral problem (see, e.g., [Kc] and [Y, pp. 23-24]): let $\Omega^1$, $\Omega_\pm^1$, $r_1$ and $\Omega^2$, $\Omega_\pm^2$, $r_2$, be defined as in Example 3.1. Assume that the problem (E) [or (E')] has the same set of eigenvalues, counting multiplicities, in $\Omega_1$ and $\Omega_2$. Under which (generic) conditions does there exist an isometry which sends $\Omega^1$, $\Omega_+^1$ and $\Omega_-^1$ onto, respectively, $\Omega^2$, $\Omega_+^2$ and $\Omega_-^2$ ? In other words, to what extent does the spectrum determine the sign of the weight?

**C. Lower bounds of the eigenvalues.** We obtain here lower bounds (resp., upper) bounds for the eigenvalues $\lambda_n^+$ (resp., $\lambda_n^-$) of the Dirichlet problem (E) studied in part A. These bounds are valid for all n and compatible with the asymptotic behavior obtained in Theorem 3.1.

**THEOREM 3.3.** Assume that $\Omega$ is a bounded domain of $\mathbb{R}^k$ with $C^2$ boundary $\partial\Omega$ and suppose that $k \geq 3$. Then, if $|\Omega_\pm| > 0$, respectively, we have for the Dirichlet problem (E) in $\Omega$:

$$|\lambda_n^\pm| \geq \Theta_k \, n^{2/k} \, \|r_\pm\|^{-1}_{L^{k/2}(\Omega)}, \qquad \forall\, n \geq 1, \tag{3.12}$$

where $\Theta_k = (k(k-2)/4e)(\omega_{k-1})^{2/k}$, $e = \exp(1)$ and $\omega_{k-1} = kB_k$ denotes the area of the unit sphere in $\mathbb{R}^k$.

**PROOF.** Evidently, we only need to find a lower bound for $\lambda_n^+$ when $|\Omega_+| > 0$. Fix $n \geq 1$ and $\varepsilon > 0$. Since $r \leq r_+ + \varepsilon$, we derive from Lemma 3.2 that

$$\lambda_n^+(r,\Omega) \geq \lambda_n^+(r_+ + \varepsilon,\Omega). \tag{3.13}$$

By [LiY, Theorem 2, p. 314] applied to the positive weight $r_+ + \varepsilon$, we have:

$$\lambda_n^+(r_+ + \varepsilon,\Omega) \geq \Theta_k \, n^{2/k} \, \|r_+ + \varepsilon\|^{-1}_{L^{k/2}(\Omega)} . \tag{3.14}$$

In view of (3.13) and (3.14), we now obtain (3.12) by letting $\varepsilon \to 0$ and applying the dominated convergence theorem. □

Since $N^+(\lambda_n^+) = n$, we deduce:

**COROLLARY 3.2.** $(k(k-2)/4e)^{k/2}\ \omega_{k-1}\ N^{\pm}(\lambda) \le \lambda^{k/2}(\int_\Omega (r_\pm)^{k/2})$, $\forall\ \lambda > 0$.

We have just used a remarkable result of Li and Yau [LiY, Theorem 2, p. 314] which improved significantly the existing upper bounds for the number of "bound states" of the Schrödinger operator in $\mathbb{R}^3$. (See [LiY, p. 311] and [Lb, p. 243] for references to previous works on this subject.) Note, however, that the method of Li and Yau does not seem a priori to apply to indefinite weights.

**REMARKS.** 1. The assumption that $|\Omega_\pm \setminus \Omega_\pm^o| = 0$ is not needed for Theorem 3.3 to hold.

2. We note that upper bounds for $|\lambda_n^\pm|$ can be deduced similarly from the known upper bounds for the eigenvalues of Dirichlet problems with positive weights. Indeed, by (3.5), $|\lambda_n^\pm(r,\Omega)| \le |\lambda_n^\pm(r,\Omega_\pm^o)|$.

3. As is the case with the result of Li and Yau, Theorem 3.3 extends to a compact Riemannian manifold with $C^2$ boundary; $\Theta_k$ must then be replaced by a constant depending on the Sobolev constant of the manifold.

4. Theorem 3.3 also extends to general self-adjoint elliptic operators of order 2; in this case, the new constant occurring in (3.12) depends on $\Theta_k$ and the constant of uniform ellipticity of the operator. This is easily seen by using a monotonicity argument (based on Lemma 4.2) and Theorem 3.3.

5. It may be worth pointing out that, for $k \ge 3$, Theorem 3.3 provides us with a new proof of the fact that $\lambda_n^\pm$ exists for all $n \ge 1$ and $\lambda_n^\pm \to \pm\infty$ when $|\Omega_\pm| > 0$. (Compare with [MM, Proof of Proposition 3, p. 290] or [dF, Proof of Proposition 1.11, p. 43].)

**4. Elliptic boundary value problems on bounded open sets.** We extend the results of paragraphs 3.A and 3.B to self-adjoint elliptic operators of order 2m.

**A. Abstract theory.** In order to obtain results on more general boundary value problems, we briefly explain the abstract theory (see, e.g., [Wn, Chapter 3] and [dF, Chapter I]). It will later be applied on bounded (§4.B) as well as unbounded (§5) open sets.

(4.1) Let H be a real or complex Hilbert space.

(4.2) Let (V, H, a) be a variational triple:

V is a dense subspace of H, with compact imbedding and a is a hermitian, bounded and coercive form defined on V (see [LM]).

(4.3) b is a hermitian form on V such that, for some positive constant $\delta$,

$$|b(u,u)| \leq \delta\, a(u,u), \quad \forall\, u \in V.$$

We consider the following variational eigenvalue problem:

$$\text{(Q)} \qquad a(u,v) = \lambda\, b(u,v), \quad \forall\, v \in V;$$

here, $\lambda$ is an eigenvalue of (Q) [or is in the spectrum of (Q)] if there exists a non zero $u \in V$ such that the last equation holds.

**REMARK.** In the example of Section 3, we consider $H = L^2(\Omega)$, $a(u,u) = \|u\|^2_{H^1(\Omega)}$, $b(u,u) = \int_\Omega r|u|^2$ and $V = H^1_0(\Omega)$ [resp., $V = H^1(\Omega)$] for the Dirichlet [resp., Neumann] boundary value problem on a bounded open set $\Omega$. In this manner, Theorem 3.1 and 3.2 will become corollaries of Theorems 4.1 and 4.2, respectively.

By (4.2), a induces in V an inner product equivalent to the original one; by the Riesz representation theorem, we define a bounded self-adjoint operator T from $(V, a(\cdot,\cdot))$ to itself by

(4.4) $$b(u,v) = a(Tu,v), \quad \forall (u,v) \in V \times V.$$

It follows that $\lambda$ is an eigenvalue of (Q) if and only if $1/\lambda$ is an eigenvalue of T: $Tu = (1/\lambda)u$, for some non zero $u \in V$.

(4.5) We now assume that T is compact.

This is the case in all the problems studied here.

It follows from the classical spectral theory of compact self-adjoint operators in Hilbert spaces that the spectrum of (Q) consists of a double sequence:

$$\ldots \leq \lambda^-_{n+1} \leq \lambda^-_n \ldots \leq \lambda^-_1 < (0) < \lambda^+_1 \leq \ldots \leq \lambda^+_n \leq \lambda^+_{n+1} \leq \ldots ;$$

(each eigenvalue has finite multiplicity and is repeated accordingly).

When it exists, $\lambda_n^{\pm}$ – which we write $\lambda_n^{\pm}(a,V)$ or $\lambda_n^{\pm}(b, V)$, if necessary – is given by the "max-min principle":

$$\text{(4.6)} \qquad \frac{1}{\lambda_n^+} = \max_{F_n \in \mathcal{F}_n} \ \min_{u \in F_n} \{b(u,u) : a(u,u) = 1\},$$

where $\mathcal{F}_n$ is the set of n-dimensional subspaces of V.
A similar formula holds for $\lambda_n^-$ since $\lambda_n^-(V,-b) = -\lambda_n^+(V,b)$.

Just as before, we deduce from (4.6) the following abstract monotonicity principles:

**LEMMA 4.1.** _If_ $b_1$, $b_2$ _are two hermitian forms satisfying_ (4.3) _and such that_ $b_1(u,u) \leq b_2(u,u)$ _for all_ $u \in V$, _then_ $\lambda_n^+(b_1,V) \geq \lambda_n^+(b_2,V)$.

**LEMMA 4.2.** _Let_ $a_1$, $a_2$ _be coercive forms satisfying_ (4.2) _and such that_ $a_1(u,u) \geq a_2(u,u)$ _for all_ $u \in V$; _then_ $\lambda_n^+(a_1,V) \geq \lambda_n^+(a_2,V)$.

**LEMMA 4.3.** _If_ $(V_1, H, a)$ _and_ $(V_2, H, a)$ _are two "variational triples" such that_ $V_1 \subset V_2$, _then_ $\lambda_n^+(b,V_1) \geq \lambda_n^+(b,V_2)$.

**B. Operators of order 2m.** In the following, we suppose that $\Omega$ is a bounded open set in $\mathbb{R}^k$; the weight function r satisfies (2.2) and the operator $A = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D^\alpha(a_{\alpha\beta} D^\beta)$ fulfills (2.1).

We first consider the Dirichlet boundary value problem: $Au = \lambda r u$, $u \in H_0^m(\Omega)$.

(4.7) We assume that $|\Omega_+ \setminus \Omega_+^o| = 0$ and $|\Omega_- \setminus \Omega_-^o| = 0$.

We can now apply the above abstract theory with $H = L^2(\Omega)$, $V = H_0^m(\Omega)$ and $a(u,v) = \int_\Omega \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} a_{\alpha\beta} D^\alpha u \overline{D^\beta v}$.

It follows from standard results on Sobolev spaces and from the uniform ellipicity of $A$(i.e., $A'(x,\xi) \geq c\, |\xi|^{2m}$, for all $(x,\xi) \in \Omega \times R^k$ and some $c > 0$) that a satisfies (4.2).

By (2.2) and the Sobolev imbedding theorem [Ad, Theorem 5.4, p. 97], we see that $b(u,v) = \int_\Omega ru\overline{v}$ satisfies (4.3).

When $k > 2m$, we also have, by using the Sobolev and the Hölder inequalities that, for $u, v \in V$:

$$|\int_\Omega ruv| \leq \|r\|_{L^p(\Omega)} \|u\|_{L^s(\Omega)} \|v\|_{L^{2^*}(\Omega)},$$

where $2^* = \frac{2k}{k-2m}$ and $\frac{1}{p} + \frac{1}{s} + \frac{1}{2^*} = 1$.

Therefore, (4.5) follows from the Rellich-Kondrachov theorem [Ad, Theorem 6.2, p. 144]. Similar results are obtained when $k \leq 2m$ by use of [Ad, Theorem 5.4, Cases B and C, p. 97] and [Ad, Theorem 6.2, Eqs. (4) and (6), p. 144]. Much like in [dF, Proof of Proposition 1.11, p. 43], one then shows that the assumption $|\Omega_\pm| > 0$ implies that $|\lambda_n^\pm| \to +\infty$ as $n \to \infty$.

We now state

**THEOREM 4.1.** *Under the hypotheses* (2.1), (2.2) and (4.7), *the estimates* (2.3) *and* (2.4) *hold and:*

$$(4.8) \qquad |\lambda_n^{\pm}| \sim n^{2m/k} \left(\int_\Omega (r_\pm(x))^{k/2m} \mu_A'(x)\ dx\right)^{-2m/k}, \quad \text{as } n \to \infty.$$

**PROOF.** The estimate (2.3) holds when r is continuous and bounded away from zero (see [Me, Theorem 5.12, p. 188]); hence, for r nonnegative, we obtain the same result by replacing r by $r + \varepsilon$ and letting $\varepsilon \to 0$. In light of Lemmas 4.1 and 4.3, we then obtain (4.8) and (2.3) exactly as in the proof of Theorem 3.1. Note that

$$\int_{\Omega_+} r^{k/2m} \mu_A' = \int_{\Omega_+^o} r^{k/2m} \mu_A' = \int_\Omega (r_+)^{k/2m} \mu_A'. \quad \square$$

Likewise, we have:

**THEOREM 4.2.** *For the Neumann problem, if* r, Ω *and* A *satisfy* (2.1), (2.2) *and* (3.8), *then* (2.3), (2.4) *and* (4.8) *hold.*

**PROOF.** We now consider $V = H^m(\Omega)$; we use, in particular, Lemma 4.3 with $V_1 = H_0^m(\Omega)$ and $V_2 = H^m(\Omega)$; and the proof goes trough as in Theorems 3.1 and 3.2. □

**5. The Schrödinger operator.** In this section, we give sufficient conditions under which a positive operator of "Schrödinger type" and with an indefinite weight has discrete "spectrum" in $\Omega = \mathbb{R}^k$ and we then determine the asymptotic distribution of its eigenvalues. We combine, in particular, the results and methods of §4 and previous works [Ro 1, 2; Me, Fl; ...] on Schrödinger operators with positive weights.

**DEFINITION 5.1.** Let $f$ be a continuous function defined on an open subset $\omega$ of $\mathbf{R}^k$. We say that $f$ satisfies $(\varepsilon)$ if there exists a positive number $\varepsilon_0$ such that $f$ can be continuously extended to $\tilde{\omega} = \{x \in \mathbf{R}^k: \text{dist}(x,\omega) < \varepsilon_0\}$ and, for all $\varepsilon \in (0,\varepsilon_0)$, there exists $\eta > 0$ such that, for all $(x,y) \in \tilde{\omega} \times \tilde{\omega}$ with $|x - y| < \eta$, we have $|f(x) - f(y)| < \varepsilon\, |f(x)|$.

**REMARKS.** 1. For instance, $f(x) = (1 + |x|^2)^\sigma$, with $\sigma > 1$, satisfies $(\varepsilon)$.

2. If a positive function $f$ satisfies $(\varepsilon)$, so does its inverse $1/f$.

In the following, we set $\Omega'_\rho = \{x \in \mathbf{R}^k: |x| > \rho\}$ for $\rho > 0$.

(5.1) Let $q$ be a nonnegative function in $L^{k/2m}_{loc}(\mathbf{R}^k)$, tending to $+\infty$ at infinity and satisfying $(\varepsilon)$ on $\Omega'_{R_1}$, for some $R_1 > 0$.

(5.2) Let $r$ be a function defined on $\mathbf{R}^k$, satisfying (2.2) and such that $q^{-1}r$ tends to 0 at infinity. Moreover, we suppose that:

a. The restriction of $r$ to $\Omega'_{R_1}$ satisfies $(\varepsilon)$.

b. $\int_{\mathbf{R}^k} (r_+)^{k/2m} = \int_{\mathbf{R}^k} (r_-)^{k/2m} = +\infty$.

(Note that r is allowed to change sign anywhere in $\mathbb{R}^k$.)

**REMARK.** (5.2) holds, for example, when $r(x) = -1$ if $|x| < 1$ and $r(x) = +1$ if $|x| > 1$.

(5.3) Let $L$ be a formally self-adjoint, uniformly elliptic operator of order 2m defined on $\mathbb{R}^k$ and verifying (2.1): $L = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D^\alpha(a_{\alpha\beta}D^\beta)$;

we suppose that for all $|\alpha| = |\beta| = m$, $a_{\alpha\beta}$ satisfies $(\varepsilon)$ on $\Omega'_{R_1}$; in addition, we assume that there exists $d > 0$ such that for all open subsets $\omega$ of $\mathbb{R}^k$,

$$\int_\omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u \overline{D^\beta u} \geq d \|u\|^2_{H^m(\omega)}, \qquad \forall u \in H^m(\omega) .$$

We study the variational eigenvalue problem

$$\text{(S)} \qquad Au = (L + q)u = \lambda ru, \quad \text{in } \mathbb{R}^k.$$

**DEFINITION 5.2.** _If $\omega$ is an open subset of $\mathbb{R}^k$, $V_q^0(\omega)$ denotes the completion of $C_0^\infty(\omega)$ with respect to the Hilbert norm_

$$\|u\|_{V_q(\omega)} = (\|u\|^2_{H^m(\omega)} + \int_\omega q|u|^2)^{1/2} .$$

_The space $V_q^1(\omega)$ is the set of restrictions to $\omega$ of elements of $V_q^0(\mathbb{R}^k)$._

We now apply the abstract theory developed in §4.A; here, $H = L^2(\mathbb{R}^k)$ is equipped with its usual inner product $(\ ,\ )$, $V = V_q^0(\mathbb{R}^k)$, $b(u,u) = (ru,u)$ and $a(u,u) = (Lu,u) + (qu,u)$, for $u \in V$.

It follows from (5.1) and (5.3) that a is coercive on $V_q^0(\mathbb{R}^k)$ and, from (5.1), that the imbedding of $V_q^0(\mathbb{R}^k)$ into $L^2(\mathbb{R}^k)$ is compact. Moreover, (4.5) holds because

$$\int_{\Omega'_\rho} r|u|^2 \leq \left( \sup_{x \in \Omega'_\rho} \left|\frac{r}{q}\right| \right) \left(\int_{\Omega'_\rho} q|u|^2\right) \leq \delta(\rho) \, \|u\|^2_{V_q(\Omega'_\rho)},$$

where $\delta(\rho)$ tends to zero as $\rho \to +\infty$; this follows since, by (5.2), $r/q$ tends to zero at infinity.

Consequently, the spectrum of (S) is discrete; moreover, the eigenvalues of (S) are characterized by the "max-min principle" (4.6).

Let $\lambda_n^+(r,\omega)$ [resp., $\Lambda_n^+(r,\omega)$] denote the nonnegative eigenvalues of the variational Dirichlet [resp., Neumann] boundary value problem:

$$Au = \lambda ru, \quad \text{in } \omega \subset \mathbb{R}^k; \quad u \in V_q^0(\omega) \text{ [resp., } u \in V_q^1(\omega)].$$

Set $N_0^+(\lambda;\, r,\omega,A) = \sum_{\lambda_n^+(r,\omega)<\lambda} 1$ and $N_1^+(\lambda;\, r,\omega,A) = \sum_{\Lambda_n^+(r,\omega)<\lambda} 1$;

one indicates in this way the dependence on the weight function r, the open set $\omega$ and the operator $A$.

Remark that

$$N_0^+(\lambda;\, r,\mathbb{R}^k,A) = N^+(\lambda),$$

where $N^+(\lambda)$ stands for the number of nonnegative eigenvalues of (S) less than or equal to $\lambda > 0$.

We shall use the subsequent lemma, which is derived from Lemma 4.3:

**LEMMA 5.1.** _If_ $\omega_1$ _and_ $\omega_2$ _are two disjoint open sets in_ $\omega$ _such that_ $\overline{\omega}_1 \cup \overline{\omega}_2 = \overline{\omega}$, _then:_

$$N_0^+(\lambda;\ r,\omega_1,A) + N_0^+(\lambda;\ r,\omega_2,A) \leq N_0^+(\lambda;\ r,\omega,A)$$

$$\leq N_1^+(\lambda;\ r,\omega,A) \leq N_1^+(\lambda;\ r,\omega_1,A) + N_1^+(\lambda;\ r,\omega_2,A).$$

This is due in particular to the inclusions

$$V_q^0(\omega_1) \oplus V_q^0(\omega_2) \subset V_q^0(\omega) \qquad \text{and} \quad V_q^1(\omega) \subset V_q^1(\omega_1) \oplus V_q^1(\omega_2).$$

Let us consider a covering of $\mathbf{R}^k$ by a countable family of disjoint open cubes $\{Q_\zeta\}_{\zeta \in \mathbf{Z}^k}$ with centers $x_\zeta$ and sides of length $\eta$:

$$\mathbf{R}^k = \bigcup_{\zeta \in \mathbf{Z}^k} \overline{Q}_\zeta.$$

Let $R > R_1$. Set $B = \{x \in \mathbf{R}^k\colon |x| < R\}$ and $G = \Omega_R' = \{x \in \mathbf{R}^k\colon |x| > R\}$; write

$$G_+ = \{x \in G\colon r(x) > 0\},\ G_- = \{x \in G\colon r(x) < 0\} \text{ and } G_0 = \{x \in G\colon r(x) = 0\}.$$

Fix $\lambda > 0$; let $G_\lambda = \{x \in G\colon \lambda r(x) - q(x) > 0\}$.

We note that $G_+$, $G_-$ and $G_\lambda$ are open; moreover, we have $G_\lambda \subset G_+$ and dist $(G_\lambda, G \setminus G_+) > 0$;

this follows since $r \geq \gamma/\lambda$ on $G_\lambda$ and r, q are continuous on G; here and thereafter, we choose $R_1$ (and $R > R_1$) so large that $q \geq \gamma$ on $\Omega'_{R_1}$ for some positive constant $\gamma$; this is possible in view of (5.1).

Put $I = \{\zeta \in \mathbf{Z}^k : \overline{Q}_\zeta \subset G_\lambda\}$ and $J = \{\zeta \in \mathbf{Z}^k : \overline{Q}_\zeta \cap \overline{G}_\lambda \neq \phi\}$.
We choose $\eta$ so small that

$$\left( \bigcup_{\zeta \in J} \overline{Q}_\zeta \right) \cap \left( G_0 \cup G_- \right) = \phi;$$

We assume in addition that B satisfies the analogue of (4.7), $G_0 \subset \overline{G_0^o}$ and

(5.4)
$$\lim_{\eta \to 0} \left( \sum_{\zeta \in J \setminus I} (r_\zeta)^{k/2m} \right) \left( \sum_{\zeta \in I} (r_\zeta)^{k/2m} \right)^{-1} = 0;$$

where $r_\zeta = r(x_\zeta)$ for $\zeta \in \mathbf{Z}^k$.

(5.5) Finally, we suppose that there exists $\lambda' > 0$ and $c > 0$ such that

$$[G_\lambda] \leq c\,[G_{\lambda/2}], \quad \forall\, \lambda \geq \lambda',$$

where $[G_\lambda] = \int_{G_\lambda} r^{k/2m}$.

We can now state

**THEOREM 5.1.** _Under the hypotheses_ (5.1) _to_ (5.5), _the spectrum of_ (S) _is discrete and the estimate_ (2.7) _holds:_

$$N^+(\lambda) \sim \int_{\Omega_+} [(\lambda r - q)_+]^{k/2m} \mu_L' , \quad \underline{\text{as}}\ \lambda \to +\infty,$$

_where_ $\Omega_+ = \{x \in \mathbb{R}^k : r(x) > 0\}$; _and similarly for_ $N^-(\lambda)$.

The idea of the proof consists in breaking $\mathbb{R}^k$ into disjoint pieces and linking the estimates so obtained with the help of Lemma 5.1. Inside the bounded open set B, we apply the results of §4.B, where r is allowed to be "singular"; outside B, r is "smooth" and positive and we can use a method similar to that of [F1].

To prove Theorem 5.1, we shall need the following results:

**PROPOSITION 5.1.** _There exist two positive constants_ c' _and_ c" _such that, for all_ $\lambda$ _sufficiently large,_

$$c' \lambda^{k/2m} [G_\lambda] \le \varphi(\lambda) \le c'' \lambda^{k/2m} [G_\lambda],$$

_where_ $\varphi(\lambda) = \int_{\Omega_+} [(\lambda r - q)_+]^{k/2m} \mu_L'$.

**PROOF OF PROPOSITION 5.1.** If follows from (5.1) to (5.3) that

$$\varphi(\lambda) = \int_{G_\lambda} (\lambda r - q)^{k/2m} \mu_L' \le c'' \int_{G_\lambda} (\lambda r)^{k/2m}$$

and, from the "Tauberian assumption" (5.5), that

$$\int_{G_\lambda} (\lambda r - q)^{k/2m}\, \mu_L' \geq \int_{G_{\lambda/2}} (\lambda r - q)^{k/2m}\, \mu_L' \geq c' \int_{G_\lambda} (\lambda r)^{k/2m};$$

note that $r = r_+$ and $\lambda r - q = (\lambda r - q)_+$ on $G_\lambda \supset G_{\lambda/2}$. □

**PROPOSITION 5.2.** _If Theorem 5.1 holds for_ $N_0^+(\lambda;\ r, \mathbb{R}^k, A')$, _then it holds for_ $N_0^+(\lambda;\ r, \mathbb{R}^k, A)$, _where_ $A' = L' + q$ _and_ $L'$ _is the principal part of_ $L$.

**PROOF OF PROPOSITION 5.2.** This is a simple consequence of interpolation theorems and of Lemma 4.2 if we notice - as in [Ro 1, Lemma 1.1, p. 353], - that

$$\varphi((1 + \varepsilon)\lambda) \leq (1 + c_1 \sqrt{\varepsilon})^{k/2m}\, \varphi(\lambda),$$

for some positive constant $c_1$.

The result now follows by letting $\varepsilon \to 0$; here and thereafter, $\varepsilon$ denotes the variable used in Definition 5.1. □

**PROPOSITION 5.3.** $$\lim_{\lambda\to+\infty} \lim_{\varepsilon\to 0} \varphi^{-1}(\lambda) \sum_{\zeta \in J} \varphi^+(\lambda,\zeta) = 1$$

$$= \lim_{\lambda\to+\infty} \lim_{\varepsilon\to 0} \varphi^{-1}(\lambda) \sum_{\zeta \in I} \varphi^-(\lambda,\zeta),$$

_where_

$$\varphi^\pm(\lambda,\zeta) = [(1 \pm \varepsilon)\, \lambda r_\zeta - q_\zeta]^{k/2m}\, \mu_\zeta\, |Q_\zeta|,$$

$$q_\zeta = q(x_\zeta) \ \underline{\text{and}}\ \mu_\zeta = (2\pi)^{-k} \int_{\{\xi \in \mathbb{R}^k:\ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_\zeta)\, \xi^{\alpha+\beta} < 1\}} d\xi.$$

This follows easily from hypothesis (5.4); observe that $\varepsilon \to 0$ implies that $\eta \to 0$.

**PROOF OF THEOREM 5.1.** In light of Proposition 5.2, it suffices to establish Theorem 5.1 for $A'$.

Now, we make use of the covering $(Q_\zeta)_{\zeta \in \mathbf{Z}^k}$ of $\mathbf{R}^k$ and of Lemma 5.1:

$$\sum_{\zeta \in I} N_0^+(\lambda;\ r, Q_\zeta, A') \leq N_0^+(\lambda;\ r, \mathbf{R}^k, A') \tag{5.6}$$

$$\leq N_1^+(\lambda;\ r, B, A') + \sum_{\zeta \in J} N_1^+(\lambda;\ r, Q_\zeta, A')$$

$$+ N_1^+(\lambda;\ r, D, A') + N_1^+(\lambda;\ r, G_-, A') + N_1^+(\lambda;\ r, G_0^o, A'),$$

where $D = G_+ \setminus (\overline{\cup_{\zeta \in J} Q_\zeta})$.

Next, we remark that $N_1^+(\lambda;\ r, G_-, A') = N_1^+(\lambda;\ r, G_0^o, A') = 0$ and $N_1^+(\lambda;\ r, D, A') = 0$;
the latter equality holds since $D \subset U_\lambda$, where
$U_\lambda = \{x \in G_+ :\ \lambda r(x) - q(x) < 0\}$ and, by (4.6), $N_1^+(\lambda;\ r, U_\lambda, A') = 0$.

By Theorems 4.1 and 4.2, we have:

$$N_i^+(\lambda;\ r, B, A') \sim \int_{B_+} [(\lambda r - q)_+]^{k/2m}\ \mu_L' \ , \quad \text{as } \lambda \to +\infty, \tag{5.7}$$

where $B_+ = \{x \in B:\ r(x) > 0\}$ and $i = 0$ or $1$ according to the boundary conditions;

in fact, by (2.2), we may choose R big enough so that $|B_+| > 0$; moreover, we observe that, because B is bounded,

$$\int_{B_+} (\lambda r)^{k/2m} \mu_L' \sim \int_{B_+} [(\lambda r - q)_+]^{k/2m} \mu_L' , \quad \text{as } \lambda \to +\infty.$$

It follows, in particular, that $\varphi^{-1}(\lambda)\, N_1(\lambda;\, r,B,A') \to 0$ as $\lambda \to +\infty$.

Thanks to (5.6), we only have to work on the domain $\bigcup_{\zeta \in J} Q_\zeta$, where r is positively bounded from below by $\lambda^{-1}\gamma$ and satisfies (ε). On each cube $Q_\zeta$, where $\zeta \in J$, we can compare $A'$ with a homogeneous operator with constant coefficients; precisely, for i = 0 or 1, it follows from condition (ε) and from Lemmas 4.1 and 4.2 that, for η small enough,

$$(5.8) \qquad N_i^+([(1 - \varepsilon)\lambda - q_\zeta]/r_\zeta;\, r_\zeta, Q_\zeta, L_\zeta) \leq N_i^+(\lambda;\, r, Q_\zeta, A')$$

$$\leq N_i^+([(1 + \varepsilon)\lambda - q_\zeta]/r_\zeta;\, r_\zeta, Q_\zeta, L_\zeta),$$

where $L_\zeta = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_\zeta)\, D^{\alpha+\beta}$ and, as above, $r_\zeta = r(x_\zeta)$;

furthermore, we know by a special case of [Me, Theorem 5.1, p. 175 and Theorem 5.12, p. 188] that

$$(5.9) \quad |N_i^+([(1 \pm \varepsilon)\lambda - q_\zeta]/r_\zeta;\, r_\zeta, Q_\zeta, L_\zeta) - \mu_\zeta\, |Q_\zeta|\, [(1 \pm \varepsilon)\lambda r_\zeta]^{k/2m}| \leq C\, \lambda^{(k-1)/2m},$$

for some suitable constant C.

In light of Propositions 5.1 to 5.3 and equations (5.6) to (5.9), we conclude the proof of Theorem 5.1 by letting $\varepsilon \to 0$ and $\lambda \to +\infty$. □

The next statement is really a corollary of the proof of Theorem 5.1:

**COROLLARY 5.1.** If, in assumption (5.2.b), we suppose instead that $\int_{\mathbb{R}^k} |r|^{k/2m}$ is finite, we must replace the conclusion of Theorem 5.1 by that of Theorem 4.1 (with $\Omega = \mathbb{R}^k$):

$$N^{\pm}(\lambda) \sim \int_{\Omega_{\pm}} (\lambda r)^{k/2m} \mu_L' , \quad \text{as } \lambda \to \pm\infty.$$

**REMARKS.** 1. The estimate for $N^{+}(\lambda)$ in Theorem 5.1 (resp., Corollary 5.1) still holds if we only assume that $\int_{\mathbb{R}^k} (r_+)^{k/2m} = +\infty$ (resp., $\int_{\mathbb{R}^k} (r_+)^{k/2m} < +\infty$) in (5.2.b); of course, an analogous remark applies to $N^{-}(\lambda)$.

2. A result similar to Theorem 5.1 and Corollary 5.1 could also be obtained by the same methods on an unbounded domain other than $\mathbb{R}^k$. In this case, mixed Dirichlet-Neumann problems could also be considered (under suitable compactness hypotheses).

3. Recently, Gurarie [Gu] has studied the asymptotic distribution of the eigenvalues of operators of Schrödinger type in $\mathbb{R}^k$ having a smooth positive weight function; in his work, based in part on the theory of pseudodifferential operators, the analogue of condition ($\varepsilon$) is the so-called "finite propagation speed" condition.

**EXAMPLE 5.1.** We now illustrate our results by considering the following eigenvalue problem:

$$\text{(ES)} \qquad Au = (-\Delta + q)u = \lambda r u, \quad \text{in } \mathbb{R}^k,$$

where

$$r(x) = \begin{cases} -1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| = 1, \\ +1 & \text{if } |x| > 1, \end{cases}$$

and where q satisfies (5.1) [e.g., $q(x) = (1 + |x|^2)^\sigma$ with $\sigma > 1$].

According to Theorem 5.1, Corollary 5.1 and Remark 1 above, we then have:

**COROLLARY 5.2.** $N^-(\lambda) \sim (2\pi)^{-k} (B_k)^2 |\lambda|^{k/2}$, <u>as</u> $\lambda \to -\infty$,

<u>and</u>

$$N^+(\lambda) \sim (2\pi)^{-k} B_k \int_{|x|>1} [(\lambda - q(x))_+]^{k/2} \, dx, \quad \underline{\text{as}}\ \lambda \to +\infty,$$

<u>where $B_k$ denotes the volume of the unit ball in $\mathbb{R}^k$.</u>

*Acknowledgments.* The second author (Michel L. Lapidus) was a member of the Mathematical Sciences Research Institute at Berkeley while part of this paper was completed. He would like to thank the Institute for Mathematics and its Applications at Minneapolis for its hospitality during the regional meeting of the American Mathematical Society at the University of Minnesota where this work was presented in the special session on 'variational methods in partial differential equations' in November 1984.

*Added note.* Since this paper was completed, the work of Birman and Solomyak on Dirichlet boundary value problems with indefinite weights on *bounded* domains was brought to our attention. In particular, there is some overlap between Theorem 4.1 herein and Theorem 2 in Birman-Solomyak "Spectral asymptotics of nonsmooth elliptic operators. II," *Trans. Moscow Math. Soc.* (AMS Transl.) 28(1973), 1-32 and Theorem 2 in Birman-Solomyak "Asymptotics of the spectrum of variational problems on solutions of elliptic equations," *Siberian Math. J.* (AMS Transl.) 20(1979), 1-15. These references will be properly acknowledged in a forthcoming version of this paper.

## REFERENCES

[Ad] R. A. Adams, Sobolev spaces, Academic Press, New York, N. Y., 1975.

[Ag] S. Agmon, Lectures on elliptic boundary value problems, Van Nostrand, Princeton, N. J., 1965.

[Bo] M. Bôcher, Boundary problems in one dimension, Proc. Fifth Intern. Congress Math. (Cambridge, 1912), Vol. I, Cambridge Univ. Press, 1913, pp. 163-195.

[Br] F. E. Browder, Le problème des vibrations pour un opérateur aux dérivées partielles self-adjoint et du type elliptique à coefficients variables, C. R. Acad. Sci. Paris Sér. A 236 (1953), 2140-2142 (June 1, 1953).

[CH] R. Courant and D. Hilbert, Methods of mathematical physics, Vol. I, English Transl., Interscience, New York, N. Y., 1953.

[DGHa] I. Dee Chang, G. W. Grube and E. Y. Harper, On the breakup of accelerating liquid drops, J. Fluid Mech. 52 (1972), 565-591.

[**dF**] D. G. de Figueiredo, Positive solutions of semilinear elliptic problems, Lecture Notes in Math., vol. 957, Springer-Verlag, Berlin, 1982, pp. 34-87.

[**Fl**] J. Fleckinger, Estimate of the number of eigenvalues for an operator of Schrödinger type, Proc. Roy. Soc. Edinburgh Sect. A **89** (1981), 355-361.

[**FlF 1**] J. Fleckinger and M. El Fetnassi, Asymptotics of eigenvalues of variational elliptic problems with indefinite weight function, Proc. May-June 1984 Workshop "Spectral theory of Sturm-Liouville Differential Operators," (Hans G. Kaper and A. Zettl eds.), ANL-84-73, Argonne National Laboratory, Argonne, Illinois, 1984, pp. 107-118.

[**FlF 2**] ____________________, Comportement asymptotique des valeurs propres de problèmes elliptiques "non définis à droite", C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), 599-602 (November 7, 1984).

[**FlMe**] J. Fleckinger and G. Métivier, Théorie spectrale des opérateurs uniformément elliptiques sur quelques ouverts irréguliers, C. R. Acad. Sci. Paris Sér A **276** (1973), 913-916.

[Ga] L. Gårding, The asymptotic distribution of the eigenvalues and eigenfunctions of elliptic operators, Math. Scand. 1 (1953), 55-72.

[GN] W. N. Gill and R. J. Nunge, Analysis of heat or mass transfer in some countercurrent flows, Intern. J. Heat Mass Transfer 8 (1965), 873-886.

[Gu] D. Gurarie, $L^p$ and spectral theory for a class of global elliptic operators on $R^n$, Preprint #84-10, Case Western Reserve Univ., Dept. of Mathematics and Statistics, Apr. 1984.

[HK] P. Hess and T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, Comm. Partial Differential Equations 5 (1980), 999-1030.

[Hi] D. Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Teubner, Leipzig, 1912; Chelsea, New York, N. Y., 1953.

[Ho] E. Holmgren, Uber Randwertaufgaben bei einer linearer Differentialgleichung zweiter ordnung, Arkiv för Mat., Astro och Fysik 1 (1904), 401-417.

[Kc] M. Kac, Can one hear the shape of a drum? , Amer. Math. Monthly (Slaught Memorial Papers, No. 11) (4) 73 (1966), 1-23.

[KKLeZ] H. G. Kaper, M. K. Kwong, C. G. Lekkerkerker and A. Zettl, Full- and half-range eigenfunctions expansions for Sturm-Liouville problems with indefinite weights, Preprint, Argonne National Laboratory, Mathematics and Computer Science Division, Jan. 1984.

[La 1] M. L. Lapidus, Valeurs propres du laplacien avec un poids qui change de signe, C. R. Acad. Sci. Paris Sér. I Math. **298** (1984), 265-268 (March 28, 1984).

[La 2] ______________, Spectral theory of elliptic problems with indefinite weights, Proc. May-June 1984 Workshop "Spectral Theory of Sturm-Liouville Differential Operators," (Hans G. Kaper and A. Zettl eds.), ANL-84-73, Argonne National Laboratory, Argonne, Illinois, 1984, pp. 159-168.

[LiY] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem, Comm. Math. Phys. **88** (1983), 309-318.

[Lb] E. Lieb, The number of bound states of one-body Schrödinger operators and the Weyl problem, Proc. Symp. Pure Math., Vol. 36, American Mathematical Society, Providence, R.I., 1980, pp. 241-252.

[LM] J. L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I, English Transl., Springer-Verlag, Berlin, 1972.

[LoS] L. H. Loomis and S. Sternberg, Advanced calculus, Addison-Wesley, Reading, Mass., 1968.

[LuR] G. S. S. Ludford and R. A. Robertson, Fully diffused regions, SIAM J. Appl. Math. 25 (1973), 693-703.

[LuW] G. S. S. Ludford and S. S. Wilson, Subcharacteristic reversal, SIAM J. Appl. Math. 27 (1974), 430-440.

[MM] A. Manes and A. M. Micheletti, Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine, Boll. Un. Mat. Ital. 7 (1973), 285-301.

[Me] G. Métivier, Valeurs propres de problèmes aux limites elliptiques irréguliers, Bull. Soc. Math. France Mém. 51-52 (1977), 125-219.

[Mi] A. B. Mingarelli, Volterra-Stieltjes integral equations and generalized ordinary differential expressions, Lecture Notes in Math., vol. 989, Springer-Verlag, Berlin, 1980.

[Pe] I. N. Pesin, Classical and modern integration theories, English Transl., Academic Press, New York, N. Y., 1970.

[Pl] Å. Pleijel, Sur la distribution des valeurs propres de problèmes régis par l'équation $\Delta u + \lambda k(x,y)u = 0$, Arkiv för Mat., Astr. och Fysik **29 B** (1942), 1-8.

[RS] M. Reed and B. Simon, Methods of modern mathematical physics, Vol. IV: Analysis of operators, Academic Press, New York, N. Y., 1978.

[Ri] R. G. D. Richardson, Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order, Amer. J. Math. **40** (1918), 283-316.

[Ro 1] G. V. Rozenbljum, Asymptotics of the eigenvalues of the Schrödinger operator, Math. USSR-Sb. 22 (1974), 349-371.

[Ro 2] ____________, Asymptotics of the eigenvalues of the Schrödinger operator, Problemy Mathematiceskogo Analiza 5 (1975), 152-166 (Russian).

[T] E. C. Tichmarsh, Eigenfunction expansions associated with second-order differential equations, Part II, Oxford Univ. Press, London, 1958.

[Wn] H. F. Weinberger, Variational methods for eigenvalue approximation, Regional Conference Series in Applied Mathematics, vol. 15, SIAM, Philadelphia, Penn., 1974.

[We] H. Weyl, Das asymptotische Verteilungsgesetz der Eingenwerte linearer partieller Differentialgleichungen, Math. Ann. 71 (1911), 441-469.

[Y] S. T. Yau, Survey on partial differential equations in differential geometry, Seminar on Differential Geometry (S. T. Yau ed.), Annals of Mathematics Studies No. 102, Princeton Univ. Press, Princeton, N. J., 1982, pp. 3-71.

**JACQUELINE FLECKINGER**
**UNIVERSITE PAUL SABATIER,**
**DEPARTEMENT DE MATHEMATIQUES IR2,**
**118, ROUTE DE NARBONNE,**
**31062, TOULOUSE, CEDEX, FRANCE.**

**MICHEL L. LAPIDUS**
**MATHEMATICAL SCIENCES RESEARCH INSTITUTE**
**223 FULTON ST., ROOM 603**
**BERKELEY, CA 94720**

*Current address:* **UNIVERSITY OF SOUTHERN CALIFORNIA,**
**DEPARTMENT OF MATHEMATICS DRB-306,**
**UNIVERSITY PARK, MC-1113,**
**LOS ANGELES, CA 90089-113, USA.**