An efficient algorithm for locating and continuing connecting orbits.

M. J. Friedman*, J. W. Demmel
Department of Mathematical Sciences, University of Alabama in Huntsville,
Huntsville, AL 35899.
Computer Science Division, University of California,
Berkeley, CA 94720-1776.

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Abstract

A successive continuation method for locating connecting orbits in parametrized systems of autonomous ODEs was considered in [9]. In this paper we present an improved algorithm for locating and continuing connecting orbits, which includes a new algorithm for the continuation of invariant subspaces based on iterative refinement techniques.

1 Introduction.

Homoclinic and *heteroclinic orbits,* also called *connecting orbits,* are trajectories connecting equilibrium points of a system of autonomous ordinary differential equations. Computation of connecting orbits is becoming increasingly important, both in dynamical systems research, such as understanding chaotic dynamics, and in a variety of applied problems, including wave propagation in combustion models, chemical reactions, neuronal interactions, solitary waves in fluid, solitons in nonlinear optical fiber, and communication processes in living cells, to name a few. The corresponding numerical problem is that of finding solutions $(u(t), \lambda)$ of the system of autonomous ODEs

$$u'(t) - f(u(t), \lambda) = 0, \quad u(\cdot), f(\cdot, \cdot) \in \mathbb{R}^n, \lambda \in \mathbb{R}^{n_\lambda}, \tag{1}$$

$$\lim_{t \to -\infty} u(t) = u_0, \quad \lim_{t \to +\infty} u(t) = u_1. \tag{2}$$

Most algorithms for the numerical analysis of connecting orbits reduce (1), (2) to a *boundary value problem* on a finite interval using linear or higher order approximations of stable and unstable manifolds near u_0 and u_1, respectively. See recent papers by Champneys,

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Kuznetsov, Sandstede [4], by Doedel, Friedman, Kunin [9] and Moore [13] for the history of the question and the bibliography. Note that in the last work an alternative approach was used based on the arclength parametrization, instead of using time t as a parameter.

The algorithms in [4] use a version of Beyn's continuation algorithm based on projection boundary conditions ([1], [2]). They were implemented in a code HomCont which is currently a part of AUTO97 [7]. HomCont has capabilities for detailed analysis of bifurcations of homoclinic orbits and some bifurcations of heteroclinic orbits, and it is user friendly. At the same time it has limited capabilities for locating connecting orbits (a simplified version of our algorithm in [9]).

The algorithms in [9] have their primary focus on locating connecting orbits and use a modification of a continuation algorithm based on projection boundary conditions (Friedman, Doedel [11]). They were implemented in an experimental code based on AUTO94 [8].

In order to have a well posed problem, it is necessary for the boundary conditions to be sufficiently smooth with respect to parameters. Both in [4] and [9], the boundary conditions are defined with respect to bases of stable or unstable eigenspaces of $f_u(u_0, \lambda)$ and $f_u(u_1, \lambda)$. The approach in [4] is to compute an orthonormal basis in the appropriate eigenspace, *at each pseudo arclength continuation step,* using a "black box" routine based on the real Schur factorization and then to adapt this basis to be smooth with respect to parameters, using a technique due to Beyn [2, App. C] which amounts to the solution of a linear system of the dimension of the eigenspace in question. The approach in [9] is to compute initially an orthonormal basis in the appropriate eigenspace via the real Schur factorization and then to continue the real Schur factorization equations (as a part of boundary conditions). We found HomCont to be unstable, while the algorithm in [9] worked well, in a situation where an eigenvalue varied rapidly with a slow change of a continuation parameter (a problem with a boundary layer type behavior). The current version of HomCont does not have safeguards against rapid changes of eigenvalues, while our new algorithm below provides several possible safeguards. At the same time precise convergence of the algorithm in [9] is not clear, and it is somewhat cumbersome to use. We also want to mention work by Dieci and Eirola [6] which provides a general differential equations framework for continuation of the block Schur factorization as well as other matrix factorizations. Reference [6] includes a comprehensive set of references for smooth matrix factorization for parameter dependent matrices.

In this paper we present an improved algorithm for locating and continuing connecting orbits, which includes a new algorithm for the continuation of invariant subspaces (CIS) based on iterative refinement techniques Stewart [14], Demmel [5]. This new algorithm is more efficient then the algorithms in [4] and [9] and is very robust. It has been implemented in an experimental code based on AUTO97 which is essentially a modification of HomCont part of AUTO97 to include the algorithm in [9] for locating and continuing connecting orbits and the CIS algorithm, while preserving the bifurcation analysis part of HomCont.

2 An improved algorithm for locating and continuing connecting orbits.

Assume, for simplicity of notation, that the fixed points u_0 and u_1 are hyperbolic, and the eigenvalues of $f_u(u_0,\lambda)$ and $f_u(u_1,\lambda)$, respectively, satisfy

$$\begin{aligned}\mathrm{Re}\,\mu_{0,n} &\leq \ldots \leq \mathrm{Re}\,\mu_{0,n_0+1} < 0 < \mu_{0,1} < \mathrm{Re}\,\mu_{0,2} \leq \ldots \leq \mathrm{Re}\,\mu_{0,n_0},\\ \mathrm{Re}\,\mu_{1,1} &\leq \ldots \leq \mathrm{Re}\,\mu_{1,n_1} < 0 < \mathrm{Re}\,\mu_{1,n_1+1} \leq \ldots \leq \mathrm{Re}\,\mu_{1,n}.\end{aligned}$$

The method extends to the case $\mu_{0,1} = 0$, as in [4]. It also extends to the cases of complex and multiple $\mu_{0,1}$ by a simple modification of Step 0, eq. (11) below, of the algorithm (see [9, Section 4.3] for a computational example). The algorithm requires evaluation of various projections associated with the eigenspaces of $f_u(u_0,\lambda)$ and $f_u(u_1,\lambda)$. Initially these projections are constructed using the real Schur factorizations [12]

$$f_u(u_0,\lambda) = Q_0 T_0 Q_0^T, \quad f_u(u_1,\lambda) = Q_1 T_1 Q_1^T.$$

The first factorization has been chosen so that the first n_0 columns $q_{0,1},\ldots,q_{0,n_0}$ of Q_0 form an orthonormal basis of the right invariant subspace S_0 of $f_u(u_0,\lambda)$, corresponding to $\mu_{0,1},\ldots,\mu_{0,n_0}$, and the remaining $n-n_0$ columns $q_{0,n_0+1},\ldots,q_{0,n}$ of Q_0 form an orthonormal basis of the orthogonal complement $S_0^\perp$. Similarly, the first n_1 columns $q_{1,1},\ldots,q_{1,n}$ of Q_1 form an orthonormal basis of the right invariant subspace S_1 of $f_u(u_1,\lambda)$, corresponding to $\mu_{1,1},\ldots,\mu_{1,n_1}$, and the remaining $n-n_1$ columns $q_{1,n_1+1},\ldots,q_{1,n}$ of Q_1 form an orthonormal basis of the orthogonal complement $S_1^\perp$. In the algorithm below the matrices $Q_0(\lambda)$ and $Q_1(\lambda)$ are assumed to be computed at each continuation step by a "black box" routine, described in Section 3, which ensures their continuity.

The approximate finite interval problem is to find a branch of solutions $(u(t),\lambda,u_0,u_1,T)$, $u \in C^1([0,1]),\mathbb{R}^n)$, $\lambda \in \mathbb{R}^{n_\lambda}$, provided $n_\lambda = n-(n_0+n_1)+2$; $n_\lambda \geq 0$, $u_0,u_1 \in \mathbb{R}^n$, $T>0$, is the length of the time interval, for some small $\epsilon_0,\epsilon_1 > 0$, of the time-scaled differential equation

$$u'(t) - T f(u(t),\lambda) = 0, \quad 0 < t < 1, \tag{3}$$

subject to left boundary conditions

$$\begin{aligned}(u(0)-u_0)\cdot q_{0,n_0+i}(u_0,\lambda) &= 0, \quad i = 1,\ldots,n-n_0, &(4)\\ |u(0)-u_0| &= \epsilon_0, &(5)\end{aligned}$$

right boundary conditions

$$\begin{aligned}(u(1)-u_1)\cdot q_{1,n_1+i}(u_1,\lambda) &= 0, \quad i = 1,\ldots,= n-n_1. &(6)\\ |u(1)-u_1| &= \epsilon_1 &(7)\end{aligned}$$

stationary state conditions

$$\begin{aligned}f(u_0,\lambda) &= 0, &(8)\\ f(u_1,\lambda) &= 0. &(9)\end{aligned}$$

Remark 1 *Initially, we perform time integration to obtain a, typically, crude orbit with initial point* $u(0) \in S_0$ *but the terminal point* $u(1) \notin S_1$, *in general. Hence* τ_i *defined by*

$$\tau_i = (u(1) - u_1) \cdot q_{1,n_1+i}(u_1, \lambda)/\epsilon_1, \quad i = 1, ..., n_\tau = n - n_1, \tag{10}$$

are, in general, nonzero, and the initial connecting orbit on the branch of connecting orbits is found via a sequence of homotopies that locate successive zero intercepts of the τ_j *in (10). In each homotopy step we compute a branch, i.e., a one-dimensional manifold, of solutions. For this we must have* $n_c - n_v = n - 1$, *where* n_c *is the number of constraints, and* n_v *is the number of scalar variables. We keep* $u(1)$ *free, while* $u\,(0)$ *is allowed to vary on the surface of the sphere in* S_0 *of radius* ϵ_0 : *(i) according to equation (12) below at steps* 0 *through* n_0, *when* λ *is fixed and* c_i *in (12) play the role of the control parameters; and (ii) according to equation (5) at steps* $n_0 + 1$ *through* $n_0 + n_\lambda$, *when* λ *varies.*

Let $S_{0,k}$, $k = 1, ..., n_0$, be the right invariant subspace of $f(u_0, \lambda_0)$ corresponding to the eigenvalues $\mu_{0,1}, ..., \mu_{0,k}$. Then the first k columns $q_{0,1}, ..., q_{0,k}$ of Q_0 form an orthonormal basis of $S_{0,k}$ and the remaining $n-k$. columns $q_{0,k+1}, ..., q_{0,n}$ of Q_0 form an orthonormal basis of the orthogonal compliment $S_{0,k}^{\perp}$.

1. Initialization.
   Step 0. Initialize the problem parameter vector λ, and set the algorithm parameters ϵ_0 and T to small, positive values, so that $u(t)$ is approximately constant on $[0, T]$. Set

   $$u(t) = u_0 + \epsilon_0 c_1 q_{01}, \quad 0 \le t \le 1, \tag{11}$$

   or

   $$u(t) = u_0 + \epsilon_0 c_1 q_{01} e^{\mathrm{Re}\,\mu_{0,1} t}, \quad 0 \le t \le 1, \quad \mathrm{Re}\,\mu_{0,1} > 0,$$

   $\epsilon_1 = |u(1) - u_1|$, $c_1 = 1$, or -1 and $c_2 = ... = c_{n_0} = 0$.

2. Locating a connecting orbit, λ is fixed.
   Step 1. Time integration to get an initial orbit. Compute a solution branch to the system (3), (4), (10), (7), and

   $$(u\,(0) - u_0) \cdot q_{0,i}(u_0, \lambda)/\epsilon_0 - c_i = 0, \quad i = 1, ..., n_c = n_0, \tag{12}$$

   in the direction of increasing T, until $u(1)$ reaches an ϵ_1-neighborhood of u_1, for some $\epsilon_1 > 0$. Scalar variables are T, $\epsilon_1 \in \mathbb{R}$, $\tau \in \mathbb{R}^{n-n_1}$. There are n differential equations with $n_c = 2n - n_1 + 1$ constrains and $n_v = n - n_1 + 2$ scalar variables, and hence $n_c - n_v = n - 1$. In practice one typically continues until ϵ_1 stops decreasing, its value being not necessarily small.
   Step k, $k = 2, ..., n_0$ (for $n_0 > 1$). Compute a branch of solutions to the system (3)-(5), (10), (7), (12) to locate a zero of, say, τ_{k-1} (while $\tau_1, ..., \tau_{k-2} = 0$, fixed). Free scalar variables are $\epsilon_1, c_1, ..., c_k, \tau_{k-1}, ..., \tau_{n-n_1}$. There are n differential equations with $n_c = 2n - n_1 + 2$ constraints and $n_v = n - n_1 + 3$ scalar variables, and hence $n_c - n_v = n - 1$.

3. Locating a connecting orbit, λ varies.
   Step k, $k = n_0 + 1, ..., n_0 + n_\lambda \equiv n - n_1 + 1$. Compute a branch of solutions to the system (3)-(5), (10), (7)-(9) to locate a zero of, say, τ_{k-1}(while $\tau_1, ..., \tau_{k-2} = 0$, fixed). Free scalar variables are $\epsilon_1, \tau_{k-1}, ..., \tau_{n-n_1}, \lambda_1, ..., \lambda_{k-n_0} \in \mathbb{R}$, $u_0, u_1 \in \mathbb{R}^n$. There are n differential equations with $n_c = 4n - n_0 - n_1 + 2$ constraints and $n_v = 3n - n_0 - n_1 + 3$ scalar variables, and hence $n_c - n_v = n - 1$.

4. Increasing the accuracy of the connecting orbit.
   Compute a branch of solutions to the system (3)-(9) in the direction of decreasing ϵ_1 until it is 'small'. Free scalar variables are $\epsilon_1, T, \lambda_1, ..., \lambda_{n_\lambda - 1} \in \mathbb{R}$, $u_0, u_1 \in \mathbb{R}^n$. As before, $n_c = 4n - n_0 - n_1 + 2$, $n_v = 3n - n_0 - n_1 + 3$.

5. Continue the connecting orbit.
   Compute a branch of solutions to the system (3)-(9). Free scalar variables are $T, \lambda_1, ..., \lambda_{n_\lambda} \in \mathbb{R}$, $u_0, u_1 \in \mathbb{R}^n$. As before, $n_c = 4n - n_0 - n_1 + 2$, $n_v = 3n - n_0 - n_1 + 3$. Alternatively, a phase condition
   $$\int_0^1 (u'(t) - q'(t)) \cdot u''(t)\, dt = 0 \tag{13}$$
   may be added if T is kept fixed and ϵ_0 and ϵ_1 are allowed to vary. Here $q(t)$ is a previously computed orbit on the branch.

Remark 2 *Our algorithm of continuation of invariant subspaces of $f_u(u_0, \lambda)$ and $f_u(u_1, \lambda)$ in Section 3 breaks only if two eigenvalues, one associated with the subspace being continued and one not, approach the same point on the imaginary axis (one from the left and one from the right). In this case the algorithm should stop anyway since a bifurcation is being approached.*

3 Continuation of Invariant Subspaces.

Let $A(\lambda) \in \mathbb{R}^{n \times n}$ denote one of the following: $f_u(u_0, \lambda)$, $f_u(u_1, \lambda)$. The basic continuation algorithm requires at each pseudo arclength continuation step computation of a right invariant (typically, stable or unstable) m-dimensional subspace $S(\lambda)$ of $A(\lambda)$. It is crucial that $S(\lambda)$ is continuous when $A(\lambda)$ is. In this section we construct a continuous orthogonal matrix $Q(\lambda) = [Q_1(\lambda)|Q_2(\lambda)] \in \mathbb{R}^{n \times n}$, $Q_1(\lambda) \in \mathbb{R}^{n \times m}$, $Q_2(\lambda) \in \mathbb{R}^{n \times (n-m)}$, where the columns of $Q_1(\lambda)$ span $S(\lambda)$, and the columns of $Q_2(\lambda)$ span the orthogonal complementary subspace $S(\lambda)^\perp$. Moreover, let $S_k(\lambda)$, $k = 1, ..., m$, be the right invariant subspace of $A(\lambda)$ corresponding to the first k eigenvalues $\mu_1, ..., \mu_k$. of A. Then the first k columns $q_1, ..., q_k$ of $Q(\lambda)$ form an orthonormal basis of $S_k(\lambda)$ and the remaining $n - k$ columns $q_{k+1}, ..., q_n$ of $Q(\lambda)$ form an orthonormal basis of the orthogonal compliment $S_k(\lambda)^\perp$.

Suppose that initially we have computed the real Schur factorization

$$A(0) = Q(0)T(0)Q^T(0), \quad Q(0) = [Q_1(0)|Q_2(0)], \tag{14}$$

where $T(0)$ is in block Schur form

$$T(0) = \begin{bmatrix} T_{11}(0) & T_{12}(0) \\ & T_{22}(0) \end{bmatrix},$$

the columns of $Q_1(0)$ span an invariant subspace $S(0)$ of $A(0)$, and the columns of $Q_2(0)$ span the orthogonal complementary subspace $S(0)^\perp$.

For λ with $|\lambda|$ small define the perturbed matrix $T(0)+E(\lambda)$ by

$$Q^T(0)A(\lambda)Q(0) = Q^T(0)[A(0)+(A(\lambda)-A(0))]Q(0) \equiv T(0)+E(\lambda) \equiv \begin{bmatrix} \widehat{T}_{11} & \widehat{T}_{12} \\ E_{21} & \widehat{T}_{22} \end{bmatrix}. \quad (15)$$

We *interpret* $S(0)$, $Q_1(0)$ and $Q_2(0)$ as *approximations* to $S(\lambda)$, $Q_1(\lambda)$ and $Q_2(\lambda)$, respectively. This allows us to use *iterative refinement techniques* to *improve the accuracy of computed invariant subspaces.* The methods are due to Stewart [14], Dongarra, Moler and Wilkinson [10], and Chatelin [3]. Even though these methods all apparently solve different equations, as shown by Demmel [5], after changing variables, they all solve the same Riccati equation in the inner loop.

Let us follow the Stewart's approach. By (15), $Q_1(0)$ spans an approximate invariant subspace $S(0)$ and $Q_2(0)$ spans $S(0)^\perp$. Let

$$Q_1(\lambda) = (Q_1(0)+Q_2(0)Y)(I+Y^TY)^{-1/2}, \quad (16)$$

and

$$Q_2(\lambda) = (Q_2(0)-Q_1(0)Y^T)(I+YY^T)^{-1/2}, \quad (17)$$

where $Y \in \mathbb{R}^{(n-m)\times m}$ is a matrix to be determined so that $Q_1(\lambda)$ spans $S(\lambda)$. It is easy to see that $Q(\lambda) = [Q_1(\lambda)|Q_2(\lambda)]$ is orthogonal. Hence $Q_1(\lambda)$ spans an invariant subspace $S(\lambda)$ if and only if $Q_2^T(\lambda)A(\lambda)Q_1(\lambda) = 0$. Writing this condition out in terms of (16) and (17) and using (15) yields the following nonlinear equation for Y :

$$\widehat{T}_{22}Y - Y\widehat{T}_{11} = -E_{21} + Y\widehat{T}_{12}Y, \quad (18)$$

which is the well-known algebraic Riccati equation. We may use the following two iterative methods to solve (18):

1. The iteration [14]:
$$\widehat{T}_{22}Y_k - Y_k\widehat{T}_{11} = -E_{21} + Y_{k-1}\widehat{T}_{12}Y_{k-1} \quad (19)$$
with $Y_0 = 0$, $k = 1, 2, \ldots, .$

2. The Newton iteration [5]:
$$(\widehat{T}_{22} - Y_{k-1}\widehat{T}_{12})Y_k - Y_k(\widehat{T}_{11} + \widehat{T}_{12}Y_{k-1}) = -E_{21} - Y_{k-1}\widehat{T}_{12}Y_{k-1} \quad (20)$$
with $Y_0 = 0$, $k = 1, 2, \ldots, .$

Therefore, we only need to solve a Sylvester equation in the inner loop of the iterative refinement. This can be effectively solved by using LAPACK routines. As shown in the convergence analysis for the iterative solvers (19) and (20) of the Riccati equation by Stewart [14] and Demmel [5], respectively, if we let

$$\kappa = \frac{\|\widehat{T}_{12}\|_F \|E_{21}\|_F}{\mathrm{sep}^2(\widehat{T}_{11}, \widehat{T}_{22})}, \quad (21)$$

then under the assumptions $\kappa < 1/4$ and $\kappa < 1/12$, the iterations (19) and (20) converge, linearly and quadratically, respectively.

The parameter κ can be interpreted as follows. Its numerator, $\|\widehat{T}_{12}\|_F \|E_{21}\|_F$ measures the quality of the initial approximate invariant subspace: it will be small when the approximation is good, and the factor $\|E_{21}\|_F$ will be zero if and only if the initial approximation is in fact correct. The function $\mathrm{sep}(\widehat{T}_{11}, \widehat{T}_{22})$ in the denominator is the smallest singular value of the operator which maps Y to $\widehat{T}_{22}Y - Y\widehat{T}_{11}$, and it measures the separation of the spectra of $\widehat{T}_{11}$ and $\widehat{T}_{22}$. If $\mathrm{sep}(\widehat{T}_{11}, \widehat{T}_{22})$ is small, it means that some eigenvalues of $\widehat{T}_{11}$ and $\widehat{T}_{22}$ can be made to merge with small changes in $\widehat{T}_{ii}$; this means that the invariant subspaces belonging to the two parts of the spectrum are unstable and hard to compute. Thus κ will be small if we start with a good initial approximate invariant subspace and if the eigenvalues associated with that subspace are well separated from the remainder of the spectrum. In summary, the both algorithms converge if (i) the spectra of $T_{11}(0)$ and $T_{22}(0)$ are far enough apart, and (ii) the perturbation $E(\lambda)$ of $T(0)$ is small enough.

Remark 3 *We point to some additional safeguards, which ensure proper performance of the algorithms (19) and (20).*

1. *Note that from eq. (15)* $\|E_{21}\|_F \leq \|E\|_F \leq \|A(\lambda) - A(0)\|_F$. *Hence by eq. (21)*

$$\|A(\lambda) - A(0)\|_F \alpha \leq \frac{1}{4}, \quad \alpha = \frac{\|\widehat{T}_{12}\|_F}{sep^2(\widehat{T}_{11}, \widehat{T}_{22})}, \tag{22}$$

   implies $\kappa < 1/4$. *Therefore we are guaranteed that for any matrix* $X \in \mathbb{R}^{n\times n}$ *with* $\|X - A(0)\|_F \alpha \leq \frac{1}{4}$ *we can find the invariant subspaces* S *and* $S^\perp$ *by Algorithm (19), say. And no eigenvalue of* $A\,|_S$ *can 'merge' with an eigenvalue from* $A\,|_{S^\perp}$, *where* $A\,|_S$ *denotes the restriction of* A *to* S, *etc. In other words, the eigenvalues of* $A\,|_S$ *and* $A\,|_{S^\perp}$ *remain separated for all matrices in a ball around* $A(0)$ *of radius* $1/(4\alpha)$. *Thus employing the safeguard (22) ensures that the iterations (19) and (20) will diverge only when a small perturbation* $A(\lambda)$ *of* $A(0)$ *will make an eigenvalue from* $S(\lambda)$ *and an eigenvalue from* $S(\lambda)^\perp$ *coalesce.*

2. *The quantity* $\|E_{21}\|_F / sep(\widehat{T}_{11}, \widehat{T}_{22})$ *can be interpreted as the tangent of the angle between the subspaces spanned by* $Q_1(0)$ *and* $Q_1(\lambda)$. *Hence the convergence of our algorithms implies that this angle always stays between* 0 *and* $\pi/2$. *It is useful to monitor this angle and, in some situations, control it. A convenient measure in this case is the sine of this angle given [12] by*

$$\mathrm{dist}(Q_1(0), Q_1(\lambda)) = \sqrt{1 - \sigma^2_{\min}\left(Q_1^T(0), Q_1(\lambda)\right)}.$$

   Substituting into this equation the expression (16) for $Q_1(\lambda)$ *gives*

$$\mathrm{dist}(Q_1(0), Q_1(\lambda)) = \sqrt{1 - \sigma^2_{\min}\left(I + Y^T Y\right)^{-1/2}} = \frac{\|Y\|_2}{\sqrt{1 + \|Y\|_2^2}} \tag{23}$$

4 Example: Heteroclinic orbits in a 4-D singular perturbation problem.

Consider the problem of finding traveling wave front solutions to the FitzHugh-Nagumo equations with two diffusive variables

$$v_t = v_{xx} + v(v-a)(1-v) - w, \quad w_t = \delta w_{xx} + \epsilon(v - \gamma w), \tag{24}$$

for small positive ϵ, while δ ranging between a small and large value. For δ small this is a singularly perturbed reaction-diffusion system. In moving coordinates, $v_1 = v(z)$, $v_2 = v'(z)$, $w_1 = w(z)$, $w_2 = w'(z)$ with $z = t + cx$, the reduced ODE is

$$\begin{aligned} v_1' &= v_2, \\ v_2' &= cv_2 - v_1(1-v_1)(v_1 - a) + w, \\ w_1' &= w_2, \\ w_2' &= [cw_2 - \epsilon(v_1 - \gamma w_1)]/\delta. \end{aligned} \tag{25}$$

In [9] we located a heteroclinic orbit of (25). Here we first reproduce this result with our new code and locate a heteroclinic orbit with $\delta = \epsilon = .001$, $\gamma = 13.8$, $a = .3$, and $c = .2572502$. In this case $n_0 = n_1 = 2$, where the relevant eigenvalues are $\mu_{0,1} = .3974$, $\mu_{0,2} = 257.304$, $\mu_{1,1} = -.4246$, and $\mu_{1,2} = -0.0676$. We then performed a two parameter continuation in (δ, c) in the direction of increasing δ. For $(\delta, c) = (.8268, .20799)$ the eigenvalues are: $\mu_{0,1} = .6958$, $\mu_{0,2} = .6542$, $\mu_{1,1} = -.4492$, and $\mu_{1,2} = -0.0889$. This computation is of interest because as δ increases, the eigenvalue $\mu_{0,2}$ rapidly decreases.

Note, that an attempt to perform this two parameter continuation with HomCont resulted in spurious solutions. See a discussion in Introduction.

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