Propagation of cracks in elastic media

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1 Formulation of the problem

In this paper we consider a quasi-stationary model of crack propagation in a two-dimensional elastic medium occupying a bounded domain Ω. The model, developed in [3], is based on earlier work [1], [4], [9], [10], [11]. The motion of the tip $X(t)$ of the crack $\Gamma(t)$ at time t is given by

$$\dot{X}(t) = \Phi(|\dot{X}(t)|)\mathbf{J}(X(t)) \tag{1.1}$$

where $\Phi(s)$ is an explicit (and rather simple) function of s and $\mathbf{J}(X(t))$ is defined in terms of the, so called, J-integral. The stress function $\varphi(x,t)$ satisfies:

$$\varphi \in H^2(\Omega) \quad \text{for } t > 0, \tag{1.2}$$

$$\Delta^2 \varphi = 0 \quad \text{in } \Omega \setminus \Gamma(t), \tag{1.3}$$

$$\varphi = \frac{\partial \varphi}{\partial n} = 0 \quad \text{from both sides of } \Gamma(t)\ , \tag{1.4}$$

$$\varphi = g, \qquad \frac{\partial \varphi}{\partial n} = h \quad \text{on } \partial\Omega, \tag{1.5}$$

and the initial portion of the crack is given by a curve which, for simplicity, we take to be a graph:

$$\begin{aligned} &\Gamma(0) = \Gamma_0 = \{x_2 = f(x_1),\ \ -x_0 \leqslant x_1 \leqslant 0\}, \\ &\quad \text{where } (-x_0, f(-x_0)) \in \partial\Omega,\ \Gamma_0 \setminus (-x_0, f(-x_0)) \in \Omega. \end{aligned} \tag{1.6}$$

Throughout this paper the normal n denotes the interior normal direction; i.e., $n = (-f'(x_1), 1)/\sqrt{1 + |f'(x_1)|^2}$ when x approaches from above Γ (denoted by Γ^+) and $n = (f'(x_1), -1)/\sqrt{1 + |f'(x_1)|^2}$ from below Γ (denoted by Γ^-).

In a recent paper [2] the present authors studied the asymptotic behavior of any solution of (1.2)–(1.4) in a neighborhood of the tip $X(t)$. Taking for simplicity $t = 0$ and assuming that

$$f(0) = 0, \quad f'(0) = 0, \tag{1.7}$$

they proved the following results:

(i) If $f \in C^{1+\alpha}[-\delta_0, 0]$, then

$$\varphi(x) = A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta) + O(r^{2-\eta}) \tag{1.8}$$

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for any $\eta > 0$ such that $\alpha + \eta > 1/2$.

(ii) If $f \in C^{2+\alpha}[-\delta_0, 0]$, then

$$\begin{aligned}\varphi(r,\theta) &= A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta) + A_3 r^2 B_3(\theta) + A_4 r^{5/2} B_4(\theta) \\ &\quad + A_5 r^{5/2} B_5(\theta) - 2A_2 r^{5/2} f''(0) \cos\frac{1}{2}\theta + O(r^{3-\eta})\end{aligned} \tag{1.9}$$

for any $\eta > 0$ such that $\alpha + \eta > 1/2$.

Here

$$\begin{aligned}B_1(\theta) = \cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta, \quad B_2(\theta) = \sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta, \quad B_3(\theta) = \sin^2\theta, \\ B_4(\theta) = \cos\frac{5}{2}\theta - 5\cos\frac{1}{2}\theta, \quad B_5(\theta) = \sin\frac{5}{2}\theta - \sin\frac{1}{2}\theta;\end{aligned} \tag{1.10}$$

note that $r^2 B_3(\theta) = x_2^2$.

Similar formulas hold near a tip $X(t)$, i.e.,

$$\varphi(x,t) = A_1(t)|x - X(t)|^{3/2} \widetilde{B}_1(\widetilde{\theta}) + A_2(t)|x - X(t)|^{3/2} \widetilde{B}_2(\widetilde{\theta}) + \cdots$$

where $\widetilde{B}_j(\widetilde{\theta})$ is obtained from $B_j(\theta)$ by rotating the coordinate systems so that the tangent to $\Gamma(t)$ at $X(t)$ is in the direction $\widetilde{\theta} = 0$.

The coefficients $A_1(t)$, $A_2(t)$ are called the *stress intensity factors*.

It was proved in [2] that if

$$A_1(0) \neq 0, \quad A_2(0) = 0 \tag{1.11}$$

then, for a $C^{1+\alpha}$ crack $\Gamma(t)$, the law (1.1) is equivalent to the condition

$$A_2(t) \equiv 0. \tag{1.12}$$

This condition means that

$$\varphi_{\tau\tau}(x,t) \sim \frac{K}{|x - X(t)|^{1/2}} \quad (K \neq 0), \qquad \varphi_{\tau n} \to 0 \tag{1.13}$$

as x approaches $X(t)$ from $\Omega \setminus \Gamma(t)$ along the tangent direction τ; n is normal to τ. This condition can also be written in terms of the stress tensor σ:

$$\sigma_{nn} \sim \frac{K}{|x - X(t)|^{1/2}} \quad (K \neq 0), \qquad \sigma_{\tau n} \to 0. \tag{1.14}$$

These are precisely the conditions which characterize *mode I fracture* (also called *opening mode*); see [8, p.24].

It was further proved in [2] that the dynamic problem (1.1)–(1.6), for some time interval $0 < t < t_0$, is equivalent to the following geometric problem:

Find a curve

$$\Gamma_{s_0} = \{x_2 = f(x_1), \ -x_0 \leqslant x_1 \leqslant s_0\}, \quad s_0 > 0 \tag{1.15}$$

and a function $\varphi(x,s)$ such that if

$$\Gamma_s = \{x_2 = f(x_1), \ -x_0 \leqslant x_1 \leqslant s\}, \quad 0 \leqslant s \leqslant s_0, \tag{1.16}$$

then

$$\varphi \in H^2(\Omega) \quad \text{for } 0 \leqslant s \leqslant s_0, \tag{1.17}$$

$$\Delta^2 \varphi = 0 \quad \text{in } \Omega \setminus \Gamma_s, \tag{1.18}$$

$$\varphi = \frac{\partial \varphi}{\partial n} = 0 \quad \text{from both sides of } \Gamma_s \ , \tag{1.19}$$

$$\varphi = g, \qquad \frac{\partial \varphi}{\partial n} = h \quad \text{on } \partial\Omega, \tag{1.20}$$

and

$$A_2(s) = 0 \quad \text{for } 0 \leqslant s \leqslant s_0 \tag{1.21}$$

where $A_2(s)$ is the stress intensity factor which arises in the asymptotic expansion about the tip Γ_s:

$$\varphi(x,s) = A_1(s)|x - X(s)|^{3/2}\widetilde{B}_1(\widetilde{\theta}) + A_2(s)|x - X(s)|^{3/2}\widetilde{B}_2(\widetilde{\theta}) + \cdots. \tag{1.22}$$

Here it is assumed that $\Gamma(t_0)$, or Γ_{s_0}, is in $C^{1+\alpha}$, Γ_s for $s = 0$ coincides with the $\Gamma(0)$ in (1.6), and the connection between $f(s)$ $(0 \leqslant s > s_0)$ and $X(t)$ is given by

$$f(s) = X_2(X_1^{-1}(s)).$$

Once $f(s)$ has been found, $X(t)$ can be obtained by solving the differential equation

$$\dot{X}(t) = \Phi(|\dot{X}(t)|)J(t, f(t)).$$

Definition. Problem (C_0) consists of finding a curve Γ_{s_0} in $C^{1+\alpha}$ and a function $\varphi(x,s)$ which solve the system (1.16)–(1.21).

In this paper we shall prove that there exists a solution to problem (C_0) with Γ_{s_0} in $C^{2+\alpha}$ for some small $\alpha > 0$.

In §2 we establish a relation between the curvature of Γ_s and stress intensity factors. To describe this relation consider, for simplicity, the case $s = 0$ and let

$$\begin{gathered} \varphi_1(x) = r^{3/2}B_1(\theta), \quad \varphi_2(x) = r^{3/2}B_2(\theta), \quad \varphi_3(x) = x_2^2, \\ \varphi_4(x) = r^{5/2}B_4(\theta), \quad \varphi_5(x) = r^{5/2}B_5(\theta). \end{gathered} \tag{1.23}$$

Denote by ψ_s the solution of (1.17)–(1.20) corresponding to the curve

$$\Gamma_0 \cup \{x_2 = \frac{\kappa}{2}x_1^2, \ \ 0 \leqslant x_1 \leqslant s\} \tag{1.24}$$

where κ is the curvature of Γ_0 at 0, and $f(0) = f'(0) = 0$. By formal asymptotic analysis (using inner and outer expansion for ψ_ε) we show that

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_1$$

where

$$\psi_0 \sim A_1\varphi_1 + A_2\varphi_2 + A_3x_2^2 + A_4\varphi_4 + A_5\varphi_5 - 2A_2\kappa r^{5/2}\cos\frac{1}{2}\theta,$$

$$\psi_1 \sim -A_1\frac{\partial\varphi_1}{\partial x_1} - A_2\frac{\partial\varphi_2}{\partial x_1} + \alpha_1\varphi_1 + \alpha_2\varphi_2$$

near 0, and

$$\begin{aligned} \frac{dA_2}{ds} &= -\frac{3}{2}\kappa A_1 + \frac{3}{2}A_5 + \alpha_2, \\ \frac{dA_1}{ds} &= -\frac{1}{2}\kappa A_2 - \frac{5}{2}A_4 + \alpha_1 \end{aligned} \tag{1.25}$$

at $s = 0$.

The same relation holds for any $s, 0 < s \leqslant s_0$, with $\kappa = \kappa(s)$ the curvature of Γ_{s_0} at $(s, f(s))$. To solve problem (C_0) we need to impose the condition $dA_2/ds = 0$, i.e.,

$$\frac{3}{2}\kappa A_1 = \frac{3}{2}A_5 + \alpha_2. \tag{1.26}$$

This relation which determines the curvature of the crack in terms of intensity factors is quite remarkable. It has however an inconvenient feature in that it involves the higher order coefficient A_5.

In §3 we give an entirely different (but also formal) derivation of a relation between the curvature and stress intensity factors. Although this relation formally coincides with (1.26), it has the significant advantage in that it is expressed by lower order coefficients. To derive this relation, we initially make a change of variables $x \to \widetilde{x}$ by translation T_ε and rotation R_ε so that the extended crack (1.24) for $s = \varepsilon$ will have its tip at the origin and the tangent at the tip in the horizontal direction. We then write

$$\psi_\varepsilon = \psi_0 + \varepsilon V \quad \text{in the variable } \widetilde{x} \tag{1.27}$$

and compute the boundary conditions for V at the original curve Γ_0. We then split V into

$$V = V_1 + V_2 \tag{1.28}$$

where, roughly speaking, V_1 corresponds to the translation T_ε and V_2 corresponds to the rotation R_ε. We show, by asymptotic analysis, that

$$\begin{aligned} V_1 &\sim \widehat{\beta}_1\varphi_1 + \widehat{\beta}_2\varphi_2 - 4\kappa A_2 r^{3/2}\cos\frac{1}{2}\theta, \\ V_2 &\sim \frac{3}{2}\kappa A_2\varphi_1 - \frac{3}{2}\kappa A_1\varphi_2 + 4\kappa A_2 r^{3/2}\cos\frac{1}{2}\theta \end{aligned} \tag{1.29}$$

where

$$\widehat{\beta}_1 = -\frac{5}{2}A_4 + \alpha_1, \qquad \widehat{\beta}_2 = \frac{3}{2}A_5 + \alpha_2. \tag{1.30}$$

Since

$$\frac{dA_1}{ds} = \widehat{\beta}_1 + \frac{3}{2}\kappa A_2, \qquad \frac{dA_2}{ds} = \widehat{\beta}_2 - \frac{3}{2}\kappa A_2, \tag{1.31}$$

we obtain again the relations (1.25); in particular, (1.26) becomes

$$\frac{3}{2}\kappa A_1 = \widehat{\beta}_2. \tag{1.32}$$

Since $\widehat{\beta}_2$ is a lower order coefficient (in the asymptotic expansion of V_1), this relation is much more useful than (1.26) for the purpose of proving existence theorems.

In §§5, 6 we shall establish (1.31) rigorously for any $C^{2+\alpha}$ curves Γ_{s_0}. The proof depends on asymptotic estimates and maximum principles for biharmonic functions obtained in [2], and an extension of one of these results which is derived in §4 of this paper.

In §7 we prove that problem (C_0) has a solution with $C^{2+\alpha}$ crack Γ_{s_0} for some small $\alpha > 0$; here we use the reformulation (1.32) of the condition $A_2(s) \equiv 0$.

We also discuss (in §7) other models of crack propagation when the condition (1.11) is not satisfied.

In §8 we briefly present the extension of our results to harmonic (instead of biharmonic) functions.

2 Formal computation of dA_2/ds (first method)

We assume that Γ_0 is a $C^{2+\alpha}$ curve given by (1.6), (1.7) and

$$\Gamma_s = \Gamma_0 \cup \{x_2 = \frac{\kappa}{2} x_1^2, \quad 0 \leqslant x_1 \leqslant s\}$$

is a $C^{2+\alpha}$ extension of Γ_0, where κ is the curvature of Γ_0 at 0. We denote by $\psi_s(x)$, or $\psi(x,s)$, the solution of (1.17)–(1.20). We want to compute $dA_2(s)/ds$ at $s=0$, where $A_2(s)$ is the second stress intensity factor of $\psi(x,s)$ (cf. (1.22)).

Recall that for $C^{2+\alpha}$ curve, given by (1.6), (1.7),

$$\psi_0 \sim A_1\varphi_1(x) + A_2\varphi_2(x) + A_3 x_2^2 + A_4\varphi_4(x) + A_5\varphi_5(x) - 2A_2\kappa|x|^{5/2}\cos\frac{1}{2}\theta, . \tag{2.1}$$

For any small $\varepsilon > 0$, set $\psi = \psi(x,\varepsilon)$ and

$$\psi = \varepsilon^{3/2} G. \tag{2.2}$$

We consider x as an outer variable, and introduce an inner variable y by $x = \varepsilon y$. We shall develop ψ in the outer variable and G in the inner variable, and go from lowest order terms to higher order terms, step by step. Note that the extension $\Gamma_\varepsilon \setminus \Gamma_0$ of the crack Γ_0 is given, in the y-variable, by

$$y_2 = \frac{1}{2}\varepsilon\kappa y_1^2, \quad 0 \leqslant y_1 \leqslant 1.$$

It will be convenient to use a coordinate system $\widetilde{y}$ for which the tip of the extended crack is at the origin and the tangent at the tip is in the $\widetilde{y}_1$-direction. This is accomplished (up to order ε of precision) by the following translation and rotation:

$$\begin{cases} y_1 = 1 + \widetilde{y}_1 - \kappa\varepsilon\widetilde{y}_2, \\ y_2 = \frac{1}{2}\kappa\varepsilon + \widetilde{y}_2 + \kappa\varepsilon\widetilde{y}_1, \end{cases} \tag{2.3}$$

or, up to the order ε terms,

$$\begin{cases} \widetilde{y}_1 = -1 + y_1 + \kappa\varepsilon y_2, \\ \widetilde{y}_2 = -\frac{1}{2}\kappa\varepsilon + y_2 - \kappa\varepsilon y_1. \end{cases} \tag{2.4}$$

We begin with the lowest order terms of ψ_0:

$$\psi_0(x) \sim A_1\varphi_1(x) + A_2\varphi_2(x).$$

By (2.2), we then have

$$\begin{aligned} G(\widetilde{y}) &\sim A_1\varphi_1(y) + A_2\varphi_2(y) \\ &\sim A_1\varphi_1(\widetilde{y}+e) + A_2\varphi_2(\widetilde{y}+e) \qquad (e = (1,0)) \\ &\sim A_1\varphi_1(\widetilde{y}) + A_2\varphi_2(\widetilde{y}) \quad \text{if } |\widetilde{y}| \to \infty, \end{aligned}$$

where lower powers of $|\widetilde{y}|$ were dropped out.

Defining $G_0(\widetilde{y})$ as the asymptotic limit of $G(\widetilde{y})$, $|\widetilde{y}| \to \infty$, we have

$$G_0 \sim A_1\varphi_1(\widetilde{y}) + A_2\varphi_2(\widetilde{y}), \quad |\widetilde{y}| \to \infty$$

and Γ_0 becomes the line $l \equiv \{\widetilde{y}_2 = 0, \widetilde{y}_1 < 0\}$ so that the zero Dirichlet boundary conditions of ψ on Γ_0 become

$$G_0 = \frac{\partial G_0}{\partial n} = 0 \quad \text{on } l.$$

Since $|G| \leqslant C|\widetilde{y}|^{3/2}$ if $|\widetilde{y}| < 1$, also

$$|G_0(\widetilde{y})| \leqslant C|\widetilde{y}|^{3/2} \quad \text{for } |\widetilde{y}| < 1$$

and clearly also $\Delta^2 G_0 = 0$ outside l. By a Liouville theorem (similar to Lemma 3.3 in [2]), we then conclude that

$$G_0(\widetilde{y}) \equiv A_1\varphi_1(\widetilde{y}) + A_2\varphi(\widetilde{y}). \tag{2.5}$$

We use this information to go back into the outer region and find the lowest order terms for ψ, in the outer variable x. We have

$$\psi = \varepsilon^{3/2} G \sim \varepsilon^{3/2} G_0 = \varepsilon^{3/2}[A_1\varphi_1(\widetilde{y}) + A_2\varphi_2(\widetilde{y})]$$

where

$$\begin{aligned}
\varphi_j(\widetilde{y}) &= \varphi_j(y_1 - 1 + \kappa\varepsilon y_2, y_2 - \frac{1}{2}\kappa\varepsilon - \kappa\varepsilon y_1) \\
&= \varphi_j(y) - \frac{\partial \varphi_j}{\partial y_1}(y) + O(\varepsilon|y||D\varphi_j|) \\
&= \frac{1}{\varepsilon^{3/2}}\varphi_j(x) - \frac{1}{\varepsilon^{1/2}}\frac{\partial \varphi_j}{\partial x_1}(x) + O\Big(\frac{1}{\varepsilon^{1/2}}|x|^{3/2}\Big)
\end{aligned}$$

for $j = 1, 2$. Hence

$$\psi \sim A_1\varphi_1(x) + A_2\varphi_2(x) - \varepsilon A_1 \frac{\partial \varphi_1(x)}{\partial x_1} - \varepsilon A_2 \frac{\partial \varphi_2(x)}{\partial x_1} + O(\varepsilon|x|^{3/2}) \tag{2.6}$$

where the "O" term is of smaller order than each of the other terms in the expansion.

We anticipate

$$\psi = \psi_0 + \varepsilon\psi_1 + \cdots \tag{2.7}$$

and what we have found so far, from (2.6), is that

$$\begin{aligned}
&\psi_0 \sim A_1\varphi_1(x) + A_2\varphi_2(x) \quad \text{as } |x| \to 0, \\
&\psi_1 \sim -A_1\frac{\partial \varphi_1(x)}{\partial x_1} - A_2\frac{\partial \varphi_2(x)}{\partial x_1} + \text{smaller terms} \quad \text{as } |x| \to 0.
\end{aligned} \tag{2.8}$$

We also have

$$\begin{aligned}
&\Delta^2\psi_1 = 0 \quad \text{in } \Omega, \\
&\psi_1 = \frac{\partial \psi_1}{\partial n} = 0 \text{ on both sides of } \Gamma_0, \\
&\psi_1 \text{ satisfies zero boundary condition on } \partial\Omega.
\end{aligned}$$

Notice that ψ_1 has $O(r^{1/2})$ singularity near 0.

In the sequel we continue to argue formally, assuming, for instance, that $\psi_1 + A_1\dfrac{\partial \varphi_1(x)}{\partial x_1} + A_2\dfrac{\partial \varphi_2(x)}{\partial x_1}$ is in H^2 near 0. Thus the asymptotic behavior of ψ_1 near 0 is

$$\psi_1 \sim -A_1\frac{\partial \varphi_1(x)}{\partial x_1} - A_2\frac{\partial \varphi_2(x)}{\partial x_1} + \alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots. \tag{2.9}$$

We need to add another term on the right-hand side of (2.9) in order to compensate for the fact that the function

$$W = \psi_1 + A_1 \frac{\partial \varphi_1(x)}{\partial x_1} + A_2 \frac{\partial \varphi_2(x)}{\partial x_1} \tag{2.10}$$

does not satisfy homogeneous boundary conditions on Γ_0. On $\Gamma_0^+ : \pi - \theta \sim \kappa|x|/2$ (we assume for simplicity that $\kappa \geqslant 0$) and the function ψ_1 has zero Dirichlet data. As for $\partial\varphi_1/\partial x_1$, since (by (1.10)) $B_1 = \dot{B}_1 = \ddot{B}_1 = 0$ at $\theta = \pi$,

$$\Big|\frac{\partial \varphi_1(x)}{\partial x_1}\Big| \leqslant C|x|^{1/2}|\theta - \pi|^3 \leqslant C|x|^{7/2},$$
$$\Big|\frac{\partial}{\partial n}\frac{\partial \varphi_1(x)}{\partial x_1}\Big| \leqslant C\frac{1}{|x|}|x|^{1/2}|\theta - \pi|^2 \leqslant C|x|^{3/2}$$

so that the contribution of $\partial\varphi_1/\partial x_1$ to the Dirichlet data of W is negligible. Next

$$\Big|\frac{\partial \varphi_2(x)}{\partial x_1}\Big| \leqslant C|x|^{1/2}|\theta - \pi|^2 \leqslant C|x|^{5/2} \qquad \text{on } \Gamma_0^+$$

which is small; however,

$$\Big|\frac{\partial}{\partial n}\frac{\partial \varphi_2(x)}{\partial x_1}\Big|$$

is of order $\dfrac{1}{|x|}|x|^{1/2}(\theta - \pi) \sim \dfrac{1}{2}|x|^{1/2}$ which is not small as $|x| \to 0$. The above considerations extend to Γ_0^-. Hence W is a biharmonic function such that

$$W \sim 0, \qquad \frac{\partial W}{\partial n} \sim \frac{\partial}{\partial n}\Big(A_2 \frac{\partial \varphi_2(x)}{\partial x_1}\Big) \quad \text{on } \Gamma_0. \tag{2.11}$$

Next, by direct computation,

$$\begin{aligned} \frac{\partial \varphi_2(x)}{\partial x_1} &= \cos\theta \frac{\partial \varphi_2}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \varphi_2}{\partial \theta} \\ &= |x|^{1/2}\Big\{\frac{3}{2}\cos\theta\Big(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta\Big) - \sin\theta\Big(\frac{3}{2}\cos\frac{3}{2}\theta + \frac{1}{2}\cos\frac{1}{2}\theta\Big)\Big\} \\ &= |x|^{1/2}\Big(\sin\frac{1}{2}\theta\Big)(1 + \cos\theta), \end{aligned} \tag{2.12}$$

so that (recall that n is the interior normal)

$$\begin{aligned} \frac{\partial}{\partial n}\frac{\partial \varphi_2(x)}{\partial x_1} &\sim \frac{\partial}{\partial x_2}\Big[|x|^{1/2}\Big(\sin\frac{1}{2}\theta\Big)(1 + \cos\theta)\Big] \\ &= -\frac{1}{|x|}|x|^{1/2}\frac{\partial}{\partial \theta}\Big[\Big(\sin\frac{1}{2}\theta\Big)(1 + \cos\theta)\Big] \\ &\sim -\frac{1}{|x|^{1/2}}\frac{\partial}{\partial \theta}\Big(\frac{1}{2}(\theta - \pi)^2\Big) = \frac{\kappa}{2}|x|^{1/2} \quad \text{on } \theta = \pi - \frac{\kappa}{2}|x|. \end{aligned} \tag{2.13}$$

Similarly (since $\partial/\partial n\Big|_{\theta = \pi - \kappa|x|/2} = -\partial/\partial n\Big|_{\theta = -\pi - \kappa|x|/2}$),

$$\frac{\partial}{\partial n}\frac{\partial \varphi_2(x)}{\partial x_1} \sim \frac{\kappa}{2}|x|^{1/2} \quad \text{on } \theta = -\pi - \frac{\kappa}{2}|x|.$$

The biharmonic function

$$\widetilde{W} = \kappa |x|^{3/2} \cos \frac{1}{2}\theta$$

satisfies the same boundary conditions:

$$\begin{aligned}
&\widetilde{W} = 0 \quad \text{ on } \theta = \pm\pi, \\
&\frac{\partial \widetilde{W}}{\partial n} = \kappa \frac{|x|^{3/2}}{|x|}(\mp 1)\frac{\partial \widetilde{W}}{\partial \theta} = \pm \frac{\kappa}{2}|x|^{1/2} \sin \frac{1}{2}\theta = \frac{\kappa}{2}|x|^{1/2} \quad \text{ for } \theta = \pm\pi,
\end{aligned}$$

and consequently

$$W = \kappa A_2 |x|^{3/2} \cos \frac{1}{2}\theta + \cdots.$$

From (2.9), (2.10) we then get the outer expansion for ψ_1:

$$\psi_1 \sim -A_1 \frac{\partial \varphi_1(x)}{\partial x_1} - A_2 \frac{\partial \varphi_2(x)}{\partial x_1} + \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \kappa A_2 |x|^{3/2} \cos \frac{1}{2}\theta + \cdots \quad \text{ as } r \to 0. \tag{2.14}$$

We finally go back to the inner expansion of G. Using (2.1), (2.14) and the definition of G in (2.2) we see that

$$\begin{aligned}
G \;\sim\;& A_1\varphi_1(y) + A_2\varphi_2(y) + A_3\varepsilon^{1/2}y_2^2 + A_4\varepsilon\varphi_4(y) + A_5\varepsilon\varphi_5(y) \\
&-A_1\frac{\partial \varphi_1}{\partial y_1}(y) - A_2\frac{\partial \varphi_2}{\partial y_1}(y) + \alpha_1\varepsilon\varphi_1(y) + \alpha_2\varepsilon\varphi_2(y) \\
&-2\kappa\varepsilon A_2|y|^{5/2}\cos\frac{1}{2}\theta + \kappa\varepsilon A_2|y|^{3/2}\cos\frac{1}{2}\theta \quad \text{ for } 1 \ll |y| \ll \frac{1}{\varepsilon}.
\end{aligned} \tag{2.15}$$

As in the analysis of G_0 above, it will be convenient to work with the variable $\widetilde{y}$ defined in (2.3). We have

$$\begin{aligned}
\varphi_j(y) \;&=\; \varphi_j(\widetilde{y}_1 + 1 - \kappa\varepsilon\widetilde{y}_2, \widetilde{y}_2 + \frac{\kappa\varepsilon}{2} + \kappa\varepsilon\widetilde{y}_1) \\
&=\; \varphi_j(\widetilde{y}_1) + \frac{\partial \varphi_j(\widetilde{y})}{\partial \widetilde{y}_1}(1 - \kappa\varepsilon\widetilde{y}_2) + \frac{\partial \varphi_j(\widetilde{y})}{\partial \widetilde{y}_2}(\frac{\kappa\varepsilon}{2} + \kappa\varepsilon\widetilde{y}_1) \quad \text{ for } 1 \ll |y| \ll \frac{1}{\varepsilon},
\end{aligned} \tag{2.16}$$

and

$$\frac{\partial \varphi_j(y)}{\partial y_1} = \frac{\partial \varphi_j(\widetilde{y})}{\partial \widetilde{y}_1} + O\Big(\frac{1}{|y|^{1/2}} + \varepsilon|y|^{1/2}\Big) \quad \text{ for } 1 \ll |y| \ll \frac{1}{\varepsilon} \tag{2.17}$$

where "O" represents smaller terms.

In the expressions $\alpha_1\varepsilon\varphi_1(y), \alpha_2\varepsilon\varphi_2(y)$, we simply replace y by $\widetilde{y}$, incurring just a small error.

Next we compute

$$Z = |y|^{5/2} \cos \frac{1}{2}\theta$$

as a function of $\widetilde{y}$. By (2.4), $y_1 = 1 + \widetilde{y}_1 + O(\varepsilon|y|)$, $y_2 = \widetilde{y}_2 + O(\varepsilon|y|)$ so that

$$Z(y) = Z(\widetilde{y}) + \frac{\partial Z}{\partial \widetilde{y}_1} + O(\varepsilon|y|^{3/2} + |y|^{1/2}) \quad \text{ for } 1 \ll |y| \ll \frac{1}{\varepsilon}.$$

Also

$$\begin{aligned}
\frac{\partial Z}{\partial \widetilde{y}_1} \;&=\; \cos\widetilde{\theta}\frac{\partial Z}{\partial r} - \frac{\sin\widetilde{\theta}}{r}\frac{\partial Z}{\partial \widetilde{\theta}} \qquad (r = |\widetilde{y}|) \\
&=\; |\widetilde{y}|^{3/2}\Big[\frac{5}{2}\cos\widetilde{\theta}\cos\frac{1}{2}\widetilde{\theta} + \frac{1}{2}\sin\widetilde{\theta}\sin\frac{1}{2}\widetilde{\theta}\Big] \\
&=\; |\widetilde{y}|^{3/2}\Big[\frac{3}{2}\Big(\cos\widetilde{\theta}\cos\frac{1}{2}\widetilde{\theta} + \sin\widetilde{\theta}\sin\frac{1}{2}\widetilde{\theta}\Big) + \Big(\cos\widetilde{\theta}\cos\frac{1}{2}\widetilde{\theta} - \sin\widetilde{\theta}\sin\frac{1}{2}\widetilde{\theta}\Big)\Big] \\
&=\; |\widetilde{y}|^{3/2}B_1(\widetilde{\theta}) - \frac{3}{2}|\widetilde{y}|^{3/2}\cos\frac{1}{2}\widetilde{\theta} + O(\varepsilon|y|^{3/2} + |y|^{1/2}) \quad \text{ for } 1 \ll |y| \ll \frac{1}{\varepsilon}.
\end{aligned}$$

Hence

$$Z(y) = |\widetilde{y}|^{5/2} \cos \frac{1}{2}\widetilde{\theta} + |\widetilde{y}|^{3/2} B_1(\widetilde{\theta}) - \frac{3}{2}|\widetilde{y}|^{3/2} \cos \frac{1}{2}\widetilde{\theta} + O(\varepsilon|y|^{3/2} + |y|^{1/2}). \tag{2.18}$$

Substituting (2.16), (2.17) and (2.18) into the right-hand side of (2.15), we get

$$\begin{aligned} G \sim\ & A_1\varphi_1(\widetilde{y}) + A_2\varphi_2(\widetilde{y}) \\ & + \kappa\varepsilon A_1\Big[\widetilde{y}_1 \frac{\partial\varphi_1}{\partial y_2}(\widetilde{y}) - \widetilde{y}_2 \frac{\partial\varphi_1}{\partial y_1}(\widetilde{y})\Big] + \kappa\varepsilon A_2\Big[\widetilde{y}_1 \frac{\partial\varphi_2}{\partial y_2}(\widetilde{y}) - \widetilde{y}_2 \frac{\partial\varphi_2}{\partial y_1}(\widetilde{y})\Big] \\ & + A_3\varepsilon^{1/2}\widetilde{y}_2^2 + A_4\varepsilon\varphi_4(\widetilde{y}) + A_5\varepsilon\varphi_5(\widetilde{y}) + A_4\varepsilon\frac{\partial\varphi_4}{\partial y_1}(\widetilde{y}) + A_5\varepsilon\frac{\partial\varphi_5}{\partial y_1}(\widetilde{y}) \\ & + \alpha_1\varepsilon\varphi_1(\widetilde{y}) + \alpha_2\varepsilon\varphi_2(\widetilde{y}) \\ & - 2\varepsilon A_2\kappa|\widetilde{y}|^{5/2}\cos\frac{1}{2}\widetilde{\theta} - 2\kappa\varepsilon A_2|\widetilde{y}|^{3/2}B_1(\widetilde{\theta}) \\ & + 4\kappa\varepsilon A_2|\widetilde{y}|^{3/2}\cos\frac{1}{2}\widetilde{\theta} \qquad\qquad \text{for } 1 \ll |\widetilde{y}| \ll \frac{1}{\varepsilon}. \end{aligned} \tag{2.19}$$

Next, by direct computation,

$$\widetilde{y}_1\frac{\partial\varphi_1}{\partial y_2}(\widetilde{y}) - \widetilde{y}_2\frac{\partial\varphi_1}{\partial y_1}(\widetilde{y}) = \frac{\partial\varphi_1}{\partial\widetilde{\theta}} = -\frac{3}{2}B_2(\widetilde{\theta})|\widetilde{y}|^{3/2} = -\frac{3}{2}\varphi_2, \tag{2.20}$$

$$\widetilde{y}_1\frac{\partial\varphi_2}{\partial\widetilde{y}_2} - \widetilde{y}_2\frac{\partial\varphi_2}{\partial\widetilde{y}_1} = \frac{\partial\varphi_2}{\partial\widetilde{\theta}} = \frac{3}{2}\varphi_1 - 4|\widetilde{y}|^{3/2}\cos\frac{1}{2}\widetilde{\theta}, \tag{2.21}$$

$$\begin{aligned} \frac{\partial\varphi_4}{\partial y_1}(\widetilde{y}) &= \cos\widetilde{\theta}\frac{\partial}{\partial r}\varphi_4 - \frac{\sin\widetilde{\theta}}{|\widetilde{y}|}\frac{\partial}{\partial\widetilde{\theta}}\varphi_4 = (\cos\widetilde{\theta})\frac{5}{2}|\widetilde{y}|^{3/2}B_4(\widetilde{\theta}) - (\sin\widetilde{\theta})|\widetilde{y}|^{3/2}\dot{B}_4(\widetilde{\theta}) \\ &= |\widetilde{y}|^{3/2}\Big(-\frac{5}{2}B_1(\widetilde{\theta})\Big) = -\frac{5}{2}\varphi_1(\widetilde{y}) \end{aligned} \tag{2.22}$$

and, similarly,

$$\frac{\partial\varphi_5}{\partial y_1}(\widetilde{y}) = \frac{3}{2}\varphi_2(\widetilde{y}). \tag{2.23}$$

Substituting these results into (2.19), we get

$$\begin{aligned} G \sim\ & A_1\varphi_1(\widetilde{y}) + A_2\varphi_2(\widetilde{y}) - \frac{3}{2}\kappa\varepsilon A_1\varphi_2 + \frac{3}{2}\kappa\varepsilon A_2\varphi_1 \\ & + A_3\varepsilon^{1/2}\widetilde{y}_2^2 + A_4\varepsilon\varphi_4(\widetilde{y}) + A_5\varepsilon\varphi_5(\widetilde{y}) + \frac{3}{2}A_5\varepsilon\varphi_2(\widetilde{y}) - \frac{5}{2}A_4\varepsilon\varphi_1(\widetilde{y}) \\ & + \alpha_1\varepsilon\varphi_1(\widetilde{y}) + \alpha_2\varepsilon\varphi_2(\widetilde{y}) \\ & - 2\kappa\varepsilon A_2|\widetilde{y}|^{5/2}\cos\frac{1}{2}\widetilde{\theta} - 2\kappa\varepsilon A_2\varphi_1(\widetilde{y}) \quad \text{for } 1 \ll |\widetilde{y}| \ll \frac{1}{\varepsilon}, \end{aligned} \tag{2.24}$$

or, more briefly,

$$G = G_0(\widetilde{y}) + \varepsilon^{1/2}G_{1/2}(\widetilde{y}) + \varepsilon G_1(\widetilde{y}) + \cdots \qquad (G_{1/2}(\widetilde{y}) = A_3\widetilde{y}_2^2). \tag{2.25}$$

We now impose the boundary conditions

$$\varepsilon G_1 = \frac{\partial(\varepsilon G_1)}{\partial n} = 0$$

at the curve $\widetilde{\Gamma}$: $\widetilde{y}_2 \sim \frac{\kappa\varepsilon}{2}\widetilde{y}_1^2$ (up to error of order $O(\varepsilon^{1+\delta})$ for some $\delta > 0$).

Since

$$G = \frac{\partial G}{\partial n} = 0 \quad \text{on } \widetilde{\Gamma},$$

$$\varepsilon G_1 = G - G_0 - \varepsilon^{1/2} G_1 = O(\varepsilon^{3/2}), \qquad i.e., G_1 = 0 \quad \text{on } \widetilde{\Gamma}.$$

Next

$$\varepsilon \frac{\partial G_1}{\partial n} = \frac{\partial G}{\partial n} - \frac{\partial G_0}{\partial n} - \varepsilon^{1/2} \frac{\partial G_{1/2}}{\partial n} = -A_1 \frac{\partial \varphi_1}{\partial n} - A_2 \frac{\partial \varphi_2}{\partial n} + O(\varepsilon^{3/2}),$$

and

$$\frac{\partial \varphi_1}{\partial n} = O(\varepsilon^2) \quad \text{since } B_1 \text{ is cubic in } \widetilde{\theta} - \pi \text{ above } \widetilde{\Gamma} \text{ and in } \widetilde{\theta} + \pi \text{ below } \widetilde{\Gamma}.$$

Hence, from above $\widetilde{\Gamma}$ ($\widetilde{\theta} \sim \pi - \kappa|\widetilde{y}|/2$),

$$\begin{aligned} \varepsilon \frac{\partial G_1}{\partial n} &= -A_2 \frac{\partial \varphi_2}{\partial y_2} + O(\varepsilon^{3/2}) \\ &= A_2 \frac{1}{|\widetilde{y}|} \frac{\partial \varphi_2}{\partial \widetilde{\theta}} + O(\varepsilon^{3/2}) = -\kappa A_2 \varepsilon |\widetilde{y}|^{3/2} + O(\varepsilon^{3/2}), \end{aligned}$$

and similarly we get the same formula for below $\widetilde{\Gamma}$ (where $\widetilde{\theta} \sim -\pi + \kappa|\widetilde{y}|/2$).

Thus the boundary conditions for G_1 are

$$G_1 = 0, \qquad \frac{\partial G_1}{\partial n} = -\kappa A_2 |\widetilde{y}|^{3/2}. \tag{2.26}$$

As $|\widetilde{y}| \to \infty$ these become boundary conditions on $\widetilde{y}_2 = 0, y_1 < 0$.

From (2.24), (2.26) we have

$$\begin{aligned} G_1 \sim \varphi_1(\widetilde{y}) \Big(-\frac{1}{2}\kappa A_2 - \frac{5}{2} A_4 + \alpha_1 \Big) + \varphi_2(\widetilde{y}) \Big(-\frac{3}{2}\kappa A_1 + \frac{3}{2} A_5 + \alpha_2 \Big) \\ + A_4 \varphi_4(\widetilde{y}) + A_5 \varphi_5(\widetilde{y}) - 2\kappa A_2 |\widetilde{y}|^{5/2} \cos \frac{1}{2} \widetilde{\theta}. \end{aligned} \tag{2.27}$$

Since the last term in G_1 satisfies the boundary conditions (2.26) on $\{\widetilde{y}_2 = 0, \widetilde{y}_1 < 0\}$ and all other terms satisfy homogeneous boundary conditions (up to a small error term), we do not need to add to G_1 any corrective terms.

We now go back to the x-coordinates in (2.27). Since the coefficient of φ_2 is clearly the derivative $dA_2/d\varepsilon$ at $\varepsilon = 0$, we get

$$\frac{dA_2}{ds} = -\frac{3}{2}\kappa A_1 + \frac{3}{2} A_5 + \alpha_2 \tag{2.28}$$

at $s = 0$, Similarly,

$$\frac{dA_1}{ds} = -\frac{1}{2}\kappa A_2 - \frac{5}{2} A_4 + \alpha_1 \tag{2.29}$$

at $s = 0$.

3 Formal computation of dA_2/ds (second method)

In this section we first make a change of coordinates from x to $\widetilde{x}$ in order to transform the extended crack

$$\Gamma_\varepsilon = \Gamma_0 \cup \{x_2 = \frac{1}{2}\kappa x_1^2, \;\; 0 \leqslant x_1 \leqslant \varepsilon\}$$

into a crack $\widetilde{\Gamma}_\varepsilon$ which has its tip at the origin and its tangent at the tip in the horizontal direction. The mapping $x \to \widetilde{x}$ is composed of translation T_ε of $(\varepsilon, \frac{1}{2}\kappa\varepsilon^2)$ into the origin and of rotation R_ε by angle θ_ε with $\tan\theta_\varepsilon = \kappa\varepsilon$; when combined, it can be written, up to order ε terms, as

$$\left\{ \begin{array}{l} x_1 = \varepsilon + \widetilde{x}_1 - \kappa\varepsilon\widetilde{x}_2, \\ x_2 = \frac{1}{2}\kappa\varepsilon^2 + \kappa\varepsilon\widetilde{x}_1 + \widetilde{x}_2 \end{array} \right. \tag{3.1}$$

or, as

$$\left\{ \begin{array}{l} \widetilde{x}_1 = -\varepsilon + x_1 + \kappa\varepsilon x_2, \\ \widetilde{x}_2 = -\kappa\varepsilon x_1 + x_2. \end{array} \right. \tag{3.2}$$

If the crack in the x-coordinates is written in the form $\Gamma_\varepsilon : \;\; x_2 = f(x_1)$, then in the $\widetilde{x}$-coordinates it becomes

$$\widetilde{\Gamma}_\varepsilon : \left\{ \begin{array}{l} \widetilde{x}_1 = x_1 - \varepsilon + \kappa\varepsilon f(x_1), \\ \widetilde{x}_2 = f(x_1) - \kappa\varepsilon x_1. \end{array} \right. \tag{3.3}$$

We seek $\psi(\widetilde{x}) \equiv \psi(\widetilde{x}, \varepsilon)$ in the form

$$\psi(\widetilde{x}) = \psi_0(\widetilde{x}) + \varepsilon V(\widetilde{x}) + \cdots \tag{3.4}$$

where we set

$$\psi_0(\widetilde{x}) = \psi_0(x_1 - \varepsilon + \kappa\varepsilon x_2, x_2 - \kappa\varepsilon x_1).$$

Then the boundary conditions

$$\psi(\widetilde{x}) = 0, \quad \frac{\partial\psi}{\partial n}(\widetilde{x}) = 0 \quad \text{for } \widetilde{x} \in \widetilde{\Gamma}_\varepsilon$$

become

$$\varepsilon V(\widetilde{x}) = -\psi_0(\widetilde{x}), \quad \varepsilon\frac{\partial V}{\partial n}(\widetilde{x}) = -\frac{\partial\psi_0}{\partial n}(\widetilde{x}) \quad \text{for } \widetilde{x} \in \widetilde{\Gamma}_\varepsilon. \tag{3.5}$$

Since $\psi_0(x) = \partial\psi_0(x)/\partial n = 0$ on Γ_ε,

$$\psi_0(\widetilde{x}) = 0 \quad \text{for } \widetilde{x} \in \widetilde{\Gamma}_\varepsilon, \quad \text{up to order } \varepsilon.$$

To compute the normal derivative of $\psi_0(\widetilde{x})$ on $\widetilde{\Gamma}_\varepsilon$, note that

$$\frac{\partial\psi_0}{\partial x_j}(\widetilde{x}) = \frac{\partial\psi_0}{\partial x_j}(x) - \varepsilon\frac{\partial}{\partial x_j}\Big(\frac{\partial\psi_0(x)}{\partial x_1}\Big) + \kappa\varepsilon x_2\frac{\partial}{\partial x_j}\Big(\frac{\partial\psi_0(x)}{\partial x_1}\Big) - \kappa\varepsilon x_1\frac{\partial}{\partial x_2}\Big(\frac{\partial\psi_0(x)}{\partial x_j}\Big).$$

Since $\partial\psi_0(x)/\partial x_j = 0$ on Γ_ε, we get

$$\begin{aligned} \nabla\psi_0(\widetilde{x}) &= \varepsilon\Big[-\frac{\partial}{\partial x_1}\Big(\nabla\psi_0(x)\Big) + \kappa x_2\frac{\partial}{\partial x_1}\Big(\nabla\psi_0(x)\Big) - \kappa x_1\frac{\partial}{\partial x_2}\Big(\nabla\psi_0(x)\Big)\Big] \\ &= \varepsilon\Big[-\nabla\Big(\frac{\partial\psi_0(x)}{\partial x_1}\Big) + \kappa x_2\nabla\Big(\frac{\partial\psi_0(x)}{\partial x_1}\Big) - \kappa x_1\nabla\Big(\frac{\partial\psi_0(x)}{\partial x_2}\Big)\Big]. \end{aligned} \tag{3.6}$$

As $\varepsilon \to 0$ the curves $\widetilde{\Gamma}_\varepsilon$ converge to Γ_0 and the limit problem for V, in the variable x, becomes:

$$\Delta^2 V = 0 \quad \text{in } \Omega \setminus \Gamma_0, \tag{3.7}$$
$$V = 0 \quad \text{on } \Gamma_0, \tag{3.8}$$
$$\frac{\partial V}{\partial n} = \frac{\partial}{\partial n}\Big(\frac{\partial \psi_0}{\partial x_1}\Big) - \kappa x_2 \frac{\partial}{\partial n}\Big(\frac{\partial \psi_0(x)}{\partial x_1}\Big) + \kappa x_1 \frac{\partial}{\partial n}\Big(\frac{\partial \psi_0(x)}{\partial x_2}\Big) \quad \text{on } \Gamma_0. \tag{3.9}$$

In addition V satisfies suitable inhomogeneous boundary conditions on $\partial\Omega$.

We shall use the expansion (2.1) for ψ_0 in order to find an expansion for V.

By (2.13) (which holds for both $\theta = \pi - \kappa|x|/2$ and $\theta = -\pi - \kappa|x|/2$)

$$\frac{\partial}{\partial n}\Big(\frac{\partial \varphi_2}{\partial x_1}\Big) \sim \frac{\kappa}{2} r^{1/2}$$

up to order $O(r^{3/2})$. Similarly,

$$\begin{aligned} \frac{\partial}{\partial n}\frac{\partial}{\partial x_1}\Big(- 2\kappa A_2 |x|^{5/2} \cos\frac{1}{2}\theta\Big) &= \frac{\partial}{\partial n}\Big[- 2\kappa A_2 |x|^{3/2} \Big(\frac{3}{2}\cos\frac{1}{2}\theta + \cos\frac{3}{2}\theta\Big)\Big] \\ &\sim \pm 2\kappa A_2 r^{1/2} \frac{\partial}{\partial\theta}\Big(\frac{3}{2}\cos\frac{1}{2}\theta + \cos\frac{3}{2}\theta\Big) \\ &= \frac{3}{2} A_2 \kappa r^{1/2} \quad \text{at } \theta = \pm\pi. \end{aligned}$$

Since the other terms in (2.1) give smaller order contributions, we get

$$\frac{\partial}{\partial n}\Big(\frac{\partial \psi_0}{\partial x_1}\Big) \sim \Big(\frac{\kappa}{2} + \frac{3}{2}\kappa\Big) A_2 r^{1/2} = 2\kappa A_2 r^{1/2} \quad \text{on both } \Gamma_0^+ \text{ and } \Gamma_0^-, \tag{3.10}$$

near the origin.

Next, by direct computation,

$$\begin{aligned} \frac{\partial \varphi_2}{\partial x_2} &= \sin\theta \frac{\partial \varphi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \varphi}{\partial \theta} \\ &= r^{1/2}\Big[\frac{5}{2}\cos\frac{1}{2}\theta - \frac{1}{2}\cos\frac{3}{2}\theta\Big] \end{aligned} \tag{3.11}$$

from which we deduce that

$$\frac{\partial}{\partial n}\Big(\frac{\partial \varphi_2}{\partial x_2}\Big) \sim -2r^{-1/2} \quad \text{on } \Gamma_0.$$

The other terms in ψ_0 give smaller order terms near $r = 0$, and so

$$\kappa x_1 \frac{\partial}{\partial n}\Big(\frac{\partial \psi_0}{\partial x_2}\Big) \sim \kappa A_2 x_1 \frac{\partial}{\partial n}\Big(\frac{\partial \varphi_2}{\partial x_2}\Big) \sim 2\kappa A_2 x_1 r^{-1/2} = -2\kappa A_2 r^{1/2} \quad \text{on } \Gamma_0 \tag{3.12}$$

near $r = 0$.

Finally

$$-\kappa x_2 \frac{\partial}{\partial n}\Big(\frac{\partial \psi_0}{\partial x_1}\Big) = O(r^{3/2}) \quad \text{on } \Gamma_0.$$

Combining this with (3.12) (3.10) we find that

$$\frac{\partial V}{\partial n} \sim O(r^{3/2}) \quad \text{on } \Gamma_0, \tag{3.13}$$

from which one can formally deduce that V has an expansion

$$V \sim \beta_1 \varphi_1 + \beta_2 \varphi_2 \quad \text{as } r \to 0 \tag{3.14}$$

From (3.4) it follows that dA_j/ds ($j = 1, 2$) is the coefficients of φ_j in V, so that, by (3.14),

$$\frac{dA_1}{ds} = \beta_1, \qquad \frac{dA_2}{ds} = \beta_2 \quad \text{at } s = 0. \tag{3.15}$$

In order to determine the β_j we write

$$V = V_1 + V_2$$

where V_1 corresponds to the translation T_ε and V_2 corresponds to the rotation R_ε. That is, V_1 and V_2 are biharmonic in $\Omega \setminus \Gamma_0$,

$$V_1 = 0, \quad \frac{\partial}{\partial n} V_1 = \frac{\partial}{\partial n} \frac{\partial \psi_0}{\partial x_1} \quad \text{on } \Gamma_0, \tag{3.16}$$

$$V_2 = 0, \quad \frac{\partial}{\partial n} V_2 = \kappa x_1 \frac{\partial}{\partial n} \Big(\frac{\partial \psi_0}{\partial x_2} \Big) - \kappa x_2 \frac{\partial}{\partial n} \Big(\frac{\partial \psi_0}{\partial x_1} \Big) \quad \text{on } \Gamma_0, \tag{3.17}$$

and V_1, V_2 satisfy appropriate boundary conditions on $\partial\Omega$.

By (3.10) and the fact that

$$\frac{\partial}{\partial n} \Big(4 r^{3/2} \cos \frac{1}{2} \theta \Big) \sim -2 r^{1/2} \sin \frac{1}{2} \theta$$

on Γ_0 we find that

$$V_1 \sim \widehat{\beta}_1 \varphi_1 + \widehat{\beta}_2 \varphi_2 - 4 \kappa A_2 r^{3/2} \cos \frac{1}{2} \theta. \tag{3.18}$$

V_2 can be computed in a similar way, or also from the fact that its boundary conditions on Γ_0 are obtained as the ε-terms of $\psi_0(R_\varepsilon x)$, i.e.,

$$V_2 = \lim_{\varepsilon \to 0} \frac{\psi_0(R_\varepsilon x) - \psi_0(x)}{\varepsilon} \tag{3.19}$$

(provided we take the boundary conditions of V_2 on $\partial\Omega$ to be defined by means of the relation (3.19)). Since

$$\psi_0 \sim A_1 \varphi_1(x) + A_2 \varphi_2(x) + A_3 x_2^2,$$

we have

$$\begin{aligned} \frac{1}{\varepsilon}[\psi_0(R_\varepsilon x) - \psi_0(x)] &= \frac{1}{\varepsilon}[A_1 \varphi_1(x_1 + \kappa\varepsilon x_2, x_2 - \kappa \varepsilon x_1) - A_1 \varphi_1(x_1, x_2)] \\ &\quad + \frac{1}{\varepsilon}[A_2 \varphi_2(x_1 + \kappa\varepsilon x_2, x_2 - \kappa\varepsilon x_1) - A_2 \varphi_2(x_1, x_2)] \\ &\sim \kappa A_1 \Big(x_2 \frac{\partial \varphi_1}{\partial x_1} - x_1 \frac{\partial \varphi_1}{\partial x_2} \Big) + \kappa A_2 \Big(x_2 \frac{\partial \varphi_2}{\partial x_1} - x_1 \frac{\partial \varphi_2}{\partial x_2} \Big) \\ &= \frac{3}{2} \kappa A_1 \varphi_2 - \frac{3}{2} \kappa A_2 \varphi_1 + 4 \kappa A_2 r^{3/2} \cos \frac{1}{2} \theta \end{aligned}$$

by (2.20), (2.21). It follows that

$$V_2 \sim \frac{3}{2} \kappa A_2 \varphi_1 - \frac{3}{2} \kappa A_1 \varphi_2 + 4 \kappa A_2 r^{3/2} \cos \frac{1}{2} \theta. \tag{3.20}$$

Combining (3.18) with (3.20), we conclude that

$$V = V_1 + V_2 = (\widehat{\beta}_1 + \frac{3}{2}\kappa A_2)\varphi_1 + (\widehat{\beta}_2 - \frac{3}{2}\kappa A_1)\varphi_2$$

and therefore, by (3.4),

$$\frac{dA_1}{ds} = \widehat{\beta}_1 + \frac{3}{2}\kappa A_2, \qquad \frac{dA_2}{ds} = \widehat{\beta}_2 - \frac{3}{2}\kappa A_1. \tag{3.21}$$

These equations have a different form than (2.28), (2.29); they have the advantage that $\widehat{\beta}_1, \widehat{\beta}_2$ are first order coefficients in the expansions of V_1, V_2 and, as a consequence, it will be much easier to analyze their regularity properties (as functions of s) than it would be for the (higher order coefficients) A_5 or A_4 in (2.28), (2.29).

We conclude this section by showing that, formally, the formulas in (3.21) agree with those in (2.28), (2.29).

Notice that

$$\psi_0 + \varepsilon\psi_1(x) \sim \psi_0(\widetilde{x}) + \varepsilon V(\widetilde{x}) \tag{3.22}$$

where x and $\widetilde{x}$ are related by (3.1) or (3.2) and $\psi_1(x)$ was defined by (2.7). Hence

$$\begin{aligned}
V(\widetilde{x}) &\sim \psi_1(x) + \frac{1}{\varepsilon}\Big[\psi_0(x) - \psi_0(\widetilde{x})\Big] \\
&\sim \psi_1(x) + \frac{1}{\varepsilon}\Big[\psi_0(x) - \psi_0(x_1 - \varepsilon + \kappa\varepsilon x_2, x_2 - \kappa\varepsilon x_2)\Big] \\
&\sim \psi_1(x) + \frac{\partial\psi_0}{\partial x_1} - \kappa x_2\frac{\partial\psi_0}{\partial x_1} + \kappa x_1\frac{\partial\psi_0}{\partial x_2} \\
&\sim \Big[- A_1\frac{\partial\varphi_1}{\partial x_1} - A_2\frac{\partial\varphi_2}{\partial x_1} + \alpha_1\varphi_1 + \alpha_2\varphi_2 + \kappa A_2|x|^{3/2}\cos\frac{1}{2}\theta\Big] \\
&\quad + A_1\frac{\partial\varphi_1}{\partial x_1} + A_2\frac{\partial\varphi_2}{\partial x_1} + A_4\frac{\partial\varphi_4}{\partial x_1} + A_5\frac{\partial\varphi_5}{\partial x_1} \\
&\quad - \kappa x_2\Big(A_1\frac{\partial\varphi_1}{\partial x_1} + A_2\frac{\partial\varphi_2}{\partial x_1}\Big) + \kappa x_1\Big(A_1\frac{\partial\varphi_1}{\partial x_2} + A_2\frac{\partial\varphi_2}{\partial x_2}\Big) \\
&\quad - 2\kappa A_2\frac{\partial}{\partial x_1}\Big(|x|^{5/2}\cos\frac{1}{2}\theta\Big) \qquad \text{(by (2.1), (2.14))}
\end{aligned}$$

Thus

$$\begin{aligned}
V(\widetilde{x}) &= \Big[\alpha_1\varphi_1 + \alpha_2\varphi_2 + \kappa A_2|x|^{3/2}\cos\frac{1}{2}\theta\Big] + A_4\frac{\partial\varphi_4}{\partial x_1} + A_5\frac{\partial\varphi_5}{\partial x_1} + \kappa A_1\frac{\partial\varphi_1}{\partial\theta} + \kappa A_2\frac{\partial\varphi_2}{\partial\theta} \\
&\quad - 2\kappa A_2\Big(\varphi_1(x) - \frac{3}{2}|x|^{3/2}\cos\frac{1}{2}\theta\Big) \\
&= \Big[\alpha_1\varphi_1 + \alpha_2\varphi_2 + \kappa A_2|x|^{3/2}\cos\frac{1}{2}\theta\Big] - \frac{5}{2}A_4\varphi_1(x) + \frac{3}{2}A_5\varphi_2(x) \\
&\quad - \frac{3}{2}\kappa A_1\varphi_2(x) + \kappa A_2\Big(\frac{3}{2}\varphi_1(x) - 4|x|^{3/2}\cos\frac{\theta}{2}\Big) \\
&\quad - 2\kappa A_2\Big(\varphi_1(x) - \frac{3}{2}|x|^{3/2}\cos\frac{1}{2}\theta\Big) \qquad \text{(by (2.20), (2.23))} \\
&= \Big(-\frac{1}{2}\kappa A_2 - \frac{5}{2}A_4 + \alpha_1\Big)\varphi_1(x) + \Big(-\frac{3}{2}\kappa A_1 + \frac{3}{2}A_5 + \alpha_2\Big)\varphi_2(x).
\end{aligned}$$

On the other hand, by (3.14),

$$V(\widetilde{x}) \sim \beta_1\psi_1(\widetilde{x}) + \beta_2\varphi_2(\widetilde{x}) \sim \beta_1\psi_1(x) + \beta_2\varphi_2(x)$$

up to order $|x|^{3/2}$, so that

$$\beta_1 = -\frac{1}{2}\kappa A_2 - \frac{5}{2}A_4 + \alpha_1, \qquad \beta_2 = -\frac{3}{2}\kappa A_1 + \frac{3}{2}A_5 + \alpha_2. \tag{3.23}$$

4 An auxiliary lemma

Notation. In the sequel we shall use the notation $[k]_{\alpha,D}$ to denote the least α-Hölder coefficient of a function k on a set D.

In [2] we derived an asymptotic behavior of solutions of $\Delta^2 w = 0$ near the tip of a $C^{2+\alpha}$ crack in case w and its normal derivative vanish along the crack. In this section we extend this result to the case where the boundary conditions of w along the crack are non-zero (but small), as well as to the case where instead of one crack we have two cracks with common tip.

Lemma 4.1 *Suppose that*

$$\Gamma_j : x_2 = f_j(x_1),\ -1 \leqslant x_1 \leqslant 0 \quad (j = 1, 2)$$

are $C^{2+\alpha}$ *curves* $(0 < \alpha < 1)$ *such that*

$$f_j(0) = f_j'(0) = 0, \quad (j = 1, 2), \qquad f_1(x_1) \leqslant f_2(x_1) \quad \text{for } -1 \leqslant x_1 < 0.$$

Let Q_Γ *be the "thin" region bounded by the curves* Γ_j, *i.e.,*

$$Q_\Gamma = \{(x_1, x_2);\ f_1(x_1) \leqslant x_2 \leqslant f_2(x_1),\ -1 \leqslant x_1 < 0\}.$$

Let $\Omega = \{x; |x| < 1\}$ *and suppose that*

$$\Delta^2 w = 0 \quad \text{in } \Omega \setminus Q_\Gamma, \tag{4.1}$$
$$|w(x)| \leqslant |x|^{3/2+\delta} \quad \text{on } \Gamma_1 \cup \Gamma_2, \tag{4.2}$$
$$|\nabla w(x)| \leqslant |x|^{1/2+\delta} \quad \text{on } \Gamma_1 \cup \Gamma_2, \tag{4.3}$$
$$[\nabla w]_{C^\sigma(\Gamma_j \cap \{|x|<R\})} \leqslant R^{1/2+\delta-\sigma} \quad \text{for some } 0 < \sigma < \delta \ \ (j = 1, 2), \tag{4.4}$$
$$|w(x)| \leqslant \min(1, M|x|^{1/2+\tau}) \quad \text{for some } \tau > 0 \text{ and } M > 0. \tag{4.5}$$

Then there exists a constant $\beta > 0$ *(independent of* α, M*) such that*

$$\left| w - \left(A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta)\right)\right| \leqslant C r^{3/2+\beta}, \tag{4.6}$$

where C *is a constant independent of* M.

Proof. We shall first prove that, for any small $\varepsilon > 0$, there exists $C = C_\varepsilon$ (independent of M) such that

$$|w(x)| \leqslant C|x|^{3/2-\varepsilon}. \tag{4.7}$$

To prove this it suffices to show that the function

$$\frac{|w(x)|}{|x|^{3/2-\varepsilon} + \delta|x|^{1/2+\tau}}$$

is bounded uniformly in δ as $\delta \to 0$. If this is not true, then there exists a sequence of functions $w = w_n$ with their corresponding $M = M_n$ and $\delta = \delta_n \to 0$, such that

$$\frac{|w_n(x_n)|}{|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau}} = \sup_{x \in \Omega \setminus Q_\Gamma} \frac{|w_n(x)|}{|x|^{3/2-\varepsilon} + \delta_n |x|^{1/2+\tau}} = C_n \to \infty,$$

as $n \to \infty$, where x_n is a sequence of points converging to zero. We define $\widetilde{w}_n(\xi)$ by

$$w_n(x) = C_n(|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau})\widetilde{w}_n(\xi), \qquad x = \xi |x_n|.$$

Then

$$|\widetilde{w}_n(e_n)| = 1 \quad \text{ for } e_n = x_n/|x_n|,$$

and, by (4.2)–(4.4),

$$\begin{aligned}
&|\widetilde{w}_n(\xi)| \leqslant \frac{(|\xi||x_n|)^{3/2+\delta}}{C_n(|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau})} \to 0, \quad \text{ for } \xi \in \widetilde{\Gamma}^n_j, \\
&\Big|\frac{\partial \widetilde{w}_n}{\partial n_\xi}\Big| \leqslant \frac{(|\xi||x_n|)^{1/2+\delta}|x_n|}{C_n(|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau})} \to 0, \quad \text{ for } \xi \in \widetilde{\Gamma}^n_j, \\
&\Big[\frac{\partial \widetilde{w}_n}{\partial n_\xi}\Big]_{C^\sigma(\{|\xi|<R\}\cap\widetilde{\Gamma}^n_j)} \leqslant \frac{(|x_n|R)^{1/2+\delta-\sigma}|x_n|}{C_n(|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau})} \to 0
\end{aligned}$$

where $\widetilde{\Gamma}^n_j$ is the image of Γ_j under the mapping $x = \xi|x_n|$. We also have

$$\begin{aligned}
|\widetilde{w}_n(\xi)| &= \Big|\frac{w_n(x)}{C_n(|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau})}\Big| \\
&\leqslant \frac{(|\xi||x_n|)^{3/2-\varepsilon} + \delta_n (|\xi||x_n|)^{1/2+\tau}}{|x_n|^{3/2-\varepsilon} + \delta_n |x_n|^{1/2+\tau}} \leqslant |\xi|^{3/2-\varepsilon} + |\xi|^{1/2+\tau} \quad \text{ for } \xi \in \widetilde{\Omega}^n \setminus \widetilde{Q}^n_\Gamma,
\end{aligned}$$

where $\widetilde{\Omega}^n$ and $\widetilde{Q}^n_\Gamma$ are the images of Ω and Q_Γ under the mapping $x = \xi|x_n|$. Notice that $\widetilde{Q}^n_\Gamma$ converges to the ray $\{\xi_2 = 0,\ \xi_1 < 0\}$. By $C^{1+\sigma}$ sub-Schauder estimates, we have, for a subsequence, $\widetilde{w}_n(\xi) \to \widetilde{w}(\xi)$ on any compact set, where $\widetilde{w}(\xi)$ is biharmonic on $\mathbb{R}^2 \setminus \{\xi_2 = 0,\ \xi_1 < 0\}$, and

$$\widetilde{w}(\xi) = \frac{\partial \widetilde{w}}{\partial n}(\xi) = 0 \quad \text{ on } \{\xi_2 = 0,\ \xi_1 < 0\}, \tag{4.8}$$

$$|\widetilde{w}(\xi)| \leqslant |\xi|^{3/2-\varepsilon} + |\xi|^{1/2+\tau}. \tag{4.9}$$

By the proof of Lemma 5.5 in [2], it then follows that $\widetilde{w}(\xi) \equiv 0$. This however is a contradiction to

$$\widetilde{w}(e_0) = \lim_{n\to\infty} \widetilde{w}_n(e_n) = 1$$

where $e_0 = \lim_{n\to\infty} e_n$ ($|e_0| = 1$).

Having proved (4.7), we can easily deduce, by rescaling,

$$|\nabla w| \leqslant C|x|^{1/2-\varepsilon}, x \in \Omega \setminus Q_\Gamma, \qquad [\nabla w]_{C^\sigma(\{|x|<R\}\setminus Q_\Gamma)} \leqslant CR^{1/2-\varepsilon-\sigma}. \tag{4.10}$$

For any $0 < \lambda < 1$ and $\varepsilon > 0$ we now introduce the function

$$w_\lambda(x) = \frac{1}{\lambda^{3/2-\varepsilon}} w(\lambda x) \quad \text{ for } |x| \leqslant 2. \tag{4.11}$$

Then, by (4.10),

$$|w_\lambda(x)| \leqslant C \quad \text{for } |x| \leqslant 2, \tag{4.12}$$

$$|\nabla w_\lambda(x)| \leqslant C \quad \text{for } |x| \leqslant 2, \tag{4.13}$$

$$[\nabla w_\lambda]_{C^\sigma(|x|<2)} \leqslant C. \tag{4.14}$$

Let $\Gamma_\lambda^j = \{x; x/\lambda \in \Gamma_j\}$. Since the curve Γ_j is in C^2, there exists a constant C such that the curve $\Gamma_\lambda^j \cap \{|x| \leqslant 2\}$ can be bounded by two line segments $\Gamma_\lambda^+ : \theta = \pi - C\lambda$ and $\Gamma_\lambda^- : \theta = -\pi + C\lambda$. On these two line segments, we have, by (4.2), (4.3) and the estimates (4.13), (4.14),

$$|w_\lambda(x)| \leqslant \lambda^{\delta+\varepsilon} + C\lambda \leqslant C\lambda^{\delta+\varepsilon}, \quad \text{for } x \in \Gamma_\lambda^\pm, \tag{4.15}$$

$$|\nabla w_\lambda(x)| \leqslant C\lambda^{\delta+\varepsilon} + C\lambda^\sigma \leqslant C\lambda^\sigma, \quad \text{for } x \in \Gamma_\lambda^\pm. \tag{4.16}$$

Define a function G_λ in $H^2(\{|x|<2\})$ by

$$\Delta^2 G_\lambda = 0 \quad \text{in } B_2, \tag{4.17}$$

$$G_\lambda = \frac{\partial G_\lambda}{\partial n} = 0 \quad \text{on } \{x_2 = 0, x_1 < 0\}, \tag{4.18}$$

$$G_\lambda = w_\lambda, \quad \frac{\partial G_\lambda}{\partial n} = \frac{\partial w_\lambda}{\partial n} \quad \text{on } \partial B_2, \tag{4.19}$$

here it is understood that $\partial(B_2 \setminus \{x_2 = 0, x_1 < 0\}$ is regularized near $(-2,0)$. Note that the estimates (4.15), (4.16) are valid also for G_λ. By applying maximum principle (Theorem 2.2 of [2]) to $G_\lambda - w_\lambda$ in the region bounded by $\Gamma_\lambda^\pm$ and $|x| = 2$, we obtain

$$\Big(\int_{|x|<2} |G_\lambda(x) - w_\lambda(x)|^p dx\Big)^{1/p} \leqslant C\lambda^\sigma.$$

Since both $G_\lambda(x)$ and $w_\lambda(x)$ are uniformly C^1 (independent of λ), we also have (by taking p large enough and using an interpolation inequality)

$$|G_\lambda(x) - w_\lambda(x)| \leqslant C\lambda^{\sigma/2} \quad \text{for } |x| \leqslant 2. \tag{4.20}$$

Clearly, G_λ has an expansion

$$G_\lambda(x) = A_1^\lambda r^{3/2} B_1(\theta) + A_2^\lambda r^{3/2} B_2(\theta) + q(x), \tag{4.21}$$

where

$$|q(x)| \leqslant Cr^2. \tag{4.22}$$

Rewriting this inequality in terms of w, we have

$$\Big|w(x) - r^{3/2} B_\lambda(\theta)\Big| \leqslant C\lambda^{3/2+\sigma/2-\varepsilon} + C\lambda^{3/2-\varepsilon}\Big(\frac{r}{\lambda}\Big)^2 \quad \text{for } r < \lambda. \tag{4.23}$$

where

$$B_\lambda(\theta) = \frac{1}{\lambda^\varepsilon}\Big[A_1^\lambda B_1(\theta) + A_2^\lambda B_2(\theta)\Big].$$

Hence

$$\Big|\frac{w(x)}{r^{3/2}} - B_\lambda(\theta)\Big| \leqslant C\lambda^{\sigma/2-\varepsilon}\Big(\frac{\lambda}{r}\Big)^{3/2} + C\lambda^{-\varepsilon}\Big(\frac{r}{\lambda}\Big)^{1/2} \quad \text{for } r < \lambda. \tag{4.24}$$

We assume that ε is sufficiently small so that

$$\eta = \frac{\sigma}{4} > 4\varepsilon.$$

Then

$$\left|\frac{w(x)}{r^{3/2}} - B_\lambda(\theta)\right| \leqslant C\lambda^{\sigma/2-\varepsilon}(\lambda^{-\eta})^{3/2} + C\lambda^{-\varepsilon+\eta/2} \leqslant C\lambda^{-\varepsilon+\eta/2} \quad \text{for } \lambda^{\eta+1} \leqslant r \leqslant 2^{\eta+1}\lambda^{\eta+1}. \tag{4.25}$$

Now take $\lambda_j = 2^{-j}$, $r = \lambda_{j+1}^{\eta+1}$. Then

$$\begin{aligned} \left|B_{\lambda_j}(\theta) - B_{\lambda_{j+1}}(\theta)\right| &\leqslant \left|\frac{w(x)}{r^{3/2}} - B_{\lambda_j}(\theta)\right| + \left|B_{\lambda_{j+1}}(\theta) - \frac{w(x)}{r^{3/2}}\right| \\ &\leqslant C\lambda_j^{-\varepsilon+\eta/2}. \end{aligned}$$

It follows that

$$\sum_j \left|B_{\lambda_j}(\theta) - B_{\lambda_{j+1}}(\theta)\right|$$

converges, and

$$\sum_{j\geqslant k} \left|B_{\lambda_j}(\theta) - B_{\lambda_{j+1}}(\theta)\right| \leqslant C\lambda_k^{-\varepsilon+\eta/2}. \tag{4.26}$$

Setting

$$B(\theta) = \lim_{\lambda_j\to 0} B_{\lambda_j}(\theta),$$

we then have

$$\left|B(\theta) - B_{\lambda_j}(\theta)\right| \leqslant C\lambda_j^{-\varepsilon+\eta/2}, \tag{4.27}$$

and it is clear that $B(\theta) = A_1B_1(\theta) + A_2B_2(\theta)$ for some constants A_1 and A_2. Since, by (4.25),

$$\left|\frac{w(x)}{r^{3/2}} - B(\theta)\right| \leqslant Cr^\beta, \qquad \beta = (-\varepsilon + \eta/2)/(\eta+1) > 0, \tag{4.28}$$

the lemma follows. $\square$

5 A rigorous derivation of the differential equation

In this section we give a rigorous statement of the derivation of the differential equations in (3.21) for a $C^{2+\alpha}$ crack

$$\Gamma_s = \Gamma_0 \cup \{x_2 = f(x_1),\ 0 \leqslant x_1 \leqslant s\}, \quad 0 \leqslant s \leqslant s_0. \tag{5.1}$$

We shall make the following assumptions:

$$\partial\Omega \quad \text{is in } C^4, \tag{5.2}$$

$$\begin{gathered} \Gamma_0 \text{ is a graph } \{x_2 = f(x_1), -x_0 \leqslant x_1 \leqslant 0\} \text{ contained in } \Omega \\ \text{except for its initial point } P_0 = (-x_0, f(-x_0)), \end{gathered} \tag{5.3}$$

$$f \in C^{2+\alpha}[-x_0, s_0] \quad \text{for some small } \alpha > 0, \tag{5.4}$$

$$f(0) = f'(0) = 0, \tag{5.5}$$

$$g \in C^3(\mathbb{R}^2), \quad h \in C^2(\mathbb{R}^2), \tag{5.6}$$

$$\text{the arc } \Gamma_0 \text{ intersects } \partial\Omega \text{ nontangentially (at } P_0), \tag{5.7}$$

and, finally,

$$\text{there is a disc } B_\mu(P_0) = \{|x - P_0| < \mu\} \text{ such that} \quad g = h = 0 \quad \text{on } B_\mu(P_0) \cap \partial\Omega. \tag{5.8}$$

Note that (by (5.5))

$$\kappa = f''(0) \tag{5.9}$$

is the curvature of Γ_{s_0} at the origin, and

$$f(x_1) = \frac{1}{2}\kappa x_1^2 + O(|x_1|^{2+\alpha}), \quad f'(x_1) = \kappa x_1 + O(|x_1|^{1+\alpha}), \quad f''(x_1) = \kappa + O(|x_1|^\alpha) \tag{5.10}$$

near the origin.

The analysis of Sections 2, 3 was carried out in the special case

$$f(x_1) = \frac{1}{2}\kappa x_1^2.$$

Here we shall use some of the same calculations for the more general case (5.10), and at the same time evaluate more carefully the incurred errors.

It will suffice to prove (3.21) at $s = 0$. Analogously to (3.2) we use a transformation

$$S_\varepsilon = R_\varepsilon T_\varepsilon : \quad x \to \widetilde{x}$$

consisting of translation T_ε and rotation R_ε:

$$\begin{aligned} \widetilde{x}_1 &= (x_1 - \varepsilon)\cos\theta_\varepsilon + (x_2 - f(\varepsilon))\sin\theta_\varepsilon, \\ \widetilde{x}_2 &= -(x_1 - \varepsilon)\sin\theta_\varepsilon + (x_2 - f(\varepsilon))\cos\theta_\varepsilon \end{aligned} \tag{5.11}$$

where

$$\theta_\varepsilon = \arctan f'(\varepsilon).$$

By (5.10),

$$\begin{aligned} &f'(\varepsilon) = \kappa\varepsilon + O(\varepsilon^{1+\alpha}), \quad \theta_\varepsilon = \kappa\varepsilon + O(\varepsilon^{1+\alpha}), \quad \frac{d\theta_\varepsilon}{d\varepsilon} = \kappa + O(\varepsilon^\alpha), \\ &\cos\theta_\varepsilon = 1 + O(\varepsilon^2), \quad \sin\theta_\varepsilon = \kappa\varepsilon + O(\varepsilon^{1+\alpha}). \end{aligned} \tag{5.12}$$

We can rewrite (5.11) also as a mapping $\widetilde{x} \to x$:

$$\begin{aligned} x_1 &= \varepsilon + \widetilde{x}_1\cos\theta_\varepsilon - \widetilde{x}_2\sin\theta_\varepsilon, \\ x_2 &= f(\varepsilon) + \widetilde{x}_1\sin\theta_\varepsilon + \widetilde{x}_2\cos\theta_\varepsilon. \end{aligned} \tag{5.13}$$

The mapping S_ε transforms Γ_ε into a new crack $\widetilde{\Gamma}_\varepsilon$ and the domain Ω into a new domain $\widetilde{\Omega}_\varepsilon$, which we shall briefly denote by $\widetilde{\Omega}$. The tip of $\widetilde{\Gamma}_\varepsilon$ is at the origin, and the tangent at the tip is in the $\widetilde{x}_1$-direction. We write

$$\widetilde{\Gamma}_\varepsilon : \quad \widetilde{x}_2 = \widetilde{f}_\varepsilon(\widetilde{x}_1), \; -\widetilde{x}_{0\varepsilon} \leqslant \widetilde{x}_1 \leqslant 0.$$

We introduce the solution ψ of the following problem:

$$\psi \in H^2(\widetilde{\Omega}), \tag{5.14}$$
$$\Delta^2\psi = 0 \quad \text{in } \widetilde{\Omega} \setminus \widetilde{\Gamma}_\varepsilon, \tag{5.15}$$
$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \widetilde{\Gamma}_\varepsilon \text{ (from both sides)}, \tag{5.16}$$
$$\psi = \widetilde{g}, \ \ \frac{\partial\psi}{\partial\widetilde{n}} = \widetilde{h} \quad \text{on } \partial\widetilde{\Omega} \tag{5.17}$$

where $\widetilde{n}$ is the normal to $\partial\widetilde{\Omega}$; here

$$\widetilde{g}(\widetilde{x}) = g(x), \quad \widetilde{h}(\widetilde{x}) = h(x). \tag{5.18}$$

We wish to compute

$$\lim_{\varepsilon\to 0} \frac{\psi - \psi_0}{\varepsilon} \tag{5.19}$$

where ψ_0 is the solution of (1.17)–(1.20) for $s = 0$. To do that we introduce a function V, as in §3, defined in Ω, and satisfying (3.7)–(3.9) and the boundary conditions on $\partial\Omega$ which are formally obtained from (5.19). Clearly,

$$\begin{aligned} V(x) &= \lim_{\varepsilon\to 0} \frac{\psi(\widetilde{x}) - \psi_0(\widetilde{x})}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\psi(\widetilde{x}) - \psi_0(x)}{\varepsilon} + \lim_{\varepsilon\to 0} \frac{\psi_0(x) - \psi_0(\widetilde{x})}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\psi_0(x) - \psi_0(S_\varepsilon x)}{\varepsilon} \quad \text{on } \partial\Omega. \end{aligned}$$

Setting

$$\mathcal{L}k = \lim_{\varepsilon\to 0} \frac{k(S_\varepsilon x) - k(x)}{\varepsilon} = -\frac{\partial k(x)}{\partial x_1} + \kappa\Big(x_2 \frac{\partial k(x)}{\partial x_1} - x_1 \frac{\partial k(x)}{\partial x_2}\Big), \tag{5.20}$$

we can write the boundary condition for V on $\partial\Omega$ in the form

$$V = -\mathcal{L}\psi_0 \quad \text{on } \partial\Omega, \tag{5.21}$$
$$\frac{\partial V}{\partial n} = -n \cdot \mathcal{L}\nabla\psi_0 - \vec{l} \cdot \nabla\psi_0 \quad \text{on } \partial\Omega, \tag{5.22}$$

where $\vec{l}$ is defined by

$$\vec{l} \cdot \nabla k = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(\frac{\partial}{\partial\widetilde{n}} - \frac{\partial}{\partial n}\Big)k, \tag{5.23}$$

$\widetilde{n}$ being the vector $S_\varepsilon n$ where n is the normal to $\partial\Omega$.

Before proving the existence and uniqueness of V we need to estimate carefully the function defined by the right-hand sides of (3.9), which we shall denote, for brevity, by k:

$$k = \frac{\partial}{\partial n}\Big(\frac{\partial\psi_0(x)}{\partial x_1}\Big) - \kappa x_2\Big(\frac{\partial\psi_0(x)}{\partial x_1}\Big) + \kappa x_1\Big(\frac{\partial\psi_0(x)}{\partial x_2}\Big) \quad \text{on } \Gamma_0. \tag{5.24}$$

Lemma 5.1 *The function k satisfies:*

$$|k| \leqslant Cr^{1/2+\alpha} \quad \textit{on } \Gamma_0, \tag{5.25}$$
$$[k]_{\alpha, B_R\cap\Gamma_0} \leqslant CR^{1/2}. \tag{5.26}$$

Proof. Recall that ψ_0 has the asymptotic behavior [2]:

$$\psi_0(x) = A_1\varphi_1(x) + A_2\varphi_2(x) + A_3x_2^2 + \widetilde{Z}, \quad |\widetilde{Z}| \leqslant C|x|^{2+\delta}$$

for any $0 < \delta \leqslant 1/2$. It will actually be more convenient to write

$$\psi_0(x) = A_1\varphi_1(x) + A_2\varphi_2(x) + A_3x_2^2 - 2A_2\kappa r^{5/2}\cos\frac{1}{2}\theta + Z \tag{5.27}$$

where

$$|Z| \leqslant C|x|^{2+\delta}. \tag{5.28}$$

Consider the function

$$\zeta = k - \frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_1}\Big) + \kappa x_2\frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_1}\Big) - \kappa x_1\frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_2}\Big) \quad \text{on } \Gamma_0. \tag{5.29}$$

We can calculate that the right-hand side is equal to

$$\mathcal{L}\Big(A_1\varphi_1(x) + A_2\varphi_2(x) + A_3x_2^2 - 2A_2\kappa r^{5/2}\cos\frac{1}{2}\theta\Big)$$

and then, the calculations that led to (3.13) show that ζ is not only $O(r^{1/2})$ but also, more precisely, that

$$|\zeta| \leqslant Cr^{1/2+\alpha} \quad \text{on } \Gamma_0, \tag{5.30}$$

and, furthermore,

$$[\zeta]_{\alpha,B_R\cap\Gamma_0} \leqslant CR^{1/2}. \tag{5.31}$$

We next need to estimates the first two derivatives of Z on Γ_0. Since $\psi_0 = \nabla\psi_0 = 0$ on Γ_0, (5.27) yields,

$$Z = -A_1\varphi_1 - A_2\varphi_2 - A_3x_2^2 + 2A_2\kappa r^{5/2}\cos\frac{1}{2}\theta \quad \text{on } \Gamma_0, \tag{5.32}$$

and

$$\nabla Z = -A_1\nabla\varphi_1 - A_2\nabla\varphi_2 - A_3\nabla x_2^2 + 2A_2\kappa\nabla\Big(r^{5/2}\cos\frac{1}{2}\theta\Big) \quad \text{on } \Gamma_0. \tag{5.33}$$

By direct computation we then get

$$|Z| \leqslant Cr^{7/2} \quad \text{on } \Gamma_0, \tag{5.34}$$

$$|\nabla Z| \leqslant Cr^{3/2} \quad \text{on } \Gamma_0, \tag{5.35}$$

$$[\nabla Z]_{\alpha,B_R\cap\Gamma_0} \leqslant CR^{3/2-\alpha}, \tag{5.36}$$

$$\Big|\frac{\partial^2 Z}{\partial\tau^2}\Big| + \Big|\frac{\partial^2 Z}{\partial\tau\partial n}\Big| \leqslant Cr^{1/2} \quad \text{on } \Gamma_0, \tag{5.37}$$

and

$$\Big[\frac{\partial^2 Z}{\partial\tau^2}\Big]_{\alpha,B_R\cap\Gamma_0} + \Big[\frac{\partial^2 Z}{\partial\tau\partial n}\Big]_{\alpha,B_R\cap\Gamma_0} \leqslant CR^{1/2-\alpha}, \tag{5.38}$$

where $\partial/\partial\tau$ is the tangential derivative along Γ_0. Recalling (5.28), we can apply $C^{2+\alpha}$ sub-Schauder estimates [2, §9] to $R^{-(2+\delta)}Z(Ry)$ near the origin. From these estimates we obtain, in particular, that

$$|\nabla^2 Z|_{B_R\setminus\Gamma_0} \leqslant CR^{\delta}, \tag{5.39}$$

$$[\nabla^2 Z]_{\alpha,B_R\setminus\Gamma_0} \leqslant CR^{\delta-\alpha}; \tag{5.40}$$

α is chosen small enough so that $\delta-\alpha$ is positive.

From (5.39), (5.40) we conclude that the last two terms in ζ (see (5.29)) can be estimated as in (5.30), (5.31). Consequently,

$$\Big|k-\frac{\partial}{\partial n}\frac{\partial Z}{\partial x_1}\Big|\leqslant Cr^{1/2+\alpha}\quad\text{ on }\Gamma_0, \tag{5.41}$$

$$\Big[k-\frac{\partial}{\partial n}\frac{\partial Z}{\partial x_1}\Big]_{\alpha,B_R\backslash\Gamma_0}\leqslant CR^{1/2}. \tag{5.42}$$

We next proceed to evaluate $\frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_1}\Big)$, making use of the inequalities

$$|f'(x_1)|\leqslant C|x_1|,\quad |f''(x_1)|\leqslant C. \tag{5.43}$$

We begin with

$$\begin{aligned}\frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_1}\Big) &= \frac{\partial}{\partial x_1}\Big[-\frac{f'(x_1)}{\sqrt{1+(f')^2}}\frac{\partial Z}{\partial x_1}+\frac{1}{\sqrt{1+(f')^2}}\frac{\partial Z}{\partial x_2}\Big]\\ &\quad +\frac{d}{dx_1}\Big(\frac{f'(x_1)}{\sqrt{1+(f')^2}}\Big)\frac{\partial Z}{\partial x_1}-\frac{d}{dx_1}\Big(\frac{1}{\sqrt{1+(f')^2}}\Big)\frac{\partial Z}{\partial x_2}\equiv I+J-K.\end{aligned}$$

Clearly

$$I=\frac{\partial}{\partial x_1}\Big(\frac{\partial Z}{\partial n}\Big)=\frac{1}{\sqrt{1+(f')^2}}\frac{\partial^2 Z}{\partial\tau\partial n}-\frac{f'(x_1)}{\sqrt{1+(f')^2}}\frac{\partial^2 Z}{\partial n^2}.$$

Using (5.39), (5.43) we find that the last term is bounded by $Cr^{1+\delta}$. By (5.35) and (5.43), J and K are bounded by $Cr^{3/2}$, so that

$$\Big|\frac{\partial}{\partial n}\frac{\partial Z}{\partial x_1}-\frac{\partial^2 Z}{\partial\tau\partial n}\Big|\leqslant Cr^{1+\delta}.$$

Now, $\partial^2 Z/\partial\tau\partial n$ was already estimated in (5.37). However, that estimate was obtained by estimating term by term in (5.33). A more careful computation (cf. §3) reveals that the most singular terms coming from $-A_2\varphi_2$ and $2A_2\kappa r^{5/2}\cos(\theta/2)$ cancel each other, so that

$$\Big|\frac{\partial^2 Z}{\partial\tau\partial n}\Big|\leqslant Cr^{1/2+\alpha}.$$

It follows that

$$\Big|\frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_1}\Big)\Big|\leqslant Cr^{1/2+\alpha}\quad\text{ on }\Gamma_0. \tag{5.44}$$

Using this in (5.41), the assertion (5.25) follows.

Similarly we can estimate the Hölder coefficient of $\frac{\partial}{\partial n}\Big(\frac{\partial Z}{\partial x_1}\Big)$ on Γ_0 and thus arrive at the inequality (5.26). □

Lemma 5.2 *There exists a unique classical solution V in $H^2(\Omega)$ satisfying (3.7)–(3.9), (5.21), (5.22) such that*

$$|V(x)|\leqslant C|x|^{3/2}. \tag{5.45}$$

Proof. In view of Lemma 5.1 we can construct $C^{2+\alpha}$ functions k_m which approximate k on Γ_0 and are supported away from the origin, such that

$$|k_m| \leqslant Cr^{1/2+\alpha} \quad \text{on } \Gamma_0, \tag{5.46}$$

$$[k_m]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant CR^{1/2+\alpha-\widetilde{\alpha}} \quad \text{if } R > \frac{1}{m}. \tag{5.47}$$

Let Φ_m be $C^{2+\alpha}(\mathbb{R}^2)$ functions such that

$$\Phi_m = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial \Phi_m}{\partial n} = k_m \quad \text{on } \Gamma_0, \quad |\nabla^2 \Phi_m| \leqslant C_m,$$

where C_m is a constant depending on m. The variational method of [7] can be applied to show that there exists a solution $v = v_m$ in $H^2(\Omega)$ to

$$\Delta^2 v = -\Delta^2 \Phi_m \in \Omega \setminus \Gamma_0,$$
$$v = \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_0$$

which takes the Dirichlet data of $V - \Phi_m$ on $\partial\Omega$. Setting $V_m = v_m + \Phi_m$ we find that V_m solves the same system as V, except that

$$\frac{\partial V_m}{\partial n} = k_m \quad \text{on } \Gamma_0.$$

Furthermore,

$$|V_m(x)| \leqslant C_m r.$$

We claim that

$$|V_m(x)| \leqslant Cr, \quad C \text{ independent of } m. \tag{5.48}$$

Indeed, otherwise we define

$$L_m = \sup_x \frac{|V_m(x)|}{|x|} = \frac{|V_m(x_m)|}{|x_m|} \qquad (x_m \in \Omega)$$

so that $L_m \to \infty$ (for a subsequence). The functions

$$\widetilde{V}_m = \frac{V_m}{L_m}$$

are uniformly bounded, and

$$\widetilde{V}_m = 0 \quad \text{on } \Gamma_0,$$
$$\Big|\frac{\partial \widetilde{V}_m}{\partial n}\Big| \leqslant \frac{C}{L_m} r^{1/2+\alpha} \quad \text{on } \Gamma_0,$$
$$\Big[\frac{\partial \widetilde{V}_m}{\partial n}\Big]_{\alpha,(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant \frac{C}{L_m} R^{1/2+\alpha} \quad \text{if } R > \frac{1}{m}.$$

By $C^{1+\alpha}$ sub-Schauder estimates [2, §9], $\widetilde{V}_m \to V^*$, where

$$\Delta^2 V^* = 0 \quad \text{in } \Omega \setminus \Gamma_0,$$
$$V^* = \frac{\partial V^*}{\partial n} = 0 \quad \text{on } \Gamma_0 \text{ and on } \partial\Omega.$$

Since further $|V^*| \leqslant Cr$, we can apply Lemma 4.1 to deduce that

$$|V^*| \leqslant Cr^{3/2}.$$

Noting that V^* is also in $C^{2+\alpha}$ near Γ_0, we may integrate by parts to get

$$0 = \int_{\Omega\setminus\Gamma_0} V^*\Delta^2 V^* = \int_{\Omega\setminus\Gamma_0} |\Delta V^*|^2$$

so that $V^* \equiv 0$. If $x^* \equiv \lim x_m$ is $\neq 0$ then $V^*(x^*) = \lim_{m\to\infty} V_m(x_m) = 1$ which is a contradiction. Thus we conclude that $x_m \to x^* = 0$.

We now rescale by introducing

$$\widehat{V}_m(x) = V_m(|x_m|x).$$

The the limit $V^* = \lim \widehat{V}_m$ is a solution of

$$\begin{aligned}
&\Delta^2 V^* = 0 \quad \text{ in } \Omega^* \equiv \mathbb{R}^2 \setminus \{-\infty < x_1 < 0, \ \ x_2 = 0\},\\
&V^* = \frac{\partial V^*}{\partial n} = 0 \quad \text{ on } \{-\infty < x_1 < 0, \ \ x_2 = 0\},\\
&|V^*| \leqslant Cr.
\end{aligned}$$

By Lemma 4.1, $|V^*| \leqslant Cr^{3/2}$ near $r = 0$ and thus, by integration by parts,

$$0 = \int_{\Omega^*} V^*\Delta^2 V^* = \int_{\Omega^*} |\Delta V^*|^2$$

so that $V^* \equiv 0$. Since $\widehat{V}_m(e_m) = 1$ where $e_m = x_m/|x_m|$, we also have $V^*(e) = 1$ where $e = \lim e_m$, which is a contradiction.

Having proved (5.48) we now take $m \to \infty$ and obtain a solution $V = \lim V_m$ of (3.7)–(3.9), (5.21), (5.22) which, by Lemma 4.1, satisfies also (5.45). To prove uniqueness we suppose that V_1 and V_2 are two solutions and let $V = V_1 - V_2$. Using the estimates (5.45) and $C^{2+\alpha}$ sub-Schauder estimates near Γ_0, we can integrate by parts to obtain

$$0 = \int_{\Omega\setminus\Gamma_0} V\Delta^2 V = \int_{\Omega\setminus\Gamma_0} |\Delta V|^2,$$

which implies that $V \equiv 0$. □

We want to analyze the asymptotic behavior of $V(x)$ near $x = 0$.

From Lemma 4.1 we know that V has an expansion

$$V(x) = \beta_1\varphi_1(x) + \beta_2\varphi_2(x) + O(r^{3/2+\delta}). \tag{5.49}$$

We next split V into $V_1 + V_2$ where V_1 is the solution corresponding to the Dirichlet data of (3.16)

$$\lim_{\varepsilon\to 0} \frac{\psi_0(T_\varepsilon x) - \psi_0(x)}{\varepsilon} \quad \text{ on } \partial\Omega,$$

and V_2 is the solution corresponding to (3.17) with Dirichlet data of

$$\lim_{\varepsilon\to 0} \frac{\psi_0(R_\varepsilon x) - \psi_0(x)}{\varepsilon} \quad \text{ on } \partial\Omega;$$

the proof of Lemma 5.2 can be applied to establish the existence and uniqueness of both V_1 and V_2.

In order to derive an asymptotic expansion for V_1 we need to work with

$$V_1 - 4\kappa A_2 r^{3/2} \cos\frac{1}{2}\theta$$

(the added term will enable the necessary cancellation with second boundary condition in (3.16)) We then obtain, upon using Lemma 4.1,

$$V_1 = \widehat{\beta}_1\varphi_1 + \widehat{\beta}_2\varphi_2 + 4\kappa A_2 r^{3/2}\cos\frac{1}{2}\theta + O(r^{3/2+\delta}). \tag{5.50}$$

Similarly

$$V_2 = \widetilde{\beta}_1\varphi_1 + \widetilde{\beta}_2\varphi_2 - 4\kappa A_2 r^{3/2}\cos\frac{1}{2}\theta + O(r^{3/2+\delta}). \tag{5.51}$$

If we substitute $\partial V_2/\partial n$ from (5.51) into (3.17) and compare the leading terms, we find that

$$\widetilde{\beta}_1 = \frac{3}{2}\kappa A_2, \quad \widetilde{\beta}_2 = -\frac{3}{2}\kappa A_1. \qquad \Box \tag{5.52}$$

We summarize:

Theorem 5.3 *There exists a $\delta > 0$ such that (5.49)–(5.51) holds and $\widetilde{\beta}_1$, $\widetilde{\beta}_2$ are given by (5.52).*

Introducing the function

$$W = \psi - \psi_0 - \varepsilon V \quad \text{in } \Omega\cap\widetilde{\Omega}, \tag{5.53}$$

we now state a fundamental result:

Theorem 5.4 *There exists a $\delta > 0$ such that*

$$|W(x)| \leqslant C\varepsilon^{1+\delta}|x|^{3/2} \quad \textit{in } \Omega\cap\widetilde{\Omega}, \tag{5.54}$$

where C is a constant independent of ε and δ; furthermore, W has the following asymptotic behavior near the origin:

$$W(x) = \gamma_1\varphi_1(x) + \gamma_2\varphi_2(x) + O(|x|^{3/2+\delta}) \tag{5.55}$$

where γ_1, γ_2 are constants, and

$$|\gamma_1| + |\gamma_2| = O(\varepsilon^{1+\delta}). \tag{5.56}$$

Theorem 5.4 can be used to compute the stress intensity factors $A_1(\varepsilon)$, $A_2(\varepsilon)$ defined in (1.22). Indeed, these coincide with the stress intensity factors of the function ψ which occurs in (5.53), and, therefore, by Theorem 5.4,

$$A_1(\varepsilon) = A_1(0) + \Big(\frac{3}{2}\kappa A_2 + \widehat{\beta}_1\Big)\varepsilon + \varepsilon\gamma_1 + \varepsilon O(|x|^\delta),$$
$$A_2(\varepsilon) = A_2(0) + \Big(-\frac{3}{2}\kappa A_2 + \widehat{\beta}_2\Big)\varepsilon + \varepsilon\gamma_2 + \varepsilon O(|x|^\delta).$$

Taking $x \to 0$ and using (5.28) we conclude that

$$\left.\frac{dA_1(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \quad \text{exists and is equal to } \frac{3}{2}\kappa A_2 + \widehat{\beta}_1, \tag{5.57}$$

and, similarly

$$\left.\frac{dA_2(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{3}{2}\kappa A_2 + \widehat{\beta}_2. \tag{5.58}$$

The proof of (5.57), (5.58) extends to the tip of the crack Γ_s, for any $0 \leqslant s \leqslant s_0$. Thus, we obtain:

Theorem 5.5 *Under the assumptions (5.1)–(5.8), the stress intensity factors* $A_1(s), A_2(s)$ *are differentiable and*

$$\frac{dA_1(s)}{ds} = \frac{3}{2}\kappa(s)A_2(s) + \widehat{\beta}_1(s), \tag{5.59}$$

$$\frac{dA_2(s)}{ds} = -\frac{3}{2}\kappa(s)A_1(s) + \widehat{\beta}_2(s), \tag{5.60}$$

where $\kappa(s)$ *is the curvature at the tip of* Γ_s *and* $\widehat{\beta}_1(s), \widehat{\beta}_2(s)$ *are obtained as in (5.50).*

The proof of Theorem 5.4 is given in the next section.

6 Proof of Theorem 5.4

We need several lemmas.

Lemma 6.1 *There holds:*

$$|\widetilde{f}_\varepsilon(x_1) - f(x_1)| \leqslant C\varepsilon|x_1|^{1+\alpha}, \quad -\min(x_0, x_{0\varepsilon}) \leqslant x_1 \leqslant 0. \tag{6.1}$$

Proof. Recall that $\widetilde{f}_\varepsilon$ was defined by

$$\widetilde{x}_2 = \widetilde{f}_\varepsilon(\widetilde{x}_1) \quad \text{where } x_2 = f(x_1) \tag{6.2}$$

and $x, \widetilde{x}$ are related by (5.11). Using (5.12) we easily find that

$$|\widetilde{f}''_\varepsilon(x_1) - f''(x_1)| \leqslant C\varepsilon^\alpha \quad \text{if } -\varepsilon \leqslant x_1 \leqslant 0.$$

Thus it remains to consider the case where $x_1 < -\varepsilon$. We can rewrite the relation (6.2) in the form

$$x_2(\widetilde{x}_1, \widetilde{f}_\varepsilon(x_1), \varepsilon) = f(x_1(\widetilde{x}_1, \widetilde{f}_\varepsilon(\widetilde{x}_1)), \varepsilon) \tag{6.3}$$

where

$$x_j = x_j(\widetilde{x}_1, \widetilde{x}_2, \varepsilon)$$

is defined by (5.12). Differentiating (6.3) in ε we easily get:

$$\frac{\partial \widetilde{f}_\varepsilon(x_1)}{\partial\varepsilon} = \frac{f'(x_1)\frac{\partial x_1}{\partial\varepsilon} - \frac{\partial x_2}{\partial\varepsilon}}{\frac{\partial x_2}{\partial\widetilde{x}_2} - f'(x_1)\frac{\partial x_1}{\partial\widetilde{x}_2}}. \tag{6.4}$$

We evaluate the denominator by using (5.13), (5.12):

$$\frac{\partial x_2}{\partial \widetilde{x}_2} - f'(x_1)\frac{\partial x_1}{\partial \widetilde{x}_2} = 1 + O(|x_1|^{1+\alpha}). \tag{6.5}$$

To evaluate the numerator we write

$$\begin{aligned} f'(x_1)\frac{\partial x_1}{\partial \varepsilon} - \frac{\partial x_2}{\partial \varepsilon} &= [f'(x_1)(1-\kappa\widetilde{x}_2) - \kappa\widetilde{x}_1 - f'(\varepsilon)] \\ &+[f'(x_1)(\frac{\partial x_1}{\partial \varepsilon} + \kappa\widetilde{x}_2 - 1)] - [\frac{\partial x_2}{\partial \varepsilon} - \kappa\widetilde{x}_1 - f'(\varepsilon)] \equiv A + B - D; \end{aligned} \tag{6.6}$$

here A is an approximation to the left-hand side whereas B and C represent error terms. Using (5.13), (5.12) we find that

$$\begin{aligned} \Big|\frac{\partial x_1}{\partial \varepsilon} + \kappa\widetilde{x}_2 - 1\Big| &\leqslant C|\widetilde{x}_1|\varepsilon + C|\widetilde{x}_2|\varepsilon^{\alpha}, \\ \Big|\frac{\partial x_2}{\partial \varepsilon} - \kappa\widetilde{x}_1 - f'(\varepsilon)\Big| &\leqslant C|\widetilde{x}_1|\varepsilon^{\alpha} + C|\widetilde{x}_2|\varepsilon^2, \end{aligned}$$

and, since $|x_1| \geqslant \varepsilon$,

$$|B| + |D| \leqslant C|x_1|^{1+\alpha}; \tag{6.7}$$

here we used the estimates $|\widetilde{x}_1| \leqslant C|x_1|$, $|\widetilde{x}_2| \leqslant C|\widetilde{x}_1|$. In the sequel we shall use the inequalities

$$|x_2| \leqslant C|x_1|^2, \quad |\widetilde{x}_2| \leqslant C|\widetilde{x}_1|^2 \tag{6.8}$$

which hold for $x_2 = f(x_1)$, $\widetilde{x}_2 = \widetilde{f}_\varepsilon(\widetilde{x}_1)$.

To estimate A we write

$$\begin{aligned} A \;\; = \;\; & [f'(x_1) - f'(\widetilde{x}_1 + \varepsilon - \kappa\varepsilon x_2)](1-\kappa\widetilde{x}_2) \\ & +[f'(\widetilde{x}_1 + \varepsilon - \kappa\varepsilon x_2)(1-\kappa\widetilde{x}_2) - \kappa\widetilde{x}_1 - f'(\varepsilon)] \equiv A_1 + A_2. \end{aligned} \tag{6.9}$$

By the mean value theorem

$$|A_1| \leqslant C|x_1 - (\widetilde{x}_1 + \varepsilon - \kappa\varepsilon x_2)| \leqslant C\varepsilon^2 + C\varepsilon|x_2| \leqslant C|x_1|^2$$

where we have used (5.13), (5.12) and (6.8), as well as the inequality $|x_1| \geqslant \varepsilon$.

From (6.8) it follows that the terms $\kappa\widetilde{x}_2$ which appears in A_2 (in the factor $(1-\kappa\widetilde{x}_2)$) is bounded by $C|x_1|^2$, so that

$$|A_2| \leqslant |\Phi(\widetilde{x}_1)| + C|x_1|^2 \tag{6.10}$$

where

$$\Phi(\widetilde{x}_1) = f'(\widetilde{x}_1 + \varepsilon - \kappa\varepsilon x_2) - \kappa\widetilde{x}_1 - f'(\varepsilon).$$

If $\widetilde{x}_1 = 0$ then $x_1 = 0$ and $x_2 = f(0) = 0$, Therefore $\Phi(0) = 0$. Also,

$$\Phi'(\widetilde{x}_1) = f''(\widetilde{x}_1 + \varepsilon - \kappa\varepsilon x_2)\Big(1 - \kappa\varepsilon\frac{\partial x_2}{\partial \widetilde{x}_2}\Big) - \kappa$$

and, since $\kappa = f''(0)$ and $\partial x_2/\partial\widetilde{x}_2 = O(\varepsilon)$ (by (5.13)), we have

$$|\Phi'(\widetilde{x}_1)| \leqslant C\varepsilon^2 + (\widetilde{x}_1 + \varepsilon - \kappa x_2)^{\alpha} \leqslant C|x_1|^{\alpha}.$$

Using the mean value theorem, we conclude that

$$|\Phi(\widetilde{x}_1)| \leqslant C|x_1|^{1+\alpha}$$

and together with (6.9), (6.10),

$$|A| \leqslant C|x_1|^{1+\alpha}.$$

Combining this estimate with (6.7) and (6.6), we find that the numerator in (6.4) is bounded by $C|x_1|^{1+\alpha}$. Recalling (6.5) we get the estimate

$$\Big|\frac{\partial \widetilde{f}_\varepsilon(x_1)}{\partial \varepsilon}\Big| \leqslant C|x_1|^{1+\alpha}.$$

Finally, by integration,

$$|\widetilde{f}_\varepsilon(x_1) - f_0(x_1)| = \Big|\int_0^\varepsilon \frac{\partial \widetilde{f}_\varepsilon(x_1)}{\partial \varepsilon} d\varepsilon\Big| \leqslant C\varepsilon|x_1|^{1+\alpha}$$

and the lemma follows. □

Lemma 6.2 *There exists a positive constant δ depending on the angles that $\partial\Omega$ and Γ_0 form at P_0 (see (5.7)) such that, if α is smaller than δ, then*

$$|W| \leqslant C\varepsilon^2 \quad on\ \partial(\Omega \cap \widetilde{\Omega}), \tag{6.11}$$

$$\Big|\frac{\partial W}{\partial n}\Big| \leqslant C\varepsilon^{1+\alpha} \quad on\ \partial(\Omega \cap \widetilde{\Omega}); \tag{6.12}$$

the constant C is independent of ε.

Proof. Denote by $\widetilde{P}_\varepsilon$ the initial point of $\widetilde{\Gamma}_\varepsilon$ and by $\widetilde{d}(x)$, $d(x)$ the distances from x to $\widetilde{P}_\varepsilon$ and P_0, respectively. We introduce the region Ω_μ^+ (Ω_μ^-) of all points in Ω with $d(x) < \mu$ which lie above (below) $\Gamma_0 \cap B_\mu(P_0)$, and, similarly, the region $\widetilde{\Omega}_\mu^+$ ($\widetilde{\Omega}_\mu^-$) of all points in $\widetilde{\Omega}$ with $\widetilde{d}(x) < \mu$ which lie above (below) $\widetilde{\Gamma}_0 \cap B_\mu(P_\varepsilon)$. If Γ_0 is in C^4 then we can apply results of Kondratév [6] to deduce that

$$|\psi_0(x)| \leqslant Cd(x)^{2+\delta}, \quad 0 < \delta < 1 \tag{6.13}$$

for $x \in \Omega_\mu^+$ where δ depends on the angles that Γ_0 and $\partial\Omega$ form at P_0. In the present case where Γ_0 and $\partial\Omega$ is only $C^{2+\alpha}$, this estimate is still true but it requires a more elaborate proof which will be given in the Appendix (§9).

Similarly, the inequality (6.13) is valid for $x \in \Omega_\mu^-$. Clearly, the inequality also holds throughout the rest of Ω.

Similarly we have

$$|\psi(x)| \leqslant C\widetilde{d}(x)^{2+\delta} \tag{6.14}$$

in $\widetilde{\Omega}$.

Consider the function

$$G(y) = \frac{1}{R^{2+\delta}}\psi(\widetilde{P}_\varepsilon + Ry)$$

for R small (say $2R < \mu/2$) and $\widetilde{P}_\varepsilon + Ry$ in $\widetilde{\Omega}_{\mu/2}^+$. From (6.14) and $C^{2+\alpha}$ sub-Schauder estimates [2, §9] we have

$$|\nabla G| + |\nabla^2 G| + [\nabla^2 G]_\alpha \leqslant C \quad \text{if } 1 < |y| < 2$$

and $\widetilde{P}_\varepsilon + Ry \in \widetilde{\Omega}^+_{\mu/2}$. The same inequality holds with respect to $\widetilde{\Omega}^-_{\mu/2}$. Hence

$$R|\nabla\psi|_{L^\infty[(B_R\setminus B_{R/2})\cap\widetilde{\Omega}^\pm_{\mu/2}]} + R^2|\nabla^2\psi|_{L^\infty[(B_R\setminus B_{R/2})\cap\widetilde{\Omega}^\pm_{\mu/2}]} + R^{2+\alpha}[\nabla\psi]_{\alpha,(B_R\setminus B_{R/2})\cap\widetilde{\Omega}^\pm_{\mu/2}} \leqslant CR^{2+\alpha}, \tag{6.15}$$

where $B_R = B_R(\widetilde{P}_\varepsilon)$. We choose α smaller than δ. Then (6.15) yields:

$$\begin{aligned} &|D^2\psi(x)| \leqslant C\widetilde{d}^\alpha(x) \quad \text{in } \widetilde{\Omega}^\pm_{\mu/2}, \\ &[D^2\psi(x)]_{\alpha,\widetilde{\Omega}^\pm_{\mu/2}} \leqslant C. \end{aligned} \tag{6.16}$$

Similarly

$$\begin{aligned} &|D^2\psi_0(x)| \leqslant Cd^\alpha(x) \quad \text{in } \Omega^\pm_{\mu/2}, \\ &[D^2\psi_0(x)]_{\alpha,\Omega^\pm_{\mu/2}} \leqslant C. \end{aligned} \tag{6.17}$$

From the definition (5.20) of the operator $\mathcal{L}h$ and the mean value theorem we get

$$\left|\mathcal{L}h - \frac{k(S_\varepsilon x) - k(x)}{\varepsilon}\right| \leqslant C\varepsilon|D^2k| \tag{6.18}$$

where $|D^2k|$ is evaluated at an intermediate point. Recalling the definition of V on $\partial\Omega$ and using (6.16), (6.17) and the estimate (6.18), we easily get

$$\frac{1}{\varepsilon}|W(x)| = \left|V(x) - \frac{\psi(\widetilde{x}) - \psi_0(x)}{\varepsilon}\right| \leqslant C\varepsilon$$

on the parts of $\partial(\Omega\cap\widetilde{\Omega})$ which lie within $\mu/4$ neighborhood of P_0 and for which $d(x) > C_0\varepsilon$; here C_0 is large enough so that $x, \widetilde{x}$ and the intermediate point (used in the mean value theorem in (6.18)) all lie either above $\Gamma_0\cup\widetilde{\Gamma}_\varepsilon$ or below $\Gamma_0\cup\widetilde{\Gamma}_\varepsilon$ (so that (6.16), (6.17) can be applied simultaneously in $\widetilde{\Omega}^+_{\mu/2}$, $\Omega^+_{\mu/2}$ or in $\widetilde{\Omega}^-_{\mu/2}$, $\Omega^-_{\mu/2}$).

Similarly we obtain

$$\frac{1}{\varepsilon}\left|\frac{\partial W}{\partial n}\right| \leqslant C\varepsilon^{\delta-\alpha} \quad \text{if } d(x) \geqslant C_0\varepsilon$$

and, upon choosing α sufficiently small, we derive both (6.11) and (6.12) (with another (smaller) δ).

If $d(x) \leqslant C_0\varepsilon$, then we can use (6.13) and interior elliptic estimates to deduce that

$$|\psi_0| \leqslant C\varepsilon^{2+\delta}, \quad |\nabla\psi_0| \leqslant C\varepsilon^{1+\delta}.$$

Similarly,

$$|\psi| \leqslant C\varepsilon^{2+\delta}, \quad |\nabla\psi| \leqslant C\varepsilon^{1+\delta},$$

and, consequently,

$$|V| \leqslant C\varepsilon^{2+\delta}, \quad |\nabla V| \leqslant C\varepsilon^{1+\delta}.$$

The estimates (6.11), (6.12) then follows from the definition of W in (5.53). □

We next estimate W and ∇W along Γ_0.

Lemma 6.3 *There exist positive constants σ, λ and $\widetilde{\alpha}$ such that, along $\Gamma_0\cap\widetilde{\Omega}$,*

$$|W(x)| \leqslant C\varepsilon^2 r^{3/2+2\alpha}, \tag{6.19}$$

$$|\nabla W(x)| \leqslant C\varepsilon^{1+\lambda} r^{1/2+\sigma}, \tag{6.20}$$

$$[\nabla W]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant C\varepsilon^{1+\lambda} r^{1/2+\sigma} \tag{6.21}$$

where C is a constant independent of ε.

Proof. Since $\psi_0 = 0$, $\nabla\psi_0 = 0$, $V = 0$ along Γ_0, we have

$$W = \psi \quad \text{on } \Gamma_0 \cap \widetilde{\Omega}, \tag{6.22}$$

and

$$\frac{\partial W}{\partial \tau} = \frac{\partial \psi}{\partial \tau}, \qquad \frac{\partial W}{\partial n} = \frac{\partial \psi}{\partial n} - \varepsilon \frac{\partial V}{\partial n} \quad \text{along } \Gamma_0 \cap \widetilde{\Omega}. \tag{6.23}$$

The function ψ satisfies

$$|\psi(x)| \leqslant C r^{3/2} \tag{6.24}$$

and it vanishes with its normal derivative along $\widetilde{\Gamma}$. We can therefore apply $C^{2+\alpha}$ sub-Schauder estimates to $\psi(Ry)/R^{3/2}$ to conclude that

$$|\psi(x)| \leqslant C r^{-1/2} d^2(x) \quad \text{on } \Gamma_0 \cap \widetilde{\Omega}, \tag{6.25}$$

$$\Big|\frac{\partial \psi(x)}{\partial \tau}\Big| \leqslant C r^{-1/2-\alpha} d(x)^{1+\alpha} \quad \text{on } \Gamma_0 \cap \widetilde{\Omega}, \tag{6.26}$$

$$\Big[\frac{\partial \psi(x)}{\partial \tau}\Big]_{\widetilde{\alpha},(B_R \setminus B_{R/2})\cap\Gamma_0} \leqslant C R^{-1/2-\alpha-\widetilde{\alpha}} d(R)^{1+\alpha-\widetilde{\alpha}} \tag{6.27}$$

where $d(x)$ is the distance from x to $\widetilde{\Gamma}_\varepsilon$ and $d(R)$ is the distance from a point in $B_R \cap \Gamma_0$ to $\widetilde{\Gamma}_\varepsilon$. Note that the estimates (6.25)–(6.27) have actually been proved only away from $\partial\Omega$; however if we use (6.14) (where $\widetilde{d}(x) = |x - \widetilde{P}_\varepsilon|$) instead of (6.24), then we can establish (in the same way) these estimates also near the initial point of Γ_0.

By Lemma 6.1,

$$d(x) \leqslant C|\widetilde{f}_\varepsilon(x) - f(x)| \leqslant C\varepsilon r^{1+\alpha}$$

and

$$d(R) \leqslant C\varepsilon R^{1+\alpha}.$$

Therefore

$$\begin{aligned} &|\psi(x)| \leqslant C\varepsilon^2 r^{3/2+2\alpha} \quad \text{on } \Gamma_0 \cap \widetilde{\Omega}, \\ &\Big|\frac{\partial \psi(x)}{\partial \tau}\Big| \leqslant C\varepsilon^{1+\alpha} r^{3/2+\alpha+\alpha^2} \quad \text{on } \Gamma_0 \cap \widetilde{\Omega}, \\ &\Big[\frac{\partial \psi(x)}{\partial \tau}\Big]_{\widetilde{\alpha},(B_R \setminus B_{R/2})\cap\Gamma_0} \leqslant C\varepsilon^{1+\alpha-\widetilde{\alpha}} r^{1+\lambda} \end{aligned} \tag{6.28}$$

for some $\lambda > 0$, and the same inequalities then hold also for W, $\partial W/\partial\tau$ (by (6.22), (6.23)). Thus it remains to estimate $\partial W/\partial n$.

Arguing as in the derivation of (3.6) but replacing the function ψ_0 by ψ, we can establish the estimate

$$\Big|\frac{\partial \psi}{\partial n} - \varepsilon\Big[\frac{\partial}{\partial n}\Big(\frac{\partial \psi}{\partial x_1}\Big) - \kappa x_2 \frac{\partial}{\partial n}\Big(\frac{\partial \psi}{\partial x_1}\Big) + \kappa x_1 \frac{\partial}{\partial n}\Big(\frac{\partial \psi}{\partial x_2}\Big)\Big]\Big| \leqslant C r^{-1/2-\alpha} \varepsilon^{1+\alpha} \quad \text{on } \Gamma_0;$$

here we used Lemma 6.1.

Introducing the notation

$$M[\psi] = \frac{\partial}{\partial n}\Big(\frac{\partial \psi}{\partial x_1}\Big) - \kappa x_2 \frac{\partial}{\partial n}\Big(\frac{\partial \psi}{\partial x_1}\Big) + \kappa x_1 \frac{\partial}{\partial n}\Big(\frac{\partial \psi}{\partial x_2}\Big), \tag{6.29}$$

we can write the last inequality in the form

$$\left|\frac{\partial\psi}{\partial n} - \varepsilon M[\psi]\right| \leqslant Cr^{-1/2-\alpha}\varepsilon^{1+\alpha} \quad \text{on } \Gamma_0. \tag{6.30}$$

Similarly one can show that

$$\left[\frac{\partial\psi}{\partial n} - \varepsilon M[\psi]\right]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant CR^{-1/2-\alpha+\widetilde{\alpha}}\varepsilon^{1+\alpha-\widetilde{\alpha}} \tag{6.31}$$

for any $0 < \widetilde{\alpha} < \alpha$.

If we write

$$\frac{\partial W}{\partial n} = \frac{\partial\psi}{\partial n} - \varepsilon\frac{\partial V}{\partial n} = \varepsilon M[\psi] - \varepsilon\frac{\partial V}{\partial n} + S \tag{6.32}$$

then, by (6.30), (6.31),

$$\begin{aligned} &|S| \leqslant Cr^{-1/2-\alpha}\varepsilon^{1+\alpha} \quad \text{on } \Gamma_0, \\ &[S]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant Cr^{-1/2-\alpha+\widetilde{\alpha}}\varepsilon^{1+\alpha-\widetilde{\alpha}}. \end{aligned} \tag{6.33}$$

Recalling the definition of $\partial V/\partial n$ on Γ_0 (in (3.9)), we can rewrite (6.32) in the form

$$\frac{\partial W}{\partial n} = \varepsilon M[\psi - \psi_0] + S. \tag{6.34}$$

We now apply $C^{2+\alpha}$ sub-Schauder estimates to ψ and ψ_0 from both sides of $\widetilde{\Gamma}_\varepsilon$ and Γ_0, respectively. Using also Lemma 6.1 we can estimate $\psi - \psi_0$ in the region bounded by $\widetilde{\Gamma}_\varepsilon$, $\widetilde{\widetilde{\Gamma}}$ and, in particular, obtain the following bounds:

$$\begin{aligned} &|\psi - \psi_0| \leqslant C\varepsilon^2 r^{3/2+2\alpha} \quad \text{on } \Gamma_0, \\ &|\nabla(\psi - \psi_0)| \leqslant C\varepsilon r^{1/2+\alpha} \quad \text{on } \Gamma_0, \\ &|\nabla^2(\psi - \psi_0)| \leqslant C\varepsilon r^{-1/2+\alpha} \quad \text{on } \Gamma_0, \\ &[\nabla^2(\psi - \psi_0)]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant C\varepsilon R^{-1/2-\widetilde{\alpha}}. \end{aligned} \tag{6.35}$$

Consequently,

$$\begin{aligned} &|\varepsilon M[\psi - \psi_0]| \leqslant C\varepsilon^2 r^{-1/2} \quad \text{on } \Gamma_0, \\ &[\varepsilon M[\psi - \psi_0]]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant C\varepsilon^2 R^{-1/2-\widetilde{\alpha}}. \end{aligned}$$

Substituting this into (6.34) and using first the estimate in (6.33), we get

$$\left|\frac{\partial W}{\partial n}\right| \leqslant C\varepsilon^{1+\alpha}r^{-1/2} \leqslant C\varepsilon^{1+\lambda}r^{1/2+\sigma}$$

for some $\lambda > 0$, $\sigma > 0$, provided $r \geqslant \varepsilon^\beta$ for some $\beta > 0$. Similarly,

$$\left[\frac{\partial W}{\partial n}\right]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant C\varepsilon^{1+\lambda}R^{1/2+\sigma}$$

provided $r \geqslant \varepsilon^\beta$ for some smaller $\beta > 0$.

It remains to estimate $\partial W/\partial n$ in case $r \leqslant \varepsilon^\beta$. We apply $C^{1+\alpha}$ sub-Schauder estimates to $\psi(Ry)/R^{3/2}$ and deduce that

$$\left|\frac{\partial\psi}{\partial n}\right| \leqslant Cr^{1/2}\frac{d}{r} \quad \text{on } \Gamma_0$$

where d is the distance from x (in Γ_0) to $\widetilde{\Gamma}_\varepsilon$. Hence, by Lemma 6.1,

$$\left|\frac{\partial\psi}{\partial n}\right| \leqslant C r^{1/2+\alpha}\varepsilon.$$

We can treat V in the same way (with $C^{1+\alpha}$ estimates), noting that

$$|\varepsilon V| \leqslant |\psi - \psi_0| \leqslant C r^{3/2}\varepsilon,$$

and deduce that,

$$\begin{aligned} &\left|\frac{\partial}{\partial n}(\varepsilon V)\right| \leqslant C r^{1/2+\alpha}\varepsilon, \\ &\left[\frac{\partial}{\partial n}(\varepsilon V)\right]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant C R^{1/2+\alpha-\widetilde{\alpha}-\lambda/\beta}. \end{aligned}$$

Recalling that $r < \varepsilon^\beta$ we deduce that

$$\begin{aligned} &\left|\frac{\partial W}{\partial n}\right| \leqslant C\varepsilon r^{1/2+\alpha} \leqslant C\varepsilon^{1+\lambda} r^{1/2+\alpha-\lambda/\beta} \quad \text{on } \Gamma_0, \\ &\left[\frac{\partial W}{\partial n}\right]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_0} \leqslant C\varepsilon^{1+\lambda} r^{1/2+\alpha-\widetilde{\alpha}-\lambda/\beta}, \end{aligned}$$

and the estimates (6.20), (6.21) for $\partial W/\partial n$ follow. □

Lemma 6.4 *There exist positive constants σ, λ, $\widetilde{\alpha}$ such that (6.19) and (6.20) hold along $\widetilde{\Gamma}_\varepsilon$ and*

$$[\nabla W]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\Gamma_\varepsilon} \leqslant C\varepsilon^{1+\lambda} R^{1/2+\sigma}. \tag{6.36}$$

Proof. It will be enough to estimate $\partial W/\partial n$ and its Hölder coefficient on Γ_ε. To do that we introduce, for any $0 < R < 1$, a cutoff function $\xi(x)$ such that $\xi(x) = 1$ if $3R/4 < r < 5R/4$ and $\xi(x) = 0$ if $r < R/2$ or $r > 3R/2$. We write $V = \widetilde{V}_1 + \widetilde{V}_2$ where

$$\begin{aligned} \frac{\partial\widetilde{V}_1}{\partial n} &= (1-\xi(x))\frac{\partial V}{\partial n} \quad \text{on } \Gamma_0, \\ \frac{\partial\widetilde{V}_2}{\partial n} &= \xi(x)\frac{\partial V}{\partial n} \quad \text{on } \Gamma_0, \end{aligned}$$

and the $\widetilde{V}_j$ are biharmonic functions in $\Omega\setminus\Gamma_0$ satisfying all the other boundary conditions as V. (Their existence is proved in the same way as for V).

Rescaling $\widetilde{V}_1(x) \to \widetilde{V}_1(Rx)$ and using $C^{2+\alpha}$ sub-Schauder estimates, we get

$$|\widetilde{V}_1| \leqslant C R^{3/2}\Big(\frac{d}{R}\Big)^2 \quad \text{on } (B_R\setminus B_{R/2})\cap\widetilde{\Gamma}_\varepsilon, \tag{6.37}$$

$$|\nabla\widetilde{V}_1| \leqslant C R^{3/2}\Big(\frac{d}{R}\Big) \quad \text{on } (B_R\setminus B_{R/2})\cap\widetilde{\Gamma}_\varepsilon, \tag{6.38}$$

$$[\nabla\widetilde{V}_1]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\widetilde{\Gamma}_\varepsilon} \leqslant C R^{3/2}\Big(\frac{d}{R}\Big)^{1-\widetilde{\alpha}}, \tag{6.39}$$

where d is the distance from a point on $\widetilde{\Gamma}_\varepsilon$ to Γ_0.

Next, by scaling $\widetilde{V}_2$ and applying $C^{1+\alpha}$ sub-Schauder estimates (making use of the estimates of $k \equiv \partial V/\partial n$ in Lemma 5.1), we get

$$|\widetilde{V}_2| \leqslant CR^{3/2+\alpha}\Big(\frac{d}{R}\Big) \quad \text{on } (B_R \setminus B_{R/2}) \cap \widetilde{\Gamma}_\varepsilon, \tag{6.40}$$

$$\Big|\nabla\widetilde{V}_2 - \nabla\widetilde{V}_2\Big|_{\Gamma_0}\Big| \leqslant CR^{1/2+\alpha}\Big(\frac{d}{R}\Big)^\alpha \quad \text{on } (B_R \setminus B_{R/2}) \cap \widetilde{\Gamma}_\varepsilon, \tag{6.41}$$

$$\Big[\nabla\widetilde{V}_2 - \nabla\widetilde{V}_2\Big|_{\Gamma_0}\Big]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\widetilde{\Gamma}_\varepsilon} \leqslant CR^{1/2+\alpha}\Big(\frac{d}{R}\Big)^{\alpha-\widetilde{\alpha}}, \quad 0 < \widetilde{\alpha} < \alpha; \tag{6.42}$$

in (6.41) the argument of $\widetilde{V}_2$ on Γ_0 is at the point nearest to the argument of $\widetilde{V}_2$ on $\widetilde{\Gamma}_\varepsilon$, and a similar convention applies to (6.42). Analogously to (6.32), we write

$$\frac{\partial W}{\partial n} = -\frac{\partial\psi_0}{\partial n} - \varepsilon\frac{\partial V}{\partial n} = \Big[\varepsilon M[\psi_0] - \varepsilon\frac{\partial V}{\partial n}\Big] + \Big[-\frac{\partial\psi_0}{\partial n} - \varepsilon M[\psi_0]\Big],$$

where $M[\psi_0]$ is defined as in (6.29).

Note that since $M[\psi_0]$ coincides with $\partial V/\partial n$ on Γ_0, it also coincides with $\partial\widetilde{V}_2/\partial n\Big|_{\Gamma_0}$ as it appears in (6.41). Using (6.41) as well as (6.39), we get

$$\Big|\frac{\partial W}{\partial n}\Big| \leqslant C\varepsilon R^{1/2+\alpha}\Big(\frac{d}{R}\Big)^\alpha + \Big|\frac{\partial\psi_0}{\partial n} - \varepsilon M[\psi_0]\Big| \quad \text{on } B_R \setminus B_{R/2}) \cap \widetilde{\Gamma}_\varepsilon.$$

By $C^{2+\alpha}$ sub-Schauder estimates, the last term is bounded by

$$C\varepsilon^{1+\alpha}r^{1/2-1-\alpha}.$$

Invoking also Lemma 6.1, we obtain

$$\Big|\frac{\partial W}{\partial n}\Big| \leqslant C\varepsilon^{1+\alpha+\alpha^2} + C\varepsilon^{1+\alpha}R^{1/2-1-\alpha} \leqslant 2C\varepsilon^{1+\alpha}R^{1/2-1-\alpha}$$

on $B_R \setminus B_{R/2}) \cap \widetilde{\Gamma}_\varepsilon$. Similarly,

$$\Big[\frac{\partial W}{\partial n}\Big]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap\widetilde{\Gamma}_\varepsilon} \leqslant C\varepsilon^{1+\alpha-\widetilde{\alpha}}R^{-1/2-\alpha+\widetilde{\alpha}}.$$

Taking $R \geqslant \varepsilon^\sigma$ for appropriate $\sigma > 0$, the asserted inequalities for $\partial W/\partial n$ and its $\widetilde{\alpha}$-Hölder coefficient on $\widetilde{\Gamma}_\varepsilon$ follow. Finally, the case $R \leqslant \varepsilon^\sigma$ can be handled by estimating $\partial\psi_0/\partial n$ and $\varepsilon\partial V/\partial n$ separately, as in Lemma 6.3. $\square$

We now proceed to prove Theorem 5.4. For clarity, we shall first consider a situation where

$$\Gamma_0 \text{ and } \widetilde{\Gamma}_\varepsilon \text{ do not intersect for all } x_1 < 0. \tag{6.43}$$

Denote by Q_ε the region bounded by Γ_0 and $\widetilde{\Gamma}_\varepsilon$. Consider the function

$$w = W/\varepsilon^{1+\lambda}.$$

We have proved so far that

$$|w| \leqslant C|x|^{3/2+\delta} \quad \text{on } \Gamma_0 \cup \Gamma_\varepsilon,$$
$$|\nabla w| \leqslant C|x|^{1/2+\delta} \quad \text{on } \Gamma_0 \cup \Gamma_\varepsilon,$$
$$[\nabla w]_{\widetilde{\alpha},(B_R\setminus B_{R/2})\cap(\Gamma_0\cup\widetilde{\Gamma}_\varepsilon)} \leqslant CR^{1/2+\delta-\sigma}$$

for some $\delta > 0$ and $0 < \sigma < \delta$, and

$$|w| \leqslant C, \quad |\nabla w| \leqslant C \quad \text{ on } \partial(\Omega \cap \widetilde{\Omega}).$$

Recall also that $W = \psi - \psi_0 - \varepsilon V$, and each of the three terms on the right-hand side is $O(|x|^{3/2})$. Hence

$$|w| \leqslant C_\varepsilon |x| \quad \text{ in } (\Omega \cap \widetilde{\Omega}) \setminus Q_\varepsilon. \tag{6.44}$$

Proceeding as in the proof of (5.48) in Lemma 5.2, we can show that (6.44) holds uniformly in ε, i.e., $|w(x)| \leqslant C|x|$. Therefore Lemma 4.1 can be applied to conclude that (5.55)–(5.57) hold.

It remains to prove Theorem 5.4 without making the assumption (6.43). We introduce a new curve

$$\widetilde{\widetilde{\Gamma}}_\varepsilon : \quad x_2 = \widetilde{\widetilde{f}}(x_1) = \widetilde{f}_\varepsilon(x_1) + C_0 \varepsilon^{2+\alpha} Q\left(\frac{x_1}{\varepsilon}\right) \tag{6.45}$$

where

$$\begin{aligned} &Q \in C^\infty(-\infty, 0), \\ &Q(\xi) \sim |\xi|^{1+\alpha} \quad \text{ if } \xi \to -\infty, \\ &Q(\xi) \sim |\xi|^{2+\alpha} \quad \text{ if } \xi \to 0- . \end{aligned}$$

In view of Lemma 6.1, the constant C_0 can be chosen such that $\widetilde{\widetilde{f}}(x_1) > f(x_1)$. Since the new curve is in $C^{2+\alpha}$, uniformly in ε, we can easily extend the estimates of Lemma 6.4 to $\widetilde{\widetilde{\Gamma}}_\varepsilon$, and then we proceed as before to complete the proof of Theorem 5.4.

7 Existence of solutions

Theorem 7.1 *If the assumptions (5.1)–(5.8) and (1.11) hold and α is small enough, then there exists a solution to problem (C_0) with Γ_{s_0} in $C^{2+\alpha}$.*

Remark. Note that the assumptions (5.1) and (5.5) are not essential and are made just for the purpose of convenient exposition. The regularity assumptions on $\partial\Omega$, f, g can also be relaxed: it suffices to assume that $\partial\Omega$ is in $C^{2+\alpha}$, $g \in C^{2+\alpha}$ and $h \in C^{1+\alpha}$.

Proof. In view of Theorem 5.5 all we need to do is to find a $C^{2+\alpha}$ curve Γ_{s_0} for which

$$-\frac{3}{2}\kappa(s) A_1(s) + \widehat{\beta}_2(s) = 0, \quad 0 \leqslant s \leqslant s_0. \tag{7.1}$$

This will be achieved by means of a fixed point theorem. We shall work with the space Y_M of all $C^{2+\alpha}$ curves

$$\Gamma_{s_0} = \Gamma_0 \cup \{x_2 = f(x_1), \ \ 0 \leqslant x_1 \leqslant s\}$$

with norm

$$\|f\|_{C^{2+\alpha}[0,s_0]} \leqslant M, \quad \text{ where } M = |f''(0)| + 1.$$

To each such f we correspond $A_1(s)$, $\widehat{\beta}_1(s)$ as in Theorem 5.5 and then define a new curve $x_2 = \widetilde{f}(x_1)$ by $\widetilde{f}(0) = \widetilde{f}'(0) = 0$ and

$$\frac{\widetilde{f}''(s)}{[1 + \widetilde{f}'(x)^2]^{1/2}} = \frac{3}{2}\frac{\widehat{\beta}_2(s)}{A_1(s)}, \quad 0 \leqslant s \leqslant s_0;$$

note that $\widetilde{f}''(0 = \kappa = f''(0)$. We denote this mapping by T.

The proof of Theorem 8.2 in [2] when applied to $C^{2+\alpha}$ curves (rather than $C^{1+\alpha}$ curves) shows that $A_1(s)$ is in $C^{1/8}$. A similar proof shows that $\widehat{\beta}_1(s)$ is in $C^{1/8}$. Consequently

$$\|\widetilde{f}\|_{C^{2+\gamma}[0,s_0]} \leqslant C(M). \tag{7.2}$$

Choosing $\alpha < 1/8$ and s_0 small enough we readily see that T maps Y_M into itself and, by (7.2), its image is precompact. Applying Schauder's fixed point theorem, we conclude that T has a fixed point, which is a solution to problem (C_0). □

Suppose next that instead of (1.11) we assume that

$$A_1(0) = 0, \quad A_2(0) \neq 0. \tag{7.3}$$

Then, as in [2], the crack propagation problem (1.1)-(1.7) then reduces to the problem of finding a $C^{2+\alpha}$ curve Γ_{s_0} such that

$$A_1(s) \equiv 0 \quad \text{if } 0 \leqslant s \leqslant s_0. \tag{7.4}$$

This corresponds to *mode II fracture*, or *sliding mode* [8, p.24], and we now have to solve the equation

$$\frac{3}{2}\kappa(x)A_2(s) + \widehat{\beta}_1(s) = 0.$$

Theorem 7.2 *Under the assumptions (5.1)–(5.8) and (7.3), there exists a $C^{2+\alpha}$ curve Γ_{s_0} such that (7.4) holds.*

More generally, if $A_1(0), A_2(0)$ are such that

$$F(A_1(0, A_2(0)) = 0$$

where F is twice continuously differentiable and $\nabla F(A_1(0, A_2(0)) \neq 0$, then we can construct a $C^{2+\alpha}$ curve Γ_{s_0} such that

$$F(A_1(s), A_2(s)) = 0 \quad \text{if } 0 \leqslant s \leqslant s_0.$$

However, this general problem, does not correspond to a model of the form (1.1).

Remark. If Γ_{s_0} is in $C^{2+\alpha}$ where $\alpha > 1/2$, then in the expansion (1.9) we can take $\eta < 1/2$ so that $O(r^{3-\eta})$ is indeed an error term. In that case we can rigorously derive the relation (1.25). In particular, if for the solution of problem (C_0), $\Gamma_{s_0} \cap \{0 \leqslant s \leqslant s_0\}$ is in $C^{2+\alpha}$ for some $\alpha > 1/2$, then

$$\frac{3}{2}\kappa A_1 = \frac{3}{2}A_5 + \alpha_2 \quad \text{along } \Gamma_{s_0}.$$

8 The case of harmonic functions

The methods of the present paper as well as of [2] can be extended to other elliptic operators. We shall consider here briefly the case where $\Delta^2\varphi = 0$ is replaced by $\Delta\varphi = 0$ and the boundary conditions in (1.4) are replaced by $\varphi = 0$ on $\Gamma(t)$.

The expansion (1.8) becomes

$$\varphi = A_1 r^{1/2} \cos\frac{1}{2}\theta + A_2 x_2 + O(r^{1+\lambda}) \quad (\lambda > 0) \tag{8.1}$$

and (1.9) becomes

$$\varphi = A_1 r^{1/2}\cos\frac{1}{2}\theta + A_2 x_2 + \frac{1}{4}A_1\kappa r^{3/2}\sin\frac{3}{2}\theta + A_3 r^{3/2}\cos\frac{3}{2}\theta + O(r^{3/2+\mu}) \quad (\mu > 0). \quad (8.2)$$

In the relation (1.27) the function V is harmonic in $\Omega \setminus \Gamma_0$ and

$$V = \frac{\partial\varphi_0}{\partial x_1} + \kappa\Big(x_1\frac{\partial\varphi_0}{\partial x_2} - x_2\frac{\partial\varphi_0}{\partial x_1}\Big) \quad \text{on } \Gamma_0. \quad (8.3)$$

We write

$$V = V_1 + \kappa V_2 \quad (8.4)$$

where V_1, V_2 are harmonic in $\Omega \setminus \Gamma_0$ and

$$V_1 = \frac{\partial\varphi_0}{\partial x_1} \quad \text{on } \Gamma_0,$$

$$V_2 = x_1\frac{\partial\varphi_0}{\partial x_2} - x_2\frac{\partial\varphi_0}{\partial x_1} \quad \text{on } \Gamma_0.$$

Then

$$V_2 = \frac{\partial\varphi_0}{\partial\theta} \quad \text{in } \Omega \setminus \Gamma_0 \quad (8.5)$$

provided we take $V_2 = \partial\varphi_0/\partial\theta$ on $\partial\Omega$, and thus

$$\kappa V_2 = -\frac{1}{2}\kappa A_1 r^{1/2}\sin\frac{1}{2}\theta + O(r). \quad (8.6)$$

Computing

$$\frac{\partial\varphi_0}{\partial x_1} \sim \pm\frac{1}{2}\kappa A_1 r^{1/2} \quad \text{on } \theta = \pm\pi,$$

we can derive the asymptotic behavior

$$V_1 \sim \frac{1}{2}\kappa A_1 r^{1/2}\sin\frac{1}{2}\theta + \alpha r^{1/2}\cos\frac{1}{2}\theta \quad \text{as } r \to 0. \quad (8.7)$$

Hence

$$V \sim \alpha r^{1/2}\cos\frac{1}{2}\theta \quad \text{as } r \to 0. \quad (8.8)$$

It follows that

$$\frac{dA_1}{ds} = \alpha, \quad (8.9)$$

and this is the analog of (3.15).

In general we do not expect to find a continuation Γ_{s_0} of the crack Γ_0 along which $A_1(s) = 0$. Consider, for example, a rectangular domain $\Omega = \{-a < x_1 < a, -b < x_2 < b\}$ with

$$\Gamma_0 = \{(x,0);\ -a < x_1 < 0\}$$

and

$$\varphi_0(x) = A_3(0) r^{3/2}\cos\frac{3}{2}\theta, \quad A_3(0) > 0.$$

Suppose such continuation is possible. Since the boundary conditions (on $\partial\Omega$) are symmetric with respect to x_2, we expect the solution to be also symmetric, i.e.,

$$\Gamma_s = \{(x_1,0); -a < x_1 < s\}, \quad \varphi(x_1,x_2,s) = \varphi(x_1,-x_2,s).$$

Thus

$$\varphi_0(x) = A_3(s) r^{3/2} \cos \frac{3}{2}\theta(s) + \cdots \tag{8.10}$$

where $(r(s), \theta(s))$ are the polar coordinates of x with respect to $(s, 0)$, and $A_3(s) \to A_3(0)$ if $s \to 0$, so that, in particular, $A_3(s) > 0$ for small s.

Note that

$$\varphi(x_1, 0, s) = A_3(0) x_1^{3/2} \cos \frac{3 \cdot 0}{2} = \varphi_0(x_1, 0). \tag{8.11}$$

From (8.10) and the inequality $A_3(s) > 0$ it follows that

$$\frac{\partial}{\partial x_2}\varphi(x_1, 0+, s) > 0, \quad \frac{\partial}{\partial x_2}\varphi(x_1, 0-, s) < 0$$

so that

$$\Delta\varphi(x, s) = -\beta(x_1)\delta_{x_1}$$

where δ_{x_1} is the Dirac function and $\beta(x_1) > 0$ if $0 < x_1 < s$, $\beta(x_1) = 0$ elsewhere. By the maximum principle we then deduce that

$$\varphi(x, s) > \varphi_0(x) \quad \text{at } (x_1, 0), 0 < x_1 < s$$

which is a contradiction to (8.11).

An alternate way of proving that $A_1(s)$ cannot be equal to zero is by directly computing that $\alpha \neq 0$ at $s = 0$.

9 Appendix. Proof of (6.13)

Let Γ_1, Γ_2 be $C^{2+\alpha}$ arcs initiating at the origin and forming an angle different from 0 and π. Let D be a bounded domain in $\mathbb{R}^2$ bounded by $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_3 is an arc disjoint from $\Gamma_1 \cup \Gamma_2$.

Theorem 9.1 *Under the above assumptions there exists a positive constant δ such that if*

$$\begin{aligned} &\psi \in H^2(D), \\ &\Delta\psi = 0 \quad \text{in } D, \\ &\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \end{aligned}$$

then

$$|\psi(x)| \leqslant C|x|^{2+\delta} \quad \text{in } D. \tag{9.1}$$

Proof. By Example 2 of [2, §9], for any $0 < \mu < 1$,

$$|\psi(r, \theta)| \leqslant C r^{1+\mu} \quad \text{in } D. \tag{9.2}$$

Hence, for any $\varepsilon > 0$, the quantity

$$M_\varepsilon = \sup_{(r,\theta)\in D} \frac{|\psi(r, \theta)|}{\varepsilon r^{1+\mu} + r^{2+\delta}}$$

is finite. If we can prove that

$$M_\varepsilon \leqslant C \tag{9.3}$$

for all $\varepsilon > 0$, where C is a constant independent of ε, then assertion (9.1) would follow. We shall assume that (9.3) does not hold and derive a contradiction.

Our assumption implies that there are sequences ε_n, R_n such that

$$M_{\varepsilon_n} = \frac{\sup_\theta |\psi(R_n, \theta)|}{\varepsilon_n R_n^{1+\mu} + R_n^{2+\delta}} \to 0.$$

We can then easily deduce that $\varepsilon_n \to 0$, $R_n \to 0$. Introduce functions G_n by

$$\psi(x) = M_n(\varepsilon_n R_n^{1+\mu} + R_n^{2+\delta}) G_n(\xi), \quad x = R_n \xi.$$

Then, as in [2, Lemma 3.2], $G_{n'} \to G$ for a subsequence $n' \to \infty$ and

$$\Delta^2 G = 0 \quad \text{in } S_\omega, \tag{9.4}$$

$$G = \frac{\partial G}{\partial n} = 0 \quad \text{on } \partial S_\omega, \tag{9.5}$$

$$|G(\xi)| \leqslant C(|\xi|^{1+\mu} + |\xi|^{2+\delta}) \quad \text{in } S_\omega, \tag{9.6}$$

and

$$G(e) = 1 \quad \text{for some } e \in S_\omega, \ |e| = 1 \tag{9.7}$$

where S_ω is the wedge

$$\{|\xi| > 0, 0 < \theta < \omega\};$$

here, for simplicity, we assumed that the tangents to Γ_1 and Γ_2 at 0 form angles $\theta = 0$ and $\theta = \omega$ with the ξ_1-axis; $0 < \omega < \pi$.

From the local bound in (9.6) and local regularity for biharmonic function in a wedge [5, p.109] we get

$$|G(\xi)| \leqslant C|\xi|^{2+\gamma(\omega)}, \quad |\xi| < 1 \tag{9.8}$$

from some $\gamma(\omega) > 0$, and in the sequel we take

$$0 < \delta < \gamma(\omega). \tag{9.9}$$

We now use a change of the radial variable, $r = e^t$. It is easily verified that G then satisfies a homogeneous elliptic equation with constant coefficients

$$\mathcal{L}(\partial_t, \partial_\theta) G = 0, \quad -\infty < t < \infty, \ 0 < \theta < \infty.$$

where

$$\mathcal{L}(\partial_t, \partial_\theta) = (\partial_{tt} + \partial_{\theta\theta})^2 - 4(\partial_{tt} + \partial_{\theta\theta})\partial_t + 4(\partial_{tt} + \partial_{\theta\theta}).$$

For any λ such that

$$2 + \delta < \lambda < 2 + \gamma(\omega), \tag{9.10}$$

we introduce the function

$$F(t, \theta) = G(t, \theta) e^{-\lambda t}.$$

In view of (9.10) and (9.6), (9.8),

$$|F(t, \theta)| \leqslant C e^{-\beta|t|}, \quad \beta = \min\{\lambda - 2 - \delta, 2 + \gamma(\omega) - \lambda\}, \tag{9.11}$$

and F satisfies an elliptic equation

$$\mathcal{L}(\partial_t, \partial_\theta; \lambda) F = 0.$$

By (9.11), the Laplace transform

$$\widetilde{F}(\widetilde{z}, \theta) = \int_{-\infty}^{\infty} F(t, \theta) e^{-\widetilde{z}t} dt$$

of F exists (and is an analytic function) for $|Re\widetilde{z}| < \beta$. It satisfies the ordinary differential equation

$$\mathcal{L}(-\widetilde{z}, \partial_\theta, \lambda)\widetilde{F} = 0 \tag{9.12}$$

with boundary conditions

$$\widetilde{F} = \frac{\partial}{\partial\theta}\widetilde{F} = 0 \quad \text{at } \theta = 0, \;\; \theta = \omega.$$

It can be easily checked (see also [5, Sect. 3.4.4] applied to the biharmonic function $G(\xi)$ near $\xi = 0$) that if the solution $\widetilde{F}$ is nontrivial then $\widetilde{z} + \lambda = z + 1$ where z is a solution of the equation

$$\sin^2(z\omega) - z^2 \sin^2 \omega = 0. \tag{9.13}$$

But since this equation has only countable number of solutions, $\widetilde{F}(\widetilde{z}, \theta)$ vanishes for all but countable $\widetilde{z}$'s, and therefore $\widetilde{F} \equiv 0$ for $|Re\widetilde{z}| < \beta$, by continuity.

In particular, taking $Re\widetilde{z} = 0$ we conclude that the Fourier transform of $F(t, \theta)$ is identically zero and therefore $G \equiv 0$, which is a contradiction to (9.7). □

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