A note on third order structure functions in turbulence

Q. Nie
Institute of Mathematics and its Applications
The University of Minnesota
Minneapolis, MN

&

S. Tanveer[0]
Department of Mathematics
The University of Chicago, Chicago, IL 60637

Dated: April 15th, '97

Abstract

Starting from the Navier-Stokes equation, we rigorously prove that a modified third order structure function $\tilde{S}_3(r)$ asymptotically equals $-4\,\epsilon\,r/3$ (ϵ is the dissipation rate) in the large Reynolds number limit of Navier-Stokes equation in an inertial regime. From this result, we rigorously confirm the Kolmogorov four-fifth law, without the Kolmogorov assumption on isotropy. Our definition of the structure function involves a solid angle averaging over all possible orientation of the displacement vector y, besides space-time averaging. Direct numerical simulation for a highly symmetric flow for a Taylor Reynolds number of 155 shows that the flow remains significantly anisotropic and that without solid angle averaging, the resulting structure functions approximately satisfy these scaling relations over some range of $r = |y|$ for some orientation of y, but not for another.

1 Introduction

Structure functions has been and continues to be a subject of much interest over the last five decades since Kolmogorov introduced it in his seminal paper [1] on turbulence. The subject is too vast reviewed properly in this short paper [See, for example, the book by Monin & Yaglom [2]].

Kolmogorov defined the $n-th$ order longitudinal structure functions to be:

$$S_{n,K}(r) = \langle |u(x+y,t) - u(x,t)|^n \; cos^n \; \theta_{\delta u,y} \rangle \tag{1}$$

where $u(x,t)$ is the Eulerian velocity field, at a location x (in $\mathcal{R}^3$) at time t, r is the magnitude of the vector y, n is a positive integer, $\theta_{\delta u,y}$ is the angle between $u(x+y,t) - u(x,t)$ and y and $\langle .. \rangle$ denotes an ensemble average. In the seminal paper, Kolmogorov [1] used essentially

[0]Permanent address: Mathematics Department, Ohio State University, Columbus, OH 43210, USA

statistical arguments to conclude that for a homogeneous isotropic flow (for which $S_{n,K}$ is only a function of r), as Reynolds number $Re \rightarrow \infty$,

$$S_{n,K}(r) \sim k_n \, (\epsilon \, r)^{\zeta_n}, \quad where \quad \zeta_n = n/3 \tag{2}$$

for $\eta << r << L$ (called the inertial scale), where L is a characteristic energy-producing-length scale and η is a viscous cut-off scale, with $\eta/L \rightarrow 0$ as the Reynolds number $Re \rightarrow \infty$. The ϵ appearing in (2) is the local dissipation rate, assumed to be finite and nonzero in the limit of $Re \rightarrow \infty$. The k_n, appearing in (2) are universal constants, by Kolmogorov's original argument. An expression for the viscous cut-off scale η can be derived from the Navier-Stokes equation by assuming *a priori* that (2) holds for $n = 2$. This gives the viscous cut-off (the so-called Kolmogorov length scale) to be $\eta = \nu^{3/4}/\epsilon^{1/4}$, where ν is the kinematic viscosity. This viscous cut-off length scale, estimated by Kolmogorov, is consistent with rigorous mathematical results [3] on the dimension of the global attractor of the Navier-Stokes dynamics (if there exists such an attractor).

Fifty years after the Kolmogorov seminal paper [1], the scope of validity of (2) remains a matter of controversy [See Frisch [4]] because for $n \neq 3$, there are no proofs or derivation of these results that uses the Navier-Stokes equation. There is some work on the Kolmogorov spectrum [5,6], associated with $n = 2$ structure function, based on modeling of the Navier-Stokes dynamics by assumed vortex structures. The results are, however, not independent of the model parameters. There also has been some rigorous [Constantin [7], Constantin et al [8]] upper bounds related to the enstrophy spectrum in two dimensions and energy spectrum in three dimensions. Fefferman & Constantin [9] also established rigorous inequality relations for different order structure functions $S_n(r)$, defined by

$$S_n(r) = \langle |u(x+y,t) - u(x,t)|^n \rangle \tag{3}$$

where $\langle . \rangle$ in their work involve space-time averaging. Note that these set of structure functions are different from that originally defined by Kolmogorov; however there exists relations between S_n and $S_{n,K}$ for isotropic [1] homogeneous flow [See Monin-Yaglom [2]] and therefore S_n will also satisfy relation (2) to the same degree as $S_{n,K}$

Experimental evidence [10,11] appear to suggest that the Kolmogorov relation needs to be corrected, at least for $n > 4$. This is popularly known as intermittency effect and has occupied the attention of many researchers in recent years. While there exists phemenological theory [12] that predicts the deviation of ζ_n from $n/3$ (for $n \neq 3$) in the relation (2), in

[1] Even without isotropy assumption, $x - t$ integration of equation (39) and a volume integration with respect to y over a sphere of radius r leads to $r^3 \; S_{2,K}(r) = \int_0^r \; \hat{r}^2 \; S_2(\hat{r}) d\hat{r}$ in the inertial regime. This gives the proportionality relation between S_2 and $S_{2,K}$ known before for isotropic flow [2]

good agreement with experiment, the relation of intermittency with Navier-Stokes dynamics remains to be understood. Much of the theoretical work in this direction involves modeling and simplification of the Navier-Stokes dynamics with a view to capturing the essential physics behind intermittency. One might expect that a simplification describing the essential physics should not violate any exact relation satisfied by the Navier-Stokes dynamics. This highlights the importance of exact relations. These are also helpful to the experimentalist by providing them with checks for consistency.

Unfortunately, there are not many exact relations known for the Navier-Stokes dynamics. Aside from inequalities mentioned before, until now, the only exact equality involving structure functions that we are aware of is that $S_{3,K} = -\frac{4}{5}\epsilon r$ as $Re \rightarrow \infty$ for r in the inertial regime. This was obtained by Kolmogorov himself [13] by using the Karman-Howarth equation [14], that was in turn derived from the Navier-Stokes equation for a statistically stationary homogeneous isotropic flow. This has been referred to in the literature as the Kolmogorov's four-fifth law.

In this paper, we present a second exact equality $\tilde{S}_3(r) = -\frac{4}{3}\, \epsilon\, r$ in an inertial regime as Reynolds number $Re \rightarrow \infty$, where $\tilde{S}_3$ is a modified third order structure function. This result is rigorously derived without making *any* assumptions about the flow structure, though the result follows in a rather straight-forward manner from the anisotropic generalization of the Karman-Howarth equation [14], attributed to Monin [see Monin & Yaglom [2], page 402]. However, to the best of our knowledge, the results for $\tilde{S}_3(r)$ do not appear anywhere in the existing literature. We also use this result on $\tilde{S}_3(r)$ to rigorously re-derive the Kolmogorov four-fifth law without the Kolmogorov assumption on flow isotropy.

A direct numerical simulation of Navier-Stokes equations for a highly symmetric periodic flow in a box, originally devised by Kida [15], was carried out upto a Taylor Reynolds number R_λ of about 155. The solid angle averaging over all possible orientation of the displacement vector y present in our definition of the the structure functions (along with space-time-averaging) makes a numerical computation of the structure functions prohibitively expensive. Instead, we computed structure functions without solid angle averaging for two independent orientation of y. We expected to get approximately the same results in both directions, as would be consistent with an isotropic flow. Instead, we found that upto $R_\lambda = 155$, the flow was far from isotropic, even for this highly symmetric flow. Without the solid angle averaging, we found the computed structure functions for $y = \frac{r}{\sqrt{3}}\,(1,\ 1,\ 1)$ displayed the theoretical large Reynolds number inertial regime dependences of $\tilde{S}_3(r)$ and $S_{3,K}(r)$ over some range in r. However, for another orientation of y, namely $y = r\,(1,0,0)$, we find no such scaling regime, at least upto $R_\lambda = 155$.

For our purposes, we define

$$\tilde{S}_3(r) = \lim_{T \to \infty} \frac{1}{T} \int \frac{d\Omega}{4\pi} \int_0^T dt \int \frac{dx}{L^3} \, |u(x+y,t) - u(x,t)|^3 \, cos \, [\theta_{y,\delta u}] \tag{4}$$

Here, L is some characteristic energy producing length scale and integration with respect to Ω refers to solid angle integration over the spherical surface $|y| = r$. Also, we redefine Kolmorov's longitudinal structure functions as

$$S_{n,K}(r) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int \frac{d\Omega}{4\pi} \int \frac{dx}{L^3} \, |u(x+y,t) - u(x,t)|^n \, cos^n \, [\theta_{y,\delta u}] \tag{5}$$

and $S_n(r)$ as

$$S_n(r) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int \frac{d\Omega}{4\pi} \int \frac{dx}{L^3} \, |u(x+y,t) - u(x,t)|^n \tag{6}$$

Note these definitions of structure functions involve space-time averaging, as well as an averaging over all possible orientation of y. Ensemble average appear in Kolmogorov's original work. If the flow is homogeneous, space integration over x is unnecessary. If the flow is isotropic, the integration over solid angle Ω is redundant. If the flow is both homogeneous and isotropic, then the definition above will reduce to the usual ensemble average if it can be assumed that for stationary turbulence, the system goes through all possible states over a long time. However, if the flow is neither homogeneous (say for an infinite geometry) nor known to be isotropic, the definitions of structure functions above are still meaningful and the results quoted in this paper remain valid.

The incompressible forced Navier-Stokes equations determining the velocity field is given by:

$$u_t(x,t) \; + \; u(x,t) \cdot \nabla u(x,t) = -\nabla p(x,t) \; + \; \nu \, \nabla^2 \, u(x,t) \; + \; f(x,t) \tag{7}$$

$$\nabla \cdot u(x,t) = 0 \tag{8}$$

It will be also assumed that the forcing $f(x,t)$ is such that both assumptions (a) and (b) below are valid:

(a) $\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int \frac{dx}{L^3} \, |f(x,t)|^2 \; \equiv \; \langle |f(x,t)|^2 \rangle \; < \; \infty$

(b) $\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int \frac{dx}{L^3} \, |\nabla f(x,t)|^2 \; \equiv \; \langle |\nabla f|^2 \rangle \; < \; \infty$

Note that we have introduced $\langle .. \rangle$ above to denote space-time averaging, rather than the ensemble average of Kolmogorov. We define a characteristic velocity scale U and a length scale L through relations:

$$U = \langle |u(x,t)|^2 \rangle^{1/2} \tag{9}$$

$$L = \frac{\epsilon}{U \, \langle |\nabla f|^2 \rangle^{1/2}} \tag{10}$$

The Reynolds number is defined by $Re \; = \; UL/\nu$.

We will rigorously prove that if smooth solutions $u(x,t)$ to (7) and (8) exist for all times (a physically reasonable assumption that is yet to be proved rigorously), then there exists a cutoff scale η_c such that the following conditions hold:
(i) $\eta_c/L \ \to \ 0$ as $Re \ \to \ \infty$

(ii)

$$\lim_{Re \ \to \ \infty, r/L \ \to \ 0, r/\eta_c \ \to \ \infty} \tilde{S}_3(r) = \ - \ \frac{4}{3} \ \epsilon \ r \tag{11}$$

(iii)

$$\lim_{Re \ \to \ \infty, r/L \ \to \ 0, r/\eta_c \ \to \ \infty} S_{3K}(r) = \ - \ \frac{4}{5} \ \epsilon \ r \tag{12}$$

The latter relation (12) is the Kolmogorov's four-fifth law. The ϵ, appearing in (11) and (12) is the time averaged normalized energy dissipation rate, defined as

$$\epsilon = \lim_{T \ \to \ \infty} \frac{1}{T} \int_0^T dt \int \frac{dx}{L^3} \ \nu \ |\nabla u(x,t)|^2 \tag{13}$$

In the standard notation of asymptotics, the result (11) and (12) can be alternately written as:

$$\tilde{S}_3(r) \ \sim \ - \ \frac{4}{3} \ \epsilon \ r \quad , \quad S_{3K}(r) \ \sim \ - \frac{4}{5} \ \epsilon \ r \tag{14}$$

for $L \ >> \ r \ >> \ \eta_c$. Since, $S_3(r) \ \geq \ |\tilde{S}_3(r)|$, it follows from (11) that

$$\lim_{Re \ \to \ \infty, r/L \ \to \ 0, r/\eta_c \ \to \ \infty} S_3(r) \ \geq \ \frac{4}{3} \ \epsilon \ r \tag{15}$$

Further, as a consequence of (15), it follows from a routine application of Holder inequality, it follows that for any $n \ > \ 3$ and $m \ < \ 3$

$$S_n^{3-m}(r) \ \ S_m^{n-3}(r) \ \geq \ \left(\frac{4}{3}\epsilon r\right)^{n-m} \tag{16}$$

with the inequality understood in the same sense as (15).

2 Derivation of Results for $\tilde{S}_3(r)$

We now proceed to derive our results for $\tilde{S}_3$. We replace argument x by $x + y$ in (7) and subtract (7) from the new equation to obtain

$$[\ \delta \ u]_t + u(x,t) \cdot \nabla[\delta u] = -\nabla \ [\delta \ p] + \ \nu \ \nabla^2 \ [\delta u] + [\delta \ f] - \frac{\partial}{\partial y_j}[\delta u_j \ \delta u] \tag{17}$$

where

$$\delta u = u(x+y,t) \ - \ u(x,t), \quad \delta p \ = \ p(x+y,t) - p(x,t), \quad \delta f = f(x+y,t) - f(x,t) \qquad (18)$$

and the subscript j denote the j-th component of the vector involved. A standard repeated index summation convention has also been used. A similar form of equations for vorticity appears in Constantin [7]. Taking the dot product of (17) with δu and integrating with respect to x over the entire volume (normalized by L^3) we obtain (after using (8) many times):

$$\frac{1}{2}\ \frac{\partial R}{\partial t}(y,t) + \nu \int \ \frac{dx}{L^3}\ \nabla(\delta u_i)\cdot\nabla(\delta u_i) \ = \nabla \ \cdot \ N(y,t) \ + \ F(y,t) \qquad (19)$$

In (19), the scalar functions $R(y,t)$ and $F(y,t)$ are defined as:

$$R(y,t) = \int \frac{dx}{L^3}\ |\delta u|^2 \qquad (20)$$

$$F(y,t) \ = \ \int \frac{dx}{L^3}\ (\delta\ f) \ \cdot \ (\delta\ u) \qquad (21)$$

and the vector function $N(y,t)$ is given by:

$$N(y,t) = \ -\frac{1}{2}\ \int \frac{dx}{L^3}\ \delta u\ |\delta\ u|^2 \qquad (22)$$

We notice that

$$\nu \int \ \frac{dx}{L^3}\ \nabla(\delta u_i)\cdot\nabla(\delta u_i) = 2\ \epsilon_1(t) \ - \ 2\ \nu \ \int \ \frac{dx}{L^3}\ (\nabla u_i(x,t))\cdot(\nabla u_i(x+y,t))$$

$$= 2\ \epsilon_1(t) \ - \ \nu\ \nabla^2\ R \qquad (23)$$

where $\epsilon_1(t)$ is the instantaneous normalized dissipation rate:

$$\epsilon_1(t) = \ \nu \ \int \ \frac{dx}{L^3}\ |\nabla u(x,t)|^2 \qquad (24)$$

Substituting (23) into (24), it is clear that

$$\frac{1}{2}\ \frac{\partial R}{\partial t}(y,t) + 2\epsilon_1(t) \ - \ \nu\ \nabla^2\ R(y,t) \ = \ \nabla \ \cdot \ N(y,t) \ + \ F(y,t) \qquad (25)$$

For isotropic flow, the dependence on y in (25) is only through $|y|$. In that case, (25) becomes the well-known Karman-Howarth equation. The anisotropic generalization in the form (25) appears to have been first derived by Monin [See Monin & Yaglom [2], page 402]. Here, we have re-derived this for the sake of completeness in the present context (where space-time averaging replaces ensemble average of an assumed statistically stationary homogeneous flow). Also, some of the the intermediate steps leading upto (25) are useful in our later derivation of the Kolmogorov four-fifth law, without the Kolmogorov assumption on isotropy.

We replace y by $\tilde{y}$ in (25) and integrate with respect to $\tilde{y}$ over a sphere of radius $r = |y|$, centered at $\tilde{y} = 0$. Using divergence theorem on the volume integrals in (26) and dividing the result by $2\pi r^2$, we obtain:

$$\frac{\partial}{\partial t} \left\{ \frac{1}{4 \pi r^2} \int_{|\tilde{y}| < r} d\tilde{y} \; R(\tilde{y}, t) \right\} + \frac{4}{3} \epsilon_1 r - 2 \nu \frac{d}{dr} T_2(r,t) - \frac{1}{2\pi r^2} \int_{|\tilde{y}| < r} d\tilde{y} F(\tilde{y}, t) = - \tilde{T}_3(r,t) \tag{26}$$

where $T_2(r,t)$ is defined by expression (6) for $n = 2$, but without any time averaging. Similarly, $\tilde{T}_3(r,t)$ is defined by the same expression as for $\tilde{S}_3$ in (4), except that time averaging is not performed. On time-integrating (26) from 0 to T and dividing the resulting expression by T and taking the limit of $T \to \infty$, it follows that

$$\frac{4}{3} \epsilon r - 2 \nu \frac{d}{dr} S_2(r) - \frac{1}{2\pi r^2} \int_{|\tilde{y}| < r} d\tilde{y} \; G(\tilde{y}) = - \tilde{S}_3(r) \tag{27}$$

where

$$\epsilon = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \; \epsilon_1(t) \quad , \quad G(\tilde{y}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \; F(\tilde{y}, t) \tag{28}$$

So far, everything is exact and involves no approximation or assumptions on the nature of the flow or the range of r. For proving rigorous results for $\tilde{S}_3(r)$ in the inertial scale, it is necessary to define a viscous cut-off scale η_c.

$$\eta_c = U \left(\frac{\nu}{\epsilon} \right)^{1/2} \tag{29}$$

From (10) and (29), it follows that

$$\frac{\eta_c}{L} = Re^{-1/2} \left(\frac{U^2 \langle |\nabla f|^2 \rangle^{1/4}}{\epsilon} \right) \tag{30}$$

Note that (7) implies:

$$\epsilon = \langle f(x,t) \cdot u(x,t) \rangle, \tag{31}$$

i.e. the time averaged work done matches time averaged dissipation. Hence the U in (30) can be thought of as determined from ϵ and f. Thus, in the limit $Re \to \infty$, (keeping ϵ and $f(x,t)$ fixed), it is clear from (30), that $\eta_c/L \to 0$.

We now present two propositions that ensures that the the second and third term on the left hand side of (27) is asymptotically negligible compared to $\epsilon \, r$, as $Re \to \infty$, when $L >> r >> \eta_c$.

Proposition 1:

If a smooth solution $u(x,t)$ satisfying (7) and (8) exists for all times, then

$$\lim_{r/L \to 0} \frac{1}{\epsilon r^3} \int_{|\tilde{y}| < r} d\tilde{y} \; G(\tilde{y}, t) = 0 \tag{32}$$

Proof: We note that on using well known triangular equality that

$$|\delta u \ \cdot \ \delta f| \ \leq \ [|u(x+y,t)| + |u(x,t)|]|\delta f|$$

On using $||\delta f|| \ \leq \ ||\nabla \ f|| \ |y|$, where $||..||$ denotes the $\mathcal{L}_2$ norm in x, and Holder's inequality, it follows that

$$|G(\tilde{y})| \ \leq \ 2 \ U \ \langle |\nabla f(x,t)|^2 \rangle^{1/2} |\tilde{y}| \tag{33}$$

It follows that

$$|\frac{1}{2\pi r^2} \int_{|\tilde{y}| \ < \ r} d\tilde{y} \ G(\tilde{y})| \leq \ U \ r^2 \ \langle |\nabla f(x,t)|^2 \rangle^{1/2} \tag{34}$$

On dividing (34) by ϵr and using the definition of L from (10), the statement of proposition 1 follows.

Proposition 2:
If a smooth solution $u(x,t)$ satisfying (5) and (6) exists for all times, then

$$\lim_{r/\eta_c \ \to \ \infty} \ \frac{\nu}{\epsilon r} \ \frac{d}{dr} \ S_2(r) \ = \ 0 \tag{35}$$

Proof: First we note that

$$\nabla_y \int \ dx \ |\delta u|^2 \ = \ \int dx (\delta u_i) \ \nabla u_i(x+y,t) \ = \ \int dx \ (u_i(x,t) - u_i(x-y,t)) \ \nabla u_i(x,t) \tag{36}$$

where the symbol ∇_y is the gradient with respect to the y variable. Therefore, using (36)

$$| \ \frac{y}{r} \cdot \nabla_y \int \frac{dx}{L^3} \left(|\delta u|^2 | \right) | \ \leq \ \int \frac{dx}{L^3} \ |u(x)||\nabla u(x,t)| + \ \int \frac{dx}{L^3} \ |u(x-y)|\nabla u(x,t)| \tag{37}$$

We notice from (6) that $S_2'(r)$ is bounded in absolute value by the time-solid-angle average of the expression on the left of (37), which from Holder's inequality is bounded by

$$\leq \ 2 \ \langle |u(x,t)|^2 \rangle^{1/2} \ \langle |\nabla u(x,t)|^2 \rangle^{1/2}$$

Therefore,

$$\nu |S_2'(r)| \ \leq \ 2 \ U \ \ \nu \ \langle |\nabla u(x,t)|^2 \rangle^{1/2} \ = \ 2 \ U \ \nu^{1/2} \epsilon^{1/2} \tag{38}$$

Proposition 2 follows from (38), if we divide it by $\epsilon \ r$ and use the definition of η_c in (29).

Remark 1: Both limits on r appearing in (32) and (35) can be satisfied if and only if $\eta_c/L \ \to \ 0$, i.e. if and only if $Re \ \to \ \infty$.
Remark 2: $r/\eta_c \ \to \ 0$ is a sufficient condition for the conclusion of proposition 2 to hold. It is not expected to be necessary. Indeed, if the Kolmogorov expression (2) is assumed valid

for $S_2(r)$ and used to evaluate $S_2'(r)$ in (35), then it is clear that (35) would remain equally valid for $r/\eta \;\to\; 0$ for $\eta = \nu^{3/4}\; \epsilon^{-1/4}$.

A direct consequence of (11) is the asymptotic inequality (15). From Holder's inequality the result (16) follows. Because of issues raised in *Remark 2*, we expect the relations (11), (15) and (16) to hold over the entire inertial scale, i.e. $\eta \; << \; r \; << L$, though our rigorous mathematical proof (which did not assume any form of Kolmogorov relation) was for the subrange $\eta_c \; << \; r \; << L$.

3 A derivation of Kolmogorov four-fifth law

We now proceed to use the result (11) for $\tilde{S}_3(r)$ to re-derive the Kolmogorov's four-fifth law, but without the Kolmogorov assumption on flow isotropy. For this purpose, it is convenient to return to (17) and take the dot product with y. This leads to

$$[y\cdot\;\delta\;u]_t + u(x,t)\cdot\nabla[y\cdot\delta u] = -\frac{\partial}{\partial x_i}[y_i\delta\;p] + \;\nu\;\nabla^2\;[y\cdot\delta u] + [y\cdot\delta\;f]$$

$$-\frac{\partial}{\partial y_j}[\delta u_j\;\delta u_i\;y_i] \;+\; [\delta u_j\;\delta u_j\;] \tag{39}$$

On multiplying (39) by $y\,\cdot\,\delta u$, integrating with respect to x over the whole volume, replacing y by $\tilde{y}$ and integrating with respect to $\tilde{y}$ over a sphere of radius $r = |y|$ in a manner similar to that shown explicitly in (17)-(26), we get

$$\frac{\partial}{\partial t}\left[2\pi\int_0^r d\tilde{r}\tilde{r}^4\;T_{2K}(\tilde{r},t)\right] + \nu Q \;=\; P - 2\pi r^4\;T_{3K}(r,t) + 4\pi\int_0^r \tilde{T}_3(\tilde{r},t)\tilde{r}^3 d\tilde{r} \;+\; \int_{|\tilde{y}|<r} d\tilde{y}\hat{F}(\tilde{y},t) \tag{40}$$

where T_{2K} and T_{3K} are given by the expression (5) (for $n = 2$ and $n = 3$), except that integration with respect to t is not performed. In (40), Q, P and $\hat{F}$ are defined by:

$$Q(r,t) = \int_{|\tilde{y}|<r} d\tilde{y}\int\frac{dx}{L^3}\;[\tilde{\delta}u]_{k,j}\;[\tilde{\delta}u]_{l,j}\;\tilde{y}_k\;\tilde{y}_l \tag{41}$$

$$P(r,t) \;=\; \int_{|\tilde{y}|<r} d\tilde{y}\int\frac{dx}{L^3}\;\left([\tilde{\delta}p]\;[\tilde{\delta}u]_{k,i}\;\tilde{y}_i\;\tilde{y}_k\right) \tag{42}$$

$$\hat{F}(\tilde{y},t) \;=\; \int\frac{dx}{L^3}\;[\tilde{\delta}u_k]\;\tilde{y}_k\;[\tilde{\delta}\;f_i]\;\tilde{y}_i \tag{43}$$

where subscript $,j$ refers to differentiation with respect to x_j and $\tilde{\delta}u$ refers to δu with argument y replaced by $\tilde{y}$ in (18) (Similarly for $\tilde{\delta}p$, etc.).

After some manipulation and using

$$\int_{|\tilde{y}|<r} d\tilde{y}\;\tilde{y}_k\;\tilde{y}_l \;=\; \frac{4\pi}{15}\;r^5\;\delta_{k,l}\quad, \tag{44}$$

where $\delta_{k,l}$ stands for the usual Kronecker delta, we find

$$\nu \, Q(r,t) \;=\; \frac{8\pi}{15} \, \epsilon_1(t) \, r^5 - 2 \, \nu \, Q_1(r,t) \tag{45}$$

where

$$Q_1(r,t) \;=\; \int_{|\tilde{y}|<r} d\tilde{y} \, \int \frac{dx}{L^3} \, u_{l,j}(x,t) \, u_{k,j}(x+\tilde{y},t) \, \tilde{y}_l \, \tilde{y}_k \tag{46}$$

On integration by parts with respect to x, and then with respect to $\tilde{y}$, one obtains

$$Q_1(r,t) = -\int \frac{dx}{L^3} \, u_l(x,t) \int d\Omega y_j y_l y_k \; r \; u_{k,j}(x+y,t) + \int \frac{dx}{L^3} u_j(x,t) \int_{|\tilde{y}|<r} d\tilde{y} \; \tilde{y}_k u_{k,j}(x+\tilde{y},t) \tag{47}$$

The latter integral term in (47) is zero from integration by parts with respect to x. Notice also that in the first integral, $y_j \; u_{k,j}(x+y,t) = r \, \frac{\partial}{\partial r} \, u_k(x+y,t)$. Therefore, it follows from (45) and (47) that

$$\nu Q(r,t) \;=\; \frac{8\pi}{15} \, \epsilon_1(t) \, r^5 - 4 \, \pi \, \nu \, r^4 \, T_{2K}'(r,t) \tag{48}$$

where superscript $'$ denotes derivative with respect to r.

We will now show that $P = 0$. For this purpose, we notice that P can be written as

$$P(r,t) = 2\int \frac{dx}{L^3} \int_{|\tilde{y}|<r} d\tilde{y} \; p(x,t) \; u_{k,i}(x,t) \; \tilde{y}_i \tilde{y}_k - 2\int \frac{dx}{L^3} \int_{|\tilde{y}|<r} d\tilde{y} \; p(x,t) \; u_{k,i}(x+\tilde{y},t) \; \tilde{y}_i \tilde{y}_k \tag{49}$$

On using (44), and the divergence condition $u_{k,k} = 0$, it follows that the first integral on the right of (49) is zero. On integrating by parts with respect to $\tilde{y}$ the second integral, we get

$$P(r,t) = -2r^3 \int \frac{dx}{L^3} \int d\Omega \; p(x,t) \; u_k(x+y,t) \; y_k \; + 8\int \frac{dx}{L^3} \int_{|\tilde{y}| \, < \, r} d\tilde{y} \; p(x,t) \; u_k(x+\tilde{y},t) \; \tilde{y}_k \tag{50}$$

The first integral in (50) is clearly zero since the surface integral

$$\int \; d\Omega \; u_k(x+y,t) \; y_k = \frac{1}{r} \int_{|\tilde{y}| \, < \, r} d\tilde{y} \; u_{k,k}(x+\tilde{y},t) = 0 \tag{51}$$

We also notice that the $2 \; \tilde{y}_k = \frac{\partial}{\partial \tilde{y}_k} \; \tilde{r}^2$, where $\tilde{r}^2 = \tilde{y}_j \; \tilde{y}_j$. Using this, the second integral in (50) can be integrated by parts with respect to $\tilde{y}$ and using (51) again, we get this to be zero as well. Thus $P = 0$.

Using the simplifications for Q and P back in (40) and using time integrating this equation from 0 to T, dividing it by T, we obtain in the limit $T \;\to\; \infty$ (after dividing by $2\pi r^4$):

$$\frac{4}{15} \;\; \epsilon \; r \;\; - \;\; \frac{2}{r^4} \int_0^r \tilde{S}_3(\tilde{r}) \; \tilde{r}^3 \; d\tilde{r} \;\; - 2 \; \nu \; S_{2K}'(r) = -S_{3K}(r) + \frac{1}{2\pi r^4} \int_{|\tilde{y}|<r} d\tilde{y} \hat{G}(\tilde{y}) \tag{52}$$

where

$$\hat{G}(\tilde{y}) = \lim_{T \;\to\; \infty} \frac{1}{T} \int_0^T \; dt \;\; \hat{F}(\tilde{y},t) \tag{53}$$

We now claim that in the inertial regime, as identified before, $\nu T_2'(r)$ and $\frac{1}{r^4}\int_{|\tilde{y}|<r} d\tilde{y}\hat{G}(\tilde{y})$ are negligible, compared to $\epsilon\ r$ in (52). We make the following propositions, whose proofs closely parallel those of propositions (1) and (2).

Proposition 3:

If a smooth solution $u(x,t)$ satisfying (5) and (6) exists for all times, then

$$\lim_{r/L\ \to\ 0}\ \frac{1}{\epsilon r^5}\int_{|\tilde{y}|<r} d\ \tilde{y}\ \ \hat{G}(\tilde{y})\ =\ 0 \tag{54}$$

Proof: We note that on using well known triangular equality that

$$|[\tilde{y}\cdot\tilde{\delta}u][\tilde{y}\cdot\tilde{\delta}f]|\ \le\ 2\ |\tilde{y}|^2\ [|u(x+\tilde{y},t)|+|u(x,t)|]|\tilde{\delta}f|$$

On using $||\tilde{\delta}f||\ \le\ ||\nabla\ f||\ |\tilde{y}|$, where $||..||$ denotes the $\mathcal{L}_2$ norm in x, and Holder's inequality, it follows that

$$|\hat{G}(\tilde{y})|\ \le\ 2\langle u^2(x,t)\rangle^{1/2}\ \langle|\nabla f(x,t)|^2\rangle^{1/2}|\tilde{y}|^3 \tag{55}$$

It follows that

$$|\frac{1}{\pi r^4}\int_{|\tilde{y}|\ <\ r} d\tilde{y}\ \hat{G}(\tilde{y})|\le\ \frac{4}{3}\ \langle|\nabla f(x,t)|^2\rangle^{1/2}\ U\ r^2 \tag{56}$$

On dividing (34) by ϵr and using the definition of L from (10), the statement of proposition 3 follows.

Proposition 4:

If a smooth solution $u(x,t)$ satisfying (5) and (6) exists for all times, then

$$\lim_{r/\eta_c\ \to\ \infty}\ \frac{\nu}{\epsilon r}\ \frac{d}{dr}\ S_{2K}(r)\ =\ 0 \tag{57}$$

Proof:

First, we note that $S_{2K}'(r)$ is the time average of $T_{2K}'(r,t)$. From (45) and (48), it clearly follows that

$$T_{2K}'(r,t)=\frac{1}{2\pi r^4}Q_1(r,t) \tag{58}$$

But, from expression (47) for $Q_1(r,t)$, we know only the first integral is nonzero and this, after integration by parts with respect to x leads to

$$T_{2K}'(r,t)=\frac{1}{2\pi r^4}\int\frac{dx}{L^3}\ u_{l,j}(x,t)\int d\Omega\ y_l y_j y_k\ r\ u_k(x+y,t) \tag{59}$$

Noticing that each of y_l, y_j and y_k are bounded by r, on long-time integration of the above equation and using Holder's inequality, it follows that

$$\nu|S_{2K}'(r)|\ \le\ 2\ \nu\ U\ \ \langle|\nabla u|^2\rangle^{1/2}\ =\ 2U\nu^{1/2}\epsilon^{1/2} \tag{60}$$

Proposition 4 follows by dividing the above expression by $\epsilon\ r$, and using the definition of η_c in (29).

Remark 3: As before with $\tilde{S}_3(r)$, Propositions 3 and 4 hold when $\eta_c << r << L$. This defines the inertial scale for the purposes of the proof, though a wider range $\eta << r << L$ is expected, if the Kolmogorov result (2) were to be valid for $n = 2$.

Given Propositions 3 and 4, it implies that in the inertial range, (52) simplifies to

$$\frac{4}{15}\epsilon r - \frac{2}{r^4}\int_0^r \tilde{S}_3(\tilde{r})\tilde{r}^3\ d\tilde{r} \ = \ -S_{3K}(r) \tag{61}$$

where the equality in (61) holds in the same sense as (11). Using the relation (11), proved before for $r/\eta_c >> 1$ and $r/L << 1$, it is not difficult to see that in the inertial range,

$$-\frac{2}{r^4}\int_0^r \tilde{S}_3(\tilde{r})\tilde{r}^3\ d\tilde{r} \ = \ \frac{8}{15}\epsilon r \tag{62}$$

The simplest way to prove this result is to institute a change in variable $s = r/\eta$, $\tilde{s} = \tilde{r}/\eta$ so that the left hand side of the above equation becomes

$$-\frac{2}{s^4}\int_0^s \ d\tilde{s}\ \tilde{S}_3(\tilde{s}\eta)\ \tilde{s}^3\ d\tilde{s} \tag{63}$$

Since $s >> 1$ (from the definition of inertial scale), it is clear from (11), that the leading asymptotic behavior of the integral above for large s is dominated by the contribution near the upper limit. This gives the result (62), where the equality is to be understood in the asymptotic sense as in (11). Using the result (62) in (61), we obtain the Kolmogorov four-fifth law, given by (12). Hence the proof is complete.

4 Numerical Simulation

Since the computation of $\tilde{S}_3$ and $S_{3,K}$ involves one three-dimensional space-integration with respect to x, a two-dimensional solid angle averaging with respect to orientations of y, as well as a one-dimensional time integration, numerical computations for $\tilde{S}_3$ and $S_{3,K}$ for sufficiently large Reynolds number was judged as prohibitively expensive.

Instead, we computed $\tilde{S}_3^c$ and $S_{3,K}^c$ defined as:

$$\tilde{S}_3^c = \frac{1}{T}\int_0^T \ dt \ \int \frac{dx}{L^3}\ |u(x+y,t) - u(x,t)|^3\ cos\ \theta_{\delta u,y} \tag{64}$$

$$S_{3,K}^c = \frac{1}{T}\int_0^T \ dt \ \int \frac{dx}{L^3}\ |u(x+y,t) - u(x,t)|^3\ cos^3\ \theta_{\delta u,y} \tag{65}$$

which are respectively modifications of $\tilde{S}_3$ and $S_{3,K}$ in that no solid angle averaging is involved. We chose two significantly different orientations of the vector y: $y = \frac{r}{\sqrt{3}}\ (1,1,1)$ and

$y = r\,(1,0,0)$. At the outset, we expected that for a highly symmetric flow, the assumptions of isotropy would actually be satisfied over some range of scales at the highest computable Reynolds number. This would make solid angle averaging moot. This would be suggested by the independence of computed $\tilde{S}_3^c/(\epsilon\, r)$ and $S_{3,K}^c/(\epsilon\, r)$ on the orientations of y. However, this did not turn out to be the case. The numerical results are presented here to indicate the degree of anisotropy of the flow and how well the linear scaling of the third order structure functions hold in some regime of r for specific orientations of y.

We solve (7-8) in a 2π-periodic cube with an initial condition of a "*high-symmetry*" as discussed in [18]. In particular, the flow at all times admits the following Fourier expansion for the x_1 component of the velocity at all times:

$$u_1(x_1,x_2,x_3,t) = \left(\sum_{\text{even } l,m,n=0}^{\infty} + \sum_{\text{odd } l,m,n=1}^{\infty} \right) \hat{u}_{1\{l,m,n\}}(t) \sin lx_1 \cos mx_2 \cos nx_3 \qquad (66)$$

The other velocity components are determined by the a permutation symmetry $u_1(x_1,x_2,x_3) = u_2(x_3,x_1,x_2) = u_3(x_2,x_3,x_1)$. The special structure of the Fourier components in (66) and the permutation relationship above saves computational time and memory [18] [19]. In our study, the initial condition and the forcing $f(x,t)$ are chosen to be the same as that in [20] [15]. Specifically,

$$u_1(x_1,x_2,x_3,t=0) = \sin x(\cos 3y \cos z - \cos y \cos 3z). \qquad (67)$$

and the forcing is chosen such that the Fourier mode $\hat{u}_{1\{1,3,1\}} = -\hat{u}_{1\{1,1,3\}} = 1$ all the time in order to imitate a constant energy supply at lower wavenumbers.

The numerical method for solving (7-8) is based on a Fourier pseudo-spectral technique. The details can be found in [21] [19]. To perform the integration in time for $\tilde{S}_3^c$, $S_{3,K}^c$ and ϵ, we use a second order Adams-Bashforth method. For the sake of saving computational time, the time-step for the integration is chosen to be $5\triangle t$ where $\triangle t$ is the time-step for solving the corresponding Navier-Stokes equations (7-8). The spatial integration in x for $\tilde{S}_3^c$, $S_{3,K}^c$ and ϵ are evaluated through summation over $N^3/64$ evenly spaced grid points in the 2π-periodic box, where N is the number of grid points in each direction of the 2π periodic domain. This quadrature is spectrally accurate. For the computational reason, we always choose y in $\tilde{S}_3^c$, $S_{3,K}^c$ on the grid points, or the periodic extension of the grid points.

For $f(x,t) = 0$, we have tested our computational results against those presented in [19]. For the forcing $f(x,t)$ we study in this paper, we have compared our results with those studied in [20] for large ν, for example, $\nu = 0.011$. We have also performed resolution study in N and $\triangle t$ for the computations of $\tilde{S}_3^c$, $S_{3,K}^c$ and ϵ. All computations are performed by using 64 bit arithmetic.

In this study, we are interested in the following quantities,

$$G_1 \equiv -\frac{\tilde{S}_3^c}{r\epsilon} \qquad G_2 \equiv -\frac{S_{3,K}^c}{r\epsilon} \tag{68}$$

as functions of r/η for various r. As shown in the previous sections, as $r/L \to 0$ and $r/\eta_c \to \infty$, $G_1 \to 4/3$ and $G_2 \to 4/5$ for isotropic flows. However, in our numerical computations for finite nonzero ν, η_c/L is a nonzero though small number and G_1 and G_2 generally are functions of T (interval for time-averaging), r as well as the orientation of y for a given Reynolds number.

In Figure 1, we plot averaged normalized averaged energy $\bar{E} = \langle |u(x,t)|^2 \rangle$ and averaged normalized energy dissipation rate ϵ as functions of T for $\nu = 0.001$ and $\nu = 0.000667$. L, the reference length scale is chosen to be 2π for computational purposes, rather than that given by (10). This differing choice makes no difference in computations of G_1 and G_2. Here, we choose $N = 256$ for both cases with $\triangle t = 0.001$ for $\nu = 0.001$ and $\triangle t = 0.0005$ for $\nu = 0.000667$ [20] [19]. It appears that $\bar{E}$ and ϵ start to settle down around $T = 10$ for $\nu = 0.001$ and $T = 9$ for $\nu = 0.000667$, respectively. Based on the equilibrated values of $\bar{E}$ and ϵ, a Taylor's micro-scale Reynolds number R_λ is defined as

$$R_\lambda = \sqrt{\frac{20}{3}} \frac{\bar{E}}{\sqrt{\nu\epsilon}} \tag{69}$$

It is known that for sufficiently large values, this Taylor Reynolds number R_λ scales as the square-root of the Reynolds number Re. The $\nu = 0.001$ calculation reported here corresponds to $R_\lambda = 134$, while for the $\nu = 0.000667$ calculation, $R_\lambda = 155$.

In Figure 2, we plot G_1 and G_2 against r/η for different T for the same cases as shown in Figure 1. Here, we choose $y = \frac{r}{\sqrt{3}}\,(1,1,1)$, where where $r = |y|$ is given. As shown in the graphs, there is a range of r (though not very large) where G_2 is approximately $4/5$, while G_1 is approximately $4/3$. The agreement for G_2 appears to be better. Given the theoretical results, it is not surprising that the range of r over which G_1 and G_2 are approximately $4/3$ and $4/5$ is larger for $R_\lambda = 155$ than that for $R_\lambda = 134$. Indeed, the fit with a constant is also better for the larger R_λ.

Similar results are presented in Figure 3 for $y = \ r\ (1,0,0)$ with r given. We see that there is no significant regime of r/η where G_1 and G_2 are constants. Further, the values are significantly smaller than $4/3$ and $4/5$ respectively, though the values are somewhat larger for $\nu = 0.00067$ than for $\nu = 0.001$. *It is possible that these approach 4/3 and 4/5 respectively as ν becomes even smaller.*

The significant differences between Figures 2 and 3 suggest a lack of isotropy in the flow. While the theoretically predicted quantities involve solid angle averages that cannot be computed with the current power of computers available, the computational results upto $R_\lambda = 155$ suggest that in some directions, one can observe an approximate linear scaling regime for $\tilde{S}_3^c$ and $\tilde{S}_{3,K}^c$ that is consistent with the rigorous large Reynolds number limiting results for $\tilde{S}_3$ and $S_{3,K}$. However, there also exists other directions for which such agreement does not exist, at least upto $R_\lambda = 155$.

5 Discussions

We conclude by noting that the rigorous equality for $\tilde{S}_3$ and $S_{3,K}$ holds for non-isotropic or inhomogeneous flows as well since the definition used here involves a space-time-solid angle averaging. Our computations even for a highly symmetric flow in a periodic box for a Taylor Reynolds number of upto 155 suggest that the assumptions on isotropy are generally not satisfied. Because of prohibitive computational expense, we are unable to assess numerically how the predicted scaling laws in the theoretical large Reynolds number limit hold for for the newly defined third order structure functions (involving space-time-solid angle averaging) at the highest Reynolds number for which computation is feasible. However, by dropping solid angle averaging for computational purposes, we noted approximate scaling regimes for some orientation of the displacement vector y, though not for others.

While the numerical computation so far is not practical for flows that are not isotropic, these relations may prove useful to experimentalists as well theoreticians seeking to model the Navier-Stokes dynamics with simpler equations.

Acknowledgments

S.T. is particularly indebted to Peter Constantin for many helpful discussions and for sharing a preprint on Littlewood Paley spectrum in 2-D turbulence. S.T. also wishes to thank Leo Kadanoff for discussions and providing me the opportunity to be at the University of Chicago. S.T also benefited very much from very helpful discussions with Philip Saffman, Dale Pullin and Paul Dimotakis. S.T. was supported by the National Science Foundation (DMS-9500986), the department of Energy (DE-FG02-92ER25119) and the University of Chicago MRSEC under award number DMR-9400379. Some of the computation are performed at Para//ab of University of Bergen, Norway. We would like to thank Professor Petter Bjorstad for his help on the computing source.

List of Figure Captions

1. Solid curves: averaged normalized energy vs time; Dotted curves: averaged normalized energy dissipation rate vs time. a) $\nu = 0.001$; b) $\nu = 0.000667$.

2. G_1 and G_2 as functions of r/η at different T for $y = \frac{r}{\sqrt{3}}(1, 1, 1)$. Dashed lines: 4/3 or 4/5. a) $\nu = 0.001$ and $T = 10 + i \times 0.5$ for $i = 0, \ldots 18$; b) $\nu = 0.000667$ and $T = 9 + i \times 0.5$ for $i = 0, \ldots 10$. Note convergence for large enough T.

3. G_1 and G_2 as functions of r/η at different T for $y = (r, 0, 0)$. Dashed lines: 4/3 or 4/5. a) $\nu = 0.001$ and $T = 10 + i \times 0.5$ for $i = 0, \ldots 18$; b) $\nu = 0.000667$ and $T = 9 + i \times 0.5$ for $i = 0, \ldots 10$. Note convergence for large enough T.

References

[1] A.N. Kolmogorov, Dokl. Akad. Nauk SSSR 30, 229, 1941.

[2] A.S. Monin & A.M. Yaglom, Statistical Fluid Mechanics (MIT Press, Cambridge, 1975), Vol. 2,

[3] P. Constantin, C. Foias, R. Temam, Memoirs of the American Mathematical Society, No. 314, V 53, 1985. D. Ruelle, Comm. Math. Phys., 87, 287, 1982. E. Lieb, Comm. Math. Phys., 92, 473, 1984.

[4] U. Frisch, Turbulence, Cambridge University Press, 1995.

[5] T.S. Lundgren, Strained Spiral Vortex Model for Turbulent Fine Structure, *Phys. Fluids* 25, 2193 (1982). D.I. Pullin & P.G. Saffman, On the Lundgren-Townsend model of turbulent fine scales, *Phys. Fluids*, A 5, 126-145.

[6] A. Bhattacharjee, C.S. Ng. & X. Wang, Phys. Rev. E. 52, 5110 (1995).

[7] P. Constantin, Littlewood-Paley spectrum for 2-D turbulence, Preprint (1997).

[8] P. Constantin, Q. Nie & S. Tanveer, under preparation (1997).

[9] P. Constantin & Ch. Fefferman, Scaling exponents in fluid turbulence, some analytic results, *Nonlinearity*, 7, 41-57 (1994).

[10] Y. Gagne, Etude experimentale' de l'itermittence et des singularite's dans le plan complexe en turbulence developpe', Thes'e de Docteur es'-Sciences Physiques, Universite' de Grenoble.

[11] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli & S. Succi, Extended self-similarity in turbulent flows, *Phys. Rev. E* 48, R29 (1993).

[12] Z.S. She & E. Leveque, Universal scaling laws in fully developed turbulence, *Phys. Rev. Lett.* 72, 336, (1994),

[13] A.N. Kolmogorov, Dissipation of energy in locally isotropic turbulence, *Dokl. Akad. Nauk SSSR* 32, 16-18 (1941) (reprinted in *Proc. R. Soc. Lond. A* 434, 15 (1991)).

[14] T. von Karman & L. Howarth, On the statistical theory of isotropic turbulence, *Proc. R. Soc. Lond. A* 164, 192 (1938).

[15] S. Kida & K. Ohkitani, Spatio-temporal intermittency and instability of a forced turbulence, *Phys. Fluids A*, 4, 1018 (1992).

[16] K.R. Sreenivasan & P. Kailasnath, Phys. Fluids A 5, 512 (1993), M. Nelkin, Adv. Phys., 43, 143 (1994).

[17] V. L'vov & I. Procaccia, Phys. Rev. E., 53, 3468 (1996)

[18] S. Kida, "Three-dimensional periodic flows with high symmetry", J. Phys. Soc. Jpn., 54, 2132 (1985).

[19] O. N. Boratav and R. B. Pelz, "Direct numerical simulation of transition to turbulence from a high-symmetry initial condition", Phys. Fluids, 6, 2757 (1994).

[20] S. Kida, M. Yamada and K. Ohkitani, "Route to Chaos in a Navier-Stokes Flow", "Lecture Notes in Num. Appl. Anal.", 10, 31 (1989).

[21] O. N. Boratav, R. B. Pelz and N. J. Zabusky, "Reconnection in orthogonally interacting vortex tubes: Direct numerical simulation and quantifications", Phys. Fluids A, 4, 581 (1992).

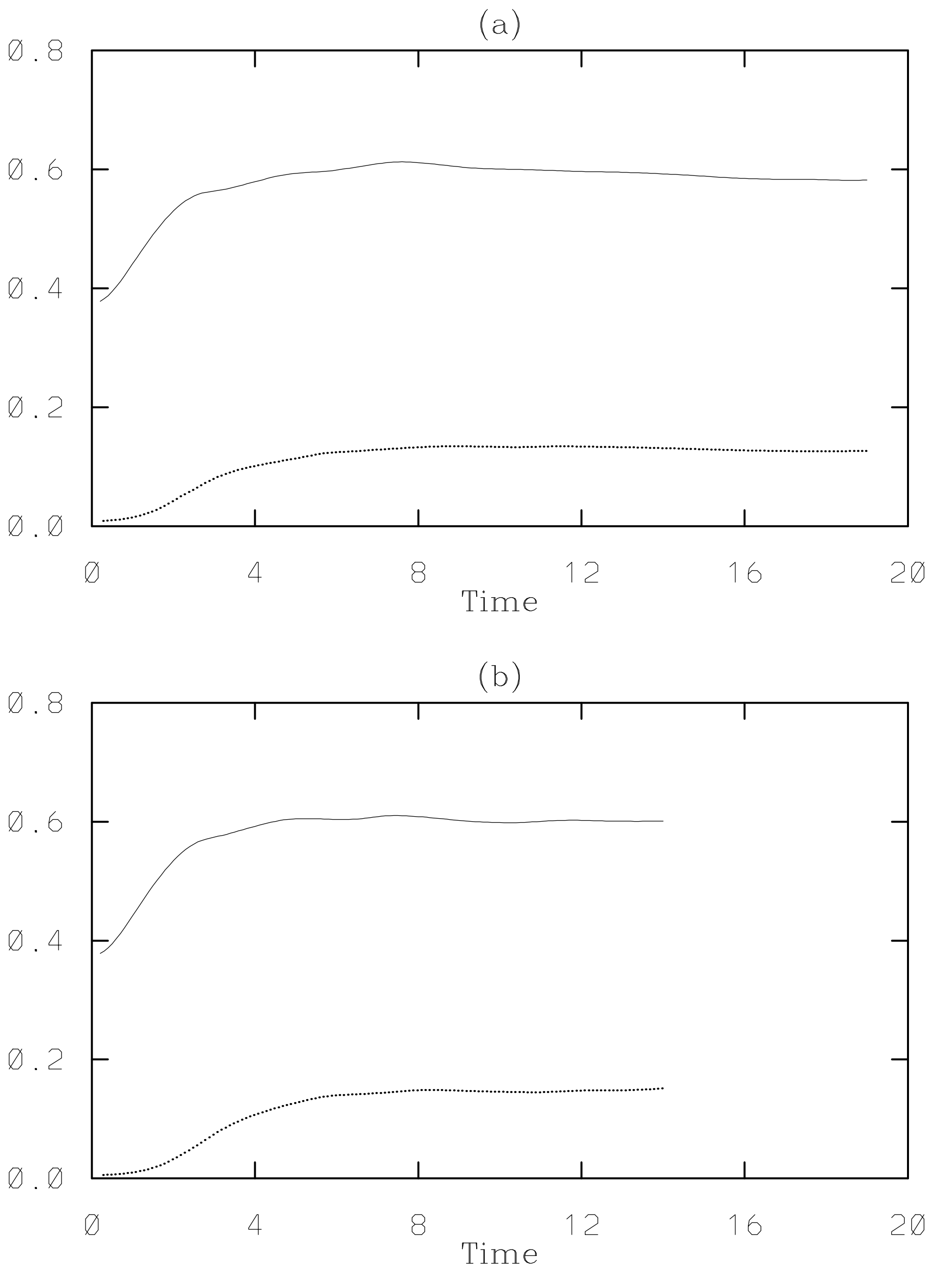


Figure 1:

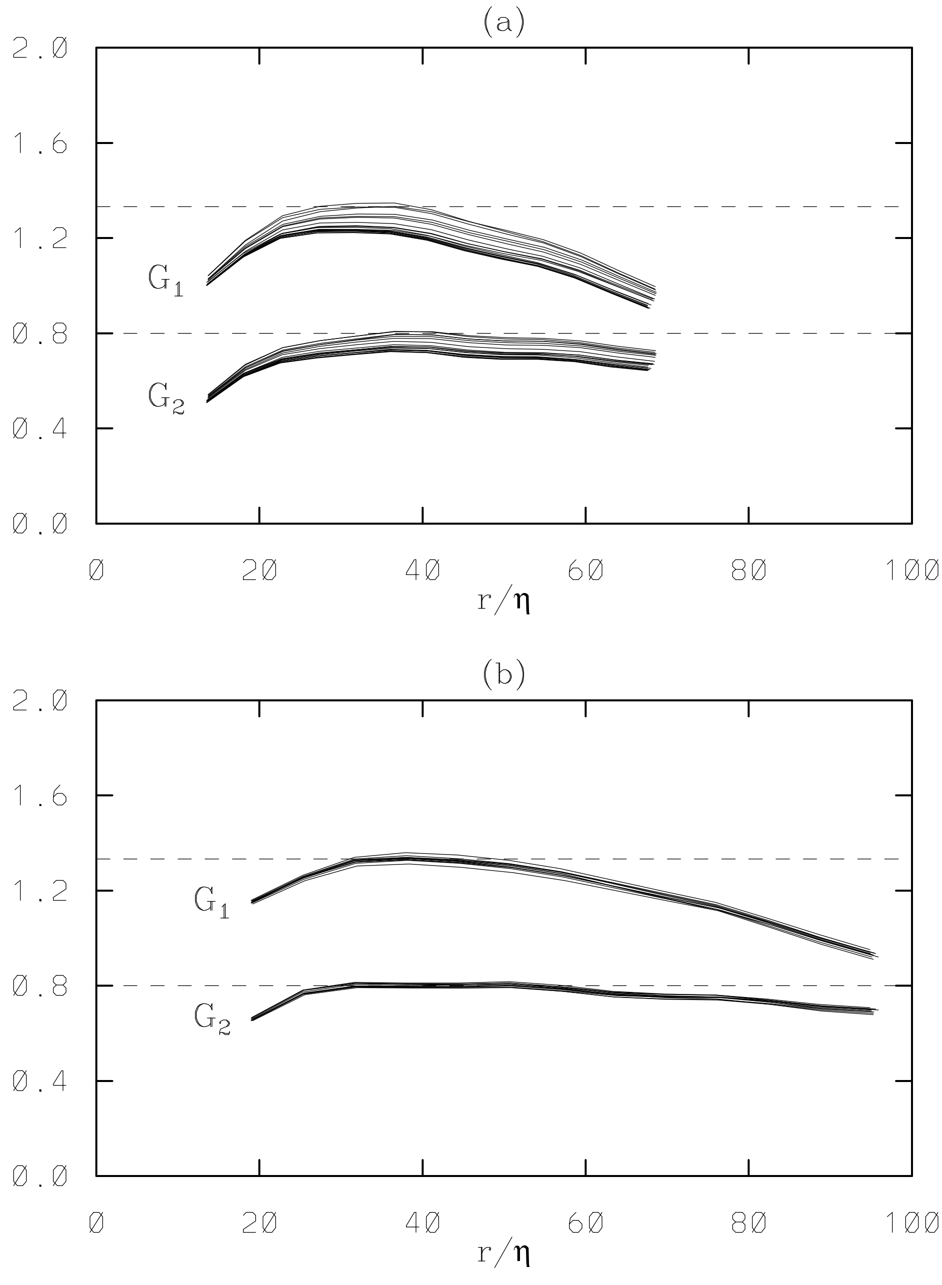
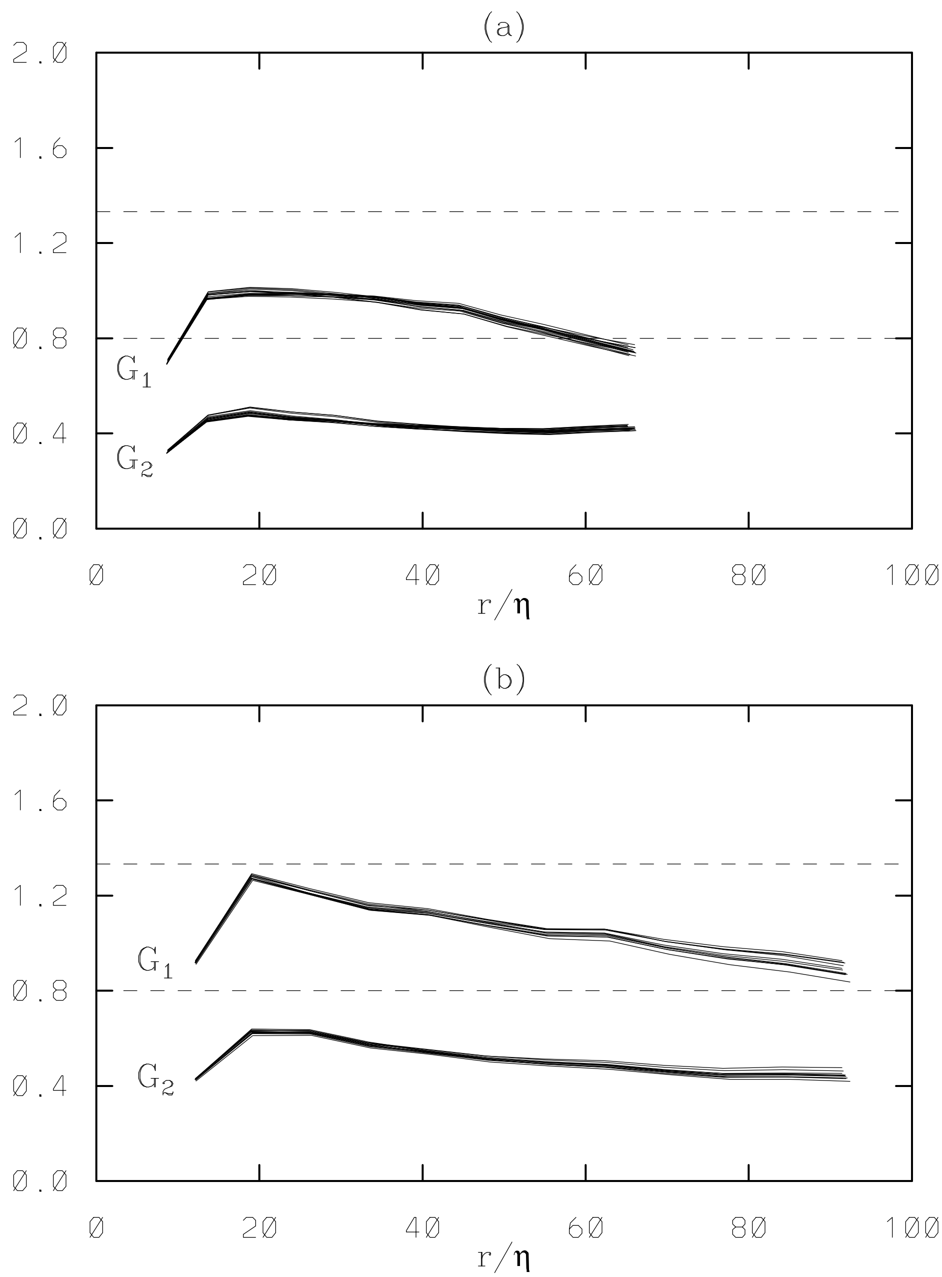

(a)
2.0
1.6
1.2
0.8
0.4
0.0
G1
G2
0
20
40
60
80
100
r/η
(b)
2.0
1.6
1.2
0.8
0.4
0.0
G1
G2
0
20
40
60
80
100
r/η

Figure 2:



(a)
2.0
1.6
1.2
0.8
0.4
0.0
G1
G2
0
20
40
60
80
100
r/η
(b)
2.0
1.6
1.2
0.8
0.4
0.0
G1
G2
0
20
40
60
80
100
r/η

Figure 3: