AN EXAMPLE OF SYMMETRY BREAKING WITH SUBMAXIMAL ISOTROPY SUBGROUP

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In [3] Golubitsky conjectured that generically the solutions to equivariant bifurcation problems have maximal isotropy subgroups as their symmetry groups. A possible physical reason for this conjecture to be true would be that breaking more symmetries requires more energy. Chossat [1] gave a class of counterexamples for representation of $SO(3)$ on $V_{2\ell}$. Especially he proved that $SO(3)$-equivariant bifurcation problems on V_4 have solutions with symmetry group D_4, which is not a maximal isotropy subgroup. In this note, using different methods, we give another class of counterexamples to this conjecture. We use the same notation as Ihrig & Golubitsky [4]. Let V be a real, finite dimensional vector space, Γ be a Lie group acting absolutely irreducibly on V. Let $g:V \times R \to V$ be a map which is equivariant (in its first variable) with respect to the action of Γ. By the absolute irreducibility, there exists a branch $S = \{(0,\lambda) | \lambda \in R\}$ of trivial solutions to the equation

$$g(v,\lambda) = 0. \tag{1}$$

Moreover, the linearization of g along S, denoted by $d_x g(0,\lambda)$ has to be of the form

$$d_x g(0,\lambda) = c(\lambda) I \tag{2}$$

where I denotes the identity of V. Cicogna's equivariant branching lemma [2], see also Vanderbauwhede [5], which is described in Ihrig & Golubitsky [4] assures the existence of a unique branch of nontrivial solutions of (1) with symmetry group Δ if

(i) Δ is a maximal isotropy subgroup and the fixed point space $V^\Delta \subseteq V$ has dimension 1,

(ii) $c(0) = 0$, $c'(0) \neq 0$.

We want to give an example, where topological considerations imply the existence of solutions of (1) with symmetry group $\Sigma \subset \Delta$, when Σ is a submaximal isotropy subgroup. Let us first describe the situation in a general setting. Assume (i), (ii) and

(iii) the normalizer of Σ, $N(\Sigma)$ is finite

(iv) there exist nonlinear Γ-equivariant mappings on V whose restriction to V^{Δ} do not vanish identically.

We consider the restriction g_{Σ} of g to V^{Σ}. Let $i_{\Sigma}(v_0,\lambda_0)$ denote the Brouwer index of a solution (v_0,λ_0) of

$$(1)_{\Sigma} \qquad g_{\Sigma}(v,\lambda) = 0.$$

For fixed $\lambda^- < 0 < \lambda^+$ we want to find information on the sum $d_0(\lambda^{\pm})$ of all indices $i_{\Sigma}(v,\lambda^{\pm})$ for all nontrivial solutions of $(1)_{\Sigma}$ near $v_0 = 0$. Fix λ to be either $\lambda = \lambda^-$ or $\lambda = \lambda^+$, $v_0 \in V^{\Sigma}$. Denote the orbit of v_0 in V by

$$(3) \qquad O(v_0) = \{hv_0 | h \in \Gamma\}$$

and let O_{Σ} be defined by

$$(4) \qquad O_{\Sigma}(v_0) = O(v_0) \cap V^{\Sigma}.$$

Important for our analysis is the set $N(\Sigma,\Delta)$ which was defined by Ihrig & Golubitsky [5], as $N(\Sigma,\Delta) = \{\gamma \in \Gamma | \gamma\Delta\gamma^{-1} \supseteq \Sigma\}$. Observe that the centralizer $C(\Sigma)$ is contained in $N(\Sigma)$, which is finite. Therefore, using lemma 5.5 in Ihrig & Golubitsky [4] we get $\dim N(\Sigma,\Delta) = \dim C(\Sigma) = 0$. Moreover $N(\Sigma,\Delta)$ is compact (lemma 5.5 in [4]), i.e. $N(\Sigma,\Delta)$ is finite. This implies $O_{\Sigma}(v_0)$ is a finite set, if $v_0 \in V^{\Delta}$. Therefore the assumption that there is no bifurcation with nonmaximal isotropy subgroup implies that the solutions of $g_{\Sigma}(v,\lambda) = 0$ are isolated and $d_0(\lambda^{\pm})$ is defined. For fixed $v_1 \in O_{\Sigma}(v_0)$ we write

(5) $$0^N_\Sigma(v_1) = \{\gamma v_1 | \gamma \in N(\Sigma)\}.$$

We observe $0^N_\Sigma(v_1) \subset 0_\Sigma(v_0)$ and more over we can write $0_\Sigma(v_0)$ as a disjoint union

$$0_\Sigma(v_0) = \bigcup_{j=1}^{m(v_0)} 0^N_\Sigma(v_j)$$ for some $v_j \in 0_\Sigma(v_0)$, where m denotes the number of different orbits under the action of $N(\Sigma)$ in $0_\Sigma(v_0)$. For each v_j we find

(6) $$|0^N_\Sigma(v_j)| = |N(\Sigma)/N(\Sigma) \cap \Gamma(v_j)|,$$ where $\Gamma(v_j) = \{h \in \Gamma | hv_j = v_j\}$.

If $v_0 \in V^\Delta$ then $\Gamma(v_j) = \gamma_j^{-1} \Delta \gamma_j$ for $\gamma_j \in \Gamma$ such that $\gamma_j \cdot v_0 = v_j$. Especially $|\Gamma(v_j)| = |\Delta|$. If these are the only solutions of $(1)_\Sigma$ having Δ (or a conjugate of Δ) as their respective symmetry groups, homotopy invariance of degree implies the following identity

(8) $$i_\Sigma(0,\lambda^-) + d_0(\lambda^-) = i_\Sigma(0,\lambda^+) + d_0(\lambda^+).$$

If $\dim V^\Sigma$ is odd we have $|i_\Sigma(0,\lambda^\pm)| = 1$, $i_\Sigma(0,\lambda^+) = -i_\Sigma(0,\lambda^-)$ (by (2), (ii)) and (8) gives

(9) $$2 = |d_0(\lambda^+) - d_0(\lambda)|.$$

If $N(\Delta)$ acts on V^Δ as Z_2, the unique branch obtained by the equivariant branching lemma will constitute a pitchfork, giving either $d_0(\lambda^+) = 0$ or $d_0(\lambda^-) = 0$. W.l.o.g. we assume $d_0(\lambda^-) = 0$. Now we want to discuss one specific example (for notation see Ihrig & Golubitsky [4]). Let $\Gamma = O(3)$, $V = V_5$ the space of spherical harmonics of order 5, $\Sigma = D_2^d$ (with $\dim V^\Sigma = 3$, see [4]), $\Delta = D_6^d$ or D_{10}^d (in both cases $\dim V^\Delta = 1$). By lemma 2.9 in [4] $N(D_2^d) = N_{SO(3)}(D_2) + Z_2^c = O + Z_2$. This implies $|N(\Sigma)| = 48$. In both cases the normalizer of Δ contains $-I$. Therefore the solutions in V^Δ constitute a pitchfork. Since $N(\Sigma) \cap \Gamma(v_j)$ is a subgroup of $N(\Sigma)$ as well as of $\Gamma(v_j)$, $|N(\Sigma) \cap \Gamma(v_j)|$ divides $|N(\Sigma)| = 48$ and $|\Gamma(v_j)| = 12$ or 20. In each case $|N(\Sigma)/N(\Sigma) \cap \Gamma(v_j)|$ can be divided by 4, i.e. we can write

(10) $$|N(\Sigma)/N(\Sigma) \cap \Gamma(v_j)| = 4 \cdot k_j \quad \text{for some } k_j \in \mathbb{N}.$$

Combining (9) and (10) we get

$$2 = \sum_{j=1}^{m(v_o)} 4a_j k_j$$

which yields a contradiction to the assumption that all solutions of $(1)_\Sigma$ have a symmetry group conjugate to Δ. Therefore there exist bifurcating solutions having D_2^d as their symmetry group.

Remark. In this note we did not show that hypothesis (iv) is true. It seems to be obvious, and moreover one can prove the existence of a large number of equivariant mappings which do not vanish on V^Δ.

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