WAVE SCATTERING IN ONE DIMENSION WITH ABSORPTION

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Abstract: Wave scattering is analyzed in a one-dimensional nonconservative medium governed by the generalized Schrödinger equation $d^2\psi/dx^2 + k^2\psi = [ikP(x) + Q(x)]\psi$, where $P(x)$ and $Q(x)$ are real, integrable potentials with finite first moments. Various properties of the scattering solutions are obtained. The corresponding scattering matrix is analyzed, and its small-k and large-k asymptotics are established. The bound states, which correspond to the poles of the transmission coefficient in the upper-half complex plane, are studied in detail. When the medium is not absorptive, it is shown that there may be bound states at complex energies, degenerate bound states, and singularities of the transmission coefficient imbedded in the continuous spectrum. Some explicit examples are provided illustrating the theory.

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1. INTRODUCTION

Wave propagation in a one-dimensional nonconservative medium is described, in the frequency domain, by the generalized Schrödinger equation

$$\psi^{+\prime\prime}(k,x)+k^2\psi^+(k,x)=[ikP(x)+Q(x)]\,\psi^+(k,x),\qquad x\in\mathbf{R}, \tag{1.1}$$

where $\mathbf{R}$ is the real line, the prime denotes the derivative with respect to the spatial coordinate x, k is the wavenumber (also known as the momentum), k^2 is the energy, $P(x)$ describes the combined effect of energy absorption and energy generation, and $Q(x)$ denotes the restoring force density. In the time domain (1.1) corresponds to the wave equation with forcing term

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}-P(x)\frac{\partial u}{\partial t}=Q(x)\,u,\qquad t,x\in\mathbf{R},$$

where the wavespeed is equal to 1. When $P(x)\le 0$, there is net absorption; however, unless otherwise stated we will not put any restriction on the sign of $P(x)$. In the sequel a significant role will be played by the associated equation

$$\psi^{-\prime\prime}(k,x)+k^2\psi^-(k,x)=[-ikP(x)+Q(x)]\,\psi^-(k,x),\qquad x\in\mathbf{R}, \tag{1.2}$$

where the sign of $P(x)$ in (1.1) has been changed.

Let $L^p_q(I)$ denote the space of measurable functions $f(x)$ such that $\int_I dx\,(1+|x|)^q\,|f(x)|^p<+\infty$, and let $L^p(I)=L^p_0(I)$. Throughout the paper we will use $||f||_1$ and $||f||_{1,1}$ to denote the $L^1(\mathbf{R})$ and $L^1_1(\mathbf{R})$ norms, $\int_{-\infty}^{\infty}dx\,|f(x)|$ and $\int_{-\infty}^{\infty}dx\,[1+|x|]\,|f(x)|$, respectively. All the results given in this paper are valid if we assume that P and Q are real valued and belong to $L^1_1(\mathbf{R})$. The conditions $P,Q\in L^1(\mathbf{R})$ are sufficient except when we consider the asymptotics of the scattering coefficients and the wavefunctions as $k\to 0$ or their values at $k=0$; then, in the generic case we assume $P\in L^1(\mathbf{R})$ and $Q\in L^1_1(\mathbf{R})$, and in the exceptional case we assume $P,Q\in L^1_1(\mathbf{R})$. We will see in Section 2 that the exceptional case occurs when the Wronskian in (2.26) vanishes; otherwise, the generic case occurs. We also prove Theorem 9.4 under the sufficient conditions $P,Q\in L^1_1(\mathbf{R})$.

The scattering solutions of (1.1) and (1.2) are those behaving like e^{ikx} or e^{-ikx} as $x\to\pm\infty$, and such solutions occur when $k^2>0$. Among the scattering solutions are the Jost solution from the left $f_l^{\pm}(k,x)$ and the Jost solution from the right $f_r^{\pm}(k,x)$ satisfying the boundary conditions

$$f_l^{\pm}(k,x)=\begin{cases} e^{ikx}+o(1), & x\to+\infty,\\ \dfrac{1}{T^{\pm}(k)}e^{ikx}+\dfrac{L^{\pm}(k)}{T^{\pm}(k)}e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{1.3}$$

$$f_r^{\pm}(k,x)=\begin{cases} \dfrac{1}{T^{\pm}(k)}e^{-ikx}+\dfrac{R^{\pm}(k)}{T^{\pm}(k)}e^{ikx}+o(1), & x\to+\infty,\\ e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{1.4}$$

where $T^{\pm}(k)$ are the transmission coefficients, and $R^{\pm}(k)$ and $L^{\pm}(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix $\mathbf{S}^{+}(k)$ associated with (1.1) is given by

$$\mathbf{S}^{+}(k)=\begin{bmatrix} T^{+}(k) & R^{+}(k) \\ L^{+}(k) & T^{+}(k) \end{bmatrix}.$$

When $P(x)\le 0$, it will be seen that $\mathbf{S}^{+}(k)$ exists for all $k\in\mathbf{R}$; however, when $P(x)\ge 0$ or when $P(x)$ has mixed sign, we will see that $\mathbf{S}^{+}(k)$ may not exist at $k=0$ or at some other real values of k. In a similar manner, we define $\mathbf{S}^{-}(k)$, the scattering matrix associated with (1.2), as

$$\mathbf{S}^{-}(k)=\begin{bmatrix} T^{-}(k) & R^{-}(k) \\ L^{-}(k) & T^{-}(k) \end{bmatrix}.$$

This paper is the first of a series aimed at solving the direct and various inverse scattering problems for (1.1). One of these inverse problems consists of the recovery of $P(x)$ and $Q(x)$ from an appropriate set of scattering data. In the radial case, when there are no bound states, Jaulent and Jean[1] presented an inversion method when $Q(x)$ is real and $P(x)$ is imaginary. They[2,3] also extended their method to solve the full-line one-dimensional inverse problem for real $Q(x)$ and imaginary $P(x)$. In this method, using the scattering data $\{R^{+}(k),R^{-}(k)\}$, a pair of two coupled Marchenko integral equations is solved and these solutions are used in a first-order ordinary differential equation whose solution leads to $P(x)$. Jaulent[4] also extended this method to the case when $P(x)$ is real although complete details and proofs were not given. When $P(x)$ is purely imaginary and $\int_{-\infty}^{\infty} dz\, P(z)=0$, Sattinger and Szmigielski[5] showed that one can simplify the method of Jaulent and Jean and recover $P(x)$ by solving an algebraic equation rather than a differential equation. Assuming that $P(x)$ and $Q(x)$ are in the Schwartz space, when $P(x)$ is real and $\int_{-\infty}^{\infty} dz\, P(z)=0$, Sattinger and Szmigielski[6] also studied the inverse scattering problem for (1.1) by analyzing an associated Riemann-Hilbert problem; however, their primary purpose is to solve the initial value problem for a pair of evolution equations, and some of the assumptions made on the reflection coefficients in Ref. 6, e.g. $R^{\pm}(k)=0$ for $|k|\ge 1$ in our notation, may severely restrict the class of potentials that can be recovered. We should also mention the study by Kaup[7] on the direct and inverse scattering problem for

$$\phi''+\left[k^2+\frac{1}{4\beta^2}\right]\phi=[ikP(x)+Q(x)]\phi,$$

where β is a constant and $P,Q\in L_1^1(\mathbf{R})$. In Refs. 6 and 7 the inverse scattering problem is analyzed by studying a Riemann-Hilbert problem on a particular Riemann surface.

There are other inverse problems for (1.1) such as the recovery of $P(x)$ in terms of appropriate scattering data when $Q(x)$ is known and the recovery of $Q(x)$ in terms of appropriate scattering data when $P(x)$ is known. In future papers we will study various inverse scattering problems for (1.1).

When $P(x)$ is purely imaginary, the methods available for selfadjoint differential operators can be employed to analyze the inverse scattering problem for (1.1); furthermore, in this case[2] the scattering matrices

$\mathbf{S}^{\pm}(k)$ are unitary, and hence the reflection coefficients cannot exceed 1 in absolute value. However, when $P(x)$ is real valued, the differential operator pertaining to (1.1) is no longer selfadjoint and the scattering matrices $\mathbf{S}^{\pm}(k)$ are no longer unitary. Consequently, the analysis of the direct and inverse scattering problems with real $P(x)$ are different and more difficult than in the case with imaginary $P(x)$. For example, the nonselfadjointness of the differential operator may lead to eigenvalues k outside the real and imaginary axes, and this operator may not be diagonalizable on the corresponding generalized eigenvector space. Further, the standard proof[7,9] of the absence of singularities of the transmission coefficient for $k \in \mathbf{R}$, which relies heavily on the selfadjointness of the differential operator, breaks down. Moreover, the reflection coefficients may not be contractive and hence the standard proof of the unique solvability of the Marchenko integral equations is no longer valid. Fortunately, when $P(x) \le 0$, some of the usual properties of the one-dimensional Schrödinger equation given in (2.17), such as the simplicity of the poles of the transmission coefficient, the confinement of these poles to the imaginary axis in the upper-half complex plane $\mathbf{C}^+$, and the absence of singularities of the transmission coefficient for real k values are still valid for (1.1), and the proofs of such properties are obtained by a variation of the arguments used for (2.17).

In the present initial paper we focus on the direct scattering problem for (1.1). Relying on techniques established in a variety of papers,[9–11] we obtain the usual analyticity properties of the Jost solutions, their small-k and large-k asymptotics, and various properties of the scattering coefficients. The bound state solutions of (1.1) and (1.2) are those nontrivial solutions belonging to $L^2(\mathbf{R})$. We will see that the bound states of (1.1) correspond to singularities of $T^+(k)$ in $\mathbf{C}^+$. When $P(x) \ge 0$ or when $P(x)$ has mixed sign, we will see that there may be bound states corresponding to some complex-k values located off the positive imaginary axis in $\mathbf{C}^+$ and that the poles of $T^+(k)$ are not necessarily simple. In the radial case when $P(x) \le 0$, under certain additional conditions on $P'(x)$, using the theory of abstract operator polynomials, Pivovarchik has shown that[12] the number of bound states is independent of $P(x)$ and that[13–15] the bound states can only occur when k is located on the positive imaginary axis and each bound state is simple. In Section 9, we will derive Pivovarchik's results in an elementary way without using the theory of abstract operator polynomials and without assuming the differentiability of $P(x)$.

This paper is organized as follows. In Section 2 we study the analyticity properties of the Jost solutions of (1.1) and analyze their asymptotics as $k \to 0$. In Section 3 we study the large-k asymptotics of the Jost solutions. In Section 4 we obtain various properties of the scattering matrices $\mathbf{S}^{\pm}(k)$ and present some examples showing that $\mathbf{S}^+(k)$ may not exist at certain k values unless $P(x) \le 0$. In Section 5 we study the small-k asymptotics of the scattering coefficients and show that in the exceptional case $\mathbf{S}^+(0)$ is jointly determined by $Q(x)$ and $P(x)$. In Section 6 we study the large-k asymptotics of the scattering coefficients. In Section 7 we study the corresponding change in the scattering coefficients when $P(x)$ and $Q(x)$ are perturbed. In Section 8 we analyze the relation between the poles of $T^+(k)$ in $\mathbf{C}^+$ and the bound states; we also study multiple poles of $T^+(k)$ in terms of Jordan chains of the differential operator given in (8.16). In Section 9 we study the bound states for (1.1) further and show that the poles of $T^+(k)$ in $\mathbf{C}^+$ can only occur in a certain region in $\mathbf{C}^+$ determined by $P(x)$ and $Q(x)$; when $P(x) \le 0$, we show that the number of bound states for (1.1) is independent of $P(x)$ and the bound states can only occur at certain negative energies, and we also obtain some lower and upper bounds for these energies. In this section we also obtain a Levinson theorem relating the number of bound states to the change in the argument of $T^+(k)$, and we show that the number of bound states is unchanged under small perturbations of $P(x)$ and $Q(x)$ in the generic case and is unchanged under small perturbations of $P(x)$ in the exceptional case. In Section 10, we show by examples that there may be bound states with complex energies and that the multiplicity of a bound state may be larger than one unless $P(x) \le 0$. In Section 11 we analyze the zeros of the Jost solutions and obtain various results concerning the number and location of these zeros and their relationship to the bound states; we also show that the number of bound states of (1.1) with real energies is greater than or equal to the number of bound states with $P(x) = 0$. Finally, in the Appendix we obtain various small-k estimates that are needed in the proof of Theorem 5.2.

2. ANALYTICITY AND SMALL-k ASYMPTOTICS OF JOST SOLUTIONS

In this section, under the sufficient conditions $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$, we show that for each $x \in \mathbf{R}$ the Jost solutions and their x-derivatives are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, and we establish their asymptotics as $k \to 0$. By $\overline{\mathbf{C}^+}$ we denote $\mathbf{C}^+ \cup \mathbf{R}$.

The Jost solutions of (1.1) and (1.2) satisfy

$$f_l^\pm(k,x) = e^{ikx} + \frac{1}{k}\int_x^\infty dy \sin k(y-x)\,[\pm ikP(y) + Q(y)]\, f_l^\pm(k,y), \tag{2.1}$$

$$f_r^\pm(k,x) = e^{-ikx} + \frac{1}{k}\int_{-\infty}^x dy \sin k(x-y)\,[\pm ikP(y) + Q(y)]\, f_r^\pm(k,y). \tag{2.2}$$

Let us define the Faddeev functions from the left $m_l^\pm(k,x)$ and from the right $m_r^\pm(k,x)$:

$$m_l^\pm(k,x) = e^{-ikx} f_l^\pm(k,x), \qquad m_r^\pm(k,x) = e^{ikx} f_r^\pm(k,x). \tag{2.3}$$

From (2.1) and (2.3) we obtain

$$m_l^\pm(k,x) = 1 + \frac{1}{2ik}\int_x^\infty dy\, [e^{2ik(y-x)} - 1]\,[\pm ikP(y) + Q(y)]\, m_l^\pm(k,y), \tag{2.4}$$

$$m_l^{\pm\prime}(k,x) = -\int_x^\infty dy\, e^{2ik(y-x)}\,[\pm ikP(y) + Q(y)]\, m_l^\pm(k,y), \tag{2.5}$$

and hence we see that

$$m_l^\pm(k,+\infty) = 1, \qquad m_l^{\pm\prime}(k,+\infty) = 0. \tag{2.6}$$

Similarly, from (2.2) and (2.3) we obtain

$$m_r^{\pm}(k,x) = 1 + \frac{1}{2ik}\int_{-\infty}^{x} dy\,[e^{2ik(x-y)}-1]\,[\pm ikP(y)+Q(y)]\,m_r^{\pm}(k,y), \tag{2.7}$$

$$m_r^{\pm\prime}(k,x) = \int_{-\infty}^{x} dy\,e^{2ik(x-y)}\,[\pm ikP(y)+Q(y)]\,m_r^{\pm}(k,y), \tag{2.8}$$

and hence

$$m_r^{\pm}(k,-\infty)=1, \qquad m_r^{\pm\prime}(k,-\infty)=0. \tag{2.9}$$

The existence of the Jost solutions and the Faddeev functions is clear from (the proof of) the following proposition, where we also establish their continuity and analyticity properties.

Proposition 2.1 Assume $P, Q \in L^1(\mathbf{R})$. Then, for each $x \in \mathbf{R}$, the functions $m_l^{\pm}(k,x)$, $m_r^{\pm}(k,x)$, $m_l^{\pm\prime}(k,x)$, and $m_r^{\pm\prime}(k,x)$ are analytic in $\mathbf{C^+}$ and continuous in $\overline{\mathbf{C^+}}\setminus\{0\}$. Consequently, for each $x \in \mathbf{R}$ the Jost solutions $f_l^{\pm}(k,x)$, $f_r^{\pm}(k,x)$ and their derivatives $f_l^{\pm\prime}(k,x)$, $f_r^{\pm\prime}(k,x)$ are analytic in $\mathbf{C^+}$ and continuous in $\overline{\mathbf{C^+}}\setminus\{0\}$. Moreover, we have

$$|m_l^{\pm}(k,x)| \le C_1 e^{C_2/|k|}, \qquad |m_r^{\pm}(k,x)| \le C_1 e^{C_2/|k|}, \qquad k \in \overline{\mathbf{C^+}}\setminus\{0\}, \tag{2.10}$$

$$|m_l^{\pm\prime}(k,x)| \le C_3(1+|k|)e^{C_2/|k|}, \qquad |m_r^{\pm\prime}(k,x)| \le C_3(1+|k|)e^{C_2/|k|}, \qquad k \in \overline{\mathbf{C^+}}\setminus\{0\}, \tag{2.11}$$

where C_1, C_2, and C_3 are constants independent of x and k.

PROOF: Iterating (2.4), we obtain $|m_l^{\pm}(k,x)| \le e^{\int_x^\infty [|P|+|Q|/|k|]}$, from which we get (2.10) for $m_l^{\pm}(k,x)$ with $C_1 = e^{||P||_1}$ and $C_2 = ||Q||_1$. Similarly, iterating (2.7) we get $|m_r^{\pm}(k,x)| \le e^{\int_{-\infty}^x [|P|+|Q|/|k|]}$, from which we have (2.10) for $m_r^{\pm}(k,x)$. Using these estimates in (2.5) and (2.8), respectively, we obtain

$$|m_l^{\pm\prime}(k,x)| \le e^{\int_x^\infty [|P|+|Q|/|k|]}\int_x^\infty [|k|\,|P|+|Q|], \qquad |m_r^{\pm\prime}(k,x)| \le e^{\int_{-\infty}^x [|P|+|Q|/|k|]}\int_{-\infty}^x [|k|\,|P|+|Q|],$$

from which we get (2.11) with $C_3 = (||P||_1+||Q||_1)e^{||P||_1}$. The analyticity and continuity results are obtained using (2.10) and (2.11). ∎

Theorem 2.2 Assume $P \in L^1(\mathbf{R})$ and $Q \in L_1^1(\mathbf{R})$. Then, for each $x \in \mathbf{R}$, the functions $m_l^{\pm}(k,x)$, $m_r^{\pm}(k,x)$, $m_l^{\pm\prime}(k,x)$, and $m_r^{\pm\prime}(k,x)$ are analytic in $\mathbf{C^+}$ and continuous in $\overline{\mathbf{C^+}}$. Consequently, for each $x \in \mathbf{R}$ the Jost solutions $f_l^{\pm}(k,x)$, $f_r^{\pm}(k,x)$ and their derivatives $f_l^{\pm\prime}(k,x)$, $f_r^{\pm\prime}(k,x)$ are analytic in $\mathbf{C^+}$ and continuous in $\overline{\mathbf{C^+}}$. Moreover, we have

$$|m_l^{\pm}(k,x)| \le C_4[1+\max\{0,-x\}], \qquad |m_r^{\pm}(k,x)| \le C_4[1+\max\{0,x\}], \qquad k \in \overline{\mathbf{C^+}}, \tag{2.12}$$

$$|m_l^{\pm\prime}(k,x)| \le C_5\,[1+|k|]\,[1+\max\{0,-x\}], \quad |m_r^{\pm\prime}(k,x)| \le C_5\,[1+|k|]\,[1+\max\{0,x\}], \qquad k \in \overline{\mathbf{C^+}}, \tag{2.13}$$

where C_4 and C_5 are constants independent of x and k.

PROOF: From (2.4), for $k \in \overline{\mathbf{C}^+}$ and $y \ge x$, using the estimates

$$(2.14) \qquad |1 - e^{2ik(y-x)}| \le 2, \qquad |1 - e^{2ik(y-x)}| \le 2|k|(y-x),$$

we obtain

$$(2.15) \qquad |m_l^+(k,x)| \le 1 + \int_x^\infty dy\,[|P(y)| + (y-x)\,|Q(y)|]\,|m_l^+(k,y)|.$$

From (2.15) we get (2.12) with $C_4 = e^{||P||_1 + ||Q||_{1,1}}$ by using Gronwall's inequality as in the proof of Theorem 2.1 of Ref. 11. Using (2.12) in (2.5) we obtain

$$|m_l^{+\prime}(k,x)| \le C_4 \int_x^\infty dy\,[|k|\,|P(y)| + |Q(y)|]\,[1 + \max\{0, -y\}],$$

and hence we have (2.13) with $C_5 = (||P||_1 + ||Q||_1)e^{||P||_1 + ||Q||_{1,1}}$ for $m_l^{+\prime}(k,x)$. Thus, $m_l^+(k,x)$ and $m_l^{+\prime}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. The proofs of (2.12) and (2.13) for $m_l^-(k,x)$, $m_r^\pm(k,x)$, and their x-derivatives are obtained in a similar way, allowing us to conclude that also these functions are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Then, using (2.3) we can conclude that $f_l^\pm(k,x)$, $f_r^\pm(k,x)$, and their x-derivatives are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$ for each $x \in \mathbf{R}$. ∎

From (2.4)-(2.8), for each $x \in \mathbf{R}$, as $k \to 0$ in $\overline{\mathbf{C}^+}$ we have

$$(2.16) \qquad m_l^\pm(k,x) = m_l^\pm(0,x) + o(1), \qquad m_l^{\pm\prime}(k,x) = m_l^{\pm\prime}(0,x) + o(1),$$

$$m_r^\pm(k,x) = m_r^\pm(0,x) + o(1), \qquad m_r^{\pm\prime}(k,x) = m_r^{\pm\prime}(0,x) + o(1).$$

Let us consider the Schrödinger equation obtained from (1.1) and (1.2) by setting $P(x) = 0$, namely

$$(2.17) \qquad \psi^{[0]\prime\prime}(k,x) + k^2\psi^{[0]}(k,x) = Q(x)\,\psi^{[0]}(k,x), \qquad x \in \mathbf{R}.$$

Let $f_l^{[0]}(k,x)$ and $f_r^{[0]}(k,x)$ denote the Jost solutions of (2.17) from the left and from the right, respectively. We have[8,9]

$$(2.18) \qquad f_l^{[0]}(0,x) = 1 + \int_x^\infty dy\,(y-x)\,Q(y)\,f_l^{[0]}(0,y),$$

$$(2.19) \qquad f_r^{[0]}(0,x) = 1 + \int_{-\infty}^x dy\,(x-y)\,Q(y)\,f_r^{[0]}(0,y),$$

$$(2.20) \qquad f_l^{[0]\prime}(0,x) = -\int_x^\infty dy\,Q(y)\,f_l^{[0]}(0,y),$$

$$(2.21) \qquad f_r^{[0]\prime}(0,x) = \int_{-\infty}^x dy\,Q(y)\,f_r^{[0]}(0,y).$$

From (2.18)-(2.21) it is seen that $f_l^{\pm}(0,x)$, $f_r^{\pm}(0,x)$, and their derivatives are determined by $Q(x)$ alone. In fact, we have

$$m_l^{\pm}(0,x)=f_l^{\pm}(0,x)=f_l^{[0]}(0,x),\qquad m_r^{\pm}(0,x)=f_r^{\pm}(0,x)=f_r^{[0]}(0,x), \tag{2.22}$$

$$m_l^{\pm\prime}(0,x)=f_l^{\pm\prime}(0,x)=f_l^{[0]\prime}(0,x),\qquad m_r^{\pm\prime}(0,x)=f_r^{\pm\prime}(0,x)=f_r^{[0]\prime}(0,x). \tag{2.23}$$

As seen from (2.17) and (2.22) we have

$$Q(x)=\frac{f_l^{\pm\prime\prime}(0,x)}{f_l^{\pm}(0,x)}=\frac{f_r^{\pm\prime\prime}(0,x)}{f_r^{\pm}(0,x)}=\frac{f_l^{[0]\prime\prime}(0,x)}{f_l^{[0]}(0,x)}=\frac{f_r^{[0]\prime\prime}(0,x)}{f_r^{[0]}(0,x)}. \tag{2.24}$$

Let $\mathbf{S}^{[0]}(k)$ denote the scattering matrix associated with (2.17):

$$\mathbf{S}^{[0]}(k)=\begin{bmatrix} T^{[0]}(k) & R^{[0]}(k)\\ L^{[0]}(k) & T^{[0]}(k)\end{bmatrix}, \tag{2.25}$$

where $T^{[0]}(k)$ is the transmission coefficient and $R^{[0]}(k)$ and $L^{[0]}(k)$ are the reflection coefficients from the right and from the left, respectively. Generically $f_l^{\pm}(0,x)$ and $f_r^{\pm}(0,x)$ are linearly independent, but in the so-called exceptional case these two functions are linearly dependent. We have[8,9]

$$[f_l^{[0]}(0,x);f_r^{[0]}(0,x)]=\int_{-\infty}^{\infty}dy\,Q(y)\,f_l^{[0]}(0,y)=\int_{-\infty}^{\infty}dy\,Q(y)\,f_r^{[0]}(0,y)=\lim_{k\to 0}\frac{-2ik}{T^{[0]}(k)}, \tag{2.26}$$

where $[f;g]=fg'-f'g$ denotes the Wronskian. Thus, we are in the generic case if $T^{[0]}(0)=0$ and in the exceptional case if $T^{[0]}(0)\neq 0$. In the exceptional case, let us define

$$\gamma=\frac{f_l^{[0]}(0,x)}{f_r^{[0]}(0,x)}. \tag{2.27}$$

Then γ is a nonzero constant determined by $Q(x)$ alone, and we have $\gamma=f_l^{[0]}(0,-\infty)=1/f_r^{[0]}(0,+\infty)$.

3. LARGE-k ASYMPTOTICS OF JOST SOLUTIONS

In this section we analyze the large-k asymptotics of the Jost solutions. We assume that $P\in L^1(\mathbf{R})$ and $Q\in L^1_1(\mathbf{R})$. The results given here will be used in Section 6 to obtain the large-k asymptotics of the scattering matrix $\mathbf{S}^+(k)$.

In terms of the Jost solutions of (1.1), let

$$\eta_l^{\pm}(k,x)=e^{\pm\zeta}m_l^{\pm}(k,x)=e^{-ikx\pm\zeta}f_l^{\pm}(k,x),\quad \eta_r^{\pm}(k,x)=e^{\pm p\mp\zeta}m_r^{\pm}(k,x)=e^{ikx\pm p\mp\zeta}f_r^{\pm}(k,x), \tag{3.1}$$

where we have defined

$$\zeta=\zeta(x)=\frac{1}{2}\int_x^{\infty}dz\,P(z),\qquad p=\frac{1}{2}\int_{-\infty}^{\infty}dz\,P(z), \tag{3.2}$$

so that $\int_{-\infty}^{x} dz\, P(z)/2 = p - \zeta$. Thus we have

$$f_l^{\pm}(k,x) = e^{ikx\mp\zeta}\eta_l^{\pm}(k,x), \qquad f_l^{\pm\prime}(k,x) = e^{ikx\mp\zeta}\left[(ik \pm P/2)\,\eta_l^{\pm}(k,x) + \eta_l^{\pm\prime}(k,x)\right], \tag{3.3}$$

$$f_r^{\pm}(k,x) = e^{-ikx\mp p\pm\zeta}\eta_r^{\pm}(k,x), \qquad f_r^{\pm\prime}(k,x) = e^{-ikx\mp p\pm\zeta}\left[(-ik \mp P/2)\,\eta_r^{\pm}(k,x) + \eta_r^{\pm\prime}(k,x)\right]. \tag{3.4}$$

Theorem 3.1 Assume $P, Q \in L^1(\mathbf{R})$. For each $x \in \mathbf{R}$, the functions $\eta_l^{\pm}(k,x)$ and $\eta_r^{\pm}(k,x)$ are analytic in $\mathbf{C^+}$, continuous in $\overline{\mathbf{C^+}} \setminus \{0\}$, and we have

$$|\eta_l^{\pm}(k,x)| \le Ce^{C/|k|}, \qquad |\eta_r^{\pm}(k,x)| \le Ce^{C/|k|}, \qquad k \in \overline{\mathbf{C^+}} \setminus \{0\}, \tag{3.5}$$

$$\eta_l^{\pm}(k,x) = 1 + o(1), \quad \eta_r^{\pm}(k,x) = 1 + o(1), \qquad k \to \infty \text{ in } \overline{\mathbf{C^+}}, \tag{3.6}$$

where C is a constant independent of x and k.

PROOF: The analyticity in $\mathbf{C^+}$, the continuity in $\overline{\mathbf{C^+}} \setminus \{0\}$, and (3.5) follow from (3.1) and Proposition 2.1. Thus, we only need to prove (3.6). Using (3.1) in (2.4), we obtain

$$\eta_l^{+}(k,x) = e^{\zeta(x)} + \frac{1}{2ik}\int_x^{\infty} dy\, [e^{2ik(y-x)} - 1]\, e^{\zeta(x)-\zeta(y)}[ikP(y) + Q(y)]\,\eta_l^{+}(k,y), \tag{3.7}$$

where $\zeta(x)$ is the quantity in (3.2). Letting $z(k,x) = \eta_l^{+}(k,x) - 1$, after some simplification, we can write (3.7) in the form

$$z(k,x) = z_0(k,x) + \frac{1}{2ik}\int_x^{\infty} dy\, [e^{2ik(y-x)} - 1]\, e^{\zeta(x)-\zeta(y)}[ikP(y) + Q(y)]\, z(k,y), \tag{3.8}$$

where we have defined

$$z_0(k,x) = \frac{1}{2ik}\int_x^{\infty} dy\, e^{2ik(y-x)}e^{\zeta(x)-\zeta(y)}[ikP(y) + Q(y)] - \frac{1}{2ik}\int_x^{\infty} dy\, e^{\zeta(x)-\zeta(y)}Q(y). \tag{3.9}$$

Using (2.14) in (3.9) we get

$$|z_0(k,x)| \le C\int_x^{\infty} dy\, [|P(y)| + |Q(y)|/|k|], \qquad k \in \overline{\mathbf{C^+}} \setminus \{0\}. \tag{3.10}$$

Iterating (3.8) with the help of (3.10), we obtain

$$|z(k,x)| \le C\left[\int_x^{\infty} dt\, [|P(t)| + |Q(t)|/|k|]\right] e^{C\int_x^{\infty} dy\, [|P(y)|+|Q(y)|/|k|]}. \tag{3.11}$$

Applying the Riemann-Lebesgue lemma to (3.9), we obtain $z_0(k,x) = o(1)$ as $k \to \pm\infty$. Using (3.10) and (3.11) we can conclude that $\eta_l^{+}(k,x)$ is uniformly bounded in $\overline{\mathbf{C^+}}$ for $|k| \ge a > 0$ for each $x \in \mathbf{R}$ and $a > 0$. Hence, by a Phragmen-Lindelöf theorem[16] we obtain (3.6) for $\eta_l^{+}(k,x)$. The proof of (3.6) for $\eta_l^{-}(k,x)$ and $\eta_r^{\pm}(k,x)$ is obtained in a similar manner. ∎

Theorem 3.2 Assume $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$. Then, for each $x \in \mathbf{R}$, the functions $\eta_l^{\pm}(k,x)$ and $\eta_r^{\pm}(k,x)$ are continuous in $\overline{\mathbf{C}^+}$, and we have

$$|\eta_l^{\pm}(k,x)| \le C[1+\max\{0,-x\}], \qquad |\eta_r^{\pm}(k,x)| \le C[1+\max\{0,x\}], \qquad k \in \overline{\mathbf{C}^+},$$

where C is a constant independent of x and k.

PROOF: The results are obtained from (3.1) and Theorem 2.2. ∎

Next, without making any additional assumptions on $P(x)$, we obtain the large-k asymptotics of the Faddeev functions. Let us define

$$(3.12) \qquad \xi_l^{\pm}(k,x) = \frac{1}{2ik}[\pm P(x)\,\eta_l^{\pm}(k,x) + 2\eta_l^{\pm\prime}(k,x)], \qquad \xi_r^{\pm}(k,x) = \frac{1}{2ik}[\mp P(x)\,\eta_r^{\pm}(k,x) + 2\eta_r^{\pm\prime}(k,x)].$$

Theorem 3.3 Assume $P, Q \in L^1(\mathbf{R})$. For each $x \in \mathbf{R}$, the functions $\xi_l^{\pm}(k,x)$ and $\xi_r^{\pm}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$, and they satisfy

$$(3.13) \qquad \xi_l^{\pm}(k,x) = o(1), \quad \xi_r^{\pm}(k,x) = o(1), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$$

PROOF: From (3.1) and (3.12) we have

$$(3.14) \qquad \xi_l^{\pm}(k,x) = \frac{1}{ik}\, m_l^{\pm\prime}(k,x)\, e^{\pm\zeta} = \frac{1}{ik}\,[f_l^{\pm\prime}(k,x) - ikf_l^{\pm}(k,x)]\, e^{-ikx\pm\zeta},$$

$$(3.15) \qquad \xi_r^{\pm}(k,x) = \frac{1}{ik}\, m_r^{\pm\prime}(k,x)\, e^{\pm p\mp\zeta} = \frac{1}{ik}\,[f_r^{\pm\prime}(k,x) + ikf_r^{\pm}(k,x)]\, e^{ikx\pm p\mp\zeta}.$$

Using (3.1), (3.14), (3.15), and Proposition 2.1, we have the continuity of $\xi_l^{\pm}(k,x)$ and $\xi_r^{\pm}(k,x)$ in k for $\overline{\mathbf{C}^+} \setminus \{0\}$ and their analyticity in k for $\mathbf{C}^+$. Thus, we only need to prove (3.13). From (2.5), (3.1), and (3.12) we get

$$(3.16) \qquad \xi_l^{+}(k,x) = -\int_x^{\infty} dy\,[P(y) - iQ(y)/k]\, e^{2ik(y-x)} e^{\zeta(x)-\zeta(y)}\, \eta_l^{+}(k,y).$$

Thus, using Theorem 3.1, we see that the right-hand side of (3.16) is uniformly bounded in $\overline{\mathbf{C}^+}$ when $|k| \ge a > 0$ for each $x \in \mathbf{R}$ and $a > 0$. Using a variant of the Riemann-Lebesgue lemma, we conclude that the right-hand side of (3.16) is of $o(1)$ as $k \to \pm\infty$. Hence, by a Phragmen-Lindelöf theorem,[16] we can conclude that the left-hand side of (3.16) is of $o(1)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, which implies (3.13). We obtain (3.13) for $\xi_l^{-}(k,x)$ and $\xi_r^{\pm}(k,x)$ in a similar manner. ∎

Using (3.13)-(3.15) we have the following:

Corollary 3.4 Assume $P, Q \in L^1(\mathbf{R})$. For each $x \in \mathbf{R}$, we have [cf. (2.11)]

$$m_l^{\pm\prime}(k,x) = o(k), \quad m_r^{\pm\prime}(k,x) = o(k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$$

Let us also mention that it is possible to study the large k-behavior of the solutions of (1.1) by converting it into a system of two coupled, first-order differential equations. We will not give the details here but refer the interested reader to Ref. 17.

4. SCATTERING COEFFICIENTS

In this section, under the assumptions $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$, we analyze various properties of the scattering coefficients. We show by examples that $\mathbf{S}^+(k)$ may not exist for some real k and $T^+(k)$ may have poles off the positive imaginary axis. In general, since $P(x)$ is assumed to be real, $\mathbf{S}^+(k)$ is not unitary; we obtain some relations when $P(x) \le 0$ and $P(x) \ge 0$, which reduce to the unitarity relations when $P(x) = 0$.

When $k \in \mathbf{R}$, the quantities $f_l^\mp(-k,x)$ and $f_r^\mp(-k,x)$ are also solutions of (1.1) and (1.2), respectively, and hence, they can be expressed as linear combinations of $f_l^\pm(k,x)$ and $f_r^\pm(k,x)$, unless the latter functions are linearly dependent. Using (1.3) and (1.4) we obtain

$$\begin{bmatrix} f_l^\mp(-k,x) \\ f_r^\mp(-k,x) \end{bmatrix} = \begin{bmatrix} T^\pm(k) & -R^\pm(k) \\ -L^\pm(k) & T^\pm(k) \end{bmatrix} \begin{bmatrix} f_r^\pm(k,x) \\ f_l^\pm(k,x) \end{bmatrix}, \qquad k \in \mathbf{R}. \tag{4.1}$$

In general, $f_l^\mp(-k,x)$ and $f_r^\mp(-k,x)$ cannot be continued to $\mathbf{C}^+$ in k because $f_l^\pm(k,x)$ and $f_r^\pm(k,x)$ usually cannot be extended to the lower-half complex plane $\mathbf{C}^-$ in k. It is already known[8] that $\mathbf{S}^{[0]}(k)$ defined in (2.25) exists for $k \in \mathbf{R} \setminus \{0\}$ if $Q \in L^1(\mathbf{R})$, and under the additional assumption $Q \in L^1_1(\mathbf{R})$ it is guaranteed that $\mathbf{S}^{[0]}(k)$ exists also at $k = 0$. We will show through examples that $\mathbf{S}^+(k)$ may not exist at $k = 0$ or at some other real values of k when $P(x) \ge 0$ or when $P(x)$ has mixed sign. We will see that when $P, Q \in L^1(\mathbf{R})$ and $P(x) \le 0$, the scattering matrix $\mathbf{S}^+(k)$ exists for all $k \in \mathbf{R} \setminus \{0\}$, and we will study $\mathbf{S}^+(k)$ at $k = 0$ in Section 5. Thus, (4.1) may not hold at certain real k unless $P(x) \le 0$.

The transformation $k \mapsto -\overline{k}$ is a reflection with respect to the imaginary axis, where the overline denotes complex conjugation. Under this transformation we have $ik \mapsto i\overline{k}$ and

$$f_l^\pm(-\overline{k},x) = \overline{f_l^\pm(k,x)}, \qquad f_r^\pm(-\overline{k},x) = \overline{f_r^\pm(k,x)}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.2}$$

Hence, for real k, we get

$$f_l^\pm(-k,x) = \overline{f_l^\pm(k,x)}, \qquad f_r^\pm(-k,x) = \overline{f_r^\pm(k,x)}, \qquad k \in \mathbf{R}. \tag{4.3}$$

Using (1.3) and (1.4) we obtain the Wronskian relation

$$[f_l^\pm(k,x); f_r^\pm(k,x)] = -\frac{2ik}{T^\pm(k)}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.4}$$

From (4.2) and (4.4) we see that

$$\frac{1}{T^\pm(-\overline{k})} = \frac{1}{\overline{T^\pm(k)}}, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}. \tag{4.5}$$

It follows from (4.5) that the zeros of $1/T^{\pm}(k)$ either lie on or are symmetrically located with respect to the imaginary axis in $\overline{\mathbf{C}^+}$; in particular, if $1/T^{+}(k_0)=0$ for some $k_0\in\mathbf{R}\setminus\{0\}$, we must also have $1/T^{+}(-k_0)=0$.

Proposition 4.1 Assume $P,Q\in L^1(\mathbf{R})$. Then, $1/T^{\pm}(k)]$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. Consequently, $T^{\pm}(k)$ cannot have any zeros in $\overline{\mathbf{C}^+}\setminus\{0\}$.

PROOF: This follows from (4.4) and Theorem 2.2. ∎

From Proposition 4.1, we obtain the following:

Corollary 4.2 If $P,Q\in L^1(\mathbf{R})$, then the zeros of $k/[(k+i)\,T^{\pm}(k)]$ in $\mathbf{C}^+$ are all isolated, and their only accumulation points, if there are any, must lie on the real axis or at infinity.

We will see in Proposition 6.2 that the zeros of $k/[(k+i)\,T^{\pm}(k)]$ in $\mathbf{C}^+$ cannot accumulate at infinity.

Using (1.3), (1.4), (2.3), (2.4), and (2.7), we obtain

$$(4.6)\quad \frac{1}{T^{\pm}(k)}=1-\frac{1}{2ik}\int_{-\infty}^{\infty}dy\,[\pm ik\,P(y)+Q(y)]\;m_l^{\pm}(k,y)=1-\frac{1}{2ik}\int_{-\infty}^{\infty}dy\,[\pm ik\,P(y)+Q(y)]\;m_r^{\pm}(k,y),$$

$$(4.7)\qquad \frac{L^{\pm}(k)}{T^{\pm}(k)}=\frac{1}{2ik}\int_{-\infty}^{\infty}dy\,e^{2iky}\,[\pm ik\,P(y)+Q(y)]\;m_l^{\pm}(k,y),$$

$$(4.8)\qquad \frac{R^{\pm}(k)}{T^{\pm}(k)}=\frac{1}{2ik}\int_{-\infty}^{\infty}dy\,e^{-2iky}\,[\pm ik\,P(y)+Q(y)]\;m_r^{\pm}(k,y).$$

Proposition 4.3 If $P,Q\in L^1(\mathbf{R})$, then $1/T^{\pm}(k)$ are bounded in the sector $\{k\in\overline{\mathbf{C}^+}:\ |k|\ge a>0\}$ for every $a>0$. If, in addition, $Q\in L^1_1(\mathbf{R})$, then $k/[(k+i)\,T^{\pm}(k)]$ are continuous and bounded in $\overline{\mathbf{C}^+}$, and $kL^{\pm}(k)/T^{\pm}(k)$ and $kR^{\pm}(k)/T^{\pm}(k)$ are continuous and bounded in $\mathbf{R}$.

PROOF: The proof is obtained by using (2.10) and (2.12) in (4.6)-(4.8). ∎

Using (1.3) and (1.4), for $k\in\mathbf{R}$ we obtain the Wronskian relations

$$(4.9)\qquad [f_l^{\pm}(k,x);f_r^{\mp}(-k,x)]=-\frac{2ik\,L^{\pm}(k)}{T^{\pm}(k)}=\frac{2ik\,R^{\mp}(-k)}{T^{\mp}(-k)},$$

$$(4.10)\qquad [f_r^{\pm}(k,x);f_l^{\mp}(-k,x)]=-\frac{2ik\,R^{\pm}(k)}{T^{\pm}(k)}=\frac{2ik\,L^{\mp}(-k)}{T^{\mp}(-k)},$$

$$(4.11)\qquad [f_l^{\pm}(k,x);f_l^{\mp}(-k,x)]=-2ik=-2ik\frac{1-L^{\pm}(k)\,L^{\mp}(-k)}{T^{\pm}(k)\,T^{\mp}(-k)},$$

$$(4.12)\qquad [f_r^{\pm}(k,x);f_r^{\mp}(-k,x)]=2ik=2ik\frac{1-R^{\pm}(k)\,R^{\mp}(-k)}{T^{\pm}(k)\,T^{\mp}(-k)}.$$

The next example shows that if we relax our conditions on $P(x)$ and $Q(x)$, the scattering matrix $\mathbf{S}^+(k)$ may not exist at all.

Example 4.4 Let $P(x)$ and $Q(x)$ have support in $(0,+\infty)$ and be given by

$$Q(x)=\theta(x)\frac{2}{(1+x)^2},\qquad P(x)=\theta(x)\frac{2}{1+x}, \tag{4.13}$$

where $\theta(x)$ is the Heaviside function. Note that $P\notin L^1(\mathbf{R})$. Two linearly independent solutions of (1.1) are given by

$$\psi_1^+(k,x)=\theta(x)\frac{e^{-ikx}}{1+x}+\theta(-x)\left[e^{-ikx}-\frac{\sin kx}{k}\right],$$

$$\psi_2^+(k,x)=\theta(x)\left[x+1+\frac{i}{k}-\frac{1}{2k^2(1+x)}\right]e^{ikx}+\theta(-x)\left[\left(\frac{1}{k}+\frac{1}{2k^3}\right)\sin kx+\left(1+\frac{i}{k}-\frac{1}{2k^2}\right)e^{ikx}\right].$$

Note that $\psi_1^+(k,x)\to 0$ and $\psi_2^+(k,x)=O(x)$ as $x\to+\infty$; hence, we cannot form a solution asymptotic to e^{ikx} as $x\to+\infty$. Although we can form a linear combination of $\psi_1^+(k,x)$ and $\psi_2^+(k,x)$ that is asymptotic to e^{-ikx} as $x\to-\infty$, the resulting function is not bounded as $x\to+\infty$. Since $\psi_1^+(-k,x)$ and $\psi_2^+(-k,x)$ constitute two linearly independent solutions of (1.2), neither (1.1) nor (1.2) have any scattering solutions. Thus, $\mathbf{S}^\pm(k)$ do not exist for the potentials given in (4.13).

Note that contrary to the case where $P(x)$ is either zero or purely imaginary, we cannot rule out any singularities of the scattering coefficients $T^\pm(k)$, $R^\pm(k)$, and $L^\pm(k)$ on the real axis a priori. The next example shows that $1/T^+(k)$ may have zeros on the real axis and that $1/T^+(k)$ may have zeros off the imaginary axis in $\mathbf{C}^+$. The numerical results were obtained by using the mathematical software Mathematica.

Example 4.5 For real parameters $a_\pm$ and $b_\pm$, let

$$P(x)=\begin{cases} b_-, & x\in(-1,0),\\ b_+, & x\in(0,1),\\ 0, & \text{elsewhere},\end{cases}\qquad Q(x)=\begin{cases} a_-, & x\in(-1,0),\\ a_+, & x\in(0,1),\\ 0, & \text{elsewhere}.\end{cases}$$

The resulting transmission coefficient can be obtained by using (4.5) of Ref. 18 and the fact that when we shift the potentials in (1.1) as $P(x)\mapsto P(x-c)$ and $Q(x)\mapsto Q(x-c)$ for some c, the resulting transmission coefficient is unchanged and the resulting reflection coefficient from the right becomes $e^{2ikc}R^+(k)$ and the resulting reflection coefficient from the left becomes $e^{-2ikc}L^+(k)$. We have

$$\frac{e^{2ik}}{T^+(k)}=e^{4ik}\left(\cos s_-+\frac{k^2+s_-^2}{2iks_-}\sin s_-\right)\left(\cos s_++\frac{k^2+s_+^2}{2iks_+}\sin s_+\right)-\left(\frac{k^2-s_-^2}{2iks_-}\sin s_-\right)\left(\frac{k^2-s_+^2}{2iks_+}\sin s_+\right),$$

where we have defined $s_\pm=\sqrt{k^2-ib_\pm k-a_\pm}$. Let us use an overline on the last digit to indicate a roundoff. When $a_-=-2.850\overline{0}$, $a_+=0$, $b_-=2.000\overline{2}$, and $b_+=1.797\overline{9}$, we find that $T^+(k)$ has a simple pole on the real axis at $k=\pm1.719\overline{4}$. For example, when $a_+=0$ and $b_+=-\sqrt{3/2}$, in order to get a zero of $1/T^+(k)$ at $k=\pm1/\sqrt{2}$, we need to let $a_-=-8.996\overline{2}$ and $b_-=3.222\overline{9}$. By using $a_-=8.630\overline{5}$, $a_+=0$, $b_-=1$, $b_+=14/5$, we obtain a zero of $1/T^+(k)$ at $k=\pm6.940\overline{5}+i$; for the same $a_\pm$ and $b_\pm$ values we obtain another zero at $k=\pm27.613\overline{5}+2.098\overline{1}i$.

In Example 4.5 we have seen that $1/T^+(k)$ may be zero on $\mathbf{R}\setminus\{0\}$. Next we give a simple example where $1/T^+(0)$ may be equal to zero.

Example 4.6 Let $Q(x)=0$ and

$$P(x)=\begin{cases} b, & x\in(-1,1),\\ 0, & \text{elsewhere},\end{cases}$$

so that $\int_{-\infty}^{\infty} dx\, P(x)=2b$. Then we have

$$\frac{1}{T^+(k)}=e^{2ik}\left[\cos 2s-(b+2ik)\frac{\sin 2s}{2s}\right],\qquad \frac{L^+(k)}{T^+(k)}=\frac{R^+(k)}{T^+(k)}=b\frac{\sin 2s}{2s},$$

where we have defined $s=\sqrt{k^2-ikb}$. Thus, as $k\to 0$, we get

$$\frac{1}{T^+(0)}=1-b,\qquad \frac{L^+(0)}{T^+(0)}=\frac{R^+(0)}{T^+(0)}=b.$$

Hence, $1/T^+(0)=0$ if and only if $b=1$.

Except at their singularities on $\mathbf{R}$, the scattering coefficients satisfy certain properties, which we present next. Using (4.3), (4.4), (4.9), and (4.10) we obtain

$$\mathbf{S}^{\pm}(-k)=\overline{\mathbf{S}^{\pm}(k)},\qquad k\in\mathbf{R}. \tag{4.14}$$

Whenever $T^{\pm}(k)$ is well defined on $\mathbf{R}$, from (4.9)-(4.12) we get

$$L^{\pm}(k)\,T^{\mp}(-k)+T^{\pm}(k)\,R^{\mp}(-k)=R^{\pm}(k)\,T^{\mp}(-k)+T^{\pm}(k)\,L^{\mp}(-k)=0,\qquad k\in\mathbf{R}, \tag{4.15}$$

$$T^{\pm}(k)\,T^{\mp}(-k)+L^{\pm}(k)\,L^{\mp}(-k)=T^{\pm}(k)\,T^{\mp}(-k)+R^{\pm}(k)\,R^{\mp}(-k)=1,\qquad k\in\mathbf{R}. \tag{4.16}$$

We can then write (4.15) and (4.16) in matrix form as

$$\mathbf{S}^{\pm}(k)\,\mathbf{S}^{\mp}(-k)^t=\mathbf{I},\qquad k\in\mathbf{R}, \tag{4.17}$$

where $\mathbf{I}$ is the 2×2 unit matrix and the superscript t denotes the matrix transpose. From (4.17) we see that for $k\in\mathbf{R}$ we have

$$T^{\mp}(k)=\frac{T^{\pm}(-k)}{T^{\pm}(-k)^2-L^{\pm}(-k)\,R^{\pm}(-k)}, \tag{4.18}$$

$$L^{\mp}(k)=\frac{-R^{\pm}(-k)}{T^{\pm}(-k)^2-L^{\pm}(-k)\,R^{\pm}(-k)},\quad R^{\mp}(k)=\frac{-L^{\pm}(-k)}{T^{\pm}(-k)^2-L^{\pm}(-k)\,R^{\pm}(-k)}. \tag{4.19}$$

Thus, although none of the scattering coefficients can in general be extended to $\mathbf{C}^-$, we see from (4.4) and (4.18) that $[T^{\pm}(k)-L^{\pm}(k)\,R^{\pm}(k)/T^{\pm}(k)]$ is analytic in $\mathbf{C}^-$. From (4.19) we conclude that the determinant of $\mathbf{S}^{\pm}(k)$ is given by

$$\det \mathbf{S}^{\pm}(k)=T^{\pm}(k)^2-L^{\pm}(k)\,R^{\pm}(k)=\frac{T^{\pm}(k)}{T^{\mp}(-k)},\qquad k\in\mathbf{R}.$$

The behavior of $\mathbf{S}^{\pm}(k)$ at $k=0$ will be analyzed in Section 5.

Contrary to the case $P(x)=0$, the scattering matrix $\mathbf{S}^{+}(k)$ is in general not determined if one of the reflection coefficients and the bound state energies are known, as illustrated by the following example.

Example 4.7 Assume $Q\in L_1^1(\mathbf{R})$, and let $\eta_l^{+}(k,x)=\eta_r^{-}(k,x)=1$ in (3.1), or equivalently

$$f_l^{+}(k,x)=e^{ikx-\int_x^\infty P/2},\qquad f_r^{-}(k,x)=e^{-ikx+\int_{-\infty}^x P/2}. \tag{4.20}$$

Comparing (4.20) with (1.3), (1.4), and (3.2), we see that

$$T^{+}(k)=e^{p},\quad L^{+}(k)=0,\quad T^{-}(k)=e^{-p},\quad R^{-}(k)=0, \tag{4.21}$$

provided the quantity p defined in (3.2) is finite. If $P\in L^1(\mathbf{R})$, we must have p finite; if p is not finite, then $P\notin L^1(\mathbf{R})$. Thus, if $p=\pm\infty$, the scattering theory developed under the assumption $P\in L^1(\mathbf{R})$ may break down. In fact, if we let $p\to+\infty$ in (4.21), we obtain

$$1/T^{+}(k)=0,\quad L^{+}(k)=0,\quad T^{-}(k)=0,\quad R^{-}(k)=0,$$

and if we let $p\to-\infty$ we get

$$T^{+}(k)=0,\quad L^{+}(k)=0,\quad 1/T^{-}(k)=0,\quad R^{-}(k)=0.$$

Note that $f_l^{[0]}(0,x)$ is uniquely determined by $Q(x)$ alone, as seen from (2.18). From (2.22) and (4.20), we obtain

$$P(x)=2\frac{f_l^{[0]\prime}(0,x)}{f_l^{[0]}(0,x)}. \tag{4.22}$$

If $P\in L^1(\mathbf{R})$, from (4.20) we see that $f_l^{+}(0,x)>0$ for $x\in\mathbf{R}$, and thus $Q(x)$ must be an exceptional potential and the corresponding Schrödinger equation (2.17) cannot have any bound states. Conversely, if $Q(x)$ is an exceptional potential belonging to $L_1^1(\mathbf{R})$, from (2.18) and (2.20) it follows that $P(x)$ defined in (4.22) must be in $L^1(\mathbf{R})$. Now let us carry the analysis further when p is finite. Using (3.2) and (4.22) we have $\int_{-\infty}^{\infty}P=-2\log\gamma$ and $e^{-p}=\gamma$, where γ is the constant defined in (2.27). From (4.19) and (4.21) we obtain $L^{-}(k)=-\gamma^2R^{+}(-k)$ for $k\in\mathbf{R}$. Now let us evaluate $L^{-}(k)$ and $R^{+}(k)$. It can be verified that the Jost solution of (1.2) from the left is given by

$$f_l^{-}(k,x)=e^{ikx+\int_x^\infty P/2}-e^{-ikx-\int_x^\infty P/2}\int_x^\infty dy\,P(y)\,e^{2iky+\int_y^\infty P}. \tag{4.23}$$

Using (4.22) we can rewrite (4.23) as

$$f_l^{-}(k,x)=\frac{1}{f_l^{[0]}(0,x)}e^{ikx}-2e^{-ikx}\,f_l^{[0]}(0,x)\int_x^\infty dy\,\frac{f_r^{[0]\prime}(0,y)}{f_r^{[0]}(0,y)^3}e^{2iky}. \tag{4.24}$$

Finally, from (1.3) and (4.24) we get

$$R^{+}(k)=2\int_{-\infty}^{\infty}dy\,\frac{f_l^{[0]\prime}(0,y)}{f_l^{[0]}(0,y)^3}e^{-2iky},\qquad L^{-}(k)=-2e^{-2p}\int_{-\infty}^{\infty}dy\,\frac{f_r^{[0]\prime}(0,y)}{f_r^{[0]}(0,y)^3}e^{2iky}. \tag{4.25}$$

In general, $\mathbf{S}^+(k)$ is not determined by only one or two of its entries. For example, as seen from (4.21), $T^+(k)$ and $L^+(k)$ cannot determine $R^+(k)$, and there are infinitely many $R^+(k)$ corresponding to these two scattering coefficients in (4.21). Therefore, the coefficients $P(x)$ and $Q(x)$ cannot in general be determined from the scattering data consisting of the transmission coefficient and only one of the reflection coefficients. The following example further illustrates this fact in the special case when $P(x)$ and $Q(x)$ have support in a half-line; thus, we can conclude that the scattering data consisting of the Jost solutions and their spatial derivatives at the boundary of the support of $P(x)$ and $Q(x)$ cannot uniquely determine $P(x)$ and $Q(x)$.

Example 4.8 Consider the same situation as in Example 4.7 but with $P(x) = Q(x) = 0$ when $x < 0$. Assume that $Q(x)$ is an exceptional potential without bound states and belongs to $L^1_1(0, +\infty)$. Then the Jost solutions of (1.1) for $x \leq 0$ are given by

$$f_l^+(k,x) = \frac{1}{T^+(k)} e^{ikx} + \frac{L^+(k)}{T^+(k)} e^{-ikx}, \quad f_r^+(k,x) = e^{-ikx}, \qquad x \leq 0,$$

$$f_l^{+\prime}(k,x) = \frac{ik}{T^+(k)} e^{ikx} - \frac{ikL^+(k)}{T^+(k)} e^{-ikx}, \quad f_r^{+\prime}(k,x) = -ike^{-ikx}, \qquad x \leq 0.$$

Hence, the two sets of scattering data $\{T^+(k), L^+(k)\}$ and $\{f_l^+(k,0), f_r^+(k,0), f_l^{+\prime}(k,0), f_r^{+\prime}(k,0)\}$ contain the same information. Using (4.21) and (4.22) we see that $L^+(k) = 0$ and $T^+(k) = 1/f_l^{[0]}(0,0)$ correspond to $P(x) = 2f_l^{[0]\prime}(0,x)/f_l^{[0]}(0,x)$ and $Q(x) = f_l^{[0]\prime\prime}(0,x)/f_l^{[0]}(0,x)^2$. We can certainly find infinitely many $f_l^{[0]}(0,x)$ such that $f_l^{[0]}(0,0)$ is a specified number and $f_l^{[0]\prime}(0,x) = 0$ when $x \leq 0$. The corresponding reflection coefficient from the right is obtained by using (4.25), namely $R^+(k) = 2\int_0^\infty dy\, e^{-2iky} f_l^{[0]\prime}(0,y)/f_l^{[0]}(0,y)^3$, from which we see that $R^+(k)$ must be analytic in $\mathbf{C}^-$ because its Fourier transform has support on a half-line.

Proposition 4.9 Assume $P, Q \in L^1(\mathbf{R})$. Then, for any $k_0 \in \mathbf{R} \setminus \{0\}$, the quantities $1/T^+(k_0)$ and $R^+(k_0)/T^+(k_0)$ cannot be zero simultaneously; similarly, $1/T^+(k_0)$ and $L^+(k_0)/T^+(k_0)$ cannot be zero simultaneously. If $1/T^+(k_0) = 0$ for some $k_0 \in \mathbf{R} \setminus \{0\}$, then none of the quantities $R^\pm(k_0)/T^\pm(k_0)$, $L^\pm(k_0)/T^\pm(k_0)$, $R^\pm(-k_0)/T^\pm(-k_0)$, $L^\pm(-k_0)/T^\pm(-k_0)$ are zero. The order of the zero of $1/T^+(k)$ at k_0 is the same as the order of the pole of $R^+(k)$ and of $L^+(k)$ at k_0.

PROOF: By Proposition 4.3 we know that $1/T^+(k)$, $R^+(k)/T^+(k)$, $L^+(k)/T^+(k)$ are continuous when $k \in \mathbf{R} \setminus \{0\}$. Morover, the right-hand sides in (4.11) and (4.12) cannot be zero when $k \neq 0$. Thus, the proof is complete. ∎

If $1/T^+(k_0) = 0$ for some $k_0 \in \mathbf{R} \setminus \{0\}$, from (4.4) we see that $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ are linearly dependent, and from (1.3) and (1.4) we obtain

$$\frac{f_l^+(k_0,x)}{f_r^+(k_0,x)} = \frac{L^+(k_0)}{T^+(k_0)} = \frac{T^+(k_0)}{R^+(k_0)} = c_0,$$

for some nonzero constant c_0, which is not necessarily real.

Proposition 4.10 Assume $P, Q \in L^1(\mathbf{R})$. Then, $R^+(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$ if and only if $1/T^+(k)$ does not vanish for $k \in \mathbf{R}\setminus\{0\}$. Similarly, $L^+(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$ if and only if $1/T^+(k)$ does not vanish for $k \in \mathbf{R}\setminus\{0\}$.

PROOF: Assume $R^+(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$; as seen from Proposition 4.9, if $1/T^+(k_0) = 0$ for some $k_0 \in \mathbf{R}\setminus\{0\}$, then $R^\pm(k_0)/T^\pm(k_0) \neq 0$; hence, writing $1/T^+(k_0) = [R^+(k_0)/T^+(k_0)][1/R^+(k_0)]$, we see that $1/T^+(k_0) = 0$ only if $|R^+(k)| \to +\infty$ as $k \to k_0$, which cannot happen if $R^+(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$. Conversely, assume $1/T^+(k_0) \neq 0$ for $k \in \mathbf{R}\setminus\{0\}$; then we see that $R^+(k)$ is given by $[R^+(k)/T^+(k)]/[1/T^+(k)]$, which is the ratio of two continuous functions with a nonzero denominator on $\mathbf{R}\setminus\{0\}$; hence $R^+(k)$ is continuous on $\mathbf{R}\setminus\{0\}$. The proof involving $L^+(k)$ is obtained in a similar manner. ∎

Proposition 4.11 Assume $P, Q \in L^1(\mathbf{R})$. For $k \in \mathbf{R}\setminus\{0\}$, the quantity $T^+(k)$ is continuous if and only if $R^+(k)$ is continuous; equivalently, $T^+(k)$ is continuous if and only if $L^+(k)$ is continuous.

PROOF: By Proposition 4.3 we know that $1/T^+(k)$ is continuous on $\mathbf{R}\setminus\{0\}$. Using $T^+(k) = 1/[1/T^+(k)]$, we see that $T^+(k)$ is continuous on $\mathbf{R}\setminus\{0\}$ if and only if $1/T^+(k)$ is nonzero on $\mathbf{R}\setminus\{0\}$. From Proposition 4.10, we see that this happens if and only if the reflection coefficients $R^+(k)$ and $L^+(k)$ are continuous on $\mathbf{R}\setminus\{0\}$. ∎

In general, the matrices $\mathbf{S}^\pm(k)$ are not unitarity. However, we can obtain some identities leading to certain useful inequalities for nonnegative or nonpositive $P(x)$. These are given in the next proposition.

Proposition 4.12 Assume $P, Q \in L^1(\mathbf{R})$. The scattering coefficients satisfy

$$\frac{1}{|T^\pm(k)|^2} = 1 + \left|\frac{L^\pm(k)}{T^\pm(k)}\right|^2 \mp \int_{-\infty}^{\infty} dx\, |f_l^\pm(k,x)|^2\, P(x), \qquad k \in \mathbf{R}\setminus\{0\},$$

$$\frac{1}{|T^\pm(k)|^2} = 1 + \left|\frac{R^\pm(k)}{T^\pm(k)}\right|^2 \mp \int_{-\infty}^{\infty} dx\, |f_r^\pm(k,x)|^2\, P(x), \qquad k \in \mathbf{R}\setminus\{0\}.$$

Hence, if $P(x) \le 0$, then $1/T^+(k)$ cannot have any zeros for $k \in \mathbf{R}$, and we have

$$|T^+(k)|^2 + |L^+(k)|^2 \le 1, \quad |T^+(k)|^2 + |R^+(k)|^2 \le 1, \qquad k \in \mathbf{R}\setminus\{0\}. \tag{4.26}$$

If $1/T^+(k)$ does not have any zeros for $k \in \mathbf{R}$ and $P(x) \ge 0$, then we have

$$|T^+(k)|^2 + |L^+(k)|^2 \ge 1, \quad |T^+(k)|^2 + |R^+(k)|^2 \ge 1, \qquad k \in \mathbf{R}\setminus\{0\}. \tag{4.27}$$

PROOF: From (1.1) and (1.2) we obtain

$$\frac{d}{dx}[f_l^\pm(-k,x); f_l^\pm(k,x)] = \pm 2ik f_l^\pm(-k,x)\, f_l^\pm(k,x)\, P(x), \tag{4.28}$$

$$\frac{d}{dx}[f_r^\pm(-k,x); f_r^\pm(k,x)] = \pm 2ik f_r^\pm(-k,x)\, f_r^\pm(k,x)\, P(x). \tag{4.29}$$

Hence, using (1.3), (1.4), (4.3), (4.14) in (4.28) and (4.29), for $k \in \mathbf{R} \setminus \{0\}$, we obtain

$$1 - \frac{1}{|T^{\pm}(k)|^2} + \left|\frac{L^{\pm}(k)}{T^{\pm}(k)}\right|^2 = \pm \int_{-\infty}^{\infty} dx \, |f_l^{\pm}(k,x)|^2 \, P(x), \tag{4.30}$$

$$1 - \frac{1}{|T^{\pm}(k)|^2} + \left|\frac{R^{\pm}(k)}{T^{\pm}(k)}\right|^2 = \pm \int_{-\infty}^{\infty} dx \, |f_r^{\pm}(k,x)|^2 \, P(x). \tag{4.31}$$

Hence, if $P(x) \le 0$, from (4.30) and (4.31) we see that $1/|T^+(k)|^2 \ge 1$ for $k \in \mathbf{R} \setminus \{0\}$, and hence $1/T^+(k)$ cannot vanish on $\mathbf{R} \setminus \{0\}$; (4.30) and (4.31) also imply (4.26) and (4.27). ∎

Proposition 4.13 Assume $P, Q \in L^1(\mathbf{R})$. If $P(x) \le 0$, then we have $\dfrac{1}{|T^+(k)|} \ge \dfrac{1}{|T^-(k)|}$ for $k \in \mathbf{R} \setminus \{0\}$.

PROOF: From (4.9), (4.10), (4.30), and (4.31) we obtain

$$\frac{1}{|T^-(k)|^2} - \frac{1}{|T^+(k)|^2} = \int_{-\infty}^{\infty} dx \, \left[|f_l^+(k,x)|^2 + |f_r^-(k,x)|^2\right] P(x), \qquad k \in \mathbf{R} \setminus \{0\}, \tag{4.32}$$

and hence the left hand side in (4.32) must be nonpositive if $P(x) \le 0$. ∎

5. SMALL-k ANALYSIS OF SCATTERING COEFFICIENTS

In this section, under the assumptions $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$ in the generic case and $P, Q \in L^1_1(\mathbf{R})$ in the exceptional case, we analyze the small-k asymptotics of $\mathbf{S}^{\pm}(k)$. In the exceptional case, we show that $\mathbf{S}^+(0)$ is not determined by $Q(x)$ alone and we obtain $\mathbf{S}^+(0)$ explicitly in terms of $P(x)$ and $Q(x)$.

From (2.22), (2.23), (2.26), and (4.4), it is seen that we have the generic case for (1.1) if and only we have the generic case for (2.17). In other words, the generic case for (1.1) is determined by $Q(x)$ alone, and the generic case occurs when the Wronskian in (2.26) is nonzero. Thus (1.1) and (1.2) are either both generic or both exceptional. From (2.3), (2.6), (2.9), and (4.4) we can conclude that

$$f_r^{\pm\prime}(0,+\infty) = -f_l^{\pm\prime}(0,-\infty) = \lim_{k\to 0} \frac{-2ik}{T^{[0]}(k)}.$$

Now let us analyze the asymptotics of $\mathbf{S}^{\pm}(k)$ as $k \to 0$ in the generic case.

Proposition 5.1 Assume $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$ and suppose that we are in the generic case. Then $R^{\pm}(0) = L^{\pm}(0) = -1$, $T^{\pm}(k)$ vanish linearly as $k \to 0$ in $\overline{\mathbf{C}^+}$, and

$$\lim_{k\to 0} \frac{2ik}{T^+(k)} = \lim_{k\to 0} \frac{2ik}{T^-(k)} = \lim_{k\to 0} \frac{2ik}{T^{[0]}(k)}. \tag{5.1}$$

Furthermore, $\det \mathbf{S}^{\pm}(0) = -1$, and

$$T^{\pm}(k) = \frac{-2ik}{\int_{-\infty}^{\infty} dy \, Q(y) \, f_l^{[0]}(0,y)} + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}. \tag{5.2}$$

PROOF: Recall that the generic case occurs if $f_l^{\pm}(0,x)$ and $f_r^{\pm}(0,x)$ are linearly independent on $\mathbf{R}$. From (4.4) we see that $T^+(k)$ must vanish linearly in order for $k/T^+(k)$ to have a nonzero limit at $k = 0$. Using

(2.22), (2.23), (2.26), and (4.4), we obtain (5.1). Since generically $R^{[0]}(0) = L^{[0]}(0) = -1$, from (4.9) and (4.10) in the limit $k \to 0$ we get $R^{\pm}(0) = L^{\pm}(0) = -1$. Then, using (4.19) we obtain $\det \mathbf{S}^{\pm}(0) = -1$. From (2.26) and (5.1) we obtain (5.2). ∎

Letting $k \to 0$ in (4.1) and using (2.22), we see that the Jost solutions would have a singularity at $k = 0$ in the generic case if a zero-energy reflection coefficient were equal to $+1$.

Now let us study the asymptotics of $\mathbf{S}^{\pm}(k)$ as $k \to 0$ in the exceptional case. To prove the following theorem, we will impose a slightly stronger condition on $P(x)$ than needed in the generic case.

Theorem 5.2 In the exceptional case, under the assumptions $P, Q \in L^1_1(\mathbf{R})$, we have

$$T^{\pm}(0) = \frac{2\gamma}{\gamma^2 + 1 \mp \int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2}, \tag{5.3}$$

$$L^{\pm}(0) = \frac{\gamma^2 - 1 \pm \int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2}{\gamma^2 + 1 \mp \int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2}, \tag{5.4}$$

$$R^{\pm}(0) = \frac{1 - \gamma^2 \pm \int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2}{\gamma^2 + 1 \mp \int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2}, \tag{5.5}$$

where γ is the constant defined in (2.27).

PROOF: We obtain $T^{+}(0)$ in (5.3) by using (A.18) and (A.19) in (A.17) and also using (4.4). The value of $T^{-}(0)$ in (5.3) is obtained by changing the sign of $P(x)$. In order to obtain $L^{+}(0)$ in (5.4), as in the displayed equation following (A32) of Ref. 11, we first derive

(5.6)
$$\begin{aligned} f_l(0,0)\,[f_l^{+}(k,x); f_r^{+}(-k,x)] =& f_r^{+}(-k,0)\left[-ikf_l(0,0) + f_l'(0,0) + \int_0^{\infty} dz\, e^{ikz}[ikP(z)+Q(z)]\tilde{\psi}(k,z)\right] \\ &- f_l^{+}(k,0)\left[-ikf_l(0,0) + f_l'(0,0) - \int_{-\infty}^{0} dz\, e^{ikz}\,[ikP(z)+Q(z)]\tilde{\psi}(k,z)\right], \end{aligned}$$

where $\tilde{\psi}(k,x)$ is the function in (A.2). Estimating various terms in (5.6) in a similar manner we estimate various terms in (A.17) as in the proof of Proposition A.4, and also by using (4.9), we obtain $L^{+}(0)$ in (5.4). Proceeding in a similar manner we obtain $R^{+}(0)$ in (5.5). The values of $L^{-}(0)$ and $R^{-}(0)$ are obtained from $L^{+}(0)$ and $R^{+}(0)$ by changing the sign of $P(x)$. ∎

Proposition 5.3 Assume that $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$ in the generic case and that $P, Q \in L^1_1(\mathbf{R})$ in the exceptional case. If any one of $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$, $L^{\pm}(k)/T^{\pm}(k)$ is continuous at $k = 0$, then all three are continuous at $k = 0$. Moreover, $1/T^{+}(k)$ and $1/T^{-}(k)$ are both continuous at $k = 0$ or both discontinuous at $k = 0$.

PROOF: Using (2.3) and (2.12) we see that the right-hand sides of (4.30) and (4.31) are continuous for $k \in \mathbf{R}$; the analysis of the left-hand sides as $k \to 0$ allows us to conclude that if any one of $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$,

$L^{\pm}(k)/T^{\pm}(k)$ is continuous at $k=0$, then all three are continuous at $k=0$. Using (2.3) and (2.12) we see that the right-hand side of (4.32) remains bounded and continuous as $k\to 0$; thus, the left-hand side must also behave this way, which allows us to conclude that $1/T^{+}(k)$ and $1/T^{-}(k)$ are both continuous at $k=0$ or both discontinuous at $k=0$. ∎

Theorem 5.4 Assume that $P\in L^1(\mathbf{R})$ and $Q\in L^1_1(\mathbf{R})$ in the generic case and that $P,Q\in L^1_1(\mathbf{R})$ in the exceptional case. Then, in the generic case the six quantities $1/T^{\pm}(k)$, $R^{\pm}(k)/T^{\pm}(k)$, and $L^{\pm}(k)/T^{\pm}(k)$ are all discontinuous at $k=0$; in the exceptional case, these quantities are all continuous at $k=0$.

PROOF: Generically, $T^{+}(0)=0$, and thus $1/T^{+}(k)$ is not continuous at $k=0$. Hence, using Proposition 5.3, we see that the remaining five quantities are also discontinuous at $k=0$. In the exceptional case, from Theorem 5.2, we see that $1/T^{+}(k)$ remains bounded and hence continuous at $k=0$. Hence, by Proposition 5.3, the remaining five quantities are also continuous at $k=0$. ∎

In the exceptional case, since $1/T^{\pm}(k)$ is continuous at $k=0$ whenever $P,Q\in L^1_1(\mathbf{R})$, with the help of Theorem 5.4, it is also possible to prove (5.3)-(5.5) as follows. First, note that (4.30) implies that in this case $T^{\pm}(0)\neq 0$, $\mathbf{S}^{\pm}(0)$ is real valued, and the zero-energy Jost solutions are also real valued. In the exceptional case, from (2.22) and (2.27) we have

$$f_l^{\pm}(0,x)=f_l^{[0]}(0,x)=\gamma\, f_r^{[0]}(0,x)=\gamma\, f_r^{\pm}(0,x).$$

Now let us consider the asymptotics of $\mathbf{S}^{\pm}(k)$ as $k\to 0$ in the exceptional case. Letting $k\to 0$ in (4.1) and using (2.22), we obtain

$$\gamma=\frac{f_l^{[0]}(0,x)}{f_r^{[0]}(0,x)}=\frac{f_l^{\pm}(0,x)}{f_r^{\pm}(0,x)}=\frac{T^{\pm}(0)}{1+R^{\pm}(0)}=\frac{T^{[0]}(0)}{1+R^{[0]}(0)}=\frac{1+L^{[0]}(0)}{T^{[0]}(0)}=\frac{1+L^{\pm}(0)}{T^{\pm}(0)}. \tag{5.7}$$

From (5.7) we have

$$\frac{1}{T^{\pm}(0)}+\frac{L^{\pm}(0)}{T^{\pm}(0)}=\gamma,\qquad \frac{1}{T^{\pm}(0)}+\frac{R^{\pm}(0)}{T^{\pm}(0)}=\frac{1}{\gamma}. \tag{5.8}$$

From (4.30) we see that

$$1-\frac{1}{T^{\pm}(0)^2}+\frac{L^{\pm}(0)^2}{T^{\pm}(0)^2}=\pm\int_{-\infty}^{\infty}dx\,P(x)\,f_l^{[0]}(0,x)^2. \tag{5.9}$$

Eliminating $L^{\pm}(0)/T^{\pm}(0)$ in (5.8) and (5.9) we obtain

$$1+\gamma^2-\frac{2\gamma}{T^{\pm}(0)}=\pm\int_{-\infty}^{\infty}dx\,P(x)\,f_l^{[0]}(0,x)^2,$$

which gives us (5.3). Then, using (5.3) in (5.8) we obtain (5.4) and (5.5).

Using (5.1) and (5.3) we obtain the following result concerning the status of $1/T^{+}(k)$ at $k=0$.

Proposition 5.5 Assume that $P\in L^1(\mathbf{R})$ and $Q\in L^1_1(\mathbf{R})$ in the generic case and that $P,Q\in L^1_1(\mathbf{R})$ in the exceptional case. In the exceptional case $1/T^{+}(k)$ vanishes at $k=0$ if and only if $\int_{-\infty}^{\infty}dx\,P(x)\,f_l^{[0]}(0,x)^2=\gamma^2+1$, where γ is the constant defined in (2.27). In the generic case, $k/T^{+}(k)$ has a nonzero limit as $k\to 0$.

PROOF: The proof in the exceptional case follows from (5.3). The proof in the generic case follows from (5.1) and the fact that $k/T^{[0]}(k)$ has a nonzero limit as $k \to 0$. ∎

In the exceptional case, since $\mathbf{S}^{\pm}(0)$ is real valued, from (5.7) it is seen that $\mathbf{S}^{\pm}(0)$ is a unitary matrix if and only of $R^{\pm}(0) = -L^{\pm}(0)$. From (5.3) and (5.4), it is seen that this occurs when $\int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2 = 0$, in which case we have $\mathbf{S}^{\pm}(0) = \mathbf{S}^{[0]}(0)$ and $\det \mathbf{S}^{\pm}(0) = 1$. Note also that the denominators in (5.3)-(5.5) are the same. Hence, when $P, Q \in L_1^1(\mathbf{R})$, in the exceptional the scattering matrices $\mathbf{S}^{\pm}(k)$ are continuous at $k = 0$ if and only if $1/T^{+}(0) \neq 0$.

When $Q(x) = 0$, we have $f_l^{[0]}(0,x) = f_r^{[0]}(0,x) = 1$, and hence $\gamma = 1$. This corresponds to the exceptional case. Using these values in (5.3)-(5.5) we see that

$$\frac{1}{T^{\pm}(0)} = 1 \mp \frac{1}{2} \int_{-\infty}^{\infty} dx\, P(x). \tag{5.10}$$

Hence, if $\int_{-\infty}^{\infty} dx\, P(x) = \pm 2$, no matter how smooth $P(x)$ is, we have

$$\frac{1}{T^{\pm}(0)} = 0, \qquad \frac{L^{\pm}(0)}{T^{\pm}(0)} = \frac{R^{\pm}(0)}{T^{\pm}(0)} = 1.$$

In this case $\mathbf{S}^{\pm}(0)$ is clearly undefined.

Theorem 5.6 Assume that $P \in L^1(\mathbf{R})$ and $Q \in L_1^1(\mathbf{R})$ in the generic case and that $P, Q \in L_1^1(\mathbf{R})$ in the exceptional case. Then either the three quantities $T^{+}(k)$, $R^{+}(k)$, $L^{+}(k)$ are all continuous on $\mathbf{R}$, or they are all discontinuous on $\mathbf{R}$.

PROOF: From Proposition 4.11 it is seen that we only need to prove the result at $k = 0$. In the exceptional case, from (5.3)-(5.5) it is seen that these three quantities are continuous at $k = 0$ if and only if $\int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2 \neq \gamma^2 + 1$. In the generic case, these three quantities are always continuous at $k = 0$ and this can be seen as follows. From Propositions 4.3 and 5.1 it follows that $T^{+}(k)$ is continuous at $k = 0$. Let

$$R^{+}(k) = \left(\frac{k\, R^{+}(k)}{T^{+}(k)} \right) \left(\frac{1}{k/T^{+}(k)} \right). \tag{5.11}$$

The first factor on the right-hand side in (5.11) is continuous at $k = 0$ because of Proposition 4.3, and the second factor is continuous at $k = 0$ because of Proposition 4.3 and (5.2). Thus, $R^{+}(k)$ is continuous at $k = 0$. The continuity of $L^{+}(k)$ at $k = 0$ in the generic case is obtained in a similar manner. ∎

In summary, if one entry in $\mathbf{S}^{\pm}(k)$ is continuous in $\mathbf{R}$, then the whole matrix $\mathbf{S}^{\pm}(k)$ is continuous at $k = 0$. A discontinuity in $\mathbf{S}^{\pm}(k)$ at some nonzero k_0 is possible if and only if $1/T^{\pm}(k_0) = 0$; the discontinuity of $\mathbf{S}^{\pm}(k)$ at $k = 0$ happens only in the exceptional case when $\int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2 = \pm(\gamma^2 + 1)$.

6. LARGE-k ANALYSIS OF SCATTERING COEFFICIENTS

In this section, under the assumptions $P, Q \in L^1(\mathbf{R})$ we analyze the large-k asymptotics of $\mathbf{S}^{\pm}(k)$.

Using (3.3), (3.4), and (4.4), we obtain

$$\frac{2ik}{T^\pm(k)}e^{\pm p}=[2ik\pm P(x)]\eta_l^\pm(k,x)\,\eta_r^\pm(k,x)+\eta_l^{\pm\prime}(k,x)\,\eta_r^\pm(k,x)-\eta_l^\pm(k,x)\,\eta_r^{\pm\prime}(k,x), \tag{6.1}$$

where p is the constant defined in (3.2). Using (3.3), (3.4), (4.9), and (4.10), we get

$$-\frac{2ik\,R^\pm(k)}{T^\pm(k)}=e^{-2ikx\mp p\pm 2\zeta}[\eta_r^\pm(k,x);\eta_l^\mp(-k,x)],\qquad k\in\mathbf{R}, \tag{6.2}$$

$$-\frac{2ik\,L^\pm(k)}{T^\pm(k)}=e^{2ikx\pm p\mp 2\zeta}[\eta_l^\pm(k,x);\eta_r^\mp(-k,x)],\qquad k\in\mathbf{R}. \tag{6.3}$$

Note that the right hand sides in (6.1)-(6.3) must be independent of x, and hence we can evaluate them at any x.

Theorem 6.1 Assume $P,Q\in L^1(\mathbf{R})$. Then

$$\frac{1}{T^\pm(k)}e^{\pm p}=1+o(1),\qquad k\to\infty \text{ in } \overline{\mathbf{C}^+}, \tag{6.4}$$

$$\frac{R^\pm(k)}{T^\pm(k)}=o(1),\quad \frac{L^\pm(k)}{T^\pm(k)}=o(1),\qquad k\to\pm\infty. \tag{6.5}$$

PROOF: Using (3.12), we can write (6.1) as

$$\frac{1}{T^\pm(k)}e^{\pm p}=\eta_l^\pm(k,x)\eta_r^\pm(k,x)+\frac{1}{2}\xi_l^\pm(k,x)\eta_r^\pm(k,x)-\frac{1}{2}\eta_l^\pm(k,x)\xi_r^\pm(k,x). \tag{6.6}$$

Using (3.6) and (3.13) in (6.6), we obtain (6.4). In a similar manner, using (3.12), we can write (6.2) and (6.3) as

$$\frac{R^\pm(k)}{T^\pm(k)}=\frac{1}{2}e^{-2ikx\mp p\pm 2\zeta}[\eta_r^\pm(k,x)\xi_l^\mp(-k,x)+\eta_l^\mp(-k,x)\xi_r^\pm(k,x)], \tag{6.7}$$

$$\frac{L^\pm(k)}{T^\pm(k)}=\frac{1}{2}e^{2ikx\pm p\mp 2\zeta}[\eta_l^\pm(k,x)\xi_r^\mp(-k,x)+\eta_r^\mp(-k,x)\xi_l^\pm(k,x)]. \tag{6.8}$$

Thus, using (3.6) and (3.13) in (6.7) and (6.8), we obtain (6.5). ∎

Note that from (6.4) it follows that e^p is known when either of $T^\pm(k)$ is known, because

$$e^p=\lim_{k\to\infty}T^+(k)=\lim_{k\to\infty}\frac{1}{T^-(k)}. \tag{6.9}$$

Proposition 6.2 Assume $P,Q\in L^1(\mathbf{R})$. If $1/T^\pm(k)$ does not vanish for $k\in\mathbf{R}$, then its number of zeros in $\mathbf{C}^+$ is finite.

PROOF: From (6.4) it follows that $1/T^\pm(k)$ approaches the nonzero constant $e^{\pm p}$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. In the exceptional case, since $1/T^\pm(k)$ is analytic in $\mathbf{C}^+$ and is assumed nonzero on $\mathbf{R}$, it follows that its zeros are confined to a bounded subset of $\mathbf{C}^+$, and hence the number of its zeros in $\mathbf{C}^+$ must be finite. In the generic case, we can apply the above argument to $k/[(k+i)T^{\pm}(k)]$ and conclude that the number of zeros of $1/T^{\pm}(k)$ in $\mathbf{C}^+$ is finite. ∎

7. PERTURBATION OF SCATTERING COEFFICIENTS

In this section we establish the stability of $1/T^+(k)$, $R^+(k)/T^+(k)$, and $L^+(k)/T^+(k)$ under small perturbations of $P(x)$ and $Q(x)$ in the norm of either $L^1(\mathbf{R})$ or $L^1_1(\mathbf{R})$. We will not state the results for $L^+(k)/T^+(k)$, since they are identical to those for $R^+(k)/T^+(k)$. In Section 9 we will apply the stability result for $1/T^+(k)$ to prove that the number of zeros of $1/T^+(k)$ in $\mathbf{C}^+$ does not change under certain perturbations of $P(x)$ and $Q(x)$.

Given two sets of potentials $P_j(x)$ and $Q_j(x)$ with $j=1,2$, we consider the generalized Schrödinger equations

$$\psi_j^{+\prime\prime}(k,x)+k^2\psi_j^+(k,x)=[ikP_j(x)+Q_j(x)]\,\psi_j^+(k,x),\qquad x\in\mathbf{R}, \tag{7.1}$$

and denote their corresponding Faddeev solutions by $m^+_{l;j}(k,x)$ and $m^+_{r;j}(k,x)$, their transmission coefficients by $T_j^+(k)$, and their reflection coefficients from the right and from the left by $R_j^+(k)$ and $L_j^+(k)$, respectively.

Proposition 7.1 Assume $P_j,Q_j\in L^1(\mathbf{R})$ for $j=1,2$. Then for $k\in\overline{\mathbf{C}^+}$ with $|k|\ge 1$, we have

$$|m^+_{l;j}(k,x)|\le a_j,\qquad |m^+_{r;j}(k,x)|\le a_j, \tag{7.2}$$

$$|m^+_{l;1}(k,x)-m^+_{l;2}(k,x)|\le a\,(||P_1-P_2||_1+||Q_1-Q_2||_1)\,, \tag{7.3}$$

$$|m^+_{r;1}(k,x)-m^+_{r;2}(k,x)|\le a\,(||P_1-P_2||_1+||Q_1-Q_2||_1)\,, \tag{7.4}$$

where $a_j=e^{||P_j||_1+||Q_j||_1}$ and $a=a_1a_2$.

PROOF: We obtain (7.2) directly from (2.10). Iterating (2.4) and by using (7.2), we get (7.3); (7.4) is obtained in a similar manner by iterating (2.7) and by using (7.2). ∎

Proposition 7.2 Assume $P_j\in L^1(\mathbf{R})$ and $Q_j\in L^1_1(\mathbf{R})$ for $j=1,2$. Then for $k\in\overline{\mathbf{C}^+}$ we have

$$|m^+_{l;j}(k,x)|\le b_j[1+\max\{0,-x\}],\qquad |m^+_{r;j}(k,x)|\le b_j[1+\max\{0,x\}], \tag{7.5}$$

$$|k\,m^+_{l;j}(k,x)|\le c_j,\qquad |k\,m^+_{r;j}(k,x)|\le c_j,\qquad |k|\le 1, \tag{7.6}$$

$$|m^+_{l;1}(k,x)-m^+_{l;2}(k,x)|\le b[1+\max\{0,-x\}]\,(||P_1-P_2||_1+||Q_1-Q_2||_{1,1})\,, \tag{7.7}$$

$$(7.8) \qquad |m^+_{r;1}(k,x)-m^+_{r;2}(k,x)|\le b[1+\max\{0,x\}]\left(||P_1-P_2||_1+||Q_1-Q_2||_{1,1}\right),$$

$$(7.9) \qquad |k\, m^+_{l;1}(k,x)-k\, m^+_{l;2}(k,x)|\le c\left(||P_1-P_2||_1+||Q_1-Q_2||_{1,1}\right), \qquad |k|\le 1,$$

$$(7.10) \qquad |k\, m^+_{r;1}(k,x)-k\, m^+_{r;2}(k,x)|\le c\left(||P_1-P_2||_1+||Q_1-Q_2||_{1,1}\right), \qquad |k|\le 1,$$

where $b_j=e^{||P_j||_1+||Q_j||_{1,1}}$, $b=b_1b_2$, $c_j=(1+b_j\,||Q_j||_{1,1})e^{||P_j||_1}$, and $c=[b||Q_2||_{1,1}+\max\{b_1,c_1\}]e^{||P_2||_1}$.

PROOF: Note that (7.5) is the same as (2.12). By iterating (2.4) with the help of the first inequality in (7.5), we get the first inequality in (7.6); the second inequality in (7.6) is obtained in a similar manner by iterating (2.7). We get (7.7) through iteration by using (2.4), (7.5), and the inequality

$$(7.11) \qquad \frac{1+\max\{0,-y\}}{1+\max\{0,-x\}}(y-x)\le 1+|y|, \qquad y\ge x.$$

We obtain (7.8) in a similar way. Using (7.5)-(7.7) and (2.4), through iteration we get (7.9); (7.10) is obtained in a similar way. ∎

Proposition 7.3 Assume $P_j,Q_j\in L^1_1(\mathbf{R})$ for $j=1,2$ and $Q_1(x)=Q_2(x)$. Then, for $k\in\overline{\mathbf{C}^+}$ we have

$$(7.12) \qquad |m^+_{l;1}(k,x)-m^+_{l;2}(k,x)|\le b\,|k|\,[1+\max\{0,-x\}]\,||P_1-P_2||_{1,1},$$

$$(7.13) \qquad |m^+_{r;1}(k,x)-m^+_{r;2}(k,x)|\le b\,|k|\,[1+\max\{0,x\}]\,||P_1-P_2||_{1,1},$$

where b is the constant defined in Proposition 7.2.

PROOF: Using (7.5), (7.11), and (2.4) we get (7.12) through iteration; (7.13) is obtained in a similar way. ∎

Proposition 7.4 Assume $P_j,Q_j\in L^1(\mathbf{R})$ for $j=1,2$. Then for $k\in\overline{\mathbf{C}^+}$ with $|k|\ge 1$, we have

$$(7.14) \qquad \left|\frac{1}{T^+_1(k)}-\frac{1}{T^+_2(k)}\right|\le C_1\left(||P_1-P_2||_1+||Q_1-Q_2||_1\right),$$

and for $k\in\mathbf{R}$ with $|k|\ge 1$, we have

$$(7.15) \qquad \left|\frac{R^+_1(k)}{T^+_1(k)}-\frac{R^+_2(k)}{T^+_2(k)}\right|\le C_1\left(||P_1-P_2||_1+||Q_1-Q_2||_1\right),$$

where $C_1=\dfrac{a_1}{2}[1+a_2\log a_2]$ with a_1 and a_2 being the constants defined in Proposition 7.1.

PROOF: We obtain (7.14) by using (7.2) and (7.3) in (4.6), and we get (7.15) by using (7.2) and (7.4) in (4.8). ∎

Proposition 7.5 Assume $P_j\in L^1(\mathbf{R})$ and $Q_j\in L^1_1(\mathbf{R})$ for $j=1,2$. Then, for $k\in\overline{\mathbf{C}^+}$ with $|k|\le 1$, we have

$$(7.16) \qquad \left|\left(\frac{k}{T^+_1(k)}-\frac{k}{T^+_2(k)}\right)\right|\le C_2\left(||P_1-P_2||_1+||Q_1-Q_2||_{1,1}\right),$$

and for $k \in \mathbf{R}$ with $|k| \le 1$, we have

$$\left|\left(\frac{kR_1^+(k)}{T_1^+(k)} - \frac{kR_2^+(k)}{T_2^+(k)}\right)\right| \le C_2\left(||P_1 - P_2||_1 + ||Q_1 - Q_2||_{1,1}\right), \tag{7.17}$$

where $C_2 = \dfrac{1}{2}[c||P_2||_1 + b||Q_2||_{1,1} + \max\{b_1, c_1\}]$ with b_1, b, c_1, and c being the constants defined in Proposition 7.2.

PROOF: We obtain (7.16) by using (7.5)-(7.7) and (7.9) in (4.6). Similarly, we obtain (7.17) by using (7.5), (7.6), (7.8), and (7.10) in (4.8). ∎

Proposition 7.6 Assume $P_j, Q_j \in L_1^1(\mathbf{R})$ for $j = 1, 2$ and $Q_1(x) = Q_2(x)$. Then, we have

$$\left|\frac{1}{T_1^+(k)} - \frac{1}{T_2^+(k)}\right| \le C||P_1 - P_2||_{1,1}, \qquad k \in \overline{\mathbf{C}^+}, \tag{7.18}$$

$$\left|\frac{R_1^+(k)}{T_1^+(k)} - \frac{R_2^+(k)}{T_2^+(k)}\right| \le C||P_1 - P_2||_{1,1}, \qquad k \in \mathbf{R}, \tag{7.19}$$

where C is a constant independent of k.

PROOF: Using (7.5), (7.7), and (7.12) in (4.6) we obtain (7.18) with $C = \dfrac{b_2}{2} + \dfrac{b}{2}(||P_1||_{1,1} + ||Q_1||_{1,1})$, where b and b_2 are the constants defined in Proposition 7.2. The proof of (7.19) is obtained in a similar manner with the same C by using (7.5), (7.8), and (7.13) in (4.8). ∎

8. BOUND STATES AND JORDAN CHAINS

Recall that the bound states of (1.1) are its nontrivial solutions belonging to $L^2(\mathbf{R})$. In this section, when $P, Q \in L^1(\mathbf{R})$ we show that the zeros of $1/T^+(k)$ in $\mathbf{C}^+$ correspond to the bound states of (1.1). We also analyze the order of each zero of $1/T^+(k)$ in $\mathbf{C}^+$ in terms of Jordan chains of the differential operator $\mathbf{W}(k)$ defined in (8.16).

It is known[8] that the transmission coefficient $T^{[0]}(k)$ corresponding to (2.17) cannot have any singularities on the real axis. We will see that we cannot rule out real zeros of $1/T^+(k)$ unless $P(x) \le 0$, and in fact some examples of such zeros are already given in Examples 4.5 and 4.6.

Proposition 8.1 Assume $P \in L^1(\mathbf{R})$ and $Q \in L_1^1(\mathbf{R})$. Then the zeros of $1/T^+(k)$ on the real axis do not correspond to the bound states of (1.1); hence (1.1) does not have any bound states at positive energies.

PROOF: Assume that $k_0 \in \mathbf{R} \setminus \{0\}$ is a zero of $1/T^+(k)$. From (1.3) and Proposition 4.9, it follows that $f_l^+(k_0, x)$ does not vanish as $x \to \pm\infty$. All solutions of (1.1) with $k = k_0$ have the form $c_+ e^{ik_0x} + c_- e^{-ik_0x} + o(1)$ as $x \to +\infty$, where c_+ and c_- are constants; thus they do not belong to $L^2(\mathbf{R})$ unless they are trivial. Since (2.17) does not have[8,9] a bound state at $k = 0$, from (2.22) and (2.23) it follows that (1.1) does not have a bound state at $k = 0$. ∎

Proposition 8.2 Assume $P, Q \in L^1(\mathbf{R})$. For each $k_0 \in \mathbf{C}^+$, the quantities $m_l^{\pm}(k_0, x)$ and $m_l^{\pm\prime}(k_0, x)$ are bounded and continuous in x, and we have

$$m_l^{\pm}(k_0,x) = \begin{cases} 1 + o(1), & x \to +\infty, \\ \dfrac{1}{T^{\pm}(k_0)} + o(1), & x \to -\infty. \end{cases} \tag{8.1}$$

Similarly, $m_r^{\pm}(k_0, x)$ and $m_r^{\pm\prime}(k_0, x)$ are bounded and continuous for each $k_0 \in \overline{\mathbf{C}^+}$, and

$$m_r^{\pm}(k_0,x) = \begin{cases} \dfrac{1}{T^{\pm}(k_0)} + o(1), & x \to +\infty, \\ 1 + o(1), & x \to -\infty. \end{cases} \tag{8.2}$$

PROOF: We proceed as in Ref. 9. First, note that from Proposition 2.1 it follows that $m_l^{\pm}(k_0, x)$ is bounded. Using (2.10) in (2.4), we obtain $|m_l^{\pm}(k_0, x)-1| \le C\int_x^{\infty}[|P|+|Q|]$, and hence $m_l^{\pm}(k_0, x) = 1+o(1)$ as $x \to +\infty$. Using (2.4) and (4.6) we obtain

$$m_l^{\pm}(k_0,x) = \frac{1}{T^{\pm}(k_0)} + A_1 + A_2 + A_3,$$

where

$$A_1 = \int_{-\infty}^{x} dy\, \left[\mp\frac{1}{2}P(y) - \frac{1}{2ik_0}Q(y)\right] m_l^{\pm}(k_0, y), \tag{8.3}$$

$$A_2 = \int_{x}^{x/2} dy\, e^{2ik_0(y-x)} \left[\pm\frac{1}{2}P(y) + \frac{1}{2ik_0}Q(y)\right] m_l^{\pm}(k_0, y), \tag{8.4}$$

$$A_3 = \int_{x/2}^{\infty} dy\, e^{2ik_0(y-x)} \left[\pm\frac{1}{2}P(y) + \frac{1}{2ik_0}Q(y)\right] m_l^{\pm}(k_0, y). \tag{8.5}$$

Using (2.10) in (8.3), we obtain $|A_1| \le C\int_{-\infty}^{x}[|P| + |Q|]$, and hence $A_1 = o(1)$ as $x \to -\infty$. Similarly, using (2.10) in (8.4), we have $|A_2| \le C\int_x^{x/2}[|P| + |Q|]$ for $x \le 0$, and hence $A_2 = o(1)$ as $x \to -\infty$. In a similar way, using (2.10) in (8.5), for $x \le 0$ we obtain

$$|A_3| \le C|e^{-ik_0x}| \int_{x/2}^{\infty} dy\, |e^{2ik_0(y-x/2)}|\,[|P(y)| + |Q(y)|] \le C|e^{-ik_0x}| \int_{x/2}^{\infty} dy\, [|P(y)| + |Q(y)|],$$

and hence $A_3 = o(1)$ as $x \to -\infty$. Thus, we have (8.1). The continuity of $m_l^{\pm}(k_0, x)$ and $m_l^{\pm\prime}(k_0, x)$ follows from (2.5). From (2.6) we have $m_l^{\pm\prime}(k_0, x) = o(1)$ as $x \to +\infty$. Using the boundedness of $m_l^{\pm}(k_0, x)$, from (2.5) we obtain $|m_l^{\pm\prime}(k_0, x)| \le C\int_x^{\infty} dy\,[|k_0P(y)| + |Q(y)|]$. Hence, $m_l^{\pm\prime}(k_0, x) = O(1)$ as $x \to -\infty$. The proof for $m_r^{\pm}(k_0, x)$ and $m_r^{\pm\prime}(k_0, x)$ is obtained in a similar manner. ∎

Proposition 8.3 Assume $P, Q \in L^1(\mathbf{R})$. If $k_0 \in \mathbf{C}^+$ is a zero of $1/T^{\pm}(k)$, then we have

$$f_l^{+}(k_0, x) = O(e^{-|x|\,\mathrm{Im}\,k_0}), \qquad f_r^{+}(k_0, x) = O(e^{-|x|\,\mathrm{Im}\,k_0}), \qquad x \to \pm\infty, \tag{8.6}$$

$$(8.7)\qquad f_l^{+\prime}(k_0,x)=O(e^{-|x|\,\mathrm{Im}\,k_0}),\qquad f_r^{+\prime}(k_0,x)=O(e^{-|x|\,\mathrm{Im}\,k_0}),\qquad x\to\pm\infty.$$

PROOF: For each $k_0\in\mathbf{C}^+$, using Proposition 8.2, (2.3), (2.6), (2.9), we obtain

$$(8.8)\qquad f_l^{\pm}(k_0,x)=\begin{cases} e^{ik_0x}+o(e^{-x\,\mathrm{Im}\,k_0}), & x\to+\infty,\\ O(e^{-x\,\mathrm{Im}\,k_0}), & x\to-\infty,\end{cases}$$

$$(8.9)\qquad f_l^{\pm\prime}(k_0,x)=\begin{cases} ik_0e^{ik_0x}+o(e^{-x\,\mathrm{Im}\,k_0}), & x\to+\infty,\\ O(e^{-x\,\mathrm{Im}\,k_0}), & x\to-\infty,\end{cases}$$

$$(8.10)\qquad f_r^{\pm}(k_0,x)=\begin{cases} O(e^{x\,\mathrm{Im}\,k_0}), & x\to+\infty,\\ e^{-ik_0x}+o(e^{x\,\mathrm{Im}\,k_0}), & x\to-\infty,\end{cases}$$

$$(8.11)\qquad f_r^{\pm\prime}(k_0,x)=\begin{cases} O(e^{x\,\mathrm{Im}\,k_0}), & x\to+\infty,\\ -ik_0e^{x\,\mathrm{Im}\,k_0}+o(e^{x\,\mathrm{Im}\,k_0}), & x\to-\infty.\end{cases}$$

When $1/T^{\pm}(k_0)=0$, $f_l^{\pm}(k_0,x)$ and $f_r^{\pm}(k_0,x)$ are linearly dependent, and hence (8.8) and (8.11) imply (8.6) and (8.7). ∎

Proposition 8.4 Assume $P,Q\in L^1(\mathbf{R})$. Then, each zero of $1/T^+(k)$ in $\mathbf{C}^+$ corresponds to a bound state of (1.1). Conversely, if (1.1) has a bound state for some $k_0\in\mathbf{C}^+$, it is necessary that $1/T^+(k_0)=0$.

PROOF: If $k_0\in\mathbf{C}^+$ is a zero of $1/T^+(k)$, we see from (4.4) that $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ are linearly dependent, and from (8.6) and the continuity of the Jost solutions in x, we can conclude that $f_l^+(k_0,\cdot)\in L^2(\mathbf{R})$. Hence, k_0 corresponds to a bound state for (1.1). Now assume that $k_0\in\mathbf{C}^+$ corresponds to a bound state for (1.1). If $1/T^+(k_0)$ were nonzero, the Jost solutions $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ would be linearly independent, and in that case from (8.1) and (8.2) we would conclude that the behavior of $f_l^+(k_0,x)$ as $x\to-\infty$ and of $f_r^+(k_0,x)$ as $x\to+\infty$ would make it impossible to form a square-integrable linear combination from these two Jost solutions. ∎

Proposition 8.5 Assume $P,Q\in L^1(\mathbf{R})$. If $k_0\in\mathbf{C}^+$ is a zero of $1/T^+(k)$, then we have

$$(8.12)\qquad \int_{-\infty}^{\infty}dx\,[P(x)+ik_0]\,f_l^+(k_0,x)\,f_l^{[0]}(0,x)=0=\int_{-\infty}^{\infty}dx\,[P(x)+ik_0]\,f_r^+(k_0,x)\,f_r^{[0]}(0,x).$$

PROOF: Evaluating (1.1) at $k=k_0$ and $k=0$, respectively, and using (2.22) and (2.23) we obtain

$$(8.13)\qquad \frac{d}{dx}[f_l^+(k_0,x);f_l^{[0]}(0,x)]=[k_0^2-ik_0P(x)]\,f_l^+(k_0,x)\,f_l^{[0]}(0,x).$$

If $k_0\in\mathbf{C}^+$ is a zero of $1/T^+(k)$, then from Proposition 8.3 it follows that $f_l^+(k_0,x)$ and $f_l^{+\prime}(k_0,x)$ decay exponentially as $x\to\pm\infty$; from (2.12) and (2.13) we see that $f_l^{[0]}(0,x)$ and $f_l^{[0]\prime}(0,x)$ are at most of $O(x)$ as $x\to\pm\infty$. Thus, integrating (8.13) on the real axis, we get

$$0=ik_0\int_{-\infty}^{\infty}dx\,[-ik_0-P(x)]\,f_l^+(k_0,x)\,f_l^{[0]}(0,x),$$

from which we have the first equality in (8.12). The proof of the second equality in (8.12) is obtained in a similar manner. ∎

Next we analyze multiple poles of $T^+(k)$. Let us differentiate (1.1) with $\psi = f_l^+(k,x)$ or $\psi = f_r^+(k,x)$ with respect to k repeatedly. Defining

$$g_{l,n}^+(k,x) = \frac{1}{n!}\left(\frac{\partial}{\partial k}\right)^n f_l^+(k,x), \qquad g_{r,n}^+(k,x) = \frac{1}{n!}\left(\frac{\partial}{\partial k}\right)^n f_r^+(k,x), \qquad n = 0,1,2,\cdots, \tag{8.14}$$

and $g_{l,n}^+(k,x) = g_{r,n}^+(k,x) = 0$ for $n = -1,-2,\cdots$, we obtain the coupled system of differential equations

$$\begin{aligned} g_{l,n}^{+\,\prime\prime}(k,x) + k^2 g_{l,n}^+(k,x) + 2k g_{l,n-1}^+(k,x) + g_{l,n-2}^+(k,x) \\ = [ikP(x) + Q(x)] g_{l,n}^+(k,x) + iP(x) g_{l,n-1}^+(k,x), \end{aligned} \tag{8.15}$$

$$\begin{aligned} g_{r,n}^{+\,\prime\prime}(k,x) + k^2 g_{r,n}^+(k,x) + 2k g_{r,n-1}^+(k,x) + g_{r,n-2}^+(k,x) \\ = [ikP(x) + Q(x)] g_{r,n}^+(k,x) + iP(x) g_{r,n-1}^+(k,x). \end{aligned}$$

Defining the differential operator

$$\mathbf{W}(k) = -k^2 - \frac{d^2}{dx^2} + ikP + Q, \tag{8.16}$$

so that $\dot{\mathbf{W}}(k) = -2k + iP$ and $\ddot{\mathbf{W}}(k) = -2\mathbf{I}$, we obtain the system of linear equations

$$\mathbf{T}(k)\, \mathbf{g}_l^+(k,x) = \mathbf{0}, \tag{8.17}$$

where $\mathbf{0}$ is the zero column vector of m entries, $\mathbf{g}_l^+(k,x)$ is the column vector $[g_{l,m-1}^+(k,x), \cdots, g_{l,0}^+(k,x)]^t$, and $\mathbf{T}(k)$ is the $m \times m$ Toeplitz matrix given by

$$\mathbf{T}(k) = \begin{bmatrix} \mathbf{W}(k) & \dot{\mathbf{W}}(k) & \frac{1}{2}\ddot{\mathbf{W}}(k) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathbf{W}(k) & \dot{\mathbf{W}}(k) & \frac{1}{2}\ddot{\mathbf{W}}(k) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mathbf{W}(k) & \dot{\mathbf{W}}(k) & \frac{1}{2}\ddot{\mathbf{W}}(k) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{W}(k) & \dot{\mathbf{W}}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mathbf{W}(k) \end{bmatrix}.$$

Using the Leibnitz formula for repeated derivatives of products, we find from (4.4)

$$\frac{1}{n!}\left(\frac{d}{dk}\right)^n \frac{-2ik}{T^+(k)} = \sum_{j=0}^{n} [g_{l,j}^+(k,x); g_{r;n-j}^+(k,x)]. \tag{8.18}$$

We call $k_0 \in \mathbf{C}^+$ an eigenvalue of $\mathbf{W}(k)$ if there exists a nontrivial $\phi \in L^2(\mathbf{R})$ such that $\mathbf{W}(k_0)\phi = 0$. Because of Proposition 8.4, this is equivalent to $1/T^+(k_0) = 0$. Further, ϕ is called an eigenfunction of $\mathbf{W}(k)$ corresponding to the eigenvalue k_0. More generally,[19] if k_0 is an eigenvalue of $\mathbf{W}(k)$, then the string of functions $\phi_0, \cdots, \phi_{m-1}$ in $L^2(\mathbf{R})$ is called a Jordan chain of length m corresponding to the eigenvalue k_0 if $\phi_0 \neq 0$ and (8.17) holds with $[g_{l,m-1}^+(k_0,x), \cdots, g_{l,0}^+(k_0,x)]^t$ replaced by the column vector $[\phi_{m-1}(k_0,x), \cdots, \phi_0(k_0,x)]^t$.

Proposition 8.6 Assume $P, Q \in L^1(\mathbf{R})$ and let $k_0 \in \mathbf{C}^+$ be an eigenvalue of $\mathbf{W}(k)$. If $\{g^+_{l,j}(k_0,\cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m at the eigenvalue k_0, then for $n = 0, 1, \cdots, m-1$ we have

$$g^+_{l,n}(k_0,x) = O[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \qquad g^+_{r,n}(k_0,x) = O[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \qquad x \to \pm\infty, \tag{8.19}$$

$$g^{+\prime}_{l,n}(k_0,x) = O[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \qquad g^{+\prime}_{r,n}(k_0,x) = O[(1+|x|)^n e^{-|x|\operatorname{Im} k_0}], \qquad x \to \pm\infty. \tag{8.20}$$

PROOF: If $k_0 \in \mathbf{C}^+$ is an eigenvalue of $\mathbf{W}(k)$, by Proposition 8.4 we have $1/T^+(k_0) = 0$. Hence $f^+_l(k_0,x)$ and $f^+_r(k_0,x)$ are linearly dependent, and thus we need to prove (8.19) and (8.20) only for $g^+_{l,n}(k_0,x)$ and $g^{+\prime}_{l,n}(k_0,x)$. For $n = 0$ these follow from (8.6) and (8.7). The following argument used to prove (8.19) and (8.20) for $n = m-1$ can be used recursively for $n = 1, \cdots, m-2$. Note that, for each $k \in \mathbf{C}^+$, (1.1) has[20] an unbounded solution $X(k,x)$ such that

$$X(k,x) = O(e^{|x|\operatorname{Im} k}), \qquad X'(k,x) = O(e^{|x|\operatorname{Im} k}), \qquad x \to \pm\infty. \tag{8.21}$$

Let us choose $X(k_0,x)$ such that $[f^+_l(k_0,x); X(k_0,x)] = 1$. Let us consider (8.15) as a second-order linear, nonhomogeneous differential equation for $g^+_{l,m-1}(k_0,x)$ and solve it by variation of parameters using the linearly independent solutions $f^+_l(k_0,x)$ and $X(k_0,x)$ of (1.1). We obtain

$$\begin{aligned} g^+_{l,m-1}(k_0,x) &= a_{m-1}\, f^+_l(k_0,x) + b_{m-1}\, X(k_0,x) \\ &- \int_0^x dy\, \Big([iP(y) - 2k_0] g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big) \left[f^+_l(k_0,x) X(k_0,y) - f^+_l(k_0,y)\, X(k_0,x)\right], \end{aligned} \tag{8.22}$$

where a_{m-1} and b_{m-1} are arbitrary constants. Since $g_{l,n}(k_0,\cdot) \in L^2(\mathbf{R})$ for $n = 0, 1, \cdots, m-2$ and $X(k_0,x)$ is unbounded as $x \to \pm\infty$, the term proportional to $X(k_0,x)$ in (8.22) must vanish. Thus, we must have

$$b_{m-1} + \int_0^\infty dy\, \Big([iP(y) - 2k_0]\, g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big)\, f^+_l(k_0,y) = 0,$$

$$b_{m-1} - \int_{-\infty}^0 dy\, \Big([iP(y) - 2k_0]\, g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big)\, f^+_l(k_0,y) = 0,$$

and hence

$$\int_{-\infty}^\infty dy\, \Big([iP(y) - 2k_0]\, g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big)\, f^+_l(k_0,y) = 0. \tag{8.23}$$

Using (8.23), we can write (8.22) as

$$\begin{aligned} g^+_{l,m-1}(k_0,x) =& A(k_0,x)\, X(k_0,x) \\ &+ \left[a_{m-1} - \int_0^x dy\, \Big([iP(y) - 2k_0]\, g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big)\, X(k_0,y)\right] f^+_l(k_0,x), \end{aligned} \tag{8.24}$$

where we have

$$A(k_0,x) = \begin{cases} \int_{-\infty}^x dy\, \Big([iP(y) - 2k_0]\, g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big)\, f^+_l(k_0,y), & x \le 0, \\ \int_x^\infty dy\, \Big([iP(y) - 2k_0]\, g^+_{l,m-2}(k_0,y) - g^+_{l,m-3}(k_0,y)\Big)\, f^+_l(k_0,y), & x \ge 0. \end{cases} \tag{8.25}$$

Using (8.19) for $n = 0, 1, \cdots, m-2$, (8.21), and (8.25), we obtain (8.19) for $g_{l,m-1}^+(k_0, x)$. Differentiating (8.24) and using (8.23), we obtain

$$g_{l,m-1}^{+\prime}(k_0, x) = \left[a_{m-1} - \int_0^x dy\, \Big([iP(y) - 2k_0]g_{l,m-2}^+(k_0, y) - g_{l,m-3}^+(k_0, y)\Big)\, X(k_0, y)\right] f_l^{+\prime}(k_0, x)$$
$$+ X'(k_0, x) \int_{-\infty}^x dy\, \Big([iP(y) - 2k_0]g_{l,m-2}^+(k_0, y) - g_{l,m-3}^+(k_0, y)\Big)\, f_l^+(k_0, y).$$

Finally, using (8.19) for $n = 0, 1, \cdots, m-2$, (8.25), and (8.21), we obtain (8.20) for $g_{l,m-1}^{+\prime}(k_0, x)$. ∎

Proposition 8.7 Assume $P, Q \in L^1(\mathbf{R})$ and let $k_0 \in \mathbf{C}^+$ be an eigenvalue of $\mathbf{W}(k)$. Then for $n = 0, 1, \cdots, m-1$ we have

$$\int_{-\infty}^{\infty} dy\, \Big([iP(y) - 2k_0]g_{l,n-1}^+(k_0, y) - g_{l,n-2}^+(k_0, y)\Big)\, f_l^+(k_0, y) = 0 \tag{8.26}$$

if and only if $\{g_{l,j}^+(k_0, \cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0.

PROOF: If (8.26) holds for $n = 1$, then we must have $f_l^+(k_0, \cdot) \in L^2(\mathbf{R})$; from Proposition 8.3 and its proof it is seen that $f_l^+(k_0, \cdot) \in L^2(\mathbf{R})$ only if $g_{l,0}^+(k_0, x) = f_l^+(k_0, x)$ is an eigenvector of $\mathbf{W}(k)$. Recursively, we can show that $g_{l,n}^+$ given in (8.14) satisfies (8.17), (8.19), and (8.20), and hence $g_{l,n}^+(k_0, \cdot) \in L^2(\mathbf{R})$ for $n = 1, \cdots, m-1$. Thus $\{g_{l,j}^+(k_0, \cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0. The converse is proved by proceeding recursively as in the proof of Proposition 8.6 leading to (8.23). ∎

Theorem 8.8 Assume $P, Q \in L^1(\mathbf{R})$ and let $k_0 \in \mathbf{C}^+$. Then the following four statements are equivalent:

(a) $\mathbf{W}(k)$ has a Jordan chain of length m corresponding to the eigenvalue k_0.

(b) $\{g_{l,j}^+(k_0, \cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0.

(c) $\{g_{r,j}^+(k_0, \cdot)\}_{j=0}^{m-1}$ is a Jordan chain of $\mathbf{W}(k)$ of length m corresponding to the eigenvalue k_0.

(d) $1/T^+(k)$ has a zero at k_0 of order at least m.

PROOF: Clearly (b) implies (a). Now assume (a) holds and let $\{\phi_j\}_{j=0}^{m-1}$ be a Jordan chain of $\mathbf{W}(k)$ of length m at the eigenvalue k_0. Then ϕ_0 must be proportional to $f_l^+(k_0, x)$ and $f_r^+(k_0, x)$ because the latter two are linearly dependent and $\phi_0(k_0, x)$ is a solution of (1.1) for $k = k_0$. Thus we have $\mathbf{W}(k_0)g_{l,0}^+(k_0, x) = 0$ and consequently (8.15) is satisfied for $n = 0, 1, \cdots, m-1$. Hence (b) holds.

Note that (b) and (c) are equivalent because, if $\mathbf{W}(k_0)g_{l,0}^+(k_0, x) = 0$ for some $k_0 \in \mathbf{C}^+$ and $g_{l,0}^+(k_0, \cdot) \in L^2(\mathbf{R})$, by Proposition 8.4 we must have $1/T^+(k_0) = 0$ and hence $g_{r,0}^+(k_0, x)$ must be a constant multiple of $g_{l,0}^+(k_0, x)$.

If (b) holds, then (8.19) and (8.20) must hold for $n = 0, 1, \cdots, m-1$ because of Proposition 8.6. Then, for $n = 0, 1, \cdots, m-1$, by evaluating the right-hand side of (8.18) at $x = -\infty$ or at $x = +\infty$, we find that its left-hand side must be zero and thus (d) holds. Now assume that (d) holds and let us show that (b) is true. By Proposition 8.7 it is sufficient to show that (8.26) is satisfied for $n = 0, 1, \cdots, m-1$. We will do this recursively. First notice that (8.26) holds for $n=0$ trivially because $g^+_{l,-1}(k_0,x)=g^+_{l,-2}(k_0,x)=0$ and that (8.19) and (8.20) hold for $g^+_{l,0}$ and $g^{+\prime}_{l,0}$, respectively, as seen from $1/T^+(k_0)=0$ and Proposition 8.3. For $n=1,\cdots,m-2$ the proof of (8.26) and of (8.19) and (8.20) with $g^+_{l,n}$ and $g^{+\prime}_{l,n}$, respectively, are similar to the case for $n=m-1$. Thus, it is sufficient to give the proofs for $n=m-1$ by assuming that these equations hold for $n=1,\cdots,m-2$. Using (1.1) for $g^+_{r,0}(k_0,x)$ and (8.15) for $g^+_{l,m-1}(k_0,x)$, we obtain the Wronskian relation

$$(8.27)\qquad -\frac{d}{dx}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]=\Big([iP(x)-2k_0]\,g^+_{l,m-2}(k_0,x)-g^+_{l,m-3}(k_0,x)\Big)\,g^+_{r,0}(k_0,x).$$

In a similar way, we obtain

$$(8.28)\qquad \frac{d}{dx}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]=\big([iP(x)-2k_0]\,g^+_{r,m-2}(k_0,x)-g^+_{r,m-3}(k_0,x)\big)\,g^+_{l,0}(k_0,x).$$

Integrating (8.27) and (8.28) we get

$$(8.29)\qquad \begin{aligned}&\int_{-\infty}^{\infty}dy\,\Big([iP(y)-2k_0]g^+_{l,m-2}(k_0,y)-g^+_{l,m-3}(k_0,y)\Big)\,g^+_{r,0}(k_0,y)\\&\quad=\lim_{x\to-\infty}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]-\lim_{x\to+\infty}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)],\end{aligned}$$

$$(8.30)\qquad \begin{aligned}&\int_{-\infty}^{\infty}dy\,\big([iP(y)-2k_0]g^+_{r,m-2}(k_0,y)-g^+_{r,m-3}(k_0,y)\big)\,g^+_{l,0}(k_0,y)\\&\quad=\lim_{x\to+\infty}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]-\lim_{x\to-\infty}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)].\end{aligned}$$

Because (8.19) and (8.20) hold for $n=0,1,\cdots,m-2$, we have

$$(8.31)\qquad \lim_{x\to\pm\infty}\sum_{j=1}^{m-2}[g^+_{l,j}(k_0,x);g^+_{r;m-j}(k_0,x)]=0.$$

Since $1/T^+(k_0)$ is assumed to have a zero of order at least m, using (8.18) for $n=m-1$ and (8.31), we obtain

$$(8.32)\qquad \lim_{x\to\pm\infty}\Big([g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0),x)]+[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]\Big)=0.$$

Using the linear dependence of $g^+_{l,0}(k_0,x)$ and $g^+_{r,0}(k_0,x)$, Proposition 8.3, and (8.19) and (8.20) with $n=0$, we obtain

$$(8.33)\qquad \lim_{x\to+\infty}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]=0,$$

$$(8.34)\qquad \lim_{x\to-\infty}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]=0.$$

From (8.32)-(8.34) we see that

$$(8.35)\qquad \lim_{x\to-\infty}[g^+_{l,m-1}(k_0,x);g^+_{r,0}(k_0,x)]=0,\qquad \lim_{x\to+\infty}[g^+_{l,0}(k_0,x);g^+_{r,m-1}(k_0,x)]=0,$$

and thus using (8.33)-(8.35) in (8.29) and (8.30), we get

$$\int_{-\infty}^{\infty} dy \left([iP(y) - 2k_0] g_{l,m-1}^{+}(k_0, y) - g_{l,m-2}^{+}(k_0, y) \right) g_{r,0}^{+}(k_0, y) = 0.$$

Thus (8.26) is proved for $n = m - 1$, and hence (b) holds. ∎

9. BOUND STATES AND POLES OF $T^{+}(k)$

In this section we further analyze the poles of $T^{+}(k)$ in $\mathbf{C}^{+}$. We show that such poles cannot occur in certain regions in $\mathbf{C}^{+}$ determined in terms of the constants defined in (9.1). When $P(x) \leq 0$, we show that such poles are confined to a certain interval on the positive imaginary axis. We analyze the change in the number of bound states when $P(x)$ and $Q(x)$ are perturbed. In the generic case we find that the number of bound states is unchanged under small perturbations of $P(x)$ and $Q(x)$; in the exceptional case we find that the number of bound states is unchanged under small perturbations of $P(x)$. When $P(x) \leq 0$ we show that the number of bound states is independent of $P(x)$. We also present a Levinson theorem relating the number of bound states to the change in the argument of $T^{+}(k)$.

Next we obtain some simple conditions on $P(x)$ and $Q(x)$ guaranteeing that there are no bound states outside certain k-regions in $\mathbf{C}^{+}$ determined by the following parameters:

$$P_{\min} = \operatorname{ess} \inf_{x \in \mathbf{R}} P(x), \qquad P_{\max} = \operatorname{ess} \sup_{x \in \mathbf{R}} P(x), \qquad Q_{\min} = \operatorname{ess} \inf_{x \in \mathbf{R}} Q(x). \tag{9.1}$$

Let us also define

$$\beta^{*} = P_{\max}/2 + \sqrt{P_{\max}^{2}/4 - Q_{\min}}.$$

Note that if $P, Q \in L^{1}(\mathbf{R})$, then it follows that $P_{\max} \geq 0$ with the equality holding if and only if $P(x) \leq 0$, that $Q_{\min} \leq 0$ with the equality holding if and only if $Q(x) \geq 0$, and that $P_{\min} \leq 0$ with the equality holding if and only if $P(x) \geq 0$. Furthermore, $\beta^{*} \geq P_{\max}$ with the equality holding if and only if $Q(x) \geq 0$. Note also that $\beta^{*} \geq 0$ with the equality holding if and only if $P(x) = Q(x) = 0$; hence, the case $\beta^{*} = 0$ is trivial.

Theorem 9.1 Assume $P, Q \in L^{1}(\mathbf{R})$, $P(x) \not\equiv 0$, and $P_{\max}$ is finite. Then the zeros of $1/T^{+}(k)$ for $P_{\max}/2 \leq \operatorname{Im} k < \beta^{*}$ can only occur on the imaginary axis, and all such zeros are simple. If, in addition, $Q_{\min}$ is finite, then there are no zeros of $1/T^{+}(k)$ in the region $\{k \in \mathbf{C}^{+} : \ (\operatorname{Im} k)^{2} - (\operatorname{Re} k)^{2} - (\operatorname{Im} k) P_{\max} \geq -Q_{\min}\}$. Consequently, $1/T^{+}(k)$ has no zeros in $\mathbf{C}^{+}$ satisfying $\operatorname{Im} k \geq \beta^{*}$.

PROOF: From (1.1), after using (4.2), we obtain

$$\frac{d}{dx}\{f_l^{+}(-\overline{k_0}, x)\, f_l^{+\prime}(k_0, x)\} = |f_l^{+\prime}(k_0, x)|^{2} + [-k_0^{2} + i k_0 P(x) + Q(x)]\, |f_l^{+}(k_0, x)|^{2}. \tag{9.2}$$

If $k_0 \in \mathbf{C}^{+}$ is a zero of $1/T^{+}(k)$, as seen from (8.8) and (8.9), the quantity $f_l^{+}(-\overline{k_0}, x)\, f_l^{+\prime}(k_0, x)$ vanishes exponentially as $x \to \pm\infty$. Thus, integrating (9.2) we obtain

$$\int_{-\infty}^{\infty} dx\, |f_l^{+\prime}(k_0, x)|^{2} = \int_{-\infty}^{\infty} dx\, [k_0^{2} - i k_0 P(x) - Q(x)]\, |f_l^{+}(k_0, x)|^{2}. \tag{9.3}$$

Letting $k_0 = \alpha + i\beta$, and separating the real and imaginary parts in (9.3), we obtain

$$i\alpha \int_{-\infty}^{\infty} dx\, [2\beta - P(x)]\, |f_l^+(k_0,x)|^2 = 0, \tag{9.4}$$

$$\int_{-\infty}^{\infty} dx\, |f_l^{+\prime}(k_0,x)|^2 = \int_{-\infty}^{\infty} dx\, [\alpha^2 - \beta^2 + \beta P(x) - Q(x)]\, |f_l^+(k_0,x)|^2. \tag{9.5}$$

From (9.4) we see that we must have $\alpha = 0$ when $P_{\max} \le 2\beta$, and hence any zero of $1/T^+(k)$ with $\operatorname{Im} k \ge P_{\max}/2$ can only occur on the positive imaginary axis. All such zeros are simple; otherwise, a zero of order two or higher would imply (8.26) with $n = 1$, i.e. $\int_{-\infty}^{\infty} dx\, [P(x) - 2 \operatorname{Im} k_0]\, |f_l^+(k_0,x)|^2 = 0$, which cannot happen if $\operatorname{Im} k_0 \ge P_{\max}/2$. From (9.5) we see that we cannot have $\alpha^2 - \beta^2 + \beta P(x) - Q(x) \le 0$. Hence, there are no zeros of $1/T^+(k)$ in $\{\alpha + i\beta \in \mathbf{C}^+ : \ \beta^2 - \alpha^2 - \beta P_{\max} \le -Q_{\min}\}$. The analysis of the corresponding region on the $\alpha\beta$-plane indicates that there cannot be any zeros of $1/T^+(k)$ on the imaginary axis when $\operatorname{Im} k \ge \beta^*$, and hence there cannot be any zeros of $1/T^+(k)$ either on or off the imaginary axis when $\operatorname{Im} k \ge \beta^*$. ∎

When $P(x) \le 0$, from Theorem 9.1 we obtain the following corollary.

Corollary 9.2 Assume $P(x) \le 0$ and $P, Q \in L^1(\mathbf{R})$. Then, the poles of $T^+(k)$ in $\mathbf{C}^+$ are all purely imaginary and simple. In addition, assume that $Q_{\min}$ defined in (9.1) is finite; then there are no zeros of $1/T^+(k)$ in $\mathbf{C}^+$ for $\operatorname{Im} k \ge \sqrt{-Q_{\min}}$.

Theorem 9.3 Assume $Q(x) = 0$ and $P \in L^1_1(\mathbf{R})$. If $\int_{-\infty}^{\infty} dx\, P(x) > 2$, then (1.1) has at least one bound state at $k = i\beta$ for some positive β. If $\int_{-\infty}^{\infty} dx\, |P(x)| \le 2$, then $1/T^+(k)$ has no zeros in $\mathbf{C}^+$.

PROOF: When $\int_{-\infty}^{\infty} dx\, P(x) > 2$, from (5.10) we see that $1/T^+(i\beta)$ is negative at $\beta = 0$ and from (6.4) we see that it is positive as $\beta \to +\infty$. Being a real-valued, continuous function of β, $1/T^+(i\beta)$ must have a zero for some positive β. Now let us prove the second statement. Assume that $k \in \mathbf{C}^+$ corresponds to a bound state; we can transform (1.1) into

$$\varphi(k,x) = \int_{-\infty}^{\infty} dy\, \mathcal{B}(k;x,y)\, \varphi(k,y), \tag{9.6}$$

where we have defined

$$\varphi(k,x) = |P(x)|^{1/2}\, \psi^+(k,x), \quad \mathcal{B}(k;x,y) = \frac{1}{2} e^{ik|x-y|} |P(x)|^{1/2}\, P(y)/|P(y)|^{1/2}.$$

When $P \in L^1(\mathbf{R})$ and $k \in \mathbf{C}^+$, the integral operator in (9.6) is Hilbert-Schmidt with the Hilbert-Schmidt norm

$$||\mathcal{B}||^2_{\text{HS}} = \frac{1}{4} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, |P(x)|\, e^{-2|x-y| \operatorname{Im} k} |P(y)|,$$

and hence, if $k \in \mathbf{C}^+$ and $\int_{-\infty}^{\infty} dx\, |P(x)| \le 2$, we have $||\mathcal{B}||_{\text{HS}} < 1$. Thus, the operator norm of that integral operator is also strictly less than 1 and hence $\varphi = 0$, implying that there cannot be any bound states of (1.1) for $k \in \mathbf{C}^+$. ∎

It is already known[8] that if $Q \in L^1_1(\mathbf{R})$, then the number of bound states for (2.17) is finite; let us denote that number by $\mathcal{N}$, and let $i\kappa_1, \cdots, i\kappa_{\mathcal{N}}$ with $0 < \kappa_1 < \cdots < \kappa_{\mathcal{N}}$ denote the zeros of $1/T^{[0]}(k)$ in $\mathbf{C}^+$. In the following, we generalize the second result of Theorem 9.3 to the case $Q(x) \not\equiv 0$.

Theorem 9.4 Assume $P, Q \in L^1_1(\mathbf{R})$. The k-values in $\mathbf{C}^+$ satisfying $\int_{-\infty}^{\infty} dx\,|ikP(x)+Q(x)| \le 2|k|$ cannot be zeros of $1/T^+(k)$. Moreover, there are no zeros of $1/T^+(k)$ in $\mathbf{C}^+\setminus\{i\kappa_1, \cdots, i\kappa_{\mathcal{N}}\}$ satisfying $|T^{[0]}(k)|\,||P||_{1,1} < 2\,e^{-||Q||_{1,1}}$.

PROOF: Let $k \in \mathbf{C}^+$ correspond to a bound state of (1.1). We can transform (1.1) into (9.6) with

$$\varphi(k,x) = |ikP(x)+Q(x)|^{1/2}\,\psi^+(k,x),$$

$$\mathcal{B}(k;x,y) = \frac{1}{2ik} e^{ik|x-y|} |ikP(x)+Q(x)|^{1/2} \frac{[ikP(y)+Q(y)]}{|ikP(y)+Q(y)|^{1/2}}.$$

A sufficient condition for the absence of the bound state is $||\mathcal{B}||_{\text{HS}} < 1$. Proceeding as in the proof of Theorem 9.3, for $k \in \mathbf{C}^+$ we obtain $||\mathcal{B}||_{\text{HS}} < \dfrac{1}{2|k|}\displaystyle\int_{-\infty}^{\infty} dx\,|ikP(x)+Q(x)|$, and hence there are no zeros of $1/T^+(k)$ at the k-values in $\mathbf{C}^+$ satisfying $\int_{-\infty}^{\infty} dx\,|ikP(x)+Q(x)| \le 2|k|$. In the special case $P(x) \equiv 0$, this implies that there are no bound states when $|k| > \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} dx\,|Q(x)|$. To prove the second part of the theorem, we note that the kernel of the resolvent $[-d^2/dx^2 + Q(x) - k^2]^{-1}$ is given by[20]

$$\mathcal{R}(k;x,y) = \frac{1}{[f_l^{[0]}(k,\cdot); f_r^{[0]}(k,\cdot)]} \left[\theta(y-x)\, f_r^{[0]}(k,x)\, f_l^{[0]}(k,y) + \theta(x-y)\, f_l^{[0]}(k,x)\, f_r^{[0]}(k,y)\right], \tag{9.7}$$

where $f_l^{[0]}(k,x)$ and $f_r^{[0]}(k,x)$ are the Jost solutions of (2.17) and we recall that $\theta(x)$ is the Heaviside function. As seen from (2.26), the Wronskian in (9.7) is equal to $-2ik/T^{[0]}(k)$, and hence we get

$$||ik|P(x)|^{1/2}\mathcal{R}(k;x,y)\,P(y)/|P(y)|^{1/2}||^2_{\text{HS}} = \frac{1}{4}|T^{[0]}(k)|^2\,C(k), \tag{9.8}$$

where we have defined

$$\begin{aligned} C(k) = &\int_{-\infty}^{\infty} dx\,|P(x)|\,|f_l^{[0]}(k,x)|^2 \int_{-\infty}^{x} dy\,|f_r^{[0]}(k,y)|^2\,|P(y)| \\ &+ \int_{-\infty}^{\infty} dx\,|P(x)||f_r^{[0]}(k,x)|^2 \int_{x}^{\infty} dy\,|f_l^{[0]}(k,y)|^2|P(y)|. \end{aligned} \tag{9.9}$$

As in (2.12) we have

$$|f_l^{[0]}(k,x)| \le (1+\max\{0,-x\})\,e^{-x\,\text{Im}k}\,e^{\int_x^{\infty} dy\,(1+|y|)\,|Q(y)|}, \qquad k \in \overline{\mathbf{C}^+}, \tag{9.10}$$

$$|f_r^{[0]}(k,x)| \le (1+\max\{0,x\})\,e^{x\,\text{Im}k}\,e^{\int_{-\infty}^{x} dy\,(1+|y|)\,|Q(y)|}, \qquad k \in \overline{\mathbf{C}^+}. \tag{9.11}$$

Using (9.10) and (9.11) in (9.9), as well as the estimates $e^{-2(x-y)\text{Im}\,k} \le 1$ for $x \ge y$, $e^{-2(y-x)\text{Im}\,k} \le 1$ for $x \le y$, and $1+\max\{0,\pm x\} \le 1+|x|$, we obtain

$$|C(k)| \le ||P||^2_{1,1} e^{2||Q||_{1,1}}. \tag{9.12}$$

From (9.8) and (9.12) we see that the Hilbert-Schmidt norm on the left-hand side of (9.8) is strictly less than 1, provided $|T^{[0]}(k)| < 2e^{-||Q||_{1,1}}/||P||_{1,1}$. Under this condition on k, there is no bound state corresponding to that $k \in \mathbf{C}^+$. ∎

Let us denote the number of bound states of (1.1), i.e. the number of zeros of $1/T^+(k)$ in $\mathbf{C}^+$ (including multiplicities) by $N(P,Q)$. In the next two propositions we obtain some stability results for $N(P,Q)$ under certain perturbations of $P(x)$ and $Q(x)$. As in Section 7, we let $T_j^+(k)$ denote the transmission coefficient corresponding to (7.1) for $j = 1, 2$.

Proposition 9.5 Assume $P_1, P_2 \in L^1(\mathbf{R})$, $Q_1, Q_2 \in L_1^1(\mathbf{R})$, $1/T_1^+(k)$ does not have any real zeros, and $Q_1(x)$ is a generic potential. If $||P_1 - P_2||_1 + ||Q_1 - Q_2||_{1,1}$ is small, i.e. if (9.14) is satisfied, then

(a) $1/T_2^+(k)$ does not have any real zeros.

(b) $N(P_2, Q_2) = N(P_1, Q_1)$.

(c) If all zeros of $1/T_1^+(k)$ are simple and purely imaginary, so are those of $1/T_2^+(k)$.

PROOF: For $a > 0$ let Γ_a be the positively-oriented contour consisting of the interval $[-a, a]$ and the semi-circle $\{k \in \overline{\mathbf{C}^+} : |k| = a\}$, and let us choose a large enough so that all zeros of $1/T_1^+(k)$ in $\mathbf{C}^+$ have absolute value less than a. Putting $F_j(k) = k/[(k+i)T_j^+(k)]$ for $j = 1, 2$, from Propositions 7.4 and 7.5 we get

$$\left|\frac{F_1(k) - F_2(k)}{F_1(k)}\right| \le \frac{C}{|F_1(k)|}\left(||P_1 - P_2||_1 + ||Q_1 - Q_2||_{1,1}\right), \tag{9.13}$$

where $C = \max\{C_1, C_2\}$ with C_1 and C_2 being the constants in Propositions 7.4 and 7.5, respectively. If $1/T_1^+(k)$ does not have any real zeros, then in the generic case $F_1(k)$ cannot have any real zeros; thus, $F_1(k)$ does not vanish on Γ_a. Moreover, by Proposition 4.3, $F_1(k)$ is continuous and bounded in $\overline{\mathbf{C}^+}$. Hence $\min_{k\in\Gamma_a} |F_1(k)| > 0$. Now choosing

$$||P_1 - P_2||_1 + ||Q_1 - Q_2||_{1,1} < \frac{1}{C}\min_{k\in\Gamma_a}\left|\frac{k}{(k+i)\,T_1^+(k)}\right|, \tag{9.14}$$

from (9.13) we obtain

$$\left|\frac{F_1(k) - F_2(k)}{F_1(k)}\right| < 1, \qquad k \in \Gamma_a. \tag{9.15}$$

Hence, from (9.15) we see $F_2(k)$ cannot vanish on Γ_a, which implies (a). Part (b) follows from (9.15) with the use of Rouché's theorem. Part (c) follows by replacing Γ_a with the union of $N(P_1, Q_1)$ small, positively-oriented circles centered at the zeros of $1/T_1^+(k)$ and applying Rouché's theorem. ∎

Proposition 9.6 Assume $Q_1 = Q_2 = Q$ in (7.1), $1/T_1^+(k)$ does not have any real zeros, $Q(x)$ is an exceptional potential, and $P_1, P_2, Q \in L_1^1(\mathbf{R})$. If $||P_1 - P_2||_{1,1}$ is small, i.e. if (9.16) is satisfied, then we have

(a) $1/T_2^+(k)$ does not have any real zeros.

(b) $N(P_2, Q) = N(P_1, Q)$.

(c) If all zeros of $1/T_1^+(k)$ are simple and purely imaginary, so are those of $1/T_2^+(k)$.

PROOF: We will proceed as in the proof of Proposition 9.5. Let us choose Γ_a as in that proof but define $F_j(k) = 1/T_j^+(k)$ for $j = 1, 2$, instead. Note that $F_1(k)$ is bounded, continuous, and nonzero on Γ_a. From (7.18) we have

$$\left|\frac{F_1(k) - F_2(k)}{F_1(k)}\right| \le C\,||P_1 - P_2||_{1,1},$$

where C is the constant defined in Proposition 7.6. From (9.8) we get

$$\left|\frac{F_1(k) - F_2(k)}{F_1(k)}\right| \le \frac{C}{|F_1(k)|}||P_1 - P_2||_{1,1},$$

and hence by choosing

$$||P_1 - P_2||_{1,1} < \frac{1}{C}\min_{k\in\Gamma_a}\left|\frac{1}{T_1^+(k)}\right|, \tag{9.16}$$

and proceeding as in the proof of Proposition 9.5, we complete the proof. ∎

Theorem 9.7 Assume $P, Q \in L^1(\mathbf{R})$ and $P(x) \le 0$. Then, either $N(P,Q)$ and $N(0,Q)$ are both infinite, or they are both finite and $N(P,Q) = N(0,Q)$. Thus, the number of bound states of (1.1) coincides with the number of bound states of (2.17).

PROOF: Since $P(x) \le 0$, by Corollary 9.2 we know that the bound states of (1.1) can only occur when k is on the positive imaginary axis. Let us write (1.1) with $k = i\beta$ as two simultaneous equations:

$$-\psi'' + V(\beta, x)\,\psi = E(\beta)\,\psi, \tag{9.17}$$

$$E(\beta) = -\beta^2, \tag{9.18}$$

where β is considered to be a parameter in the potential $V(\beta, x) = Q(x) - \beta\,P(x)$ of the Schrödinger equation (9.17), and $E(\beta)$ denotes the corresponding energy for each β. Each bound-state energy $-\kappa_j^2$ of (2.17) gives rise to an eigenvalue branch $E_j(\beta)$. From (9.17) we have

$$E'(\beta) = \frac{\langle\psi, -P\psi\rangle}{\langle\psi, \psi\rangle}, \tag{9.19}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $L^2(\mathbf{R})$. If $P(x) \le 0$, from (9.19) we see that $E'(\beta) \ge 0$ and hence each $E_j(\beta)$ is a nondecreasing function of β. Therefore, for $\beta \ge 0$, the graph of $E_j(\beta)$ must intersect the parabola $E = -\beta^2$ at exactly one point, say $(\beta_j, -\beta_j^2)$, and each $E_j(\beta)$ gives rise to exactly one solution of (9.18). The number $N(P,Q)$ is equal to the number of intersections of the eigenvalue branches $E_j(\beta)$ with the parabola given in (9.18) for $j \ge 1$. Since each of the $N(0,Q)$ branches is responsible for exactly one intersection, we conclude that $N(P,Q) = N(0,Q)$. Note that if $Q \in L^1(\mathbf{R})$ but $Q \notin L_1^1(\mathbf{R})$, it is possible that $N(0,Q) = +\infty$, but then we also have $N(P,Q) = +\infty$. ∎

If $P,Q\in L^1(\mathbf{R})$ and $P(x)\le 0$, then either $N(P,Q)=+\infty$, in which case the set of bound-state energies of (1.1) consists of a strictly decreasing sequence of negative numbers converging to 0, or $N(P,Q)$ is finite and equal to $\mathcal{N}$, in which case we let $k=i\beta_j$ for $j=1,\cdots,\mathcal{N}$ with $0<\beta_1<\cdots<\beta_{\mathcal{N}}$ denote the zeros of $1/T^+(k)$ in $\mathbf{C}^+$. Since the condition $Q\in L^1_1(\mathbf{R})$ guarantees the finiteness of $N(0,Q)$, from Theorem 9.1 we obtain the following:

Corollary 9.8 Assume $P(x)\le 0$, $Q(x)\ge 0$, $P\in L^1(\mathbf{R})$, and $Q\in L^1_1(\mathbf{R})$. Then, there are no zeros of $1/T^+(k)$ in $\overline{\mathbf{C}^+}$.

In the next theorem, when $P(x)\le 0$, using the constant $P_{\min}$ defined in (9.1), we obtain some upper and lower bounds on each bound-state energy of (1.1).

Theorem 9.9 Assume $N(0,Q)$ is finite and nonzero, $P_{\min}$ is finite, $P(x)\le 0$, and $P,Q\in L^1(\mathbf{R})$; let $k=i\kappa_j$ correspond to the bound states of (2.17) for $j=1,\cdots,\mathcal{N}$. Then, the zeros of $1/T^+(k)$ in $\mathbf{C}^+$ occur at $k=i\beta_j$ satisfying $\beta_*\le\beta_j\le\kappa_j$ for $j=1,\cdots,\mathcal{N}$, where $\beta_*=P_{\min}/2+\sqrt{P_{\min}^2/4+\kappa_1^2}$. In particular, $\beta_1\ge\beta_*$ and $\beta_{\mathcal{N}}\le\kappa_{\mathcal{N}}$, with the equalities holding if and only if $P(x)=0$.

PROOF: At a bound state with $k=i\beta_j$ of (1.1), replacing k_0 in (9.5) by $0+i\beta_j$, we get

$$\int_{-\infty}^{\infty}dx\, f_l^{+\prime}(i\beta_j,x)^2=\int_{-\infty}^{\infty}dx\,[-\beta_j^2+\beta_jP(x)-Q(x)]\,f_l^+(i\beta_j,x)^2. \tag{9.20}$$

On the other hand, since $-\kappa_{\mathcal{N}}^2$ is the lowest bound-state energy for (2.17), we have

$$-\kappa_{\mathcal{N}}^2\le\frac{\int_{-\infty}^{\infty}dx\,[f_l^{+\prime}(i\beta_j,x)^2+Q(x)\,f_l^+(i\beta_j,x)^2]}{\int_{-\infty}^{\infty}dx\,f_l^+(i\beta_j,x)^2}, \tag{9.21}$$

with the equality holding if and only if $f_l^{[0]}(i\kappa_{\mathcal{N}},x)$ and $f_l^+(i\beta_j,x)$ are linearly dependent. From (9.20) and (9.21) we obtain

$$-\kappa_{\mathcal{N}}^2\le-\beta_j^2+\beta_j\frac{\int_{-\infty}^{\infty}dx\,P(x)\,f_l^+(i\beta_j,x)^2}{\int_{-\infty}^{\infty}dx\,f_l^+(i\beta_j,x)^2}. \tag{9.22}$$

Since $P(x)\le 0$, from (9.22) we see that $\beta_j\le\kappa_{\mathcal{N}}$ with the equality holding if and only if $P(x)=0$ and $j=\mathcal{N}$. Thus, $\beta_j\in(0,\kappa_{\mathcal{N}}]$ for $j=1,\cdots,\mathcal{N}$. Now let us improve the bounds on β_j. From the proof of Theorem 9.7, recall that each eigenvalue branch $E_j(\beta)$ gives rise to exactly one solution of (9.18) starting with $-\kappa_j^2$ at $\beta=0$ and ending with $-\beta_j^2$ at $\beta=\beta_j$. Since $E_j(\beta)$ is an increasing function of β, we get $-\kappa_j^2=E_j(0)\le E_j(\beta_j)=-\beta_j^2$, and hence $\beta_j\le\kappa_j$. Now consider $E_1(\beta)$, the eigenvalue branch corresponding to β_1. From (9.19) using $P_{\min}\le P(x)$, we obtain $0\le E_1'(\beta)\le -P_{\min}$; more specifically, $0<E_1'(\beta)<-P_{\min}$ unless $P(x)=0$. Since $E_1(0)=-\kappa_1^2$ and $E_1(\beta)$ is nondecreasing, we get $E_1(\beta)\le-\beta P_{\min}-\kappa_1^2$. Thus, from the inequality $-\beta_1^2\le-\beta_1P_{\min}-\kappa_1^2$, we get $\beta_1\ge\beta_*$. Note that the equality in $\beta_1\ge\beta_*$ holds if and only if $P(x)=0$ because $\beta_*\ge\kappa_1$ with the equality holding if and only if $P(x)\ge 0$. ∎

From the proof of Theorem 9.9 we get the following:

Corollary 9.10 Assume $P, Q \in L^1(\mathbf{R})$, $P(x) \le 0$, and $N(0,Q) = +\infty$, and let $\{\mathcal{E}_j\}$ and $\{\mathcal{E}_j^{[0]}\}$ for $j \ge 1$ denote the bound-state energies of (1.1) and (2.17), respectively, ordered such that $\mathcal{E}_j < \mathcal{E}_{j+1}$ and $\mathcal{E}_j^{[0]} < \mathcal{E}_{j+1}^{[0]}$. Then, we have $\mathcal{E}_j^{[0]} \le \mathcal{E}_j < 0$ for $j \ge 1$, and hence the bound-state energies of (1.1) cannot occur below the lowest bound-state energy of (2.17).

Recall that the Levinson theorem[21] relates the number of bound states for the Schrödinger equation to the change in the phase of the transmission coefficient. Next we generalize the Levinson theorem to (1.1).

Theorem 9.11 Assume that $P \in L^1(\mathbf{R})$ in the generic case and $P \in L^1_1(\mathbf{R})$ in the exceptional case and that $Q \in L^1_1(\mathbf{R})$, and suppose $1/T^+(k)$ does not have any real zeros. Then, the number of bound states of (1.1) is related to the principal argument of $T^+(k)$ as

$$\arg T^+(0+) = \pi \left[N(P,Q) - \frac{d}{2} \right], \tag{9.23}$$

where $d = 0$ in the exceptional case and $d = 1$ in the generic case.

PROOF: For $b > a > 0$, let $\Gamma_{a,b}$ be the positively-oriented contour consisting of the circular arcs $\{k \in \overline{\mathbf{C}^+} : |k| = a\}$ and $\{k \in \overline{\mathbf{C}^+} : |k| = b\}$ and the segments $[-b, -a]$ and $[a, b]$. Let us choose a and b so that all zeros of $1/T^+(k)$ in $\mathbf{C}^+$ are enclosed by $\Gamma_{a,b}$. By the argument principle we have

$$N(P,Q) = -\frac{1}{2\pi i} \int_{\Gamma_{a,b}} dk\, \frac{\dot{T}(k)}{T(k)} = -\frac{1}{2\pi}\, \Delta_{\Gamma_{a,b}} \left[\arg T(k)\right], \tag{9.24}$$

where $\Delta_{\Gamma_{a,b}}[\arg T(k)]$ indicates the change in the argument of $T(k)$ when $\Gamma_{a,b}$ is traversed once. This change is independent of a and b, and hence we evaluate it by letting $a \to 0$ and $b \to +\infty$. By Theorem 6.1 the contribution to that change from the large semicircle $\{k \in \overline{\mathbf{C}^+} : |k| = b\}$ vanishes as $b \to +\infty$. In view of (5.2) and (5.3), we see that the contribution from the small semicircle $\{k \in \overline{\mathbf{C}^+} : |k| = a\}$ in the limit $a \to 0$ is equal to 0 in the exceptional case and $-\pi$ in the generic case. The contribution from the interval $(0, +\infty)$ is given by

$$\arg T^+(+\infty) - \arg T^+(0+) = -\arg T^+(0+), \tag{9.25}$$

provided we define $\arg T^+(+\infty) = 0$, which, according to Theorem 6.1, amounts to taking the principle value of the argument. By (4.14), the contribution from the interval $(-\infty, 0)$ is the same as (9.25). Hence, the right-hand side in (9.24) is equal to $\dfrac{1}{\pi} \arg T^+(0+)$ in the exceptional case and $\dfrac{1}{2} + \dfrac{1}{\pi} \arg T^+(0+)$ in the generic case, which gives us (9.23). ∎

Finally, let us show that in the special case when $P(x)$ and $Q(x)$ have support on a half-line, we can relate the poles of the transmission coefficient to the poles of a reflection coefficient. Since there is no loss of generality in choosing our half-lines as $\mathbf{R}^+ = (0, +\infty)$ or $\mathbf{R}^- = (-\infty, 0)$ instead of $(a, +\infty)$ or $(-\infty, b)$, respectively, for some constants a and b, we will state the following proposition using $\mathbf{R}^\pm$.

Proposition 9.12 Assume $P(x) = Q(x) = 0$ for $x \in \mathbf{R}^-$ and $P, Q \in L^1(\mathbf{R}^+)$. Then $L^+(k)$ is meromorphic in $\mathbf{C}^+$ having poles coinciding with the poles of $T^+(k)$. Furthermore, none of the zeros of $L^+(k)$ coincide with the poles of $T^+(k)$ in $\mathbf{C}^+$. These assertions remain valid if $\mathbf{R}^-$ and $L^+(k)$ are replaced by $\mathbf{R}^+$ and $R^+(k)$, respectively.

PROOF: If $P(x) = Q(x) = 0$ for $x \in \mathbf{R}^-$, from Proposition 2.1, we see that $f_l^+(k,0)$ and $f_l^{+\prime}(k,0)$ are analytic in $\mathbf{C}^+$. Hence, using (1.3) we can conclude that $L^+(k)/T^+(k)$ is analytic in $\mathbf{C}^+$, allowing us to conclude that the poles of $L^+(k)$ and $T^+(k)$ must coincide in $\mathbf{C}^+$. Since $f_l^+(k,0)$ and $f_l^{+\prime}(k,0)$ cannot vanish simultaneously, it follows that $1/T^+(k)$ and $L^+(k)/T^+(k)$ cannot vanish simultaneously in $\mathbf{C}^+$, and hence the zeros of $L^+(k)$ and the poles of $T^+(k)$ cannot coincide in $\mathbf{C}^+$. The proof when $P(x)$ and $Q(x)$ have support in $\mathbf{R}^-$ is obtained in a similar manner. ∎

10. EXAMPLES

In this section we illustrate the number and location of the poles of $T^+(k)$ in $\mathbf{C}^+$ with some explicit examples. In Example 10.3 we present an explicit case where the corresponding $T^+(k)$ has a double pole on the positive imaginary axis. Thus, the bound states of (1.1) are not necessarily simple unless $P(x) \le 0$.

Example 10.1 Let $P(x) = b$ in $(0,1)$ and zero elsewhere and let $Q(x) = a$ in $(0,1)$ and zero elsewhere, where a, b are some real parameters. We can explicitly solve the direct scattering problem and obtain

$$\frac{1}{T^+(k)} = e^{ik}\left[\cos s + \frac{k^2+s^2}{2iks}\sin s\right], \qquad e^{-ik}\frac{L^+(k)}{T^+(k)} = e^{ik}\frac{R^+(k)}{T^+(k)} = \frac{k^2-s^2}{2iks}\sin s, \tag{10.1}$$

where we have defined $s = \sqrt{k^2 - ibk - a}$. The Jost solutions of (1.1) are given by

$$f_l^+(k,x) = \begin{cases} e^{ik(1+x)}\cos s + e^{ik}\dfrac{\sin s}{2iks}\left[(k^2+s^2)e^{ikx} + (k^2-s^2)e^{-ikx}\right], & x \le 0, \\ e^{ik}\left[\cos s(1-x) - \dfrac{ik}{s}\sin s(1-x)\right], & x \in [0,1], \\ e^{ikx}, & x \ge 1, \end{cases}$$

$$f_r^+(k,x) = \begin{cases} e^{-ikx}, & x \le 0, \\ \cos sx - \dfrac{ik}{s}\sin sx, & x \in [0,1], \\ e^{ik(1-x)}\cos s + \dfrac{\sin s}{2iks}\left[(k^2+s^2)e^{ik(1-x)} + (k^2-s^2)e^{ik(x-1)}\right], & x \ge 1. \end{cases}$$

When $b = 0$ and $a < 0$, we have a square-well potential, and in this case it is already known from quantum mechanics that (1.1) has $\mathcal{N}$ bound states whenever

$$(\mathcal{N}-1)\pi < \sqrt{-a} \le \mathcal{N}\pi. \tag{10.2}$$

The following numerical values were obtained by using the mathematical software Maple. When $a = -100$ and $b = 0$, from (10.2) we have $\mathcal{N} = 4$. The four bound states occur at $k = i\kappa_j$ where

$$\kappa_1 = 1.9\overline{3}, \quad \kappa_2 = 6.4\overline{1}, \quad \kappa_3 = 8.5\overline{5}, \quad \kappa_4 = 9.6\overline{5}. \tag{10.3}$$

When $a = -100$ and $b = -10$, there are four bound states at $k = i\beta_j$, where

$$\beta_1 = 0.7\overline{6}, \quad \beta_2 = 3.5\overline{5}, \quad \beta_3 = 5.1\overline{1}, \quad \beta_4 = 5.9\overline{2}.$$

When $a = -100$ and $b = -100$, there are still four bound states with

$$\beta_1 = 0.1\overline{1}, \quad \beta_2 = 0.5\overline{8}, \quad \beta_3 = 0.8\overline{6}, \quad \beta_4 = 0.9\overline{7}. \tag{10.4}$$

Comparing (10.4) with (10.3), we see that when $a = -100$ and $b = -100$, all four $\beta_j \in (0, \kappa_1)$. As we make b more negative the bound-state energies are pushed toward zero. Now let us see what happens when $b > 0$. By Theorem 9.3, if $a = 0$ and $b > 2$, we must have a bound state at $k = i\beta$ for some positive β. Letting $a = 0, b = 21/10$, we obtain a bound state at $k = 0.1\overline{5}i$. Choosing $a = 0$ and $b = 10$, we obtain three bound states at $k = i\beta_j$ with

$$\beta_1 = 2.1\overline{4}, \quad \beta_2 = 5.9\overline{6}, \quad \beta_3 = 9.2\overline{7}. \tag{10.5}$$

Choosing $a = 0, b = 100$, when k is on the positive imaginary axis we obtain thirty-one bound states with

$$\beta_1 = 0.1\overline{0}, \quad \beta_2 = 0.4\overline{1}, \quad \beta_3 = 0.9\overline{3}, \quad \beta_4 = 1.6\overline{7}, \quad \beta_5 = 2.6\overline{4}, \quad \beta_6 = 3.8\overline{5}, \quad \beta_7 = 5.3\overline{3},$$

$$\beta_8 = 7.0\overline{9}, \quad \beta_9 = 9.1\overline{9}, \quad \beta_{10} = 11.6\overline{9}, \quad \beta_{11} = 14.6\overline{3}, \quad \beta_{12} = 18.2\overline{0}, \quad \beta_{13} = 22.6\overline{1},$$

$$\beta_{14} = 28.4\overline{3}, \quad \beta_{15} = 37.6\overline{3}, \quad \beta_{16} = 60.4\overline{1}, \quad \beta_{17} = 69.6\overline{9}, \quad \beta_{18} = 75.6\overline{0}, \quad \beta_{19} = 80.1\overline{1},$$

$$\beta_{20} = 83.7\overline{7}, \quad \beta_{21} = 86.8\overline{3}, \quad \beta_{22} = 89.4\overline{2}, \quad \beta_{23} = 91.6\overline{3}, \quad \beta_{24} = 93.5\overline{2}, \quad \beta_{25} = 95.1\overline{2},$$

$$\beta_{26} = 96.4\overline{6}, \quad \beta_{27} = 97.5\overline{7}, \quad \beta_{28} = 98.4\overline{6}, \quad \beta_{29} = 99.1\overline{4}, \quad \beta_{30} = 99.6\overline{2}, \quad \beta_{31} = 99.9\overline{1}.$$

Note that the bound states may occur even when $a > 0$ and $b > 0$. For example, when $a = 1$ and $b = 10$, we obtain four bound states on the positive imaginary axis with

$$\beta_1 = 0.1\overline{3}, \quad \beta_2 = 2.5\overline{0}, \quad \beta_3 = 5.6\overline{3}, \quad \beta_4 = 9.1\overline{6}.$$

Note that when $b > 0$ we cannot exclude the possibility of bound states at certain k-values off the positive imaginary axis. Thus, in addition to the bound states listed in this example, there may be some bound states with complex energies. For example, when $a = -93/10$ and $b = 4$, we find bound states at $k = \pm 0.976\overline{4} + 0.023\overline{3}i$. Note also that there may be zeros of $1/T^+(k)$ for real k; for example, when $a = -9.273\overline{8}$ and $b = 3.970\overline{8}$, we obtain a zero of $1/T^+(k)$ at $k = 1$.

Example 10.2 Let

$$\eta_l^{\pm}(k, x) = 1 - \frac{2i(1 \pm b)\epsilon}{k + i\epsilon} \frac{ce^{-2\epsilon x}}{1 + ce^{-2\epsilon x}}, \qquad x \geq 0, \tag{10.6}$$

$$\eta_r^{\pm}(k, x) = 1 - \frac{2i(1 \pm b)\epsilon}{k + i\epsilon} \frac{ce^{2\epsilon x}}{1 + ce^{2\epsilon x}}, \qquad x \leq 0, \tag{10.7}$$

where c, ϵ are positive parameters and b is a real parameter. Using (1.1), (1.2), (3.3), (3.4), (10.6), and (10.7), we obtain

$$P(x) = \frac{4b\epsilon ce^{-2\epsilon|x|}}{1 + ce^{-2\epsilon|x|}}, \tag{10.8}$$

$$Q(x) = \frac{4\epsilon^2 c e^{-2\epsilon|x|}[-3b - 2 + b^2 c e^{-2\epsilon|x|}]}{(1 + ce^{-2\epsilon|x|})^2}. \tag{10.9}$$

Note that $P(x)$ is continuous at $x = 0$ with $P(0) = 4b\epsilon c/(1+c)$, and both $P(x)$ and $Q(x)$ are even functions. Furthermore, the sign of $P(x)$ is the same as the sign of b. Writing (10.8) as

$$P(x) = -2b\theta(x)\frac{d}{dx}\ln(1 + ce^{-2\epsilon x}) + 2b\theta(-x)\frac{d}{dx}\ln(1 + ce^{2\epsilon x}),$$

we obtain

$$\frac{1}{2}\int_x^\infty dz\, P(z) = \ln(1 + ce^{-2\epsilon x})^b, \qquad x \geq 0,$$

$$\frac{1}{2}\int_{-\infty}^x dz\, P(z) = \ln(1 + ce^{2\epsilon x})^b, \qquad x \leq 0.$$

Hence, we have $e^p = (1+c)^{2b}$. Using (6.1), (6.4), and (6.5), we get

$$\frac{2ike^{\pm p}}{T^\pm(k)} = \left(2ik \pm \frac{4b\epsilon c}{1+c}\right)\left(1 - \frac{2i(1 \pm b)\epsilon}{k + i\epsilon}\frac{c}{1+c}\right)^2 + \frac{4i\epsilon}{k + i\epsilon}\frac{2(1 \pm b)\epsilon c}{(1+c)^2}\left(1 - \frac{2i(1 \pm b)\epsilon}{k + i\epsilon}\frac{c}{1+c}\right),$$

$$-2ik\frac{L^\pm(k)}{T^\pm(k)} = 2ik\frac{R^\mp(-k)}{T^\mp(-k)} = \frac{8(1 \pm b)\epsilon^3 c(-1 + c \pm 2bc)}{(k + i\epsilon)(k - i\epsilon)(1+c)^3}.$$

Hence, we obtain

$$T^\pm(k) = \frac{k(k + i\epsilon)^2 e^{\pm p}}{(k - k_1^\pm)(k - k_2^\pm)(k - k_3^\pm)}, \tag{10.10}$$

where we have defined

$$k_1^\pm = i\frac{\epsilon}{1+c}[-1 + c \pm 2bc], \tag{10.11}$$

$$k_2^\pm = \frac{i\epsilon}{2(1+c)}\left[(-1 + c \pm 4bc) + \sqrt{1 + c^2 + 14c \pm 16bc}\right], \tag{10.12}$$

$$k_3^\pm = \frac{i\epsilon}{2(1+c)}\left[(-1 + c \pm 4bc) - \sqrt{1 + c^2 + 14c \pm 16bc}\right]. \tag{10.13}$$

Let us now analyze the poles of $T^+(k)$. From (10.11)-(10.13) it is seen that ϵ appears as a multiplicative factor in the poles of $T^+(k)$. We can divide the half-plane $\{(c, b) : c > 0\}$ into four separate regions by using the three nonintersecting curves $b = \Gamma_+(c)$, $b = \Gamma_0(c)$, and $b = \Gamma_-(c)$, where

$$\Gamma_0(c) = \frac{1-c}{2c}, \qquad \Gamma_\pm(c) = -\frac{c-3}{4c} \pm \sqrt{\frac{(c-3)^2}{16c^2} + \frac{1}{c}}.$$

The exceptional case occurs on these three curves; note that $k_3^+ = 0$ on Γ_+, $k_2^+ = 0$ on Γ_-, and $k_1^+ = 0$ on Γ_0. The number of bound states changes by one as we cross each of these three curves; otherwise, we are in the generic case. Note that:

(i) If $b > \Gamma_+(c)$, then k_1^+, k_2^+, and k_3^+ all lie on the positive imaginary axis, and hence we have three bound states.

(ii) If $\Gamma_0(c) < b \le \Gamma_+(c)$, then k_1^+ and k_2^+ lie on the positive imaginary axis, but $k_3^+ \notin \mathbf{C}^+$; hence we have two bound states.

(iii) If $\Gamma_-(c) < b \le \Gamma_0(c)$, then there is exactly one bound state because k_2^+ lies on the positive imaginary axis but k_1^+ and k_3^+ are not in $\mathbf{C}^+$.

(iv) There are no bound states when $b \le \Gamma_-(c)$ because none of k_1^+, k_2^+, and k_3^+ lie in $\mathbf{C}^+$. In this case, k_1^+ is always located on the imaginary axis; k_2^+ and k_3^+ lie on the imaginary axis when $b \ge -(1/c + c + 14)/16$ and they lie in $\mathbf{C}^-$ symmetrically located with respect to the imaginary axis.

Next we present an example where $T^+(k)$ has a double pole on the positive imaginary axis.

Example 10.3 Consider the functions $P(x)$ and $Q(x)$ given in (10.8) and (10.9), respectively, with $\epsilon > 0$, but $b = -(c^2 + 14c + 1)/(16c)$ and $c \in (-1, -5 + \sqrt{20})$. Thus, both c and b are negative, and as seen from (10.8) we have $P(x) > 0$. From (10.12) and (10.13) we see that $k_2^+ = k_3^+ = -i(c^2 + 10c + 5)/[8(1 + c)]$, and hence $T^+(k)$ given in (10.10) has a double pole on the positive imaginary axis for any $c \in (-1, -5 + \sqrt{20})$. Note also that when $b = (1 - c)/(4c)$ and $c \in (-1, -5 + \sqrt{20})$, although k_1^+ is located on the negative imaginary axis, k_2^+ and k_3^+ are symmetrically located on the real axis; thus, in this case $T^+(k)$ has poles on the real axis. When $b = (1 - c)/4$ and $c = -5 + \sqrt{20}$, both k_2^+ and k_3^+ become equal to zero, and hence we get a simple pole for $T^+(k)$ at $k = 0$; this corresponds to the exceptional case.

11. EIGENVALUE CURVES AND ZEROS OF JOST SOLUTIONS

In this section we study the zeros of the Jost solutions of (1.1) for a fixed $k \in \overline{\mathbf{C}^+}$ and analyze the number of such zeros in relation to the bound states of (1.1) and (2.17). As in Section 9, we let $N(P, Q)$ denote the number of bound states of (1.1). When $P(x) \le 0$ we show that the number of zeros of the Jost solutions of (1.1) is related to $N(P, Q)$ in a simple manner, and we present some examples showing that this relation does not hold in general. We establish the connection between the results of Section 8 on Jordan chains and certain zeros of the Jost solutions of (1.1). This connection uses the eigenvalue branches introduced in the proof of Theorem 9.7. We also show that the number of bound states of (1.1) with real energies is greater than or equal to $N(0, Q)$.

In the first proposition we collect some results about the oscillation properties of solutions of generalized Schrödinger equations related by inequalities involving the coefficients. Although the methods for proving such results are familiar,[20,22] we include a proof for the convenience of the reader.

Consider the pair of generalized Schrödinger equations

$$\chi_j''(\mu, x) - \mu^2 \chi_j(\mu, x) = V_j(\mu, x)\,\chi_j(\mu, x), \qquad \mu \ge 0, \qquad j = 1, 2. \tag{11.1}$$

Note that if we let $V_j(\mu, x) = -\mu P(x) + Q(x)$ in (11.1), we get (1.1) for $k = i\mu$.

Proposition 11.1 Assume $V_j(\mu, \cdot) \in L^1(\mathbf{R})$ if $\mu > 0$, $V_j(0, \cdot) \in L^1_1(\mathbf{R})$, and $V_1(\mu_2, x) \leq V_2(\mu_1, x)$ if $0 \leq \mu_1 \leq \mu_2$. Let $\chi_1(\mu_1, x)$ and $\chi_2(\mu_2, x)$ denote two nontrivial solutions of (11.1) with the corresponding coefficients $V_1(\mu_1, x)$ and $V_2(\mu_2, x)$, respectively. Then:

(i) Suppose $\chi_2(\mu_2, x)$ has two successive zeros a and b with $a < b$. If $0 \leq \mu_1 < \mu_2$, then $\chi_1(\mu_1, x)$ has at least one zero in (a, b). If $0 \leq \mu_1 = \mu_2$ and $V_1(\mu_1, x) \not\equiv V_2(\mu_1, x)$ on (a, b), then $\chi_1(\mu_1, x)$ has at least one zero in (a, b). If $0 \leq \mu_1 = \mu_2$, $V_1(\mu_1, x) \equiv V_2(\mu_1, x)$ on (a, b), and $\chi_1(\mu_1, x)$ and $\chi_2(\mu_1, x)$ are linearly independent in (a, b), then $\chi_1(\mu_1, x)$ has exactly one zero in (a, b).

(ii) Suppose $\chi_2(\mu_2, x)$ remains bounded as $x \to +\infty$. Let a denote the largest zero of $\chi_2(\mu_2, x)$, and set $b = +\infty$. Then the assertions of (i) remain true if we replace the interval (a, b) by $(a, +\infty)$.

(iii) If $0 \leq \mu_1 \leq \mu_2$, $\chi_2(\mu_2, x)$ is bounded as $x \to +\infty$, and $\chi_1(\mu_1, x)$ has no zeros in $\mathbf{R}$, then $\chi_2(\mu_2, x)$ has no zeros in $\mathbf{R}$ either.

(iv) If $\chi_2(\mu_2, x)$ is bounded as $x \to -\infty$ and a is the smallest zero of $\chi_2(\mu_2, x)$, then the assertion of (iii) holds, and the assertions in (i) remain true if we replace the interval (a, b) by $(-\infty, a)$.

PROOF: We omit the proof of (i) since on a finite interval such results are known (e.g. Theorem 1.1 on p. 208 of Ref. 20). Moreover, our proof of (ii) is easily modified to prove (i). The proof of (iv) is analogous to the proofs of (ii) and (iii), and hence we will only prove (ii) and (iii).

(ii) The proof can be given using contradiction. Without loss of generality we may assume that $\chi_1(\mu_1, x)$ and $\chi_2(\mu_2, x)$ are strictly positive in $(a, +\infty)$. When $b > a$, where a is the largest zero of $\chi_2(\mu_2, x)$, from (11.1) we get

$$\begin{aligned} &\chi_2(\mu_2, b)\, \chi_1'(\mu_1, b) - \chi_2'(\mu_2, b)\, \chi_1(\mu_1, b) + \chi_2'(\mu_2, a)\, \chi_1(\mu_1, a) \\ &\qquad = \int_a^b dx\, [V_1(\mu_1, x) - V_2(\mu_2, x) + \mu_1^2 - \mu_2^2]\, \chi_1(\mu_1, x)\, \chi_2(\mu_2, x). \end{aligned} \tag{11.2}$$

Note that, by the asymptotic properties of the solutions [cf. (8.8)-(8.11)] and their assumed positivity in $(a, +\infty)$, we have for $\mu_2 > 0$ and some $c_2 > 0$

$$\chi_2(\mu_2, x) = c_2 e^{-\mu_2 x} + o(e^{-\mu_2 x}), \qquad \chi_2'(\mu_2, x) = -c_2 \mu_2 e^{-\mu_2 x} + o(e^{-\mu_2 x}), \qquad x \to +\infty. \tag{11.3}$$

Furthermore, if $\chi_1(\mu_1, x)$ is unbounded as $x \to +\infty$, then for some $c_1 > 0$ we have

$$\chi_1(\mu_1, x) = c_1 e^{\mu_1 x} + o(e^{\mu_1 x}), \qquad \chi_1'(\mu_1, x) = c_1 \mu_1 e^{\mu_1 x} + o(e^{\mu_1 x}), \qquad x \to +\infty. \tag{11.4}$$

If $\mu_1 = 0$ and $\chi_1(0, x)$ is unbounded as $x \to +\infty$, then for some $\tilde{c}_1 > 0$

$$\chi_1(0, x) = \tilde{c}_1 x + o(x), \qquad \chi_1'(0, x) = \tilde{c}_1 + o(1), \qquad x \to +\infty. \tag{11.5}$$

If $\chi_2(0,x)$ is bounded as $x \to +\infty$, then for some $\tilde{c}_2 > 0$ we have

$$\chi_2(0,x) = \tilde{c}_2 + o(1), \qquad \chi_2'(0,x) = o\left(\frac{1}{x}\right), \qquad x \to +\infty. \tag{11.6}$$

Using (11.3)-(11.6) we will let $b \to +\infty$ in (11.2). When $0 \le \mu_1 \le \mu_2$, the limit as $b \to +\infty$ of the right-hand side of (11.2) exists and is nonpositive; it is equal to zero precisely when $\mu_1 = \mu_2$ and $V_1(\mu_1,x) \equiv V(\mu_2,x)$ on $(a,+\infty)$. If $0 \le \mu_1 < \mu_2$, the limit of the left-hand side of (11.2) is equal to $\chi_2'(\mu_2,a)\,\chi_1(\mu_1,a)$, which is nonnegative. Hence we have a contradiction, and thus $\chi_1(\mu_1,x)$ must have a zero in $(a,+\infty)$. If $0 < \mu_1 = \mu_2$ and $\chi_1(\mu_1,x)$ is unbounded, then the limit on the left-hand side of (11.2) is equal to $2c_1c_2\mu_1 + \chi_2'(\mu_1,a)\,\chi_1(\mu_1,a)$, which is strictly positive; the right-hand side is nonpositive and so again we have a contradiction. If $0 < \mu_1 = \mu_2$ and $\chi_1(\mu_1,x)$ is bounded, then the limit of the left-hand side is equal to $\chi_2'(\mu_2,a)\,\chi_1(\mu_1,a)$, which is nonnegative. If also $V_1(\mu_1,x) \not\equiv V_2(\mu_2,x)$ on $(a,+\infty)$, then the right-hand side is strictly negative and we have a contradiction. If $V_1(\mu_1,x) \equiv V_2(\mu_2,x)$ on $(a,+\infty)$, then $\chi_1(\mu_1,a) > 0$ due to the linear independence of $\chi_1(\mu_1,x)$ and $\chi_2(\mu_2,x)$, and so the left-hand side of (11.2) is strictly positive while its right-hand side is zero. In this case, by (i), there can only be one zero of $\chi_1(\mu_1,x)$ in $(a,+\infty)$. If $0 = \mu_1 = \mu_2$ and $\chi_1(0,x)$ is unbounded, then because of (11.6) the left-hand side of (11.2) approaches $\tilde{c}_1\,\tilde{c}_2 + \chi_2'(0,a)\,\chi_1(0,a)$, which is again strictly positive. If $\mu_2 = \mu_1 = 0$ and $\chi_1(0,x)$ is bounded, then the limit of the left-hand side of (11.2) is $\chi_2'(0,a)\,\chi_1(0,a)$, which is nonnegative. If $V_1(0,x) \not\equiv V_2(0,x)$ on $(a,+\infty)$, then the right-hand side of (11.2) is strictly negative, while if $V_1(0,x) \equiv V_2(0,x)$ on $(a,+\infty)$, then its right-hand side is zero and its left-hand side is strictly positive due to the linear independence of $\chi_1(0,x)$ and $\chi_2(0,x)$. In both cases we arrive at a contradiction. As in (i), if $V_1(0,x) \equiv V_2(0,x)$ on $(a,+\infty)$, we conclude that there is exactly one zero of $\chi_1(0,x)$ in $(a,+\infty)$.

(iii) Suppose $\chi_2(\mu_2,x)$ does have some zeros, the largest of which is a. Then, under the assumptions made in (i) and (ii), it follows that $\chi_1(\mu_1,x)$ has a zero to the right of a, contradicting the assumptions of (iii). The only situation not covered by (i) and (ii) is when $\mu_2 = \mu_1$, $V_1(\mu_1,x) \equiv V_2(\mu_2,x)$ on $(a,+\infty)$, and $\chi_1(\mu_1,x)$ and $\chi_2(\mu_1,x)$ are linearly dependent in $(a,+\infty)$, but then $\chi_2(\mu_1,a) = 0$ implies $\chi_1(\mu_1,a) = 0$, which is again a contradiction. ∎

From Proposition 11.1 we obtain the following:

Corollary 11.2 Assume $P, Q \in L^1(\mathbf{R})$ and let β be a positive constant. Then $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ have the same number of zeros. If $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ are linearly independent and if they have two or more zeros, their zeros are separated, i.e. between two successive zeros of $f_l^+(i\beta,x)$ there is a zero of $f_r^+(i\beta,x)$ and vice versa. If $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ are linearly independent and if they have at least one zero, to the right of the largest zero of $f_l^+(i\beta,x)$ there exists a zero of $f_r^+(i\beta,x)$, and to the left of the smallest zero of $f_r^+(i\beta,x)$ there exists a zero of $f_l^+(i\beta,x)$. A bound state at $k = i\beta$ occurs if and only if the zeros of $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ coincide.

Our next result concerns the zeros of the Jost solutions of (2.17). Since some theorems of this type have already been proved elsewhere (see e.g. Theorem 14.10 of Ref. 22 or Theorem XIII.8 on p. 90 of Ref. 23), we only comment on certain details that may not be obvious from those references. Recall that $N(0,Q)$ denotes the number of bound states of (2.17).

Proposition 11.3 (i) Suppose $Q \in L^1(\mathbf{R})$ and $\beta > 0$. Then the number of zeros of $f_l^{[0]}(i\beta, x)$ is equal to the number of bound states of (2.17) with energies contained in the interval $(-\infty, -\beta^2)$.

(ii) Suppose further that $Q \in L_1^1(\mathbf{R})$. Then, the number of zeros of $f_l^{[0]}(0,x)$ is equal to $N(0,Q)$.

PROOF: (i) Since we only assume $Q \in L^1(\mathbf{R})$, there may be infinitely many bound states of (2.17) with energies accumulating at zero. All such energies are negative, and let us denote them by $-\gamma_j^2$ with $\gamma_j > \gamma_{j+1} > 0$ for $j \geq 1$. It is known (Theorem 14.10 of Ref. 22) that $f_l^{[0]}(i\gamma_j, x)$ has exactly $(j-1)$ zeros. Hence, we only need to consider the zeros of $f_l^{[0]}(i\beta, x)$ when β is not equal to any γ_j. If $\beta > \gamma_1$, then from Proposition 11.1 (iii) with $V_1 = V_2 = Q$, $\mu_1 = \gamma_1$, $\mu_2 = \beta$, $\chi_1(\mu_1, x) = f_l^{[0]}(i\gamma_1, x)$, and $\chi_2(\mu_2, x) = f_l^{[0]}(i\beta, x)$, it follows that $f_l^{[0]}(i\beta, x)$ has no zeros. If $\beta \in (\gamma_{j+1}, \gamma_j)$, then, by Proposition 11.1 (i) and (ii) with $\mu_1 = \beta$ and $\mu_2 = \gamma_j$, we see that $f_l^{[0]}(i\beta, x)$ has at least j zeros. On the other hand, using Proposition 11.1, we can conclude $f_l^{[0]}(i\beta, x)$ cannot have more than j zeros because the number of its zeros is nondecreasing as β decreases and $f_l^{[0]}(i\gamma_{j+1}, x)$ has exactly j zeros. Thus $f_l^{[0]}(i\beta, x)$ has exactly j zeros when $\beta \in (\gamma_{j+1}, \gamma_j)$. This proves (i) when $N(0,Q) = +\infty$ because the bound states of (2.17) can only occur when k is on the positive imaginary axis. If $N(0,Q)$ is finite and is denoted by $\mathcal{N}$, then we must still consider the case when $\beta \in (0, \gamma_{\mathcal{N}})$. Then using Lemma 1 on p. 91 of Ref. 23 we conclude that $f_l^{[0]}(i\beta, x)$ has exactly $\mathcal{N}$ zeros because if it had more than $\mathcal{N}$ zeros one could find a subspace of dimension at least $(\mathcal{N}+1)$ on which the expectation value of $(-d^2/dx^2 + Q - \beta P)$ is less than or equal to $-\beta^2$, and this would imply the existence of at least $(\mathcal{N}+1)$ eigenvalues less than or equal to $-\beta^2$.

(ii) In this case the condition $Q \in L_1^1(\mathbf{R})$ guarantees that $N(0,Q)$ is finite. It only remains to consider the case $\beta = 0$. Note that $f_l^{[0]}(0,x)$ cannot have more than $\mathcal{N}$ zeros; this is because $f_l^{[0]}(i\beta, x)$ has exactly $\mathcal{N}$ zeros when β is sufficiently small and by (2.16) we see that as $\beta \to 0$ we have $f_l^{[0]}(i\beta, x) \to f_l^{[0]}(0,x)$ uniformly on compact x-intervals. On the other hand, in (11.1) by setting $\mu_1 = 0$, $\mu_2 = \beta$, $V_1(\mu_1, x) = Q(x)$, $V_2(\mu_2, x) = Q(x) - \beta P(x)$, $V_1(\mu_1, x) = Q(x)$, and $\chi_2(\mu_2, x) = f_l^{[0]}(\beta, x)$, and using Proposition 11.1, we see that $f_l^{[0]}(0,x)$ has at least $\mathcal{N}$ zeros. Hence $f_l^{[0]}(0,x)$ must have exactly $\mathcal{N}$ zeros. ∎

If $Q \in L_1^1(\mathbf{R})$, then $N(0,Q)$ is finite and as in Section 9 we let $k = i\kappa_j$ for $j = 1, \cdots, \mathcal{N}$ denote the bound states of (2.17). From Theorems 9.7 and 9.9, when $P \in L^1(\mathbf{R})$, $Q \in L_1^1(\mathbf{R})$, and $P(x) \leq 0$, we already know that the bound states of (1.1) occur at $k = i\beta_j$ satisfying $\beta_j \leq \kappa_j$ for $j = 1, \cdots, \mathcal{N}$. In the next theorem, we extend Proposition 11.3 to (1.1) and analyze the number of zeros of the Jost solutions of (1.1) when k is on the positive imaginary axis.

Theorem 11.4 Assume that $P \in L^1(\mathbf{R})$, $Q \in L_1^1(\mathbf{R})$, and $P(x) \leq 0$. Then, for each $\beta \geq 0$, the functions $f_l^+(i\beta, x)$ and $f_r^+(i\beta, x)$ have the same number of zeros, and this number is equal to the number of bound states of (1.1) with energies contained in the interval $(-\infty, -\beta^2)$.

PROOF: From Proposition 11.1 (i) and (ii), we see that $f_l^+(i\beta, x)$ and $f_r^+(i\beta, x)$ have the same number of zeros. Since $P(x) \le 0$, from the proof of Theorem 9.7 it follows that, for any fixed $\beta > 0$, the number of eigenvalues of the operator $(-d^2/dx^2 + Q - \beta P)$ below $-\beta^2$ is equal to the number of $E_j(\beta)$ values that lie below $-\beta^2$. Note that if $\beta \in [\beta_j, \beta_{j+1})$ for $j = 1, \cdots, \mathcal{N} - 1$, then the $(\mathcal{N} - j)$ values $E_{\mathcal{N}}(\beta), E_{\mathcal{N}-1}(\beta), \cdots, E_{j+1}(\beta)$ lie strictly below $-\beta^2$; if $\beta \in [\beta_{\mathcal{N}}, +\infty)$ then there are no eigenvalues below $-\beta^2$, and if $\beta \in [0, \beta_1)$ then exactly $\mathcal{N}$ eigenvalues lie below $-\beta^2$. Using Proposition 11.3 when the potential $Q(x)$ in (2.17) is replaced by $Q(x) - \beta P(x)$, we can conclude that $f_l^+(i\beta, x)$ has no zeros for $\beta \in [\beta_{\mathcal{N}}, +\infty)$, $\mathcal{N}$ zeros for $\beta \in [0, \beta_1)$, and $(\mathcal{N} - j)$ zeros for $\beta \in [\beta_j, \beta_{j+1})$ for $j = 1, \cdots, \mathcal{N} - 1$. ∎

From Proposition 11.1 and Theorem 11.4 we have the following:

Corollary 11.5 Assume that $P, Q \in L^1(\mathbf{R})$ and that $P(x) \le 0$. The zeros of $f_l^+(i\beta_j, x)$ separate the zeros of $f_l^+(i\beta_{j+1}, x)$, i.e. between two consecutive zeros of $f_l^+(i\beta_j, x)$ there is exactly one zero of $f_l^+(i\beta_{j+1}, x)$, where $k = i\beta_j$ for $j = 1, \cdots, \mathcal{N}$ correspond to the bound states of (1.1). The zeros of $f_r^+(i\beta_j, x)$ separate the zeros of $f_r^+(i\beta_{j+1}, x)$. Two successive zeros of $f_l^+(i\beta_j, x)$ cannot coincide with two successive zeros of $f_l^+(i\beta_{j+1}, x)$; equivalently, two successive zeros of $f_r^+(i\beta_j, x)$ cannot coincide with two successive zeros of $f_r^+(i\beta_{j+1}, x)$.

When we no longer have $P(x) \le 0$, as Example 4.5 shows there may be bound states of (1.1) with complex energies, and as Example 10.1 shows $N(P, Q)$ may be larger than $N(0, Q)$. In the next theorem, when $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$, we analyze the bound states of (1.1) when k is on the positive imaginary axis, establish the connection between Theorem 8.8 and the zeros of $f_l^+(i\beta, x)$, and also consider multiple zeros of $1/T^+(k)$ on the positive imaginary axis.

Theorem 11.6 Suppose $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$. Then:

(i) If (2.17) has $\mathcal{N}$ bound states with $\mathcal{N} \ge 1$, then (1.1) has at least $\mathcal{N}$ bound states with real (negative) energies.

(ii) $1/T^+(i\beta)$ has a zero of order m at some positive β_0 if and only if the function $E_0(\beta) + \beta^2$ has a zero of order m at β_0, where $E_0(\beta)$ denotes the unique eigenvalue branch of the operator $(-d^2/dx^2 + Q - \beta P)$ satisfying $E_0(\beta) \to -\beta_0^2$ as $\beta \to \beta_0$. If $m = 1$, then the graph of $E_0(\beta)$ and the graph of the parabola $E = -\beta^2$ intersect with different slopes at β_0. If $m \ge 2$ and m is even, then the graphs touch at β_0 but do not cross each other. If $m \ge 3$ and m is odd, then the graphs cross smoothly such that at the point of intersection they have the same slopes.

(iii) For each eigenvalue branch, we have $E_0''(\beta) \le 0$ for $\beta > 0$ with the equality holding if and only if $P(x) \equiv 0$. Hence the graph of each eigenvalue branch is concave down if $P(x) \not\equiv 0$; in the trivial case $P(x) \equiv 0$, the graph of each eigenvalue branch is a horizontal line.

(iv) The number of zeros of $f_l^+(i\beta, x)$ behaves in the following manner as β is increased from $\beta_0 - \epsilon$ to $\beta_0 + \epsilon$

when ϵ is sufficiently small: If m is even, then the number of zeros is either constant throughout the interval $(\beta_0-\epsilon,\beta_0+\epsilon)$ or it is constant in $(\beta_0-\epsilon,\beta_0)\cup(\beta_0,\beta_0+\epsilon)$ but one less at β_0. If m is odd, then the number of zeros either increases or decreases by one as β crosses β_0. The number of zeros of $f_l^+(i\beta,x)$ can only change at a β value corresponding to a bound state of (1.1).

PROOF: (i) Since we are only interested in the bound states corresponding to $k=i\beta$ for $\beta>0$, we first derive a lower bound for $E_{\mathcal{N}}(\beta)$, which will show that $E_{\mathcal{N}}(\beta)=o(\beta^2)$ as $\beta\to+\infty$. Let us indicate the Fourier transform by a caret:

$$\hat{\psi}(q)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\,e^{iqx}\psi(x),\qquad \psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\,e^{-iqx}\hat{\psi}(q).$$

Then, letting $||\cdot||_2$ denote the norm on $L^2(\mathbf{R})$, for $a>0$ we get (cf. Theorem IX.28 of Ref. 24)

$$|\psi(x)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\,|\hat{\psi}(q)|\le\frac{1}{\sqrt{2a}}\left[\int_{-\infty}^{\infty}dq\,(a^2+q^2)\,|\hat{\psi}(q)|^2\right]^{1/2}\le\frac{1}{\sqrt{2a}}\left(||\psi'||_2^2+a^2||\psi||_2^2\right)^{1/2},\tag{11.7}$$

where we have used the Schwarz inequality, $||\psi||_2=||\hat{\psi}||_2$, $\int_{-\infty}^{\infty}dq/(a^2+q^2)=\pi/a$, and $\widehat{\psi'}=-iq\hat{\psi}$. Incidentally, the equality in (11.7) holds if and only if $\psi(x)=e^{-a|x|}$. Next we use (11.7) to estimate the quadratic forms $\langle Q\psi,\psi\rangle$ and $\langle P\psi,\psi\rangle$. From (11.7) we obtain

$$\int_{-\infty}^{\infty}dx\,|Q(x)|\,|\psi(x)|^2\le\frac{1}{2a}\left(||\psi'||_2^2+a^2||\psi||_2^2\right)\left(\int_{-\infty}^{\infty}dx\,|Q(x)|\right).\tag{11.8}$$

If $P(x)\equiv 0$, then (1.1) and (2.17) become identical, and in this trivial case both equations have exactly $\mathcal{N}$ bound states. Thus, there is no loss of generality in assuming $P(x)\not\equiv 0$. In order to estimate the integral $\int_{-\infty}^{\infty}dx\,|P(x)|\,|\psi(x)|^2$, we split it into two parts: one over the region $\{x:|P(x)|>M\}$ and the other over the region $\{x:|P(x)|\le M\}$, where the constant $M\ge 0$ is arbitrary for the moment but it will be fixed later. Then, for any $b>0$, (11.7) implies

$$\int_{-\infty}^{\infty}dx\,|P(x)|\,|\psi(x)|^2\le\frac{1}{2b}\left(||\psi'||_2^2+b^2||\psi||_2^2\right)\left(\int_{\{|P(x)|>M\}}dx\,|P(x)|\right)+M\,||\psi||_2^2.\tag{11.9}$$

Combining (11.8) and (11.9) we get

$$\begin{aligned}\langle-\psi''+Q\,\psi-\beta P\,\psi,\psi\rangle\ge&||\psi'||_2^2\left(1-\frac{1}{2a}\int_{-\infty}^{\infty}dx\,|Q(x)|-\frac{\beta}{2b}\int_{\{|P(x)|>M\}}dx\,|P(x)|\right)\\&-||\psi||_2^2\left(\frac{a}{2}\int_{-\infty}^{\infty}dx\,|Q(x)|+\frac{\beta b}{2}\int_{\{|P(x)|>M\}}dx\,|P(x)|+\beta\,M\right).\end{aligned}$$

We now set

$$a=\int_{-\infty}^{\infty}dx\,|Q(x)|,\qquad b=\beta\int_{\{|P(x)|>M\}}dx\,|P(x)|,$$

and assume that ψ is a normalized eigenfunction corresponding to the eigenvalue $E_{\mathcal{N}}(\beta)$. Then the left-hand side of (11.9) is equal to $E_{\mathcal{N}}(\beta)$ and hence

$$E_{\mathcal{N}}(\beta)\ge-\frac{1}{2}\left(\int_{-\infty}^{\infty}dx\,|Q(x)|\right)^2-\frac{\beta^2}{2}\left(\int_{\{|P(x)|>M\}}dx\,|P(x)|\right)^2-\beta M.\tag{11.10}$$

Since by choosing M large enough we can make the second term on the right-hand side of (11.10) as small as we please, it follows that $E_{\mathcal{N}}(\beta) = o(\beta^2)$ as $\beta \to +\infty$. Thus $E_{\mathcal{N}}(\beta) > -\beta^2$ for β sufficiently large, while $E_{\mathcal{N}}(0) = -\kappa_{\mathcal{N}}^2 < 0$. Hence, by the intermediate value theorem, the equation $E_{\mathcal{N}}(\beta) = -\beta^2$ has at least one solution. A similar argument shows that each of the remaining eigenvalue branches $E_j(\beta)$ for $j = 1, \cdots, \mathcal{N}-1$ must intersect the parabola $E = -\beta^2$ at least once. Since each intersection increases the number of negative-energy bound states of (1.1), the proof of (i) is complete. Note that if an eigenvalue branch $E_j(\beta)$ touches or intersects the parabola $E = -\beta^2$ at other points, such additional points are also responsible for additional negative-energy bound states of (1.1). Moreover, there may be other eigenvalue branches $E(\beta)$ starting at $(\tilde{\beta}, 0)$ for some $\tilde{\beta} > 0$ and intersecting or touching the parabola $E = -\beta^2$ at one or more points; again, each of such points also increases the number of negative-energy bound states of (1.1).

(ii) If $P(x) \equiv 0$, each eigenvalue branch $E_0(\beta)$ becomes the horizontal line $E_0(\beta) = -\beta_0^2$ for $\beta \geq 0$, and hence $E_0''(\beta) = 0$ for $\beta > 0$. Thus, in the rest of the analysis we can assume that $P(x) \not\equiv 0$. Associated with the eigenvalue $E_0(\beta)$ there exists[25] a real-valued, analytic eigenvector $\psi(\beta, x)$. Near $\beta = \beta_0$ we have the convergent expansions

$$E_0(\beta) = \sum_{n=0}^{\infty} a_n \, (\beta - \beta_0)^n, \qquad \psi(\beta, x) = \sum_{n=0}^{\infty} \psi_n(x) \, (\beta - \beta_0)^n, \tag{11.11}$$

with $\psi_n \in L^2(\mathbf{R})$ for $n \geq 0$. Substituting (11.11) in (1.1) we get the following set of equations (see pp. 333–334 of Ref. 26) for $n \geq 0$:

$$\psi_n''(x) - \beta_0^2 \, \psi_n(x) + a_1 \, \psi_{n-1}(x) + a_2 \, \psi_{n-2}(x) = [-\beta_0 \, P(x) + Q(x)] \, \psi_n(x) - P(x) \, \psi_n(x) - \sum_{j=3}^{n} a_j \, \psi_{n-j}(x), \tag{11.12}$$

where it is assumed that $a_{-n} = \psi_{-n}(x) = 0$ if $n \geq 1$. From (9.18) and (9.19) we see that

$$a_0 = E_0(\beta_0), \qquad a_1 = -\frac{1}{||\psi_0||_2^2} \int_{-\infty}^{\infty} dx \, P(x) \, \psi_0(x)^2. \tag{11.13}$$

We may choose $\psi_0(x) = f_l^+(i\beta_0, x)$. It suffices to prove that $1/T^+(i\beta)$ has a zero of order at least m at β_0 if and only if $E_0(\beta) + \beta^2$ has a zero of order at least m at β_0. From Proposition 8.4 we know that this is true when $m = 1$. If β_0 is a zero of $E_0(\beta) + \beta^2$ of order m for some $m \geq 2$, then the coefficients a_n in (11.3) are determined for $n = 0, 1, \cdots, m-1$ by expanding $E_0(\beta) + \beta^2$ about β_0. Thus, for $m = 2$ we get $a_0 = -\beta_0^2$, $a_1 = -2\beta_0$; for $m = 3$ we get $a_0 = -\beta_0^2$, $a_1 = -2\beta_0$, $a_2 = -1$; for $m \geq 4$ we get $a_0 = -\beta_0^2$, $a_1 = -2\beta_0$, $a_2 = -1$, and $a_3 = \cdots = a_{m-1} = 0$. Then, comparing (11.12) and (8.15) and using the fact that the functions $g_{l,n}^+(i\beta_0, x)$ are uniquely determined as solutions of (8.15) by the requirement that $g_{l,n}^+(i\beta_0, \cdot) \in L^2(\mathbf{R})$, we obtain

$$\psi_n(x) = i^n g_{l,n}^+(i\beta_0, x), \qquad n = 0, \cdots, m-1. \tag{11.14}$$

Thus, by Theorem 8.8, we see that $1/T^+(i\beta)$ has a zero of order at least m at β_0. Conversely, suppose $1/T^+(i\beta)$ has a zero of order at least m at β_0. From (11.12) one can derive (see p. 334 of Ref. 26) the following recursion formula for the coefficients a_n :

$$a_n = -\frac{1}{||\psi_0||_2^2} \int_{-\infty}^{\infty} dx\, \psi_0(x) \left(P(x)\, \psi_{n-1}(x) + \sum_{j=1}^{n-1} a_j\, \psi_{n-j}(x) \right), \qquad n \geq 2. \tag{11.15}$$

Now assume $m \geq 2$. Since the functions $\{g_{l,n}^{+}(i\beta_0, \cdot)\}_{n=1}^{m-1}$ form a Jordan chain of length m, using (8.26) with $n = 1$ and (11.13) we get $a_1 = -2\beta_0$. Hence $E_0(\beta) + \beta^2$ has a zero of order at least 2 at β_0. If $m \geq 3$, then using (8.26) with $n = 2$, (11.14) and (11.15) we obtain $a_2 = -1$ and this, in turn, implies that $E_0(\beta) + \beta^2$ has a zero of order at least 3 at β_0. If $m \geq 4$, then (8.26) and (11.15) give $a_3 = 0$ and then $a_j = 0$ for all $j = 3, \cdots, m-1$. As a result, $E_0(\beta) + \beta^2$ has a zero of order at least m at β_0. If m is even, then $E_0(\beta)$ touches the parabola $E = -\beta^2$ at β_0 but stays either above or below the parabola; if m is odd, then $E_0(\beta)$ intersects the parabola $E = -\beta^2$ by crossing from one side to the other.

(iii) We need to show that a_2 in (11.11) is negative for any $\beta_0 > 0$. We have (see p. 334 of Ref. 26)

$$\psi_1(x) = -\psi_0(x) \int_{x_0}^{x} \frac{dt}{\psi_0(t)^2} \int_{-\infty}^{t} ds\, \psi_0(s)\, [P(s)\, \psi_0(s) + a_1\, \psi_0(s)], \tag{11.16}$$

where the constant x_0 is arbitrary; however, since changing x_0 amounts to adding a constant multiple of $\psi_0(x)$ to $\psi_1(x)$, with the help of (11.6) one can show that the value of a_n given in (11.15) is independent of x_0. Using (11.16) in (11.15) with $n = 2$, after performing an integration by parts, we get

$$a_2 = -\frac{1}{||\psi_0||_2^2} \int_{-\infty}^{\infty} \frac{dx}{\psi_0(x)^2} \left(\int_{-\infty}^{x} dt\, \psi_0(t)^2\, [P(t) + a_1] \right)^2 < 0.$$

Thus $a_2 < 0$ in (11.11) for any $\beta_0 > 0$, and hence we have $E_0''(\beta) < 0$ for any $\beta > 0$.

(iv) Let us consider the number of zeros of $f_l^{+}(i\beta, x)$ in relation to the behavior of the eigenvalue branch $E_0(\beta)$ near β_0. From Proposition 11.3 (i), when $Q(x)$ in (2.17) is replaced by $Q(x) - \beta\, P(x)$, we know that the number of zeros of $f_l^{+}(i\beta, x)$ is equal to the number of eigenvalue branches lying below $-\beta^2$. Let I_{β_0} denote the interval $(\beta_0 - \epsilon, \beta_0 + \epsilon)$ and let J_{β_0} denote $(\beta_0 - \epsilon, \beta_0) \cup (\beta_0, \beta_0 + \epsilon)$ for sufficiently small $\epsilon > 0$, and let us consider the number of eigenvalue branches below $-\beta^2$ when $\beta \in I_{\beta_0}$. If m is even, then $E_0(\beta)$ touches the parabola $E = -\beta^2$ at β_0 but stays either above or below that parabola; in the former case $E_0(\beta) > -\beta^2$ for $\beta \in J_{\beta_0}$ and hence the number of zeros of $f_l^{+}(i\beta, x)$ remains unchanged for $\beta \in I_{\beta_0}$; in the latter case $E_0(\beta) < -\beta^2$ for $\beta \in J_{\beta_0}$ and hence the number of zeros of $f_l^{+}(i\beta, x)$ for $\beta \in J_{\beta_0}$ is exactly one more than the number of zeros of $f_l^{+}(i\beta_0, x)$. If m is odd, then $E_0(\beta)$ intersects the parabola $E = -\beta^2$ by crossing from one side to the other of that parabola; if $E_0(\beta) < -\beta^2$ on $(\beta_0 - \epsilon, \beta_0)$, then the number of zeros of $f_l^{+}(i\beta, x)$ decreases by one as β increases through β_0; if $E_0(\beta) > -\beta^2$ on $(\beta_0 - \epsilon, \beta_0)$, then the number of zeros increases by one as β increases through β_0. In order to prove that the number of zeros of $f_l^{+}(i\beta, x)$ can only change if β corresponds to a bound state of (1.1) with real (negative) energy, we can proceed as follows. If β_1 and β_2 with $\beta_1 < \beta_2$ correspond to two consecutive real bound-state energies of (1.1), then no eigenvalue branch can intersect the parabola $E = -\beta^2$ for $\beta \in (\beta_1, \beta_2)$. Hence the number of eigenvalue branches that lie below $-\beta^2$ is constant for $\beta \in (\beta_1, \beta_2)$, or equivalently, the number of zeros of $f_l^+(i\beta, x)$ is constant for $\beta \in (\beta_1, \beta_2)$. ∎

The next example illustrates Theorem 11.6 and shows how the intersection of the eigenvalue curves $E_j(\beta)$ and the parabola $E = -\beta^2$ affects the bound states and the number of zeros of the Jost solutions of (1.1) when k is on the positive imaginary axis. The numerical values in this example were obtained by using Mathematica.

Example 11.7 Consider the same $P(x)$ and $Q(x)$ as in Example 10.1 where the parameters a and b will be chosen below.

(a) Let $a = 0$ and $b = 10$, and hence $Q(x) = 0$ and $P(x) \geq 0$. Note that $N(0, Q) = 0$. ¿From (9.17) we obtain [cf. (10.1)]

$$2\Delta\sqrt{-E}\cos\Delta = (\Delta^2 + E)\sin\Delta, \tag{11.17}$$

where $E = E(\beta)$ is the energy in (9.17) and $\Delta = \sqrt{\beta b - a + E}$. Using the half-angle formula for the tangent function, we can write (11.17) as a pair of equations determining the eigenvalue curves:

$$\tan\left(\frac{\Delta}{2}\right) = \frac{\sqrt{-E}}{\Delta}, \qquad \tan\left(\frac{\Delta}{2}\right) = -\frac{\Delta}{\sqrt{-E}}. \tag{11.18}$$

The eigenvalue curves in (11.18) can be plotted in the (β, E) plane for $\beta \geq 0$. As seen in the proof of Theorem 9.7, the bound states of (1.1) with real energies correspond to the β values where the eigenvalue curves intersect the parabola $E = -\beta^2$. When $a = 0$ and $b = 10$, from (11.18) we obtain two eigenvalue branches intersecting the parabola $E = -\beta^2$. Let $E_2(\beta)$ denote the eigenvalue branch responsible for the lowest real bound-state energy. We see that $E_2(\beta)$ emerges from $(0, 0)$ and intersects the parabola $E = -\beta^2$ at $\beta_3 = 9.27\overline{3}$. The second eigenvalue branch, $E_1(\beta)$, emerges from zero at $\beta = \pi^2/10$ and then intersects the parabola $E = -\beta^2$ at $\beta_1 = 2.14\overline{4}$ and at $\beta_2 = 5.96\overline{3}$. These eigenvalue branches and the parabola $E = -\beta^2$ are plotted in Fig. 1.

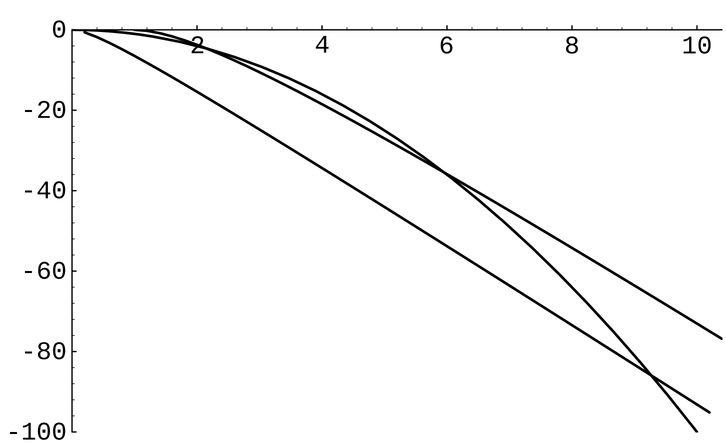

Fig. 1 The parabola $E = -\beta^2$ intersecting the eigenvalue curves $E_1(\beta)$ and $E_2(\beta)$

As seen from (10.5), the values β_1, β_2, and β_3 correspond to the simple zeros of $1/T^+(i\beta)$. If $\beta \geq \beta_3$, then $f_l^+(i\beta,x)$ has no zeros. If $\beta \in [\beta_2,\beta_3)$, then $f_l^+(i\beta,x)$ has one zero because $E_2(\beta)$ is the only eigenvalue below $-\beta^2$. If $\beta \in (\beta_1,\beta_2)$, then $f_l^+(i\beta,x)$ has two zeros because both $E_2(\beta)$ and $E_1(\beta)$ lie below $-\beta^2$. If $\beta \in (0,\beta_1]$ then $f_l^+(i\beta,x)$ has one zero, and if $\beta=0$ then $f_l^+(0,x)$ has no zeros because (2.17) with $Q(x)=0$ has no bound states.

(b) When $a=0$, one can choose the parameter b such that the branch $E_1(\beta)$ just touches the parabola $E=-\beta^2$ at β_1. Then, the slope of the eigenvalue curve at β_1 must be equal to $-2\beta_1$, and this happens when

$$\tan\left(\frac{\sqrt{b^2-4}}{4}\right) = -\frac{\sqrt{b+2}}{\sqrt{b-2}},$$

from which we get $b=9.206\overline{6}$, leading to $\beta_1=3.60\overline{3}$, and β_1 corresponds to a double zero of $1/T^+(i\beta)$. The eigenvalue branch $E_2(\beta)$ intersects the parabola $E=-\beta^2$ at β_2; we have $\beta_2=8.43\overline{3}$, which is responsible for the lowest real bound-state energy. In this case, $f_l^+(i\beta,x)$ has no zeros for $\beta=0$, one zero for $\beta\in(0,\beta_2)$, and no zeros for $\beta\in[\beta_2,+\infty)$, where $\beta_2=8.43\overline{3}$. We show the two eigenvalue branches and the parabola $E=-\beta^2$ in Fig. 2.

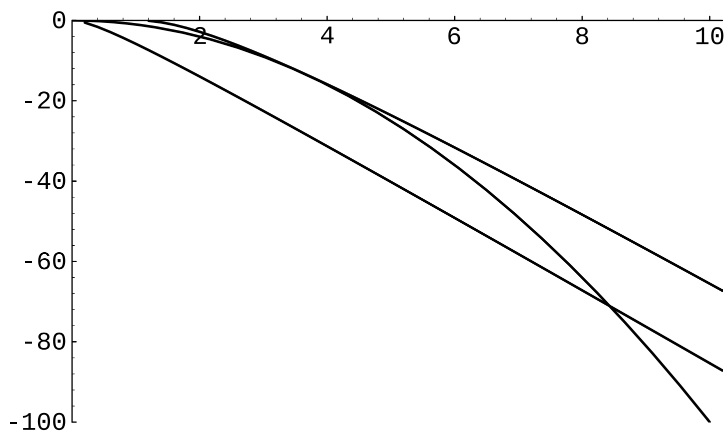

Fig. 2 The parabola $E=-\beta^2$ touching $E_1(\beta)$ and intersecting $E_2(\beta)$

(c) Let $b=10$ and let us adjust a such that the lowest real bound-state energy corresponds to a double zero of $1/T^+(i\beta)$. Proceeding as in (b), we obtain

$$a=\frac{10\,\beta_1(\beta_1-4)}{\beta_1-3},$$

where β_1 is obtained by solving

$$\tan\left(\frac{\sqrt{10\beta+3\beta^2-\beta^3}}{2\sqrt{\beta-3}}\right) = \frac{\beta\sqrt{\beta-3}}{\sqrt{10\beta+3\beta^2-\beta^3}}. \tag{11.19}$$

From (11.19) we get $\beta_1=4.72\overline{4}$ and hence $a=19.85\overline{2}$. In this case $f_l^+(i\beta,x)$ has no zeros for any $\beta\geq 0$. The eigenvalue curve $E_1(\beta)$ and the parabola $E=-\beta^2$ are plotted in Fig. 3, and it is seen that there are no other real bound-state energies besides $-\beta_1^2$.

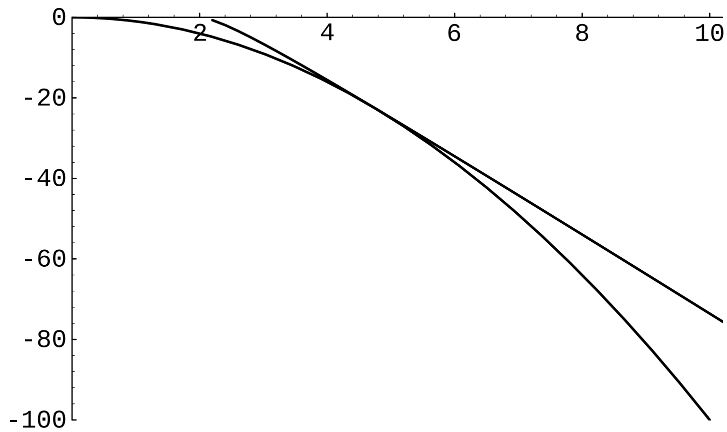

Fig. 3 The parabola $E=-\beta^2$ touching the eigenvalue curve $E_1(\beta)$

In the next example, when k is on the positive imaginary axis, we evaluate the number of zeros of the Jost solutions of (1.1) corresponding to some of the potentials considered in Example 10.1. The numerical values in this example were obtained by using Maple.

Example 11.8 Consider the same $P(x)$ and $Q(x)$ studied in Example 10.1. For various specific values of a and b listed in that example, the zeros of $1/T^+(i\beta)$ were all simple. Hence, as Theorem 11.6 (iv) states, we expect the number of zeros of $f_l^+(i\beta,x)$ and $f_r^+(i\beta,x)$ to change by ±1 at each zero of $1/T^+(i\beta)$ as β varies in $(0,+\infty)$. For example, when $a=0$ and $b=21/10$, one finds that $f_l^+(i\beta,x)$ has one zero for $\beta\in(0,\beta_1)$ and no zeros for $\beta\in(\beta_1,+\infty)$, where $\beta_1=0.1\overline{5}$; moreover, $f_l^+(i\beta,x)$ has no zeros when $\beta=0$ because $N(0,Q)=0$. When $a=0$ and $b=100$, one finds that $f_l^+(i\beta,x)$ has no zeros for $\beta=0$, no zeros for $\beta\in(\beta_{31},+\infty)$, one zero for $\beta\in(0,\beta_1)$ and one zero for $\beta\in(\beta_{30},\beta_{31})$, j zeros for $\beta\in(\beta_{j-1},\beta_j)$ and j zeros for $\beta\in(\beta_{31-j},\beta_{32-j})$ with $j=2,3,\cdots,15$, and sixteen zeros for $\beta\in(\beta_{16},\beta_{17})$.

The next proposition concerns the zeros of the Jost solutions when k lies off the positive imaginary axis.

Proposition 11.9 Assume $P,Q\in L^1(\mathbf{R})$ and let $k_0=\alpha+i\beta$ for some $\alpha\neq0$ and $\beta>0$. If $P(x)\le2\beta$, then $f_l^+(k_0,x)$ and $f_r^+(k_0,x)$ cannot vanish for any $x\in\mathbf{R}$.

PROOF: Using (4.2) in (4.28) we obtain

$$\frac{d}{dx}[f_l^+(-\overline{k_0},x);f_l^+(k_0,x)]=2i\alpha\,[P(x)-2\beta]\,|f_l^+(k_0,x)|^2. \tag{11.20}$$

Suppose $f_l^+(k_0,x)$ has at least one zero and let d be the right-most zero of $f_l^+(k_0,x)$. Note that, as seen from (4.2), the zeros of $f_l^+(-\overline{k_0},x)$ and $f_l^+(k_0,x)$ coincide. Integrating (11.20) over $(d,+\infty)$ and using (2.3), Proposition 8.2, and the boundedness of $1/T^+(k)$ for each $k\in\mathbf{C}^+$, we obtain

$$2i\alpha\int_d^\infty dx\,[P(x)-2\beta]\,|f_l^+(k_0,x)|^2=0.$$

This is impossible if $\alpha\neq0$ and $P(x)\le2\beta$; note that $P(x)=2\beta$ on a semi-infinite interval would contradict $P\in L^1(\mathbf{R})$. Hence, $f_l^+(k_0,x)$ cannot vanish for any $x\in\mathbf{R}$. The proof for $f_r^+(k_0,x)$ is analogous. ∎

APPENDIX: SMALL-k ESTIMATES

In this appendix, proceeding as in Refs. 10 and 11, we obtain various small-k estimates that are needed in the proof of Theorem 5.2.

In the exceptional case, let $\tilde{\psi}(k,x)$ be the solution of (1.1) satisfying the initial conditions

$$\tilde{\psi}(k,0)=f_l(0,0),\qquad \tilde{\psi}'(k,0)=f_l'(0,0),\qquad k\in\mathbf{R}. \tag{A.1}$$

Note that $\tilde{\psi}(0,x)=f_l(0,x)$, and hence $\tilde{\psi}(0,x)$ is bounded in such a way that $\tilde{\psi}(0,+\infty)=1$ and $\tilde{\psi}(0,-\infty)=\gamma$, where γ is the constant defined in (2.27). We have

$$\tilde{\psi}(k,x)=f_l(0,0)\,\cos kx+f_l'(0,0)\frac{\sin kx}{k}+\frac{1}{k}\int_0^x dy\,\sin k(x-y)\,[ikP(y)+Q(y)]\,\tilde{\psi}(k,y). \tag{A.2}$$

Let $\psi_1(k,x)$ denote the solution of (1.1) with $P(x)=0$ and satisfying (A.1). We have

$$\psi_1(k,x)=f_l(0,0)\,\cos kx+f_l'(0,0)\frac{\sin kx}{k}+\frac{1}{k}\int_0^x dy\,\sin k(x-y)\,Q(y)\,\psi_1(k,y). \tag{A.3}$$

Note that $\tilde{\psi}(0,x)=\psi_1(0,x)$.

Proposition A.1 Assume $Q\in L^1_1(\mathbf{R})$. For $k,x\in\mathbf{R}$, we have

$$|\psi_1(k,x)-\psi_1(0,x)|\le C\left[\left(\frac{|kx|}{1+|kx|}\right)^2+|k|\frac{|kx|}{1+|kx|}\right],\qquad |\psi_1(k,x)|\le C(1+|k|), \tag{A.4}$$

where C is a constant independent of x and k.

PROOF: Note that $\psi_1(0,x)=f_l(0,x)$ and hence it is uniformly bounded for $x\in\mathbf{R}$. Furthermore,

$$\psi_1(0,x)=f_l(0,0)+x\,f_l'(0,0)+\int_0^x dy\,(x-y)\,Q(y)\,\psi_1(0,y). \tag{A.5}$$

Subtracting (A.5) from (A.3) and iterating the resulting integral equation as in the proof of Proposition A.1 of Ref. 11, we obtain the first inequality in (A.4). Using that inequality and the boundedness of $\psi_1(0,x)$, we obtain the second inequality in (A.4). ∎

Let us choose a second linearly independent solution of (1.1) with $P(x)=0$ such that the Wronskian $[\psi_1(k,x);\psi_2(k,x)]$ is equal to 1. For example, we can choose $\psi_2(k,x)$ satisfying the initial conditions $\psi_2(k,0)=0$ and $\psi_2'(k,0)=1/f_l(0,0)$; note that there is no loss of generality in assuming $f_l(0,0)\ne 0$, and the case $f_l(0,0)=0$ can be handled by a shift of the origin. We have

$$\psi_2(k,x)=\frac{\sin kx}{k\,f_l(0,0)}+\frac{1}{k}\int_0^x dy\,\sin k(x-y)\,Q(y)\,\psi_2(k,y). \tag{A.6}$$

Proposition A.2 Assume $Q\in L^1_1(\mathbf{R})$. Then, for $x,k\in\mathbf{R}$ we have

$$|\psi_2(k,x)|\le\frac{C|x|}{1+|kx|},\qquad |\psi_2(k,x)-\psi_2(0,x)|\le C|x|\left(\frac{|kx|}{1+|kx|}\right)^2, \tag{A.7}$$

where C is a constant independent of x and k.

PROOF: Iterating (A.6) as in the proof of Proposition A.1 of Ref. 11, we obtain the first inequality in (A.7). Note that from (A.6) we have

$$\psi_2(0,x)=\frac{x}{f_l(0,0)}+\int_0^x dy\,(x-y)\,Q(y)\,\psi_2(0,y). \tag{A.8}$$

Subtracting (A.8) from (A.6) and iterating the resulting integral equation, we obtain the second inequality in (A.7). ∎

Proposition A.3 Assume $P,Q\in L_1^1(\mathbf{R})$. Then, for $x\in\mathbf{R}$ and as $k\to 0$ in $\mathbf{R}$, we have

$$\begin{aligned}\tilde\psi(k,x)-\psi_1(k,x)=&-ik\psi_1(0,x)\int_0^x dz\,\psi_2(0,z)\,P(z)\,\psi_1(0,z)+ik\psi_2(0,x)\int_0^x dz\,P(z)\,\psi_1(0,z)^2\\&+O\left(|k|\frac{|kx|}{1+|kx|}\right)+O\left(|kx|\left[\frac{|kx|}{1+|kx|}\right]^2\right).\end{aligned} \tag{A.9}$$

PROOF: Recall that $\psi_1(k,x)$ and $\psi_2(k,x)$ are two linearly independent solutions of (1.1) when $P(x)=0$. Using variation of parameters on (1.1), we obtain

$$\tilde\psi(k,x)-\psi_1(k,x)=-ik\,\psi_1(k,x)\int_0^x dz\,\psi_2(k,z)\,P(z)\,\tilde\psi(k,z)+ik\,\psi_2(k,x)\int_0^x dz\,\psi_1(k,z)\,P(z)\,\tilde\psi(k,z). \tag{A.10}$$

Let us write (A.10) as

$$\tilde\psi(k,x)-\psi_1(k,x)=A_1+A_2+A_3+A_4+A_5+A_6+A_7+B_1+B_2, \tag{A.11}$$

where we have defined

$$B_1=-ik\,\psi_1(k,x)\int_0^x dz\,\psi_2(k,z)\,P(z)\,[\tilde\psi(k,z)-\psi_1(k,z)],$$

$$B_2=ik\,\psi_2(k,x)\int_0^x dz\,\psi_1(k,z)\,P(z)\,[\tilde\psi(k,z)-\psi_1(k,z)],$$

$$A_1=-ik\psi_1(0,x)\int_0^x dz\,\psi_2(0,z)\,P(z)\,\psi_1(0,z),$$

$$A_2=-ik[\psi_1(k,x)-\psi_1(0,x)]\int_0^x dz\,\psi_2(k,z)\,P(z)\,\psi_1(k,z),$$

$$A_3=-ik\psi_1(0,x)\int_0^x dz\,[\psi_2(k,z)-\psi_2(0,z)]\,P(z)\,\psi_1(k,z),$$

$$A_4=-ik\psi_1(0,x)\int_0^x dz\,\psi_2(0,z)\,P(z)\,[\psi_1(k,z)-\psi_1(0,z)],$$

$$A_5=ik\psi_2(0,x)\int_0^x dz\,P(z)\,\psi_1(0,z)^2,$$

$$A_6=ik[\psi_2(k,x)-\psi_2(0,x)]\int_0^x dz\,P(z)\,\psi_1(k,z)^2,$$

$$A_7 = ik\psi_2(0,x)\int_0^x dz\,[\psi_1(k,z)-\psi_1(0,z)]\,[\psi_1(k,z)+\psi_1(0,z)]\,P(z)\psi_1(k,z).$$

Using the estimates in (A.4) and (A.7), we obtain

$$|A_2| \le C|k|(1+|k|)\left[\left(\frac{|kx|}{1+|kx|}\right)^2 + |k|\frac{|kx|}{1+|kx|}\right]\int_0^{|x|} dt\,|t\,P(t)|, \tag{A.12}$$

$$|A_3| \le C|k|\left(\frac{|kx|}{1+|kx|}\right)^2 \int_0^{|x|} dt\,|t\,P(t)|, \tag{A.13}$$

$$|A_4| \le C|k|\left[\left(\frac{|kx|}{1+|kx|}\right)^2 + |k|\frac{|kx|}{1+|kx|}\right]\int_0^{|x|} dt\,|t\,P(t)|, \tag{A.14}$$

$$|A_6| \le C|kx|(1+|k|)^2\left(\frac{|kx|}{1+|kx|}\right)^2\int_0^{|x|} dt\,|P(t)|, \tag{A.15}$$

$$|A_7| \le C(1+|k|)\frac{|kx|}{1+|kx|}\int_0^{|x|} dz\,\left[\left(\frac{|kz|}{1+|kz|}\right)^2 + |k|\frac{|kz|}{1+|kz|}\right]|P(z)|. \tag{A.16}$$

Iterating the integral equation for $\tilde\psi(k,x)-\psi_1(k,x)$ given in (A.11) and using (A.12)-(A.16), we obtain (A.9). ∎

In order to estimate the small-k asymptotics of $T^+(k)$, we will use (4.4). Note that as in (A24) of Ref. 11 we have

(A.17)
$$f_l(0,0)\,[f_l^+(k,x);f_r^+(k,x)] = f_r^+(k,0)\left[-ikf_l(0,0)+f_l'(0,0)+\int_0^\infty dz\,e^{ikz}[ikP(z)+Q(z)]\tilde\psi(k,z)\right]$$
$$-\,f_l^+(k,0)\left[ikf_l(0,0)+f_l'(0,0)-\int_{-\infty}^0 dz\,e^{-ikz}\,[ikP(z)+Q(z)]\tilde\psi(k,z)\right].$$

Proposition A.4 Assume $P,Q\in L^1_1(\mathbf{R})$. Then, as $k\to 0$ we have

$$\int_0^\infty dz\,e^{ikz}\,[ikP(z)+Q(z)]\,\tilde\psi(k,z) = -f_l'(0,0)+ikf_l(0,0)-ik+ik\int_0^\infty dz\,P(z)\,f_l(0,z)^2+o(|k|), \tag{A.18}$$

$$\int_{-\infty}^0 dz\,e^{-ikz}[ikP(z)+Q(z)]\,\tilde\psi(k,z) = f_l'(0,0)-ik\gamma+ikf_l(0,0)+(ik/\gamma)\int_{-\infty}^0 dz\,P(z)\,f_l(0,z)^2+o(|k|), \tag{A.19}$$

where γ is the constant defined in (2.27).

PROOF: Let us write

$$\int_0^\infty dz\,e^{ikz}\,[ikP(z)+Q(z)]\,\tilde\psi(k,z) = J_1+J_2+J_3+J_4+J_5+J_6, \tag{A.20}$$

where

$$J_1 = \int_0^\infty dz\, Q(z)\, \psi(0,z), \qquad J_2 = \int_0^\infty dz\, Q(z)\, [e^{ikz} - 1]\, \psi(0,z),$$

$$J_3 = ik \int_0^\infty dz\, e^{ikz} P(z)\, \psi(0,z),$$

$$J_4 = \int_0^\infty dz\, e^{ikz}\, Q(z)\, [\psi_1(k,z) - \psi(0,z)],$$

$$J_5 = ik \int_0^\infty dz\, e^{ikz} P(z)\, [\tilde{\psi}(k,z) - \psi(0,z)],$$

$$J_6 = \int_0^\infty dz\, e^{ikz}\, Q(z)\, [\tilde{\psi}(k,z) - \psi_1(k,z)].$$

As in (A25) and (A26) of Ref. 11 we have $J_1 = -f_l'(0,0)$ and

$$J_2 = ik[f_l(0,0) - 1] + o(|k|), \qquad k \to 0.$$

As $k \to 0$, using (A.4) we obtain $J_4 = o(|k|)$ and

$$J_3 = ik \int_0^\infty dz\, P(z)\, \psi(0,z) + o(|k|),$$

and using (A.9) we have $J_5 = o(|k|)$ and

$$\text{(A.21)} \quad J_6 = ik \int_0^\infty dz\, Q(z) \left[-\psi_1(0,z) \int_0^z dt\, \psi_2(0,t)\, P(t)\, \psi_1(0,t) + \psi_2(0,z) \int_0^z dt\, P(t)\, \psi_1(0,t)^2 \right] + o(|k|).$$

Note that $[\psi_1(k,x); \psi_2(k,x)] = 1$ and $Q(z)\psi_s(0,z) = \psi_s''(0,z)$ for $s = 1,2$. Hence, using $\psi_2'(0,+\infty) = 1$, $\psi'(0,z) = o(z^{-1})$ as $z \to +\infty$, and integration by parts twice in (A.21), we obtain

$$J_6 = -ik \int_0^\infty dz\, P(z)\, \psi(0,z) + ik \int_0^\infty dz\, P(z)\, \psi(0,z)^2 + o(|k|), \qquad k \to 0.$$

Thus, from (A.20) we obtain (A.18). Similarly, using $\psi_1'(0,-\infty) = 0$ and $\psi_1(0,-\infty) = \gamma$, we get (A.19). ∎

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