SEVERAL PROOFS OF IHARA'S THEOREM

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ABSTRACT. We give three proofs that the reciprocal of Ihara's zeta function can be expressed as a simple polynomial times a determinant involving the adjacency matrix of the graph. The first proof, for regular graphs, is based on representing radial symmetric eigenfunctions on regular trees in terms of certain polynomials. The second proof, also for regular graphs, is a consequence of the fact that the resolvent of the adjacency operator on regular trees is exponential. A third proof,in many ways simpler than the rest, works for irregular graphs as well.

INTRODUCTION

We present three proofs of a well known theorem. Ihara's theorem for graphs expresses Ihara's zeta function (an analogue, for finite graphs, of both Riemann's and Selberg's zeta function) as the reciprocal of a polynomial involving the graph's adjacency matrix. The regular case was first proven by Ihara in 1966 [3]. Our first proof of the regular case (theorem 5) utilizes what are probably standard techniques involving spherical functions on trees. Our second proof, otherwise quite similar to one by Sunada [5], uses a result of the author that the resolvent for the adjacency operator on regular trees is "purely exponential" (see [4] or appendix). Generalizations of Ihara's theorem to possibly irregular graphs have appeared in papers by Bass [1], Hashimoto [2], Venkov and Nitikin [7], and Stark and Terras [6]. We give an elementary and short proof of this general theorem (theorem 8). Our proof is similar to that in [6] but is even more concise. We also give a short proof that if G and H are finite and H "covers" G, then the zeta function of H divides that of G (an analogue of [6, corollary 3]).

PRELIMINARIES

Let G be a connected simple graph which is not a tree. We also let G denote the vertex set of the graph. We write $x_E y$ if x and y are *adjacent*; that is x and y share an edge. The condition that the graph be simple is that two vertices share at most one edge. A *path* is a sequence of vertices such that consecutive vertices share an edge (we do not require that all vertices are distinct). A path is said to be *non-backtracking* if a subsequence of the form $..., x, y, x, ...$ does not appear. The *length* of the path is one less than the number of vertices in the path sequence. For vertices x and y, we define the *distance* between x and y, denoted $d(x, y)$, to be the

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length of the shortest path containing x and y. Thus d is a metric. We also use d to denote *vertex degree*; i.e. $d(x)$ is the number of vertices adjacent to x. A graph is *regular* if $d(x)$ is constant. In particular, a graph is *d-regular* if $d(x) = d$. The *adjacency matrix* A of G is defined by $A(x, y) = 1$ or 0 according to whether x and y are adjacent or not.

Given two graphs G and H, a *covering* is a surjection of H onto G which preserves adjacency and vertex degree. In this case, we say that H *covers* G. The covering relation is a partial order and a maximal element of the connected poset containing G is necessarily a tree T which is unique up to isomorphism. Equivalently, we may define T to be the set of all non-backtracking paths starting at a particular vertex where we say that two such paths are adjacent if the addition of a vertex on the end of one gives the other. Let θ denote the covering of T onto G. The covering also defines an equivalence relation on T (we write $x \sim y$ if $\theta(x) = \theta(y)$) and we denote the equivalence class of a vertex x by $[x]$.

Let $\widehat{A}$ and A denote the adjacency matrices of T and G respectively. Since θ preserves adjacency, they are related by the following equation:

$$A(\theta(x), \theta(y)) = \sum_{z \in [y]} \widehat{A}(x, z). \tag{1}$$

In general, we say that $\widehat{M}$ *covers* M if (1) holds for M in place of A. It is easy to verify that the covering relation is preserved by matrix multiplication (i.e. $\widehat{MN} = \widehat{M}\widehat{N}$). Also, if f is a "vector" (i.e. function on G) and $\widehat{f} = f \circ \theta$, then $\widehat{Af} = \widehat{A}\widehat{f}$. For $a, b \in G$, let $D(a, b) = d(a)I(a, b)$. Define $Q = D - I$ and let $\widehat{D}$ and $\widehat{Q}$ be defined analogously for T. Note that $\widehat{D}$ covers D and $\widehat{Q}$ covers Q.

Let $S_n(x)$ denote the metric sphere, in T, of radius n and center x. For vertices a and b in G, let $\alpha_n(a, b)$ be the number of non-backtracking paths of length n from a to b. Then

$$\alpha_n(\theta(x), \theta(y)) = |S_n(x) \cap [y]|.$$

Let K_u denote the corresponding power series in a real variable u:

$$K_u(a, b) := \sum_{n=0}^{\infty} \alpha_n(a, b) u^n.$$

Then K_u is covered by $\widehat{K}_u(x, y) := u^{d(x,y)}$. Note that

$$\begin{aligned}
\widehat{A}\widehat{K}_u(x, y) &= \sum_{zEx} u^{d(z,y)} \\
&= u^{d(x,y}[(d(x) - 1)u + 1/u - (1/u - u)\hat{I}(x, y)] \\
&= \widehat{K}_u(x, y)[d(x)u + (1/u - u)(1 - \hat{I}(x, y))]
\end{aligned}$$

and so $\widehat{A}\widehat{K}_u = u\widehat{D}\widehat{K}_u + (1/u - u)(\widehat{K}_u - \hat{I})$. Hence $AK_u = uDK_u + (1/u - u)(K_u - I)$ and so

$$(I - uA + u^2 Q)K_u = (1 - u^2)I. \tag{2}$$

We let ν denote the number of vertices in G and ϵ the number of edges in G. A *cycle* is a finite path whose first and last vertices agree. We do not distinguish a starting point (for example, we consider the cycles (x, y, z, x) and (y, z, x, y) to be the same). A cycle (considered as a sequence of *edges*) can be cancatenated with itself several times giving rise to another cycle. Such cycles are called *multiples* of the original cycle. A cycle is called *prime* if it all its multiples are non-backtracking and it is not a multiple of a strictly smaller cycle. Since G is finite, the number of such cycles is finite. Furthermore, since we assume that G is not a tree, there are *some* prime cycles. Let $C_1, C_2, \dots$ denote these cycles and $\omega_1, \omega_2, \dots$ their respective lengths. We define the zeta function to be

$$Z(u) = \prod_i (1 - u^{\omega_i})^{-1}.$$

Let a_n be the number of non-backtracking cycles of length n (hence $a_n = Tr(\alpha_n)$). We note that even though such a cycle may be non- backtracking, its multiples may backtrack. In this case, the graph is said to have a *tail.* That is, the cycle can be written as

$$(x_0, \dots, x_k = y_0, y_1, \dots, y_{n-2k} = x_k, x_{k-1}, \dots, x_0)$$

where the cycle $(y_0, \dots, y_{n-2k})$ is a multiple of a prime cycle.

Let b_n denote the number of cycles of length n which are multiples of prime cycles and which have designated starting points. One may think of these as the cycles of length n with tails of length 0. A formula for b_n is

$$b_n = \sum_{\omega_i | n} \omega_i.$$

The following lemma relates the sequence (b_i) to the zeta function.

Lemma 1. $u\frac{d}{du}logZ(u) = \sum_{n=1}^{\infty} b_n u^n$.

Proof. By the definition of Z,

$$logZ(u) = -\sum_i log(1 - u^{\omega_i})$$

and thus

$$u\frac{d}{du}logZ(u) = \sum_i \frac{\omega_i u^{\omega_i}}{1 - u^{\omega_i}} = \sum_i \sum_{k=1}^{\infty} \omega_i u^{k\omega_i} = \sum_{n=1}^{\infty} \sum_{\omega_i | n} \omega_i u^n. \quad \square$$

We now give a combinatorial argument relating (a_n) and (b_n).

Lemma 2. *Let q=d-1. Then*

$$\sum_{n=1}^{\infty} b_n u^n = \frac{1-qu^2}{1-u^2} \sum_{n=1}^{\infty} a_n u^n$$

Proof. Note that a_n, the number of non-backtracking cycles of length n, is the sum, for k less than $n/2$, of the number of such cycles with a tail of length k. This latter quantity is the number of multiples of prime cycles with total length $n-2k$ (with designated starting point) *times* the number of tails of length k attached to the starting point. That is,

$$a_n = \sum_{k=0}^{[n/2]-1} b_{n-2k} c_k$$

where c_k is the number of tails with length k. Using $b_1 = 0$,

$$\sum_{n=1}^{\infty} a_n u^n = (\sum_{k=0}^{\infty} c_k u^{2k})(\sum_{n=1}^{\infty} b_n u^n).$$

From a point on a multiple of a prime cycle, there is exactly one tail of length 0 (so $c_0 = 1$). There are $q-1$ ways to choose the first edge of a tail (since two of the d possible choices would result in backtracking) but,after that,there are q ways to choose further edges. Hence $c_n = \frac{q-1}{q} q^n$ for all $n \geq 1$ and it is easily verified that

$$\sum_{k=0}^{\infty} c_k u^{2k} = \frac{1-u^2}{1-qu^2}.$$

The lemma follows. □

Radial Functions

We consider *radial* functions (i.e. functions which depend only on the distance between the variable and a fixed point) on homogeneous trees. Given any function, by averaging over concentric spheres, we get a *radialization.* Since the "radialization operator" commutes with the adjacency operator, the radialization of an eigenfunction is still an eigenfunction. The radial eigenfunction is then determined by a sequence for which there is a three-term recurrence relation (depending on the eigenvalue). It follows that there is a sequence of polynomials which, when evaluated at the eigenvalue, give the values of the normalized radial eigenfunction.

Given a function f and two vertices x and y on the d-regular tree T, we define $\pi_x f(y)$ to be the average of f on the metric sphere containing y and centered at x. That is,

$$\pi_x f(y) = \frac{1}{|S_{\hat{d}(x,y)}(x)|} \sum_{z \in S_{\hat{d}(x,y)}(x)} f(z).$$

It is crucial to our argument that A and π_x commute.

Lemma 3. *For all x and y, $A\pi_x f(y) = \pi_x Af(y)$.*

Proof. There are two cases: $x = y$ and $x \neq y$.

Since $\pi_x f(x) = f(x)$,

$$\pi_x Af(x) = Af(x) = \sum_{yEx} f(y) = \sum_{yEx} \pi_x f(y) = A\pi_x f(x).$$

Suppose $d(x, y) = n > 0$. Let $S_k = S_k(x)$ denote the metric sphere with center x and radius k. Then

$$\begin{aligned}
\pi_x Af(y) &= \frac{1}{|S_n|} \sum_{z \in S_n} Af(z) \\
&= \frac{1}{|S_n|} [\sum_{u \in S_{n+1}} f(u) + \sum_{v \in S_{n-1}} f(v) \begin{cases} d & \text{if } n = 1 \\ d-1 & \text{if } n > 1 \end{cases}] \\
&= \frac{d-1}{|S_{n+1}|} \sum_{u \in S_{n+1}} f(u) + \frac{1}{|S_{n-1}|} \sum_{v \in S_{n-1}} f(v) \\
&= A\pi_x f(y).
\end{aligned}$$

□

Given x, if ϕ is a an eigenfunction of A, then so is $\pi_x \phi$. Since $\pi_x \phi$ is radially symmetric around x, it is determined by a sequence $s_0, s_1, \ldots$ defined by

$$s_{d(x,y)} = \pi_x \phi(y).$$

The equation $A\pi_x \phi = c\pi_x \phi$ then translates into the difference equation

$$s_{n+2} = \frac{c}{d-1} s_{n+1} - \frac{1}{d-1} s_n; s_0 = \phi(x), s_1 = c\phi(x).$$

We define some polynomials inductively:

$$p_0(t) = 1, p_1(t) = t, p_{n+2}(t) = \frac{t}{d-1} p_{n+1}(t) - \frac{1}{d-1} p_n(t).$$

Hence

$$\pi_x \phi(y) = \phi(x) p_{\hat{d}(x,y)}(c). \tag{3}$$

Suppose (ϕ_n) is an orthonormal basis of $L^2(G)$ and that (c_n) are the corresponding eigenvalues. That is,

$$A\phi_n = c_n \phi_n.$$

The characteristic function of the equivalence class of y has an eigenfunction expansion:

$$\chi_{[y]}(z) = \chi_{\theta(y)}(\theta(z)) = \sum_n \langle \chi_{\theta(y)}, \phi_n \rangle \phi_n(\theta(z)) = \sum_n \phi_n(\theta(z)) \overline{\phi_n(\theta(y))}.$$

Note that,by equation (3) above,

$$\langle \phi_n \circ \theta, \chi_{S_N(x)} \rangle = |S_N(x)| \pi_x(\phi_n \circ \theta) = |S_N(x)| \phi_n(\theta(x)) p_N(c_n)$$

and so

$$\begin{aligned} |S_N(x) \cap [y]| &= \langle \chi_{[y]}, \chi_{S_N(x)} \rangle \\ &= \sum_n \overline{\phi_n(\theta(y))} \langle \phi_n \circ \theta, \chi_{S_N(x)} \rangle \\ &= |S_N(x)| \sum_n p_N(c_n) \phi(\theta(x)) \overline{\phi(\theta(y))}. \end{aligned}$$

Letting $y = x$ and summing over all x in a representative set of G, we get:

Proposition 4. *If a_n is the number of non-backtracking cycles of length n,and $|S_n| = |S_n(x)|$, then*

$$a_n = |S_n| \sum_{\lambda \in Spec(A)} p_n(\lambda).$$

IHARA'S THEOREM

Theorem 5 (Ihara).

$$Z(u) = \frac{1}{(1-u^2)^{\epsilon - \nu} det(I - uA + qu^2 I)}.$$

Proof. In general, a three-term recurrence (with constant coefficients) for a family of orthogonal polynomials (p_n) gives a rational generating function. In particular, if, given p_0 and p_1, $p_{n+2} = ap_{n-1} - bp_n$, then

$$\sum_{n=1}^{\infty} p_n(u) u^n = \frac{p_1(u)u - bp_0(u)u^2}{1 - au + bu^2}.$$

Applying this to the case above (we also use the fact that $|S_n| = \frac{q+1}{q} q^n$ for $n \geq 1$) we have

$$\sum_{n=1}^{\infty} |S_n| p_n(\lambda) u^n = \frac{(\lambda - u)(q+1)u}{1 - \lambda(q+1)u + qu^2}. \tag{4}$$

Summing over λ and using the results above, we have

$$\begin{aligned} u \frac{d}{du} log Z(u) &= \sum_{n=1}^{\infty} b_n u^n \\ &= \sum_{\lambda \in Spec(A)} \frac{(1 - qu^2)(\lambda - du)u}{(1-u^2)(1 - \lambda u + u^2(d-1))}. \end{aligned}$$

Using partial fractions and integrating (and $q = d-1$),

$$\int \frac{(1-qu^2)(\lambda - du)}{(1-u^2)(1-\lambda u + qu^2)} du$$

$$= -\frac{q-1}{2} log(1-u^2) - log(1-\lambda u + qu^2)$$

and so, using $\frac{q-1}{2}\nu = \epsilon - \nu$,and summing over λ,

$$log Z(u) = log(\frac{1}{(1-u^2)^{\epsilon-\nu} det(I - uA + qu^2 I)}$$

(constants arising from integration must vanish by comparing values when u=0).

□

An alternative proof of formula (4) and, thus, of theorem 5 avoids the radial function machinery by using the fact that the resolvent on the homogeneous tree is "purely exponential" (see [4]). For completeness, we present a proof of this fact in an appendix.

Let R_λ denote the resolvent of A on G. That is, let

$$R_\lambda(a,b) = \sum_{n=0}^{\infty} \frac{A^{(n)}(a,b)}{\lambda^{n+1}}$$

where $A^{(n)}$ denotes the n^{th} power of A. We let $\widehat{R}_\lambda$ denote the corresponding kernel on T and note that, by (1) and induction,

$$R_\lambda(\theta(x), \theta(y)) = \sum_{z\in[y]} \widehat{R}_\lambda(x,z).$$

The result from [4] is that on the d-regular tree, if σ is the least real root of $(d-1)\sigma = \lambda - \frac{1}{\sigma}$, then

$$\widehat{R}_\lambda(x,y) = \frac{\sigma^{\hat{d}(x,y)}}{\lambda - d\sigma}. \tag{5}$$

Hence

$$R_\lambda = \frac{1}{\lambda - d\sigma(\lambda)} K_{\sigma(\lambda)}.$$

Then

$$\sum_{a\in G} R_t(a,a) = Tr(R_t) = \sum_{\lambda \in Spec(A)} \frac{1}{t-\lambda}$$

and thus

$$\sum_{n=0}^{\infty} a_n \sigma(t)^n = Tr(K_{\sigma(t)}) = [t - d\sigma(t)]Tr(R_t) = \sum_{\lambda\in Spec(A)} \frac{t - d\sigma(t)}{t - \lambda}.$$

Subtracting the first term from the left, 1 from each term on the right ($a_0 = \nu = |Spec(A)|$) and letting $u = \sigma(t)$, one finds

$$\sum_{n=1}^{\infty} a_n u^n = \sum_{\lambda \in Spec(A)} \frac{\lambda - du}{\sigma^{-1}(u) - \lambda}.$$

By lemma 1 and the fact that $\sigma^{-1}(u) = \frac{1+(d-1)u^2}{u}$, formula (4) follows.

The General Case

By equating determinants of both sides of both sides of equation (2), and using theorem 5, we get

$$Z(u) = det(K_u)/(1-u^2)^{\epsilon}.$$

Remarkably, this formula holds for irregular graphs as well.

With a_n and b_n defined as above, we have the following generalization of lemma 4.

Lemma 6.

$$\sum_{n=1}^{\infty}(a_n - b_n)u^n = \frac{u^2}{1-u^2}Tr((Q-I)(K_u - I)).$$

Proof. Let C be a cycle starting at a of length n. Suppose we wish to attach a tail of length 1 to the cycle at a so as to get a *non-backtracking* cycle of length $n+2$. If C is a multiple of a prime cycle, then there are $d(a)-2$ ways to do this (since 2 choices of a tail necessarily cause backtracking) whereas if C has a tail with end a, there are $d(a)-1$ ways to do this (since, this time, only one choice causes backtracking). Then $(d(a)-2)\alpha_n(a,a)$ undercounts the number of cycles of length $n+2$ with non-trivial tails attached at a by the number of such cycles of length n. Summing over a,

$$a_{n+2} - b_{n+2} = \sum_a (d(a)-2)\alpha_n(a,a) + a_n - b_n.$$

By induction,

$$a_n - b_n = \sum_a (d(a)-2) \sum_{1 \le k < n/2} \alpha_{n-2k}(a,a).$$

Taking the power series and rearranging terms, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - b_n)u^n &= \sum_a \sum_{n=1}^{\infty} \sum_{1 \le k < n/2} (d(a)-2)\alpha_{n-2k}(a,a)u^n \\
&= \sum_a \sum_{k=1}^{\infty} \sum_{n=2k+1}^{\infty} (d(a)-2)\alpha_{n-2k}(a,a)u^n \\
&= \sum_a \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (d(a)-2)\alpha_n(a,a)u^{n+2k} \\
&= \frac{u^2}{1-u^2} \sum_a (d(a)-2) \sum_{n=1}^{\infty} \alpha_n(a,a)u^n
\end{aligned}$$

which is the desired result. □

Another lemma is required. Given a square matrix $M = (m_{ij})$ with differentiable entries, let $M' = (m'_{ij})$.

Lemma 7. *Let M be a matrix with differentiable entries. When M is invertible,*

$$(log|detM|)' = Tr(M'M^{-1}).$$

Proof. Let $M_k = (m_{k;ij})$ denote the matrix M with the k^{th} row replaced by its corresponding derivatives. Note that the ij^{th} entry of $M_k M^{-1}$ is $\sum_\ell m_{k;i\ell} m_{\ell j}^{(-1)}$ which is $\sum_\ell m'_{k\ell} m_{\ell j}^{(-1)}$ or δ_{ij} according to whether $i = k$ or not. Thus, $det(M_k M^{-1}) = \sum_\ell m'_{k\ell} m_{\ell k}^{(-1)}$. Then

$$det(M)' = \sum_\pi sgn(\pi)(\prod_i m_{i\pi(i)})' = \sum_j \sum_\pi sgn(\pi) \prod_i m_{j;i\pi(i)} = \sum_j det(M_j)$$

and so $(log|detM|)' = \sum_j det(M_j M^{-1}) = \sum_{j,\ell} m'_{j\ell} m_{\ell j}^{(-1)} = Tr(M'M^{-1})$. □

Theorem 8.

$$Z(u) = \frac{1}{(1-u^2)^{\epsilon-\nu} det(I - uA + u^2 Q)}.$$

Proof. Let $B = I - uA + u^2 Q$ and $K = K_u$. Then $B' = -A + 2uQ$ and, by equation (2), $B^{-1} = \frac{1}{1-u^2} K$. Hence

$$\begin{aligned} uB'B^{-1} &= \frac{1}{1-u^2}(2u^2QK - uAK) \\ &= \frac{1}{1-u^2}(u^2QK + (u^2QK - uAK + K) - K) \\ &= \frac{1}{1-u^2}((u^2Q - I)K + BK) \\ &= \frac{1}{1-u^2}(u^2Q - I)K + I. \end{aligned}$$

By rewriting the statement of lemma 6,

$$\sum_{n=1}^{\infty}(a_n - b_n)u^n = Tr([K - I] - [\frac{u^2}{1-u^2}(Q - I)] + [\frac{1}{1-u^2}(u^2Q - I)K + I]).$$

Since $Tr(K - I) = \sum_{n=1}^{\infty} a_n u^n$, we have

$$u\frac{d}{du} log Z(u) = \sum_{n=1}^{\infty} b_n u^n = \frac{u^2}{1-u^2} Tr(Q - I) - uTr(B'B^{-1}).$$

The theorem follows from lemma 7 and the fact that $Tr(Q - I) = \sum_a (d(a) - 2) = 2(\epsilon - \nu)$. □

The following result is an analogue of [6, Corollary 3].

Corollary 9. *If H and G are finite graphs and H covers G, then $\widehat{Z}$ divides Z where Z and $\widehat{Z}$ are the zeta functions for G and H respectively.*

Proof. With A and Q defined as above, let $\widehat{A}$ and $\widehat{Q}$ be the corresponding matrices for H. Since

$$det(I - uA + u^2Q) = \prod_{\lambda \in Spec(A+uQ)} (1 - u\lambda)$$

and $\epsilon(G) - \nu(G) = \frac{1}{2}Tr(Q - I) \leq \frac{1}{2}Tr(\widehat{Q} - I) = \epsilon(H) - \nu(H)$ it is enough to show that $Spec(A + uQ) \subset Spec(\widehat{A} + u\widehat{Q})$. Suppose $Af + uQf = \lambda f$. Let $\widehat{f} = f \circ \theta$ where θ is the covering map $G \to H$. Since $\widehat{A}$ and $\widehat{Q}$ *cover* A and Q respectively (in the sense of equation (1)),

$$\widehat{A}\widehat{f} + u\widehat{Q}\widehat{f} = \widehat{Af + uQf} = \lambda\widehat{f}$$

and the corollary follows. □

APPENDIX

Proof of formula 5. Since T is regular, $\widehat{R}_\lambda$ is radially symmetric. That is, there exists a sequence $s_0, s_1, \ldots$ such that

$$s_{\hat{d}(x,y)} = \widehat{R}_\lambda(x, y).$$

By the definition of $\widehat{R}_\lambda$,

$$\widehat{A}\widehat{R}_\lambda = \lambda\widehat{R}_\lambda - I$$

and thus $ds_1 = \lambda s_0 - 1$ and, for $n \geq 1$,

$$s_{n+1} = \frac{\lambda s_n}{d-1} - \frac{s_{n-1}}{d-1}.$$

Letting $r_n = \frac{s_{n+1}}{s_n}$, we have, for $n \geq 1$,

$$r_{n+1} = \frac{\lambda}{d-1} - \frac{1}{(d-1)r_n}.$$

A function like $f(x) = a - b/x$ $(a, b > 0)$ has two fixed points. If they are real, then the interates of f are either convergent to the larger or fixed at the smaller. Let σ and τ be the solutions of the equation $(d-1)t = \lambda - \frac{1}{t}$. If $\lambda > 2\sqrt{d-1}$, then the solutions, $\sigma < \tau$, are real. Furthermore, σ and τ are decreasing and increasing respectively (as functions of λ) and are equal at $2\sqrt{d-1}$. For $\lambda > 2\sqrt{d-1}$, if $r_n(\lambda)$ is not identically $\sigma(\lambda)$, then $s_n^{1/n}(\lambda)$ converges to $\tau(\lambda)$. If we perturb λ slightly, we see that $s_n(\lambda - \epsilon) \geq s_n(\lambda)$ but

$$\lim_{n\to\infty} s_n^{1/n}(\lambda - \epsilon) \geq \lim_{n\to\infty} s_n^{1/n}(\lambda) = \tau(\lambda) > max(\tau(\lambda - \epsilon), \sigma(\lambda - \epsilon))$$

– a contradiction. Hence r_n is identically σ and thus $s_n = c\sigma^n$ for some constant c uniquely determined by the conditions above. □

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