Uniform Decay Rates of Solutions to a Structural Acoustics Model with Nonlinear Dissipation

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Abstract

In this work, the asymptotic behavior of solutions to a coupled hyperbolic/parabolic–like system is investigated. It is shown that with both components of the equation being subjected to nonlinear damping (boundary damping for the wave component, interior for the beam), a global uniform stability is attained for all (weak) solutions.

1 Introduction

1.1 Statement of Problem

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n = 2$ or 3, with Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_0$, and with both Γ_i being open and nonempty. We further specify that Γ_0 be a simply connected segment of $\partial\Omega$, a smooth (C^2) manifold with boundary $\partial\Gamma_0$. In this paper, we investigate the stability properties of functions $\overrightarrow{z} = [z(t,x), z_t(t,x)]$ and $\overrightarrow{v} = [v(t,x), v_t(t,x)]$ which solve the following coupled system consisting of a coupled semilinear wave and "elastic" equation on finite time T:

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } \Omega \times (0,T) \\ \dfrac{\partial z}{\partial \nu} = v_t - g_1(z_t) \text{ on } \Gamma_0 \times (0,T) \\ z = 0 \text{ on } \Gamma_1 \times (0,T) \\ \overrightarrow{z}(t=0) = \overrightarrow{z_0} \in H^1(\Omega) \times L^2(\Omega); \end{cases} \tag{1}$$

$$\begin{cases} v_{tt} = -\mathbf{\mathring{A}} v - \mathbf{\mathring{A}}^{\frac{1}{2}} g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}} v_t\right) - z_t \text{ on } \Gamma_0 \times (0,T) \\ \overrightarrow{v}(t=0) = \overrightarrow{v_0} \in D\left(\mathbf{\mathring{A}}^{\frac{1}{2}}\right) \times L^2(\Gamma_0), \end{cases} \tag{2}$$

where $\mathbf{\mathring{A}}$: $D(\mathbf{\mathring{A}}) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ is a positive definite self–adjoint operator, with its fractional powers therefore being well–defined, and the g_i are functions which satisfy the following assumptions for $i = 1, 2$:

(H1) (i) $g_i(s)$ is continuous and monotone increasing;

(ii) $g_i(s)s > 0$ for $s \neq 0$;

(iii) $m_i s \leq g_i(s) \leq M_i s$ for $|s| > 1$.

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In addition, we shall need the following geometrical assumptions:

(H2) There exists a $\left[C^2\left(\overline{\Omega}\right)\right]^n$–vector field $h = [h_1(x), h_2(x), ..., h_n(x)] \in \left[C^2(\overline{\Omega})\right]^n$ such that

(i) $h \cdot \nu \leq 0$ on Γ_1, where ν is the unit normal of Γ pointing toward the exterior of Ω;

(ii) The Jacobian matrix $H(x)$ of $h(x)$ is uniformly positive definite on $\overline{\Omega}$.

The coupled system is a nonlinear version of the "structural acoustics" model derived by H.T. Banks et al. in [3] and [4] to describe the active control of acoustic pressure in a chamber through the placement and implementation of piezoelectric ceramic patches on one of the (flexible) chamber walls. The modelling partial differential equation is a wave coupled to an elastic equation, with the coupling being accomplished through the velocity terms of the solution $[z, v]$. In these papers above, the abstract operator $\mathring{\mathbf{A}}$ is taken to be the biharmonic Δ^2 with appropriate homogeneous boundary conditions, and $\mathring{\mathbf{A}}^{\frac{1}{2}}$ will be the Laplacian operator Δ. The implemented control term for this particular "smart materials" application takes the form of a linear combination of derivatives of delta functions placed on the beam component (2) so as to mathematically describe the bending moments induced through the chamber wall Γ_0 by the piezoelectric patches. In this paper, we deal strictly with the uncontrolled pde model (1)–(2), and our objective here is to demonstrate that the "energy" E of the system, defined by

$$E\left(\overrightarrow{z}, \overrightarrow{v}, t\right) \equiv \frac{1}{2}\int_{\Omega}\left[|\nabla z(t)|^2 + |z_t(t)|^2\right] d\Omega + \frac{1}{2}\left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v(t)\right\|^2_{L^2(\Gamma_0)} + \frac{1}{2}\|v_t(t)\|^2_{L^2(\Gamma_0)}, \qquad (3)$$

decays uniformly as $t \to \infty$.

1.2 Preliminaries

Before dealing with the coupled system (1)–(2), we will consider its wave component as an equivalent abstraction, for which we will need the following background material:

- Let the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be defined by

$$Az = -\Delta z;\ D(A) = \left\{ z \in H^2(\Omega) \ \ni\ z|_{\Gamma_1} = 0,\ \left.\frac{\partial z}{\partial \nu}\right|_{\Gamma 0} = 0 \right\}. \qquad (4)$$

Note that A is self-adjoint, positive definite, and hence the fractional powers of A are well defined.

- By [10], we have the following characterization:

$$D(A^{\frac{1}{2}}) = H^1_{\Gamma_1}(\Omega) = \left\{ z \in H^1(\Omega) \ \ni\ z = 0 \text{ on } \Gamma_1 \right\},$$
$$\text{with } \|z\|^2_{D(A^{\frac{1}{2}})} = \left\|A^{\frac{1}{2}} z\right\|^2_{L^2(\Omega)} = \int_{\Omega} |\nabla z|^2\, d\Omega = \|z\|^2_{H^1_{\Gamma_1}}\ \forall\, z \in D(A^{\frac{1}{2}}), \qquad (5)$$

where the last equality in (5) follows from Poincaré's inequality.

- We define the map N by

$$\phi = N\psi \Longleftrightarrow \begin{cases} \Delta\phi = 0 & \text{on } \Omega \\ \phi|_{\Gamma_1} = 0 & \text{on } \Gamma_1 \\ \left.\dfrac{\partial \phi}{\partial \nu}\right|_{\Gamma_0} = \psi & \text{on } \Gamma_0; \end{cases} \qquad (6)$$

elliptic theory will then yield that

$$N \in \mathcal{L}(L^2(\Gamma_1), D(A^{\frac{1}{2}})). \tag{7}$$

- Let $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma_0)$ be the restriction to Γ_0 of the familiar Sobolev trace map; viz.

$$\forall\, z \in H^1(\Omega), \gamma(z) = \begin{cases} z\,|_{\Gamma_0} & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_1. \end{cases} \tag{8}$$

It can then be shown easily that

$$N^* A = \gamma(z) \;\; \forall\, z \in D(A^{\frac{1}{2}}). \tag{9}$$

- We define the energy spaces

$$H_1 \equiv D(A^{\frac{1}{2}}) \times L^2\,(\Omega)\,; \tag{10}$$
$$H_0 \equiv D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times L^2(\Gamma_0). \tag{11}$$

With these definitions in hand, the system (1)–(2) is equivalent to the following abstract representation:

$$z_{tt} = -Az - ANg_1\,(N^* A z_t) + ANv_t \;\; \text{on } (0,\infty) \times \Omega; \tag{12}$$
$$v_{tt} = -\mathring{\mathbf{A}}v - \mathring{\mathbf{A}}^{\frac{1}{2}} g_2 \left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t\right) - NA^* z_t \;\; \text{on } (0,\infty) \times \Gamma_0. \tag{13}$$

We will hence proceed to look for solutions $[\overrightarrow{z}, \overrightarrow{v}]$ of (1)–(2) (and hence (12)–(13)) existing in $H_1 \times H_0$ *a.e.* in time, and subsequently study their decay properties. Note that the pointwise defined terms $N^* A z_t$ and $g_2 \left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t\right)$ are intially only formal representations, inasmuch as $[z_t, v_t]$ are taken to live pointwise in $L^2(\Omega)\times$ $L^2(\Gamma_0)$ only.

1.3 Literature

The uniform stability for each of the individual wave and beam components of (1)–(2) has been well–established these past few years, and we spell out here a short (necessarily incomplete) list of results. Concerning the (linear) uncoupled elastic equation, we have the result of S. Chen and Triggiani in [9] that the associated abstract operator which models the pde generates an analytic semigroup, which automatically provides for the exponential decay of the solution $[v, v_t]$ of the second–order system

$$\begin{aligned} v_{tt} &= -\mathring{\mathbf{A}}v - \mathring{\mathbf{A}}v_t \text{ on } (0,\infty) \times \Gamma_1 \\ [v(0), v_t(0)] &= \overrightarrow{v_0} \in H_0. \end{aligned} \tag{14}$$

For the wave equation with $L^2(0,T; L^2(\Omega))$–Neumann feedback control; viz.

$$\begin{aligned} & z_{tt} = \Delta z && \text{on } (0,\infty) \times \Omega \\ & z(0,x) = z_0, \;\; z_t(0,x) = z_1 && \text{on } \Omega \\ & z(t,x) = 0 && \text{on } (0,\infty) \times \Gamma_1 \\ & \frac{\partial z(t,x)}{\partial \nu} = -g_1\,(z_t(t,x)) && \text{on } (0,\infty) \times \Gamma_0; \end{aligned} \tag{15}$$

G. Chen in [8] proved the exponential stability of solutions (15), in the (linear) case that $g_1(s) = s$, under the geometrical conditions that Ω be "star–shaped". Lagnese in [11], and subsequently Triggiani, in [18] through an alternate proof, showed the uniform stabilization of (15) with again $g_1(s) = s$, under the lessened constraint that there exists a $\left[C^2(\overline{\Omega})\right]^n$–vector field $h(x)$ such that

(h.i) $h \cdot \nu \leq 0$ on Γ_1

where ν denotes the unit–normal vector to Γ;

(h.ii) h is parallel to ν on $\Gamma_{0;}$

(h.iii) The Jacobian matrix $H(x)$ of $h(x)$ is uniformly positive definite on $\overline{\Omega}$.

Also, C. Bardos, G. Lebeau and J. Rauch in [6] have derived stability results for linear wave equations with more general boundary conditions than those in (15), under the assumptions of geometric optics; however the techniques used in the proofs therein are not easily adaptable to our particular situation, based as they are on microlocal analysis and the propagation of singularities. We pay particular attention here to the result of I. Lasiecka and R. Triggiani in [15], who have shown the exponential decay of solutions of (15) without the constraint (h.ii). This result is proved by using the standard multipliers $h \cdot \nabla z$ and $z\, div\, h$, and invoking a (pseudodifferential) trace estimate which we state here for future reference:

Lemma A(*see* [15]). *Let $\epsilon > 0$ be arbitrarily small. Let z solve an arbitrary second–order hyperbolic equation on $(0, T)$ with smooth space–dependent coefficients. Then with $Q_T \equiv (0,T) \times \Omega$,*

$$\int_{\epsilon}^{T-\epsilon} \int_{\Gamma_0} \left(\frac{\partial z}{\partial \tau}\right)^2 d\Gamma_0 dt < C_{T,\epsilon} \left\{ \int_0^T \int_{\Gamma_0} \left\{ \left(\frac{\partial z}{\partial \nu}\right)^2 + z_t^2 \right\} d\Gamma_0 dt + \|z\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} \right\}, \tag{16}$$

where the ϵ on the left of (16) need not be the same as the ϵ for the Q_T–norm on the right, and where $\dfrac{\partial}{\partial \tau}$ denotes the tangential, and $\dfrac{\partial}{\partial \nu}$ the co–normal derivative.

In addition, the works in [20], [19] and [14] all deal with boundary stabilization of the wave equation (15) with the full nonlinearity g_1 in place, and varying geometrical conditions being assumed in each paper. The approach adopted in this paper will essentially be that employed in [14], an approach which in turn originated in [12]. Namely, we shall generate appropriate estimates for the energy functional $\int_0^T E\left(\overrightarrow{z}, \overrightarrow{v}, t\right) dt$, as opposed to the classical method of constructing a particular Lyapunov function for a generally nonlinear equation, and subsequently proving differential inequalities with respect to this Lyapunov function. Our initial estimate for the energy functional will be "tainted" with lower order terms, these having to be removed through a compactness/uniqueness argument. In attaining the preliminary estimate for our energy functional, the use of Lemma A will be critical here, as it was in obtaining uniform decay for the linearization of (1)–(2) in [1], under the geometric assumptions posted above in (H2) (The strong stability of the linearization of (1)–(2) was ascertained in [2])). Note that the assumptions in (H2) include no restrictions on the active portion of the boundary Γ_0. It is owing to the control of the tangential derivative provided by Lemma A that one is allowed to forego additional geometric assumptions imposed upon the active portion of the boundary Γ_0; e.g. $h \cdot \nu \geq 0$ on Γ_0, or h being parallel to ν on Γ_0.

1.4 Statement of Main Results

Theorem 1 *(i) With the initial data $[\overrightarrow{z_0}, \overrightarrow{v_0}] \in H_1 \times H_0$, the system (1)–(2) has a unique solution $[\overrightarrow{z}, \overrightarrow{v}] \in C([0,\infty); H_1 \times H_0)$.*

(ii) In addition, the velocity terms of the solution have the following regularity:

$$z_t \in L^2_{loc}(0,\infty; L^2(\Gamma_0)), \tag{17}$$

$$v_t \in L^2_{loc}\left(0,\infty; D\left(\mathring{\boldsymbol{A}}^{\frac{1}{2}}\right)\right) \tag{18}$$

(consequently, $g_1(N^*Az_t)$, $g_2\left(\mathring{\boldsymbol{A}}^{\frac{1}{2}}v_t\right) \in L^2_{loc}(0,\infty;L^2(\Gamma_0))$ *by (H1)(iii)).*
(iii) Furthermore, the solution $[\overrightarrow{z}, \overrightarrow{v}]$ *satisfies the following energy relation:*

$$E(\overrightarrow{z}, \overrightarrow{v}, 0) = E(\overrightarrow{z}, \overrightarrow{v}, T)$$

$$+\int_0^T \left[\left(g_1(N^*Az_t), N^*Az_t\right)_{L^2(\Gamma_0)} + \left(g_2\left(\mathring{\boldsymbol{A}}^{\frac{1}{2}}v_t\right), \mathring{\boldsymbol{A}}^{\frac{1}{2}}v_t\right)_{L^2(\Gamma_0)} \right] dt. \tag{19}$$

Before stating our stability result, we must first define some needed functions. Let $f(x)$ be defined by

$$f(x) \equiv f_1(x) + f_2(x), \tag{20}$$

where the f_i are concave, strictly increasing functions, with $f_i(0) = 0$, and such that

$$f_i(sg_i(s)) \geq s^2 + g_i^2(s) \ \text{ for } |s| \leq 1; \tag{21}$$

note that such function can be straightforwardly constructed, given the hypotheses on the g_i in (H1) (see [14]). With this function, we then define

$$\widetilde{f}(x) \equiv f\left(\frac{x}{meas(\Sigma_{0T})}\right), \tag{22}$$

where $\Sigma_{0T} \equiv (0,T) \times \Gamma_0$. As $\widetilde{f}$ is monotone increasing, then $cI + \widetilde{f}$ is invertible for all $c \geq 0$. For K a positive constant, we then set

$$p(x) \equiv \left(cI + \widetilde{f}\right)^{-1}(Kx); \tag{23}$$

the function p is easily seen to be positive, continuous and strictly increasing with $p(0) = 0$. Finally, let

$$q(x) \equiv x - (I+p)^{-1}(x); \tag{24}$$

We can now proceed to state our stability result.

Theorem 2 *Assume that the hypotheses (H1)–(H2) are in place. Let* $[\overrightarrow{z}, \overrightarrow{v}]$ *be the weak solution of the coupled system (1)–(2), assured by Theorem 1. With the energy* $E(\overrightarrow{z}, \overrightarrow{v}, t)$ *as defined in (3), there then exists a* $T_0 > 0$ *such that*

$$E(\overrightarrow{z}, \overrightarrow{v}, t) \leq \mathcal{S}\left(\frac{t}{T_0} - 1\right) \ \text{for } t > T_0, \tag{25}$$

with $\lim_{t\to\infty} \mathcal{S}(t) = 0$*, where the contraction semigroup* $\mathcal{S}(t)$ *is the solution of the differential equation*

$$\frac{d}{dt}\mathcal{S}(t) + q(\mathcal{S}(t)) = 0, \ \ \mathcal{S}(0) = E(\overrightarrow{z}, \overrightarrow{v}, 0) \tag{26}$$

(where q *is as given in (24)). Here, the constant* K *(from definition (23)) will depend on* $E(\overrightarrow{z}, \overrightarrow{v}, 0)$ *and time* T_0*, and the constant* c *(from definition (23)) is taken here to be* $c \equiv \dfrac{m_1^{-1} + M_1 + m_2^{-1} + M_2}{meas(\Sigma_{0T_0})}$.

Remark 1 *The assumptions (H2) on the vector field h will be satisfied if Γ_1 is a sufficiently small portion of the the boundary $\partial\Omega$; viz. if $meas(\Gamma_1) \leq \frac{1}{2} meas(\partial\Omega)$.*

Since the heart of the matter here is the stability of the system (1)–(2), we relegate the proof of well–posedness (Theorem 1) to the Appendix below, and commence with the proof of uniform stability.

2 Proof of Theorem 2

Throughout, we will make use of the denotations $Q_T \equiv (0,T) \times \Omega$, $\Sigma_T \equiv (0,T) \times \Gamma$ and $\Sigma_{iT} \equiv (0,T) \times \Gamma_i$, $i = 0, 1$, where $T > 0$ is arbitrary. The proof of Theorem 2 proceeds through several steps.

Proposition 1 *There exists a constant C, independent of time $0 < T < \infty$, such that for all $\epsilon > 0$*

$$\int_\epsilon^{T-\epsilon} \left[\left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v \right\|^2_{L^2(\Gamma_0)} + \|v_t\|^2_{L^2(\Gamma_0)} \right] dt$$

$$\leq C \left[\left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v(T-\epsilon) \right\|^2_{L^2(\Gamma_0)} + \|v_t(T-\epsilon)\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v(\epsilon) \right\|^2_{L^2(\Gamma_0)} + \|v_t(\epsilon)\|^2_{L^2(\Gamma_0)} \right.$$

$$\left. + \int_0^T \left(\|N^* A z_t\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right\|^2_{L^2(\Gamma_0)} + \left\| g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right) \right\|^2_{L^2(\Gamma_0)} \right) dt + \|v\|^2_{L^2(\Sigma_{0T})} \right]$$

Proof: As $\mathring{\mathbf{A}}$ is boundedly invertible, we easily have that

$$\int_0^T \|v_t\|^2_{L^2(\Gamma_0)} dt \leq \left\| \mathring{\mathbf{A}}^{-\frac{1}{2}} \right\| \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right\|^2_{L^2(\Gamma_0)} dt. \tag{27}$$

Moreover, multiplying (13) by v and integrating from 0 to T yields

$$\begin{aligned} \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v \right\|^2_{L^2(\Gamma_0)} dt &= -\int_0^T \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right), \mathring{\mathbf{A}}^{\frac{1}{2}} v \right)_{L^2(\Gamma_0)} dt \\ &+ \int_0^T \left[\|v_t\|^2_{L^2(\Gamma_0)} - (N^* A z_t, v)_{L^2(\Gamma_0)} \right] dt - \left[(v_t, v)_{L^2(\Gamma_0)} \right]_0^T ; \end{aligned} \tag{28}$$

using (27) and Cauchy–Schwarz on the right hand side of (28) will then give

$$\begin{aligned} (1 - \tfrac{\epsilon}{2}) \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v \right\|^2_{L^2(\Gamma_0)} dt &\leq C_\epsilon \left[\int_0^T \left(\|N^* A z_t\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right\|^2_{L^2(\Gamma_0)} dt \right) + \int_0^T \left\| g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right) \right\|^2_{L^2(\Gamma_0)} dt \right. \\ &\left. + \|v\|^2_{L^2(\Sigma_{0T})} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v(T) \right\|^2_{L^2(\Gamma_0)} + \|v_t(T)\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v(0) \right\|^2_{L^2(\Gamma_0)} + \|v_t(0)\|^2_{L^2(\Gamma_0)} \right], \end{aligned} \tag{29}$$

where C_ϵ is independent of time. The result follows upon coupling (27) and (29), and replacing the interval $(0,T)$ above with $(\epsilon, T-\epsilon)$. ■

We must likewise estimate the wave component $\overrightarrow{z}$ of (1)–(2).

Lemma 1 *Let* $\overrightarrow{w} \equiv [w, w_t] \in C([0,T]; H_1)$ *satisfy the following wave equation:*

$$\begin{cases} w_{tt} = \Delta w \ \ on \ Q_T; \\ [w(t=0), w_t(t=0)] = \overrightarrow{w_0} \in H_1; \\ w = 0 \ \ on \ \Sigma_{1T}, \end{cases} \tag{30}$$

with $w_t|_{\Gamma_0}$ *and* $\left.\frac{\partial w}{\partial \nu}\right|_{\Gamma_0} \in L^2(0,T; L^2(\Gamma_0))$. *Then for every* $\epsilon > 0$, w *satisfies the following estimate*

$$\int_{\epsilon}^{T-\epsilon} \left[\left\| A^{\frac{1}{2}} w \right\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)} \right] dt \le C_T \left[\int_{\Sigma_{0T}} \left[\left(\frac{\partial w}{\partial \nu} \right)^2 + w_t^2 \right] d\Sigma_{0T} + \|w\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} \right.$$
$$\left. + \, C \left[\left\| A^{\frac{1}{2}} w(T-\epsilon) \right\|_{L^2(\Omega)} + \|w_t(T-\epsilon)\|_{L^2(\Omega)} + \left\| A^{\frac{1}{2}} w(\epsilon) \right\|_{L^2(\Omega)} + \|w_t(\epsilon)\|_{L^2(\Omega)} \right] \right], \tag{31}$$

where the constant C_T *depends upon time* T, *but constant* C *does not.*

Proof: By Lemma 2.2 of [14], there exists a sequence $\left\{ w^{(n)} \right\}_{n=1}^{\infty} \in C([0,T]; H^2(\Omega)) \cap C^1([0,T]; D(A^{\frac{1}{2}}))$ which satisfies

$$\begin{cases} w_{tt}^{(n)} = \Delta w^{(n)}; \\ \left[w^{(n)}, w_t^{(n)} \right] \to [w, w_t] \ \text{ in } C([0,T]; H_1); \\ N^* A w_t^{(n)} \to N^* A w_t \ \text{ in } L^2(0,T; L^2(\Gamma_0)); \\ \frac{\partial w^{(n)}}{\partial \nu} \to \frac{\partial w}{\partial \nu} \ \text{ in } L^2(0,T; L^2(\Gamma_0)), \end{cases} \tag{32}$$

and it will thus suffice to prove the inequality (31) for a smooth solution $w \in C([0,T]; H^2(\Omega)) \cap C^1([0,T]; D(A^{\frac{1}{2}}))$. With the given vector field $h(x)$ satisfying (H2)(i)–(ii), we have upon multiplying the wave equation in (30) by $h \cdot \nabla w$ the standard identity (see [18], Appendix A):

$$\int_{Q_T} H \nabla w \cdot \nabla w \, dQ_T = \int_{\Sigma_T} \frac{\partial w}{\partial \nu} h \cdot \nabla w \, d\Sigma_T \tag{33}$$
$$+ \frac{1}{2} \int_{\Sigma_T} w_t^2 \, h \cdot \nu \, d\Sigma_T - \frac{1}{2} \int_{\Sigma_T} |\nabla w|^2 \, h \cdot \nu \, d\Sigma_T$$
$$- \frac{1}{2} \int_{Q_T} \left\{ w_t^2 - |\nabla w|^2 \right\} div \, h \, dQ_T - \left[(w_t, h \cdot \nabla w)_{L^2(\Omega)} \right]_0^T.$$

As $[w, w_t] \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$, we then note that

$$\text{on } \Sigma_{1T}: \ \ w_t = 0; \ \left| \frac{\partial w}{\partial \nu} \right| = |\nabla w|; \ h \cdot \nabla w = h \cdot \nu \frac{\partial w}{\partial \nu};$$

and thus

$$\int_{\Sigma_{1T}} \frac{\partial w}{\partial \nu} h \cdot \nabla w \, d\Sigma_{1T} + \frac{1}{2} \int_{\Sigma_{1T}} w_t^2 \, h \cdot \nu \, d\Sigma_{1T} - \frac{1}{2} \int_{\Sigma_{1T}} |\nabla w|^2 \, h \cdot \nu \, d\Sigma_{1T} \tag{34}$$
$$= \frac{1}{2} \int_{\Sigma_{1T}} |\nabla w|^2 \, h \cdot \nu \, d\Sigma_{1T} \le 0,$$

after using the condition (H2)(i). Inserting the inequality (34) into (33) will therefore yield

$$\begin{aligned}\int_{Q_T} H\nabla w\cdot\nabla w\,dQ_T \le & \int_{\Sigma_{0T}} \frac{\partial w}{\partial\nu} h\cdot\nabla w\,d\Sigma_{0T} + \frac{1}{2}\int_{\Sigma_{0T}} w_t^2\, h\cdot\nu\,d\Sigma_{0T} \\ & -\frac{1}{2}\int_{\Sigma_{0T}} |\nabla w|^2\, h\cdot\nu\,d\Sigma_{0T} - \frac{1}{2}\int_{Q_T}\left\{w_t^2 - |\nabla w|^2\right\} div\, h\,dQ_T \\ & -\left[(w_t, h\cdot\nabla w)_{L^2(\Omega)}\right]_0^T. \end{aligned} \tag{35}$$

Hence, the implementation of condition (H2)(ii) and Cauchy–Schwarz on the estimate (35) will yield

$$\begin{aligned}\rho\int_{Q_T}|\nabla w|^2\,dQ_T \le C & \left[\int_{\Sigma_{0T}}\left[\left(\frac{\partial w}{\partial\nu}\right)^2 + w_t^2\right] d\Sigma_{0T} + \int_{\Sigma_{0T}}|\nabla w|^2\,d\Sigma_{0T}\right. \\ & \left. + \left\|A^{\frac{1}{2}}w(T)\right\|_{L^2(\Omega)} + \|w_t(T)\|_{L^2(\Omega)} + \|\overrightarrow{w}(0)\|_{H_1} - \frac{1}{2}\int_{Q_T}\left\{w_t^2 - |\nabla w|^2\right\} div\, h\,dQ_T\right]. \end{aligned} \tag{36}$$

Now, to handle the last term on the right hand side of (36), we multiply the wave equation (30) by $w\,div\,h$ and integrate by parts to obtain

$$\begin{aligned}\int_{Q_T}\left\{w_t^2 - |\nabla w|^2\right\} div\, h\,dQ_T = & \left[(w_t, w\,div\,h)_{L^2(\Omega)}\right]_0^T \\ & + \int_{Q_T} w\,\nabla\,(div\,h)\cdot\nabla w\,dQ_T - \int_{\Sigma_{0T}}\frac{\partial w}{\partial\nu} w\,div\,h\,d\Sigma_{0T}, \end{aligned} \tag{37}$$

after using Green's Theorem and the identity $\nabla(w\,div\,h)\cdot\nabla w = w\nabla(div\,h)\cdot\nabla w + |\nabla w|^2\,div\,h$. We thus have upon majorizing the right hand side of (37) with the use of Trace Theory and Poincaré's inequality,

$$\begin{aligned}\left|\int_{Q_T}\left\{w_t^2 - |\nabla w|^2\right\} div\, h\,dQ_T\right| \le C_1 & \left[\int_{Q_T} w^{\,2} dQ_T + \int_{\Sigma_{0T}}\left(\frac{\partial w}{\partial\nu}\right)^2 d\Sigma_{0T}\right. \\ & \left. + \left\|A^{\frac{1}{2}}w(T)\right\|_{L^2(\Omega)} + \|w_t(T)\|_{L^2(\Omega)} + \|\overrightarrow{w}(0)\|_{H_1}\right] + \frac{\epsilon}{4}\int_{Q_T}|\nabla w|^2\,dQ_T, \end{aligned} \tag{38}$$

where $\epsilon > 0$ is arbitrarily small, and where the noncrucial dependence of C_1 upon ϵ has not been noted. Thus for ϵ small enough, adding the inequalities (36) and (38) together yields

$$\begin{aligned}(\rho - \tfrac{\epsilon}{8})\int_{Q_T}|\nabla w|^2\,dQ_T \le C_1 & \left[\int_{\Sigma_{0T}}\left[\left(\frac{\partial w}{\partial\nu}\right)^2 + w_t^2\right] d\Sigma_{0T} + \int_{\Sigma_{0T}}|\nabla w|^2\,d\Sigma_{0T}\right. \\ & \left. + \left\|A^{\frac{1}{2}}w(T)\right\|_{L^2(\Omega)} + \|w_t(T)\|_{L^2(\Omega)} + \|\overrightarrow{w}(0)\|_{H_1} + \int_{Q_T} w^{\,2} dQ_T\right]. \end{aligned} \tag{39}$$

Moreover, (38) and (39) together gives

$$\begin{aligned}(\rho - \tfrac{\epsilon}{8})\int_{Q_T} w_t^2\,dQ_T \le C_1 & \left[\int_{\Sigma_{0T}}\left[\left(\frac{\partial w}{\partial\nu}\right)^2 + w_t^2\right] d\Sigma_{0T} + \int_{\Sigma_{0T}}|\nabla w|^2\,d\Sigma_{0T}\right. \\ & \left. + \left\|A^{\frac{1}{2}}w(T)\right\|_{L^2(\Omega)} + \|w_t(T)\|_{L^2(\Omega)} + \|\overrightarrow{w}(0)\|_{H_1} + \int_{Q_T} w^{\,2} dQ_T\right] \end{aligned} \tag{40}$$

(where the constants C_0 and C_1 above are not necessarily the same throughout). Replacing the interval $(0, T)$ by $(\epsilon, T-\epsilon)$ in the estimates (39) and (40), using the fact that on Γ, $|\nabla z|^2 =$

$\left(\frac{\partial z}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2$, and a subsequent application of Lemma A will give the desired estimate (31). The dependence on time of the constant C_T results then from the use of the microlocal estimate Lemma A. ∎

Upon the combination of Proposition 1 and Lemma 1 (applied now to the solution component $\overrightarrow{z}$ of (1)–(2)), we then have the preliminary estimate

$$\int_\epsilon^{T-\epsilon} E(\overrightarrow{z}, \overrightarrow{v}, t)\, dt \le C_T \left[\int_0^T \left(\|N^* A z_t\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right\|^2_{L^2(\Gamma_0)} + \|g_1(N^* A z_t)\|^2_{L^2(\Gamma_0)} \right) dt \right.$$
$$\left. + \int_0^T \left\| g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right) \right\|^2_{L^2(\Gamma_0)} dt + \|z\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} + \|v\|^2_{L^2(\Sigma_{0T})} \right]$$
$$+ C\left[E(\overrightarrow{z}, \overrightarrow{v}, T-\epsilon) + E(\overrightarrow{z}, \overrightarrow{v}, \epsilon) \right] \tag{41}$$

where C_T depends upon T, but C does not. Applying the dissipativity property inherent in the relation (19), viz. $\forall\, T \ge t_0 \ge 0$

$$E(\overrightarrow{z}, \overrightarrow{v}, t_0) = E(\overrightarrow{z}, \overrightarrow{v}, T)$$
$$+ \int_{t_0}^T \left[(g_1(N^* A z_t), N^* A z_t)_{L^2(\Gamma_0)} + \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right), \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right)_{L^2(\Gamma_0)} \right] dt, \tag{42}$$

we obtain

Proposition 2 *For time T large enough, the following estimate holds for the solution $[\overrightarrow{z}, \overrightarrow{v}]$ of (1)–(2):*

$$E(\overrightarrow{z}, \overrightarrow{v}, T) \le \frac{C_T + 2C}{T - 2C} \left[\int_0^T \left(\|N^* A z_t\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right\|^2_{L^2(\Gamma_0)} + \|g_1(N^* A z_t)\|^2_{L^2(\Gamma_0)} \right) dt \right.$$
$$\left. + \int_0^T \left\| g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right) \right\|^2_{L^2(\Gamma_0)} dt + \|z\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} + \|v\|^2_{L^2(\Sigma_{0T})} \right]. \tag{43}$$

Via a "nonlinear" compactness/uniqueness argument we now proceed to eliminate the lower order terms $\|v\|^2_{L^2(\Sigma_{0T})}$ and $\|z\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)}$.

Lemma 2 *With T sufficiently large, the inequality (43) implies that there exists a nonnegative constant $C(E(\overrightarrow{z}, \overrightarrow{v}, 0))$ such that the solution $[\overrightarrow{z}, \overrightarrow{v}]$ of (1)–(2) obeys the following inequality:*

$$\|v\|^2_{C([0,T];L^2(\Gamma_0))} + \|z\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} \le C(E(\overrightarrow{z}, \overrightarrow{v}, 0)) \left\{ \int_0^T \left[\left\| z_t|_{\Gamma_0} \right\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right\|^2_{L^2(\Gamma_0)} \right.\right.$$
$$\left.\left. + \|g_1(N^* A z_t)\|^2_{L^2(\Gamma_0)} + \left\| g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \right) \right\|^2_{L^2(\Gamma_0)} \right] dt \right\}, \tag{44}$$

where the constant $C(E(\overrightarrow{z}, \overrightarrow{v}, 0))$ remains bounded for bounded values of $E(\overrightarrow{z}, \overrightarrow{v}, 0)$.

Proof: If the lemma is false, there then exists a sequence $\left\{\overrightarrow{z_0^{(n)}}, \overrightarrow{v_0^{(n)}}\right\}_{n=1}^{\infty}$ and a corresponding sequence $\left\{\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right\}_{n=1}^{\infty}$ which satisfies for all n,

$$\begin{cases} z_{tt}^{(n)} = \Delta z^{(n)} \quad \text{on } \Omega \times (0,T) \\ \dfrac{\partial z^{(n)}}{\partial \nu} = v_t^{(n)} - g_1\left(z_t^{(n)}\right) \text{ on } \Gamma_0 \times (0,T) \\ z^{(n)} = 0 \text{ on } \Gamma_1 \times (0,T) \\ \overrightarrow{z^{(n)}}(t=0) = \overrightarrow{z_0^{(n)}} \in H^1(\Omega) \times L^2(\Omega); \end{cases} \tag{45}$$

$$\begin{cases} v_{tt}^{(n)} = -\mathring{\mathbf{A}} v^{(n)} - \mathring{\mathbf{A}}^{\frac{1}{2}} g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) - z_t^{(n)} \text{ on } \Gamma_0 \times (0,T) \\ \overrightarrow{v^{(n)}}(t=0) = \overrightarrow{v_0^{(n)}} \in D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right) \times L^2(\Gamma_0), \end{cases} \tag{46}$$

with

$$\lim_{n\to\infty} \frac{\left\|v^{(n)}\right\|^2_{C([0,T];L^2(\Gamma_0))} + \left\|z^{(n)}\right\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)}}{\int_0^T \left[\left\|z_t^{(n)}\Big|_{\Gamma_0}\right\|^2_{L^2(\Gamma_0)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right\|^2_{L^2(\Gamma_0)} + \left\|g_1\left(N^* A z_t^{(n)}\right)\right\|^2_{L^2(\Gamma_0)} + \left\|g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right)\right\|^2_{L^2(\Gamma_0)}\right] dt} = \infty, \tag{47}$$

while the sequence of initial energy $\left\{E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, 0\right)\right\}_{n=1}^{\infty}$ is uniformly bounded in n. By the energy relation (19), the sequence $\left\{E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, t\right)\right\}_{n=1}^{\infty}$ is also bounded uniformly for $0 \le t \le T$, and consequently there exists a subsequence, still denoted by $\left\{\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right\}_{n=1}^{\infty}$, and $[\overrightarrow{z}, \overrightarrow{v}] = [z, z_t, v, v_t]$ such that

$$\overrightarrow{z^{(n)}} \to \overrightarrow{z} \text{ in } L^\infty(0,T;H_1) \text{ weak star;} \tag{48}$$

$$\overrightarrow{v^{(n)}} \to \overrightarrow{v} \text{ in } L^\infty(0,T;H_0) \text{ weak star.} \tag{49}$$

We also have that $z^{(n)} \to z$ weakly in $H^1(Q_T)$, and consequently by a classic compactness theorem (see [16], p. 99, Theorem 16.1) and Sobolev Trace Theory,

$$z^{(n)} \to z \text{ in } H^{\frac{1}{2}+\epsilon}(Q_T) \text{ strongly.} \tag{50}$$

Moreover, we deduce from (49) and a compactness result of Simon's (see [17], Corollary 4) that

$$v^{(n)} \to v \text{ in } C([0,T];L^2(\Gamma_0)) \text{ strongly.} \tag{51}$$

We now consider two possibilities:

Case I. $[\overrightarrow{z}, \overrightarrow{v}] \neq 0$. Then with this assumption, the inequality (47) implies that $\left\{N^* A z_t^{(n)}\right\}$, $\left\{\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right\}$, $\left\{g_1\left(N^* A z_t^{(n)}\right)\right\}$ and $\left\{g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right)\right\}$ each converge to 0 in $L^2(\Sigma_{0T})$. Upon passage to the limit in (45)–(46), we then have that $[\overrightarrow{z}, \overrightarrow{v}]$ satisfies the system

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } \Omega \times (0,T) \\ \dfrac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma_0 \times (0,T) \\ z = 0 \text{ on } \Gamma_1 \times (0,T) \end{cases} \tag{52}$$

$$0 = \mathbf{\mathring{A}} v \text{ on } \Gamma_0 \times (0,T) \tag{53}$$

As $\mathbf{\mathring{A}}$ is invertible, we immediately have that

$$v = 0. \tag{54}$$

Moreover, if we make the change of variable, $w = z_t$, then w solves

$$\begin{cases} w_{tt} = \Delta w \quad \text{on } \Omega \times (0,T) \\ \dfrac{\partial w}{\partial \nu} = w = 0 \text{ on } \Gamma_0 \times (0,T) \\ w = 0 \text{ on } \Gamma_1 \times (0,T). \end{cases} \tag{55}$$

By Holmgren's Uniqueness Theorem, we have that $z_t = 0$ on Q_T, and consequently

$$z = 0, \tag{56}$$

after using the ellipticity of A. So $[\overrightarrow{z}, \overrightarrow{v}] = 0$, which contradicts our opening assumption.

Case II. $[\overrightarrow{z}, \overrightarrow{v}] = 0$. In this case, denoting

$$C_n \equiv \left(\left\| v^{(n)} \right\|^2_{C([0,T];L^2(\Gamma_0))} + \left\| z^{(n)} \right\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} \right)^{\frac{1}{2}}; \tag{57}$$

$$\left[\overrightarrow{\widetilde{z}^{(n)}}, \overrightarrow{\widetilde{v}^{(n)}} \right] \equiv \frac{1}{C_n} \left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}} \right]; \tag{58}$$

then

$$\left\| \widetilde{v}^{(n)} \right\|^2_{C([0,T];L^2(\Gamma_0))} + \left\| \widetilde{z}^{(n)} \right\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} = 1 \quad \text{for every } n, \tag{59}$$

and as $[\overrightarrow{z}, \overrightarrow{v}] = 0$, we have from (50) and (51) that $\lim_{n\to\infty} C_n = 0$. Also, one has *a fortiori* that

$\left[\overrightarrow{\widetilde{z}^{(n)}}, \overrightarrow{\widetilde{v}^{(n)}}\right]$ satisfies

$$\begin{cases} \widetilde{z}_{tt}^{(n)} = \Delta \widetilde{z}^{(n)} \quad \text{on } \Omega \times (0,T) \\ \dfrac{\partial \widetilde{z}^{(n)}}{\partial \nu} = \widetilde{v}_t^{(n)} - \dfrac{g_1\left(\widetilde{z}_t^{(n)}\right)}{C_n} \text{ on } \Gamma_0 \times (0,T) \\ \widetilde{z}^{(n)} = 0 \text{ on } \Gamma_1 \times (0,T) \\ \overrightarrow{\widetilde{z}^{(n)}}(t=0) = \overrightarrow{\widetilde{z}_0^{(n)}} \in H^1(\Omega) \times L^2(\Omega); \end{cases} \tag{60}$$

$$\begin{cases} \widetilde{v}_{tt}^{(n)} = -\mathring{\mathbf{A}} \widetilde{v}^{(n)} - \dfrac{\mathring{\mathbf{A}}^{\frac{1}{2}} g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} \widetilde{v}_t^{(n)}\right)}{C_n} - \widetilde{z}_t^{(n)} \text{ on } \Gamma_0 \times (0,T) \\ \overrightarrow{\widetilde{v}^{(n)}}(t=0) = \overrightarrow{\widetilde{v}_0^{(n)}} \in D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right) \times L^2(\Gamma_0), \end{cases} \tag{61}$$

with $\left[\overrightarrow{\widetilde{z}_0^{(n)}}, \overrightarrow{\widetilde{v}_0^{(n)}}\right] = \frac{1}{C_n}\left[\overrightarrow{z_0^{(n)}}, \overrightarrow{v_0^{(n)}}\right]$. In addition, (58) and (47) imply that

$$N^* A \widetilde{z}_t^{(n)},\ \mathring{\mathbf{A}}^{\frac{1}{2}} \widetilde{v}_t^{(n)} \to 0 \text{ in } L^2(0,T; L^2(\Gamma_0)) \text{ as } n \to \infty. \tag{62}$$

Moreover, using the dissipative relation (42) (applied to $\left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right]$), followed by the estimate (43), we have for all $t \in (0,T]$,

$$E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, t\right) dt \le C_T \int_0^T \left[\left\| \left. z_t^{(n)} \right|_{\Gamma_0} \right\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)} \right\|^2_{L^2(\Gamma_0)} + \left\| g_1\left(N^* A z_t^{(n)}\right) \right\|^2_{L^2(\Gamma_0)} \right.$$
$$\left. + \left\| g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) \right\|^2_{L^2(\Gamma_0)} + \left\| v^{(n)} \right\|^2_{C([0,T];L^2(\Gamma_0))} + \left\| z^{(n)} \right\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} \right] \tag{63}$$

(where the constant C_T here is different from that in (43)). Dividing both sides of this inequality by C_n, we then have that $E\left(\overrightarrow{\widetilde{z}^{(n)}}, \overrightarrow{\widetilde{v}^{(n)}}, t\right)$ is uniformly bounded for $0 \le t \le T$, and thus there is a subsequence $\left\{\left[\overrightarrow{\widetilde{z}^{(n)}}, \overrightarrow{\widetilde{v}^{(n)}}\right]\right\}$ and $\left[\overrightarrow{\widetilde{z}}, \overrightarrow{\widetilde{v}}\right]$ such that

$$\begin{aligned} \overrightarrow{\widetilde{z}^{(n)}} &\to \overrightarrow{\widetilde{z}} \text{ in } L^\infty(0,T;H_1) \text{ weak star;} \\ \overrightarrow{v^{(n)}} &\to \overrightarrow{v} \text{ in } L^\infty(0,T;H_0) \text{ weak star;} \\ \widetilde{z}^{(n)} &\to \widetilde{z} \text{ in } H^{\frac{1}{2}+\epsilon}(Q_T) \text{ strongly;} \\ \left.\widetilde{z}^{(n)}\right|_{\Gamma_0} &\to \left.\widetilde{z}\right|_{\Gamma_0} \text{ in } L^2\left(\Sigma_{0T}\right) \text{ strongly;} \\ \widetilde{v}^{(n)} &\to \widetilde{v} \text{ in } C([0,T];L^2(\Gamma_0)) \text{ strongly.} \end{aligned}$$

The last two convergences above and (59) yield that

$$\|\widetilde{v}\|^2_{C([0,T];L^2(\Gamma_0))} + \|\widetilde{z}\|^2_{H^{\frac{1}{2}+\epsilon}(Q_T)} = 1. \tag{64}$$

But as $\dfrac{g_1\left(N^*Az_t^{(n)}\right)}{C_n}, \dfrac{g_2\left(\mathbf{\mathring{A}}v_t^{(n)}\right)}{C_n} \to 0$ in $L^2(0,T;L^2(\Gamma_0))$, by (47), we can then pass to the limit in (60)–(61), after recalling the convergences in (62); and subsequently invoking ellipticity and Holmgren's Theorem, as was done in the final part of Case I, we arrive at $\overrightarrow{z} = 0$ and $\overrightarrow{v} = 0$, a conclusion which contradicts (64).

The proof of Lemma 2 is hence complete.

We thus have upon combining Proposition and Lemma 2

Proposition 3 *For $T > 0$ large enough, the solution $[\overrightarrow{z}, \overrightarrow{v}]$ of (1)–(2) satisfies*

$$E\left(\overrightarrow{z}, \overrightarrow{v}, T\right) \leq C_T\left(E\left(\overrightarrow{z}, \overrightarrow{v}, 0\right)\right)\int_0^T \left[\left\|N^*Az_t\right\|^2_{L^2(\Gamma_0)} + \left\|\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right\|^2_{L^2(\Gamma_0)} + \left\|g_1\left(N^*Az_t\right)\right\|^2_{L^2(\Gamma_0)} + \left\|g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\right\|^2_{L^2(\Gamma_0)}\right] dt, \quad (65)$$

where the constant $C_T\left(E\left(\overrightarrow{z}, \overrightarrow{v}, 0\right)\right)$ remains bounded for bounded values of $E\left(\overrightarrow{z}, \overrightarrow{v}, 0\right)$.

2.1 Conclusion of Theorem 2.

Let

$$\Sigma_\alpha \equiv \left\{(t,x) \in \Sigma_{0T} \ni |v_t| > 1 \ a.e.\right\};$$

$$\Sigma_\beta \equiv \Sigma_{0T} \setminus \Sigma_\alpha.$$

Then using hypothesis (H1)(iii), we obtain

$$\int_{\Sigma_\alpha}\left(g_2^2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right) + \left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)^2\right) d\Sigma_\alpha \leq \left(m_2^{-1} + M_2\right)\int_{\Sigma_\alpha} g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t d\Sigma_\alpha. \quad (66)$$

Moreover, from (21)

$$\int_{\Sigma_\beta}\left(g_2^2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right) + \left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)^2\right) d\Sigma_\beta \leq \int_{\Sigma_\beta} f_2\left(g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right) d\Sigma_\beta. \quad (67)$$

Then by Jensen's inequality,

$$\begin{aligned}\int_{\Sigma_\beta} f_2\left(g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right) d\Sigma_\beta &\leq meas\left(\Sigma_{0T}\right) f_2\left(\frac{1}{meas\left(\Sigma_{0T}\right)}\int_{\Sigma_\beta} g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t d\Sigma_\beta\right) \\ &= meas\left(\Sigma_{0T}\right)\widetilde{f}_2\left(\int_{\Sigma_\beta} g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t d\Sigma_\beta\right), \quad (68)\end{aligned}$$

where $\widetilde{f}_2\left(s\right) \equiv f_2\left(\dfrac{s}{meas\left(\Sigma_{0T}\right)}\right)$. Thus,

$$\int_{\Sigma_{0T}}\left(g_2^2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right) + \left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)^2\right) d\Sigma_{0T} \leq \left(m_2^{-1} + M_2\right)\int_{\Sigma_{0T}} g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t d\Sigma_{0T} + meas\left(\Sigma_{0T}\right)\widetilde{f}_2\left(\int_{\Sigma_{0T}} g_2\left(\mathbf{\mathring{A}}^{\frac{1}{2}}v_t\right)\mathbf{\mathring{A}}^{\frac{1}{2}}v_t d\Sigma_{0T}\right). \quad (69)$$

In the very same way as was done above, we have

$$\int_{\Sigma_{0T}}\left(g_1^2\left(N^*Az_t\right)+\left(N^*Az_t\right)^2\right)d\Sigma_{0T}\le\left(m_1^{-1}+M_1\right)\int_{\Sigma_{0T}}g_1\left(N^*Az_t\right)N^*Az_t d\Sigma_{0T}$$
$$+meas\left(\Sigma_{0T}\right)\widetilde{f}_1\left(\int_{\Sigma_{0T}}g_1\left(N^*Az_t\right)N^*Az_t d\Sigma_{0T}\right), \tag{70}$$

where $\widetilde{f}_1(s)\equiv f_1\left(\frac{s}{meas\left(\Sigma_{0T}\right)}\right)$. Splicing together (65) and (69)–(70), and further recalling the definition (22), we have

$$E\left(\overrightarrow{z},\overrightarrow{v},T\right)$$
$$\le C_T\left(E\left(\overrightarrow{z},\overrightarrow{v},0\right)\right)\left[M\int_{\Sigma_{0T}}\left[g_1\left(N^*Az_t\right)N^*Az_t d\Sigma_{0T}+g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t\right)\mathring{\mathbf{A}}^{\frac{1}{2}}v_t\right]d\Sigma_{0T}\right.$$
$$\left.+meas\left(\Sigma_{0T}\right)\widetilde{f}\left(\int_{\Sigma_{0T}}\left[g_1\left(N^*Az_t\right)N^*Az_t+g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t\right)\mathring{\mathbf{A}}^{\frac{1}{2}}v_t\right]d\Sigma_{0T}\right)\right], \tag{71}$$

where $M\equiv m_1^{-1}+M_1+m_2^{-1}+M_2$. Setting

$$K \equiv \frac{1}{C_T\left(E\left(\overrightarrow{z},\overrightarrow{v},0\right)\right)meas\left(\Sigma_{0T}\right)};$$
$$c \equiv \frac{M}{meas\left(\Sigma_{0T}\right)},$$

we then obtain

$$p\left[E\left(\overrightarrow{z},\overrightarrow{v},T\right)\right] \le \int_{\Sigma_{0T}}\left[g_1\left(N^*Az_t\right)N^*Az_t+g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t\right)\mathring{\mathbf{A}}^{\frac{1}{2}}v_t\right]d\Sigma_{0T}$$
$$= E\left(\overrightarrow{z},\overrightarrow{v},0\right)-E\left(\overrightarrow{z},\overrightarrow{v},T\right), \tag{72}$$

where the function p is as defined in (23). To finish the proof of Theorem 2, we invoke the following result from [14]:

Lemma B. *Let p be a positive, increasing function such that $p(0)=0$. Since p is increasing, we can define an increasing function q, $q(x)\equiv x-(I+p)^{-1}(x)$. Consider a sequence s_n of positive numbers which satisfies*

$$s_{m+1}+p(s_{m+1})\le s_m.$$

Then $s_m\le\mathcal{S}(m)$, where $\mathcal{S}(t)$ is a solution of the differential equation

$$\frac{d}{dt}\mathcal{S}(t)+q\left(\mathcal{S}(t)\right)=0,\ \ \mathcal{S}(0)=s_0.$$

Moreover, if $p(x)>0$ for $x>0$, then $\lim_{n\to\infty}\mathcal{S}(t)=0$.

With this result in mind, we replace T (resp. 0) in (72) with $n(T+1)$ (resp. nT) to obtain

$$E\left(\overrightarrow{z},\overrightarrow{v},n(T+1)\right)+p\left(E\left(\overrightarrow{z},\overrightarrow{v},n(T+1)\right)\right)\le E\left(\overrightarrow{z},\overrightarrow{v},nT\right), \tag{73}$$

for $n = 0, 1, ...$ Applying Lemma B with $s_m \equiv E(\vec{z}, \vec{v}, mT)$ thus results in

$$E(\vec{z}, \vec{v}, nT) \leq \mathcal{S}(n), \quad n = 0, 1, ... \tag{74}$$

Finally, using the dissipativity of $E(\vec{z}, \vec{v}, t)$ inherent in the relation (42), we have for $t = nT + \tau$, $0 \leq \tau \leq T$,

$$E(\vec{z}, \vec{v}, t) \leq E(\vec{z}, \vec{v}, nT) \leq \mathcal{S}(n) \leq \mathcal{S}\left(\frac{t-\tau}{T}\right) \leq \mathcal{S}\left(\frac{t}{T} - 1\right) \text{ for } t > T, \tag{75}$$

where we have used above the fact that $\mathcal{S}(\cdot)$ is dissipative.

The proof of Theorem 2 is now complete.

3 Appendix–Proof of Theorem 1

Proof of Theorem 1(i). For the proof of well–posedness for the system (1)–(2), we invoke here nonlinear semigroup theory (see [5] for a general treatise). To wit, for $[\vec{z}, \vec{v}] \equiv [z_1, z_2, v_1, v_2] \in H_1 \times H_0$, we will define the operator $\mathcal{A} : D(\mathcal{A}) \subset H_1 \times H_0 \to H_1 \times H_0$ as

$$\mathcal{A}\begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} z_2 \\ -Az_1 - ANg_1(N^*Az_2) + ANv_2 \\ v_2 \\ -N^*Az_2 - \mathring{\mathbf{A}}v_1 - \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) \end{bmatrix}, \tag{76}$$

with $D(\mathcal{A}) = \Big\{[\vec{z}, \vec{v}] \in \left[D\left(A^{\frac{1}{2}}\right)\right]^2 \times \left[D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right)\right]^2 \ni z_1 + Ng_1(N^*Az_2) - Nv_2 \in D(A)$ and $v_1 - g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) \in D\left(\mathring{\mathbf{A}}\right)\Big\}$, and proceed to show that $-\mathcal{A}$ is maximal monotone.

For the monotonicity: If $[\overrightarrow{z}, \overrightarrow{v}]$ and $\left[\overrightarrow{\widetilde{z}}, \overrightarrow{\widetilde{v}}\right] \in D(\mathcal{A})$, then

$$
\begin{aligned}
&\left(\mathcal{A}\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} - \mathcal{A}\begin{bmatrix} \overrightarrow{\widetilde{z}} \\ \overrightarrow{\widetilde{v}} \end{bmatrix}, \begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} - \begin{bmatrix} \overrightarrow{\widetilde{z}} \\ \overrightarrow{\widetilde{v}} \end{bmatrix}\right)_{H_1\times H_0} = \\
&\left(\begin{bmatrix} z_2 - \widetilde{z}_2 \\ -Az_1 - ANg_1\left(N^*Az_2\right) + ANv_2 - \left(-A\widetilde{z}_1 - ANg_1\left(N^*A\widetilde{z}_2\right) + AN\widetilde{v}_2\right) \\ v_2 - \widetilde{v}_2 \\ -N^*Az_2 - \mathring{\mathbf{A}}v_1 - \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) - \left(-N^*A\widetilde{z}_2 - \mathring{\mathbf{A}}\widetilde{v}_1 - \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{v}_2\right)\right) \end{bmatrix}, \begin{bmatrix} z_1 - \widetilde{z}_1 \\ z_2 - \widetilde{z}_2 \\ v_1 - \widetilde{v}_1 \\ v_2 - \widetilde{v}_2 \end{bmatrix}\right)_{H_1\times H_0} \\
= \;& \left(A^{\frac{1}{2}}(z_2 - \widetilde{z}_2), A^{\frac{1}{2}}(z_1 - \widetilde{z}_1)\right)_{L^2(\Omega)} + \langle AN(v_2 - \widetilde{v}_2), (z_2 - \widetilde{z}_2)\rangle_{[D(A^{\frac{1}{2}})]'\times D(A^{\frac{1}{2}})} \\
&- \left(A^{\frac{1}{2}}(z_1 - \widetilde{z}_1), A^{\frac{1}{2}}(z_2 - \widetilde{z}_2)\right)_{L^2(\Omega)} - \langle ANg_1(N^*Az_2) - ANg_1(N^*A\widetilde{z}_2), z_2 - \widetilde{z}_2\rangle_{[D(A^{\frac{1}{2}})]'\times D(A^{\frac{1}{2}})} \\
&+ \left(\mathring{\mathbf{A}}^{\frac{1}{2}}(v_2 - \widetilde{v}_2), \mathring{\mathbf{A}}^{\frac{1}{2}}(v_1 - \widetilde{v}_1)\right)_{L^2(\Gamma_0)} - \left(N^*Az_2 - N^*A\widetilde{z}_2), (v_2 - \widetilde{v}_2)\right)_{L^2(\Gamma_0)} \\
&- \left(\mathring{\mathbf{A}}^{\frac{1}{2}}(v_1 - \widetilde{v}_1), \mathring{\mathbf{A}}^{\frac{1}{2}}(v_2 - \widetilde{v}_2)\right)_{L^2(\Gamma_0)} - \left\langle \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) - \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{v}_2\right), v_2 - \widetilde{v}_2\right\rangle_{[D(\mathring{\mathbf{A}}^{\frac{1}{2}})]'\times D(\mathring{\mathbf{A}}^{\frac{1}{2}})} \\
= \;& -\left(g_1(N^*Az_2) - g_1(N^*A\widetilde{z}_2), N^*Az_2 - N^*A\widetilde{z}_2\right)_{L^2(\Omega)} - \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) - g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{v}_2\right), \mathring{\mathbf{A}}^{\frac{1}{2}}v_2 - \mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{v}_2\right)_{L^2(\Gamma_0)} \\
\leq \;& 0,
\end{aligned}
$$

after using the hypotheses (H1) (note in particular that (H1)(iii) imply that $g_1(N^*Az_2)$, $g_2(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2) \in L^2(\Gamma_0)$ for $z_2 \in D\left(A^{\frac{1}{2}}\right)$ and $v_2 \in D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right)$). $-\mathcal{A}$ is thus monotone.

To show that $-\mathcal{A}$ is maximal, we will verify the equivalent statement $\mathcal{R}(I - \mathcal{A})) = H_1 \times H_0$: For arbitrary $\left[\overrightarrow{\phi}, \overrightarrow{\psi}\right] \in H_1 \times H_0$, if there exists $[\overrightarrow{z}, \overrightarrow{v}] \in D(\mathcal{A})$ such that

$$\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} - \mathcal{A}\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} = \begin{bmatrix} \overrightarrow{\phi} \\ \overrightarrow{\psi} \end{bmatrix}, \tag{77}$$

then

$$\begin{cases} z_1 - z_2 = \phi_1 \\ z_2 + Az_1 + ANg_1(N^*Az_2) - ANv_2 = \phi_2 \\ v_1 - v_2 = \psi_1 \\ v_2 + N^*Az_2 + \mathring{\mathbf{A}}v_1 + \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) = \psi_2 \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} z_2 + Az_2 + ANg_1(N^*Az_2) - ANv_2 = \phi_2 - A\phi_1 \\ v_2 + N^*Az_2 + \mathring{\mathbf{A}}v_2 + \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right) = \psi_2 - \mathring{\mathbf{A}}\psi_1 \end{cases}$$

$$\Longleftrightarrow$$

$$(\mathbf{F} + \mathbf{G})\begin{bmatrix} z_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \phi_2 - A\phi_1 \\ \psi_2 - \mathring{\mathbf{A}}\psi_1 \end{bmatrix}, \tag{78}$$

where $\mathbf{F}$, $\mathbf{G}$: $D\left(A^{\frac{1}{2}}\right) \times D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right) \to \left[D\left(A^{\frac{1}{2}}\right)\right]' \times \left[D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right)\right]'$ are defined by

$$\mathbf{F} \equiv \begin{bmatrix} \mathbf{I} + A & -AN \\ N^*A & \mathbf{I} + \mathring{\mathbf{A}} \end{bmatrix} \text{ and } \mathbf{G}\begin{bmatrix} z \\ v \end{bmatrix} \equiv \begin{bmatrix} ANg_1(N^*Az) \\ \mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v\right) \end{bmatrix}. \tag{79}$$

In regard to the sum of these operators, we have the following:

Proposition 4 $\mathcal{R}(\boldsymbol{F} + \boldsymbol{G}) = \left[D\left(A^{\frac{1}{2}}\right)\right]' \times \left[D\left(\mathring{\boldsymbol{A}}^{\frac{1}{2}}\right)\right]'$.

Proof: *A fortiori*, $\mathbf{F} \in \mathcal{L}\left(D\left(A^{\frac{1}{2}}\right) \times D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right), \left[D\left(A^{\frac{1}{2}}\right)\right]' \times \left[D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right)\right]'\right)$ is coercive, so if we can establish that $\mathbf{G}$ is maximal monotone, the asserted result will promptly follow upon application of Corollary 1.3 of [5] (p. 48). To this end, we first consider the component $ANg_1(N^*Az)$ of $\mathbf{G}$: As g_1 is monotone increasing, then $g_1(\cdot) = \partial\Phi(\cdot)$ as a mapping from $L^2(\Gamma_0)$ into itself, where Φ is some proper, convex, lower semicontinuous functional on $L^2(\Gamma_0)$ (see [7], p. 37). In particular then,

$$g_1(N^*A(\cdot)) = \partial\Phi(N^*A(\cdot))$$

as a mapping from $D\left(A^{\frac{1}{2}}\right)$ into $L^2(\Gamma_0)$. Since $N^*A : D\left(A^{\frac{1}{2}}\right) \to H^{\frac{1}{2}}(\Gamma_0)$ is surjective, by (9) and Sobolev trace theory, we then can invoke Lemma 2.1 of [13] to obtain

$$\partial(\Phi N^*A(\cdot)) = ANg_1(N^*A(\cdot)).$$

This above equality directly yields that $ANg_1(N^*A(\cdot))$ is maximal monotone. By an almost identical line of reasoning, $\mathring{\mathbf{A}}^{\frac{1}{2}}g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}(\cdot)\right)$ is also maximal monotone, thereby giving that the entire structure $\mathbf{G}$ is maximal monotone. The proof of Proposition 4 is concluded. ■

Proposition 1 thus provides the existence of $[z_2, v_2] \in D\left(A^{\frac{1}{2}}\right) \times D\left(\mathring{\mathbf{A}}^{\frac{1}{2}}\right)$ which solves the operator equation (78), and setting

$$z_1 \equiv z_2 + \phi_1$$

$$v_1 \equiv v_2 + \psi_1,$$

then *a fortiori* $[\overrightarrow{z}, \overrightarrow{v}] = [z_1, z_2, v_1, v_2] \in D(\mathcal{A})$, and using the equivalence between the equations (77) and (78), we deduce that $\mathcal{R}(I - \mathcal{A}) = H_1 \times H_0$. Consequently, $-\mathcal{A}$ generates a nonlinear semigroup of contractions $\{\mathcal{T}(t)\}_{t \geq 0}$ on $\overline{D(\mathcal{A})} = H_1 \times H_0$, and so for initial data $[\overrightarrow{z_0}, \overrightarrow{v_0}] \in H_1 \times H_0$, $[\overrightarrow{z}(t), \overrightarrow{v}(t)] \equiv \mathcal{T}(t) \begin{bmatrix} \overrightarrow{z_0} \\ \overrightarrow{v_0} \end{bmatrix}$ will be the (weak) solution of (1)–(2), thereby proving Theorem 1(i).

Proposition 5 *Suppose, in addition to the hypotheses (H1), that g_1 and g_2 are coercive; viz. for all s_1, $s_2 \in \mathbb{R}$, there exist positive constants α_i, $i = 1, 2$, such that $(g_i(s_1) - g_i(s_2))(s_1 - s_2) \geq \alpha_i |s_1 - s_2|^2$. Then the following regularity properties hold true for the respective velocities of the solution $[z, v]$ to (1)–(2):*

$$z_t|_{\Gamma_0} \in L^2(0, \infty; L^2(\Gamma_0)); \tag{80}$$

$$v_t \in L^2(0, \infty; D(\mathring{\mathbf{A}}^{\frac{1}{2}})). \tag{81}$$

Consequently, because of (H1)(iii), we have also that $g_1(N^* A z_t)$ and $g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t\right) \in L^2(0, \infty; L^2(\Gamma_0))$.

Proof: If $[\overrightarrow{z_0}, \overrightarrow{v_0}] \in D(\mathcal{A})$, then by the semigroup property the corresponding solution $[\overrightarrow{z}, \overrightarrow{v}]$ satisfies

$$\frac{d}{dt} \begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} \in L^\infty(0, \infty; H_1 \times H_0), \tag{82}$$

with $[\overrightarrow{z}(t), \overrightarrow{v}(t)] \in D(\mathcal{A})$ for $t \geq 0$. So as $N^* A \in \mathcal{L}\left(D(A^{\frac{1}{2}}), H^{\frac{1}{2}}(\Gamma_0)\right)$, we deduce that

$$z_t|_{\Gamma_0} \in L^\infty(0, \infty; L^2(\Gamma_0)). \tag{83}$$

(82) further implies that

$$\mathring{\mathbf{A}}^{\frac{1}{2}} v_t \in L^\infty(0, \infty; L^2(\Gamma_0)); \tag{84}$$

and moreover the equation (13) and an application of $\mathring{\mathbf{A}}^{\frac{1}{2}}$ give

$$g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t\right) = -\mathring{\mathbf{A}}^{-\frac{1}{2}} v_{tt} - \mathring{\mathbf{A}}^{\frac{1}{2}} v - \mathring{\mathbf{A}}^{-\frac{1}{2}} N^* A z_t \in L^\infty(0, \infty; L^2(\Gamma_0)). \tag{85}$$

Because of (H1)(iii) and (83), we also have readily

$$g_1(N^* A z_t) \in L^2_{loc}(0, \infty; L^2(\Gamma_0)). \tag{86}$$

Thus, mindful of (83)–(86), we multiply the equation (12) by z_t, (13) by v_t, integrate in time and space, subsequently integrate by parts and invoke the coercivity assumption to obtain

$$E(\overrightarrow{z}, \overrightarrow{v}, T) + \int_0^T \left[\alpha_1 \|N^* A z_t\|^2_{L^2(\Gamma_0)} + \alpha_2 \left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v_t\right\|^2_{L^2(\Gamma_0)}\right] dt \leq E(\overrightarrow{z}, \overrightarrow{v}, 0), \tag{87}$$

where $E(\overrightarrow{z}, \overrightarrow{v}, t)$ is as defined in (3). Since $\overline{D(\mathcal{A})} = H_1 \times H_0$, the inequality above can be extended to all $[\overrightarrow{z_0}, \overrightarrow{v_0}] \in H_1 \times H_0$ so as to provide the regularity (80) and (81). Moreover, the assumption (H1)(iii), (80) and (81) will give $g_1(N^* A z_t)$, $g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t\right) \in L^2(0,\infty; L^2(\Gamma_0))$. ■

Proof of Theorem 1(ii): Following a theme played in [14], we will approximate the solution $[\overrightarrow{z}, \overrightarrow{v}]$ of (1)–(2), corresponding to fixed initial data $[z_0, v_0] \in H_1 \times H_0$, by the solution $\left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right]$ to the following parametrized system:

$$z_{tt}^{(n)} = -Az^{(n)} - ANg_1\left(N^* A z_t^{(n)}\right) - \frac{1}{n} ANN^* A z_t^{(n)} + ANv_t^{(n)} \text{ on } (0,\infty) \times \Omega; \tag{88}$$

$$v_{tt}^{(n)} = -\mathring{\mathbf{A}} v^{(n)} - \mathring{\mathbf{A}}^{\frac{1}{2}} g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) - \frac{1}{n}\mathring{\mathbf{A}} v_t^{(n)} - NA^* z_t^{(n)} \text{ on } (0,\infty) \times \Gamma_0; \tag{89}$$

$$[\overrightarrow{z}(t=0), \overrightarrow{v}(t=0)] = [\overrightarrow{z_0}, \overrightarrow{v_0}]. \tag{90}$$

As the functions $g_i^{(n)}(s) \equiv g_i(s) + \dfrac{s}{n}$, $i = 1, 2$, satisfy the assumptions (H1), there is indeed a solution $\left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right] \in C([0,\infty); H_1 \times H_0)$ to (88), and as each $g_i^{(n)}$ is coercive, Proposition 5 gives the additional regularity

$$\left. z_t^{(n)} \right|_{\Gamma_0} \in L^2(0,\infty; L^2(\Gamma_0)); \tag{91}$$

$$v_t^{(n)} \in L^2(0,\infty; D(\mathring{\mathbf{A}}^{\frac{1}{2}})). \tag{92}$$

We also have, after the use of (91), (92) and the hypotheses (H1)(iii), that

$$g_1(N^* A z_t^{(n)}) \quad \in \quad L^2(0,\infty; L^2(\Gamma_0)); \tag{93}$$

$$g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) \quad \in \quad L^2(0,\infty; L^2(\Gamma_0)). \tag{94}$$

We work now to obtain a bound on $E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, t\right)$, for $0 \le t \le T$ and fixed $T > 0$, which is uniform with respect to n. First off, the inclusions (91), (92) and (94) will allow for the equality in (89) to be taken in $C\left([0,\infty); \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'\right)$; so taking the duality pairing of both sides of (89) with respect to $v_t^{(n)}$ and subsequently integrating from 0 to t, $0 \le t \le T$, followed by an integration by parts (this procedure entirely justified by the regularity posted in (92)) will then give the relation

$$2\int_0^T \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) + \frac{1}{n}\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}, \mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right)_{L^2(\Gamma_0)} = -\int_0^T \left(N^* A z_t^{(n)}, v_t^{(n)}\right)_{L^2(\Gamma_0)} dt \tag{95}$$
$$+ \|\overrightarrow{v_0}\|_{H_0}^2 - \left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v^{(n)}(T)\right\|_{L^2(\Gamma_0)}^2 - \left\|v_t^{(n)}(T)\right\|_{L^2(\Gamma_0)}^2 dt.$$

The same sort of energy method cannot be directly applied to the wave component of $E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, T\right)$, since $\overrightarrow{z^{(n)}}$ need not smooth enough to justify an integration by parts. However, since $\left. z_t^{(n)}\right|_{\Gamma_0} \in L^2(0,\infty; L^2(\Gamma_0))$ and $g_1(N^* A z_t^{(n)}) \in L^2(0,\infty; L^2(\Gamma_0))$ for all n, Proposition 2.1 of [14] (p. 512) does in fact allow the desired energy relation for all $t > 0$

$$\|z_0\|_{H_1}^2 = 2\int_0^t \left[\left(g_1\left(N^* A z_t^{(n)}\right) + \frac{1}{n} N^* A z_t^{(n)}, N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} + \left(v_t^{(n)}, N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} \right] d\tau$$

$$+ \left\| A^{\frac{1}{2}} z^{(n)}(t) \right\|_{L^2(\Omega)}^2 + \left\| z_t^{(n)}(t) \right\|_{L^2(\Omega)}^2 dt. \tag{96}$$

Adding (95) and (96), we obtain for all n and t, $0 \le t \le T$,

$$E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, 0\right) = E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, t\right) + \int_0^t \left(g_1\left(N^* A z_t^{(n)}\right) + \frac{1}{n} N^* A z_t^{(n)}, N^* A z_t^{(n)} \right)_{L^2(\Gamma_0)} d\tau$$

$$+ \int_0^t \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) + \frac{1}{n} \mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}, \mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)} \right)_{L^2(\Gamma_0)} d\tau. \tag{97}$$

The energy relation above immediate gives that for all n and t, $0 \le t \le T$,

$$E\left(\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}, t\right) \le \|\overrightarrow{z_0}\|_{H_1} + \|\overrightarrow{v_0}\|_{H_0}. \tag{98}$$

Moreover, the same relation (97) and (H1)(iii) yield that for all n

$$\left\| N^* A z_t^{(n)} \right\|_{L^2(\Sigma_{0T})} + \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)} \right\|_{L^2(\Sigma_{0T})} \le C\left(\|\overrightarrow{z_0}\|_{H_1} + \|\overrightarrow{v_0}\|_{H_0} \right). \tag{99}$$

Thus, $\left\{ \left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}} \right] \right\}_{n=1}^{\infty}$ is bounded in $L^\infty(0,T; H_1 \times H_0)$ and $\left\{ \left[N^* A z_t^{(n)}, \mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)} \right] \right\}$ in $L^2(\Sigma_{0T})$; consequently there exists a subsequence, still denoted as $\left\{ \left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}} \right] \right\}_{n=1}^{\infty}$, and $\left[\overrightarrow{\widetilde{z}}, \overrightarrow{\widetilde{v}} \right] \in L^\infty(0,T; H_1 \times H_0)$ such that

$$\left[z^{(n)}, z_t^{(n)} \right] \quad \to \quad [\widetilde{z}, \widetilde{z}_t] \text{ weakly in } L^\infty(0,T; H_1); \tag{100}$$

$$v^{(n)} \quad \to \quad \widetilde{v} \text{ weakly in } L^\infty\left(0,T; D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right). \tag{101}$$

$$v_t^{(n)} \quad \to \quad \widetilde{v}_t \text{ weakly in } L^2\left(0,T; D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right). \tag{102}$$

Moreover, by Simon's compactness result (see [17]) $z^{(n)} \to \widetilde{z}$ in $L^\infty(0,T; H^{1-\epsilon}(\Omega))$ strongly $\forall\, \epsilon > 0$, and this convergence coupled with Sobolev Trace Theory gives us that

$$z^{(n)}\Big|_{\Gamma_0} \quad \to \quad \widetilde{z}|_{\Gamma_0} \text{ strongly in } L^\infty(0,T; L^2(\Gamma_0));$$

$$z_t^{(n)}\Big|_{\Gamma_0} \quad \to \quad \widetilde{z}_t|_{\Gamma_0} \text{ weakly in } L^2(0,T; L^2(\Gamma_0)). \tag{103}$$

In addition, $g_1\left(N^* A z_t^{(n)}\right)$ and $g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right)$ are both uniformly bounded in $L^2(0,T; L^2(\Gamma_0))$, using the hypotheses (H1)(iii) and (99), and so there are subsequences $\left\{ g_1\left(N^* A z_t^{(n)}\right) \right\}_{n=1}^{\infty}$, $\left\{ g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) \right\}_{n=1}^{\infty}$ and ω_1, $\omega_2 \in L^2(0,T; L^2(\Gamma_0))$, say, which satisfy

$$g_1\left(N^* A z_t^{(n)}\right) \quad \to \quad \omega_1 \text{ weakly in } L^2(0,T; L^2(\Gamma_0)); \tag{104}$$

$$g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}} v_t^{(n)}\right) \quad \to \quad \omega_2 \text{ weakly in } L^2(0,T; L^2(\Gamma_0)). \tag{105}$$

Letting $\left[\overrightarrow{z^{(m)}}, \overrightarrow{v^{(m)}}\right]$ and $\left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right]$ be elements of the solution sequence, and using the energy relation (97), this time applied to the solution $\left[\overrightarrow{\widehat{z}}, \overrightarrow{\widehat{v}}\right] \equiv \left[\overrightarrow{z^{(m)}} - \overrightarrow{z^{(n)}}, \overrightarrow{v^{(m)}} - \overrightarrow{v^{(n)}}\right]$ of the system

$$\widehat{z}_{tt} = -A\widehat{z} - AN\left(g_1\left(N^*Az_t^{(m)}\right) - g_1\left(N^*Az_t^{(n)}\right)\right) - ANN^*A\left(\frac{1}{m}z_t^{(m)} - \frac{1}{n}z_t^{(n)}\right) + AN\widehat{v}_t; \tag{106}$$

$$\widehat{v}_{tt} = -\mathring{\mathbf{A}}\widehat{v} - \mathring{\mathbf{A}}^{\frac{1}{2}}\left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(m)}\right) - g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(n)}\right)\right) - \mathring{\mathbf{A}}\left(\frac{1}{m}v_t^{(m)} - \frac{1}{n}v_t^{(n)}\right) - NA^*\widehat{z}_t; \tag{107}$$

$$\left[\overrightarrow{\widehat{z}}(t=0), \overrightarrow{\widehat{v}}(t=0)\right] = [0,0],$$

we obtain

$$\begin{aligned}
&\left\|A^{\frac{1}{2}}\left(z^{(m)}(t) - z^{(n)}(t)\right)\right\|^2_{L^2(\Omega)} + \left\|z_t^{(m)}(t) - z_t^{(n)}(t)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\left(v^{(m)}(t) - v^{(n)}(t)\right)\right\|^2_{L^2(\Gamma_0)} \\
&+ \left\|v_t^{(m)}(t) - v_t^{(n)}(t)\right\|^2_{L^2(\Gamma_0)} + \int_0^t \left(g_1\left(N^*Az_t^{(m)}\right) - g_1\left(N^*Az_t^{(n)}\right), N^*A(z_t^{(m)} - z_t^{(n)}\right)_{L^2(\Gamma_0)} dt \\
&+ \int_0^t \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(m)}\right) - g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(n)}\right), \mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(m)} - \mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(n)}\right)_{L^2(\Gamma_0)} dt \\
\leq\ &\left(\frac{1}{m} + \frac{1}{n}\right)\left[\left\|N^*Az_t^{(m)}\right\|^2_{L^2(\Sigma_{0T})} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(m)}\right\|^2_{L^2(\Sigma_{0T})} + \left\|N^*Az_t^{(n)}\right\|^2_{L^2(\Sigma_{0T})} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(n)}\right\|^2_{L^2(\Sigma_{0T})}\right].
\end{aligned} \tag{108}$$

The estimate above thus implies that actually $\left[\overrightarrow{z^{(n)}}, \overrightarrow{v^{(n)}}\right] \to \left[\overrightarrow{\widetilde{z}}, \overrightarrow{\widetilde{v}}\right]$ strongly in $C\left([0,T]; H_1 \times H_0\right)$, and of particular pertinence here,

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t \left[\left(g_1\left(N^*Az_t^{(n)}\right), N^*Az_t^{(n)}\right)_{L^2(\Gamma_0)} + \int_0^t \left(g_2\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(n)}\right), \mathring{\mathbf{A}}^{\frac{1}{2}}v_t^{(n)}\right)_{L^2(\Gamma_0)}\right] dt \\
&= \int_0^t (\omega_1, N^*A\widetilde{z}_t)_{L^2(\Gamma_0)} + \left(\omega_2, \mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{v}_t\right)_{L^2(\Gamma_0)},
\end{aligned} \tag{109}$$

where ω_1 (resp. ω_2) is the weak limit posted in (104) ((resp. 105)). With the limits (100)–(105) and (109), and the fact that each g_i is monotonic increasing, we can then invoke [7] (p. 37) and Lemma 1.3 of [5] (p.42) to infer that

$$\begin{aligned}
\omega_1 &= g_1(N^*A\widetilde{z}_t); \\
\omega_2 &= g_2(\mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{v}_t).
\end{aligned}$$

With these convergences, along with those posted in (100)–(105), one can then pass to the limit in (88)–(89) so as to obtain the deduction that the weak limit $\left[\overrightarrow{\widetilde{z}}, \overrightarrow{\widetilde{v}}\right] = [\overrightarrow{z}, \overrightarrow{v}]$, where again $[\overrightarrow{z}, \overrightarrow{v}]$ is the solution of (1)–(2). In particular then, we have by (103) and (102) the additional regularity

$$z_t| \in L^2(0,T; L^2(\Gamma_0)); \;\; v_t \in L^2(0,T; D(\mathring{\mathbf{A}}^{\frac{1}{2}})),$$

for arbitrary $T > 0$. The proof of the Theorem 1(ii) is concluded.

Proof of Theorem 1(iii): The details here are identical to the derivation of the parametrized energy relations (95)–(97), and so will not be repeated.

The proof of Theorem 1 is now complete.

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