# Invariance Principles for Parabolic Equations with Random Coefficients

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ABSTRACT. A general Hilbert-space-based stochastic averaging theory is brought forth in this note for arbitrary-order parabolic equations with (possibly long range dependent) random coefficients. We start with regularity conditions on the coefficients and Cauchy data of

$$\partial_t u^\varepsilon(x,t) = \sum_{0\le|k|\le 2p} A_k(x,t/\varepsilon,\omega)\partial_x^k u^\varepsilon(x,t),\ u^\varepsilon(x,0)=\varphi(x) \tag{1}$$

which are only slightly stronger than those required to prove pathwise existence and uniqueness for the above equation. Next, we impose on the coefficients of (1) a pointwise (in $x$ and $t$) weak law of large numbers and a weak invariance principle

$$\left\{\varepsilon^h \int_0^{t\varepsilon^{-1}} A_k(x,s) - A_k^0(x)\,ds\right\}_{|k|\le 2p} \Rightarrow \left\{\widehat{\Theta}_k\right\}_{|k|\le 2p} \tag{2}$$

in $C([0,T],\mathcal{H}_1),\mathcal{H}_1$ being a separable Hilbert space of functions and $h\in(0,1)$ denoting any fixed constant. ($h=1/2$ provides the weighting for the classical

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functional central limit theorem whereas $h \in (0, 1/2)$ allows for long range time dependence in the coefficients.) Then, under these extraordinarily general conditions, we infer a weak invariance principle of the form $\varepsilon^{h-1}(u^\varepsilon - u) \Rightarrow \widehat{y}$. Here $u$ is the non-random, $\varepsilon$-homogeneous solution of the equation

$$\partial_t u(x,t) = \sum_{0\le|k|\le 2p} A_k^0(x)\,\partial_x^k u(x,t), \quad u(x) = \varphi(x) \tag{3}$$

and $\widehat{y}$ is a Hilbert-space-valued stochastic evolution satisfying the linear stochastic partial differential equation

$$\partial_t \widehat{y}(t,x) = \sum_{|k|\le 2p} A_k^0(x)\,\partial_x^k \widehat{y}(t,x)dt + \sum_{|k|\le 2p} \widehat{\Theta}_k(dt,x)\,u_k(t,x) \tag{4}$$

mildly, where $u_k \doteq \partial_x^k u$. We have striven for singular generality by avoiding all Gaussian, semimartingale, Markov, or other conditions on $\left\{\widehat{\Theta}_k\right\}_{|k|\le 2p}$ and not requiring any mixing, moment, Markov or martingale-type assumptions on the coefficients but rather only the natural weak convergence conditions stated above. This generality covers the case where the coefficients have long range dependence in $t$ and the limit object is, for example, fractional Brownian motion as well as the classical situation where the dependence decays fast enough for a Hilbert space version of the Donsker-type functional central limit theorem to hold. In such situations $\left\{\widehat{\Theta}_k\right\}_{|k|\le 2p}$ is a Gaussian process and our main hypothesis (2) is satisfied if $\left\{\widehat{\Theta}_k(t)\right\}_{|k|\le 2p}$ has covariance which is trace class with respect to our Hilbert space $\mathcal{H}_1$ for all $t$. In particular, this trace class condition can be validated under the conditions of the previous work of Watanabe [Probab. Th. Rel. Fields, 77:359-378, 1988].

## 1. Introduction

Questions involving the asymptotic behavior (as $\varepsilon \to 0$) of a system of ordinary differential equations:

$$\dot{Z}^\varepsilon(t) = F(Z^\varepsilon(t), t/\varepsilon), \quad \varepsilon > 0, \quad Z^\varepsilon(0) = z_0, \tag{5}$$

were apparently first encountered in problems of celestial mechanics and have since become important in several areas of physics and engineering. The additional regularity which justifies the anticipation of some kind of asymptotic limit as $\varepsilon \to 0$ is that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(x,t)\,dt \doteq \bar{F}(x) \tag{6}$$

exists for each $x \in \Re^d$. Under this and other regulatory conditions, Bogoliubov (see [3]), Gikhman [12], and Besjes [2] proved versions of the classical *averaging principle* which states that the solution of (5) converges uniformly over intervals like $[0,T]$ to the solution of

$$\dot{Z}(t) = \bar{F}(Z(t)), \qquad Z(0) = z_0 \tag{7}$$

as $\varepsilon \to 0$. Nevertheless, some of the richest motivational sources for averaging require that $F(x,t)$ in (5) is a random field and the so-called *stochastic averaging principle* was borne out of a desire to retain the non-random nature of the asymptotic solution (7). Indeed, Khas'minskii [14] suggested comparing (5) to (7) when $\bar{F}$ is defined by "double averaging" i.e.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T EF(x,t)\,dt \doteq \bar{F}(x) \tag{8}$$

and established a functional central limit theorem for $\frac{1}{\sqrt{\varepsilon}}(Z^\varepsilon - Z)$. Freidlin [10], Kouritzin and Heunis [18], and Kouritzin and Heunis [19] have since established complimentary large deviation bounds, Prokhorov distance bounds, and a functional law of the iterated logarithm for $\frac{1}{\sqrt{\varepsilon}}(Z^\varepsilon - Z)$.

Whereas a theory for the random ordinary differential equations in (5) parallel to the classical weak and almost sure fluctuation results for partial sums of random elements appears to be unfolding, little has been done on systems of random parabolic partial differential equations. Khas'minskii [13], Bensoussan et al. [1], Zhikov et al. [23], Kurtz [21] (as an application of an abstract theorem), and Kouritzin [15], [16] established averaging principles for parabolic equations and, in the latest two cases, for their derivatives. However, the fluctuation problem for the stochastic averaging of parabolic partial differential equations has hardly been addressed. Suppose $u^\varepsilon$ and $u$ are the continuous, bounded, $\Re$-valued solutions to second-order parabolic equations with the specific forms

$$\partial_t u^\varepsilon(t,x) = \sum_{i,j} a_{ij}(\frac{t}{\varepsilon},x,\omega)\frac{\partial u^\varepsilon(t,x)}{\partial x_i \partial x_j} + \sum_i b_i(\frac{t}{\varepsilon},x,\omega)\frac{\partial u^\varepsilon(t,x)}{\partial x_i}, \quad \varepsilon > 0, \tag{9}$$

$$\partial_t u(t,x) = \sum_{i,j} \bar{a}_{ij}(x)\frac{\partial u(t,x)}{\partial x_i \partial x_j} + \sum_i \bar{b}_i(x)\frac{\partial u(t,x)}{\partial x_i}, \tag{10}$$

subject to $u^\varepsilon(0,x) = u(0,x) = \varphi(x)$ and

$$\bar{a}_{ij}(x) \doteq Ea_{ij}(t,x), \qquad \bar{b}_i(x) \doteq Eb_i(t,x)\ . \tag{11}$$

Then, Watanabe [22] shows under strict stationarity and many other conditions that $\frac{1}{\sqrt{\varepsilon}}(u^\varepsilon - u)$ converges weakly in $C([0,\infty);\mathcal{S}')$, $\mathcal{S}'$ being the space of tempered distributions, to a generalized Ornstein-Uhlenbeck process. In fact, Watanabe's theorem requires a limiting technical assumption (see assumption (A.VII)' in his paper) and uniform boundedness of the coefficients with respect to $\omega$. In this note, we explore a far more general weak convergence theory for the classical (see e.g. Chapters 1 and 9 of Friedman [11] or Chapter 1 of Eidel'man [7]) arbitrary-order parabolic partial differential equations

$$\partial_t u^\varepsilon(t,x,\omega) = \sum_{0\le|k|\le 2p} A_k(t/\varepsilon,x,\omega)\,\partial_x^k u^\varepsilon(t,x,\omega), \quad u^\varepsilon(0,x,\omega)=\varphi(x), \tag{12}$$

where $p\in\mathbb{N}$, and $\{A_k(\tau,x),\ \tau\ge 0\}$ is a $\mathbb{C}^{N\times N}$-valued stochastic process for each $x\in\Re^d$ and $k\in\mathbb{N}^d$ such that $0\le|k|\doteq k_1+k_2+...+k_d\le 2p$. Suppose $h$ is any constant in the interval $(0,1)$ and $u$ is defined by

$$\partial_t u(t,x) = \sum_{0\le|k|\le 2p} A_k^0(x)\,\partial_x^k u(t,x), \quad u(0,x)=\varphi(x). \tag{13}$$

Then, a weak invariance principle for $\varepsilon^{h-1}(u^\varepsilon-u)$ in $C([0,\infty);H_2)$, $H_2$ being a Hilbert space of functions, will be established without the need for any specific dependence, moment, or stationarity conditions. Our main result states that a weak invariance principle for $\varepsilon^{h-1}(u^\varepsilon-u)$ exists provided there is enough regularity to prove there are pathwise unique continuous, bounded solutions to (12) and (13) and for the coefficients $A_k, A_k^0$ to satisfy the following standard weak convergence results:

$$\begin{array}{ll}\text{pointwise LLN} & \int_0^t \partial_y^k[A_m\left(\frac{s}{\varepsilon},y,\omega\right)-A_m^0(y)]ds \Rightarrow 0\ \forall|k|\le 2p+1, |m|\le 2p \\ \text{invariance in } \mathcal{H}_1 & \left\{\varepsilon^h\int_0^{\cdot\varepsilon^{-1}} A_m(\tau)-A_m^0\,d\tau\right\}_{|m|\le 2p} \Rightarrow \left\{\widehat{\Theta}_m\right\}_{|m|\le 2p}\end{array} \tag{14}$$

as $\varepsilon\to 0$, $\mathcal{H}_1$ being a Hilbert space to be defined in Section 2. Under these conditions, $\varepsilon^{h-1}(u^\varepsilon-u)$ will be shown to converge in distribution to the mild solution of

$$\partial_t\widehat{y}(t,x) = \sum_{|k|\le 2p} A_k^0(x)\,\partial_x^k\widehat{y}(t,x)dt + \sum_{|m|\le 2p}\widehat{\Theta}_m(dt,x)\partial_x^m u(t,x). \tag{15}$$

Unfortunately, neither the classical theory for parabolic equations nor the classical Sobolev spaces are quite appropriate for this general invariance principle transfer method. For example, to obtain $\varepsilon$-independent bounds on the fundamental solutions to (12) from the theory in e.g. Friedman [11] Chapter 9, one would require an assumption like:

**(A)** For each $|m|=q$, (the principle coefficient) $A_m(t/\varepsilon,x)$ is continuous in $t$ uniformly with respect to $(x,t,\varepsilon)\in\Re^d\times[0,\infty)\times(0,1]$.

This would not allow our principle coefficients to depend on $t$ or $\varepsilon$. Fortunately, it is shown in Kouritzin [15] that Assumption (A) can be avoided if one imposes a slightly stronger parabolic condition on (12). Similarly, the classical Sobolev-type spaces were found inappropriate for $\mathcal{H}_1$ and we were forced to choose a new Hilbert space that can be thought of as an extension of the fractional Sobolev spaces on bounded domain (see Section 6.8 of Kufner et. al. [20]) to $\Re^d$.

The only condition imposed on our limit object $\left\{\widehat{\Theta}_m\right\}_{|m|\leq 2p}$ is that it belongs to $C([0,T];\mathcal{H}_1)$. Hence, we must define what we mean by stochastic integration with respect to $\widehat{\Theta}_m$ and by mild solutions to (15). An important advantage of our general conditions and this "invariance principle transfer" approach is that the analysis includes non-semimartingale, non-Markov limit objects $\left\{\widehat{\Theta}_m\right\}_{|m|\leq 2p}$ like fractional Brownian motions which are typical for long range (in $t$) dependent coefficients. In the special case where semimartingale conditions prevail the limit can be represented as a classical stochastic integral or by standard notions of linear stochastic partial differential equations (see Kouritzin [17]).

Our approach is to bend a few powerful theorems from the general theory of parabolic partial differential equations and from contemporary probability theory with a modest amount of analysis and Khas'minskii's method of decomposing stochastic averaging processes like $\varepsilon^{h-1}(u^\varepsilon - u)$ into a principle part $z^\varepsilon$ and an "error" process $v^\varepsilon \doteq z^\varepsilon - \varepsilon^{h-1}(u^\varepsilon - u)$. With the appropriate definitions and the weak invariance principle assumption in (14), we find that the convergence of the principle part follows relatively easily and our real challenge is to show that (14) implies that the error process $v^\varepsilon$ converges in distribution to zero. To this end; $v^\varepsilon$ is expressed in terms of the fundamental solution of (12); our regularity, the weak law of large numbers hypothesis, and a theorem (from Kouritzin [15]) on averaging for fundamental solutions are used to replace this fundamental solution with the fundamental solution for (13); and, finally, a constructive argument based on both weak convergence hypotheses and Skorokhod's representation theorem is developed to show that the modified error converges to zero.

Our proof is sketched in Subsection 3.1 and then proved in Subsections 3.2-3.3. Many of the details for these proofs have been placed into the lemmas of Section 4.

## 2. Notation, Conditions, and Result

Throughout this note, $p, N$, and $d$ will be fixed positive integers,

$$q \doteq 2p, \tag{16}$$

and $\gamma, h$ will be fixed constants such that $0 < \gamma < \frac{1}{4}$, $0 < h < 1$. Moreover, $|\cdot|$ will be used to denote absolute value as well as Euclidean distance in $\mathbb{C}^N$ and $\Re^d$, and $\|\cdot\|$ will be used to denote the $|\cdot|$-induced norm for $\mathbb{C}^{N\times N}$ matrices. However, for vectors

$k = (k_1, k_2, ..., k_d)$ of non-negative integers,

$$|k| \doteq k_1 + k_2 + \cdots + k_d \tag{17}$$

and "$\sum_{|k|\le q}$" denotes the summation over all possible $d$-tuples $k$ of non-negative integers such that $|k| \le q$. Furthermore, following Schwartz multi-index notation for our differential operators, we define

$$\partial_x^k \doteq \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_d}^{k_d} \tag{18}$$

and, letting $\{e_1, e_2, ..., e_d\}$ denote the standard basis for $\Re^d$, we set

$$d_x = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \cdots + \partial_{x_d} e_d. \tag{19}$$

Finally, $\mathcal{L}(X)$ will be used to denote the law or distribution of a random variable $X$ and $a_{m,n} \overset{n,m}{\ll} b_{m,n}$ will imply that there is a constant $c > 0$ such that $|a_{m,n}| \le c\,|b_{m,n}|$ for all $n, m$. The latest notation is a natural extension to the Vinogradov symbol.

As opposed to the classical work for parabolic equations where the coefficients are deterministic, we assume that $(\Omega, \mathcal{F}, P)$ is a probability space and $\{A_m(s,x), s \ge 0\}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ for each $0 \le |m| \le q, x \in \Re^d$. Furthermore, we assume for almost all $\omega \in \Omega$ that:

**(C1)** The system (12) is uniformly parabolic in the sense that

$$-\sup_{t\ge 0} \sup_{x\in\Re^d} \max_j \sup_{|\xi|=1} \lambda_j(\xi; x, t; \omega) > 0, \tag{20}$$

where $\{\lambda_j(\xi; x, t)\}_{j=1}^{2N}$ are the (real) roots of the polynomial

$$\det\left(\sum_{|m|=q} \begin{bmatrix} \mathrm{Re}[A_m(t,x) + A_m^T(t,x)] & -\mathrm{Im}[A_m(t,x) - A_m^T(t,x)] \\ \mathrm{Im}[A_m(t,x) - A_m^T(t,x)] & \mathrm{Re}[A_m(t,x) + A_m^T(t,x)] \end{bmatrix} (i\xi)^m - \lambda\, I_{2N}\right) \tag{21}$$

for all $\xi, x \in \Re^d$, and $t \ge 0$, $I_{2N}$ being the identity matrix in $\Re^{2N\times 2N}$.

**(C2)** (13) is uniformly parabolic in the sense that

$$-\sup_{x\in\Re^d} \max_l \sup_{|\xi|=1} \mathrm{Re}\left\{\lambda_l^0(\xi; x)\right\} > 0, \tag{22}$$

where $\{\lambda_l^0(\xi; x)\}_{l=1}^N$ are the roots of the polynomial

$$\det\left(\sum_{|m|=q} A_m^0(x)\,(i\xi)^m - \lambda\, I_N\right) \tag{23}$$

for all $\xi, x \in \Re^d$, $I_N$ being the identity matrix in $\mathbb{C}^{N\times N}$.

**(C3)** $\partial_x^k A_m$ and $\partial_x^k A_m^0$ exist and are continuous and uniformly bounded on $[0,\infty)\times\Re^d$ respectively $\Re^d$ for all $0 \le |m| \le q, 0 \le |k| \le q+2$.

**(C4)** $\partial_x^k \varphi$ exists and is a bounded, continuous function on $\Re^d$ for all $0 \le |k| \le q+2$.

In preparation for stating our main result we define our objects of study and the spaces in which they live. We start our definitions with the Hilbert space on which we will prove our desired central limit theorem.

**Definition 1.** *We define* $(H_2, <,>_2)$ *to be the Hilbert space of weighted* $\mathbb{C}^N$*-valued* $L^2(\Re^d)$ *-functions* $f$ *such that*

$$|f|_2 \doteq \sqrt{\int_{\Re^d} (1+|x|^2)^{-2d}\ |f(x)|^2\ dx} < \infty. \tag{24}$$

As previously advertised, we do not impose any specific dependence, moment, Markov, or martingale approximation conditions but rather only assume natural weak convergence results. Hence, we will also require a space on which to postulate one of our two weak convergence hypotheses. The space we utilize can be thought of as an extension to $\Re^d$ of the classical $W^{k,2}(\Omega)$, $k \notin \mathbb{N}$ Sobolev spaces for bounded domains $\Omega$. It is clear from the proofs in the sequel that the weights $w_2(x) \doteq (1+|x|^2)^{-2d}$ and $w_1(x) \doteq (1+|x|^2)^{-d}$ in the definitions of $H_2$ above and $H_1$ below could be replaced with any $w_2(x) \doteq (1+|x|^2)^{-a}$ and $w_1(x) \doteq (1+|x|^2)^{-b}$ provided $b, a-b > d/2$.

**Definition 2.** *Suppose* $B_x$ *denotes the open unit ball of* $\Re^d$ *centered at* $x$*. Then,* $(H_1, <,>_1)$ *and* $(H_1^v, <,>_1)$ *denote the separable Hilbert spaces (c.f. Lemma 12 of Section 4) of weighted* $\mathbb{C}^{N\times N}$*-valued respectively* $\mathbb{C}^N$*-valued* $L^2(\Re^d)-$*functions* $f$ *which are also Hölder continuous on average in the sense that*

$$\|f\|_1 \doteq \sqrt{\int_{\Re^d} |1+|x|^2|^{-d} \left\{ \|f(x)\|^2 + \int_{B_x} |x-\xi|^{-d-\gamma} \|f(\xi)-f(x)\|^2 d\xi \right\} dx} < \infty \tag{25}$$

*or likewise with* $|\cdot|$ *replacing* $\|\cdot\|$*. Moreover, we define* $\mathcal{H}_1 \doteq \bigotimes_{|k|\le q} H_1$ *to be the Hilbert space of all possible* $\{f_k\}_{|k|\le q}$ *such that* $f_k \in H_1$ *for all* $0 \le |k| \le q$ *and give it norm* $\left\|\{f_k\}_{|k|\le q}\right\|_1^2 \doteq \sum_{|k|\le q} \|f_k\|_1^2$.

Now, we are interested in functional results so we require an additional definition.

**Definition 3.** *$C([0,T],H)$ and $E([0,T],H)$ refer to the class of continuous respectively càglàd (left-continuous, right-hand-limit) functions $f$ such that $f(0)=0$ and $f(t)\in H$ for all $0\le t\le T$, with $H$ being $H_1,\mathcal{H}_1,H_2$, or $H_2\times\mathcal{H}_1$. We will always use the topology generated by $\sup_{[0,T]}|\cdot|_H$ for these spaces.*

Next, to ease the notation in the sequel, we define

$$\overline{A}_m(\tau,x)\doteq A_m(\tau,x)-A_m^0(x)\quad \forall x\in\Re^d,\tau\ge 0,\ 0\le|m|\le q \tag{26}$$

and

$$u_k(\tau,x)\doteq\partial_x^k u(\tau,x)\quad \forall x\in\Re^d,\tau\ge 0,\ 0\le|k|\le 2q+1. \tag{27}$$

Then, we can define our stochastic processes of interest:

**Definition 4.** *For all $x\in\Re^d,t\in[0,\infty),\varepsilon\in(0,1],|m|\le q$, and $|k|\le q+1$, we define*

$$\alpha^\varepsilon_{m,k}(t,x)\doteq\varepsilon\int_0^{t\varepsilon^{-1}}\partial_x^k\overline{A}_m(\tau,x)\,d\tau \tag{28}$$

$$\mathcal{A}^\varepsilon_m(t,x)\doteq\varepsilon^h\int_0^{t\varepsilon^{-1}}\overline{A}_m(\tau,x)\,d\tau. \tag{29}$$

*Furthermore, $\mathcal{A}^\varepsilon$ will denote the $\mathcal{H}_1$-valued (see Condition (C3)) process $\{\mathcal{A}^\varepsilon_m\}_{|m|\le q}$.*

**Definition 5.** *For each $x\in\Re^d,t\in[0,\infty),\varepsilon\in(0,1]$, we define*

$$y^\varepsilon(t,x)\doteq\left[u^\varepsilon(t,x)-u(t,x)\right]/\varepsilon^{1-h}, \tag{30}$$

*where $u^\varepsilon,u$ are the unique continuous bounded $\mathbb{C}^N$-valued solutions to (12) and (13).*

Under a variety of probabilistic conditions, one finds that $\alpha^\varepsilon_{m,k}(t,x)$ satisfies a weak law of large numbers (for each fixed $(m,k,x,t)$) and $\mathcal{A}^\varepsilon$ satisfies a weak invariance principle in $\mathcal{H}_1$. The main theses of this note are that these two weak convergence assumptions are sufficient to establish that $y^\varepsilon$ converges in distribution to $\widehat{y}$ (say) and to characterize $\widehat{y}$ as a stochastic integral of the limit of the $\mathcal{A}^\varepsilon$s. For notational convenience in the remainder of this note we let $\Rightarrow$ denote convergence in distribution.

**Theorem 6.** *Suppose Regularity Conditions (C1-C4) hold, $\Gamma$ is the fundamental solution to (13), $T_s,s\ge 0$ are the continuous operators from $H_1^v\to H_2$ (c.f. Lemma 16 (i) of Section 4) defined by*

$$T_sf(x)\doteq\int_{\Re^d}\Gamma(x,s,\xi)\,f(\xi)\,d\xi\qquad\forall\ x\in\Re^d,\ s\in[0,T], \tag{31}$$

*and* $\widehat{\Theta}$ *is a* $(C[0,T];\mathcal{H}_1)$*-valued process on some probability space* $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$. *Then,* $u_k$ *satisfies the bounds (107) in Theorem 13 of Section 4 and the stochastic integral*

$$\{\widehat{y}(t), 0\le t\le T\} \doteq \left\{-\sum_{|k|\le q}\int_0^t T_{t-\tau}\, d\widehat{\Theta}_k(\tau)\, u_k(\tau), 0\le t\le T\right\} \quad \text{a.s.}, \tag{32}$$

*interpreted in the sense of Definition 10 (to follow), exists. Moreover, assume that*

$$\mathcal{A}^\varepsilon \Rightarrow \widehat{\Theta} \quad in \;\; C([0,T];\mathcal{H}_1) \quad as \;\; \varepsilon\to 0 \tag{33}$$

*and, for each* $|m|\le q$, $|k|\le q+1, x\in\Re^d$ *and some (whence all)* $t\in(0,T]$, *that*

$$\alpha^\varepsilon_{m,k}(t,x)\Rightarrow 0 \quad as \;\; \varepsilon\to 0. \tag{34}$$

*Then, it follows that*

$$\begin{bmatrix} y^\varepsilon \\ \mathcal{A}^\varepsilon\end{bmatrix} \Rightarrow \begin{bmatrix}\widehat{y}\\ \widehat{\Theta}\end{bmatrix} \quad in \;\; C([0,T];H_2\times\mathcal{H}_1) \quad as \;\; \varepsilon\to 0. \tag{35}$$

**Remark 1.** *Suppose we let* $(C([0,\infty);H),\rho)$, $H$ *being a separable Hilbert space, denote the complete, separable metric space of continuous* $H$*-valued functions on* $[0,\infty)$ *with a metric of uniform convergence on compacts e.g.*

$$\rho(f^r,f) = \sum_{i=1}^\infty 2^{-i}\sup_{t\in[0,i]}|f^r(t)-f(t)|_H\,. \tag{36}$$

*Then, as a trivial corollary of Theorem 6, one can replace* $C([0,T];\mathcal{H}_1)$ *in (33) and* $C([0,T];H_2\times\mathcal{H}_1)$ *in (35) with respectively* $C([0,\infty);\mathcal{H}_1)$ *and* $C([0,\infty);H_2\times\mathcal{H}_1)$.

We have not imposed any semimartingale or like assumption on $\widehat{\Theta}_k$. Hence, we should explain the integration in (32). However, before we can proceed with our discussion on integration we must define vector spaces of bounded, continuously differentiable functions which will play a role in this discussion as well as the proofs in the sequel.

**Definition 7.** $C^1_u$ $(C^{1,v}_u)$ *denotes the space of continuous* $\mathbb{C}^{N\times N}$*-valued* ($\mathbb{C}^N$*-valued) functions* $f$ *such that* $\partial_{x_i}f(t,x)$ *exists and is continuous on* $[0,T]\times\Re^d$ *for* $i=1,...,d$ *and there exists a* $c_f>0$ *such that*

$$\|f(t,x)\|\le c_f \text{ and } \|\partial_{x_i}f(t,x)\|\le c_f \;\;\forall\, x\in\Re^d, t\in[0,T], i=1,...,d. \tag{37}$$

*Moreover,* $E^1_u,(E^{1,v}_u)$ *will be used for the space of* $\mathbb{C}^{N\times N}$*-valued* ($\mathbb{C}^N$*-valued) functions* $f$ *that are càglàd in* $t$ *and continuous in* $x$ *such that* $\partial_{x_i}f(t,x)$ *exists and is also càglàd in* $t$ *and continuous in* $x$ *and (37) holds.*

Now, suppose $f \in C_u^1$. It follows from Theorem 13 (i) with $b = k$ and $b = k + e_i$, $i = 1, ..., d$ and Lemma 16 (iii) of Section 4 that

$$\int_0^t |T_{t-\tau} f(\tau) \partial_\tau u_k(\tau)|_2 \, d\tau, \int_0^t |\partial_\tau T_{t-\tau} f(\tau) \, u_k(\tau)|_2 \, d\tau < \infty \; \forall \, t \in [0, T] \, , 0 \leq |k| \leq q \quad (38)$$

so $T_{t-\tau} f(\tau) \partial_\tau u_k(\tau)$ and $\partial_\tau T_{t-\tau} f(\tau) u_k(\tau)$ are Riemann integrable (Lemma 1.1.4 of Ethier and Kurtz [8] and Lemma 16 (ii)) for all $t \in [0, T]$. Dominated convergence, the substitution $s = t - \tau$, Theorem 13 (i),and Lemma 16 (iii) establish that

$$\int_0^t T_{t-\tau} f(\tau) \, \partial_\tau u_k(\tau) d\tau, \int_0^t (\partial_\tau T_{t-\tau}) \; f(\tau) \, u_k(\tau) d\tau \in C([0, T]; H_2). \quad (39)$$

**Remark 2.** *The difficulty that has arisen is that due to singularity*

$$\int_0^t (\partial_\tau T_{t-\tau}) \, f(\tau) \, u_k(\tau) d\tau \, (x) = \int_0^t \int_{\Re^d} (\partial_\tau \Gamma(x, t - \tau, \xi)) \, f(\tau, \xi) \, u_k(\tau, \xi) \, d\xi \, d\tau \quad (40)$$

*only exists as an iterated integral when $f \in C_u^1$ yet we require sensible integrals with $f \in C([0, T]; H_1)$. Consequently, we define our integral first for $f \in C_u^1$ and then extend by continuity and completeness.*

**Definition 8.** *For $f \in C_u^1$ with $f(0) \equiv 0$ and $|k| \leq q$, we define the integral of $(T, u_k)$ with respect to $f$ to be the $C([0, T]; H_2)$ object defined for all $t \in [0, T]$ by*

$$\int_0^t T_{t-\tau} f(d\tau) \, u_k(\tau) \doteq f(t) u_k(t) - \int_0^t \partial_\tau T_{t-\tau} f(\tau) u_k(\tau) d\tau - \int_0^t T_{t-\tau} \, f(\tau) \partial_\tau u_k(\tau) d\tau. \quad (41)$$

It will often be the case that $\widehat{\Theta}_k \notin C_u^1$. However, $\widehat{\Theta}_k \in C([0, T]; H_1)$ for almost all $\widehat{\omega} \in \widehat{\Omega}$ and Lemma 12 (i) of Section 4 states that $C_u^1$ is dense in $C([0, T]; H_1)$. Furthermore, one finds from (31), Theorem 13 (i), Lemma 14 (i), Remark 3 (following the statement of Lemma 15), and Lemma 15 (i,ii) (with $a = q - \gamma$ in (i)) that

$$f \to \int_0^t T_{t-\tau} \, f(\tau) \, \partial_\tau u_k(\tau) \, d\tau \text{ and } f \to \int_0^t \partial_\tau T_{t-\tau} \, f(\tau) \, u_k(\tau) \, d\tau \quad (42)$$

are continuous mappings from $(C_u^1, |\cdot|_{C([0,T];H_1)})$ to $C([0, T]; H_2)$ for each $|k| \leq q$.

**Definition 9.** *Suppose now that $0 \leq |k| \leq q$, $f \in C([0, T]; H_1)$ and $\{f_n\}_{n=1}^\infty \subset C_u^1$ is such that $f_n \to f$ in $C([0, T]; H_1)$ and $f(0) = f_n(0) \equiv 0$. Then, we define the integral of $(T, u_k)$ with respect to $f$ to be the $C([0, T]; H_2)$ object defined by*

$$\left\{ \int_0^t T_{t-\tau} f(d\tau) u_k(\tau), t \in [0, T] \right\} \doteq \lim_{n \to \infty} \left\{ \int_0^t T_{t-\tau} f_n(d\tau) u_k(\tau), t \in [0, T] \right\}. \quad (43)$$

*(Obviously, the definition does not depend on the approximating sequence $\{f_n\}_{n=1}^\infty$).*

Finally, noting that our integral is a continuous mapping, we are in a position to define our stochastic integrals and our mild solution to the limiting stochastic partial differential equation.

**Definition 10.** *When* $|k| \leq q$ *and* $\left\{\widehat{\theta}(t), t \in [0,T]\right\}$ *is a* $C([0,T];H_1)-$*valued random variable on some probability space* $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$*, we define the stochastic integral of* $(T, u_k)$ *with respect to* $\widehat{\theta}$ *to be the* $C([0,T];H_2)$*-valued random variable defined for each* $\widehat{\omega} \in \widehat{\Omega}$ *as in the above definition and denote this integral by*

$$\left\{\int_0^t T_{t-\tau}\, d\widehat{\theta}(\tau)\, u_k(\tau), t \in [0,T]\right\}. \tag{44}$$

**Definition 11.** *When* $\widehat{\Theta}$ *is a* $(C[0,T];\mathcal{H}_1)$*-valued process on* $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$*, we define the mild solution to the linear stochastic partial differential equation*

$$\partial_t \widehat{y}(t,x) = \sum_{|k|\leq q} A_k^0(x)\, \partial_x^k \widehat{y}(t,x)dt + \sum_{|k|\leq q} \widehat{\Theta}_k(dt,x)u_k(t,x) \tag{45}$$

*to be* $\left\{-\sum_{|k|\leq q} \int_0^t T_{t-\tau}\, d\widehat{\Theta}_k(\tau)\, u_k(\tau), t \in [0,T]\right\}$.

Of course this is a natural definition because $T_s, s \geq 0$ has the same form as the semi-group associated with the differential operator in (45) living on some larger Sobolev-type space. Moreover, it is established in Kouritzin [17] that this definition is equivalent to weak form definitions and the stochastic integral in (44) can be interpreted in the standard way when $\left\{\widehat{\Theta}_t,\, t \in [0,T]\right\}$ is a semimartingale.

## 3. Proof of Main Result.

To avoid unnecessary notational encumbrances, we take $T = 1$, define $I \doteq [0,1]$, let $\{\varepsilon_r\}_{r=1}^{\infty} \subset (0,1]$ be an arbitrary sequence such that $\varepsilon_r \overset{r\to\infty}{\to} 0$ monotonically, and set

$$y^r \doteq y^{\varepsilon_r}, \quad A_m^r(t,x) \doteq A_m(t/\varepsilon_r, x),\ \overline{A}_m^r(t,x) \doteq \overline{A}_m(t/\varepsilon_r, x),\ \alpha_{m,k}^r \doteq \alpha_{m,k}^{\varepsilon_r} \tag{46}$$

for all $x \in \Re^d, t \in I, r = 1,2,..., 0 \leq |k| \leq q+1$, and $0 \leq |m| \leq q$. Clearly, to validate Theorem 6 it suffices to show that all such sequences $\{\varepsilon_r\}_{r=1}^{\infty}$ have subsequences $\{\varepsilon_{r_i}\}_{i=1}^{\infty}$satisfying Theorem 6.

**3.1. Sketch of Proof.** As mentioned in the introduction, our approach uses a few powerful theorems from the general theory of parabolic partial differential equations. In particular, motivated by Theorems 9.4.2, 9.4.3, and 9.5.6 of Friedman [11] and Theorems A and 1 of Kouritzin [15], we state the following two theorems which can be proved easily by following the development of these other five theorems.

*Theorem A: Suppose Regularity Conditions (C1-C4) hold. Then, there exist unique continuous, bounded solutions to (12) and (13) on $I \times \Re^d$ which are given in terms of the fundamental solutions to (12) and (13) by*

$$u^\varepsilon(t,x) \doteq \int_{\Re^d} \Gamma^\varepsilon(x,t;\xi,0)\varphi(\xi)d\xi \quad \textit{and} \quad u(t,x) \doteq \int_{\Re^d} \Gamma(x,t;\xi,0)\varphi(\xi)d\xi. \tag{47}$$

*Moreover, if $f$ is a continuous, bounded function on $I\times\Re^d$ which is Hölder continuous in $x$ uniformly over bounded sets then*

$$\zeta^\varepsilon(t,x) \doteq \int_0^t \int_{\Re^d} \Gamma^\varepsilon(x,t;\xi,\tau)f(\tau,\xi)d\xi d\tau,\ \zeta(t,x) \doteq \int_0^t \int_{\Re^d} \Gamma(x,t;\xi,\tau)f(\tau,\xi)d\xi d\tau \tag{48}$$

*are the unique continuous, bounded solutions to*

$$\partial_t \zeta^\varepsilon(t,x) = \sum_{|k|\le q} A_k(\frac{t}{\varepsilon},x)\partial_x^k \zeta^\varepsilon(t,x) - f(t,x),\ \partial_t\zeta(t,x) = \sum_{|k|\le q} A_k^0(x)\partial_x^k\zeta(t,x) - f(t,x). \tag{49}$$

*Theorem B: Suppose that (C1-C3) of Section 2 hold, $\{\varepsilon_l\}_{l=1}^\infty \subset (0,1]$ is a sequence decreasing to zero and for each $(t,y) \in I\times\Re^d$, $0 \le |k| \le q$ we have that*

$$\left\| \int_0^t A_k\left(\frac{s}{\varepsilon_l},y\right) - A_k^0(y)\,ds \right\| \to 0 \ \text{ as } l\to\infty. \tag{50}$$

*Then, for any $0<\chi,\nu<1$ there exists a positive constant $\widetilde{c} = \widetilde{c}_{\chi,\nu}$ and a sequence $\{\beta_l = \beta_l(\chi,\nu)\}_{l=1}^\infty$ satisfying $\lim_{l\to\infty}\beta_l = 0$ such that*

$$\left\|\partial_x^b[\Gamma^l(x,t;\xi,\tau) - \Gamma(x,t;\xi,\tau)]\right\| \le \frac{\beta_l\,|1+|\xi|^2|^\nu}{(t-\tau)^{\frac{d+|b|+\chi}{2p}}} \exp\left[-\widetilde{c}\left|\frac{|x-\xi|^{2p}}{t-\tau}\right|^{\frac{1}{2p-1}}\right] \tag{51}$$

*for all $0\le|b|<q$, $l=1,2,...,$ $0\le\tau\le t\le 1$ and $x,\xi\in\Re^d$, where $\Gamma^\varepsilon$ and $\Gamma$ are the fundamental solutions of (12) respectively (13) above and $\Gamma^l \doteq \Gamma^{\varepsilon_l}$.*

We also mentioned that we will utilize a standard decomposition method of stochastic averaging problems due to Khas'minskii [14]. Thus $y^\varepsilon$ will be treated as the sum of a "principle" part $z^\varepsilon \doteq \sum_{|k|\le q}\left\{-\int_0^t T_{t-\tau}\,d\mathcal{A}_k^\varepsilon(\tau)\,u_k(\tau),\ t\in I\right\}$ and an "error" process $v^\varepsilon$. **(i)** We first sketch how we will prove that $v^\varepsilon$ converges in distribution to zero. Indeed, it follows from integration by parts and Theorem A that

$$v^{\varepsilon_r}(t,x) = \sum_{|k|\le q}\left\{\sum_{j=1}^3 \widetilde{F}_r^j(\mathcal{A}_k^{\varepsilon_r}, z^{\varepsilon_r})(t,x)\right. \tag{52}$$

$$+\sum_{|m|\le q}\int_0^t\int_{\Re^d}\left[\Gamma^r(x,t;\xi,\tau)-\Gamma(x,t-\tau,\xi)\right]\mathcal{A}_k^{\varepsilon_r}(\tau,\xi)\,\partial_\xi^k\Big\{\overline{A}_m^r(\tau,\xi)\,u_m(\tau,\xi)\Big\}\,d\xi d\tau$$

$$\left.+\sum_{|m|\le q}\int_0^t\int_{\Re^d}\Gamma(x,t-\tau,\xi)\,\mathcal{A}_k^{\varepsilon_r}(\tau,\xi)\,\partial_\xi^k\left\{\overline{A}_m^r(\tau,\xi)\,u_m(\tau,\xi)\right\}\,d\xi d\tau\right\},$$

where the $\widetilde{F}_r^j$ are bilinear forms to be defined on (57), (58), and (60) of Subsection 3.2 in terms of the above fundamental solutions. Next, in (64) of Subsection 3.2 we use classical type bounds (see Lemma 14) in conjunction with Lemma 15 of Section 4 and tightness for $\{\mathcal{A}_k^{\varepsilon_r},\ r=1,2,...\}$ to reduce convergence of the first term of (52) to showing that $z^{\varepsilon_r}$ converges in distribution to zero in the appropriate sense. The proof of this in turn follows largely from the work in Kouritzin [16] (see Theorem 13 of this note). For the second term in (52) we employ a subsequence method to replace our weak law of large numbers assumption (34) with almost sure convergence and use Theorem B pathwise along with Lemma 15 of Section 4 to conclude in (68) of Subsection 3.2 that this term also converges in distribution to zero. For the last term in (52) we let $k=k_1+k_2$ and find by Skorokhod's representation theorem and a constructive argument that we can redefine a subsequence of $\left\{\left(\mathcal{A}_k^{\varepsilon_r},\partial_\xi^{k_1}A_m^r\right),\ r=1,2,...\right\}$ on a new probability space where our weak invariance principle is replaced with almost sure convergence and we have desirable almost sure bounds. Then, the third term is also shown to converge to zero by continuity, approximation, and integration techniques.

**(ii)** In Subsection 3.3 we show that the principle part $z^r$ converges in distribution to $\widehat{y}$ by showing $z^r$ is a continuous function of $\mathcal{A}^{\varepsilon_r}$ and using our imposed weak invariance principle assumption again with the continuous mapping theorem. Due to the way that we have defined our stochastic integral, the limit of the $\mathcal{L}(z^r)$ is $\mathcal{L}(\widehat{y})$.

Our proof will follow these two steps in Subsections 3.2 and 3.3 respectively. Many of the details have been relegated to the supporting lemmas of Section 4.

**3.2. Convergence of the error process.** Motivated to a large degree by Equation (3.14) of Khas'minskii [14], we define our approximation to $y^r$ via $z^r\doteq z_0^r$, where for each $|m|\le q$, $|l|\le q+1$, $r=1,2,...$, $x\in\Re^d$, and $t\in I$

$$z_{m+l}^r(t,x)\doteq-\varepsilon_r^{h-1}\partial_x^l\int_0^t\int_{\Re^d}\partial_x^m\Gamma(x,t-\tau,\xi)\sum_{|k|\le q}\overline{A}_k^r(\tau,\xi)\,u_k(\tau,\xi)\,d\xi\ d\tau,\tag{53}$$

and denote our error term by $v^r(t,x)\doteq z^r(t,x)-y^r(t,x)$. (When $|m|=q$ the integration in (53) must be interpreted as an iterated integral due to singularity.) Then, it follows easily from Theorem 13 (ii), Condition (C3), Theorem A, (30), (12), (13), (46), (26-27), and (53) that $v^r$ is the unique continuous, bounded solution of

$$\partial_t v^r(t,x)=\sum_{|k|\le q}A_k^r(t,x)\partial_x^k v^r(t,x)-\sum_{|k|\le q}\overline{A}_k^r(t,x)\,z_k^r(t,x),\quad v^r(0)\equiv 0\tag{54}$$

i.e. $v^r(t,x) = \sum_{|k|\le q} v_k^r(t,x)$ and

$$v_k^r(t,x) = \int_0^t \int_{\Re^d} \Gamma^r(x,t;\xi,\tau)\, \overline{A}_k^r(\tau,\xi)\, z_k^r(\tau,\xi)\, d\xi\ d\tau \tag{55}$$

for all $0 \le |k| \le q,\ x \in \Re^d, t \in I$. Then, using (55), integration by parts, (46), (29), the definition $\mathcal{A}^r \doteq \mathcal{A}^{\varepsilon_r}$, Condition (C3), Theorem 13 (ii) with $b = k$ and $b = k + e_i$, $i = 1,..,d$, and Lemma 14 (iii,iv), we find that

$$\begin{aligned} v_k^r(t,x) &= \lim_{v \nearrow t} \int_{\Re^d} \int_0^v \Gamma^r(x,t;\xi,\tau)\, \overline{A}_k^r(\tau,\xi)\ z_k^r(\tau,\xi)\ d\tau\ d\xi \qquad (56)\\ &= \sum_{j=1}^2 F_r^j\left(\varepsilon_r^{1-h}\mathcal{A}_k^r z_k^r\right)(t,x) - \varepsilon_r^{1-h}\int_0^t\int_{\Re^d}\Gamma^r(x,t;\xi,\tau)\,\mathcal{A}_k^r(\tau,\xi)\,\partial_\tau z_k^r(\tau,\xi)\,d\xi\,d\tau \end{aligned}$$

for all $0 \le |k| \le q,\ x \in \Re^d, t \in I$, where for all $\rho \in C_u^{1,v}$

$$F_r^1(\rho)(t,x) \doteq \rho(t,x) \tag{57}$$

$$F_r^2(\rho)(t,x) \doteq -\int_0^t \int_{\Re^d} \partial_\tau \Gamma^r(x,t;\xi,\tau)\, \rho(\tau,\xi)\, d\xi\ d\tau. \tag{58}$$

(One can see that the integration in $F_r^2$ makes sense as an iterated integral by Lemma 15 (ii) of Section 4.) Moreover, it follows by (53), Condition (C3), and Theorem 13 (i,ii) with $l = m$ and $b = k, k + e_i,\ i = 1,..,d$ that

$$\begin{aligned} &\varepsilon_r^{1-h}\int_0^t\int_{\Re^d}\Gamma^r(x,t;\xi,\tau)\,\mathcal{A}_k^r(\tau,\xi)\,\partial_\tau z_k^r(\tau,\xi)\,d\xi\,d\tau \qquad (59)\\ = &\sum_{|m|\le q}\Big\{F_r^3\left(\varepsilon_r^{1-h}\mathcal{A}_k^r\ \partial_\xi^k\left[A_m^0\ z_m^r\right]\right)(t,x)\\ + &\int_0^t\int_{\Re^d}\left[\Gamma^r(x,t;\xi,\tau)-\Gamma(x,t-\tau,\xi)\right]\mathcal{A}_k^r(\tau,\xi)\,\partial_\xi^k\Big\{\overline{A}_m^r(\tau,\xi)\,u_m(\tau,\xi)\Big\}\,d\xi d\tau\\ + &\sum_{k_1+k_2=k} c_{k_1,k_2}\int_0^t\int_{\Re^d}\Gamma(x,t-\tau,\xi)\,\mathcal{A}_k^r(\tau,\xi)\,\partial_\xi^{k_1}\overline{A}_m^r(\tau,\xi)\,u_{m+k_2}(\tau,\xi)\,d\xi d\tau\Big\} \end{aligned}$$

for some collection of integers $\{c_{k_1,k_2}\}$ and all $0 \le |k| \le q$, where $\sum_{k_1+k_2=k}$ denotes the summation over all vectors $k_1, k_2$ such that $k_1 + k_2 = k$ and

$$F_r^3(\rho)(t,x) \doteq \int_0^t \int_{\Re^d} \Gamma^r(x,t;\xi,\tau)\, \rho(\tau,\xi)\, d\xi\ d\tau \quad \text{for all } \tau^{1-\frac{\gamma}{q}}\rho \in C_u^{1,v}. \tag{60}$$

(To avoid defining more spaces, we will use the notation $\tau^a\rho \in C_u^{1,v}$ to denote the space of $\rho$ such that $(\tau,x) \to \tau^a\rho(\tau,x) \in C_u^{1,v}$.) Next, fixing $0 < \nu < \frac{1}{4}$, letting $C_3$ ($C_3^v$)

denote the space of all continuous $\mathbb{C}^{N\times N}$-valued ($\mathbb{C}^N$-valued) functions $g$ on $I\times\Re^d$ such that $(1+|x|^2)^{-\nu}g(t,x)$ vanishes at infinity with norm

$$\|g\|_3 \doteq \sup_{x,t}(1+|x|^2)^{-\nu}\,\|g(t,x)\| \quad (|g|_3 \doteq \sup_{x,t}(1+|x|^2)^{-\nu}\,|g(t,x)|), \tag{61}$$

and defining the Hölder continuity norms $\|\cdot\|_4$ ($|\cdot|_4$)

$$\|\rho\|_4 \doteq \sup_{t,x}\frac{\|\rho(t,x)\|}{(1+|x|^2)^{\nu}} + \sup_{t,0<|x-\xi|<1}\frac{\|\rho(t,x)-\rho(t,\xi)\|}{|x-\xi|^{\gamma}\,(1+|x|^2)^{\nu}} \quad \forall\,\rho\in C_3, \tag{62}$$

(likewise with $|\cdot|$ replacing $\|\cdot\|$ for $\rho\in C_3^v$), one finds from (57-58), (60), Condition (C3), Lemma 14 (iii), Lemma 15 (i,ii,iii) (with $a=(|m|-\gamma/2)\vee 0$ in (i)) and Remark 3 of Section 4 that

$$\begin{aligned}&\left|\sum_{j=1}^{2}F_r^j\Big(\varepsilon_r^{1-h}\mathcal{A}_k^r\,z_k^r\Big)+F_r^3\Big(\varepsilon_r^{1-h}\mathcal{A}_k^r\,\partial_\xi^k\Big[A_m^0\,z_m^r\Big]\Big)\right|_{C(I;H_2)} \\ \le\;& C\,|\mathcal{A}_k^r|_{C(I;H_1)}\,\varepsilon_r^{1-h}\left[|z_k^r|_4+\sum_{k_1}|z_{k_1,m}^r|_3\right] \quad \forall r=1,2,...,\;0\le|k|,|m|\le q.\end{aligned} \tag{63}$$

Here $z_{k_1,m}^r(t,x)\doteq t^{\frac{(|m|-\gamma/2)\vee 0}{q}}z_{k_1+m}^r(t,x)$ $\forall(t,x)\in I\times\Re^d$, $C$ is a $\Re$-valued random variable, and $\sum_{k_1}$ represents the summation over $k_1\in\mathbb{N}_0^d$ such that there exists a $k_2\in\mathbb{N}^d$ with $k=k_1+k_2$. Hence, using the tightness of $\left\{|\mathcal{A}^r|_{C(I;H_1)},\,r=\,1,2,3,...\right\}$, Condition (C3), and Theorem 13 (iii,iv), we find for any $\lambda,\delta>0$ that

$$\begin{aligned}&P\left\{\max_{k,m}\left|\sum_{j=1}^{2}F_r^j\Big(\varepsilon_r^{1-h}\mathcal{A}_k^r\,z_k^r\Big)+F_r^3\Big(\varepsilon_r^{1-h}\mathcal{A}_k^r\,\partial_\xi^k\Big[A_m^0 z_m^r\Big]\Big)\right|_{C(I;H_2)}>\lambda\right\} \\ \le\;& P\Big\{C\,|\mathcal{A}^r|_{C(I;\mathcal{H}_1)}>K_\delta\Big\}+P\left\{\varepsilon_r^{1-h}\max_{k,m}\left[|z_k^r|_4+\sum_{k_1}\left|z_{k_1,m}^r\right|_3\right]>\frac{\lambda}{K_\delta}\right\}<\delta\end{aligned} \tag{64}$$

for all large enough $K_\delta>0$ and large enough $r$.

Now, we consider the second term in (59). Suppose $\{(t_j,x_j)\}_{j=1}^{\infty}$ is a dense subset of $I\times\Re^d$. Then, it follows from our hypothesis (34) and a simple triangle argument that there exists a (monotonic) subsequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of $\{\varepsilon_r\}_{r=1}^{\infty}$such that

$$\alpha_{m,0}^i(t_j,x_j)\to 0 \;\text{ as }\; i\to\infty \;\;\forall\,j=1,2,...,\,0\le|m|\le q \quad \text{a.s.} \tag{65}$$

Hence, it follows from the local equicontinuity of $\left\{\alpha_{m,0}^i\right\}_{i=1}^{\infty}$ (c.f. (26), (28), and Condition (C3)) that for almost all $\omega$

$$\alpha_{m,0}^i(t,x)\to 0 \quad \text{as}\;\; i\to\infty \;\;\forall\,x\in\Re^d, t\in I, 0\le|m|\le q. \tag{66}$$

Therefore, fixing an $\omega$ such that (66) holds, we find from (26), (28), and Theorem B of Subsection 3.1 that there exists a constant $c > 0$ and a sequence $\{\beta_i\}_{i=1}^\infty$ both independent of $(x, t; \xi, \tau)$ and satisfying $\lim_{i\to\infty} \beta_i = 0$ such that

$$\left\|\Gamma^i(x,t;\xi,\tau) - \Gamma(x,t-\tau,\xi)\right\| \le \frac{\beta_i\,(1+|\xi|^2)^\nu}{(t-\tau)^{(d+1/2)/q}} \exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} \tag{67}$$

for all $x, \xi \in \Re^d, 0 \le \tau \le t \le 1$, and $i = 1, 2, 3, ...$ Hence, it follows from (67), Theorem 13 (i) (with $|b| = |k| + 1$, $|l| = |m| - 1$), Lemma 15 (i) (with $a = q - 1$), Condition (C3) as well as the argument in (64) that

$$\left|\int_0^t \int_{\Re^d} [\Gamma^i(x,t;\xi,\tau) - \Gamma(x,t-\tau,\xi)]\mathcal{A}_k^i(\tau,\xi)\,\partial_\xi^k\left\{\overline{A}_m^i(\tau,\xi) u_m(\tau,\xi)\right\} d\xi d\tau\right|_{C(I;H_2)} \tag{68}$$
$$\le \quad C\,\beta_i\,\left|\mathcal{A}^i\right|_{C(I;\mathcal{H}_1)} \overset{i\to\infty}{\Rightarrow} 0 \quad \forall\, 0 \le |k|, |m| \le q.$$

It remains to handle the third term in (59). By a successive subsequence argument it suffices to let $k_1, k_2$, and $m$ be arbitrarily fixed vectors of non-negative integers such that $k = k_1 + k_2$ and $0 \le |k|, |m| \le q$, and then show that

$$\int_0^t \int_{\Re^d} \Gamma(x,t-\tau,\xi)\,\mathcal{A}_k^i(\tau,\xi)\,\partial_\xi^{k_1}\overline{A}_m^i(\tau,\xi)\,u_{m+k_2}(\tau,\xi)\,d\xi d\tau \Rightarrow 0 \tag{69}$$

along some subsequence. This in turn will be done by redefining $\{\mathcal{A}_k^i,\ i = 1, 2, ...\}$ and $\left\{\alpha_{m,k_1}^i,\ i = 1, 2, ...\right\}$ on a new probability space where the convergence in distribution for $\mathcal{A}_k^i$ (as $i \to \infty$) is replaced with almost sure convergence, repeating an argument like (65-66) to establish almost sure convergence for $\alpha_{m,k_1}^i$, and using continuity as well as integration by parts. However, due care must be taken because we will require almost sure bounds on subsequences of $\left\{\partial_\xi^b \partial_\xi^{k_1} A_m^i\right\}$ and we must handle singular integrals to make the arguments work. Suppose

$$\|f\|_\infty \doteq \sup_{t\in I, x\in\Re^d} \|f(t,x)\| \qquad \forall f \in C_3. \tag{70}$$

Then, on the basis of monotonicity and boundedness (c.f. (46) and Condition (C3)) one can find random variables $\{f_b\}_{0\le|b|\le 2}$ such that

$$\left\|\partial_\xi^b \partial_\xi^{k_1} A_m^i\right\|_\infty \nearrow f_b \ \text{ as } i \to \infty\ \forall 0 \le |b| \le 2 \ \text{ a.s.} \tag{71}$$

A simple argument then establishes that $\left\{\left(\mathcal{A}_k^i, \left\{\left\|\partial_\xi^b \partial_\xi^{k_1} A_m^i\right\|_\infty\right\}_{0\le|b|\le 2}\right),\ i = 1, 2, ...\right\}$

is tight in $C(I;H_1)\times\Re^{d^2+d+1}$. Hence, there exists a subsequence $\{\varepsilon_j\}_{j=1}^{\infty}$ and a probability measure $Q$ with marginals $\mathcal{L}(\widehat{\Theta}_k)$ and $\mathcal{L}\left(\{f_b\}_{0\le|b|\le 2}\right)$ such that

$$\begin{bmatrix} \mathcal{A}_k^j \\ \left\{\left\|\partial_\xi^b\partial_\xi^{k_1}A_m^j\right\|_\infty\right\}_{0\le|b|\le 2}\end{bmatrix}\Rightarrow Q \qquad \text{as} \qquad j\to\infty. \tag{72}$$

Now, Lemma 17 states that $(C_3,\|\cdot\|_3)$ is complete and separable so Skorokhod's representation theorem and Lemma 19 yield $C(I;H_1)$-valued random variables $\{\widetilde{a}^j, j=1,2,3,...\},\widetilde{\Theta}$; $\Re^{d^2+d+1}$-valued random variables $\left\{\left\{\widetilde{f}_b^j\right\}_{0\le|b|\le 2},\ j=1,2,...\right\}$, $\left\{\widetilde{f}_b\right\}_{0\le|b|\le 2}$; and $(C_3)^{d^2+d+1}$-valued random variables $\left\{\{\widetilde{A}_b^j\}_{0\le|b|\le 2}), j=1,2,3,...\right\}$ on some probability space $\left(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P}\right)$ such that

$$\text{(i) } \mathcal{L}\begin{pmatrix} \widetilde{a}^j \\ \left\{\widetilde{f}_b^j\right\}_{0\le|b|\le 2} \\ \left\{\widetilde{A}_b^j\right\}_{0\le|b|\le 2}\end{pmatrix} = \mathcal{L}\begin{pmatrix} \mathcal{A}_k^j \\ \left\{\left\|\partial_\xi^b\partial_\xi^{k_1}A_m^j\right\|_\infty\right\}_{0\le|b|\le 2} \\ \left\{\partial_\xi^b\partial_\xi^{k_1}A_m^j\right\}_{0\le|b|\le 2}\end{pmatrix}\ j=1,2,...; \tag{73}$$

$$\text{(ii) } \begin{pmatrix} \mathcal{L}(\widetilde{\Theta}) \\ \mathcal{L}(\left\{\widetilde{f}_b\right\}_{0\le|b|\le 2})\end{pmatrix} = \begin{pmatrix} \mathcal{L}(\widehat{\Theta}_k) \\ \mathcal{L}(\{f_b\}_{0\le|b|\le 2})\end{pmatrix}; \tag{74}$$

$$\text{(iii) } \widetilde{a}^j\to\widetilde{\Theta}\ , \quad \text{and} \quad \widetilde{f}_b^j\to\widetilde{f}_b \quad \forall 0\le|b|\le 2 \quad \text{as} \quad j\to\infty \qquad \text{a.s.} \tag{75}$$

Furthermore, we find by (46), (26), (73), Theorem 13 (i), and Lemmas 14 (i), 15 (i) $(a=q-1)$ that

$$\begin{aligned} &\mathcal{L}\left(\int_0^t\int_{\Re^d}\Gamma(x,t-\tau,\xi)\,\mathcal{A}_k^j(\tau,\xi)\,\partial_\xi^{k_1}\overline{A}_m^j(\tau,\xi)\,u_{m+k_2}(\tau,\xi)\,d\xi d\tau\right) \quad (76)\\ =\ &\mathcal{L}\left(\int_0^t\int_{\Re^d}\Gamma(x,t-\tau,\xi)\,\widetilde{a}^j(\tau,\xi)\left[\widetilde{A}_0^j(\tau,\xi)-\partial_\xi^{k_1}A_m^0(\xi)\right]u_{m+k_2}(\tau,\xi)\,d\xi d\tau\right)\end{aligned}$$

on $C(I;H_2)$ and thereby it suffices to expose a further subsequence $\{\varepsilon_l\}_{l=1}^{\infty}$ such that

$$\left|\int_0^t\int_{\Re^d}\Gamma(x,t-\tau,\xi)\,\widetilde{a}^l(\tau,\xi)\ \left[\widetilde{A}_0^l(\tau,\xi)-A'(\xi)\right]u'(\tau,\xi)\,d\xi d\tau\right|_{C(I;H_2)} \overset{l\to\infty}{\to} 0 \tag{77}$$

for almost all $\widetilde{\omega}\in\widetilde{\Omega}$, where

$$u'(\tau,\xi)\doteq u_{m+k_2}(\tau,\xi),\ A'(\xi)\doteq\partial_\xi^{k_1}A_m^0(\xi)\ \forall\xi\in\Re^d,\tau\in I. \tag{78}$$

First, on the basis of (73), (61), and (28) we know that for all $j = 1, 2, 3, \ldots$

$$\text{(i) } \tilde{\alpha}^j \stackrel{D}{=} \alpha^j_{m,k_1} \qquad \text{on} \qquad (C_3, \|\cdot\|_3), \tag{79}$$

$$\text{(ii) } \partial^b_x \tilde{\alpha}^j = \left\{ \int_0^t (\tilde{A}^j_b(\tau) - \partial^b_x A') d\tau,\, t \in I \right\} \text{ on } (C_3, \|\cdot\|_3)\ \forall 0 \le |b| \le 2 \text{ a.s.}, \tag{80}$$

$$\text{(iii) } \partial_{x_n} \tilde{\alpha}^j \stackrel{D}{=} \alpha^j_{m,k_1+e_n} \text{ on } \qquad (C_3, \|\cdot\|_3)\ \forall n = 1, 2, \ldots, d \tag{81}$$

where

$$\tilde{\alpha}^j(t,x) \doteq \int_0^t (\tilde{A}^j_0(\tau, x) - A'(x))\, d\tau. \tag{82}$$

Moreover, it follows from (73), (75), and the measurability of $\|\cdot\|_\infty$ with respect to $(C_3, \|\cdot\|_3)$ (see Lemma 17) that

$$\sup_j \left\| \tilde{A}^j_b \right\|_\infty = \sup_j \tilde{f}^j_b < \infty \quad \forall\, 0 \le |b| \le 2 \qquad \text{a.s.} \tag{83}$$

Thus, using (79-83), and repeating the argument in (65-66), we find a subsequence $\{\varepsilon_l\}_{l=1}^\infty$ such that for almost all $\omega$

$$\left\| \tilde{\alpha}^l(t,x) \right\| \vee \max_{1 \le n \le d} \left\| \partial_{x_n} \tilde{\alpha}^l(t,x) \right\| \stackrel{l \to \infty}{\to} 0 \quad \forall t \in I,\, x \in \Re^d \tag{84}$$

and $\left\{ \tilde{\alpha}^l \right\}_{l=1}^\infty \bigcup \left\{ \partial_{x_n} \tilde{\alpha}^l \right\}_{l=1, n=1}^{\infty, d}$ is uniformly bounded and equicontinuous on $I \times \Re^d$. Then, by (62), (84), and $d+1$ applications of Lemma 18, one finds that

$$\left\| \tilde{\alpha}^l \right\|_4 \le \left\| \tilde{\alpha}^l \right\|_3 + 2^{2\nu} \sum_{n=1}^d \left\| \partial_{x_n} \tilde{\alpha}^l \right\|_3 \stackrel{l \to \infty}{\to} 0 \qquad \text{a.s.} \tag{85}$$

Hence, if one fixes an $\tilde{\omega}$ such that (75) and (85) hold, one finds by Lemma 12 that there exists $\left\{ \tilde{\Theta}^n \right\}_{n=1}^\infty \subset C^1_u$ such that

$$\tilde{\Theta}^n \stackrel{n \to \infty}{\to} \tilde{\Theta}(\tilde{\omega}) \qquad \text{in } C(I; H_1) \tag{86}$$

and equicontinuity of $t \to \tilde{\Theta}_n(t)$ establishes that

$$\left| \tilde{\Theta}_n - \tilde{\Theta}(\tilde{\omega}) \right|_{C(I;H_1)} \stackrel{n \to \infty}{\to} 0 \tag{87}$$

as well, where

$$\tilde{\Theta}_n(t,x) \doteq \tilde{\Theta}^n \left( \frac{[tn]}{n}, x \right) \qquad \forall x \in \Re^d, t \in I. \tag{88}$$

Thus, utilizing Condition (C3), (78), (83), Theorem 13 (i) with $b = k_2$, $l = m$, and Lemmas 14 (i), 15 (i) with $a = q - \gamma$ as well as (75) and (87), one concludes that

$$\left| F_l^4\left(\widetilde{a}^l - \widetilde{\Theta}_n\right)\right|_{C(I;H_2)} \overset{l,n}{\ll} \left|\widetilde{a}^l - \widetilde{\Theta}\right|_{C(I;H_1)} + \left|\widetilde{\Theta} - \widetilde{\Theta}_n\right|_{C(I;H_1)} \to 0 \tag{89}$$

as $l, n \to \infty$ where for all $\rho \in E\,(I; H_1)\,, x \in \Re^d, t \in I$

$$F_l^4(\rho)(t,x) \doteq \int_0^t \int_{\Re^d} \Gamma(x, t-\tau, \xi)\rho(\tau,\xi)\left[\widetilde{A}_0^l(\tau,\xi) - A'(\xi)\right] u'(\tau,\xi)\, d\xi d\tau. \tag{90}$$

Now, using (131-133) (to follow) as well as the argument in (134) of the proof of Lemma 15 (with $a = q-\gamma$, $\nu = b = 0$, $\overline{\Gamma} = \Gamma \cdot \tau^{\frac{\gamma - q}{q}}$, $f = \widetilde{\Theta}_n$, and $g = \tau^{\frac{q-\gamma}{q}}[\widetilde{A}_0^l - A']u'$) and availing ourselves of (78), Theorem 13 (i), and Lemma 14 (i), (87), (83), and Condition (C3), one finds that

$$\begin{aligned} &\left| \int_0^{\delta\wedge t} \int_{\Re^d} \Gamma(x, t-\tau, \xi)\widetilde{\Theta}_n(\tau,\xi)\left[\widetilde{A}_0^l(\tau,\xi) - A'(\xi)\right] u'(\tau,\xi)\, d\xi d\tau \right|_{C(I;H_2)} \\ &\overset{l,n,\delta}{\ll} \sup_t \int_0^{\delta \wedge t} \tau^{\frac{\gamma - q}{q}} d\tau \overset{l,n,\delta}{\ll} \delta^{\frac{\gamma}{q}} \end{aligned} \tag{91}$$

for all $l, n = 1, 2, ...,$ and $\delta \in I$ (small). On the other hand, Stieltjes-type integration by parts, (82), Theorem 13 (i), Lemma 14 (i,ii,v), and (88) yields

$$\begin{aligned} &\lim_{z \nearrow t} \int_{\Re^d} \int_\delta^z \Gamma(x, t-\tau, \xi)\widetilde{\Theta}_n(\tau,\xi)\left[\widetilde{A}_0^l(\tau,\xi) - A'(\xi)\right] u'(\tau,\xi)\, d\tau d\xi \\ = \;& \widetilde{\Theta}_n\,(t,x)\, \widetilde{\alpha}^l(t,x)\, u'(t,x) - \int_{\Re^d} \Gamma(x, t-\delta, \xi)\widetilde{\Theta}_n(\delta,\xi)\, \widetilde{\alpha}^l(\delta,\xi)\, u'(\delta,\xi) d\xi \\ &- \int_\delta^t \int_{\Re^d} \partial_\tau \Gamma(x, t-\tau,\xi)\widetilde{\Theta}_n\,(\tau,\xi)\, \widetilde{\alpha}^l(\tau,\xi)\, u'(\tau,\xi) d\xi\, d\tau \\ &- \int_\delta^t \int_{\Re^d} \Gamma(x, t-\tau,\xi)\widetilde{\Theta}_n\,(\tau,\xi)\, \widetilde{\alpha}^l(\tau,\xi)\, \partial_\tau u'(\tau,\xi) d\xi\, d\tau \\ &- \sum_{i=\lceil \delta n\rceil}^{\lceil tn \rceil -1} \int_{\Re^d} \Gamma(x, t - \frac{i}{n}, \xi)\left[\widetilde{\Theta}^n\!\left(\frac{i+1}{n}, \xi\right) - \widetilde{\Theta}^n\!\left(\frac{i}{n}, \xi\right)\right] \widetilde{\alpha}^l(\frac{i}{n}, \xi) u'(\frac{i}{n}, \xi) d\xi \end{aligned} \tag{92}$$

for all $x \in \Re^d, 0 < \delta < t \le 1$. (The purpose behind the approximation in (86) is now manifest. Since $\widetilde{\Theta}^n \in C_u^1$ the iterated integral in the third term on the right of (92) is well defined for each $n = 1, 2, 3, ...$) Again, borrowing (131-133) as well as the arguments in (134) (with $f \doteq \widetilde{\Theta}_n$ or $f \doteq \widetilde{\Theta}^n(\frac{i+1}{n}) - \widetilde{\Theta}^n(\frac{i}{n})$, and $\overline{\Gamma} \doteq \Gamma \cdot \tau^{\frac{\gamma - q}{q}}$,

$g \doteq \tau^{\frac{q-\gamma}{q}} \widetilde{\alpha}^l u'$ or $\overline{\Gamma} \doteq \Gamma \cdot \tau^{\frac{\gamma-2q}{q}}$, $g \doteq \tau^{\frac{2q-\gamma}{q}} \widetilde{\alpha}^l \partial_\tau u'$) and using (78), Theorem 13 (i) and Lemma 14 (i), (87), (82-83), and Condition (C3), one finds that

$$\max_{t\geq\delta} \left| \int_{\Re^d} \Gamma(x,t-\delta,\xi) \widetilde{\Theta}_n(\delta,\xi)\, \widetilde{\alpha}^l(\delta,\xi)\, u'(\delta,\xi) d\xi \right|_{C(I;H_2)} \overset{n,l}{\ll} \left\| \widetilde{\alpha}^l \right\|_3 \tag{93}$$

$$\max_{t\geq\delta} \left| \int_\delta^t \int_{\Re^d} \Gamma(x,t-\tau,\xi) \widetilde{\Theta}_n(\tau,\xi)\, \widetilde{\alpha}^l(\tau,\xi)\, \partial_\tau u'(\tau,\xi) d\xi \right|_{C(I;H_2)} \overset{n,l}{\ll} \left\| \widetilde{\alpha}^l \right\|_3 \tag{94}$$

$$\max_{t\geq\delta} \left| \sum_{i=\lceil\delta n\rceil}^{\lceil tn\rceil-1} \int_{\Re^d} \Gamma(x,t-\frac{i}{n},\xi)[\widetilde{\Theta}^n(\frac{i+1}{n},\xi) - \widetilde{\Theta}^n(\frac{i}{n},\xi)] \widetilde{\alpha}^l(\frac{i}{n},\xi) u'(\frac{i}{n},\xi) d\xi \right|_{C(I;H_2)} \overset{\delta,n,l}{\ll} n \left\| \widetilde{\alpha}^l \right\|_3 . \tag{95}$$

Similarly, from the argument in and following (135) as well as (136-137), (142) (with $\tau^{\frac{\gamma-q}{q}} \partial_\tau \Gamma$ instead of $\partial_\tau \overline{\Gamma}$, $\rho \doteq \widetilde{\Theta}_n$, $g \doteq \tau^{\frac{q-\gamma}{q}} \widetilde{\alpha}^l u'$) as well as (78), Theorem 13 (i) with $b = k_2$, $k_2 + e_i$ (each $i = 1, ..., d$) and Lemma 14 (i,ii), (87), (82-83), and Condition (C3), it follows easily that

$$\max_{t\geq\delta} \left| \int_\delta^t \int_{\Re^d} \partial_\tau \Gamma(x,t-\tau,\xi) \widetilde{\Theta}_n(\tau,\xi)\, \widetilde{\alpha}^l(\tau,\xi)\, u'(\tau,\xi) d\xi \right|_{C(I;H_2)} \overset{n,l}{\ll} \left\| \widetilde{\alpha}^l \right\|_4 \tag{96}$$

for all $l, n = 1, 2, ...$ Therefore, we find by (90), (91), (92), and (93-96) that

$$\left| F_l^4(\widetilde{\Theta}_n) \right|_{C(I;H_2)} \leq \widetilde{C}(\delta^{\frac{\gamma}{q}} + n \left\| \widetilde{\alpha}^l \right\|_3 + \widetilde{c}_\delta \left\| \widetilde{\alpha}^l \right\|_4) \tag{97}$$

for $\Re$-valued random variables $\widetilde{C}$ (independent of $l, n, \delta$) and $\widetilde{c}_\delta$ (independent of $l, n$). Finally, it follows by (89), (90), (97), and (85) that

$$\begin{aligned} &\left| \int_0^t \int_{\Re^d} \Gamma(x,t-\tau,\xi)\, \widetilde{a}^l(\tau,\xi) \left[ \widetilde{A}_0^l(\tau,\xi) - A'(\xi) \right] u'(\tau,\xi) d\xi\, d\tau \right|_{C(I;H_2)} \\ \leq &\left| F_l^4\left(\widetilde{a}^l - \widetilde{\Theta}_n\right) \right|_{C(I;H_2)} + \left| F_l^4\left(\widetilde{\Theta}_n\right) \right|_{C(I;H_2)} \overset{l\to\infty}{\to} 0 \qquad \text{a.s.} \end{aligned} \tag{98}$$

by choosing small enough $\delta > 0$, large enough $n$, and then letting $l \to \infty$.

**3.3. Convergence of the Approximation Process.** Recalling the definition of $z^r$ in (53) and following the arguments in (56-58), one finds that $z^r = \sum_{|k|\leq q} z_{0,k}^r$ ($\doteq \sum_{|k|\leq q} \left\{ -\int_0^t T_{t-\tau}\, d\mathcal{A}_k^r(\tau)\, u_k(\tau), t \in [0,T] \right\}$), where

$$z_{0,k}^r(t,x) = \sum_{j=5}^{6} F^j(\mathcal{A}_k^r u_k)(t,x) + F^7(\mathcal{A}_k^r \partial_\tau u_k)(t,x) \tag{99}$$

for all $x \in \Re^d, t \in I$, where

$$F^5(\rho)(t,x) \doteq -\rho(t,x) \quad \forall \rho \in C_u^{1,v} \tag{100}$$

$$F^6(\rho)(t,x) \doteq \int_0^t \int_{\Re^d} \partial_\tau \Gamma(x, t-\tau, \xi)\, \rho(\tau, \xi)\, d\xi\, d\tau \quad \forall \rho \in C_u^{1,v} \tag{101}$$

$$F^7(\rho)(t,x) \doteq \int_0^t \int_{\Re^d} \Gamma(x, t-\tau, \xi)\, \rho(\tau, \xi)\, d\xi\, d\tau \quad \forall \tau^{1-\frac{\gamma}{q}} \rho \in C_u^{1,v}. \tag{102}$$

Next, we define

$$F(\rho) \doteq \sum_{j=5}^{6} F^j(\rho\, u_k) + F^7(\rho\, \partial_\tau u_k) \quad \forall \rho \in C_u^1 \tag{103}$$

and show that $F$ defines a continuous map from $\left(C_u^1, |\cdot|_{C(I;H_1)}\right)$ to $C(I;H_2)$. Indeed, (100-103), and (39-41) of Section 2 establish $F(\rho) \in C(I;H_2)$ for all $\rho \in C_u^1$. Moreover, it follows from Theorem 13, and Lemmas 14, 15 (i,ii) that

$$|F(\rho)|_{C(I;H_2)} \overset{\rho}{\ll} |\rho|_{C(I;H_1)} \qquad \text{for all} \qquad \rho \in C_u^1 \tag{104}$$

so Lemma 12 and a simple Cauchy sequence argument allow one to extend the definition of $F$ to $C(I;H_1)$ while retaining its continuous nature. Thus, it follows by (99), (103), (104), Hypothesis (33) and the continuous mapping theorem that

$$z^r \Rightarrow \sum_{|k|\le q} F(\widehat{\Theta}_k) \qquad \text{in} \qquad C(I;H_2) \qquad \text{as} \qquad r \to \infty. \tag{105}$$

However, it follows from (103) and Definition 10 that for $\widehat{P}-$almost all $\widehat{\omega} \in \widehat{\Omega}$

$$\sum_{|k|\le q} F(\widehat{\Theta}_k) = \left\{ -\sum_{|k|\le q} \int_0^t T_{t-\tau}\, d\widehat{\Theta}_k(\tau)\, u_k(\tau), 0 \le t \le T \right\}. \quad \blacksquare \tag{106}$$

## 4. Supporting Results

This section contains eight previously-referenced results, many of which are partially proven elsewhere. In this case, to avoid duplication, we will only supply the necessary references and changes. Our first lemma is used in (42-43) of Section 2, (72-75) and (86) of Subsection 3.2, (104) of Subsection 3.3, and (131) of Section 4. It is proved in Kouritzin [17].

**Lemma 12.** *$H_1$, the pre-Hilbert space of Definitions 2, is complete and separable. Furthermore, (i) $C_u^1$, $(E_u^{1,v})$ the complex vector space of Definition 7, is a dense subset of $C(I;H_1)$ $(E(I;H_1^v)$ with norm $|\cdot|_{C(I;H_1^v)})$ and (ii) $C(I;H_1)$ is a separable Banach space.*

The following lemma is used in the statement of Theorem 6 and Definitions 7-10 of Section 2; and (54), (56), (64), (68), (89), (91), and (93-96) of Subsection 3.2.

**Theorem 13.** *Under Condition (C1-C4) of Section 2, it follows that (i) $u$ is continuously differentiable with respect to $\partial_x^{b+l}$, $\partial_t\partial_x^{b+l}$ and*

$$|u_{b+l}(t,x)| \overset{x,t}{\ll} t^{\frac{(\gamma-|l|)\wedge 0}{q}} \text{ and } |\partial_t u_{b+l}(t,x)| \overset{x,t}{\ll} t^{\frac{\gamma-|l|}{q}-1}\ \forall\, |b|\le q+1, |l|\le q, \tag{107}$$

*$t\in I, x\in\Re^d$ and (ii) $z_r$ is continuously differentiable with respect to $\partial_x^{b+l}$ and there exist $\Re$-valued random variables $C_r$ independent of $t,x$ such that*

$$\left|z_{b+l}^r(t,x)\right| \le C_r\, t^{\frac{(\gamma-|l|)\wedge 0}{q}} \text{ and } |\partial_t z_b^r(t,x)| \le C_r t^{\frac{\gamma}{q}-1} \tag{108}$$

*for all $0\le |b|\le q+1, 0\le |l|\le q, t\in I, x\in\Re^d$. Now, suppose in addition Hypothesis (34) of Section 2 holds and $|\cdot|_3, |\cdot|_4$ are as defined in (61) and (62). Then, it follows that (iii) $\varepsilon_r^{1-h}|z_{b,l}^r|_3 \Rightarrow 0$ and (iv) $\varepsilon_r^{1-h}|z_b^r|_4 \Rightarrow 0$ as $r\to\infty$ for all $0\le |b|,|l|\le q$, where $z_{b,l}^r(t,x) \doteq t^{\frac{(|l|-\gamma/2)\vee 0}{q}} z_{b+l}^r(t,x)\ \forall t\in I, x\in\Re^d$.* ■

**Proof.** The bounds in (i) follow from the proof of Lemma 2 (i) in [16]. Specifically, we use $\partial_x^l\Gamma(x,s,\xi)$ and $\partial_s\partial_x^l\Gamma(x,s,\xi)$ instead of $\partial_x^l\Gamma^r(x,s,\xi,0)$ and avail ourselves (c.f. Lemma 3 of Kouritzin [16]) of the bounds

$$\|\partial_y^m\Gamma_k(y,s,y+w)\| \vee s\cdot\|\partial_s\partial_y^m\Gamma_k(y,s,y+w)\| \le \frac{C}{s^{\frac{d+|k|}{q}}}\exp\left\{-c\left|\frac{|w|^q}{s}\right|^{\frac{1}{q-1}}\right\} \tag{109}$$

$$\|\partial_z^m\Gamma_k(z,s,z+w)-\partial_y^m\Gamma_k(y,s,y+w)\| \le \frac{C\,|z-y|^{\overline{\gamma}}}{s^{\frac{d+|k|}{q}}}\exp\left\{-c\left|\frac{|w|^q}{s}\right|^{\frac{1}{q-1}}\right\} \tag{110}$$

$$\|\partial_s[\partial_z^m\Gamma_k(z,s,z+w)-\partial_y^m\Gamma_k(y,s,y+w)]\| \le \frac{C\,|z-y|^{\overline{\gamma}}}{s^{\frac{d+|k|+q}{q}}}\exp\left\{-c\left|\frac{|w|^q}{s}\right|^{\frac{1}{q-1}}\right\} \tag{111}$$

for all $y,z\in\mathbb{R}^d$, $s\in I$, $0\le|k|\le q$ and $0\le|m|\le q+1$, where $\overline{\gamma}\doteq 3\gamma$ and

$$\Gamma_k(x,s,\xi)\doteq\partial_x^k\Gamma(x,s,\xi).$$

Then, (i) follows from exactly the same method as Lemma 2 (i) of Kouritzin [16]. Moreover, using (109) and (110) as well as the proof of Lemma 2 (ii) of [16], one finds that

$$|u_{b+l}(t,x)-u_{b+l}(t,x')| \overset{x,x',t}{\ll} |x-x'|^{2\gamma}\, t^{\frac{(\gamma-|l|)\wedge 0}{q}} \tag{112}$$

for all $x, x' \in \mathbb{R}^d$, $t \in I$, $0 \le |b| \le q+1$ and $0 \le |l| \le q$. From (109-112) together with (53) of Subsection 3.2, it follows that $\varepsilon_r^{1-h} z^r = \sum_{|m|\le q} B_m^r u$ (with $B_m^r$ being defined in (25) of Kouritzin [16]) and the first bound in (ii) follows from the proof of Theorem 1 of Kouritzin [16]. Of course, the second bound in (ii) follows from the first, (i), Condition (C3), and

$$\partial_t z^r(t,x) = \sum_{|k|\le q} A_k(x)\, \partial_x^k z^r(t,x) + \sum_{|k|\le q} \overline{A}_k^r(t,x)\, u_k(t,x). \tag{113}$$

Turning to (iii) and (iv), we find that for any subsequence $\{\varepsilon_i\}_{i=1}^\infty$ of $\{\varepsilon_r\}_{r=1}^\infty$ there exists by the method of (65-66) further subsequence $\{\varepsilon_j\}_{j=1}^\infty$ such that

$$\alpha_{m,k}^j(t,x) \overset{j\to\infty}{\to} 0 \quad \forall t \in I,\, x \in \Re^d,\ |m| \le q, |k| \le q+1 \quad \text{a.s.} \tag{114}$$

Now, comparing (62) with (10) in [16], one can see that the proof of Theorem 1 of Kouritzin [16] will also establish that

$$\left|\varepsilon_j^{1-h} z_{b,l}^j\right|_4 \le \left|\sum_{|m|\le q} \partial_x^{l+b} B_m^j u\right|_{l,\gamma} \overset{j\to\infty}{\to} 0 \quad \text{a.s.} \tag{115}$$

Hence, $\varepsilon_r^{1-h}\left|z_{b,l}^r\right|_3 \Rightarrow 0$ and $\varepsilon_r^{1-h}\left|z_b^r\right|_4 \Rightarrow 0$ as $r \to \infty$. ■

The following lemma is used in (42) of Section 2; (56), (63), (76), (89), (91), (92), and (93-96) of Subsection 3.2; (99) and (104) of Subsection 3.3; and in Lemma 16 to follow.

**Lemma 14.** *Under Condition (C1-C3) of Section 2, there exist constants $c, C > 0$ and random variables $m, M > 0$ all independent of $x, t, \xi, \tau$ and $\varepsilon$ such that*

$$\text{(i) } \|\partial_\xi^b \Gamma(x, t-\tau, \xi)\| \le \frac{C}{(t-\tau)^{\frac{d+|b|}{q}}} \exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} \tag{116}$$

$$\text{(ii) } \|\partial_\tau \Gamma(x, t-\tau, \xi)\| \le \frac{C}{(t-\tau)^{\frac{d+q}{q}}} \exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} \tag{117}$$

$$\text{(iii) } \|\partial_\xi^b \Gamma^\varepsilon(x, t; \xi, \tau)\| \le \frac{M}{(t-\tau)^{\frac{d+|b|}{q}}} \exp\left\{-m\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} \tag{118}$$

*for all $0 \le |b| \le q$. Moreover, for any $\rho \in E(I; H_1^v)$ we have that*

$$\text{(iv) } \lim_{z \nearrow t} \int_{\Re^d} \Gamma^\varepsilon(x, t; \xi, z) \rho(z, \xi)\, d\xi = \rho(t, x) \tag{119}$$

*for almost all $x, \omega$ and (v) the same result holds with $\Gamma$ replacing $\Gamma^\varepsilon$.*

**Proof.** (i) and (ii) follow from Friedman [11] pp. 260-1. Unfortunately, immediate application of the classical theory for (iii) would not allow our principle coefficients to depend on $t$ or $\varepsilon$ (see the introduction). Therefore, the classical results can not be used directly. Still, it is unnecessary to develop a completely new theory. Indeed, suppose we used the notation $\zeta^k \doteq \zeta_1^{k_1}\,\zeta_2^{k_2}...\zeta_d^{k_d}$ for $\zeta \in \mathbb{C}^d$, defined

$$V^\varepsilon(t,\tau;y,\zeta) = I + \int_\tau^t \sum_{|k|=q} A_k(\frac{s}{\varepsilon},y)(i\zeta)^k\, V^\varepsilon(s,\tau;y,\zeta)\,ds \tag{120}$$

for all $0 \le \tau \le t \le 1$, $y \in \Re^d$, $\zeta \in \mathbb{C}^d$, and $\varepsilon \in (0,1]$ and showed that

$$\left\| \partial_y^s V^\varepsilon(t,\tau;y,\zeta) \right\| \le C_s \exp\left\{[\lambda_s|\beta|^q - \delta_s|\alpha|^q](t-\tau)\right\} \tag{121}$$

for some constants $C_s, \lambda_s, \delta_s > 0$ and all $0 \le |s| \le q$, $0 \le \tau \le t \le 1$, $y \in \Re^d$, $\zeta = \alpha + i\beta \in \mathbb{C}^d$, and $\varepsilon \in (0,1]$. Then, the bounds (3.15) on p. 250 of Friedman [11] would follow (for $0 \le |s| \le q$ without Assumption A of the introduction) from the considerations on p. 249 (top) and pp. 245-6 of [11]. (iii) would follow from the arguments on pp. 261-3 (top) of Friedman [11] in the special case $a = m = 0$ in his proof. Hence, it remains to show (121). However, the case $s \equiv 0$ has already been proved in Equation (152) of Kouritzin [15] with $\delta_0 \doteq \frac{\delta}{2}$, $\lambda_0 \doteq \frac{\lambda'}{2}$, and $\delta, \lambda'$ defined by (35), (158) of [15]. Moreover, suppose we define for all $\mathbb{C}^{N\times N}$ matrices $B$

$$|||B||| = \sqrt{\sum_{m,n=1}^N |B_{m,n}|^2} \tag{122}$$

and let $\delta, \lambda$ be $\delta, \lambda'$ as in (35), (158) of [15]. Then, following (156-159) of Kouritzin [15], using (120), and recalling the inequality $ab \le \frac{a^2+b^2}{2}$, we find

$$\begin{aligned}
&\left|\left|\left| \partial_{y_j} V^\varepsilon(t,\tau;y,\zeta) \right|\right|\right|^2 \exp\left\{[\delta^*|\alpha|^q - \lambda^*|\beta|^q](t-\tau)\right\} \qquad (123)\\
\le\ & \int_\tau^t [(\delta^* - \delta)|\alpha|^q + (\lambda - \lambda^*)|\beta|^q] \left|\left|\left| \partial_{y_j} V^\varepsilon(u,\tau;y,\zeta) \right|\right|\right|^2 \exp\left\{[\delta^*|\alpha|^q - \lambda^*|\beta|^q](u-\tau)\right\} du \\
& + C\,|\zeta|^q \int_\tau^t \left|\left|\left| \partial_{y_j} V^\varepsilon(u,\tau;y,\zeta) \right|\right|\right| \cdot |||V^\varepsilon(u,\tau;y,\zeta)||| \exp\left\{[\delta^*|\alpha|^q - \lambda^*|\beta|^q](u-\tau)\right\}\, du \\
\le\ & \int_\tau^t [(\delta^* - \delta + \gamma)|\alpha|^q + (\lambda - \lambda^* + \gamma)|\beta|^q] \left|\left|\left| \partial_{y_j} V^\varepsilon(u,\tau;y,\zeta) \right|\right|\right|^2 \exp\{[\delta^*|\alpha|^q - \lambda^*|\beta|^q](u-\tau)\} du \\
& + C'\,|\zeta|^q \int_\tau^t |||V^\varepsilon(u,\tau;y,\zeta)|||^2 \exp\left\{[\delta^*|\alpha|^q - \lambda^*|\beta|^q](u-\tau)\right\}\, du
\end{aligned}$$

for any $\delta^*, \lambda^*, \gamma > 0$ and some constants $C, C' > 0$ which depend on $\delta^*, \lambda^*, \gamma$. Now, we choose $\gamma > 0$ (small), set $\delta^* = \delta - \gamma = 2\delta_0 - \gamma$, $\lambda^* = \lambda + \gamma = 2\lambda_0 + \gamma$ and use the inequality $(t-\tau)|\zeta|^q \overset{t,\tau,\zeta}{\ll} \exp\{\gamma[|\alpha|^q+|\beta|^q](t-\tau)\}$ to find that the second term on the far right of (123) is bounded by a constant $C''$. Hence, (123) becomes

$$\left\|\partial_{y_j} V^\varepsilon(t,\tau;y,\zeta)\right\| \le (C'')^{1/2}\exp\{1/2[\delta^*|\alpha|^q-\lambda^*|\beta|^q](t-\tau)\} \tag{124}$$

for all $j=1,2,...,d$, $y\in\Re^d$, $\zeta=\alpha+i\beta\in\mathbb{C}^d$, $0\le\tau\le t\le 1$ and $\varepsilon\in(0,1]$. The remainder of the bounds (121) can be established through induction and the same argument.

(iv) and (v) follow from a development similar to Friedman [11] p. 255 (and the referenced portions of Chapter 1) when $\rho\in E_u^{1,v}$. Then, (v) follows in the general case from (31), Lemma 16 (iii), and the fact that $E_u^{1,v}$ is dense in $E(I;H_1^v)$ with norm $|\cdot|_{C(I;H_1^v)}$ (c.f. Lemma 12). (iv) can be proved in a similar manner. ■

The following lemma is used in (42) of Section 2; (58), (63), (76), and (89) of Subsection 3.2; and (104) of Subsection 3.3. We describe $C^*$ because $C'$ is easier to derive than $C^*$ and it is important in (63) that $C'$ and $C^*$ form random variables.

**Lemma 15.** *Let $|\cdot|_3$ and $|\cdot|_4$ be as defined in (61) and (62) and $I=[0,T]$. (i) Suppose $a,b\ge 0$ are such that $a+b<q$ and $\overline{\Gamma}$ is any $\mathbb{C}^{N\times N}$-valued kernel satisfying*

$$\|\overline{\Gamma}(x,t;\xi,\tau)\| \le \frac{C\,\tau^{-\frac{a}{q}}(1+|\xi|^2)^\nu}{(t-\tau)^{\frac{d+b}{q}}}\exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} \tag{125}$$

*for some $C,c>0$ and $0\le\nu<d/8$. Then, there exists a constant $C'>0$ such that*

$$\left|\int_0^t\int_{\Re^d}\overline{\Gamma}(x,t;\xi,\tau)\,f(\tau,\xi)g(\tau,\xi)\,d\xi\,d\tau\right|_{C(I;H_2)} \le C'|g|_3\,|f|_{C(I;H_1)} \tag{126}$$

*for all $f\in E(I;H_1)$ and continuous, bounded $\mathbb{C}^N$-valued $g$. (ii) On the other hand, if $\overline{\Gamma}$ is continuously differentiable with respect to $\xi$ up to order $q$,*

$$\|\partial_\xi^m\overline{\Gamma}(x,t;\xi,\tau)\| \le \frac{C}{(t-\tau)^{\frac{d+|m|}{q}}}\exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} \tag{127}$$

*for all $0\le|m|\le q$, and*

$$\partial_\tau\overline{\Gamma}(x,t;\xi,\tau) = -\sum_{|k|\le q}(-1)^{|k|}\partial_\xi^k\left[\overline{\Gamma}(x,t;\xi,\tau)\cdot B_k(\tau,\xi)\right] \tag{128}$$

*for some $\{B_m\}_{|m|\le q}$ such that $\partial_\xi^k B_m$ is continuous and bounded on $I\times\Re^d$ for all $0\le|k|\le|m|\le q$. Then, the iterated integral on the left of (129) makes sense and*

$$\left|\int_0^t\int_{\Re^d}\partial_\tau\overline{\Gamma}(x,t;\xi,\tau)\,\rho(\tau,\xi)\,g(\tau,\xi)\,d\xi\,d\tau\right|_{C(I;H_2)} \le C^*|g|_4\,|\rho|_{C(I;H_1)} \tag{129}$$

*for some constant $C^*>0$ and all $\rho\in E_u^1$ and $g\in C_u^{1,v}$. (iii) $C^*$ can be chosen as*

$$C^* \doteq \sup_t \int_0^t \Bigg\{ \sum_{|k|<q} \sup_x \int_{B_x} \frac{\left|1+|\xi|^2\right|^\nu}{\left|1+|x|^2\right|^{d/2}} \left\|\partial_\xi^k[\overline{\Gamma}(x,t;\xi,\tau)B_k(\tau,\xi)]\right\| \, d\xi \tag{130}$$
$$+ \sum_{|k|=q} \sup_x \left\| \int_{B_x} \partial_\xi^k[\overline{\Gamma}(x,t;\xi,\tau)B_k(\tau,\xi)]\, d\xi \right\|$$
$$+ \sum_{|k|\le q} \Bigg[ \sqrt{\int_{\Re^d}\int_{B_x^c} \frac{|1+|\xi|^2|^{d+2\nu}}{|1+|x|^2|^{2d}} \left\|\partial_\xi^k[\overline{\Gamma}(x,t;\xi,\tau)B_k(\tau,\xi)]\right\|^2 d\xi\, dx}$$
$$+ \sqrt{\sup_x \int_{B_x} \frac{\left|1+|z|^2\right|^{2\nu}}{\left|1+|x|^2\right|^{d/2}} \|\partial_\xi^k[\overline{\Gamma}(x,t;z,\tau)B_k(\tau,z)]\|\, dz \,\cdot}$$
$$\overline{\sup_{|x-\xi|<1} \frac{|x-\xi|^{d+\gamma}}{(t-\tau)^{\frac{d+\gamma}{q}}} \frac{\|\partial_\xi^k[\overline{\Gamma}(x,t;\xi,\tau)B_k(\tau,\xi)]\|}{\left|1+|x|^2\right|^{d/2}}} \left[1+\int_{B_0}|z|^{-d+\gamma}dz\right]\Bigg]\Bigg\}\, d\tau.$$

**Remark 3.** *In our applications of (ii) (in (42) of Section 2, (58) and (63) in Subsection 3.2, and (104) in Subsection 3.3) $\overline{\Gamma}$ is a fundamental solution for (12) or (13) and (128) follows from the adjoint problem (see e.g. Friedman [11] pp. 258-9).*

**Proof.** (i) First, one finds from (125), (61), Cauchy-Schwarz, and the bound $\int_{B_x} s^{-\frac{d}{q}} \exp\left\{-c\left|\frac{|x-\xi|^q}{s}\right|^{\frac{1}{q-1}}\right\} d\xi \overset{x,s,\xi}{\ll} 1$ that

$$\int_{\Re^d} |1+|x|^2|^{-2d} \left| \int_{B_x^c} \overline{\Gamma}(x,t;\xi,\tau) f(\tau,\xi) g(\tau,\xi) d\xi \right|^2 dx \tag{131}$$
$$\overset{t,\tau}{\ll} \int_{\Re^d} \frac{|g|_3^2}{|1+|x|^2|^{2d}} \left[ \int_{B_x^c} \frac{|f(\tau,\xi)||1+|\xi|^2|^{2\nu}}{|t-\tau|^{\frac{d+b}{q}} \tau^{\frac{a}{q}}} \exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{\frac{1}{q-1}}\right\} d\xi \right]^2 dx$$
$$\overset{t,\tau}{\ll} \int_{\Re^d} \frac{|g|_3^2\, \tau^{-\frac{a}{p}}}{|1+|x|^2|^{2d}} \int_{B_x^c} \frac{|1+|\xi|^2|^{d+4\nu}}{|t-\tau|^{\frac{d+b}{p}}} \left|\frac{t-\tau}{|x-\xi|^q}\right|^{\frac{4d+b}{q}} d\xi dx \cdot \|f(\tau)\|_1^2$$
$$\overset{t,\tau}{\ll} \frac{|g|_3^2\, |f|_{C(I;H_1)}^2}{|t-\tau|^{\frac{b-2d}{q}} \tau^{\frac{a}{p}}} \int_{\Re^d}\int_{B_x^c} \frac{|1+|x|^2|^{d+2\nu} + |x-\xi|^{2d+8\nu}}{|1+|x|^2|^{2d}\, |x-\xi|^{4d+b}} d\xi\, dx \overset{t,\tau}{\ll} \frac{|g|_3^2\, |f|_{C(I;H_1)}^2}{|t-\tau|^{\frac{b-2d}{q}} \tau^{\frac{a}{p}}}$$

for all $0\le\tau\le t\le 1$,

$$\int_{\Re^d} |1+|x|^2|^{-2d} \left| \int_{B_x} \overline{\Gamma}(x,t;\xi,\tau) f(\tau,x) g(\tau,\xi)\, d\xi \right|^2 dx \tag{132}$$

$$\stackrel{t,\tau}{\ll} \int_{\Re^d} \frac{|g|_3^2\,\tau^{-\frac{a}{p}}}{|1+|x|^2|^{2d}} \left[ \int_{B_x} \frac{|1+|\xi|^2|^{2\nu}}{|t-\tau|^{\frac{d+b}{q}}} \exp\left\{ -c \left| \frac{|x-\xi|^q}{t-\tau} \right|^{\frac{1}{q-1}} \right\} d\xi \; \|f(\tau,x)\| \right]^2 dx$$

$$\stackrel{t,\tau}{\ll} \tau^{-\frac{a}{p}} |t-\tau|^{-\frac{b}{p}} |g|_3^2 \, |f|^2_{C(I;H_1)} \qquad \forall\, 0 \le \tau \le t \le 1,$$

and

$$\int_{\Re^d} |1+|x|^2|^{-2d} \left| \int_{B_x} \overline{\Gamma}(x,t;\xi,\tau)[f(\tau,\xi)-f(\tau,x)]\, g(\tau,\xi) d\xi \right|^2 dx \tag{133}$$

$$\stackrel{t,\tau}{\ll} \int \left[ \int_{B_x} \frac{|g|_3\, |1+|\xi|^2|^{2\nu}}{\tau^{\frac{a}{q}} |t-\tau|^{\frac{d+b}{q}}} \exp\left\{ -c \left| \frac{|x-\xi|^q}{t-\tau} \right|^{\frac{1}{q-1}} \right\} \frac{\|f(\tau,\xi)-f(\tau,x)\|}{|1+|x|^2|^d} d\xi \right]^2 dx$$

$$\stackrel{t,\tau}{\ll} \int \int_{B_x} \frac{|g|_3^2 (|1+|x|^2|^{4\nu} + |\xi-x|^{8\nu})}{\tau^{\frac{a}{p}} |t-\tau|^{\frac{d+2b}{q}}} \exp\left\{ -c \left| \frac{|x-\xi|^q}{t-\tau} \right|^{\frac{1}{q-1}} \right\} \frac{\|f(\tau,\xi)-f(\tau,x)\|^2}{|1+|x|^2|^{2d}} d\xi dx$$

$$\stackrel{t,\tau}{\ll} \int \frac{|g|_3^2\,\tau^{-\frac{a}{p}} |t-\tau|^{\frac{\gamma-2b}{q}}}{|1+|x|^2|^d} \int_{B_x} \frac{\|f(\tau,\xi)-f(\tau,x)\|^2}{|x-\xi|^{d+\gamma}} d\xi dx \stackrel{t,\tau}{\ll} \frac{|g|_3^2\, |f|^2_{C(I;H_1)}}{\tau^{\frac{a}{p}} |t-\tau|^{\frac{2b-\gamma}{q}}}$$

for all $0 \le \tau \le t \le 1$. Therefore, it follows from (131), (132), (133), and convexity of norm that

$$\left| \int_0^t \int_{\Re^d} \overline{\Gamma}(x,t;\xi,\tau)\, f(\tau,\xi)\, g(\tau,\xi) d\xi \; d\tau \right|_{C(I;H_2)} \tag{134}$$

$$\stackrel{f,g}{\ll} \sup_t \int_0^t (t-\tau)^{-\frac{b}{q}} \tau^{-\frac{a}{q}} d\tau \cdot |g|_3 \; |f|_{C(I;H_1)} \stackrel{f,g}{\ll} |g|_3 \, |f|_{C(I;H_1)} \,.$$

(ii) It follows from (128) that

$$\int_0^t \int_{\Re^d} \partial_\tau \overline{\Gamma}(x,t;\xi,\tau)\, \rho(\tau,\xi)\, g(\tau,\xi)\, d\xi\; d\tau \tag{135}$$

$$= \; -\sum_{|k|\le q} (-1)^{|k|} \int_0^t \int_{\Re^d} \partial_\xi^k [\overline{\Gamma}(x,t;\xi,\tau) \cdot B_k(\tau,\xi)] \left| 1+|\xi|^2 \right|^\nu \rho(\tau,\xi) \frac{g(\tau,\xi)}{|1+|\xi|^2|^\nu} d\xi\; d\tau$$

and the terms $|k| < q$ can be handled by (i). For the terms $|k| = q$ in (135), we follow (134) substituting the following development for (131-133). First, we let $k$ such that $|k| = q$ be fixed, set $\Gamma(x,t;\xi,\tau) \doteq \overline{\Gamma}(x,t;\xi,\tau) \cdot B_k(\xi,\tau)$, and find by repeating the arguments in (131) that

$$\int_{\Re^d} |1+|x|^2|^{-2d} \left| \int_{B_x^c} \partial_\xi^k \Gamma(x,t;\xi,\tau) f(\tau,\xi) g(\tau,\xi) d\xi \right|^2 dx \stackrel{t,\tau}{\ll} \frac{|g|_3^2\, |f|^2_{C(I;H_1)}}{|t-\tau|^{2-\frac{d}{p}}}. \tag{136}$$

Similarly, following the argument in (133), we find that

$$\int_{\Re^d}\left|\int_{B_x}\partial_\xi^k\Gamma(x,t;\xi,\tau)[f(\tau,\xi)g(\tau,\xi)-f(\tau,x)g(\tau,x))]d\xi\right|^2\frac{dx}{|1+|x|^2|^{2d}} \quad (137)$$
$$\overset{t,\tau}{\ll} |t-\tau|^{\frac{\gamma-2q}{q}}|g|_4^2\,|f|_{C(I;H_1)}^2 \qquad \text{for all} \quad 0\le\tau\le t\le 1.$$

However, for our final bound we must employ more elaborate methods. Suppose $n$ is an arbitrary $d$-vector of non-negative integers such that $\partial_\xi^k=\partial_{\xi_i}\partial_\xi^n$ for some $1\le i\le d$. Then, $B_x$ is a regular domain in $\Re^d$ and its boundary $\partial B_x$ is a $(d-1)$-manifold. Therefore, noting that

$$d\left\{\partial_\xi^n\Gamma(x,t;\xi,\tau)\,d\xi_1\wedge d\xi_2\wedge\cdots\wedge d\xi_{i-1}\wedge d\xi_{i+1}\wedge\cdots\wedge d\xi_d\right\} \quad (138)$$
$$= (-1)^{i-1}\partial_{\xi_i}\partial_\xi^n\Gamma(x,t;\xi,\tau)\,d\xi_1\wedge\cdots\wedge d\xi_{i-1}\wedge d\xi_i\wedge d\xi_{i+1}\wedge\cdots\wedge d\xi_d,$$

one has by the version of the divergence theorem in Fleming [9] p. 359 that

$$\left\|\int_{B_x}\partial_\xi^k\Gamma(x,t;\xi,\tau)d\xi\right\|=\left\|\int_{\partial B_x^0}\partial_\xi^n\Gamma(x,t;\xi,\tau)d\xi_1\wedge\cdots\wedge d\xi_{i-1}\wedge d\xi_{i+1}\wedge\cdots\wedge d\xi_d\right\|, \quad (139)$$

where $\partial B_x^0$ denotes the boundary of $B_x$ with either orientation. Hence, noting that

$$\left\|\partial_\xi^n\Gamma(x,t;\xi,\tau)\right\| \overset{x,t;\xi,\tau}{\ll} \frac{1}{|t-\tau|^{1+\frac{d-1}{q}}}\exp\left\{-c'|t-\tau|^{\frac{-1}{q-1}}\right\} \overset{x,t;\xi,\tau}{\ll} 1 \quad (140)$$

for $\xi\in\partial B_x$ and observing that the surface area $S_{d-1}(\partial B_x)=S_{d-1}(\partial B_0)$, we find by (4) on the top of p. 357 of Fleming [9], (139) and (140) that

$$\left|\int_{B_x}\partial_\xi^k\Gamma(x,t;\xi,\tau)d\xi\right| \overset{x,t,\tau}{\ll} 1 \qquad \forall x\in\Re^d, 0\le\tau\le t\le 1, |k|=q. \quad (141)$$

Therefore, by (141)

$$\int_{\Re^d}|1+|x|^2|^{-2d}\left|\sum_{|k|=q}\int_{B_x}\partial_\xi^k\Gamma(x,t;\xi,\tau)\,d\xi f(\tau,x)g(\tau,x)\right|^2 dx \overset{t,\tau}{\ll} |g|_3^2\,|f|_{C(I;H_1)}^2 \quad (142)$$

for all $0\le\tau\le t\le 1$ and (ii) follows from (135), (136-137), and (142). ■

The following lemma is used in (31), (38), and (39) of Section 2, and in Lemma 14 (iv) above. For the statement and proof of this result we define $C^1(\Re^d)$ to be the Banach space (see Kufner et. al. [20] p. 26) of continuous $\mathbb{C}^N$-valued functions $g$ such that $\partial x_i g(x)$ exists and is continuous on $\Re^d$ for $i=1,...,d$ and

$$|g|_* \doteq \sup_{x\in\Re^d,1\le i\le d}|g(x)|\vee|\partial_{x_i}g(x)|<\infty. \quad (143)$$

**Lemma 16.** *Under Conditions (C1-C3) of Section 2, one has that*

*(i)* $T_s : H_1^v \to H_2$ and $\partial_s T_s : C^1(\Re^d) \to H_2$ are continuous $\forall s > 0$,
*(ii)* $T_s f$ and $s^{1-\frac{1}{2q}} \partial_s T_s g : [0,1] \to H_2$ *are* continuous $\forall f \in H_1^v, g \in C^1(\Re^d)$,
*(iii)* $\|T_s\|_{1,2} \overset{s}{\ll} 1$ *and* $\|\partial_s T_s\|_{*,2} \overset{s}{\ll} s^{\frac{1}{2q}-1}$ $\forall s \geq 0$,

*where* $\|\cdot\|_{1,2}$ *denotes the operator norm from* $H_1^v$ *to* $H_2$, $\|\cdot\|_{*,2}$ *denotes the operator norm from* $C^1(\Re^d)$ *to* $H_2$ *and* $T_s, s \geq 0$ *are the operators defined in (31) of Section 2.*

**Proof.** (i) follows from the arguments in (131-133) and (135-142) with $t - \tau = s$ and $b = a = 0$. (iii) It follows easily from the bounds in 14 (i,ii) and the arguments in (131-133) and (135-142) with $\gamma = 1$ in (137) that $s \to T_s f$ and $s \to s^{1-\frac{1}{2q}} \partial_s T_s g$ are bounded on $[0,1]$ for all $f \in H_1^v$ and $g \in C^1(\Re^d)$. Then, (iii) follows from (i), this boundedness, and the Banach-Steinhaus theorem. (ii) Suppose $s = 0$ and $\{s_n\}_{n=1}^{\infty} \subset (0,1]$ is such that $s_n \to 0$. Then since (iii) has already been established we may use Lemma 14 (v) to determine that

$$\lim_{n\to\infty} T_{s_n} f(x) = \lim_{n\to\infty} T_{1-(1-s_n)} f(x) = f(x) \tag{144}$$

for almost all $x \in \Re^d$. Moreover, by the adjoint problem (see Friedman [11] pp. 258-9)

$$\partial_\sigma \Gamma(x, \sigma, \xi) = \sum_{|k|\leq q} (-1)^{|k|} \partial_\xi^k [\Gamma(x, \sigma, \xi)\, A_k^0(\xi)] \tag{145}$$

so by Lemma 14 (i), Condition (C3), and the divergence theorem

$$\begin{aligned} |s_n^{1-\frac{1}{2q}} \partial_s T_{s_n} g(x)| &\leq s_n^{1-\frac{1}{2q}} \left\{ \int_{B_x^c} \|\partial_s \Gamma(x, s_n, \xi)\| \, |g(\xi)| \, d\xi \right. \\ &+ \sum_{|k|<q} \int_{B_x} \|\partial_\xi^k [\Gamma(x, s_n, \xi)\, A_k^0(\xi)]\| \, |g(\xi)| \, d\xi + \sum_{|k|=q} \| \int_{B_x} \partial_\xi^k [\Gamma(x, s_n, \xi)\, A_k^0(\xi)] d\xi \| \, |g(x)| \\ &+ \left. \sum_{|k|=q} \int_{B_x} \|\partial_\xi^k [\Gamma(x, s_n, \xi)\, A_k^0(\xi)]\| \, |g(\xi) - g(x)| d\xi \right\} \\ &\overset{s_n,x}{\ll} s_n^{1-\frac{1}{2q}} \left\{ \int_{\Re^d} s_n^{-(s+q-1)/q} \exp\left\{ -c' \left| \frac{|x-\xi|^q}{s_n} \right| \right\} d\xi + 1 \right\} \overset{s_n,x}{\ll} s_n^{\frac{1}{2q}} \to 0. \end{aligned} \tag{146}$$

On the other hand, suppose $s > 0$ and $\{s_n\} \subset [\frac{s}{2}, 1 \wedge 2s]$ is such that $s_n \to s$. Then, for all $x \in \Re^d$

$$\|\Gamma(x, s_n, \xi)\| \vee s_n^{1-\frac{1}{2q}} \|\partial_s \Gamma(x, s_n, \xi)\| \overset{s_n,\xi,x}{\ll} s^{-\frac{2d+1}{2q}} \exp\left\{ -c \left| \frac{|x-\xi^q}{s} \right|^{\frac{1}{q-1}} \right\} \tag{147}$$

which is $\xi$-integrable so dominated convergence establishes that

$$\lim_{n\to\infty} T_{s_n} f(x) = T_s f(x) \text{ and } \lim_{n\to\infty} s_n^{1-\frac{1}{2q}} \partial_s T_{s_n} g(x) = s^{1-\frac{1}{2q}} \partial_s T_s g(x). \tag{148}$$

Hence, we have pointwise convergence almost everywhere by (144), (146), and (148). Moreover, by Lemma 14 (i) one finds that

$$|T_{s_n} f(x)| \overset{s_n,x}{\ll} \int_{\Re^d} s_n^{-d/q} \exp\left\{-c\left|\frac{|x-\xi|^q}{s_n}\right|^{\frac{1}{q-1}}\right\} d\xi \overset{s_n,x}{\ll} 1 \tag{149}$$

and the lemma follows by this pointwise convergence, (146), (149), and dominated convergence. ■

The following lemma is used in (73-75) and (83) of Subsection 3.2.

**Lemma 17.** *$(C_3, \|\cdot\|_3)$ as defined in (61) is separable and complete and $\|\cdot\|_\infty$ is $(C_3, \|\cdot\|_3)-$measurable, where*

$$\|f\|_\infty \doteq \sup_{t\in I, x\in\Re^d} \|f(t,x)\|. \tag{150}$$

**Proof.** $(C_3, \|\cdot\|_3)$ is complete and separable by isometry to the space of continuous functions on $I \times \Re^d$ which vanish at infinity with sup norm. Completeness and separability for this second space are well known. Defining the domain $D^N \doteq I \times [-N, N]^d$ and the norms

$$\|f\|_N \doteq \sup_{t\in I, x\in D^N} \|f(t,x)\|, \tag{151}$$

we find $\|\cdot\|_N$ is continuous with respect to $\|\cdot\|_3$. Hence, $\|\cdot\|_\infty = \lim_{N\to\infty} \|\cdot\|_N$ is measurable. ■

The following lemma is used in (85) of Subsection 3.2. Its proof follows from the argument in lines (66-77) of Kouritzin [15] with the definition $\phi^l(t,x) \doteq \frac{\beta^l(t,x)}{(1+|x|^2)^\nu}$.

**Lemma 18.** *Suppose $\|\cdot\|_3$ is as defined in (61) and $\left\{\beta^l\right\}_{l=1}^\infty$ is a uniformly (in $l, t, x$) bounded, equicontinuous family on $I \times \Re^d$ such that*

$$\beta^l(t,x) \to 0 \quad \text{as } l \to \infty \quad \forall t \in I,\, x \in \Re^d. \tag{152}$$

*Then, $\|\beta^l\|_3 \to 0$ as $l \to \infty$.*

The following lemma is used in (73-75) of Subsection 3.2.

**Lemma 19.** *Let $\mathcal{X}, \mathcal{Y}$ be complete separable metric spaces. Suppose that $\{X_n, n = 0,1,2,...\}$ is a $\mathcal{X}$-valued process defined on a probability space $(\Omega, \mathcal{F}, P)$ and for each $n \geq 0$, $(\widetilde{X}_n, \tilde{Y}_n)^T$ is a $\mathcal{X} \times \mathcal{Y}$-valued random vector defined on probability space $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ such that: $\mathcal{L}(\widetilde{X}_n) = \mathcal{L}(X_n)$ for $n = 0,1,2,...$ Then, there exists a $\mathcal{X} \times \mathcal{Y}$-valued process $\left\{\left(\widehat{X}_n, \hat{Y}_n\right)^T, n = 0,1,2,...\right\}$ defined on a common probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that:*

$$\begin{array}{ll} (a) & \mathcal{L}(\widehat{X}_0\, \widehat{X}_1\, \widehat{X}_2 ...) = \mathcal{L}(X_0\, X_1\, X_2 ...) \quad \text{on } \prod_{k\in\mathbb{N}_0} \mathcal{B}(\mathcal{X}) \\ (b) & \mathcal{L}\binom{\widehat{X}_n}{\hat{Y}_n} = \mathcal{L}\binom{\widetilde{X}_n}{\tilde{Y}_n} \qquad \text{for all } n = 0,1,2,... \end{array}$$

**Proof.** The key to proving this result is the construction of a consistent family of probability measures $\left\{\hat{P}_n\right\}_{n=0}^{\infty}$ such that $\hat{P}_n$ is defined on $\mathcal{B}\left((\mathcal{X} \times \mathcal{Y})^{n+1}\right)$;

$$\hat{P}_n(\Gamma_0 \times \mathcal{Y} \times \Gamma_1 \times \mathcal{Y} \times \cdots \times \Gamma_n \times \mathcal{Y}) = P(X_0 \in \Gamma_0, X_1 \in \Gamma_1, \ldots, X_n \in \Gamma_n) \qquad (153)$$

for all $\Gamma_j \in \mathcal{B}(\mathcal{X}), j = 0,1,2,...,n$; and

$$\hat{P}_n((\mathcal{X} \times \mathcal{Y})^i \times (\Gamma_i \times \Gamma_i') \times (\mathcal{X} \times \mathcal{Y})^{n-i}) = \tilde{P}_i(\widetilde{X}_i \in \Gamma_i, \tilde{Y}_i \in \Gamma_i') \qquad (154)$$

for all $\Gamma_i \in \mathcal{B}(\mathcal{X}), \Gamma_i' \in \mathcal{B}(\mathcal{Y}), i = 0,1,2,...,n$; followed by application of the consistency theorem (see e.g. III.51 of Dellacherie and Meyer [5]). However; recalling (see e.g. Dudley and Philipp [6] Lemma 2.13) the fact:

*Let $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ be complete separable metric spaces. Suppose $P_1$ and $P_2$ are probability measures on $\mathcal{B}(\mathcal{Z}_1 \times \mathcal{Z}_2)$ and $\mathcal{B}(\mathcal{Z}_2 \times \mathcal{Z}_3)$ such that $P_1$ and $P_2$ have the same marginal on $\mathcal{B}(\mathcal{Z}_2)$. Then, there exists a probability measure $Q_{P_1,P_2}$ on $\mathcal{B}(\mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3)$ such that the marginal of $Q_{P_1,P_2}$ on $\mathcal{B}(\mathcal{Z}_1 \times \mathcal{Z}_2)$ is $P_1$ and the marginal on $\mathcal{B}(\mathcal{Z}_2 \times \mathcal{Z}_3)$ is $P_2$.*

and defining $\left\{\hat{P}_n\right\}_{n=0}^{\infty}$ inductively via

$$\hat{P}_0 \doteq \mathcal{L}\begin{pmatrix}\widetilde{X}_0 \\ \tilde{Y}_0\end{pmatrix}, \quad \hat{P}_{n+1} \doteq Q_{\hat{\mu}, \mathcal{L}\binom{\widetilde{X}_{n+1}}{\tilde{Y}_{n+1}}}, \qquad (155)$$

where

$$\begin{aligned} & \hat{\mu}(\Gamma_0 \times \Gamma_0' \times \cdots \times \Gamma_n \times \Gamma_n' \times \Gamma_{n+1}) \qquad (156) \\ \doteq\ & Q_{\mu, \mathcal{L}(X_0\, X_1 \ldots X_{n+1})}(\Gamma_0' \times \cdots \times \Gamma_n' \times \Gamma_0 \times \cdots \times \Gamma_n \times \Gamma_{n+1}) \end{aligned}$$

and

$$\mu(\Gamma_0' \times \cdots \times \Gamma_n' \times \Gamma_0 \times \cdots \times \Gamma_n) \doteq \hat{P}_n(\Gamma_0 \times \Gamma_0' \times \cdots \times \Gamma_n \times \Gamma_n') \qquad (157)$$

for all $\Gamma_i \in \mathcal{B}(\mathcal{X}), i = 0,1,2,...,n+1$; $\Gamma_i' \in \mathcal{B}(\mathcal{Y}), i = 0,1,2,...,n$; one can easily verify consistency, (153), and (154) for $\left\{\hat{P}_n\right\}_{n=1}^{\infty}$. ■

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