CONVERGENCE OF SOLITARY-WAVE SOLUTIONS IN A PERTURBED BI-HAMILTONIAN DYNAMICAL SYSTEM. I. COMPACTONS AND PEAKONS.

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Abstract. We investigate how the non-analytic solitary wave solutions — peakons and compactons — of an integrable biHamiltonian system arising in fluid mechanics, can be recovered as limits of classical solitary wave solutions forming analytic homoclinic orbits for the reduced dynamical system. This phenomenon is examined to understand the important effect of linear dispersion terms on the analyticity of such homoclinic orbits.

1. Introduction. Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytic formulae (typically a sech^2 function or variants thereof) and serve as prototypical solutions that model physical localized waves. In the case of integrable systems, the solitary waves interact cleanly, and are known as solitons. For many examples, localized initial data ultimately breaks up into a finite collection of solitary wave solutions; this fact has been proved analytically for certain integrable equations such as the Korteweg-deVries equation, [2], and is observed numerically in many others. More recently, the appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations, including peakons, [6], [13], which have a corner at their crest, cuspons, [25], having a cusped crest, and, compactons, [17], [18], [20], which have compact support, has vastly increased the menagerie of solutions appearing in model equations, both integrable and non-integrable. The distinguishing feature of the systems admitting non-analytic solitary wave solutions is that, in contrast to the classical nonlinear wave equations, they all include a nonlinear dispersion term, meaning that the highest order derivatives (characterizing the dispersion relation) do not occur linearly in the system, but are typically multiplied by a function of the dependent variable.

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The first and most important of the nonlinearly dispersive, integrable equations is the equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + 3\gamma u u_x + \gamma\nu(u u_{xxx} + 2u_x u_{xx}). \tag{1.1}$$

Here α, β, γ and ν are real constants and $u(x,t)$ is the unknown function depending on the temporal variable t and the spatial variable x. This equation contains both linear dispersion terms νu_{xxt}, βu_{xxx}, and the nonlinear dispersion terms $u u_{xxx}$. Equation (1.1) can, in certain parameter regimes, be regarded an an integrable perturbation of the well-known BBM (or regularized long wave) equation

$$u_t + u_x + u u_x - u_{xxt} = 0, \tag{1.2}$$

which was originally proposed, [4], as an alternative to the celebrated Korteweg-deVries (KdV) equation

$$u_t + u_x + u u_x + u_{xxx} = 0, \tag{1.3}$$

in the modeling of dispersive nonlinear wave phenomena. Although the integrability of the KdV equation makes it a more mathematically significant equation (see [2]), the BBM equation has better analytical properties, including the more desirable linear dispersion relation for fluid modeling, [4].

Equation (1.1) first appears (albeit with a slight error in the coefficients) in the work of Fuchssteiner (ref. [8], Equation (5.3)). Camassa and Holm, [6], rederived (1.1), for certain values of the coefficients including $\nu < 0$, as a model for water waves, and established an associated linear scattering problem. They began a systematic study of the solutions of (1.1), discovering that its soliton solutions are only piecewise analytic, having a corner at their crest, and hence named them peakons. Although not classical solutions, peakons do form weak solutions of (1.1). Like classical solitons, there exist multi-peakon solutions of (1.1) with cleanly interacting peakons; see [3] for a detailed analysis of their behavior. At the same time, Rosenau, [18], discovered a wide variety of nonlinear wave equations with nonlinear dispersion that admit compactly supported solitary wave solutions — epitomized by the (presumably) nonintegrable family of equations

$$u_t \pm (u^m)_x + (u^n)_{xxx} = 0, \tag{1.4}$$

depending on positive integers m, n. If $n \geq 2$, (1.4) admits a one-parameter family of compactons, the parameter being the wave speed (which also governs its amplitude). Inspired by this discovery, Rosenau proposed the alternative form of Equation (1.1), with $\nu > 0$, and claimed that it provided an example of an integrable system supporting compacton solutions. However, very few of Rosenau's proposed compactons are actually solutions of (1.1) in the weak sense — indeed we shall prove that there is at most a single wave speed for which (1.1) with $\nu > 0$ admits a compacton solution vanishing at infinity. An additional difficulty with this equation is that, for $\nu > 0$, the dispersion relation for (1.1) has singularities which raises questions about it being a well-posed initial value problem.

The nonlinear wave Equation (1.1) is just one of a wide variety of examples of "dual Hamiltonian systems" which can be constructed from classical soliton equations such as the KdV equation by either a Lagrange transformation, [19], or, more generally, a rearrangement of the operators appearing in their bi- (or, rather, multi-) Hamiltonian structure; the basic method appears in [9], and has been extensively developed in [7], [16], to which we refer the reader for many additional interesting examples. We remark that certain versions of the Harry Dym equation derived by this dualization procedure do admit parametrized families of compactons and are integrable (at least in the sense that they appear in a Hamiltonian hierarchy).

Although one can characterize the compactons and peakons as weak solutions, we remark that they are, in fact, solutions in a considerably stronger (although not quite classical) sense than the more traditionally studied weak solutions such as shock waves. Indeed, they are piecewise analytic, satisfying the equation in a classical sense away from singularities, and, moreover, each term (or certain combinations of terms) in the equation have well defined limits at the singularities. Indeed, one does not require any entropy condition to prescribe the type of singularity. We propose an apparently new and potentially useful definition of such "pseudo-classical" solutions that will handle a wide variety of such non-analytic solitary wave solutions, and distinguish them from shock waves.

For classical, linearly dispersive systems, the characterization of solitary waves and more complicated solutions by their analyticity properties has been the focus of a significant amount of study. Kruskal, [12], proposed analyzing the interactive properties of solitary wave solutions by the behavior of their poles in the complex plane. The Painlevé test for integrability of nonlinear systems, [1], [23], [26], is based on the analyticity of their solutions. The convergence of the general Painlevé series expansions has been studied in [11]. Recently Bona and Li, [5], showed that for various types of linearly dispersive systems, including equations of both KdV and BBM type, all weak solitary wave solutions which are essentially bounded and decay to zero at infinity are necessarily classical solutions, and can be analytically extended to a horizontal strip in the complex plane containing the real axis. In contrast, we will demonstrate that the system (1.1) not only has solitary wave solutions that are restrictions of analytic functions defined on a horizontal strip, but also admits both compactons, whose second order derivatives are discontinuous, and peakons, whose first order derivatives have a discontinuity, as weak (or, rather, pseudo-classical) solutions. The existence of such types of non-analytic solutions requires that the linear dispersion term vanishes for certain function values, and these are precisely the values at which the discontinuities of the solution appear. This fact indicates the important role played by the linear dispersion terms in the formation of analytic travelling wave solutions, and the significant influence of nonlinear terms on the behavior of these solutions.

The appearance of non-analytic solutions thus draws our attention to a more detailed understanding of the effects of both linear and nonlinear dispersion terms on travelling wave solutions, especially, on solitary wave solutions. In this paper, we shall discuss how such non-analytic solitary wave solutions can appear as the limits of classical, analytic solitary wave solutions. (This observation is not as paradoxical as it might initially seem — a classical instance of such loss of analyticity occurs in the convergence of Fourier series.) Such limits can be effected in two different, but essentially equivalent ways. First one can add to the equation a small linear dispersion term, having the effect of forcing analyticity of the perturbed solitary wave solution, and then allowing the coefficient of this additional term to vanish. In (1.1), the coefficient ν has this effect, provided we compensate by setting $\gamma = \tilde{\gamma}/\nu$ to leave the nonlinearly dispersive term uu_{xxx} intact. This approach is, in part, motivated by the convergence properties of the KdV equation as the dispersion term (meaning the coefficient of u_{xxx}) goes to zero; the convergence of classical solutions to the KdV equation to non-analytic shock wave solutions of the resulting dispersionless Burgers' equation $u_t + uu_x = 0$ was analyzed in great detail by Lax, Levermore and Venakides, [14], [15], [24]. Of course, our situation is analytically simpler since the limiting equation does not have shocks, and, besides, we are only attempting to analyze solitary wave solutions. Alternatively, one can replace the vanishing condition at infinity by the condition $u \to a$ as $|x| \to \pm\infty$, meaning that the wave appears as a disturbance on a fluid of uniform depth a. For most values of a, this has the effect of eliminating the effect of the nonlinearly dispersive terms, and again one can investigate how the associated analytic solitary wave solutions lose their analyticity as the undisturbed depth $a \to 0$. Of course, these two approaches are closely related — one can replace u by $\hat{u} = u - a$ to eliminate the non-zero asymptotic depth; the resulting equation will then include an additional linearly dispersive term depending on the small parameter a (as well as additional nonlinear terms). In this paper we investigate both strategies for (1.1) — here the transformation $\hat{u} = u - a$ merely redefines the parameters $\nu, \alpha, \beta, \gamma$ appropriately. Our main result is that, in all cases, analytic solitons converge to non-analytic peakons and compactons *provided* they are weak (i.e. pseudo-classical) solutions to the system. Thus the convergence of analytic solitary wave solutions under vanishing linear dispersion is to pick out those non-analytic solitary wave solutions which are "genuine" in the sense that they are weak, or pseudo-classical, solutions. We anticipate that this will form a rather general convergence phenomena, applicable to both integrable and nonintegrable systems, including (1.4), alike.

Our analysis breaks into three parts. First, methods from the theory of dynamical systems — in particular center manifold theory — will be employed to produce a preliminary analysis of the ordinary differential equations describing travelling wave solutions to Equation (1.1). This allows us to determine the precise parameter regimes for which (1.1) admits solitary wave solutions, which can always be characterized as the limit of periodic travelling wave solutions, as well as peakons and compactons, which are manifested by particular types of singularities in the phase plane associated with the (integrated form of) the dynamical system. To proceed further, we shall need to determine the analytic continuation of the resulting solutions in the complex plane. In contrast to the KdV equation, whose sech^2 solitons have a unique extension to a single-valued meromorphic function, the solitary wave solutions of (1.1) extend to multiply-valued analytic functions, with quite complicated branching behavior. The second part of this paper is devoted to a detailed analysis of these complex analytic extensions. To determine the convergence properties, we must restrict our attention to a region supporting a single-valued extension; branching implies that the extension is not unique, but depends on how the branch cuts are arranged in the complex plane. Interestingly, the choice of branch cuts, and hence single-valued extension, affects the convergence of the solution in the complex plane, leading to different non-analytic solitary wave solutions in the limit, which, nevertheless, restrict to the same peakon or compacton on the real axis. In the final part of the paper, we shall study the properties of branch points of these solutions, and their behavior as the corresponding solutions are converging to compactons, peakons or solitary wave solutions in order to understand how singularities influence properties of these solutions during the process of convergence.

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2. Notation. We let $C^k = C^k(\mathbb{R})$ denote the space of k times continuously differentiable functions defined on the real axis. The space of all infinitely differentiable functions with compact support in $\mathbb{R}$ is denoted by $C_c^\infty = C_c^\infty(\mathbb{R})$. The space $L^p = L^p(\mathbb{R})$ with $1 \leq p \leq \infty$

consists of all pth-power Lebesgue-integrable functions defined on the real line $\mathbb{R}$ with the usual modification if $p = \infty$. The standard norm of a function $f \in L^p$ will be denoted by $\| f \|_p$. The inner product of two functions f and g in L^2 is the integral

$$\langle f, g \rangle = \int_{\infty}^{\infty} f(x)\overline{g(x)}dx,$$

where the overbar denotes complex conjugation. For any integer $k \geq 0$ and constant $p \geq 1$, the Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R})$ consists of all tempered distributions f such that $f^{(m)} \in L^p$ for all $0 \leq m \leq k$. The space $W^{k,2}$ is usually denoted by H^k.

A (classical) *travelling wave solution*[1] of Equation (1.1) of wave speed c is a solution of class C^3, having the particular form $u = \phi(x - ct)$. A travelling wave solution is called a *solitary wave* if ϕ has a well-defined limit $\lim_{|x| \to \infty} \phi(x)$, which is the same at both $\pm\infty$; the limiting value represents the undisturbed depth of the fluid. A solitary wave solution of Equation (1.1), or its corresponding homoclinic orbit, is said to be analytic if the solution is a real analytic function defined on the real axis. Analytic solitary wave solutions of integrable evolution equations, such as the KdV equation, are known as *solitons*, which indicates that they emerge from collisions unchanged in form, save for a phase shift; see [2], [10].

By a *fixed point* of a dynamical system $x' = f(t, x)$, where $x \in \mathbb{R}^n$, we mean a point x_0 such that $f(t, x_0) = 0$ for all $t \in \mathbb{R}$. The fixed point is called *quasi-hyperbolic* of *degree one* if the linearized mapping derived from the system near this point has eigenvalues with their real parts different from zero except one eigenvalue with zero real part. By a *singularity* of the dynamical system $x' = f(t, x)$, we indicate a point x_0 such that $f(t, x)$ is not analytic at (t, x_0) for some t in $\mathbb{R}$.

3. Dynamical systems for solitary waves. The soliton solutions to the KdV equation can be viewed as the limits of the periodic cnoidal wave solutions; see [2], [10]. Let us review this well-known fact from a dynamical systems point of view. Substituting the travelling wave solution $u(x, t) = \phi(x - ct)$, for constant wave speed c, into the KdV equation (1.3), one obtains the ordinary differential equation

$$-(c - 1)\phi' + \phi\phi' + \phi''' = 0. \tag{3.1}$$

[1] Here we give a standard definition of classical solutions of Equation (1.1) versus the definition of pseudo-classical solutions we have proposed in Section 1, which is needed to include a new class of travelling wave solutions — compactons and peakons of (1.1).

The transformation $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \phi \\ \phi' \\ \phi'' \end{pmatrix}$ reduces (3.1) to the dynamical system

$$\vec{y}\,' = \begin{pmatrix} y_2 \\ y_3 \\ (c-1)y_2 - y_1 y_2 \end{pmatrix}. \tag{3.2}$$

The fixed points of system (3.2) consists of all points of the y_1-axis. To observe properties of travelling wave solutions near each fixed point $(a, 0, 0)$ on the y_1-axis, we let $\vec{y} = \vec{\xi} + \vec{a} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ and substitute the transformation into (3.2), leading to the system

$$\vec{\xi}\,' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & c-1-a & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\xi_1 \xi_2 \end{pmatrix}.$$

If $a < c - 1$, then there are a one-dimensional center manifold, a one-dimensional stable manifold and a one-dimensional unstable manifold near $(a, 0, 0)$ with a unique homoclinic orbit represented by the function

$$\phi(x) = a + 3(c-a-1)\,\mathrm{sech}^2\, \frac{\sqrt{c-a-1}\; x}{2} \tag{3.3}$$

which is the limit of periodic cnoidal solutions of Equation (3.1). On the other hand, if $a \geq c - 1$, then there is a three-dimensional center manifold near the fixed point $(a, 0, 0)$ and $(c - 1, 0, 0)$ is a bifurcation point of the system.

Another property of Equation (3.1) worth mentioning is that it induces a homeomorphism of $(-\infty, c-1)$, the set of fixed points having homoclinic orbits, onto the interval $(c-1, \infty)$, the set of fixed points where there are periodic orbits. This fact can be verified as follows. For each $a < c - 1$, we substitute $\phi = \psi + a$ into (3.1), integrate the resulting equation once and take the integration constant to be zero, leading to the equation

$$-(c-a-1)\psi + \frac{\psi^2}{2} + \psi'' = 0. \tag{3.4}$$

The dynamical system (3.4) has two fixed points. One is the origin which supports a one-dimensional stable manifold and a one-dimensional unstable manifold with a unique homoclinic orbit representing the solitary wave solution expressed by (3.3). At the other fixed point $(2(c-a-1), 0)$, there is a two-dimensional center manifold where there exist periodic orbits converging to the homoclinic orbit at the origin as sketched in Figure 1.

Substituting $\psi = \varphi + 2(c-a-1)$ into (3.4) and comparing the resulting equation

$$-\big(c-1-(2c-a-2)\big)\varphi + \frac{\varphi^2}{2} + \varphi'' = 0$$

with (3.1), then one may realize that periodic orbits near the point $(2(c-a-1), 0)$ of system (3.4) can be regarded as those near the fixed point $(2c-a-2, 0, 0)$. In consequence, the homeomorphism $\Phi\colon (-\infty, c-1) \longrightarrow (c-1, \infty)$ is naturally defined by

$$\Phi(a) = 2c - 2 - a. \tag{3.5}$$

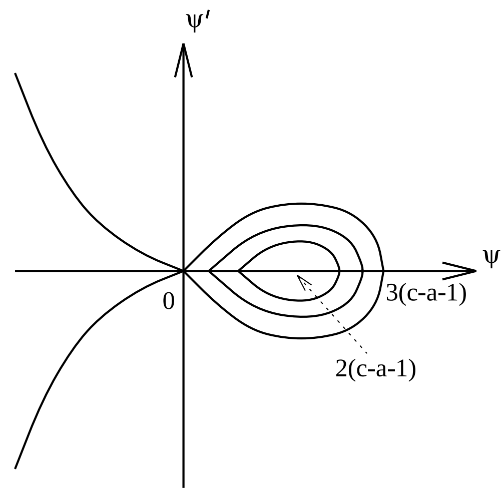

Fig. 1. The phase plane of system (3.4) with $a < c-1$

This not only shows that the quasi-hyperbolic points of system (3.1) are in one-to-one correspondence with its three-dimensional center manifolds, but also indicates that for each quasi-hyperbolic point, the corresponding three-dimensional center manifold contains a sequence of periodic orbits converging to the homoclinic orbit at the quasi-hyperbolic point. We shall see that there is a similar mapping Ψ for the dynamical system obtained by reduction from Equation (1.1) which may be used to illustrate its more complicated, but more interesting properties.

Now let us consider the nonlinearly dispersive Equation (1.1), and assume that $\gamma\nu \neq 0$. Replacing u by $u/(\gamma\nu)$, we reduce (1.1) to the simpler equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + \frac{3}{\nu} u u_x + u u_{xxx} + 2 u_x u_{xx}. \tag{3.6}$$

The resulting ordinary differential equation for travelling wave solutions $u(x,t) = \phi(x - ct)$ of speed c is

$$(\alpha + c)\phi' + (\beta + c\nu + \phi)\phi''' + \frac{3}{\nu}\phi\phi' + 2\phi'\phi'' = 0. \tag{3.7}$$

Using the same transformation $\vec{y} = \vec{\xi} + \vec{a}$ as before yields the system of equations

$$\vec{\xi}\,' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\dfrac{\alpha + c + \frac{3a}{\nu}}{\beta + c\nu + a} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\frac{3}{\nu}\xi_1\xi_2 + 2\xi_2\xi_3}{\beta + c\nu + a} + \dfrac{\xi_1[(\alpha + c + \frac{3a}{\nu})\xi_2 + \frac{3}{\nu}\xi_1\xi_2 + 2\xi_2\xi_3]}{(\beta + c\nu + a)(\beta + c\nu + a + \xi_1)} \end{pmatrix}. \tag{3.8}$$

Clearly, the set of fixed points and singularities of Equation (3.7) also consists of all points of the y_1-axis. Next, we discuss properties of each fixed point or singularity $(a, 0, 0)$ of system (3.7) in different cases.

Case I. When $\nu > 0$ and $\beta + c\nu > \frac{\nu(\alpha+c)}{3}$.

The constants $-(\beta + c\nu)$ and $-\frac{\nu}{3}(\alpha + c)$ divide the y_1-axis into three intervals

$$(-\infty, -(\beta + c\nu)), \quad (-(\beta + c\nu), -\frac{\nu}{3}(\alpha + c)) \text{ and } (-\frac{\nu}{3}(\alpha + c), \infty).$$

For any $a \in (-\infty, -(\beta + c\nu)) \cup [-\frac{\nu}{3}(\alpha + c), \infty)$, the system (3.8) shows that (3.7) has a three-dimensional center manifold near the point $(a, 0, 0)$ and $(-\frac{\nu}{3}(\alpha + c), 0, 0)$ is a bifurcation point. On the other hand, if $a \in (-(\beta + c\nu), -\frac{\nu}{3}(\alpha + c))$, then there is a one-dimensional center manifold, a one-dimensional stable manifold, and a one-dimensional unstable manifold at the point $(a, 0, 0)$, near which there is a unique, analytic, homoclinic orbit represented by the solution

$$\phi(x - ct) = a + \xi(x - ct).$$

Such a solution is obtained as the limit, as $\delta \to 0$, of periodic solutions

$$\phi_\delta\left(x - \left(c + \frac{\delta}{2\nu}\right)t\right) = a - \frac{\delta}{2} + \xi_\delta\left(x - \left(c + \frac{\delta}{2\nu}\right)t\right).$$

The period of $\phi_\delta(x)$ is

$$T = 2\int_\delta^{\nu B} \frac{\sqrt{\nu(A + \zeta)}\,d\zeta}{\sqrt{\zeta(\nu B - \zeta)(\zeta - \delta)}},$$

where $A = \beta + c\nu + a$, $B = -(\alpha + c + \frac{3a}{\nu})$, and $0 < \delta < \nu B$ is a constant. Note that the functions $\xi(x)$ and $\xi_\delta(x)$ satisfy the respective differential equations

$$(\xi')^2 = \frac{\xi^2(\nu B - \xi)}{\nu(\xi + A)}, \qquad \text{and} \qquad (\xi_\delta')^2 = \frac{\xi_\delta(\nu B - \xi_\delta)(\xi_\delta - \delta)}{\nu(\xi_\delta + A)}.$$

The point $(-(\beta+c\nu),0,0)$ forms a singular point of system (3.7), providing the compacton solution

$$\phi_0(x)=\begin{cases} -(\beta+c\nu)+\big(3(\beta+c\nu)-\nu(\alpha+c)\big)\cos^2\dfrac{x}{2\sqrt{\nu}}, & \text{if } |x|\le\sqrt{\nu}\,\pi \\ -(\beta+c\nu), & \text{otherwise}\end{cases} \tag{3.9}$$

occurs as a weak solution of (3.7) in the following sense.

Definition 3.1. *A solitary wave $\phi(x)$ with undisturbed depth $a=\lim_{|x|\to\infty}\phi(x)$ is a weak solution of the ordinary differential equation (3.7) if and only if $\xi=\phi-a\in H^1$, and*

$$\left\langle(\alpha+c)\phi+\frac{3\phi^2}{2\nu}-\frac{(\phi')^2}{2},\,g'\right\rangle+\left\langle\big(\beta+c\nu+\frac{\phi}{2}\big)\phi,\,g'''\right\rangle=0, \tag{3.10}$$

for any $g\in C_c^\infty(\mathbb{R})$.

We are interested in studying the behavior of the solitary wave solution $\phi(x)=a+\xi(x)$ as its asymptotic amplitude a approaches the singular value $-(\beta+c\nu)$.

Equation (3.7) also implicitly suggests a one-to-one mapping of its quasi-hyperbolic fixed points to its three-dimensional center manifolds. However, unlike the KdV equation (3.1), the resulting mapping is not surjective. To find the required mapping, one may use a procedure similar to that of deriving (3.5). For each $a\in(-(\beta+c\nu),-\frac{\nu}{3}(\alpha+c))$, we substitute $\phi=\psi+a$ into (3.7); integrating the resulting equation once and setting the integral constant to zero, we obtain

$$\big(\alpha+c+\frac{3a}{\nu}\big)\psi+(\beta+c\nu+a+\psi)\psi''+\frac{3\psi^2}{2\nu}+\frac{(\psi')^2}{2}=0. \tag{3.11}$$

The system (3.11) has two fixed points — the origin and $\big(-2(\frac{\nu}{3}(\alpha+c)+a),0\big)$. The origin is a saddle point whose unique homoclinic orbit represents an analytic solitary wave solution. Near the point $\big(-2(\frac{\nu}{3}(\alpha+c)+a),0\big)$, there exists a two-dimensional center manifold having periodic orbits converging to the homoclinic orbit at the origin as sketched in Figure 2. Substituting $\psi=\xi-2\big(\frac{\nu}{3}(\alpha+c)+a\big)$ into (3.11) and comparing the resulting equation

$$\big(\alpha+c-2(\alpha+c)-\frac{3a}{\nu}\big)\xi+\big(\beta+c\nu-\frac{2\nu}{3}(\alpha+c)-a+\xi\big)\xi''+\frac{3\xi^2}{2\nu}+\frac{(\xi')^2}{2}=0$$

with (3.7), one may recognize that periodic orbits near the fixed point $\big(-2(\frac{\nu}{3}(\alpha+c)+a),0\big)$ of (3.11) come from the center manifold of the fixed point $\big(-\frac{2\nu}{3}(\alpha+c)-a,0,0\big)$ in system (3.7). Therefore the homeomorphism

$$\Psi(a)=-\frac{2\nu}{3}(\alpha+c)-a \tag{3.12}$$

of $\left(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\right)$ onto $\left(-\frac{\nu}{3}(\alpha+c), \beta+c\nu-\frac{2\nu}{3}(\alpha+c)\right)$ determines a one-to-one mapping from the set $\{(a,0,0); a \in \left(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\right)\}$ of quasi-hyperbolic points to the set of points $\{(\Psi(a),0,0); a \in \left(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\right)\}$ whose center manifolds contain periodic orbits converging to homoclinic orbits at the corresponding quasi-hyperbolic fixed points. One may also notice that the mapping Ψ is defined in such a way that the points $(0,0)$ and $(\Psi(a)-a,0)$ always appear as a pair of fixed points in (3.11).

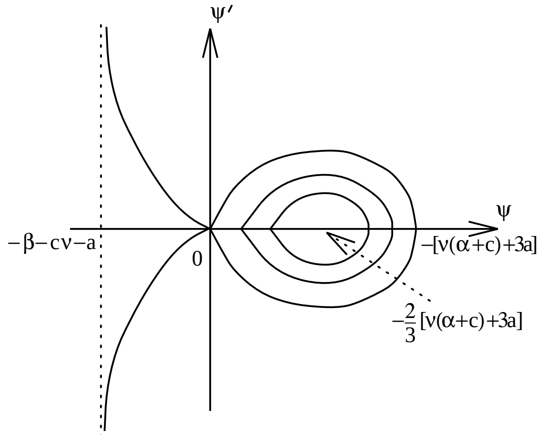

Fig. 2. The phase plane of system (3.11) with $-\beta-c\nu < a < -\frac{\nu}{3}(\alpha+c)$ in *Case I*

When $a < -\beta-c\nu$, both $(0,0)$ and $(\Psi(a)-a,0)$ have a two-dimensional center manifold with periodic orbits at each of the points, but they are separated by the singular point $(-\beta-c\nu-a,0)$ as sketched in Figure 3. On the other hand, if $a \in \left(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\right)$, the two points $(a,0)$ and $(\Psi(a)-a,0)$ always stay on the right-hand side of the singular point $(-\beta-c\nu-a,0)$ as shown in Figure 2. The case $a=-\beta-c\nu$ is the most unusual since the singular point $(-\beta-c\nu-a,0)$ is at the origin where a periodic orbit passes through, from which the compacton is defined as a weak solution of Equation (3.11); while the fixed point $(\Psi(a)-a,0)$ still has a two-dimensional center manifold containing periodic orbits.

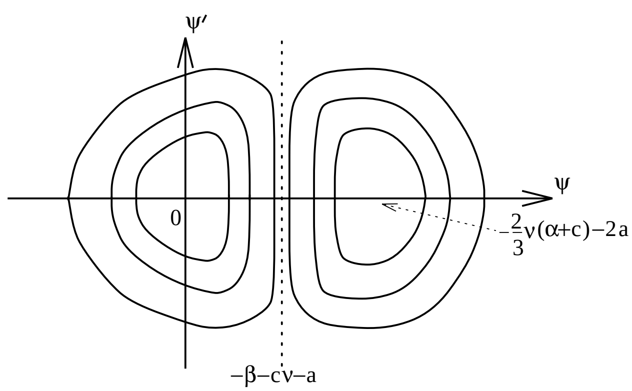

Fig. 3. The phase plane of Equation (3.11) when $a < -\beta-c\nu$ in *Case I*

Case II. When $\nu < 0$ and $\beta + c\nu > \frac{\nu(\alpha+c)}{3}$.

In this case, the interval $(-(\beta + c\nu), -\frac{\nu}{3}(\alpha + c)]$ on the y_1-axis consists of fixed points supporting three-dimensional center manifolds. The other two intervals, $(-\infty, -(\beta + c\nu))$ and $(-\frac{\nu}{3}(\alpha+c), \infty)$, consist of quasi-hyperbolic points, and $(-\frac{\nu}{3}(\alpha+c), 0, 0)$ is a bifurcation point. Unlike *Case I*, at the singular point $(-(\beta+c\nu), 0, 0)$, there exists only the stationary solution $\phi(x) \equiv -(\beta + c\nu)$ and compactons do not occur.

The mapping Ψ defined in (3.12) also offers a convenient way to describe this case as follows. When $a \in (-\infty, -\beta - c\nu)$, $\Psi(a) \in \big(\beta + c\nu - \frac{2\nu}{3}(\alpha + c), \infty\big)$. Both $(a, 0, 0)$ and $(\Psi(a), 0, 0)$ are quasi-hyperbolic points without homoclinic orbits. However, there exist cuspon solutions in which $(0, 0)$ and $(\Psi(a) - a, 0)$ appear as a pair of fixed points of (3.11), corresponding to $(a, 0, 0)$ and $(\Psi(a), 0, 0)$ of the system (3.7), respectively, and having the singular point $(-\beta - c\nu - a, 0)$ between them. This is illustrated in Figure 4.

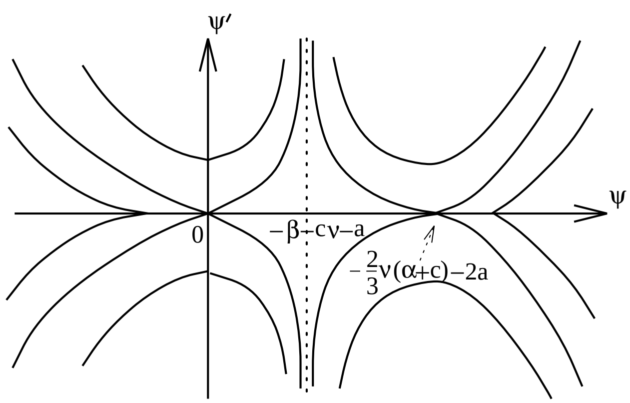

Fig. 4. The phase plane of Equation (3.11) when $a < -\beta - c\nu$ in *Case II*

If $a = -\beta - c\nu$, then the singular point $(-\beta - c\nu - a, 0)$ and the origin merge together, and as a consequence, the cuspon ceases to exist, although there is still a cuspon associated with the point $(\Psi(a) - a, 0)$.

For each $a \in \big(-\beta - c\nu, -\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c)\big]$, the value of $\Psi(a)$ lies in the interval $\big[\frac{1}{2}(\beta - \nu\alpha), \beta + c\nu - \frac{2\nu}{3}(\alpha + c)\big)$ and the origin of system (3.11) changes its property to possess a two-dimensional center manifold with periodic orbits. Even though the singular point $(-\beta - c\nu - a, 0)$ is on the left-hand side of both the origin and the saddle point $(\Psi(a) - a, 0)$, there is still no homoclinic orbit, but a cuspon at the point $(\Psi(a) - a, 0)$ as illustrated in Figure 5.

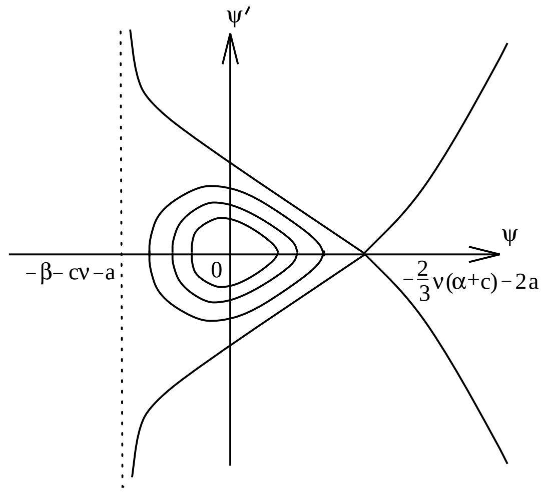

Fig. 5. The phase plane of Equation (3.11) in *Case II*
with $-\beta - c\nu < a < -\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c)$

It is worth mentioning that at $a = \frac{1}{2}(\beta - \nu\alpha)$, the peakon

$$\phi_{\boldsymbol{p}}(x) = \frac{\beta - \nu\alpha}{2} - \big[\frac{3}{2}(\beta + c\nu) - \frac{\nu}{2}(\alpha + c)\big]e^{-(-\nu)^{-1/2}|x|},$$

forms a weak solution, meaning that it satisfies (3.10). Later, we shall prove that the homoclinic orbits at fixed points $(a, 0, 0)$ of system (3.7) converge to the peakon $\phi_{\boldsymbol{p}}$ as $a \in (-\frac{\nu}{3}(\alpha + c), \frac{\beta-\nu\alpha}{2})$ approaches the endpoint $\frac{\beta-\nu\alpha}{2}$.

If $a \in \big(-\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c), -\frac{\nu}{3}(\alpha + c)\big)$, then $\Psi(a) \in \big(-\frac{\nu}{3}(\alpha + c), \frac{1}{2}(\beta - \nu\alpha)\big)$ and there is a homoclinic orbit at the saddle point $(\Psi(a) - a, 0)$ which is the limit of periodic orbits contained in the center manifold at the origin, as displayed in Figure 6.

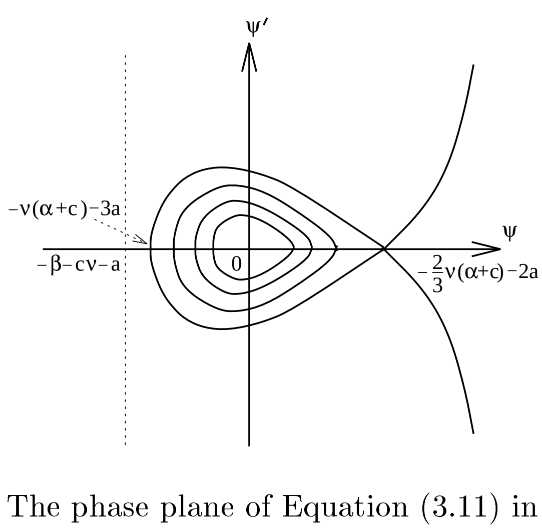

Fig. 6. The phase plane of Equation (3.11) in *Case II*
with $-\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c) < a < -\frac{\nu}{3}(\alpha + c)$

Remark. A naïve explanation for the existence of so many cuspons in this case is that the family of quasi-hyperbolic points of system (3.7) outnumbers the fixed points having three-dimensional center manifolds, so that the mapping Ψ associates a great number of quasi-hyperbolic points to those of the same kind. The cuspons are present there because of the strong effect of the singular point $(-\beta - c\nu, 0, 0)$. Furthermore, we can use the equation

$$(\psi')^2 = -\frac{\psi^2(\psi + \nu(\alpha + c) + 3a) + d}{\nu(\psi + \beta + c\nu + a)} \tag{3.13}$$

derived from (3.11) by integration, where d is the integration constant, to sketch the phase plane of (3.11) for different values of a. Based on this, one may show that a necessary condition for a homoclinic orbit to exist at the point $(a, 0, 0)$ is that the mapping Ψ associates the quasi-hyperbolic point $(a, 0, 0)$ to a three-dimensional center manifold, *i.e.* if $(a, 0, 0)$ is a quasi-hyperbolic point and $(\Psi(a), 0, 0)$ does not have a three-dimensional center manifold, then homoclinic orbits can not exist at $(a, 0, 0)$. In contrast, there is a surplus of three-dimensional center manifolds in *Case I*, so that the mapping Ψ is able to associate every quasi-hyperbolic point to a three-dimensional center manifold. In addition, a homoclinic orbit is formed at each quasi-hyperbolic point because of the smaller effect of the singular point $(-\beta - c\nu, 0, 0)$ in this case than *Case II*.

Compared with system (3.7), the KdV system seems to be perfect, because the number of its quasi-hyperbolic points is balanced with the number of three-dimensional center manifolds, *i.e.* the mapping Φ defined in (3.5) is a one-to-one and onto mapping, and there are no singularities. Therefore, studying properties of one quasi-hyperbolic point of the KdV system is sufficient to understand properties of other fixed points in the system, whereas for system (3.7), we need to consider different cases in which it is also necessary to investigate fixed points in different intervals on the y_1-axis.

We summarize the remaining two cases to conclude this section.

Case III. When $\nu > 0$ *and* $\beta + c\nu = \frac{\nu(\alpha+c)}{3}$.

For any a with $a \neq -\frac{\nu}{3}(\alpha + c)$, there is a three-dimensional center manifold at $(a, 0, 0)$ with periodic orbits going around this point. Moreover, $(-\frac{\nu}{3}(\alpha+c), 0, 0)$ is a singular point of system (3.7) without periodic orbits.

Case IV. When $\nu < 0$ *and* $\beta + c\nu = \frac{\nu(\alpha+c)}{3}$.

Each $a \neq -\frac{\nu}{3}(\alpha + c)$, supports a one-dimensional center manifold, a one-dimensional stable manifold and a one-dimensional unstable manifold at $(a, 0, 0)$ without homoclinic orbits. This is not surprising since homoclinic orbits are usually accompanied by periodic orbits converging to them, but there is neither a center manifold nor a periodic orbit in this case. However we shall show that there is a cuspon at the point $(a, 0, 0)$. On the other hand, if $a = -\frac{\nu}{3}(\alpha + c)$, then $(a, 0, 0)$ is a singular point, without cuspons.

As we have seen in the above discussion, Equation (1.1) has analytic solitary wave solutions in *Cases I* and *II*, which are illustrated as homoclinic orbits in Figures 2 and 6, respectively. A question arises naturally as how these homoclinic orbits behave when the singularity $(-\beta - c\nu - a, 0)$ is close to them. The answer is that in the first case, the solitary wave solutions at points $(a, 0, 0)$ converge to the compacton ϕ_0 given by (3.9) when $a \to -\beta - c\nu$ and $-\beta - c\nu < a < -\frac{\nu}{3}(\alpha + c)$; in the second case, the solitary wave solutions at $(a, 0, 0)$ converge to the peakon at the fixed point $\left(\frac{1}{2}(\beta - \nu\alpha), 0, 0\right)$ as $a \to \frac{1}{2}(\beta - \nu\alpha)$ with $-\frac{\nu}{3}(\alpha + c) < a < \frac{1}{2}(\beta - \nu\alpha)$.

Summarizing, we let the constant c be the speed of propagation of the indicated travelling wave solutions of Equation (1.1), and a the undisturbed depth. Any such solitary wave solution takes the form $a + \phi_a(x - ct)$, where the function $\psi = \phi_a$ satisfies the ordinary differential Equation (3.11) with asymptotic boundary conditions $\lim_{|x|\to\infty} \phi_a(x) = 0$.

Theorem 3.1. *If the coefficients of Equation* (1.1) *satisfy the inequalities*

$$\gamma \neq 0, \qquad \nu > 0, \qquad \textit{and} \qquad \beta + c\nu > \frac{\nu}{3}(\alpha + c),$$

then there exists an orbitally unique and analytic solitary wave solution $a + \phi_a(x - ct)$ for each $a \in \left(-\beta - c\nu, -\frac{\nu}{3}(\alpha + c)\right)$. Moreover, as a approaches $-\beta - c\nu$, the sequence of the solitary wave solutions $\{a + \phi_a(x)\}$ converges to the compacton solution given in (3.9).

Proof. Let $\varepsilon = a + \beta + c\nu$, $B = -(\alpha + c + \frac{3a}{\nu})$ and $B_0 = \frac{3}{\nu}(\beta + c\nu) - (\alpha + c)$. Then Equation (3.11) for the solitary wave solution reduces to

$$\nu(\phi_a + \varepsilon)(\phi_a')^2 = \phi_a^2(\nu B - \phi_a). \tag{3.14}$$

Using the inequality $0 \leq \phi_a(x) \leq \nu B$ valid for all $x \in \mathbb{R}$, one may show that sequences of functions $\{\phi_a'\}$ and $\{\phi_a''\}$ are uniformly bounded on the real axis. Therefore, the Ascoli-Arzelà Theorem shows that, as $a \to -\beta - c\nu$, there exist subsequences of the families $\{\phi_a\}$ and $\{\phi_a'\}$, without loss of generality still denoted by $\{\phi_a\}$ and $\{\phi_a'\}$, which are uniformly convergent to a function ϕ and its derivative ϕ', respectively, on any compact set of $\mathbb{R}$. Here we are relying on the fact that each ϕ_a is an even function, since ϕ_a is symmetric with respect to its elevation and translation invariant. Taking the limit on both sides of (3.14) as $a \to -\beta - c\nu$, or as $\varepsilon \to 0$ leads to the equation

$$\nu\phi{\phi'}^2 = \phi^2(\nu B_0 - \phi) \tag{3.15}$$

satisfied by the function ϕ. Since $\lim_{\varepsilon\to 0}\max_{x\in\mathbb{R}}\phi_a(x) = \lim_{\varepsilon\to 0}\nu B = \nu B_0 > 0$ and each ϕ_a is even, monotone on each side of the origin and exponentially decaying to zero at infinity, the limiting function ϕ is a nontrivial solution of of (3.15). Thus, ϕ satisfies the equation $\nu {\phi'}^2 = \phi(\nu B_0 - \phi)$. Therefore, as an even and monotone decreasing function on the positive real axis, $\phi = \phi_0 + \beta + c\nu$, that is to say, $\phi_0 = \phi - \beta - c\nu$ is the compacton solution (3.9). □

The corresponding result for peakons follows.

Theorem 3.2. *If the coefficients of Equation* (1.1)*satisfy the inequalities*

$$\gamma \neq 0, \qquad \nu < 0, \qquad \textit{and} \qquad \beta + c\nu > \frac{\nu}{3}(\alpha + c),$$

then for each $a \in \left(-\frac{\nu}{3}(\alpha+c), \frac{1}{2}(\beta - \nu\alpha)\right)$, *there exists an orbitally unique and analytic solitary wave solution* $a + \phi_a(x - ct)$. *Moreover, as* $a \to \frac{1}{2}(\beta - \nu\alpha)$, *the sequence of the solitary wave solutions* $\{\phi_a(x)\}$ *is convergent to the peakon solution*

$$\phi(x) = -\left[\frac{3}{2}(\beta + c\nu) - \frac{\nu}{2}(\alpha + c)\right] e^{-(-\nu)^{-1/2}|x|}. \tag{3.16}$$

Proof. One can straightforwardly show that the first order derivatives $\{\phi_a'\}$ of the solitary wave solutions $\{a + \phi_a\}$ at points $(a, 0, 0)$ are uniformly bounded for all $a \in \left(-\frac{\nu}{3}(\alpha + c), \frac{1}{2}(\beta - \nu\alpha)\right)$. Therefore, there exists a sequence of even functions monotonically decreasing on the positive axis, still denoted by $\{\phi_a\}$, satisfying the equation

$$\nu(\phi_a + \beta + c\nu + a)(\phi_a')^2 = -\phi_a^2(\phi_a + 3a + \nu(\alpha + c)) \tag{3.17}$$

and converging to the function ϕ as $a \to \frac{1}{2}(\beta - \nu\alpha)$. This may be derived by solving (3.17) to obtain an implicit expression of the function ϕ_a and then taking the limit as $a \to \frac{1}{2}(\beta - \nu\alpha)$; see [13] for details. □

Remark. As we pointed out in the previous discussion, solitary wave solutions do not exist if $\beta + c\nu = \frac{\nu}{3}(\alpha + c)$. In case $\beta + c\nu < \frac{\nu}{3}(\alpha + c)$, we can replace u by $-u$ in Equation (1.1), which has the effect of changing the sign of the coefficients α and β, and the wave speed c. Note that this transformation will change waves of elevation moving to the right ($c > 0$) into waves of depression, moving to the left. Otherwise, the conclusions in Theorems 3.1 and 3.2 also apply to the above equation. Therefore, if $\nu > 0$, and $a \in \left(-\frac{\nu}{3}(\alpha + c), -(\beta + c\nu)\right)$, Equation (3.7) admits a solitary wave solution in the form

$a + \phi_a(x)$, such that the sequence of solitary wave solutions $\{a + \phi_a(x)\}$ converges to a compacton as a weak solution of (3.7) when $a \to -(\beta + c\nu)$. On the other hand, if $\nu < 0$, then for each $a \in \left(\frac{1}{2}(\beta - \nu\alpha), -\frac{\nu}{3}(\alpha + c)\right)$, there is a solitary wave of elevation $a + \phi_a(x)$, such that the sequence $\{a + \phi_a(x)\}$ converges to a peakon, also as a weak solution of (3.7), as $a \to \frac{1}{2}(\beta - \nu\alpha)$. In either case, ϕ_a satisfies (3.11). In the remaining part of this paper, we shall only consider the case $\beta + c\nu > \frac{\nu}{3}(\alpha + c)$, since any result in this case can be directly applied to the case $\beta + c\nu < \frac{\nu}{3}(\alpha + c)$.

To understand how analytic solitary wave solutions converge to functions, such as compactons and peakons, having singularities on the real axis $\mathbb{R}$, we shall extend solitary wave solutions mentioned in the last two theorems to functions defined in the complex plane to study singularity distribution of these functions. This method not only provides another way to prove the last two theorems, but also makes it clear that singularities of solitary wave solutions are approaching the real axis in the process of convergence, or roughly speaking, singularities of compactons or peakons come from those of analytic solitary wave solutions, which are close to the real axis in the complex plane. We shall consider the solitary wave solutions in *Case I* and *Case II* separately because of their different structures as functions defined on the complex plane.

CONVERGENCE OF SOLITARY-WAVE SOLUTIONS IN A PERTURBED BI-HAMILTONIAN DYNAMICAL SYSTEM. II. COMPLEX ANALYTIC BEHAVIOR.

Y. A. Li[1] and P. J. Olver[1,2]

Abstract. In this part, we prove that the solitary wave solutions investigated in part I are extended as analytic functions in the complex plane, except at most countably many branch points and branch lines. We describe in detail how the limiting behavior of the complex singularities allows the creation of non-analytic solutions with corners and/or compact support.

This is the second in a series of three papers investigating the solitary wave solutions of the integrable model wave equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + \frac{3}{\nu} u u_x + u u_{xxx} + 2 u_x u_{xx}. \tag{3.6}$$

(We adopt the notation and numbering of statements from part I.) The ordinary differential equation for travelling wave solutions $u(x,t) = \phi(x - ct)$ is

$$(\alpha + c)\phi' + (\beta + c\nu + \phi)\phi''' + \frac{3}{\nu}\phi\phi' + 2\phi'\phi'' = 0. \tag{3.7}$$

Substituting $\phi = \phi_a + a$, where a is the undisturbed fluid depth for our solitary wave solutions, and integrating the resulting equation twice, leads to the first order equation

$$\nu(\phi_a + \beta + c\nu + a)(\phi_a')^2 = -\phi_a^2(\phi_a + 3a + \nu(\alpha + c)) \tag{3.17}$$

To understand why analytic solitary wave solutions converge to non-analytic functions, such as compactons and peakons, having singularities on the real axis, we shall extend the solitary wave solutions described in Theorems 3.1 and 3.2 in Part I to functions defined in the complex plane to study singularity distribution of these functions. This method not only provides another way to prove the last two theorems, but also makes it clear

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that singularities of solitary wave solutions are approaching the real axis in the process of convergence, or roughly speaking, singularities of compactons or peakons come from those of analytic solitary wave solutions, which are close to the real axis in the complex plane.

The explicit form (3.3) of solitary wave solutions of the KdV equation shows that they are restriction to the real axis of meromorphic functions with countably many poles in the complex plane so that their analytic extension is unique. In contrast to these functions, extensions of solitary wave solutions under our consideration do not have poles but branch points. These branch points play an important role in the formation of singularities of compactons and peakons which are continuous functions but have discontinuous first or second order derivative. The main purpose of analytic extensions for these solitary wave solutions is to use them explaining how compactons or peakons lose analyticity although they are limits of solitary wave solutions of Equation (3.7) as restrictions of holomorphic functions defined on a strip containing the real axis in the complex plane.

We shall consider the solitary wave solutions in *Case I*, when $\nu > 0$, and *Case II*, when $\nu < 0$, separately because of their different structures as functions defined on the complex plane.

For any complex number $z \in \mathbb{C}$, the real part and the imaginary part of z are denoted by $\Re z$ and $\Im z$, respectively. The real part $u(x,y)$ and imaginary part $v(x,y)$ of an analytic function $w = F(z) = F(x+iy) = u(x,y) + iv(x,y)$ which is defined on the domain $\Omega \subset \mathbb{C}$ will be called the velocity potential and stream function respectively. The level sets of the velocity potential, $u(x,y) = u_0$, and the stream function, $v(x,y) = v_0$, are called the equipotentials and streamlines of F, respectively. Finally, $\log w$ is the single-valued branch of the natural logarithmic function $\operatorname{Log} w$, defined as $\log w = \log |w| + i\arg w$ with $-\pi < \arg w \leq \pi$.

4. Analytic extensions of solitary wave solutions for $\nu > 0$.

Under the assumption of Theorem 3.1, Equation (3.17) has an orbitally unique and analytic solitary wave solution ϕ_a. Rescaling (3.17) by using the transformation $\phi_a(x) = -\nu(\alpha + c + \frac{3a}{\nu})\varphi(x)$, we reduce it to the equation

$$(\delta\varphi + \epsilon)(\varphi')^2 = \varphi^2(1-\varphi), \tag{4.1}$$

where $\delta = \nu$ and $\epsilon = -(\beta + c\nu + a)/(\alpha + c + \frac{3a}{\nu})$. The phase plane portrait of (4.1) indicates that its solitary wave solution φ is a positive, even function with unit amplitude and decaying to zero at infinity. Therefore, as $x > 0$, the solution φ satisfies the integral equation

$$x = \int_{\varphi}^{1} \frac{1}{\zeta}\sqrt{\frac{\delta\zeta+\epsilon}{1-\zeta}}d\zeta = \int_{\varphi}^{1} \frac{\delta\zeta+\epsilon}{\zeta}\frac{d\zeta}{\sqrt{(\delta\zeta+\epsilon)(1-\zeta)}}.$$

Making the substitution

$$\zeta = \frac{\delta-\epsilon}{2\delta} + \frac{\delta+\epsilon}{2\delta}\sin\theta, \tag{4.2}$$

into the preceding integral yields the equation

$$-x = \sqrt{\epsilon}\,\log\frac{\tan\frac{\theta}{2}+\tan\frac{\theta_0}{2}}{1+\tan\frac{\theta_0}{2}\tan\frac{\theta}{2}} + \sqrt{\delta}\,\big(\theta-\frac{\pi}{2}\big), \tag{4.3}$$

where θ_0 is a constant satisfying $\sin\theta_0 = \frac{\delta-\epsilon}{\delta+\epsilon}$ and $|\theta_0| < \frac{\pi}{2}$. Equation (4.3) expresses θ implicitly as a function of x, with range $-\theta_0 \le \theta \le \pi+\theta_0$. Another expression satisfied by the function θ is

$$\exp\left[-\frac{1}{\sqrt{\epsilon}}\big(x+\sqrt{\delta}(\theta-\frac{\pi}{2})\big)\right] = \frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}}. \tag{4.4}$$

We shall use (4.3) and (4.4) to discuss properties of the function θ in the following lemma. Then the transformation (4.2) will help find extension of the solitary wave solution φ to the complex plane.

Lemma 4.1. *The function θ has an extension $\Theta(z)$ which is a holomorphic function on the strip $\{z\in\mathbb{C};\ |\Im z| < \sqrt{\epsilon}\,\pi\}$ and continuous up to its boundary, such that $\Theta(z)$ maps the line segment $\{z = iy;\ y\in(-\sqrt{\epsilon}\,\pi, \sqrt{\epsilon}\,\pi)\}$ onto the line segment $\{\theta = \frac{\pi}{2}+i\eta;\ \eta\in(-\eta_\epsilon,\eta_\epsilon)\}$ with $\Theta(\sqrt{i\epsilon}\,\pi) = \frac{\pi}{2}-i\eta_\epsilon$, $\Theta(0) = \frac{\pi}{2}$ and $\Theta(-i\sqrt{\epsilon}\,\pi) = \frac{\pi}{2}+i\eta_\epsilon$ for some $\eta_\epsilon > 0$.*

Proof. If we consider the right-hand side of (4.3) as a function of θ, denoted by $-\Sigma(\theta)$, then $\Sigma(\theta)$ maps the interval $(-\theta_0, \pi+\theta_0)$ homeomorphically onto the real axis $\mathbb{R}$ with $\Sigma(-\theta_0) = \infty$, $\Sigma(\frac{\pi}{2}) = 0$ and $\Sigma(\pi+\theta_0) = -\infty$. Substituting $\theta = \frac{\pi}{2}+i\eta$ into the right-hand side of Equation (4.3) to extend the function Σ to the line $\{\theta = \frac{\pi}{2}+i\eta;\ \eta\in\mathbb{R}\}$, it follows that

$$\Sigma\big(\frac{\pi}{2}+i\eta\big) = -i\left(\sqrt{\delta}\,\eta + 2\sqrt{\epsilon}\tan^{-1}\big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta}{2}\big)\right), \tag{4.5}$$

which indicates that $i\Sigma(\frac{\pi}{2}+i\eta)$ is an odd and increasing function of η, mapping the real axis to itself homeomorphically. In consequence, Θ maps $\{z = iy;\ y\in[0,\sqrt{\epsilon\pi}]\}$ to the line segment $\{\theta = \frac{\pi}{2}+i\eta;\ \eta\in[-\eta_\epsilon, 0]\}$ and it maps $\{z = iy;\ y\in[-\sqrt{\epsilon\pi}, 0]\}$ to the line segment $\{\theta = \frac{\pi}{2}+i\eta;\ \eta\in[0,\eta_\epsilon]\}$, where $\Sigma(\frac{\pi}{2}+i\eta_\epsilon) = -i\sqrt{\epsilon\pi}$.

Replacing x with $x+iy$ and substituting $\theta=\xi+i\eta$ into Equation (4.4), one obtains the equation

$$\frac{\sin\theta_0\cosh\eta+\sin\xi+i\cos\theta_0\sinh\eta}{\cosh\eta+\cos(\xi-\theta_0)}=\exp\left[\frac{-1}{\sqrt{\epsilon}}\big(x+\sqrt{\delta}(\xi-\frac{\pi}{2})+i(y+\sqrt{\delta}\eta)\big)\right]. \quad (4.6)$$

Comparing norms and angles on both sides of (4.6) leads to the equations,

$$\frac{\cosh\eta-\cos(\xi+\theta_0)}{\cosh\eta+\cos(\xi-\theta_0)}=\exp\left[-\frac{2}{\sqrt{\epsilon}}\big(x+\sqrt{\delta}(\xi-\frac{\pi}{2})\big)\right], \quad (4.7)$$

and

$$\frac{\cos\theta_0\sinh\eta}{\sin\theta_0\cosh\eta+\sin\xi}=-\tan\frac{y+\sqrt{\delta}\eta}{\sqrt{\epsilon}},$$

or equivalently,

$$\sin\xi=-\sin\theta_0\cosh\eta-\cos\theta_0\sinh\eta\cot\frac{y+\sqrt{\delta}\eta}{\sqrt{\epsilon}}. \quad (4.8)$$

These will be used to determine the equipotentials and streamlines of the function Σ.

First, we consider the streamline (4.8) of the function Σ for $y\in[-\sqrt{\epsilon}\,\pi,\sqrt{\epsilon}\,\pi]$. Since the graph of the streamline (4.8) is symmetric with respect to the line $\{\frac{\pi}{2}+i\eta;\eta\in\mathbb{R}\}$ for each $y\in(0,\sqrt{\epsilon}\,\pi)$ and the streamline for $y=y_0$ is the reflection of the streamline for $y=-y_0$ with respect to the real axis, it is sufficient to study streamlines for $y\in(0,\sqrt{\epsilon}\,\pi]$ with the restriction $\Re(\theta)=\xi\le\frac{\pi}{2}$. If $0<y_0\le\sqrt{\epsilon}\,\pi$, then it follows from (4.5) that the streamline passes through the point $(\frac{\pi}{2},\eta_0)$ for some η_0 with $-\eta_\epsilon<\eta_0<0$ and

$$0<y_0+\sqrt{\delta}\,\eta_0=-2\sqrt{\epsilon}\tan^{-1}\big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta_0}{2}\big)<\sqrt{\epsilon}\,\pi. \quad (4.9)$$

Let

$$f(\eta)=-\sin\theta_0\cosh\eta-\cos\theta_0\sinh\eta\cot\frac{y_0+\sqrt{\delta}\,\eta}{\sqrt{\epsilon}}.$$

Then the derivative of f with respect to η is

$$f'(\eta)=\sinh\eta\left(\frac{\sqrt{\delta}\,\cos\theta_0}{\sqrt{\epsilon}\,\sin^2\frac{1}{\sqrt{\epsilon}}(y_0+\sqrt{\delta}\,\eta)}-\sin\theta_0\right)-\cos\theta_0\cosh\eta\cot\frac{1}{\sqrt{\epsilon}}(y_0+\sqrt{\delta}\,\eta). \quad (4.10)$$

Applying the inequality $\sinh\eta<0$ for $\eta<0$ and the estimate

$$\frac{\sqrt{\delta}\,\cos\theta_0}{\sqrt{\epsilon}\,\sin^2\frac{1}{\sqrt{\epsilon}}(y_0+\sqrt{\delta}\,\eta)}-\sin\theta_0=\frac{2\delta}{(\delta+\epsilon)\sin^2\frac{1}{\sqrt{\epsilon}}(y_0+\sqrt{\delta}\,\eta)}-\frac{\delta-\epsilon}{\delta+\epsilon}\ge 1$$

obtained from the definition of θ_0 in (4.3), as well as Inequality (4.9) to (4.10), one concludes that $f'(\eta)<0$ holds if $y_0\in(0,\sqrt{\epsilon}\,\pi/2]$ and $\eta\in(\eta_0,0)$. Moreover, $\lim_{\eta\to 0} f(\eta)=-\sin\theta_0$. In consequence, the streamline (4.8) is a decreasing curve from the point $(-\theta_0,0)$ to the point $(\frac{\pi}{2},\eta_0)$, and then it is an increasing curve connecting the points $(\frac{\pi}{2},\eta_0)$ and $(\pi+\theta_0,0)$ in the complex plane.

When $y_0\in(\frac{\sqrt{\epsilon}\,\pi}{2},\sqrt{\epsilon}\,\pi)$, we write (4.10) in another equivalent expression

$$f'(\eta)=\cosh\eta\left(\frac{\sqrt{\delta}\,\cos\theta_0}{\sqrt{\epsilon}\,\sin^2 A}-\sin\theta_0\right)\left(\tanh\eta-\frac{\cos\theta_0\cot A}{1+\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\cot^2 A}\right),$$

where $A=\frac{1}{\sqrt{\epsilon}}(y_0+\sqrt{\delta}\,\eta)$. It follows from the inequalities $f'\big(\frac{1}{\sqrt{\delta}}(\frac{\sqrt{\epsilon}\,\pi}{2}-y_0)\big)<0$ and $f'(0)>0$ that there is an $\tilde\eta\in\big(\frac{1}{\sqrt{\delta}}(\frac{\sqrt{\epsilon}\,\pi}{2}-y_0),0\big)$ such that $f'(\tilde\eta)=0$, or

$$\tanh\tilde\eta-\frac{\cos\theta_0\cot\tilde A}{1+\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\cot^2\tilde A}=0, \tag{4.11}$$

where $\tilde A=\frac{1}{\sqrt{\epsilon}}(y_0+\sqrt{\delta}\,\tilde\eta)$. Substituting (4.11) into the function f yields the following estimate

$$\begin{aligned} f(\tilde\eta)&=-\cosh\tilde\eta\left(\sin\theta_0+\frac{\cos^2\theta_0\cot^2\tilde A}{1+\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\cot^2\tilde A}\right)\\ &=-\frac{\frac{\delta-\epsilon}{\delta+\epsilon}+\frac{2\delta}{\delta+\epsilon}\cot^2\tilde A}{\sqrt{1+\frac{4\delta^2}{(\delta+\epsilon)^2}\cot^2\tilde A+\frac{4\delta^2}{(\delta+\epsilon)^2}\cot^4\tilde A}}>-1 \end{aligned} \tag{4.12}$$

Moreover, the substitution of (4.11) into the second order derivative

$$f''(\eta)=-\sin\theta_0\cosh\eta-\cos\theta_0\sinh\eta\cot A+\frac{2\sqrt{\delta}\cos\theta_0\cosh\eta}{\sqrt{\epsilon}\sin^2 A}-\frac{2\delta\cos\theta_0\sinh\eta\cot A}{\epsilon\sin^2 A}$$

of f leads to the inequality

$$\begin{aligned} f''(\tilde\eta)=\cosh\tilde\eta&\left(-\sin\theta_0-\frac{\cos^2\theta_0\cot^2\tilde A}{1+\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\cot^2\tilde A}+2\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0(1+\cot^2\tilde A)+\right.\\ &\left.-\frac{2\delta\cos^2\theta_0\cot^2\tilde A(1+\cot^2\tilde A)}{\epsilon(1+\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\cot^2\tilde A)}\right)=\frac{(3\delta+\epsilon+2\delta\cot^2\tilde A)\cosh\tilde\eta}{(\delta+\epsilon)\big(1+\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\cot^2\tilde A\big)}>0, \end{aligned} \tag{4.13}$$

which implies that f' has only one zero $\tilde{\eta}$ in the interval $\left(\frac{1}{\sqrt{\delta}}(\frac{\sqrt{\epsilon}\,\pi}{2} - y_0), 0\right)$. Therefore, the function f is decreasing on the interval $(\eta_0, \tilde{\eta})$ and it is increasing on the interval $(\tilde{\eta}, 0)$ with $\lim_{\eta\to 0} f(\eta) = -\sin\theta_0$, which characterizes the streamline (4.8) connecting the points $(-\theta_0, 0)$ and $\left(\frac{\pi}{2}, \eta_0\right)$ for $y \in (\frac{\sqrt{\epsilon}\,\pi}{2}, \sqrt{\epsilon}\,\pi)$ and $|\xi| \le \frac{\pi}{2}$.

If $y_0 = \sqrt{\epsilon}\,\pi$, then (4.9) and (4.10) indicate that $f'(\eta_0) < 0$. As a matter of fact, f' also keeps the negative sign on the interval $(\eta_0, 0)$. Assume that $f'(\tilde{\eta}) = 0$ for some $\tilde{\eta} \in (\eta_0, 0)$. Then inequalities (4.12) and (4.13) imply that $f(\tilde{\eta}) > -1$, $f''(\tilde{\eta}) > 0$ and $f'(\eta) > 0$ for any $\eta \in (\tilde{\eta}, 0)$. Thus $f(\tilde{\eta})$ is a local minimum and $f(\tilde{\eta}) \le \lim_{\eta\to 0} f(\eta) = -\sin\theta_0 - \sqrt{\frac{\epsilon}{\delta}}\cos\theta_0 = -1$, which contradicts Inequality (4.12). Therefore f is a strictly decreasing function. Using the Taylor expansion

$$f(\eta) = -1 + \frac{\eta^2}{6} + \cdots,$$

of f at $\eta = 0$, one can also compute

$$\lim_{\substack{\eta\to 0\\ \eta<0}} \frac{d}{d\eta}[\sin^{-1} f(\eta)] = \lim_{\substack{\eta\to 0\\ \eta<0}} \frac{f'(\eta)}{\sqrt{1 - f^2(\eta)}} = \lim_{\substack{\eta\to 0\\ \eta<0}} \frac{\frac{\eta}{3} + \cdots}{\sqrt{\frac{\eta^2}{3} + \cdots}} = -\frac{1}{\sqrt{3}}.$$

Therefore, the streamline in the region $\{\xi + i\eta;\ |\xi| \le \frac{\pi}{2}, \eta < 0\}$ is a decreasing curve connecting the points $\xi + i\eta = -\frac{\pi}{2}$ and $\frac{\pi}{2} + i\eta_0$. In addition, (4.6) shows that when $\eta = 0$, $y = \sqrt{\epsilon}\,\pi$ and $-\frac{\pi}{2} < \xi < -\theta_0$,

$$\frac{\sin\xi + \sin\theta_0}{1 + \cos(\xi - \theta_0)} = -e^{-\frac{1}{\sqrt{\epsilon}}\left(x + \sqrt{\delta}(\xi - \frac{\pi}{2})\right)}. \tag{4.14}$$

In consequence, by tracing along the streamline for $y = \sqrt{\epsilon}\,\pi$ from the lower-half plane up to the real axis, Σ is seen to be a one-to-one mapping of the line segment $\{\xi; \xi \in (-\frac{\pi}{2}, -\theta_0)\}$ onto the line $\{x + i\sqrt{\epsilon}\,\pi; x \in (\sqrt{\delta}\,\pi, \infty)\}$ with $\Sigma(-\frac{\pi}{2}) = \sqrt{\delta}\,\pi$, which may be verified by showing that the derivative of $\Sigma(\xi)$ determined by (4.14) is positive and $\lim_{\xi\to-\theta_0} \Sigma(\xi) = \infty$ when $\xi \in (-\frac{\pi}{2}, -\theta_0)$.

In addition, the Cauchy-Riemann equations

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta}, \qquad \frac{\partial x}{\partial \eta} = -\frac{\partial y}{\partial \xi} \tag{4.15}$$

imply that the derivative of the function $x = x(\xi(\eta), \eta)$ can be expressed as

$$\frac{dx}{d\eta} = x_\xi \xi_\eta + x_\eta = \frac{x_\xi^2 + x_\eta^2}{x_\eta},$$

in which $\xi(\eta) = \sin^{-1} f(\eta)$ is determined by the streamline $y_0 = y(\xi, \eta)$ for all $\eta \in (\eta_0, 0)$ with the value $\xi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and some fixed $y_0 \in (0, \sqrt{\epsilon}\pi)$. The derivative of $z = \Sigma(\theta) = \Sigma(\xi + i\eta) = x(\xi, \eta) + iy(\xi, \eta)$, taking the form

$$\begin{aligned} \frac{dz}{d\theta} = &-\sqrt{\delta} - \frac{\sqrt{\epsilon}\cos\theta_0(\sin\xi\cosh\eta + \sin\theta_0)}{|\sin\xi\cosh\eta + \sin\theta_0 + i\cos\xi\sinh\eta|^2} + \\ &+ \frac{i\sqrt{\epsilon}\cos\theta_0\cos\xi\sinh\eta}{|\sin\xi\cosh\eta + \sin\theta_0 + i\cos\xi\sinh\eta|^2}, \end{aligned} \tag{4.16}$$

implies that $\frac{dx}{d\eta} > 0$ if $\eta < 0$ and $|\xi| < \frac{\pi}{2}$, which together with the symmetry of the streamline with respect to the line $\xi = \frac{\pi}{2}$ and the equality $-x(\xi, \eta) = x(\pi - \xi, \eta)$ implies that Σ is a one-to-one mapping of the streamline connecting the points $(-\theta_0, 0)$ and $(\pi + \theta_0, 0)$ in the lower-half plane onto the line $\{x + iy_0; x \in \mathbb{R}\}$ for each $y_0 \in (0, \sqrt{\epsilon}\pi)$ with $\Sigma(\pi + \theta_0) = -\infty + iy_0$, $\Sigma(\frac{\pi}{2} + i\eta_0) = iy_0$ and $\Sigma(-\theta_0) = \infty + iy_0$. Using a similar argument, one can also show that the streamline $\sqrt{\epsilon}\pi = y(\xi, \eta)$ of the function Σ, consisting of the line segments $\{\xi;\ \xi \in [-\frac{\pi}{2}, -\theta_0)\}$ and $\{\xi;\ \xi \in (\pi + \theta_0, \frac{3\pi}{2}]\}$ and the curve in the lower-half plane connecting the points $(-\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ as shown in Figure 7, is homeomorphic to the line $\{x + i\sqrt{\epsilon}\pi; x \in \mathbb{R}\}$. Then the symmetry of the streamlines with respect to the real axis leads to the conclusion that the streamline $y_0 = y(\xi, \eta)$ connecting the points $(-\theta_0, 0)$ and $(\pi + \theta_0, 0)$ on the upper-half plane is homeomorphic to the line $\{x + iy_0;\ x \in \mathbb{R}\}$ for each $y_0 \in (-\sqrt{\epsilon}\pi, 0)$ and this result can be easily extended to the streamline $-\sqrt{\epsilon}\pi = y(\xi, \eta)$ as well.

Next, we investigate the equipotential expressed by (4.7) which can also be written in the following equivalent form

$$\cosh\eta = -\sin\theta_0\sin\xi + \cos\theta_0\cos\xi\coth B, \tag{4.17}$$

where $B = \frac{1}{\sqrt{\epsilon}}(x + \sqrt{\delta}(\xi - \frac{\pi}{2}))$ for some $x \in (-\infty, 0) \cup (0, \infty)$. Since the equipotential $x(\xi, \eta) = 0$ is the line $\xi = \frac{\pi}{2}$ which is mapped homeomorphically to the imaginary axis $x = 0$ by Σ as demonstrated in (4.5), and the equipotential satisfies the relations $x(\xi, \eta) = x(\xi, -\eta)$ and $x(\xi, \eta) = -x(\pi - \xi, \eta)$, it is sufficient to consider the case when $x \in (0, \infty)$ with the restrictions $\eta < 0$ and $\xi < \frac{\pi}{2}$.

Let

$$g(\xi, x) = -\sin\theta_0\sin\xi + \cos\theta_0\cos\xi\coth B.$$

For each fixed $x \in (0, \sqrt{\delta}\,\pi)$, the explicit formula

$$\frac{\partial}{\partial\xi}\, g(\xi, x) = -\sin\theta_0\cos\xi - \cos\theta_0\sin\xi\coth B - \sqrt{\frac{\delta}{\epsilon}}\frac{\cos\theta_0\cos\xi}{\sinh^2 B} \tag{4.18}$$

shows that $\frac{\partial g}{\partial \xi} \to -\infty$ as $\xi \to \frac{\pi}{2} - \frac{x}{\sqrt{\delta}}$ and $\xi > \frac{\pi}{2} - \frac{x}{\sqrt{\delta}}$. Then g is a decreasing function of ξ in the interval $(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}}, \tilde{\xi})$ for some $\tilde{\xi}$ and $\lim_{\xi \to (\frac{\pi}{2} - \frac{x}{\sqrt{\delta}})^+} g(\xi, x) = \infty$. As a matter of fact, there is an $x_0 \in (0, \sqrt{\delta}\pi)$ such that for each $x \in (0, x_0]$, the inequality $\frac{\partial}{\partial \xi} g(\xi, x) \leq 0$ holds for any $\xi \in (\frac{\pi}{2} - \frac{x}{\sqrt{\delta}}, \xi_x)$, where $\xi_x \in (-\theta_0, \frac{\pi}{2})$ and $g(\xi_x, x) = 1$. This may be verified by considering solutions (ξ, x) of the equations

$$\frac{\partial}{\partial \xi} g(\xi, x) = 0, \qquad \frac{\partial^2}{\partial \xi^2} g(\xi, x) = 0,$$

for $|\xi| < \frac{\pi}{2}$ and $x \in (0, \infty)$. Substituting

$$\tan \xi = \frac{\tanh B}{\cos \theta_0} - \sqrt{\frac{\delta}{\epsilon}} \coth B \tag{4.19}$$

obtained from the first equation into the second equation yields the expression

$$\frac{\partial^2}{\partial \xi^2} g(\xi, x) = \frac{\cos \xi \left(\frac{2\delta}{\delta+\epsilon} - \sinh^2 B\right)}{\cos \theta_0 \sinh B \cosh B} = 0. \tag{4.20}$$

Then the resulting equivalent system of equations

$$\tan \xi = \frac{\tanh B}{\cos \theta_0} - \sqrt{\frac{\delta}{\epsilon}} \coth B, \qquad \sinh^2 B = \frac{2\delta}{\delta + \epsilon},$$

gives the unique solution

$$\xi_0 = -\tan^{-1} \frac{\sqrt{2}\,\delta}{\sqrt{\epsilon(3\delta + \epsilon)}}, \qquad x_0 = \sqrt{\delta} \left(\frac{\pi}{2} + \frac{1}{2}\sqrt{\frac{\epsilon}{\delta}} \log \frac{L+1}{L-1} - \xi_0 \right),$$

with $L = \sqrt{\frac{3\delta+\epsilon}{2\delta}}$. One may also estimate x_0 to show that $0 < x_0 < \sqrt{\delta}\,\pi$. Since $\frac{\partial^3}{\partial \xi^3} g(\xi_0, x_0) = -\frac{\delta+\epsilon}{\epsilon} \cos \xi_0 < 0$, $\frac{\partial}{\partial \xi} g(\xi, x_0)$ is non-positive in a neighbourhood of ξ_0, *i.e.* $g(\xi, x_0)$ is decreasing in the same neighbourhood with an inflection point at $\xi = \xi_0$. Suppose that $\frac{\partial}{\partial \xi} g(\xi_1, x_0) = 0$ for some $\xi_1 \in (\frac{\pi}{2} - \frac{x_0}{\sqrt{\delta}}, \xi_0)$. Then (ξ_1, x_0) also satisfies (4.19) and it follows from (4.20) that

$$\begin{aligned}\frac{\partial^2}{\partial \xi^2} g(\xi_1, x_0) &= \frac{2\cos \xi_1 \left(\frac{2\delta}{\delta+\epsilon} - \sinh^2 \sqrt{\frac{\delta}{\epsilon}} \left(\frac{x_0}{\sqrt{\delta}} + \xi_1 - \frac{\pi}{2}\right) \right)}{\cos \theta_0 \sinh\left(2\sqrt{\frac{\delta}{\epsilon}}\left(\frac{x_0}{\sqrt{\delta}} + \xi_1 - \frac{\pi}{2}\right)\right)} \\ &> \frac{2\cos \xi_1 \left(\frac{2\delta}{\delta+\epsilon} - \sinh^2 \sqrt{\frac{\delta}{\epsilon}} \left(\frac{x_0}{\sqrt{\delta}} + \xi_0 - \frac{\pi}{2}\right) \right)}{\cos \theta_0 \sinh\left(2\sqrt{\frac{\delta}{\epsilon}}\left(\frac{x_0}{\sqrt{\delta}} + \xi_1 - \frac{\pi}{2}\right)\right)} = 0.\end{aligned}$$

This implies that $\frac{\partial}{\partial \xi} g(\xi, x_0) > 0$ for all $\xi \in (\xi_1, \xi_0)$ and thus leads to a contradiction to the fact $\frac{\partial}{\partial \xi} g(\xi, x_0) \leq 0$ in a neighbourhood of ξ_0. On the other hand, if $\frac{\partial}{\partial \xi} g(\xi_2, x_0) = 0$ for some $\xi_2 \in (\xi_0, \xi_{x_0})$, then $\frac{\partial^2}{\partial \xi^2} g(\xi_2, x_0) < 0$ which implies that $\frac{\partial}{\partial \xi} g(\xi, x_0) > 0$ for all $\xi \in (\xi_0, \xi_2)$ and also leads to the same contradiction. Therefore, one can conclude that $\frac{\partial}{\partial \xi} g(\xi, x_0) < 0$ for all $\xi \in (\frac{\pi}{2} - \frac{x_0}{\sqrt{\delta}}, \xi_0) \cup (\xi_0, \xi_{x_0})$.

Because

$$\frac{\partial}{\partial x}\Big(\frac{\tanh B}{\cos\theta_0} - \sqrt{\frac{\delta}{\epsilon}}\coth B\Big) = \frac{1}{\sqrt{\epsilon}}\Big(\frac{1}{\cos\theta_0 \cosh^2 B} + \sqrt{\frac{\delta}{\epsilon}}\frac{1}{\sinh^2 B}\Big) > 0,$$

as $0 < x < x_0$, we have

$$\begin{aligned}\frac{\partial}{\partial \xi} g(\xi, x) &= -\cos\theta_0 \cos\xi \coth B \left(\tan\xi - \frac{\tanh B}{\cos\theta_0} + \sqrt{\frac{\delta}{\epsilon}}\coth B\right) \\ &< -\cos\theta_0 \cos\xi \coth B \left(\tan\xi - \frac{\tanh\big(\sqrt{\frac{\delta}{\epsilon}}(x_0 + \xi - \frac{\pi}{2})\big)}{\cos\theta_0} + \right. \\ &\left. + \sqrt{\frac{\delta}{\epsilon}}\coth\Big(\sqrt{\frac{\delta}{\epsilon}}(x_0 + \xi - \frac{\pi}{2})\Big)\right) \leq 0.\end{aligned}$$

Therefore, the equipotential (4.7), which can be described by the function

$$\eta = -\cosh^{-1} g(\xi, x)$$

in the lower-half plane, is a monotone increasing curve on the interval $(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}}, \xi_x)$ with $\lim\limits_{\xi \to \left(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}}\right)^+} \eta = -\infty$ and $\eta(\xi_x, x) = 0$ for each fixed $x \in (0, x_0]$.

Now let $x \in (x_0, \sqrt{\delta}\pi)$. Since

$$\tan\xi_x = \frac{1 - \sin\theta_0 \cosh\big(\sqrt{\frac{\delta}{\epsilon}}(\frac{x}{\sqrt{\delta}} + \xi_x - \frac{\pi}{2})\big)}{\cos\theta_0 \sinh\big(\sqrt{\frac{\delta}{\epsilon}}(\frac{x}{\sqrt{\delta}} + \xi_x - \frac{\pi}{2})\big)}$$

for some $\xi_x \in (-\theta_0, \frac{\pi}{2})$, we find

$$\frac{\partial}{\partial \xi} g(\xi_x, x) = -\cos\xi_x \frac{\cosh\big(\sqrt{\frac{\delta}{\epsilon}}(\frac{x}{\sqrt{\delta}} + \xi_x - \frac{\pi}{2})\big) + 1}{\sinh^2\big(\sqrt{\frac{\delta}{\epsilon}}(\frac{x}{\sqrt{\delta}} + \xi_x - \frac{\pi}{2})\big)} < 0.$$

Moreover, $\frac{\partial}{\partial\xi}g(\xi_0,x) > \frac{\partial}{\partial\xi}g(\xi_0,x_0) = 0$, and $\lim_{\xi\to(\frac{\pi}{2}-\frac{x}{\sqrt{\delta}})^+}\frac{\partial}{\partial\xi}g(\xi,x) = -\infty$. Therefore, $\frac{\partial}{\partial\xi}g(\xi,x)$ has at least two zeros ξ_1 and ξ_2 such that $\xi_1 \in (\frac{\pi}{2}-\frac{x}{\sqrt{\delta}},\xi_0)$ and $\xi_2 \in (\xi_0,\xi_x)$. Then it follows from the identities

$$\frac{\partial^2}{\partial\xi^2}g(\xi_i,x) = \frac{2\cos\xi_i\big(\frac{2\delta}{\delta+\epsilon}-\sinh^2\big(\sqrt{\frac{\delta}{\epsilon}}(\frac{x}{\sqrt{\delta}}+\xi_i-\frac{\pi}{2})\big)\big)}{\cos\theta_0\sinh\big(2\sqrt{\frac{\delta}{\epsilon}}(\frac{x}{\sqrt{\delta}}+\xi_i-\frac{\pi}{2})\big)},$$

for $i=1,2$, that ξ_1 and ξ_2 are only zeros of $\frac{\partial g}{\partial\xi}$ in the interval $(\frac{\pi}{2}-\frac{x}{\sqrt{\delta}},\xi_x)$ and

$$\frac{\partial^2}{\partial\xi^2}g(\xi_2,x) < 0 < \frac{\partial^2}{\partial\xi^2}g(\xi_1,x).$$

Furthermore,

$$g(\xi,x) = \frac{\sqrt{\frac{\delta}{\epsilon}}\cos\theta_0\coth^2 B-\sin\theta_0}{\sqrt{1-\frac{\delta}{\epsilon}\cos^2\theta_0\coth^2 B+\frac{\delta}{\epsilon}\cos^2\theta_0\coth^4 B}} > 1,$$

whenever $\frac{\partial}{\partial\xi}g(\xi,x) = 0$ and $\xi \in (\frac{\pi}{2}-\frac{x}{\sqrt{\delta}},\xi_x)$. Hence, the equipotential (4.17) in the lower-half plane described by the function

$$\eta = -\cosh^{-1}g(\xi,x)$$

is seen increasing on the intervals $(\frac{\pi}{2}-\frac{x}{\sqrt{\delta}},\xi_1)$ and (ξ_2,ξ_x), decreasing on the interval (ξ_1,ξ_2) and having the asymptote $\xi = \frac{\pi}{2}-\frac{x}{\sqrt{\delta}}$.

If $x=\sqrt{\delta}\pi$, then $\lim_{\xi\to-\frac{\pi}{2}}g(\xi,\sqrt{\delta}\pi) = 1$ which combined with the estimates

$$\frac{\partial}{\partial\xi}g(\xi_{\sqrt{\delta}\pi},\sqrt{\delta}\pi) < 0, \quad \frac{\partial}{\partial\xi}g(\xi_0,\sqrt{\delta}\pi) > 0 \text{ and } \lim_{\xi\to-\frac{\pi}{2}^+}\frac{\partial}{\partial\xi}g(\xi,\sqrt{\delta}\pi) = 0$$

implies that there is a $\xi_3 \in (-\frac{\pi}{2},\xi_{\sqrt{\delta}\pi})$ such that g is increasing on the interval $(-\frac{\pi}{2},\xi_3)$ and decreasing on the interval $(\xi_3,\xi_{\sqrt{\delta}\pi})$. Hence, the equipotential $\sqrt{\delta}\pi = x(\xi,\eta)$ in the lower-half plane connects the points $\theta=-\frac{\pi}{2}$ and $\xi_{\sqrt{\delta}\pi}$, decreasing on the interval $(-\frac{\pi}{2},\xi_3)$ and increasing on the interval $(\xi_3,\xi_{\sqrt{\delta}\pi})$, and it also contains the line segment $\{-\frac{\pi}{2}+i\eta;\ -\infty<\eta\le 0\}$ which is mapped onto the line $\{\sqrt{\delta}\,\pi+iy;\ y\ge\sqrt{\epsilon}\,\pi\}$ by Σ.

In a similar way, one may show that for each $x \in (\sqrt{\delta}\pi,\infty)$, there exists ξ_4 and ξ_5 such that $-\frac{\pi}{2} < \xi_4 < \xi_5 < -\theta_0 < \xi_x$ and the equipotential $x = x(\xi,\eta)$ in the lower-half plane connects the points $\theta = \xi_4$ and ξ_x,, which is decreasing on the interval (ξ_4, ξ_5) and increasing on the interval (ξ_5, ξ_x).

Finally, we apply Cauchy-Riemann equations (4.15) to the derivative

$$\frac{dy}{d\xi} = y_\xi + y_\eta \eta_\xi = -\frac{x_\eta^2 + x_\xi^2}{x_\eta}$$

of the stream function $y = y(\xi, \eta(\xi))$. The function $\eta(\xi) = -\cosh^{-1} g(\xi, x)$ is determined by the equipotential (4.17) for $\eta \leq 0$ and $|\xi| \leq \frac{\pi}{2}$. Using (4.16) to obtain the estimate $\frac{dy}{d\xi} < 0$ for $\eta < 0$ and taking into account of the symmetry of the equipotential with respect to the real axis yield conclusions that Σ maps the equipotential $x = x(\xi, \eta)$ homeomorphically to the line $\{x + iy;\ y \in (-\infty, \infty)\}$ for each fixed $x \in (0, \sqrt{\delta}\,\pi)$ with $\xi \in (\frac{\pi}{2} - \frac{x}{\sqrt{\delta}}, \xi_x)$, $\Sigma(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}} - i\infty) = x + i\infty$ and $\Sigma(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}} + i\infty) = x - i\infty$. Also, Σ is a one-to-one mapping of the equipotential $x = x(\xi, \eta)$ onto the line segment $\{x + iy;\ y \in (-\sqrt{\epsilon}\,\pi, \sqrt{\epsilon}\,\pi)\}$ for each $x \in [\sqrt{\delta}\,\pi, \infty)$ with $\xi \in (-\frac{\pi}{2}, \xi_x)$. Moreover, it follows from the the identity $x(\xi, \eta) = -x(\pi - \xi, \eta)$ that the above conclusions also apply to the equipotential $x = x(\xi, \eta)$ for each $x \in (-\infty, 0)$ with the restriction $\xi \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

As a result of the above discussion, the function $\Sigma(\theta)$ is seen as a conformal mapping of the domain Ω onto the strip $\{x + iy;\ -\infty < x < \infty, |y| < \sqrt{\epsilon}\,\pi\}$, where Ω is bounded above by the streamline $-\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ consisting of the line segments $\{\xi;\ \xi \in [-\frac{\pi}{2}, -\theta_0)\}$ and $\{\xi;\ \xi \in (\pi + \theta_0, \frac{3\pi}{2})\}$, and the thickest solid curve in the upper-half plane shown in Figure 7, and Ω is bounded below by the streamline $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ also consisting of the line segments $\{\xi;\ \xi \in [-\frac{\pi}{2}, -\theta_0)\}$ and $\{\xi;\ \xi \in (\pi + \theta_0, \frac{3\pi}{2})\}$, as well as the thickest solid curve in the lower-half plane illustrated in Figure 7. Therefore, the inverse $\theta(x)$ of the the function $\Sigma(\theta)$ has an analytic extension $\Theta(z)$ to the strip $\{x + iy;\ -\infty < x < \infty, |y| < \sqrt{\epsilon}\,\pi\}$ and continuous up to its boundary ([21], Thm. 14.18). Then the transformation $\varphi = \frac{\delta - \epsilon}{2\delta} + \frac{\delta + \epsilon}{2\delta} \sin\theta$ leads to the conclusion that the solitary wave solution φ of (4.1) has an analytic extension to the same strip. □

An immediate consequence of Lemma 4.1 is the existence of a cuspon as a weak solution of Equation (4.1). One may have noticed that (4.14) implicitly determines the value of the function Σ on the line segment $\{\xi;\ \xi \in (-\frac{\pi}{2}, -\theta_0)\}$ which is mapped to the line $\{x + i\sqrt{\epsilon}\,\pi;\ x \in (\sqrt{\delta}\,\pi, \infty)\}$. Extending the function Σ on the real axis from $\xi = -\frac{\pi}{2}$ to the left up to the point $\xi = -\pi + \theta_0$, one may also realize that Σ maps $\{\xi;\ \xi \in (-\pi + \theta_0, -\theta_0)\}$ homeomorphically to the line $\{x + i\sqrt{\epsilon}\,\pi;\ x \in (-\infty, \infty)\}$. This leads us to the discovery of the cuspon solution of Equation (4.1), as stated in the following corollary.

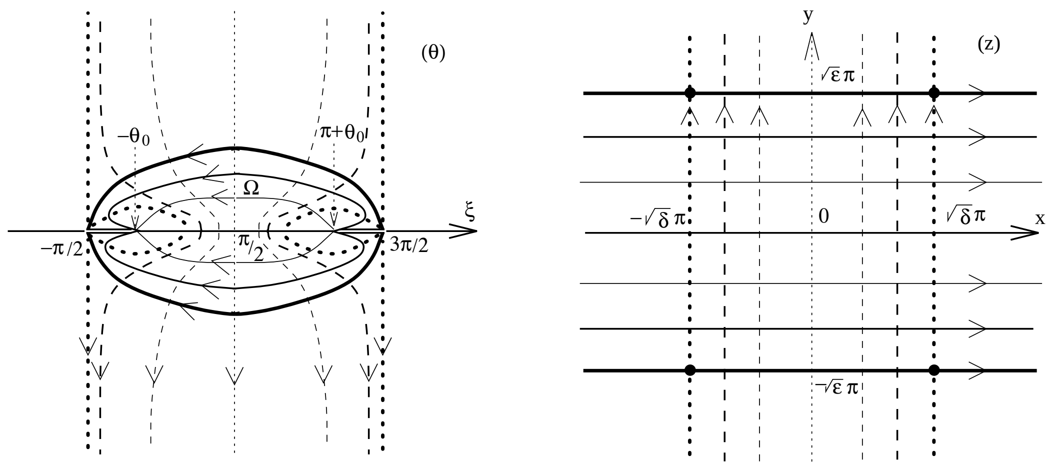

Fig. 7. Streamlines and equipotentials of the function $z = \Sigma(\theta)$

Corollary 4.2. *Equation (4.1) has a weak solution φ_p in the sense of Definition 3.1. Moreover, φ_p is an even and negative function, continuous on the real axis, monotonically increasing on the positive x-axis and approaching zero at infinity, thereby representing a wave of depression. The derivative φ_p' has a discontinuity at the minimum of φ_p.*

Proof. It follows from (4.14) that

$$x = -\sqrt{\delta}(\xi - \frac{\pi}{2}) - \sqrt{\epsilon}\log(\frac{-(\sin\xi + \sin\theta_0)}{1+\cos(\xi - \theta_0)})$$

which determines a function of ξ, denoted by $x = \Sigma_p(\xi)$ for $-\pi + \theta_0 < \xi < -\theta_0$. Since

$$\frac{dx}{d\xi} = -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0}, \tag{4.21}$$

$\frac{dx}{d\xi} > 0$ on the interval $(-\pi + \theta_0, -\theta_0)$, and thus Σ_p is an increasing function whose graph is symmetric with respect to the point $(-\frac{\pi}{2}, \sqrt{\delta}\,\pi)$, having an inflection point at $\xi = -\frac{\pi}{2}$ and asymptotes $\xi = -\pi + \theta_0$ and $\xi = -\theta_0$. Therefore, the inverse of Σ_p, denoted by $\xi = \xi(x)$, is also an increasing function, symmetric with respect to the point $(\sqrt{\delta}\,\pi, -\frac{\pi}{2})$ with $\xi'(\sqrt{\delta}\,\pi) = \infty$. Then the transformation

$$\varphi_1(x) = \frac{\delta - \epsilon}{2\delta} + \frac{\delta + \epsilon}{2\delta}\sin\big(\xi(x)\big) \tag{4.22}$$

yields the function $\varphi_1(x)$. Because of the symmetry $-\pi - \xi(x) = \xi(2\sqrt{\delta}\,\pi - x)$, the graph of φ_1 is symmetric with respect to the line $x = \sqrt{\delta}\,\pi$ and a cusp is formed at $x = \sqrt{\delta}\,\pi$ such that $\lim_{x\to(\sqrt{\delta}\,\pi)^-} \varphi_1'(x) = -\infty$ and $\lim_{x\to(\sqrt{\delta}\,\pi)^+} \varphi_1'(x) = \infty$. But φ_1 is a continuous function itself. Since $x = \Sigma_p(\xi)$ satisfies (4.21), substituting the transformation (4.22) into (4.21) yields Equation (4.1) which φ_1 satisfies every where except at the point $x = \sqrt{\delta}\,\pi$. Let $\varphi_p(x) = \varphi_1(x + \sqrt{\delta}\,\pi)$. Then the continuity of φ_1 and the translation invariant property of Equation (4.1) show that φ_p is the cuspon solution to Equation (4.1) satisfying properties stated in this lemma. It is also worth noticing that so called "cuspon solution" in this case is represented by two unbounded orbits illustrated in Figure 2. □

In order to find further extension of $\Theta(z)$, we need to study other properties of the function Σ defined on the complex plane, such as its singularities, streamlines $y = y(\xi, \eta)$ for $|y| > \sqrt{\epsilon}\,\pi$ and zeros of its derivative given by (4.21). We summarize these properties as follows.

(*i*) *Singularities of* Σ *and zeros of* $\frac{d\Sigma}{d\theta}$. In Figure 7, there are four points noticeable on the ξ-axis, $\xi = -\frac{\pi}{2}, -\theta_0, \pi + \theta_0$ and $\frac{3\pi}{2}$. At $\xi = -\frac{\pi}{2}$ and $\frac{3\pi}{2}$, the function Σ is not angle-preserving, since they are zeros of $\frac{d\Sigma}{d\theta}$. While $\xi = -\theta_0$ and $\pi + \theta_0$ are singularities of Σ. As a matter of fact, there are two countable sets of points on the ξ-axis. One is the set of zeros of $\frac{d\Sigma}{d\theta}$, $\mathcal{O} = \{-\frac{\pi}{2} + 2n\pi;\ n = 0, \pm 1, \pm 2, \cdots\}$; the other one is the set of singularities of Σ, $\mathcal{S} = \{-\theta_0 + 2n\pi, \theta_0 + (2n+1)\pi;\ n = 0, \pm 1, \pm 2, \cdots\}$. Integrating $\frac{d\Sigma}{d\theta}$ along a closed Jordan curve Γ counterclockwise once, whose interior contains only the singular point $\xi = -\theta_0 + 2n\pi$, but not other points in $\mathcal{S}$, yields

$$\int_\Gamma -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0} d\xi = -i2\pi\sqrt{\epsilon}.$$

On the other hand, integrating $\frac{d\Sigma}{d\theta}$ along a closed path Γ_1 counterclockwise once such that the interior of Γ_1 contains only the singular point $\xi = \theta_0 + (2n+1)\pi$, but not other points in $\mathcal{S}$, yields

$$\int_{\Gamma_1} -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0} d\xi = i2\pi\sqrt{\epsilon}.$$

Therefore, singularities of Σ are branch points of infinite order.

(*ii*) *A single-valued branch* Σ_0 *of* Σ. Notice that if we integrate $\frac{d\Sigma}{d\xi}$ on a closed path Γ_2 whose interior contains only two adjacent singularities $(2n-1)\pi + \theta_0$ and $2n\pi - \theta_0$ of Σ for some integer n, then

$$\int_{\Gamma_2} -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0} d\xi = -i2\pi\sqrt{\epsilon} + i2\pi\sqrt{\epsilon} = 0,$$

i.e. the value of the function Σ does not change after a complete circuit around the two singular points $(2n-1)\pi + \theta_0$ and $2n\pi - \theta_0$. Therefore, to find a single-valued branch Σ_0

of the multi-valued function Σ, one may define countably many branch lines on the ξ-axis by $\{\xi;\ (2n-1)\pi+\theta_0 \le \xi \le 2n\pi-\theta_0\}$ for $n = 0, \pm 1, \pm 2, \cdots$. Let Σ_0 be defined to take the same value as the function Σ on the domain Ω as illustrated in Figure 7. Then using the property that $x+iy = \Sigma(\xi+i\eta)$ if and only if $x - 2n\pi\sqrt{\delta} + iy = \Sigma(\xi + i\eta + 2n\pi)$, one can extend the function Σ_0 to the domain

$$\Omega_n = \{\xi + i\eta + 2n\pi;\ \xi + i\eta \in \Omega\} \qquad \text{for} \qquad n = \pm 1, \pm 2, \cdots,$$

where Ω is the domain bounded by the streamlines $-\sqrt{\epsilon}\,\pi = y(\xi,\eta)$ and $\sqrt{\epsilon}\,\pi = y(\xi,\eta)$, and contained in the strip $\{\xi+i\eta;\ -\frac{\pi}{2} < \xi < \frac{3\pi}{2}, \eta \in \mathbb{R}\}$ as illustrated in Figure 7. To get a complete portrait of the function Σ_0, let us consider its streamlines $y_0 = y(\xi,\eta)$ for $|y_0| > \sqrt{\epsilon}\,\pi$. Using an argument similar to that used to consider the streamline $\sqrt{\epsilon}\,\pi = y(\xi,\eta)$, one may show that the streamline is decreasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and increasing on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$ with $\frac{d}{d\xi}\eta(-\frac{\pi}{2}) = \frac{d}{d\xi}\eta(\frac{\pi}{2}) = \frac{d}{d\xi}\eta(\frac{3\pi}{2}) = 0$, where η is a function of ξ determined by the streamline (4.8) and $\eta(\frac{\pi}{2})$ is determined by the equation $y_0 = -i\Sigma_0\big(\frac{\pi}{2} + i\eta\big)$ as shown in (4.5). Then the property of the stream function $y(\xi,\eta) = y(\xi+2\pi,\eta)$ shows that for each $y_0 \in (-\infty, -\sqrt{\epsilon}\,\pi) \cup (\sqrt{\epsilon}\,\pi, \infty)$, the streamline $y_0 = y(\xi,\eta)$ determines a function $\eta = \eta(\xi)$ defined on the real axis such that $\eta(\xi)$ is a continuous and periodic function with the period 2π. Using the Cauchy-Riemann equations and (4.16) again leads to the conclusion that Σ_0 is a one-to-one mapping of the streamline $y_0 = y(\xi,\eta)$ onto the line $y = y_0$.

Remark. For any integer n, we find that Σ maps $\overline{\Omega}_n$ onto the strip $\{x+iy;\ x \in \mathbb{R}, |y| \le \sqrt{\epsilon}\,\pi\}$. This fact indicates that to create a Riemann surface on which Σ is defined as a conformal mapping onto a manifold containing the strip $\{x+iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$ so that the inverse of this conformal mapping is an extension of the function $\Theta(z)$ defined in Lemma 4.1, Ω_n must be excluded from the construction of the Riemann surface, for $n = \pm 1, \pm 2, \cdots$.

Now we are in the position to extend $\Theta(z)$ to a function defined on the complex plane. First of all, it is important to realize that $\Theta(z)$ has four singularities on the boundary of the strip $\{x+iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$, even though Θ can be extended up to the boundary of the strip continuously. We demonstrate this fact in the following lemma.

Lemma 4.3. *$\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ and $-\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ are singularities of the function $\Theta(z)$ and each of them is a branch point of order three.*

Proof. We choose a closed path γ in the θ-plane, as illustrated in Figure 8, which starts from the point $\theta = \xi_0 + i\eta_0$ on the streamline $y_0 = y(\xi,\eta)$ such that $-\frac{\pi}{2} < \xi_0 < 0$, $\eta_0 < 0$,

$x_0 + iy_0 = \Sigma_0(\xi_0 + i\eta_0)$ with $0 < x_0 < \sqrt{\delta}\,\pi$ and $0 < y_0 < \sqrt{\epsilon}\,\pi$, and both $\sqrt{\delta}\,\pi - x_0$ and $\sqrt{\epsilon}\,\pi - y_0$ are sufficiently small. We take the streamline $y_0 = y(\xi, \eta)$ starting at the point $\xi_0 + i\eta_0$ to the left until reaching to the intersection of the streamline $y_0 = y(\xi, \eta)$ and the equipotential $2\sqrt{\delta}\,\pi - x_0 = x(\xi, \eta)$. Then we go up along this equipotential to cross the ξ-axis and get to the intersection of the equipotential $2\sqrt{\delta}\,\pi - x_0 = x(\xi, \eta)$ and the streamline $2\sqrt{\epsilon}\,\pi - y_0 = y(\xi, \eta)$. Taking this streamline to the right leads to the intersection point of the streamline $2\sqrt{\epsilon}\,\pi - y_0 = y(\xi, \eta)$ and the equipotential $x_0 = x(\xi, \eta)$, from which we take the equipotential $x_0 = x(\xi, \eta)$, going up to cross the streamline $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ and arriving at the intersection of $x_0 = x(\xi, \eta)$ and $y_0 = y(\xi, \eta)$. In this way, the corresponding values of the function Σ describes the rectangle $\Sigma(\gamma)$ with four vertices $x_0 + iy_0$, $2\sqrt{\delta}\,\pi - x_0 + iy_0$, $2\sqrt{\delta}\,\pi - x_0 + i(2\sqrt{\epsilon} - y_0)$ and $x_0 + i(2\sqrt{\epsilon} - y_0)$ in the z-plane as indicated in Figure 8. If one keeps going along the path γ counterclockwise for a complete circuit, then the corresponding trace described by the function Σ in the z-plane finishes three complete circuits of the rectangle $\Sigma(\gamma)$. Since the path γ is contained in the strip $\{\xi + i\eta;\ -\pi < \xi < 0, \eta \in \mathbb{R}\}$, and sine function $\sin\theta$ has the period 2π, $\sqrt{\delta}\,\pi + i\sqrt{\epsilon}\,\pi$ is also a branch point of order three of the function $\varphi = \frac{\delta+\epsilon}{2\delta}(\sin\Theta + \sin\theta_0)$. In the similar way, one may show that the other three points $\sqrt{\delta}\,\pi - i\sqrt{\epsilon}\,\pi$ and $-\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ are also branch points of order three of both functions Θ and φ. □

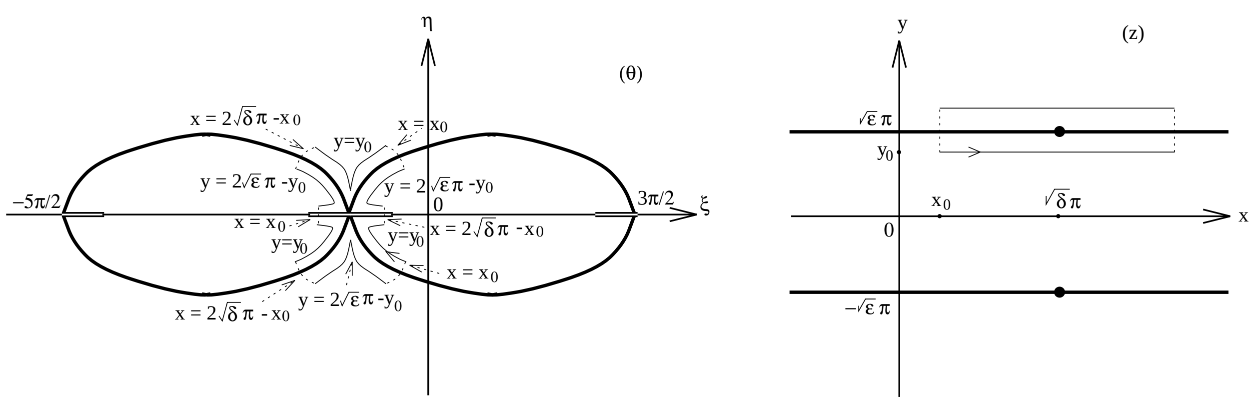

Fig. 8. A sketch of the path γ in the θ-plane and the corresponding path $\Sigma(\gamma)$ in the z-plane

One may also question whether Θ has other singularities on the lines $\{x \pm i\sqrt{\epsilon}\,\pi;\ x \in \mathbb{R}\}$. As a matter of fact, it follows from the derivative of Σ shown in (4.21) and the streamlines $-\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ and $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ portrayed in Figure 7 that $\Theta(z)$ has a local analytic extension from the interior of the strip $\{z \in \mathbb{C};\ |\Im z| < \sqrt{\epsilon}\,\pi\}$ to each point on the lines

$\{x+i\sqrt{\epsilon}\,\pi;\ x\in\mathbb{R}\}$ and $\{x-i\sqrt{\epsilon}\,\pi;\ x\in\mathbb{R}\}$ except the four branch points $\sqrt{\delta}\,\pi\pm i\sqrt{\epsilon}\,\pi$ and $-\sqrt{\delta}\,\pi\pm i\sqrt{\epsilon}\,\pi$. Therefore, to extend $\Theta(z)$ beyond these two lines, one needs to define branch lines connecting these branch points. Different definitions of branch lines lead to different extensions of Θ to the complex plane. In the following theorems, we discuss two distinct extensions of $\Theta(z)$ beyond the strip $\{z\in\mathbb{C};\ |\Im z|\le\sqrt{\epsilon}\,\pi\}$.

Theorem 4.4. *Let Σ_0 be the single-valued branch of Σ defined in (ii), and let*

$$D_0=\bigcup_{n\neq 0}\overline{\Omega}_n\cup\{\xi;\ -\infty<\xi\le-\theta_0\}\cup\{\xi;\ \pi+\theta_0\le\xi<\infty\},$$

where $\overline{\Omega}_n$ is the same as defined in (ii). Then the restriction of Σ_0 to the Riemann surface $X_0=\mathbb{C}\setminus D_0$ is a conformal mapping of X_0 onto the Riemann surface

$$Y_0=\mathbb{C}\setminus\big(\{x+i\sqrt{\epsilon}\,\pi;\ |x|\ge\sqrt{\delta}\,\pi\}\cup\{x-i\sqrt{\epsilon}\,\pi;\ |x|\ge\sqrt{\delta}\,\pi\}\big),$$

and the inverse Θ_0 of the function Σ_0 is an analytic extension of $\Theta(z)$ to the manifold Y_0 such that $\Theta_0(z)$ has countably many branch points $(2k+1)\sqrt{\delta}\,\pi\pm i\sqrt{\epsilon}\,\pi$ for integers $k=0,\pm1,\pm2\cdots$ and when $k\neq-1,0$, the singularities $(2k+1)\sqrt{\delta}\,\pi\pm i\sqrt{\epsilon}\,\pi$ are located at the upper side of the branch line $\{x+i\sqrt{\epsilon}\,\pi;\ |x|\ge\sqrt{\delta}\,\pi\}$ and the lower side of the branch line $\{x-i\sqrt{\epsilon}\,\pi;\ |x|\ge\sqrt{\delta}\,\pi\}$, satisfying $\Theta_0\big((2k+1)\sqrt{\delta}\,\pi\pm i\sqrt{\epsilon}\,\pi\big)=-(2k+1)\pi+\frac{\pi}{2}$.

Proof. It follows from Lemma 4.1 an the discussion in (ii) that $\Sigma_0|_{X_0}$ is a one-to-one mapping of X_0 onto Y_0. Therefore, the inverse of $\Sigma_0|_{X_0}$, denoted by $\Theta_0(z)$, is an analytic function on Y_0, and its restriction to the strip $\{x+iy;\ x\in\mathbb{R},|y|<\sqrt{\epsilon}\,\pi\}$ is $\Theta(z)$. □

Theorem 4.5. *Let Y_1 be the manifold*

$$Y_1=\mathbb{C}\setminus\bigcup_{n=-\infty}^{\infty}\{x+i\sqrt{\epsilon}\,(2n+1)\pi;\ |x|\le\sqrt{\delta}\,\pi\},$$

and let X_1 be the Riemann surface formed by infinitely many layers of the domain Ω defined in Lemma 4.1, having branch lines $\{\xi;\ -\frac{\pi}{2}\le\xi\le-\theta_0\}$ and $\{\xi;\ \pi+\theta_0\le\xi\le\frac{3\pi}{2}\}$, and being pasted in such a way that on any layer of the Riemann surface, if one goes across any of the two branch lines from the lower half plane, one gets to the next lower layer of the Riemann surface; whereas if one goes across any of the branch lines from the upper-half plane, one arrives at the adjacent upper layer of the Riemann surface. Then there exists a conformal mapping of X_1 onto the Riemann surface Y_1 such that its inverse Θ_1 is an analytic extension of Θ from the strip $\{x+iy;\ x\in\mathbb{R},|y|<\sqrt{\epsilon}\,\pi\}$ to Y_1, and

Θ_1 is continuous up to the boundary of Y_1. Moreover, Θ_1 has infinitely many branch points $\pm\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi$ for $n = 0, \pm1, \pm2, \cdots$, such that each pair of branch points $\pm\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi$ are connected by the branch line

$$b_n = \{x + i(2n+1)\sqrt{\epsilon}\,\pi;\ |x| \leq \sqrt{\delta}\,\pi\}$$

which is regarded as a line segment having an upper side and a lower side.

Proof. It follows from Lemma 4.1 and (ii) that Θ is a homeomorphism of the strip $\{x + iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$ onto Ω, whose inverse is Σ_0. Let $\Omega^0 = \Omega$ represent the image of this strip mapped by Θ. As we have pointed out in (i) that $-\theta_0$ and $\pi + \theta_0$ are branch points of infinite order of the multi-valued function Σ. If we take a path going across the branch line $\{\xi;\ -\frac{\pi}{2} \leq \xi \leq -\theta_0\}$ from the lower-half plane and extend the value of Σ_0 continuously after crossing this line, we obtain another single-valued branch of Σ defined on Ω, provided that the extension is restricted inside Ω. We denote this single-valued branch by Σ_1 which maps Ω homeomorphically onto the strip $\{x + iy;\ x \in \mathbb{R}, \sqrt{\epsilon}\,\pi < y < 3\sqrt{\epsilon}\,\pi\}$. Extending Σ_1 continuously to the boundary of Ω, one obtains values of Σ_1 at its upper boundary and the upper side of line segments $\{\xi;\ -\frac{\pi}{2} \leq \xi \leq -\theta_0\}$ and $\{\xi;\ \pi+\theta_0 \leq \xi \leq \frac{3\pi}{2}\}$ as the stream line $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ such that $\Sigma_1(-\theta_0) = \infty + i\sqrt{\epsilon}\,\pi$, $\Sigma_1(-\frac{\pi}{2}) = \sqrt{\delta}\,\pi + i\sqrt{\epsilon}\,\pi$, $\Sigma_1(\frac{3\pi}{2}) = -\sqrt{\delta}\,\pi + i\sqrt{\epsilon}\,\pi$ and $\Sigma_1(\pi + \theta_0) = -\infty + i\sqrt{\epsilon}\,\pi$; while at the lower boundary of Ω and the lower side of line segments $\{\xi;\ -\frac{\pi}{2} \leq \xi \leq -\theta_0\}$ and $\{\xi;\ \pi + \theta_0 \leq \xi \leq \frac{3\pi}{2}\}$, one has the streamline $3\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ of Σ_1 such that $\Sigma_1(-\theta_0) = \infty + i3\sqrt{\epsilon}\,\pi$, $\Sigma_1(-\frac{\pi}{2}) = \sqrt{\delta}\,\pi + i3\sqrt{\epsilon}\,\pi$, $\Sigma_1(\frac{3\pi}{2}) = -\sqrt{\delta}\,\pi + i3\sqrt{\epsilon}\,\pi$ and $\Sigma_1(\pi + \theta_0) = -\infty + i3\sqrt{\epsilon}\,\pi$. Notice that if we choose a path, denoted by γ_1 as shown in Figure 9, and extend the value of Σ_0 by starting from a point on γ_1 in the lower half plane, going clockwise along γ_1 and coming back to the same point after a complete circuit, then the corresponding values form a complete circuit of a rectangle in the z-plane. Therefore, if one define the branch line $\{x + i\sqrt{\epsilon}\,\pi, |x| \leq \sqrt{\delta}\,\pi\}$, then one can extend the function Θ from the strip $\{x + iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$ to the domain $\{x + iy; -\sqrt{\epsilon}\,\pi < y < 3\sqrt{\epsilon}\,\pi\} \setminus \{x + i\sqrt{\epsilon}\,\pi, |x| \leq \sqrt{\delta}\,\pi\}$ uniquely and analytically, such that Θ is a one-to-one mapping of the strip $S_1 = \{x + iy; \sqrt{\epsilon}\,\pi < y < 3\sqrt{\epsilon}\,\pi\}$ onto Ω, whose inverse is Σ_1. We denote the image $\Theta(S_1)$ of the strip S_1 by Ω^1. Using a similar argument, we obtain countably many single-valued branches of the function Σ, denoted by Σ_n for $n = \pm1, \pm2, \cdots$, such that the values of the function Σ_{n+1} on Ω are defined by extending the function Σ_n continuously along any path in Ω, which starts from a point on either the branch line $\{\xi;\ -\frac{\pi}{2} \leq \xi \leq -\theta_0\}$ or the branch line $\{\xi;\ \pi + \theta_0 \leq \xi \leq \frac{3\pi}{2}\}$, and goes up to the upper-half plane; while values of Σ_{n-1} on Ω is defined by extending the function Σ_n continuously along any path in Ω, starting from a point on either the branch line $\{\xi;\, -\frac{\pi}{2} \le \xi \le -\theta_0\}$ or the branch line $\{\xi;\, \pi+\theta_0 \le \xi \le \frac{3\pi}{2}\}$ and going down to the lower half plane. Moreover, for any integer n, Σ_n maps Ω homeomorphically to the strip $S_n = \{x+iy;\, x \in \mathbb{R}, (2n-1)\sqrt{\epsilon}\,\pi < y < (2n+1)\sqrt{\epsilon}\,\pi\}$, whose inverse is an analytic extension of Θ to the strip S_n. We denote $\Theta(S_n)$ by Ω^n. Then the Riemann surface X_1 is formed by countably many domains Ω^n, for $n = 0, \pm 1, \pm 2, \cdots$, in such a way that for any integer n, Ω^n is placed on the top of Ω^{n+1} and they are pasted as described in this theorem. Thus, we obtain an analytic extension of Θ, denoted by Θ_1, as a conformal mapping of Y_1 onto the Riemann surface X_1, which maps each strip S_n homeomorphically to Ω^n contained in the Riemann surface X_1. Furthermore, for each integer n, if we extend the function Θ_1 from the strip S_n to the branch line b_n, then values of Θ_1 on the lower side of b_n form the lower boundary of Ω^n which is a portion of the streamline $(2n+1)\sqrt{\epsilon}\,\pi = y_n(\xi,\eta)$ of the function Σ_n, with $\Theta_1(-\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi) = \frac{3\pi}{2}$ and $\Theta_1(\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi) = -\frac{\pi}{2}$; if we extend the function Θ_1 from the strip S_{n+1} to the branch line b_n, then values of Θ_1 defined on the upper side of b_n form the upper boundary of Ω^{n+1} as a part of the streamline $(2n+1)\sqrt{\epsilon}\,\pi = y_{n+1}(\xi,\eta)$ of the function Σ_{n+1}, also satisfying relations $\Theta_1(-\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi) = \frac{3\pi}{2}$ and $\Theta_1(\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi) = -\frac{\pi}{2}$. Therefore, if we regard each branch line b_n as a cut with an upper side and a lower side, then Θ_1 also has a continuous extension to the boundary $\bigcup_{n=-\infty}^{\infty} b_n$ of Y_1. □

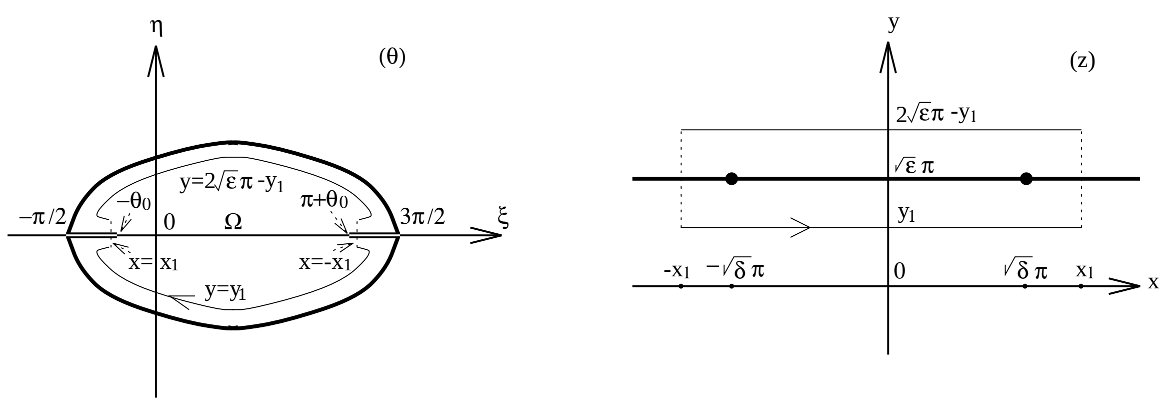

Fig. 9. A sketch of the path γ_1 in the θ-plane and the corresponding path $\Sigma(\gamma_1)$ in the z-plane

Remark. It follows from the definition of the function Θ_1 in the last theorem that Θ_1 can also be regarded as a periodic function with the period $T = i2\sqrt{\epsilon}\,\pi$, mapping each strip $\overline{S_n}$ homeomorphically onto the domain $\overline{\Omega}$. In Part III, we will use this property to discuss how solitary wave solutions extended as analytic functions defined on a Riemann surface converge to either a compacton or a solitary wave solution of the KdV equation as ϵ or δ approaches zero.

As we have seen in Section 3, the dynamical structure of Equation (1.1) changes noticeably when the sign of the parameter ν changes from positive to negative. In the next section, we shall show that solitary wave solutions of Equation (1.1) in case $\nu < 0$ also have different singularity distribution from that of solitary wave solutions in case $\nu > 0$ when they are extended as analytic functions on a Riemann surface.

5. Analytic extensions of solitary wave solutions for $\nu < 0$.

Suppose that the coefficients of Equation (3.17) satisfy the conditions

$$-(\beta + c\nu) < -\frac{\nu}{3}(\alpha + c) < a < \frac{1}{2}(\beta - \nu\alpha) \quad \text{and} \quad \nu < 0.$$

Then (3.17) has an orbitally unique and analytic solitary wave solution ϕ_a defined on the real axis as demonstrated in Section 3. Rescaling (3.17) by the transformation $\phi_a = [3a + \nu(\alpha + c)]\varphi$ reduces it to the equation

$$(\delta\varphi + \rho)(\varphi')^2 = \varphi^2(\varphi + 1), \tag{5.1}$$

where $\delta = -\nu > 0$, $\rho = \frac{-\nu(\beta + c\nu + a)}{3a + \nu(\alpha + c)} > \delta$. Then the corresponding solitary wave solution of (5.1) is symmetric with respect to its depression, having the amplitude $A = 1$ with $\varphi < 0$ and monotonically approaching zero at infinity. Since solutions to (5.1) are translation invariant, without loss of generality, we let the solitary wave solution φ be an even function, which satisfies the integral equation

$$-x = \int_{-1}^{\varphi} \frac{\sqrt{\delta\varphi + \rho}\, d\varphi}{\varphi\sqrt{\varphi + 1}}$$

for $x \geq 0$. Let

$$\varphi = \frac{\rho - \delta}{2\delta}\cosh t - \frac{\rho + \delta}{2\delta}. \tag{5.2}$$

Substituting it in the above integral yields the following expressions

$$\begin{aligned} -x &= \int_0^t \left[\sqrt{\delta} + \frac{\sqrt{\rho}\sinh t_0}{\cosh t - \cosh t_0}\right] dt = \int_0^t \frac{\sqrt{\delta}(\cosh t + 1)}{\cosh t - \cosh t_0}\, dt \\ &= \sqrt{\delta}\, t + \sqrt{\rho}\log\frac{\tanh\frac{t_0}{2} - \tanh\frac{t}{2}}{\tanh\frac{t_0}{2} + \tanh\frac{t}{2}}, \end{aligned} \tag{5.3}$$

where $\cosh t_0 = \frac{\rho+\delta}{\rho-\delta}$, $t_0 > 0$. Let

$$\Delta(t) = -\big(\sqrt{\delta}\, t + \sqrt{\rho} \log \frac{\tanh \frac{t_0}{2} - \tanh \frac{t}{2}}{\tanh \frac{t_0}{2} + \tanh \frac{t}{2}}\big). \tag{5.4}$$

We shall extend the function Δ to the complex plane, which in turn will provide information to find an analytic extension of the inverse $t = \Xi(x)$ of Δ, satisfying Equation (5.3).

Since

$$\frac{\tanh \frac{t_0}{2} - \tanh \frac{t}{2}}{\tanh \frac{t_0}{2} + \tanh \frac{t}{2}} = e^{-\frac{x+\sqrt{\delta}t}{\sqrt{\rho}}},$$

replacing t and x with $\xi + i\eta$ and $x + iy$ in the above equation, respectively, and rewriting its left-hand side as a sum of its real part and imaginary part, one obtains

$$\frac{\cosh t_0 \cos \eta - \cosh \xi - i \sinh t_0 \sin \eta}{\cosh(t_0 + \xi) - \cos \eta} = e^{-\frac{x+\sqrt{\delta}\xi + i(y+\sqrt{\delta}\eta)}{\sqrt{\rho}}}.$$

Comparing angles and norms on both sides of this equation yields the two relations

$$\frac{\sinh t_0 \sin \eta}{\cosh t_0 \cos \eta - \cosh \xi} = \tan \frac{y + \sqrt{\delta}\, \eta}{\sqrt{\rho}}, \tag{5.5}$$

and

$$\frac{\cosh(t_0 - \xi) - \cos \eta}{\cosh(t_0 + \xi) - \cos \eta} = e^{-\frac{2(x+\sqrt{\delta}\,\xi)}{\sqrt{\rho}}}, \tag{5.6}$$

which provide equations of streamlines and equipotentials of the function Δ, respectively. Similar to the technique used to study the function Σ, starting from the imaginary axis $t = i\eta$ to extend $\Delta(t)$, one has the expression

$$y = -i\Delta(i\eta) = -\sqrt{\delta}\, \eta + \sqrt{\rho}\, 2n\pi + 2\sqrt{\rho} \tan^{-1}\big(\sqrt{\frac{\rho}{\delta}} \tan \frac{\eta}{2}\big), \tag{5.7}$$

for any $\eta \in \big((2n-1)\pi, (2n+1)\pi\big]$ with $n = 0, \pm 1, \pm 2, \cdots$, which leads to the determination of streamlines $y_0 = y(\xi, \eta)$ of the function Δ. As illustrated in Figure 10, when $y_0 \in (-(\sqrt{\rho} - \sqrt{\delta}\,)\pi, (\sqrt{\rho} - \sqrt{\delta}\,)\pi)$, the streamline $y_0 = y(\xi, \eta)$ is a smooth curve connecting the points $-t_0$ and t_0 on the ξ-axis, and Δ maps the streamline diffeomorphically to the line $\{x + iy_0,\, x \in \mathbb{R}\}$, on which $\Delta(-t_0) = -\infty + iy_0$, $\Delta(i\eta_0) = iy_0$, and $\Delta(t_0) = \infty + iy_0$, where y_0 and η_0 satisfy (5.7) with $n = 0$. If $y_0 = (\sqrt{\rho} - \sqrt{\delta}\,)\pi$, the streamline is a curve in the upper-half plane connecting points $t = -t_0, \pi i$ and t_0, having a corner at the point $t = \pi i$, and Δ is a homeomorphism of this streamline to the line $\{x + i(\sqrt{\rho} - \sqrt{\delta}\,)\pi;\, x \in \mathbb{R}\}$. On the other hand, for $y_0 = -(\sqrt{\rho} - \sqrt{\delta}\,)\pi$, the streamline is a curve symmetric to the streamline for $y_0 = (\sqrt{\rho} - \sqrt{\delta}\,)\pi$ with respect to the ξ-axis. For each $x_0 > 0$, the equipotential $x_0 = x(\xi, \eta)$ is a closed loop in the right-half plane which starts from a point to the right of the point t_0 on the ξ-axis, going around t_0 clockwise once for a complete circuit. In this way, Δ is a one-to-one map of the equipotential onto the line segment $\{x_0 + iy;\ |y| < \sqrt{\rho}\pi\}$; while the equipotential $-x_0 = x(\xi, \eta)$ is symmetric to the equipotential $x_0 = x(\xi, \eta)$ with respect to the η-axis. The proof of these facts and the following lemma concerning analytic extensions of Δ and its inverse function will be shown in the appendix of this paper to avoid tedious details here.

Lemma 5.1. *Let Δ be the function defined in (5.4). Then as a homeomorphism of the line segment $\{\xi;\ -t_0 < \xi < t_0\}$ onto the real axis, Δ has an analytic extension to the region D bounded by the streamline $(\sqrt{\rho} - \sqrt{\delta}\,)\pi = y(\xi, \eta)$, connecting points $t = -t_0, \pi i$ and t_0, and the streamline $-(\sqrt{\rho} - \sqrt{\delta}\,)\pi = y(\xi, \eta)$, connecting points $t = -t_0, -\pi i$ and t_0. Moreover, the extension of Δ is a one-to-one mapping of D onto the strip*

$$\mathcal{S}_0 = \{x + iy;\ x \in \mathbb{R}, |y| < (\sqrt{\rho} - \sqrt{\delta}\,)\pi\},$$

whose inverse $t = \Xi(z)$ is an analytic function defined on the strip $\mathcal{S}_0$ and continuous up to its boundary such that $t = \Xi(z)$ satisfies Equation (5.3). In consequence, the transformation (5.2) leads to the analytic extension of the solitary wave solution φ of (5.1) to the strip $\mathcal{S}_0$ as well.

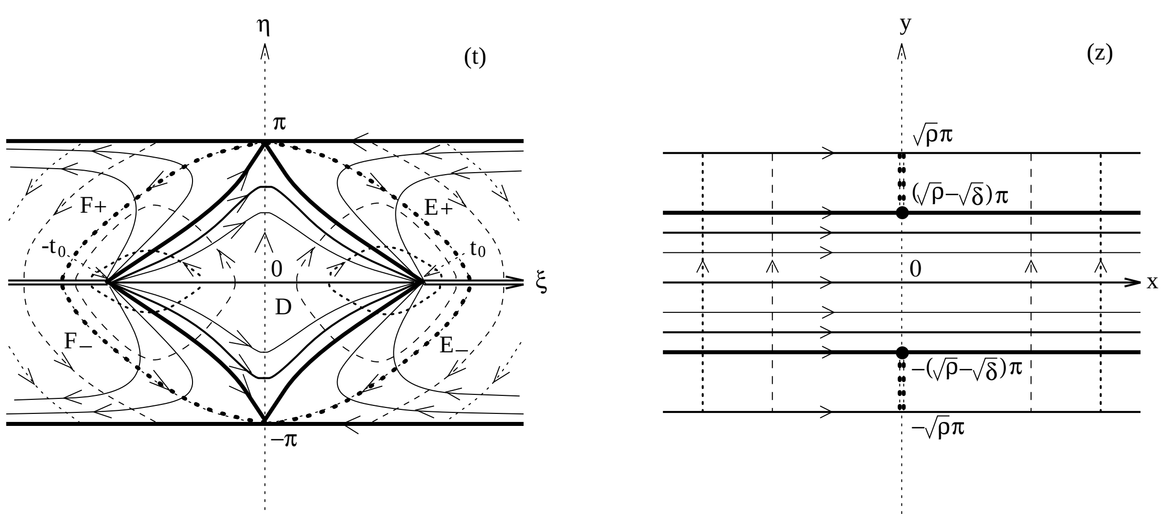

Fig. 10. Streamlines and equipotentials of the function Δ when $\delta < \rho \leq 3\delta$

Similar to the discussion of solitary wave solutions in Section 4, now we summarize properties of the function Δ, including its singularities and zeros of it derivatives, and single-valued branches, which will lead us to find singularities of the function Ξ on the boundary of the trip $\mathcal{S}_0$ and its further extension to the complex plane.

(a) *Singularities of* Δ *and zeros of* $\frac{d\Delta}{dt}$. It follows from the derivative $\frac{d\Delta}{dt} = \frac{\sqrt{\delta}(\cosh t+1)}{\cosh t_0 - \cosh t}$ that Δ has singularities $t = \pm t_0 + 2n\pi i$ for $n = 0, \pm 1, \pm 2, \cdots$. If integrating $\frac{d\Delta}{dt}$ along a closed Jordan curve Γ counterclockwise once, whose interior contains only one singularity $t = t_0 + 2n\pi i$ of Δ for some integer n, then one obtains

$$\int_\Gamma \frac{d\Delta}{dt} dt = \int_\Gamma \frac{\sqrt{\delta}(\cosh t + 1)}{\cosh t_0 - \cosh t} dt = -2\pi\sqrt{\rho}\, i.$$

If $\widetilde{\Gamma}$ is a closed path whose interior contains only the singularity $t = -t_0 + 2n\pi i$ of Δ, then the integration of $\frac{d\Delta}{dt}$ along $\widetilde{\Gamma}$ counterclockwise once yields the result

$$\int_{\widetilde{\Gamma}} \frac{d\Delta}{dt} dt = \int_{\widetilde{\Gamma}} \frac{\sqrt{\delta}(\cosh t + 1)}{\cosh t_0 - \cosh t} dt = 2\pi\sqrt{\rho}\, i.$$

Therefore, $t = \pm t_0 + 2n\pi i$ are branch points of infinite order of Δ, and thus Δ is a multi-valued function. On the other hand, zeros of $\frac{d\Delta}{dt}$ are $t = (2k+1)\pi i$ for $k = 0, \pm 1, \pm 2, \cdots$. Because of this reason, Δ is not angle-preserving at these points. As illustrated in Figure 10, the streamlines $(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ and $-(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ joining the points $t = \pm t_0$ and forming the boundary of the region D have corners at $t = \pi i$ and $t = -\pi i$, respectively. As a matter of fact, Δ is not one-to-one in some neighbourhood of these points either, because $\eta = \pi i$ is also the streamline $(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ such that $\Delta(-\infty + \pi i) = \infty + (\sqrt{\rho} - \sqrt{\delta})\pi i$, $\Delta(\pi i) = (\sqrt{\rho} - \sqrt{\delta})\pi i$ and $\Delta(\infty + \pi i) = -\infty + (\sqrt{\rho} - \sqrt{\delta})\pi i$; while $\eta = -\pi i$ is another streamline $-(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ such that $\Delta(-\infty - \pi i) = \infty - (\sqrt{\rho} - \sqrt{\delta})\pi i$, $\Delta(-\pi i) = -(\sqrt{\rho} - \sqrt{\delta})\pi i$ and $\Delta(\infty - \pi i) = -\infty - (\sqrt{\rho} - \sqrt{\delta})\pi i$, which indicates that the open set we shall seek and use to construct the Riemann surface on which Δ is a conformal mapping has to be contained in the strip $\{\xi + \eta;\ \xi \in \mathbb{R}, |\eta| < \pi\}$.

Next, we determine a single-valued branch Δ_0 of the function Δ defined on the complex plane, whose restriction to the line segment $\{t = \xi;\ |\xi| < t_0\}$ is the same as the function defined in (5.4). Then other single-value branches may be found by using Δ_0.

(b) *A single-valued branch* Δ_0 *of* Δ. Since Δ has branch points $\pm t_0 + 2n\pi i$, first of all, we define branch lines by $l_n = \{\xi + 2n\pi i;\ |\xi| \geq t_0\}$ for $n = 0, \pm 1, \pm 2, \cdots$, so that the integration of $\frac{d\Delta}{dt}$ along any closed path contained in the open set $\mathbb{C} \setminus \bigcup_{-\infty}^{\infty} l_n$ is zero, *i.e.* values of the function Δ_0 can be uniquely determined on the manifold $\mathbb{C} \setminus \bigcup_{-\infty}^{\infty} l_n$. Because $D \subset \mathbb{C} \setminus \bigcup_{-\infty}^{\infty} l_n$, the function Δ_0 is shown to be a homeomorphism of D onto the strip $\mathcal{S}_0$ in Lemma 5.1, whose streamlines and equipotentials are sketched in Figure 10. It is worth mentioning that in Figure 10 the curve E_- connecting the point $t = -\pi i$ and a point to the right of $t = t_0$ is part of the equipotential $0 = x(\xi,\eta)$ such that $\Delta_0(E_-) = \{iy;\ -\sqrt{\rho}\,\pi < y < -(\sqrt{\rho}-\sqrt{\delta}\,)\pi\}$, and the curve E_+ symmetric to E_- with respect to the ξ-axis is also a portion of the equipotential $0 = x(\xi,\eta)$ such that Δ_0 is a one-to-one mapping of E_+ onto the line segment $\{iy;\ (\sqrt{\rho}-\sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi\}$. On the other hand, the curves F_- and F_+ symmetric to E_- and E_+ with respect to the imaginary axis have the same image as E_- and E_+, respectively, *i.e.* both $E_-\bigcup\{i\eta;\ |\eta| \le \pi\} \cup E_+$ and $F_-\bigcup\{i\eta;\ |\eta| \le \pi\} \cup F_+$ are the equipotential $0 = x(\xi,\eta)$ such that Δ_0 maps each of them homeomorphically onto the line segment $\{i\eta;\ |\eta| \le \sqrt{\rho}\,\pi\}$. In addition, solid lines passing through the curves E_+ and F_+ in Figure 10 are streamlines $y_0 = y(\xi,\eta)$ for some $y_0 \in ((\sqrt{\rho}-\sqrt{\delta}\,)\pi, \sqrt{\rho}\,\pi)$, and solid lines passing through the curves E_- and F_- are streamlines $y_0 = y(\xi,\eta)$ for some $y_0 \in (-\sqrt{\rho}\,\pi, -(\sqrt{\rho}-\sqrt{\delta}\,)\pi)$. For each of them, Δ_0 is a homeomorphism onto the line $\{x+iy_0;\ x \in \mathbb{R}\}$. Dotted and dashed lines to the right of curves E_- and E_+ are equipotentials $x_0 = x(\xi,\eta)$ for some $x_0 < 0$, which are mapped homeomorphically onto the lines $\{x_0+iy;\ -\sqrt{\rho}\,\pi < y < -(\sqrt{\rho}-\sqrt{\delta}\,)\pi\}$ and $\{x_0+iy;\ (\sqrt{\rho}-\sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi\}$, respectively. Symmetrically, dotted and dashed lines to the left of curves F_- and F_+ are equipotentials $x_0 = x(\xi,\eta)$ for some $x_0 > 0$, whose images are also line segments $\{x_0+iy;\ -\sqrt{\rho}\,\pi < y < -(\sqrt{\rho}-\sqrt{\delta}\,)\pi\}$ and $\{x_0+iy;\ (\sqrt{\rho}-\sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi\}$, respectively.

To determine values of Δ_0 as a function defined on the complex plane, one may use the following two properties of its streamlines and equipotentials: 1) the equation $y_0 = y(\xi,\eta)$ holds if and only if the equation $y_0+2(\sqrt{\rho}-\sqrt{\delta}\,)\pi = y(\xi,\eta+2\pi)$ is valid, and 2) $x_0 = x(\xi,\eta)$ if and only if $x_0 = x(\xi,\eta+2\pi)$. In consequence, for any $\xi+i\eta \in \mathbb{C}\setminus \bigcup_{-\infty}^{\infty} l_n$, there is an integer n such that $(2n-1)\pi \le \eta < (2n+1)\pi$, and $\Delta_0(\xi+i\eta) = \Delta_0\big(\xi+i(\eta-2n\pi)\big)+2n(\sqrt{\rho}-\sqrt{\delta}\,)\pi i$. Moreover, if each of the branch lines l_n, for $n = 0, \pm1, \pm2, \cdots$, is regarded as having an upper side and a lower side, then Δ_0 also has a continuous extension to these lines as the boundary of the manifold $\mathbb{C} \setminus \bigcup_{-\infty}^{\infty} l_n$.

Before finding a further extension of the function Ξ beyond the strip $\mathcal{S}_0$, we need to point out in the following lemma that Ξ has singularities on the boundary of $\mathcal{S}_0$, which has an effect on how to determine a further extension for Ξ.

Lemma 5.2. *The function Ξ has singularities at $z = \pm i(\sqrt{\rho}-\sqrt{\delta}\,)\pi$ which are branch points of order three.*

Proof. Since $\Xi\big((\sqrt{\rho}-\sqrt{\delta}\,)\pi i\big) = \pi i$ and $\Xi|_{\mathcal{S}_0} = (\Delta_0|_D)^{-1}$, to show that $z = (\sqrt{\rho}-\sqrt{\delta}\,)\pi i$

is a branch point of Ξ, we choose a closed path γ_2 consisting of segments of streamlines and equipotentials of the single-valued branch Δ_0 such that its interior contains $t = \pi i$ as sketched in Figure 11. We observe corresponding values of Δ_0 when the value of t changes along the path γ_2 in counterclockwise direction. Because the trace described by $\Delta_0(t)$ forms three complete circuits of a rectangle in the z-plane with its interior containing the point $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ after t has finished going along the path γ_2 for a complete circuit, it follows that the point $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is a branch point of order three of the function Ξ. Noticing that $\cosh t$ is a periodic function with the period $T = 2\pi i$, the solitary wave solution to (5.1) is given by the transformation $\varphi = \frac{\rho-\delta}{2\delta}(\cosh t - \cosh t_0)$ with $t = \Xi(z)$ and the path γ_2 is contained in the strip $\{\xi + i\eta;\ \xi \in \mathbb{R},\ \frac{\pi}{2} < \eta < \frac{3\pi}{2}\}$, one concludes that $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is also a branch point of order three of the solitary wave solution φ. To verify that $z = -(\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is a branch point of order three, one may construct a path symmetric to the path γ_2 with respect to the ξ-axis and then use a similar argument to get the conclusion. $\square$

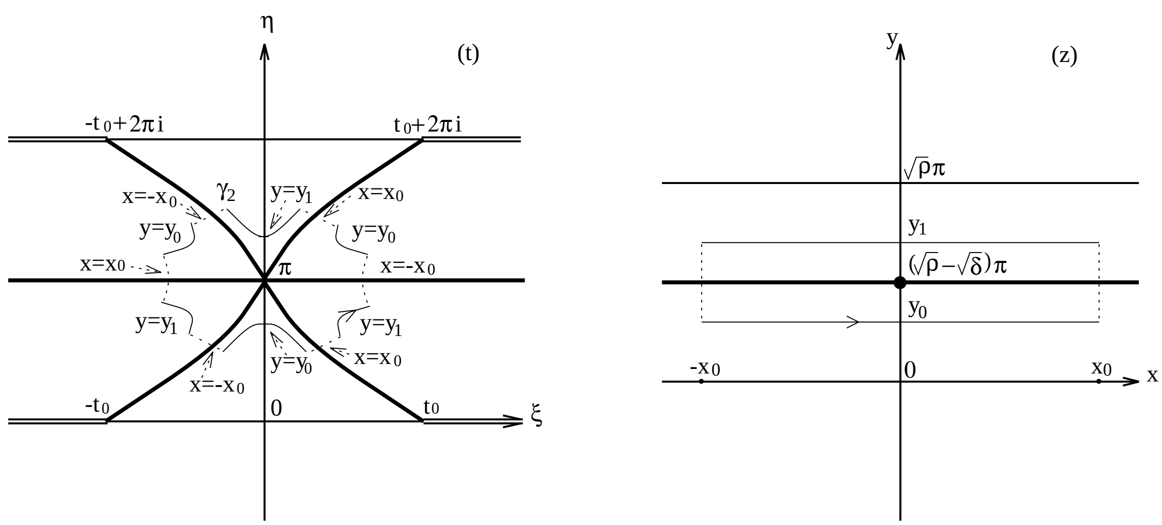

Fig. 11. The path γ_2 and the corresponding rectangle in the z-plane

It follows from Lemma 5.2 that any extension of solitary wave solutions of Equation (5.1) beyond the strip $\mathcal{S}_0$ will depend on how branch lines are defined in the complex plane. In the following theorems, we give two different definitions of branch lines and consequent extensions of $\Xi(z)$.

Theorem 5.3. *Let $\mathcal{D}_0$ be the open set bounded by the curves $E_\pm$, $F_\pm$ and branch lines $s_+ = \{t = \xi;\ t_0 \le \xi \le \xi_0\}$ and $s_- = \{t = \xi;\ -\xi_0 \le \xi \le -t_0\}$, where ξ_0 is the intersection* *of E_+ and the ξ-axis as shown in Figure 12. Then the function $t = \Xi(z)$ has a further extension $t = \Xi_1(z)$ to the manifold*

$$\mathcal{Y}_1 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{iy;\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le \big((2n+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\},$$

such that $t = \Xi_1(z)$ is a homeomorphism of the open set

$$\begin{aligned}\mathcal{E}_n =&\{x+iy;\ x \in \mathbb{R},\ (2n-1)\sqrt{\rho}\,\pi < y < (2n+1)\sqrt{\rho}\,\pi\}\\ &\setminus \{iy;\ (2n-1)\sqrt{\rho}\,\pi \le y \le \big((2n-1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi,\\ &\qquad or\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le (2n+1)\sqrt{\rho}\,\pi\}\end{aligned}$$

onto $\mathcal{D}_n = \mathcal{D}_0$ for integers $n = 0, \pm 1, \pm 2, \cdots$, and it is a conformal mapping of $\mathcal{Y}_1$ onto the Riemann surface $\mathcal{X}_1$ which is constructed by pasting countably many domains $\mathcal{D}_n$ as layers in such a way that for any integer n, on the layer $\mathcal{D}_n$ of $\mathcal{X}_1$, if one goes across any of the two branch lines $s_\pm$ from the lower half plane, one gets to the next lower layer $\mathcal{D}_{n-1}$ of the Riemann surface; whereas if one goes across any of the branch lines $s_\pm$ from the upper-half plane, one arrives at the adjacent upper layer $\mathcal{D}_{n+1}$ of the Riemann surface. Furthermore, if each branch line

$$\{iy;\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le \big((2n+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\}$$

connecting the branch points $\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,\big)\pi$ of Ξ_1, for $n = 0, \pm 1, \pm 2, \cdots$, is regarded as a cut with a left side and a right side, then the function Ξ_1 is also continuous up to the boundary $\bigcup_{n=-\infty}^{\infty} \{iy;\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le \big((2n+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\}$ of $\mathcal{Y}_1$.

Proof. The proof is similar to that of Theorem 4.5. We begin with the single-valued branch Δ_0 which is shown in *(b)* to be a homeomorphism of $\overline{\mathcal{D}_0}$ onto $\overline{\mathcal{E}_0}$ such that Δ_0 is a one-to-one mapping of F_+ and E_+ onto the left side and the right side of the line segment $\{iy;\ (\sqrt{\rho} - \sqrt{\delta}\,)\pi \le y \le \sqrt{\rho}\,\pi\}$, respectively, and it is also a one-to-one mapping of F_- and E_- onto the left side and the right side of the line segment $\{iy;\ -\sqrt{\rho}\,\pi \le y \le -(\sqrt{\rho} - \sqrt{\delta}\,)\pi\}$, respectively. Moreover, Δ_0 maps the upper side and the lower side of the branch line s_- onto the lines $\{x + i\sqrt{\rho}\,\pi;\ x \le 0\}$ and $\{x - i\sqrt{\rho}\,\pi;\ x \le 0\}$, respectively, and it maps the upper side and the lower side of the branch line s_+ onto the lines $\{x + i\sqrt{\rho}\,\pi;\ x \ge 0\}$ and $\{x - i\sqrt{\rho}\,\pi;\ x \ge 0\}$, respectively. If we take any path going across either the line s_- or the line s_+ from the upper half domain $\mathcal{D}_0$ to the lower half domain $\mathcal{D}_0$ and extend values of Δ_0 continuously after crossing the line, then we obtain another single-valued branch Δ_1 of the function Δ defined on $\mathcal{D}_0$ such that

$\Delta_1 = \Delta_0 + 2\sqrt{\rho}\,\pi i$, Δ_1 is a homeomorphism of $\overline{\mathcal{D}_0}$ onto $\overline{\mathcal{E}_1}$ and its inverse as a function defined on $\overline{\mathcal{E}_1}$ is an extension of $t = \Xi(z)$ from $\overline{\mathcal{E}_0}$ to $\overline{\mathcal{E}_0 \cup \mathcal{E}_1}$, denoted by $\Xi_1(z)$. Noticing that values of $\Delta(t)$ along the path γ_3, which starts from a point t_1 in the upper half plane with $\Delta(t_1) = \Delta_0(t_1)$ and proceeds in clockwise direction for a full circuit, form of a rectangle in the z-plane as illustrated in Figure 12, one concludes that the value of $\Xi_1(z)$ does not change after z goes around the rectangle once, whose interior contains branch points $(\sqrt{\rho} \pm \sqrt{\delta}\,)\pi$. Therefore, if we define $\{iy;\, (\sqrt{\rho} - \sqrt{\delta}\,)\pi \le y \le (\sqrt{\rho} + \sqrt{\delta}\,)\pi\}$ as a branch line, then the extension Ξ_1 of Ξ to the manifold $\overline{\mathcal{E}_0 \cup \mathcal{E}_1}$ is a continuous and single-valued function and analytic in the interior of the manifold. In a similar way, one may use the single-valued branch Δ_n of Δ, which is a homeomorphism of $\overline{\mathcal{D}_0}$ onto the manifold $\overline{\mathcal{E}_n}$, to define the extension $\Xi_1(z)$ to $\mathcal{E}_n$ for any integer n, such that $\Xi_1|_{\overline{\mathcal{E}_n}} = (\Delta_n|_{\overline{\mathcal{D}_0}})^{-1}$, Ξ_1 is a conformal mapping of $\mathcal{Y}_1$ onto $\mathcal{X}_1$ and is continuous up to the boundary of $\mathcal{Y}_1$. $\square$

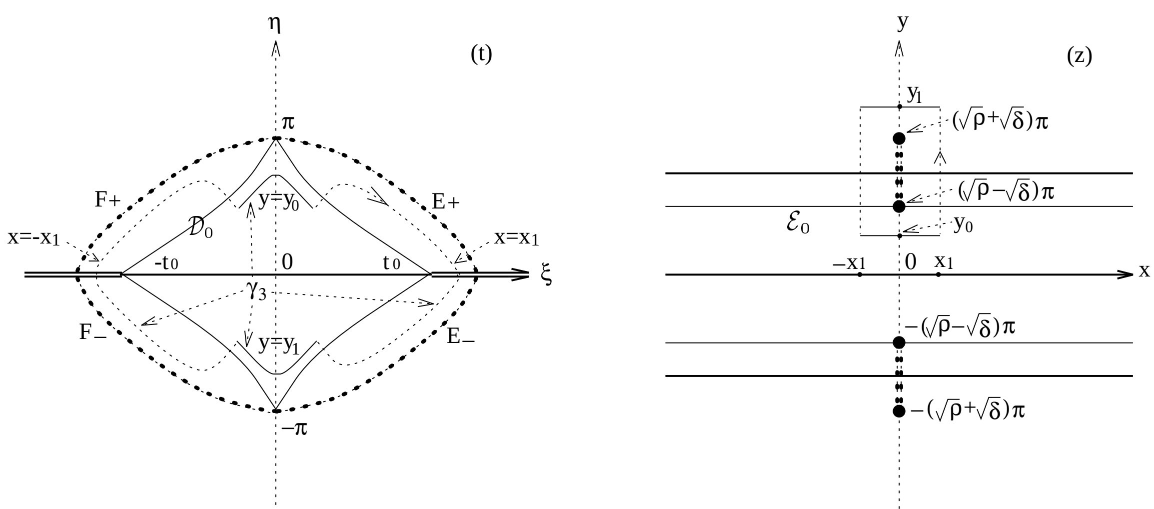

Fig. 12. The path γ_3 and the corresponding rectangle in the z-plane

Another extension of $\Xi(z)$ is also obtained by using single-valued branch of Δ as follows. Let $\mathcal{G}_0$ be the open set bounded by the lines $\{\xi + i\pi;\, \xi \ge 0\}$, $\{\xi - i\pi;\, \xi \ge 0\}$ and $\{t = \xi;\, t_0 \le \xi < \infty\}$, as well as part of the streamline $(\sqrt{\rho} - \sqrt{\delta}\,)\pi = y(\xi, \eta)$ connecting points $t = -t_0, \pi i$ and part of the streamline $-(\sqrt{\rho} - \sqrt{\delta}\,)\pi = y(\xi, \eta)$ connecting points $t = -t_0, -\pi i$ in the left half plane as illustrated in Figure 13. As we have pointed out in Lemma 5.1 that Δ_0 is a conformal mapping of the open set $D \subset \mathcal{G}_0$ onto the strip $\{x + iy;\, x \in \mathbb{R}, |y| < (\sqrt{\rho} - \sqrt{\delta}\,)\pi\}$. In addition, Δ_0 is a one-to-one mapping of the streamlines $l_\pm$ onto the lines $\{x \pm i(\sqrt{\rho} - \sqrt{\delta}\,)\pi;\, x \in \mathbb{R}\}$, respectively, where l_+ is the streamline $(\sqrt{\rho} - \sqrt{\delta}\,)\pi = y(\xi, \eta)$ connecting points $t = -t_0, \pi i$ and t_0, and l_- is the streamline $-(\sqrt{\rho}-\sqrt{\delta}\,)\pi = y(\xi,\eta)$ connecting points $t=-t_0, -\pi i$ and t_0. If extending Δ_0 to the entire open set $\mathcal{G}_0$, one finds that on the right-hand side of the streamline l_+, streamlines $y_0 = y(\xi,\eta)$ coming from infinity and ending up at the point $t=t_0$ are those for some $y_0 \in \big((\sqrt{\rho}-\sqrt{\delta}\,)\pi, \sqrt{\rho}\,\pi\big)$ such that Δ_0 is a homeomorphism of $y_0 = y(\xi,\eta)$ onto the line $\{x+iy_0;\, x\in\mathbb{R}\}$. In addition, the upper side of the line $\{t=\xi;\, t_0<\xi<\infty\}$ is the streamline $\sqrt{\rho}\,\pi = y(\xi,0)$ of Δ_0. Symmetrically, on the right-hand side of the streamline l_-, streamlines are those for some $y_0 \in \big(-\sqrt{\rho}\,\pi, -(\sqrt{\rho}-\sqrt{\delta}\,)\pi\big)$ and the lower side of the line $\{t=\xi;\, t_0<\xi<\infty\}$ is the streamline $-\sqrt{\rho}\,\pi = y(\xi,0)$ such that Δ_0 maps each of these streamlines homeomorphically onto the line $\{x+iy_0;\, x\in\mathbb{R}\}$. Furthermore, Δ_0 maps the boundary $\{\xi+\pi i;\, \xi\geq 0\}$ of $\mathcal{G}_0$ onto the line $\{x+(\sqrt{\rho}-\sqrt{\delta}\,)\pi i;\, x\leq 0\}$ with $\Delta_0(i\pi) = (\sqrt{\rho}-\sqrt{\delta}\,)\pi i$ and $\Delta_0(\infty+\pi i) = -\infty+(\sqrt{\rho}-\sqrt{\delta}\,)\pi i$. It also maps the boundary $\{\xi-\pi i;\, \xi\geq 0\}$ onto the line $\{x-(\sqrt{\rho}-\sqrt{\delta}\,)\pi i;\, x\leq 0\}$ with $\Delta_0(-i\pi) = -(\sqrt{\rho}-\sqrt{\delta}\,)\pi i$ and $\Delta_0(\infty-\pi i) = -\infty-(\sqrt{\rho}-\sqrt{\delta}\,)\pi i$. Therefore, Δ_0 is a conforming mapping of $\mathcal{G}_0$ onto the open set

$$\mathcal{H}_0 = \{x+iy;\, x\in\mathbb{R}, |y|<\sqrt{\rho}\,\pi\} \setminus \{x\pm(\sqrt{\rho}-\sqrt{\delta}\,)\pi i;\, x\leq 0\}$$

where $\{x-(\sqrt{\rho}-\sqrt{\delta}\,)\pi i;\, x\leq 0\}$ and $\{x+(\sqrt{\rho}-\sqrt{\delta}\,)\pi i;\, x\leq 0\}$ are branch lines of the function $\Xi(z)$ and are considered as lines with upper sides and lower sides. In consequence, we extend the function $\Xi(z)$ to the closed set $\overline{\mathcal{H}_0}$, whose inverse is Δ_0 defined on $\overline{\mathcal{G}_0}$ such that the extension Ξ_2 of Ξ is a conformal mapping of $\mathcal{H}_0$ onto $\mathcal{G}_0$ and continuous up to the boundary of $\mathcal{H}_0$.

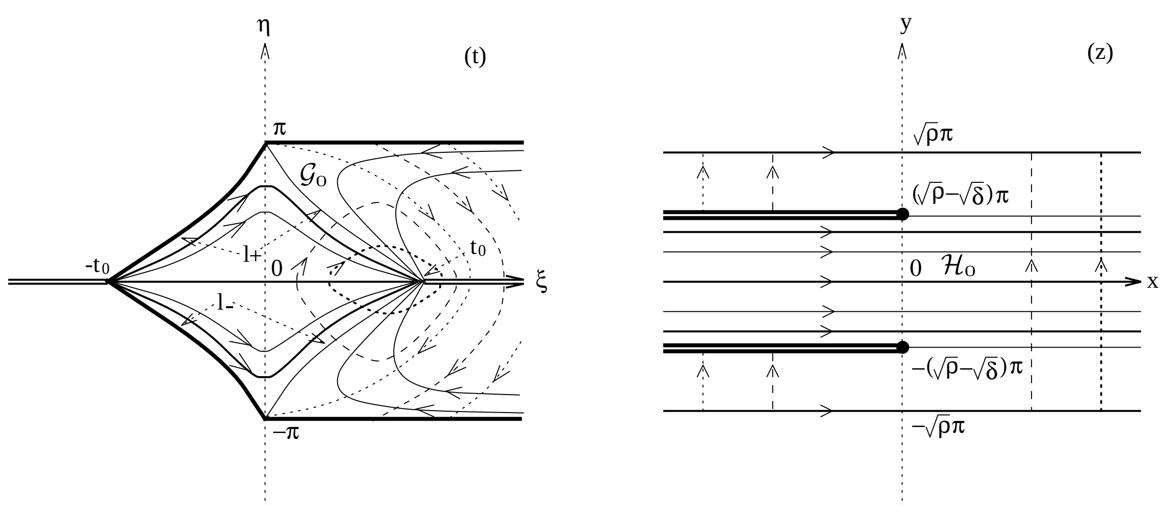

Fig. 13. A sketch of sets $\mathcal{G}_0$ in the t-plane and $\mathcal{H}_0$ in the z-plane

Using the single-valued branch Δ_0 of the function Δ, one may obtain other single-valued branches of Δ by extending Δ_0 along a path going across the branch line $\{t = \xi;\, t_0 \le \xi < \infty\}$. Precisely speaking, for any integer $n > 0$, the value of the single-valued branch Δ_n at any point $\tilde{t} \in \mathcal{G}_0$ is defined by continuous extension of the value of Δ_0 along any closed path in $\mathcal{G}_0$, whose interior contains t_0 and starts from the point $\tilde{t}$, going around the path clockwise for n complete circuits and then coming back to the point $\tilde{t}$. It follows from the discussion on singularities of Δ in (a) that $\Delta_n(\tilde{t}) = \Delta_0(\tilde{t}) + 2\sqrt{\rho}\, n\pi i$. Similarly, the value of the single-valued branch Δ_{-n} at the point $\tilde{t}$ is defined by continuous extension of the value of Δ_0 along any closed path in $\mathcal{G}_0$, whose interior contains t_0 and starts from the point $\tilde{t}$, but going around the path counterclockwise for n complete circuits and then coming back to the point $\tilde{t}$. Then it follows from the discussion in (a) again that $\Delta_{-n}(\tilde{t}) = \Delta_0(\tilde{t}) - 2\sqrt{\rho}\, n\pi i$. In consequence, the function Δ_n is a conformal mapping of $\mathcal{G}_n = \mathcal{G}_0$ onto the open set

$$\mathcal{H}_n = \{x + iy;\, x \in \mathbb{R}, |y - 2\sqrt{\rho}\, n\pi| < \sqrt{\rho}\, \pi\} \setminus \{x + i\big(2n\sqrt{\rho} \pm (\sqrt{\rho} - \sqrt{\delta}\,)\big)\pi;\, x \le 0\},$$

where $\mathcal{G}_n$ is used to specify the domain of Δ_n for $n = 0 \pm 1, \pm 2, \cdots$. Furthermore, the inverse of Δ_n and the inverse of Δ_{n+1} defined on $\overline{\mathcal{H}_n}$ and $\overline{\mathcal{H}_{n+1}}$, respectively, share the same value at any point on the line $\{x + i(2n+1)\sqrt{\rho}\, \pi;\, x > 0\}$ whose image is the line segment $\{t = \xi;\, t_0 \le \xi < \infty\}$ for $n = 0, \pm 1, \pm 2, \cdots$. This fact implies a further analytic extension Ξ_2 of the function Ξ to the complex plane specified in the following theorem.

Theorem 5.4. *Let $\mathcal{Y}_2$ be the manifold defined as*

$$\mathcal{Y}_2 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{x + i\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,\big)\pi;\, x \le 0\}.$$

Then the function $\Xi(z)$ defined in Lemma 5.1 has an analytic extension, denoted by $\Xi_2(z)$, to the manifold $\mathcal{Y}_2$ such that Ξ_2 is a homeomorphism of the open set $\mathcal{H}_n$ onto the open set $\mathcal{G}_n$ with $\Xi_2|_{\mathcal{H}_n} = (\Delta_n|_{\mathcal{G}_n})^{-1}$ for any integer n, and it is a conformal mapping of $\mathcal{Y}_2$ onto the Riemann surface $\mathcal{X}_2$ which is constructed by pasting countably many sets $\mathcal{G}_n$ as layers in such a way that for any integer n, the upper side of the line $l_b = \{t = \xi;\, t_0 \le \xi < \infty\}$ in $\mathcal{G}_n$ is glued to the lower side of the line l_b in $\mathcal{G}_{n+1}$ which is placed right on the top of $\mathcal{G}_n$; while the lower side of the line l_b in $\mathcal{G}_n$ is glued to the upper side of the line l_b in $\mathcal{G}_{n-1}$ which is positioned right below $\mathcal{G}_n$. Furthermore, Ξ_2 also has a continuous extension to the boundary $\bigcup_{n=-\infty}^{\infty} \{x + i\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\big)\pi;\, x \le 0\}$ of $\mathcal{Y}_2$, if for any integer n the branch *line* $\{x+i\big((2n+1)\sqrt{\rho}-\sqrt{\delta}\,\big)\pi;\ x\le 0\}$ *or* $\{x+i\big((2n+1)\sqrt{\rho}+\sqrt{\delta}\,\big)\pi;\ x\le 0\}$ *of* Ξ_2 *is considered as one with an upper side and a lower side.*

We have shown by concrete constructions that analytic extension of any solitary wave solution to Equation (3.7) is not unique, which is a distinct property different from solitary wave solutions of the KdV equation. There are also some other different extensions of solitary wave solutions of (3.7). We leave them to interested readers to find out. What we will focus on in the next section is the convergence of solitary wave solutions of (3.7) as functions extended to the complex plane to a compacton, a peakons or a solitary wave solution of the KdV equation, as well as the explanation of why compactons and peakons are weak solutions of Equation (3.7) and why they have singularities on the real axis.

Appendix.

Proof of Lemma 5.1. Similar to the proof of Lemma 4.1, we investigate streamlines and equipotentials of the function Δ to describe its properties on the strip $\{t=\xi+i\eta\in\mathbb{C};\ |\Im t|\le\pi\}$. Beginning with the equation of streamlines

$$\cosh\xi=\cosh t_0\cos\eta-\sinh t_0\sin\eta\cot\frac{y+\sqrt{\delta}\,\eta}{\sqrt{\rho}} \tag{A1}$$

derived from Equation (5.5), we let f be the function

$$f(\eta,y)=\cosh t_0\cos\eta-\sinh t_0\sin\eta\cot A$$

defined on the domain $\{(\eta,y);\ 0\le\eta<\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}},0<y<\sqrt{\rho}\,\pi\}$, where $A=\frac{y+\sqrt{\delta}\,\eta}{\sqrt{\rho}}$. Then computations show that if the derivative

$$\begin{aligned}\frac{\partial}{\partial\eta}f(\eta,y)&=-\cosh t_0\sin\eta-\sinh t_0\cos\eta\cot A+\sinh t_0\sin\eta\frac{\sqrt{\frac{\delta}{\rho}}}{\sin^2 A}\\&=\sinh t_0\sin\eta\cot A\left(\sqrt{\frac{\delta}{\rho}}\cot A-\frac{\tan A}{\sinh t_0}-\cot\eta\right)\end{aligned}$$

has a zero at the point η_0 with $0<\eta_0<\pi$, then

$$\frac{\partial^2}{\partial\eta^2}f(\eta_0,y)=\frac{\sin\eta}{\sinh t_0\cos A_0\sin A_0}\big(\sin^2 A_0-\frac{2\delta}{\rho-\delta}\big), \tag{A2}$$

where $0<A_0=\frac{y+\sqrt{\delta}\,\eta_0}{\sqrt{\rho}}<\pi$. Therefore, when $\rho>3\delta$, the system of equations

$$\begin{cases}\dfrac{\partial}{\partial\eta}f(\eta,y)=0,\\[2mm]\dfrac{\partial^2}{\partial\eta^2}f(\eta,y)=0\end{cases} \tag{A3}$$

has a unique solution

$$\eta_0 = \cot^{-1} \frac{2\delta}{\sqrt{2\rho(\rho - 3\delta)}}, \qquad y_0 = -\sqrt{\delta}\,\eta_0 + \sqrt{\rho}\left(\pi - \sin^{-1}\sqrt{\frac{2\delta}{\rho - \delta}}\right), \tag{A4}$$

satisfying the conditions $0 < \eta_0 < \pi/2$, $\max\{\frac{\sqrt{\rho}\,\pi}{2}, (\sqrt{\rho} - \sqrt{\delta}\,)\pi\} < y_0 < \sqrt{\rho}\,\pi$, $f(\eta_0, y_0) > 1$ and $\frac{\partial^3}{\partial\eta^3} f(\eta_0, y_0) = \frac{\sqrt{(\rho-\delta)(\rho-3\delta)}}{\sqrt{\rho(\rho-2\delta)}} > 0$. It also follows from (5.7) that when $0 < y \leq (\sqrt{\rho} - \sqrt{\delta}\,)\pi$, there is a unique $\eta_y \in (0, \pi]$ such that η_y and y satisfy Equation (5.7) for $n = 0$, especially, when $y = (\sqrt{\rho} - \sqrt{\delta}\,)\pi$, $\eta_y = \pi$. That is to say $1 = f(\eta_y, y)$, or the streamline passes through the point $t = i\eta_y$. In addition, $\frac{\partial}{\partial\eta} f(\eta_y, y) < 0$ if $0 < y < (\sqrt{\rho} - \sqrt{\delta}\,)\pi$, $\lim_{\eta\to\pi} \frac{\partial}{\partial\eta} f(\eta, (\sqrt{\rho} - \sqrt{\delta}\,)\pi) = 0$ and $\lim_{\eta\to\pi} \frac{\partial^2}{\partial\eta^2} f\big(\eta, (\sqrt{\rho} - \sqrt{\delta}\,)\pi\big) = 1/3$, which means that $f(\eta, y)$ is decreasing for any η sufficiently close to η_y with $\eta < \eta_y$. We shall use these results to consider the function $f(\eta, y)$ in three cases.

(i) If $0 < \delta < \rho \leq 3\delta$, then for any y with $0 < y \leq (\sqrt{\rho} - \sqrt{\delta}\,)\pi$, $\lim_{\eta\to 0} \frac{\partial}{\partial\eta} f(\eta, y) = -\sinh t_0 \cot \frac{y}{\sqrt{\rho}} < 0$ and $\frac{\partial}{\partial\eta} f(\eta_y, y) \leq 0$. Assume that there is an $\tilde{\eta} \in (0, \eta_y)$ such that $\frac{\partial}{\partial\eta} f(\tilde{\eta}, y) = 0$ and $\frac{\partial}{\partial\eta} f(\eta, y) < 0$ for all $\eta \in (0, \tilde{\eta})$. It follows from (A2) that $\tilde{A} \neq \pi/2$ and

$$\frac{\partial^2}{\partial\eta^2} f(\tilde{\eta}, y) = \frac{\sin\eta}{\sinh t_0 \cos\tilde{A} \sin\tilde{A}} \big(\sin^2\tilde{A} - \frac{2\delta}{\rho - \delta}\big) \begin{cases} < 0, & \text{if } \tilde{A} < \pi/2, \\ > 0, & \text{if } \tilde{A} > \pi/2 \end{cases}$$

where $\tilde{A} = \frac{y + \sqrt{\delta}\,\tilde{\eta}}{\sqrt{\rho}}$. It is impossible for $\frac{\partial^2}{\partial\eta^2} f(\tilde{\eta}, y)$ to be negative, otherwise the inequality $\frac{\partial}{\partial\eta} f(\eta, y) > 0$ would be valid for any η sufficiently close to $\tilde{\eta}$ with $\eta < \tilde{\eta}$. $\frac{\partial^2}{\partial\eta^2} f(\tilde{\eta}, y)$ cannot be positive either since it would imply that $f(\eta, y)$ were increasing on the interval $(\tilde{\eta}, \eta_y)$ and resulted in the contradiction $1 < f(\tilde{\eta}, y) \leq f(\eta_y, y) = 1$. Hence, $f(\eta, y)$ is decreasing on the interval $(0, \eta_y)$ for any y with $0 < y \leq (\sqrt{\rho} - \sqrt{\delta}\,)\pi$. When $(\sqrt{\rho} - \sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi/2$, $\lim_{\eta\to 0} \frac{\partial}{\partial\eta} f(\eta, y) = -\sinh t_0 \cot \frac{y}{\sqrt{\rho}} < 0$ and $\lim_{\eta\to \frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}} \frac{\partial}{\partial\eta} f(\eta, y) = \infty$. Hence, there is an $\eta_1 \in \big(0, \frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}\big)$ such that $\frac{\partial}{\partial\eta} f(\eta, y) < 0$ for all $\eta \in (0, \eta_1)$ and $\frac{\partial}{\partial\eta} f(\eta_1, y) = 0$. It follows from (A2) that η_1 is the only zero of $\frac{\partial f}{\partial\eta}$ and $\frac{\partial}{\partial\eta} f(\eta, y) > 0$ for all $\eta \in \big(\eta_1, \frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}\big)$. Therefore, $f(\eta, y)$ is decreasing on the interval $(0, \eta_1)$, and increasing on the interval $\big(\eta_1, \frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}\big)$. For any y with $y \in [\sqrt{\rho}\,\pi/2, \sqrt{\rho}\,\pi)$, $\lim_{\eta\to 0} \frac{\partial}{\partial\eta} f(\eta, y) = -\sinh t_0 \cot \frac{y}{\sqrt{\rho}} \geq 0$ and $\lim_{\eta\to \frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}} \frac{\partial}{\partial\eta} f(\eta, y) = \infty$. Then (A2), the estimates

$$\lim_{\eta\to 0} \frac{\partial^2}{\partial\eta^2} f(\eta, y) \begin{cases} > 0, & \text{if } \ y \geq \dfrac{\sqrt{\rho}\,\pi}{2} \ \text{ and } \ \rho < 3\delta \\ = 0, & \text{if } \ y = \dfrac{\sqrt{\rho}\,\pi}{2} \ \text{ and } \ \rho = 3\delta, \end{cases}$$

$\lim_{\eta\to 0}\frac{\partial^3}{\partial\eta^3}f(\eta,\frac{\sqrt{\rho}\,\pi}{2})=0$ and $\lim_{\eta\to 0}\frac{\partial^4}{\partial\eta^4}f(\eta,\frac{\sqrt{\rho}\,\pi}{2})>0$ imply that f is an increasing function on the interval $\big(0,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$.

(ii) If $3\delta<\rho\le 4\delta$, then using a method similar to the proof in (i), one may show that $f(\eta,y)$ is a decreasing function on the interval $(0,\eta_y)$ when $0<y\le(\sqrt{\rho}-\sqrt{\delta}\,)\pi$, and there is an $\eta_1\in\big(0,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$ such that $f(\eta,y)$ is decreasing on the interval $(0,\eta_1)$ and increasing on the interval $\big(\eta_1,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$ for any $y\in\big((\sqrt{\rho}-\sqrt{\delta}\,)\pi,\sqrt{\rho}\,\pi/2\big]$ with $\rho\le 4\delta$. The difference between Case (i) and Case (ii) is when $y\in(\sqrt{\rho}\,\pi/2,y_0)$, where y_0 takes the form as expressed in (A4), which together with η_0 solves System (A3). Since

$$\sqrt{\frac{\delta}{\rho}}\cot\frac{y+\sqrt{\delta}\,\eta_0}{\sqrt{\rho}}-\frac{\tan\frac{y+\sqrt{\delta}\,\eta_0}{\sqrt{\rho}}}{\sinh t_0}>\sqrt{\frac{\delta}{\rho}}\cot\frac{y_0+\sqrt{\delta}\,\eta_0}{\sqrt{\rho}}-\frac{\tan\frac{y_0+\sqrt{\delta}\,\eta_0}{\sqrt{\rho}}}{\sinh t_0}=\cot\eta_0>0,\quad\text{(A5)}$$

it follows that $\frac{\partial}{\partial\eta}f(\eta_0,y)<0$. In addition, as we have pointed out in (i) that $\frac{\partial}{\partial\eta}f(\eta,y)$ is positive for any η sufficiently close to zero or sufficiently close to $\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}$ with $0<\eta<\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}$. Therefore, $\frac{\partial f}{\partial\eta}$ has at least two zeros in the interval $\big(0,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$, denoted by η_2 and η_3, satisfying the condition $\eta_2<\eta_0<\eta_3$. It follows from (A2) that η_2 and η_3 are only zeros of $\frac{\partial f}{\partial\eta}$, and thus $f(\eta,y)$ is increasing on the intervals $(0,\eta_2)$ and $\big(\eta_3,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$ and decreasing on the interval (η_2,η_3). When $y\in[y_0,\sqrt{\rho}\,\pi)$, the inequality

$$\sqrt{\frac{\delta}{\rho}}\cot\frac{y+\sqrt{\delta}\,\eta}{\sqrt{\rho}}-\frac{\tan\frac{y+\sqrt{\delta}\,\eta}{\sqrt{\rho}}}{\sinh t_0}\le\sqrt{\frac{\delta}{\rho}}\cot\frac{y_0+\sqrt{\delta}\,\eta}{\sqrt{\rho}}-\frac{\tan\frac{y_0+\sqrt{\delta}\,\eta}{\sqrt{\rho}}}{\sinh t_0}\le\cot\eta\qquad\text{(A6)}$$

holds for any $\eta\in\big(0,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$ which means $\frac{\partial f}{\partial\eta}$ is nonnegative on the same interval. Hence, f is an increasing function.

(iii) If $4\delta<\rho$, then similar to Case (ii), f in a decreasing function on the interval $(0,\eta_y)$ for any $y\in(0,\sqrt{\rho}\,\pi/2]$. When $y\in(\sqrt{\rho}\,\pi/2,(\sqrt{\rho}-\sqrt{\delta}\,)\pi]$, $\frac{\partial}{\partial\eta}f(0,y)=-\sinh t_0\cot\frac{y}{\sqrt{\rho}}>0$ and $\frac{\partial}{\partial\eta}f(\eta,y)<0$ for any η sufficiently close to η_y with $0<\eta<\eta_y$. Hence, there is an $\eta_4\in(0,\eta_y)$ such that f is increasing on the interval $(0,\eta_4)$ and decreasing on the interval (η_4,η_y). If $(\sqrt{\rho}-\sqrt{\delta}\,)\pi<y<y_0$, then Inequality (A5) holds and $\frac{\partial}{\partial\eta}f(\eta_0,y)<0$. Similar to the proof in (ii), one may show that $\frac{\partial f}{\partial\eta}$ has two zeros, η_2 and η_3 such that f is increasing on the intervals $(0,\eta_2)$ and $\big(\eta_3,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$ and decreasing on the interval $\big(\eta_2,\eta_3\big)$. Because Inequality (A6) holds for any $\eta\in\big(0,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$ if $y_0\le y<\sqrt{\rho}\,\pi$, f becomes an increasing function on the interval $\big(0,\frac{\sqrt{\rho}\,\pi-y}{\sqrt{\delta}}\big)$.

Now we summarize properties of streamlines using the fact that streamlines are symmetric with respect to the imaginary axis *i.e.* $y(\xi,\eta)=y(-\xi,\eta)$. For each $y\in(0,(\sqrt{\rho}-\sqrt{\delta}\,)\pi]$, the streamline $y = y(\xi,\eta)$ is a curve starting from the point $t = -t_0$, passing though the point $t = i\eta_y$ and ending at the point $t = t_0$. The Cauchy-Riemann equation

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{\partial x}{\partial \xi} + i\frac{\partial y}{\partial \xi} = \frac{\partial y}{\partial \eta} - i\frac{\partial x}{\partial \eta}\\ &= \frac{\sqrt{\delta}\left((\cosh\xi\cos\eta + 1)(\cosh\xi\cos\eta - \cosh t_0) - i(\cosh t_0 + 1)\sinh\xi\sin\eta\right)}{|\cosh(\xi + i\eta) - \cosh t_0|^2}\end{aligned}$$

and the derivative $\frac{D}{D\eta}x(\xi(\eta),\eta) = \frac{(\frac{\partial x}{\partial \xi})^2 + (\frac{\partial x}{\partial \eta})^2}{\frac{\partial x}{\partial \eta}}$ shows that Δ maps the streamline homeomorphically to the line $\{z \in \mathbb{C};\ \Re z \in \mathbb{R}, \Im z = y\}$ such that $\Delta(-t_0) = -\infty + iy$, $\Delta(i\eta_y) = iy$ and $\Delta(t_0) = \infty + iy$, where $\xi(\eta)$ is implicitly determined by the streamline. It is worth noticing that these streamlines are smooth except $y = (\sqrt{\rho} - \sqrt{\delta}\,)\pi$ which has a corner at the point $t = i\pi$ since computation shows that

$$\lim_{\eta\to\pi}\frac{d\xi}{d\eta} = \lim_{\eta\to\pi}\frac{\frac{\partial}{\partial\eta}f\left(\eta,(\sqrt{\rho}-\sqrt{\delta}\,)\pi\right)}{\sqrt{f^2\left(\eta,(\sqrt{\rho}-\sqrt{\delta}\,)\pi\right) - 1}} = -\frac{1}{\sqrt{3}},$$

a consequence of $\frac{d\Delta}{dt} = 0$ at $t = i\pi$. Using the Cauchy-Riemann equation, one may also show that Δ is a one-to-one mapping of the streamline $y = y(\xi,\eta)$ on the right-side of the imaginary axis onto the line $\{z \in \mathbb{C};\ \Re z \in \mathbb{R}, \Im z = y\}$ for each $y \in \left((\sqrt{\rho} - \sqrt{\delta}\,)\pi, \sqrt{\rho}\,\pi\right)$ such that $\Delta(t_0) = \infty + iy$ and $\Delta\left(\infty + i\frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}\right) = -\infty + iy$. Symmetrically, Δ is also a one-to-one mapping of the streamline $y = y(\xi,\eta)$ on the left-side of the imaginary axis onto the line $\{z \in \mathbb{C};\ \Re z \in \mathbb{R}, \Im z = y\}$ for each $y \in \left((\sqrt{\rho} - \sqrt{\delta}\,)\pi, \sqrt{\rho}\,\pi\right)$ such that $\Delta(-t_0) = -\infty + iy$ and $\Delta\left(-\infty + i\frac{\sqrt{\rho}\,\pi - y}{\sqrt{\delta}}\right) = \infty + iy$. The upper side of the branch line $\{t \in \mathbb{R};\ t_0 \le t < \infty\}$ as defined in (b) Section 5 is the streamline $\sqrt{\rho}\,\pi = y(\xi,\eta)$ and the lower side of this branch line is the streamline $-\sqrt{\rho}\,\pi = y(\xi,\eta)$, such that Δ maps them homeomorphically to the lines $\{z \in \mathbb{C};\ \Im z = \sqrt{\rho}\,\pi\}$ and $\{z \in \mathbb{C};\ \Im z = -\sqrt{\rho}\,\pi\}$, respectively. Because of the symmetry property $f(\eta, y) = f(-\eta, -y)$ of the function f, for each $y \in (0, (\sqrt{\rho} - \sqrt{\delta}\,)\pi)$, the streamline $-y = y(\xi,\eta)$ in the lower half plane is symmetric to the streamline $y = y(\xi,\eta)$ with respect the real axis and thus can be described in a similar way.

Next we consider equipotentials of the function Δ, whose equation can be expressed as

$$\cos\eta = \cosh t_0\cosh\xi - \sinh t_0\sinh\xi\coth B, \tag{A7}$$

where $B = \frac{x + \sqrt{\delta}\,\pi}{\sqrt{\rho}}$. Let

$$g(\xi, x) = \cosh t_0\cosh\xi - \sinh t_0\sinh\xi\coth B.$$

Then

$$\frac{\partial}{\partial\xi}g(\xi,x) = \cosh t_0 \sinh\xi - \sinh t_0 \cosh\xi \coth B + \sqrt{\frac{\delta}{\rho}}\sinh t_0 \sinh\xi / \sinh^2 B$$

$$= \cosh\xi\left(1+\sqrt{\frac{\delta}{\rho}}\sinh t_0 \coth^2 B\right)\left[\tanh\xi - \frac{\sinh t_0 \coth B}{1+\sqrt{\frac{\delta}{\rho}}\sinh t_0 \coth^2 B}\right].$$

It follows that when $x > 0$, $\lim_{\xi\to\frac{-x}{\sqrt{\delta}}^+}\frac{\partial g}{\partial\xi} = -\infty$ and $\lim_{\xi\to\infty}\frac{\partial g}{\partial\xi} = \infty$. Hence, $\frac{\partial g}{\partial\xi}$ has a zero, denoted by ξ_0, in the interval $\left(\frac{-x}{\sqrt{\delta}},\infty\right)$ and then

$$\tanh\xi_0 = \frac{\sinh t_0 \coth B_0}{1+\sqrt{\frac{\delta}{\rho}}\sinh t_0 \coth^2 B_0}, \tag{A8}$$

where $B_0 = \frac{x+\sqrt{\delta}\,\xi_0}{\sqrt{\rho}}$. Substituting (A8) into the second order derivative of g, one obtains

$$\frac{\partial^2}{\partial\xi^2}g(\xi_0,x) = \frac{\cosh\xi_0(1+\sqrt{\frac{\delta}{\rho}}\sinh t_0/\sinh^2 B_0)}{1+\sqrt{\frac{\delta}{\rho}}\sinh t_0 \coth^2 B_0} > 0, \tag{A9}$$

which implies that ξ_0 is the only zero of $\frac{\partial g}{\partial\xi}$. On the other hand, substituting (A8) into the function g yields the expression

$$g(\xi_0,x) = \frac{\cosh t_0 - \sqrt{\frac{\delta}{\rho}}\sinh t_0 \coth^2 B_0}{\sqrt{1-\frac{\delta}{\rho}\sinh^2 t_0 \coth^2 B_0 + \frac{\delta}{\rho}\sinh_0^t \coth^4 B_0}}.$$

Hence, $|g(\xi_0,x)| < 1$, and the graph of $g-1$ intersects with real axis at two points, denoted by ξ_1 and ξ_2 which are determined by the equations $\tanh\frac{\xi}{2} = \tanh\frac{x+\sqrt{\delta}\,\xi}{\sqrt{\rho}}\tanh\frac{t_0}{2}$ and $\tanh\frac{\xi}{2} = \coth\frac{x+\sqrt{\delta}\,\xi}{\sqrt{\rho}}\tanh\frac{t_0}{2}$, respectively. It follows that $0 < \xi_1 < t_0 < \xi_2$. If $x = 0$, then $\lim_{\xi\to 0^+} g(\xi,0) = -1$, $\lim_{\xi\to 0^+}\frac{\partial g}{\partial\xi} = 0$, $\lim_{\xi\to 0^+}\frac{\partial^2 g}{\partial\xi^2} = 1/3$ and $\lim_{\xi\to\infty}\frac{\partial g}{\partial\xi} = \infty$, which together with (A9) implies that $g(\xi,0)$ is an increasing function on the interval $(0,\infty)$. When $x < 0$, $\lim_{\xi\to\frac{-x}{\sqrt{\delta}}^+}\frac{\partial g}{\partial\xi} = \infty$ and $\lim_{\xi\to\infty}\frac{\partial g}{\partial\xi} = \infty$. It follows from (A9) that $g(\xi,x)$ is also an increasing function on the interval $\left(\frac{-x}{\sqrt{\delta}},\infty\right)$.

Because Equation (A7) determines η as a function of ξ given by $\eta = \cos^{-1} g(\xi,x)$ with $0 \le \eta \le \pi$, we can use this relation between g and η, the Cauchy-Riemann equation of the function Δ and the symmetry properties $x(\xi,\eta) = x(\xi,-\eta)$ and $x(\xi,\eta) = -x(-\xi,\eta)$ of equipotentials to describe their graphs. When $x = 0$, the equipotential $0 = x(\xi,\eta)$ consists of the curves E_-, E_+ and the line segment $\{t = i\eta;\ \eta \in (-\pi,\pi)\}$ as illustrated in Figure 10, such that Δ is a one-to-one mapping of the equipotential onto the line segment $\{z = iy;\ y \in [-\sqrt{\rho}\,\pi, \sqrt{\rho}\,\pi]\}$ with $\lim_{\substack{\Im t\to 0\\ t\in E_-}} \Delta(t) = -i\sqrt{\rho}\,\pi$, $\Delta(-i\pi) = -i(\sqrt{\rho}-\sqrt{\delta}\,)\pi$, $\Delta(0) = 0$, $\Delta(\pi) = i(\sqrt{\rho}-\sqrt{\delta}\,)\pi$ and $\lim_{\substack{\Im t\to 0\\ t\in E_+}} \Delta(t) = i\sqrt{\rho}\,\pi$. When $x > 0$, the equipotential $x = x(\xi,\eta)$ is a loop, starting from the point ξ_2, going clockwise through the point ξ_1 and returning to the point ξ_2, such that Δ maps the equipotential homeomorphically onto the line segment $\{z \in \mathbb{C};\ \Re z = x, \Im z \in [-\sqrt{\rho}\,\pi, \sqrt{\rho}\,\pi]\}$ with $\lim_{\substack{t\to\xi_2\\ \Im t<0}} \Delta(t) = x - i\sqrt{\rho}\,\pi$, $\Delta(\xi_1) = x$ and $\lim_{\substack{t\to\xi_2\\ \Im t>0}} \Delta(t) = x + i\sqrt{\rho}\,\pi$. If $x < 0$, the equipotential $x = x(\xi,\eta)$ in the right-half plane consists of two curves symmetric to each other with respect to the real axis. One is decreasing connecting the points $t = \xi_3 + i\pi$ and $t = \xi_4$, the other one is increasing connecting the points $t = \xi_3 - i\pi$ and $t = \xi_4$ with $0 < \xi_3 < \xi_4$, such that Δ maps the previous curve to the line segment $\{z \in \mathbb{C};\ \Re z = x, \Im z \in [(\sqrt{\rho}-\sqrt{\delta}\,)\pi, \sqrt{\rho}\,\pi]\}$ with $\Delta(\xi_3+i\pi) = x+i(\sqrt{\rho}-\sqrt{\delta}\,)\pi$ and $\Delta(\xi_4) = x+i\sqrt{\rho}\,\pi$, and it maps the latter one to the line segment $\{z \in \mathbb{C};\ \Re z = x, \Im z \in [-\sqrt{\rho}\,\pi, -(\sqrt{\rho}-\sqrt{\delta}\,)\pi]\}$ with $\Delta(\xi_3-i\pi) = x-i(\sqrt{\rho}-\sqrt{\delta}\,)\pi$ and $\Delta(\xi_4) = x-i\sqrt{\rho}\,\pi$. Symmetrically, one may describe equipotentials of Δ in the left-half plane in a similar way.

As a result, the restriction of the function Δ to the region bounded by streamlines $(\sqrt{\rho}-\sqrt{\delta}\,)\pi = y(\xi,\eta)$ and $-(\sqrt{\rho}-\sqrt{\delta}\,)\pi = y(\xi,\eta)$, denoted by l_+ and l_- in Figure 13, respectively, is a homeomorphism to the strip $\{z \in \mathbb{C};\ |\Im z| \le (\sqrt{\rho}-\sqrt{\delta}\,)\pi\}$ such that one may use its inverse to define the analytic function Ξ on the strip as discussed in Section 5. $\square$

CONVERGENCE OF SOLITARY-WAVE SOLUTIONS IN A PERTURBED BI-HAMILTONIAN DYNAMICAL SYSTEM. III. CONVERGENCE TO NON-ANALYTIC SOLUTIONS.

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Abstract. In this part, we prove that the solitary wave solutions investigated in part I are extended as analytic functions in the complex plane, except at most countably many branch points and branch lines. We describe in detail how the limiting behavior of the complex singularities allows the creation of non-analytic solutions with corners and/or compact support.

This is the third in a series of three papers investigating the solitary wave solutions of the integrable model wave equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + \frac{3}{\nu} u u_x + u u_{xxx} + 2 u_x u_{xx}. \tag{3.6}$$

As in part II, we adopt the notation and numbering of statements from part I.

Under the assumptions stated in Theorems 3.1 or 3.2, the ordinary differential equation for travelling wave solutions $u(x,t) = \phi(x - ct)$, which is

$$(\alpha + c)\phi' + (\beta + c\nu + \phi)\phi''' + \frac{3}{\nu}\phi\phi' + 2\phi'\phi'' = 0, \tag{3.7}$$

can be reduced to either (4.1) or (5.1) by the simple linear transformation $\phi = a - |\nu|\big(\alpha + c + \frac{3a}{\nu}\big)\varphi$. Therefore, it suffices to show convergence of solitary wave solutions of Equations (4.1) and (5.1). We will focus on the convergence of solitary wave solutions of (3.7) as functions extended to the complex plane to a compacton, a peakons or a solitary wave solution of the KdV equation, as well as the explanation of why compactons and peakons are weak solutions of Equation (3.7) and why they have singularities on the real axis.

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6. Convergence to compactons.

In Section 4, we have constructed two different analytic extensions $\Theta_0(z)$ and $\Theta_1(z)$ defined on the manifolds Y_0 and Y_1, respectively, for the solution $\theta(x)$ satisfying Equation (4.3) such that $\varphi(x) = \frac{\delta+\epsilon}{2\delta}\sin\theta(x) + \frac{\delta-\epsilon}{2\delta}$ is the solitary wave solution of (4.1) for $x \in \mathbb{R}$, $\delta > 0$ and $\epsilon > 0$. We denote $\Theta_{\epsilon,\delta,0} = \Theta_0$, $\Theta_{\epsilon,\delta,1} = \Theta_1$,

$$Y_{\epsilon,\delta,0} = Y_0 = \mathbb{C} \setminus \{x \pm i\sqrt{\epsilon}\,\pi; |x| \geq \sqrt{\delta}\,\pi\}$$

and

$$Y_{\epsilon,\delta,1} = Y_1 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{x + i(2n+1)\sqrt{\epsilon}\,\pi;\ |x| \leq \sqrt{\delta}\,\pi\}$$

to indicate that these functions and manifolds depend on the parameters ϵ and δ. Then we have their convergent property stated in the following theorems.

Theorem 6.1. *The sequence of the functions $\{\Theta_{\epsilon,\delta,0}(z)\}$ converges to the function $\Theta_0(z)$ defined by*

$$\Theta_0(z) = \begin{cases} \dfrac{\pi}{2} - \dfrac{z}{\sqrt{\delta}}, & \text{if } \Im z \neq 0, \text{ or } \Im z = 0 \text{ and } |\Re z| < \sqrt{\delta}\,\pi \\ -\dfrac{\pi}{2}, & \text{if } \Im z = 0 \text{ and } \Re z \geq \sqrt{\delta}\,\pi \\ \dfrac{3\pi}{2}, & \text{if } \Im z = 0 \text{ and } \Re z \leq -\sqrt{\delta}\,\pi \end{cases}$$

as $\epsilon \to 0$. Therefore, as functions extended to the manifold $\overline{Y_{\epsilon,\delta,0}}$, solitary wave solutions $\Phi_{\epsilon,\delta,0}(z) = \frac{\delta+\epsilon}{2\delta}\sin\Theta_{\epsilon,\delta,0}(z) + \frac{\delta-\epsilon}{2\delta}$ satisfying Equation (4.1) converge to the function

$$\Phi_0(z) = \begin{cases} \cos^2 \dfrac{z}{2\sqrt{\delta}}, & \text{if } \Im z \neq 0, \text{ or } \Im z = 0 \text{ and } |\Re z| < \sqrt{\delta}\,\pi \\ 0, & \text{if } \Im z = 0, \text{ and } |\Re z| \geq \sqrt{\delta}\,\pi \end{cases}$$

when $\epsilon \to 0$, and $\Phi_0(z)$ is an analytic function on the manifold

$$\mathbb{C} \setminus \{z = x;\ |x| \geq \sqrt{\delta}\,\pi\},$$

having line segments $\{z = x;\ x \geq \sqrt{\delta}\,\pi\}$ and $\{z = x;\ x \leq -\sqrt{\delta}\,\pi\}$ as its natural boundary.

Proof. To show the convergence of functions $\{\Theta_{\epsilon,\delta,0}\}$ on the open set $\mathbb{C} \setminus \{z = x;\ |x| \geq \sqrt{\delta}\,\pi\}$, we use the property that for any compact set K of $\mathbb{C}\setminus \{z = x;\ |x| \geq \sqrt{\delta}\,\pi\}$, $\{\Theta_{\epsilon,\delta,0}\}$ has a normal family defined in K. Let M, N, r and $\tilde{r}$ be any constants such that $0 < \tilde{r} < M$ and $0 < r < \sqrt{\delta}\,\pi < N$, and let K be the compact set whose boundary consists of line segments with vertices $z = -N + iM, -N + i\tilde{r}, -r + i\tilde{r}, -r - i\tilde{r}, -N - i\tilde{r}, -N - iM, N - iM, N - i\tilde{r}, r - i\tilde{r}, r + i\tilde{r}, N + i\tilde{r}$ and $N + iM$ as sketched in Figure 14.

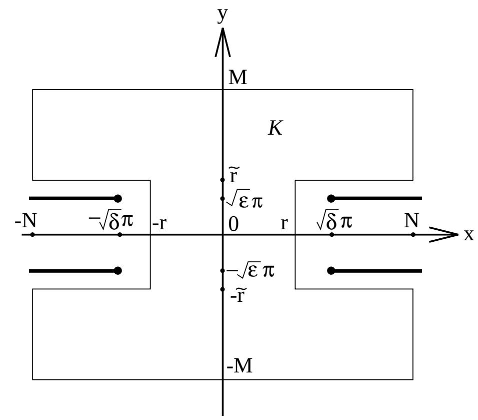

Fig. 14. A sketch of the compact set K

It follows from the discussion in *(ii)* of Section 4 that $\Theta_{\epsilon,\delta,0}$ maps vertical boundary of K to equipotentials of the function $\Sigma_{\epsilon,\delta,0} = \Sigma_0$ and horizontal lines to streamlines of $\Sigma_{\epsilon,\delta,0}$. The maximum and minimum values of $\eta_{\epsilon,\delta,0}(z) = \Im(\Theta_{\epsilon,\delta,0}(z))$ on K is attained on the streamlines $\pm M = y(\xi,\eta)$ of $\Sigma_{\epsilon,\delta,0}$ such that $-M = y(\frac{\pi}{2}, \eta_{\epsilon,\delta,0,M})$, where $\eta_{\epsilon,\delta,0,M} = \max_{z\in K}|\eta_{\epsilon,\delta,0}(z)|$. Since $(y,\eta) = (-M, \eta_{\epsilon,\delta,0,M})$ satisfies the equation

$$y = -i\Sigma_{\epsilon,\delta,0}\big(\frac{\pi}{2} + i\eta\big) = -\left(\sqrt{\delta}\,\eta + 2\sqrt{\epsilon}\tan^{-1}\big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta}{2}\big)\right) \tag{6.1}$$

as demonstrated in Lemma 4.1 and

$$\lim_{\epsilon\to 0}\left(\sqrt{\delta}\,\eta + 2\sqrt{\epsilon}\tan^{-1}\big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta}{2}\big)\right) = \sqrt{\delta}\,\eta,$$

it follows that

$$\lim_{\epsilon\to 0}\eta_{\epsilon,\delta,0,M} = \lim_{\epsilon\to 0}\eta_{\epsilon,\delta,0}(0,-M) = \frac{M}{\sqrt{\delta}}.$$

On the other hand, the extremum of $\xi_{\epsilon,\delta,0}(z) = \Re(\Theta_{\epsilon,\delta,0}(z))$ on K is attained on the equipotentials given by the equation $\pm N = x(\xi,\eta)$ of $\Sigma_{\epsilon,\delta,0}$. Let n be the smallest integer greater than or equal to $N/(2\sqrt{\delta}\,\pi)$. Then $\xi_{\epsilon,\delta,0,N} = \max_{z\in K}|\xi_{\epsilon,\delta,0}(z)| \le 2(n+1)\pi$ because of the property $\xi_{\epsilon,\delta,0}(x + 2k\sqrt{\delta}\,\pi, y) = \xi_{\epsilon,\delta,0}(x,y) - 2k\pi$ for any integer k. Therefore, the functions $\{\Theta_{\epsilon,\delta,0}(z)\}$ are uniformly bounded on K for any ϵ sufficiently small which implies that there exists a subsequence of $\{\Theta_{\epsilon,\delta,0}\}$, for the sake of simplicity, still denoted by $\{\Theta_{\epsilon,\delta,0}\}$, analytic in K and uniformly convergent to an analytic function on any compact subset of K as $\epsilon \to 0$. Then Equation (6.1) leads to the definition of the limiting function on the imaginary axis

$$\lim_{\epsilon\to 0}\Theta_{\epsilon,\delta,0}(iy) = \lim_{\epsilon\to 0}\big(\frac{\pi}{2} + i\eta_{\epsilon,\delta,0}(iy)\big) = \frac{\pi}{2} - \frac{iy}{\sqrt{\delta}}$$

for any y with $|y| < M$. The fact that the limiting function is holomorphic in K and its restriction to the imaginary axis is the same as that of the function $\Theta_0(z) = \frac{\pi}{2} - \frac{z}{\sqrt{\delta}}$ leads to the conclusion that the limiting function defined in K is the same as $\Theta_0(z)$. Since any convergent subsequence of $\{\Theta_{\epsilon,\delta,0}\}$ has the same limit, this implies that the sequence $\{\Theta_{\epsilon,\delta,0}(z)\}$ itself converges to $\Theta_0(z)$ in the interior of K when $\epsilon \to 0$.

Next, we need to show the convergence of $\{\Theta_{\epsilon,\delta,0}\}$ on lines $\{z = x;\ |x| \geq \sqrt{\delta}\,\pi\}$ which will be carried out by using Equation (4.4) in the form

$$e^{-\frac{1}{\sqrt{\epsilon}}\left(x+\sqrt{\delta}(\theta-\frac{\pi}{2})\right)} = \frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}}, \tag{6.2}$$

where $\theta = \Theta_{\epsilon,\delta,0}(x)$ is implicitly determined by Equation (6.2) with $\sin\theta_0 = \frac{\delta-\epsilon}{\delta+\epsilon}$ and $-\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$. Equation (4.4) is relabeled as (6.2), because of frequent use in this section.

For any ν with $0 < \nu < \pi$, let θ_1 be a fixed constant such that $0 < \theta_1 + \frac{\pi}{2} < \frac{\nu}{2}$. Because

$$\lim_{\epsilon\to 0} e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta_1+\frac{\pi}{2})} = 0, \quad \lim_{\epsilon\to 0}\theta_0 = \lim_{\epsilon\to 0}\arcsin\frac{\delta-\epsilon}{\delta+\epsilon} = \frac{\pi}{2} \quad \text{and} \quad \lim_{\epsilon\to 0}\frac{\sin\frac{\theta_1+\theta_0}{2}}{\cos\frac{\theta_1-\theta_0}{2}} = 1,$$

there is an $\epsilon_0 > 0$, such that if $0 < \epsilon < \epsilon_0$,

$$e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta_1+\frac{\pi}{2})} < \frac{1}{3}, \quad \frac{\sin\frac{\theta_1+\theta_0}{2}}{\cos\frac{\theta_1-\theta_0}{2}} > \frac{1}{2} \quad \text{and} \quad -\frac{\pi}{2} < -\theta_0 < \theta_1.$$

Then using the properties that $\frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}}$ is an increasing function of θ and $e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})}$ is a decreasing function of θ leads to the estimates

$$\frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}} > \frac{\sin\frac{\theta_1+\theta_0}{2}}{\cos\frac{\theta_1-\theta_0}{2}} > e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta_1+\frac{\pi}{2})} > e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})}$$

for any $\theta \in (\theta_1, \pi+\theta_0)$ and any $\epsilon \in (0, \epsilon_0)$. Therefore, the solution $\theta_{\epsilon,\delta,0,\frac{\pi}{2}}$ of the equation

$$\frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}} = e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})}$$

must satisfy the condition $-\theta_0 < \theta_{\epsilon,\delta,0,\frac{\pi}{2}} < \theta_1 < \nu - \frac{\pi}{2}$. It follows that $\theta_{\epsilon,\delta,0,\frac{\pi}{2}} = \Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi)$ and $\lim_{\epsilon\to 0}\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi) = -\frac{\pi}{2}$. Because for any $x > \sqrt{\delta}\,\pi$,

$$e^{-\frac{1}{\sqrt{\epsilon}}\left(x+\sqrt{\delta}(\theta-\frac{\pi}{2})\right)} < e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})} \quad \text{and} \quad \frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}} > \frac{\sin\frac{\theta_{\epsilon,\delta,0,\frac{\pi}{2}}+\theta_0}{2}}{\cos\frac{\theta_{\epsilon,\delta,0,\frac{\pi}{2}}-\theta_0}{2}}$$

hold for any θ with $\theta_{\epsilon,\delta,0,\frac{\pi}{2}} \le \theta < \pi + \theta_0$, the solution $\theta_{\epsilon,\delta,0,x}$ of Equation (6.2) satisfies the conditions $-\theta_0 < \theta_{\epsilon,\delta,0,x} < \Theta_{\epsilon,\delta,0}(\sqrt{\delta}\pi)$ and $\theta_{\epsilon,\delta,0,x} = \Theta_{\epsilon,\delta,0}(x)$, which implies that functions $\Theta_{\epsilon,\delta,0}(x)$ are uniformly convergent to $-\frac{\pi}{2}$ on the interval $[\sqrt{\delta}\,\pi, \infty)$ as $\epsilon \to 0$. Then the equality $\Theta_{\epsilon,\delta,0}(-x) = \pi - \Theta_{\epsilon,\delta,0}(x)$ leads to the conclusion that the sequence of the functions $\Theta_{\epsilon,\delta,0}$ is uniformly convergent to $\frac{3\pi}{2}$ on the interval $(-\infty, -\sqrt{\delta}\,\pi]$. As a result, functions $\Phi_{\epsilon,\delta,0}(z)$ converge to $\varPhi_0(z)$ for any $z \in \mathbb{C}$ when $\epsilon \to 0$. □

Theorem 6.2. *The solutions $\{\Theta_{\epsilon,\delta,1}(z)\}$ of Equation (4.3) converge to the function defined as*

$$\varTheta_1(z) = \begin{cases} \dfrac{\pi}{2} - \dfrac{x}{\sqrt{\delta}}, & \text{if } \Re z = x \text{ and } |x| \le \sqrt{\delta}\,\pi \\ -\dfrac{\pi}{2}, & \text{if } \Re z > \sqrt{\delta}\,\pi \\ \dfrac{3\pi}{2}, & \text{if } \Re z < -\sqrt{\delta}\,\pi \end{cases}$$

when $\epsilon \to 0$. Hence, solitary wave solutions $\Phi_{\epsilon,\delta,1}(z) = \frac{\delta+\epsilon}{2\delta}\sin\Theta_{\epsilon,\delta,1}(z) + \frac{\delta-\epsilon}{2\delta}$ of Equation (4.1) converge to the function given by

$$\varPhi_1(z) = \begin{cases} \cos^2\dfrac{x}{2\sqrt{\delta}}, & \text{if } \Re z = x \text{ and } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{if } |\Re z| > \sqrt{\delta}\,\pi \end{cases}$$

as $\epsilon \to 0$, such that $\varPhi_1(z)$ is an analytic function on the manifold

$$\mathbb{C} \setminus \{z \in \mathbb{C};\ |\Re z| \le \sqrt{\delta}\,\pi\},$$

having the natural boundary $\{\pm\sqrt{\delta}\,\pi + iy;\ y \in \mathbb{R}\}$.

Proof. It follows from the definition of $\Theta_{\epsilon,\delta,1}$ that for any integer n, the restriction of $\Theta_{\epsilon,\delta,1}(z)$ to the strip $\{x+iy;\ x \in \mathbb{R}, (2n-1)\sqrt{\epsilon}\,\pi \le y \le (2n+1)\sqrt{\epsilon}\,\pi\}$ is the inverse of the single-valued branch Σ_n such that $\Sigma_n(\theta) = \Sigma_0(\theta) + i2n\pi\sqrt{\epsilon}$ for any $\theta \in \overline{\Omega}$, where Σ_0 is the single-valued branch of Σ defined in *(i)*. That means for any

$$x + iy \in \{x+iy;\ x \in \mathbb{R}, (2n-1)\sqrt{\epsilon}\,\pi \le y \le (2n+1)\sqrt{\epsilon}\,\pi\},$$

$\Theta_{\epsilon,\delta,1}(x+iy) = \Theta_{\epsilon,\delta,1}(x+i(y-2n\pi\sqrt{\epsilon})) \in \overline{\Omega}$. Let $\eta_{\epsilon,\delta,1}(x,y) = \Im\big(\Theta_{\epsilon,\delta,1}(z)\big)$ for $z = x+iy$. Then $\eta_\epsilon = \max\limits_{x+iy\in Y_{\epsilon,\delta,1}} |\eta_{\epsilon,\delta,1}(x,y)| = \max\limits_{x\in\mathbb{R},|y|\le\sqrt{\epsilon}\,\pi} |\eta_{\epsilon,\delta,1}(x,y)|$, and it follows from lemma 4.1 that $(y,\eta) = (\sqrt{\epsilon}\,\pi, -\eta_\epsilon)$ satisfies Equation (6.1), *i.e.*

$$\sqrt{\epsilon}\,\pi = \sqrt{\delta}\,\eta_\epsilon + 2\sqrt{\epsilon}\tan^{-1}\Big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta_\epsilon}{2}\Big).$$

Therefore, $\lim\limits_{\epsilon\to 0}|\eta_{\epsilon,\delta,1}(x,y)| \le \lim\limits_{\epsilon\to 0}\eta_\epsilon = 0$ for any $x+iy\in\mathbb{C}$, which implies that the sequence $\{\eta_{\epsilon,\delta,1}(x,y)\}$ is uniformly convergent to 0 in the complex plane $\mathbb{C}$ as $\epsilon\to 0$. Now we only need to show that functions $\xi_{\epsilon,\delta,1}(x,y) = \Re(\Theta_{\epsilon,\delta,1}(x+iy))$ converge to $\mathit{\Theta}_1(x+iy)$ when $\epsilon\to 0$.

For any fixed $x\in[0,\sqrt{\delta}\,\pi]$, it follows from Lemma 4.1 that $\frac{\pi}{2} - \frac{x}{\sqrt{\delta}} \le \xi_{\epsilon,\delta,1}(x,y) \le \xi_{\epsilon,\delta,1}(x)$, where y is any fixed real number, $\Theta_{\epsilon,\delta,1}(x) = \xi_{\epsilon,\delta,1}(x)$ holds for any $x\in\mathbb{R}$ and $(x,\theta) = (x,\xi_{\epsilon,\delta,1}(x))$ satisfies Equation (6.2) with $-\theta_0 < \xi_{\epsilon,\delta,1}(x) \le \frac{\pi}{2}$. Because the implicit differentiation of Equation (6.2) with respect to x yields the estimate

$$\frac{d}{dx}\left(\xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}\right) = \frac{1}{\sqrt{\delta}} - \frac{\sin\xi_{\epsilon,\delta,1}(x) + \sin\theta_0}{\sqrt{\delta}\,(\sin\xi_{\epsilon,\delta,1}(x)+1)} > 0,$$

$\xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}$ is an increasing function of x on the interval $[0,\infty)$, and thus

$$0 = \xi_{\epsilon,\delta,1}(0) - \frac{\pi}{2} \le \xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2} \le \xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi) + \frac{\pi}{2}$$

for any $\epsilon > 0$. Hence, it follows from the identity $\Theta_{\epsilon,\delta,0}(x) = \Theta_{\epsilon,\delta,1}(x) = \xi_{\epsilon,\delta,1}(x)$ for any $x\in\mathbb{R}$ and the limit $\lim\limits_{\epsilon\to 0}\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi) = -\frac{\pi}{2}$ obtained in Theorem 6.1 that

$$\begin{aligned}0 &\le \lim_{\epsilon\to 0}\left(\xi_{\epsilon,\delta,1}(x,y) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}\right) \le \lim_{\epsilon\to 0}\left(\xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}\right)\\ &\le \lim_{\epsilon\to 0}\left(\xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi) + \frac{\pi}{2}\right) = 0,\end{aligned}$$

that is to say, functions $\xi_{\epsilon,\delta,1}(x,y)$ are uniformly convergent to the function $\mathit{\Theta}_1(x+iy)$ on the strip $\{x+iy;\, 0\le x\le\sqrt{\delta}\,\pi, y\in\mathbb{R}\}$ as $\epsilon\to 0$.

If $x\in[\sqrt{\delta}\,\pi,\infty)$, then $-\frac{\pi}{2} \le \xi_{\epsilon,\delta,1}(x+iy) \le \xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi)$. Hence,

$$-\frac{\pi}{2} \le \underline{\lim}_{\epsilon\to 0}\,\xi_{\epsilon,\delta,1}(x+iy) \le \overline{\lim_{\epsilon\to 0}}\,\xi_{\epsilon,\delta,1}(x+iy) \le \lim_{\epsilon\to 0}\xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi) = -\frac{\pi}{2},$$

i.e. $\xi_{\epsilon,\delta,1}(x,y)$ are uniformly convergent to $-\frac{\pi}{2}$ on the strip $\{x+iy;\, x\in[\sqrt{\delta}\,\pi,\infty), y\in\mathbb{R}\}$. Using the identity $\xi_{\epsilon,\delta,1}(x,y) = \pi - \xi_{\epsilon,\delta,1}(-x,y)$ for any $x+iy\in\mathbb{C}$ also leads to the conclusion that $\xi_{\epsilon,\delta,1}(x,y)$ are uniformly convergent to $\varTheta_1(x+iy)$ in the half plane $\{x+iy;\ x\le 0, y\in\mathbb{R}\}$. In consequence, the sequence of functions $\{\Phi_{\epsilon,\delta,1}(z)\}$ is uniformly convergent to the function $\varPhi_1(z)$ in the complex plane $\mathbb{C}$ as $\epsilon\to 0$, and for any integer $n>0$, the n-th order derivatives $\Phi^{(n)}_{\epsilon,\delta,1}(z)$ of $\Phi_{\epsilon,\delta,1}(z)$ are uniformly convergent to the derivative $\varPhi^{(n)}_1(z)\equiv 0$ on any compact set contained in the half planes $\{z\in\mathbb{C};\ \Re z<-\sqrt{\delta}\,\pi\}$ and $\{z\in\mathbb{C};\ \Re z>\sqrt{\delta}\,\pi\}$. □

Remark. Since $\xi_{\epsilon,\delta,1}$ are continuous functions and uniformly convergent to the function $\varPhi_1$ in $\mathbb{C}$ as $\epsilon\to 0$, $\varPhi_1$ is inevitably a continuous function defined on the complex plane. On the other hand, even though $\Phi_{\epsilon,\delta,1}$ also converge to $\varPhi_1$ uniformly, unlike any of functions $\Phi_{\epsilon,\delta,1}$, $\varPhi_1$ does not possess analyticity on a strip containing the real axis. This is because when ϵ becomes smaller, the function $\Phi_{\epsilon,\delta,1}$ has more an more singularities located at points $\pm\sqrt{\delta}\,\pi+i(2n+1)\sqrt{\epsilon}\,\pi$, for $n=0,\pm1,\pm2,\cdots$, and the strip $\{z\in\mathbb{C};\ |\Im z|<\sqrt{\epsilon}\,\pi\}$ on which $\Phi_{\epsilon,\delta,1}$ is a holomorphic function becomes narrower with its width eventually approaching zero as $\epsilon\to 0$. In addition, $\Phi_{\epsilon,\delta,1}$ has infinitely many branch lines $\{x+i(2n+1)\sqrt{\epsilon}\,\pi;\ |x|\le\sqrt{\delta}\,\pi\}$ together with the branch points of $\Phi_{\epsilon,\delta,1}$ causing the formation of the strip $\{z\in\mathbb{C};\ |\Re z|\le\sqrt{\delta}\,\pi\}$ where $\Phi_1(z)$ loses analyticity completely.

After demonstrating convergence of solitary wave solutions to the functions $\varPhi_0$ and $\varPhi_1$, respectively, whose restrictions to the real axis are identically equal, called a compacton, it is time to show why the compacton is a week solution of the equation

$$-\varphi'+3\varphi\varphi'+\delta(2\varphi'\varphi''+\varphi\varphi''')=0 \tag{6.3}$$

which is the limiting equation of the perturbed equation

$$-\varphi'+\epsilon\varphi'''+3\varphi\varphi'+\delta(2\varphi'\varphi''+\varphi\varphi''')=0 \tag{6.4}$$

when $\epsilon\to 0$ with $\epsilon>0$ and $\delta>0$. We have verified that Equation (6.4) possesses a solitary wave solution, denoted by $\varphi_{\epsilon,\delta}$, which is defined on the real axis and has two different analytic extensions $\Phi_{\epsilon,\delta,0}(z)$ and $\Phi_{\epsilon,\delta,1}(z)$ under our consideration, that means $\varphi_{\epsilon,\delta}(x)=\Phi_{\epsilon,\delta,0}(x)=\Phi_{\epsilon,\delta,1}(x)$ for any $x\in\mathbb{R}$. It follows from proof of Theorem 6.2 that $\varphi_{\epsilon,\delta}(x)$ is uniformly convergent to the compacton $\varPhi_0(x)=\varphi_0(x)$ on the real axis. Since $\varphi_{\epsilon,\delta}(x)$ satisfies Equation (4.1),

$$\varphi'_{\epsilon,\delta}(x)=-\frac{\operatorname{sign}x\,\varphi_{\epsilon,\delta}(x)\sqrt{1-\varphi_{\epsilon,\delta}(x)}}{\sqrt{\delta\varphi_{\epsilon,\delta}(x)+\epsilon}}. \tag{6.5}$$

It follows from the inequality $0 \le \varphi_{\epsilon,\delta} \le 1$ that the estimate

$$
\begin{aligned}
&\left|\varphi'_{\epsilon,\delta}(x) + \frac{\operatorname{sign} x}{\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)(1-\varphi_{\epsilon,\delta}(x))}\right| \\
&= \left|\frac{\operatorname{sign} x}{\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)(1-\varphi_{\epsilon,\delta}(x))}\left(1 - \frac{\sqrt{\varphi_{\epsilon,\delta}(x)}}{\sqrt{\varphi_{\epsilon,\delta}(x)+\frac{\epsilon}{\delta}}}\right)\right| \\
&= \left|\frac{\epsilon \operatorname{sign} x \sqrt{\varphi_{\epsilon,\delta}(x)(1-\varphi_{\epsilon,\delta}(x))}}{\delta\sqrt{\delta\varphi_{\epsilon\delta}(x)+\epsilon}\left(\sqrt{\varphi_{\epsilon,\delta}(x)} + \sqrt{\varphi_{\epsilon,\delta}(x)+\frac{\epsilon}{\delta}}\right)}\right| \le \frac{\sqrt{\epsilon}}{\delta}
\end{aligned}
$$

holds for any $x \in \mathbb{R}$, which implies that functions $\varphi'_{\epsilon,\delta}(x)$ are also uniformly convergent, and

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\varphi'_{\epsilon,\delta}(x) = -\frac{\operatorname{sign} x}{\sqrt{\delta}}\sqrt{\varphi_0(x)\big(1-\varphi_0(x)\big)} \\
&= \varphi'_0(x) = \begin{cases} -\dfrac{1}{2\sqrt{\delta}}\sin\dfrac{x}{\sqrt{\delta}}, & \text{if } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{if } |x| > \sqrt{\delta}\,\pi. \end{cases}
\end{aligned}
$$

As a matter of fact, one may also obtain L^p-convergence of the functions $\varphi_{\epsilon,\delta}$, $\varphi'_{\epsilon,\delta}$, $\varphi''_{\epsilon,\delta}$, $\varphi_{\epsilon,\delta}\varphi'''_{\epsilon,\delta}$ and $\epsilon\varphi'''_{\epsilon,\delta}$ for any p with $p \ge 1$. Notice that as a distribution φ'''_0 is not a L^p-function, but $\varphi_0\varphi'''_0$ belongs to the space L^p as a well-defined distribution in the sense that $\varphi_0\varphi'''_0$ can be expressed as $\varphi_0\varphi'''_0 = \frac{1}{2}(\varphi_0^2)''' - 3\varphi'_0\varphi''_0$ which is the difference of the continuous functions

$$
\frac{1}{2}\big(\varphi_0^2(x)\big)''' = \begin{cases} \frac{1}{4\delta\sqrt{\delta}}(2\sin\frac{2x}{\sqrt{\delta}} + \sin\frac{x}{\sqrt{\delta}}), & \text{if } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{otherwise} \end{cases}
$$

and

$$
3\varphi'_0(x)\varphi''_0(x) = \begin{cases} \frac{3}{8\delta\sqrt{\delta}}\sin\frac{2x}{\sqrt{\delta}}, & \text{if } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{otherwise} \end{cases}
$$

such that $({\varphi_0}')^2 \in H^2$ and $\varphi_0^2 \in H^4$. We summarize results about uniform convergence and L^p-convergence to the compacton φ_0 and its derivatives in the following theorem.

Theorem 6.3. *Let p be any constant with $p \ge 1$. Then as $\epsilon \to 0$, the compacton φ_0 and its derivatives up to the second order are limits of the functions $\varphi_{\epsilon,\delta}$ and their derivatives $\varphi'_{\epsilon,\delta}$ and $\varphi''_{\epsilon,\delta}$ in L^p-norm, respectively, and the functions $\varphi_{\epsilon,\delta}\varphi'''_{\epsilon,\delta}$ and $\epsilon\varphi'''_{\epsilon,\delta}$ also converge to $\varphi_0\varphi'''_0$ and 0 in L^p, respectively. Furthermore, when $\epsilon \to 0$, $\varphi_{\epsilon,\delta}$ and $\varphi'_{\epsilon,\delta}$ converge to φ_0 and φ'_0 uniformly on the real axis, and $\varphi^{(n)}_{\epsilon,\delta}$ converge to $\varphi^{(n)}_0$ uniformly on any compact* *set contained in the open set $(-\infty, -\sqrt{\delta}\,\pi) \cup (-\sqrt{\delta}\,\pi, \sqrt{\delta}\,\pi) \cup (\sqrt{\delta}\,\pi, \infty)$ fro any integer n, such that φ_0 and φ_0' are continuous functions on the real axis with $\varphi_0(x) = \varphi_0'(x) = 0$ for any x with $|x| \geq \sqrt{\delta}\,\pi$. In consequence, the compactons φ_0 and $\phi_0 = -(\beta + c\nu) + \big(3(\beta + c\nu) - \nu(\alpha + c)\big)\varphi_0$ given by (3.9) satisfy Equations (6.3) and (3.7) every where, respectively, with $\varphi_0 \in W^{2,p}$.*

Proof. Because for any fixed $x \in \mathbb{R}$, the derivative of $\theta = \Theta_{\epsilon,\delta,0}(x)$ with respect to ϵ obtained by implicit differentiation of (6.2) takes the form

$$\frac{\partial \theta}{\partial \epsilon} = \frac{x + \sqrt{\delta}(\theta - \frac{\pi}{2})}{2\epsilon\sqrt{\delta}} \, \frac{\sin\theta + \sin\theta_0}{\sin\theta + 1} + \frac{\cos\theta}{(\delta + \epsilon)(\sin\theta + 1)},$$

and $\varphi_{\epsilon,\delta} = \frac{\delta+\epsilon}{2\delta}\sin\theta + \frac{\delta-\epsilon}{2\delta}$, it follows that

$$\begin{aligned} \frac{\partial \varphi_{\epsilon,\delta}(x)}{\partial \epsilon} &= \frac{\delta+\epsilon}{2\delta}\,\frac{\partial\theta}{\partial\epsilon}\cos\theta + \frac{\sin\theta - 1}{2\delta} \\ &= \frac{(\delta+\epsilon)\big(x + \sqrt{\delta}(\theta - \frac{\pi}{2})\big)}{4\epsilon\delta\sqrt{\delta}} \, \frac{\cos\theta(\sin\theta + \sin\theta_0)}{\sin\theta + 1}. \end{aligned} \tag{6.6}$$

Since Equation (6.2) shows that $-\theta_0 \leq \theta \leq \frac{\pi}{2}$ and $x + \sqrt{\delta}(\theta - \frac{\pi}{2}) \geq 0$ when $x \geq 0$, and $\frac{\pi}{2} \leq \theta \leq \pi + \theta_0$ and $x + \sqrt{\delta}(\theta - \frac{\pi}{2}) \leq 0$ if $x \leq 0$, one concludes that

$$\frac{\partial}{\partial\epsilon}\varphi_{\epsilon,\delta}(x) \geq 0$$

for any $x \in \mathbb{R}$, which means

$$\varphi_0(x) \leq \varphi_{\epsilon_1,\delta}(x) \leq \varphi_{\epsilon_2,\delta}(x) \tag{6.7}$$

if $\epsilon_1 < \epsilon_2$. In addition, for any $\epsilon > 0$, $\delta > 0$ and $p > 0$, using the integral transformation $x = \varphi_{\epsilon,\delta}^{-1}(y)$, one obtains the estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |\varphi_{\epsilon,\delta}(x)|^p dx &= 2\int_0^{\infty} |\varphi_{\epsilon,\delta}(x)|^p dx \\ &= 2\int_0^1 \frac{y^{p-1}\sqrt{\delta y + \epsilon}}{\sqrt{1-y}} dy < \infty, \end{aligned}$$

Hence, $\{\int_{-\infty}^{\infty} |\varphi_{\epsilon,\delta}(x) - \varphi_0(x)|^p dx\}$ is a bounded, increasing sequence with respect to ϵ if $0 < \epsilon \leq \epsilon_0 < \infty$ for some $\epsilon_0 > 0$. In consequence, (6.7) and the uniform convergence of $\varphi_{\epsilon,\delta}$ to φ_0 leads to the conclusion that the limit

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} |\varphi_{\epsilon,\delta}(x) - \varphi_0(x)|^p dx = 0$$

exists for any $p > 0$, which combined with the fact of uniform convergence of $\varphi'_{\epsilon,\delta}$ to φ'_0 and the following estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\varphi'_{\epsilon,\delta}(x) - \varphi'_0(x)|^p dx \\
=& 2\int_0^{\sqrt{\delta}\,\pi} |\varphi'_{\epsilon,\delta}(x) - \varphi'_0(x)|^p dx + 2\int_{\sqrt{\delta}\,\pi}^{\infty} \left| \frac{\varphi_{\epsilon,\delta}(x)\sqrt{1-\varphi_{\epsilon,\delta}(x)}}{\sqrt{\delta\varphi_{\epsilon,\delta}(x)+\epsilon}} \right|^p dx \\
\le& 2\int_0^{\sqrt{\delta}\,\pi} |\varphi'_{\epsilon,\delta}(x) - \varphi'_0(x)|^p dx + \frac{2}{\delta^{\frac{p}{2}}}\int_{\sqrt{\delta}\,\pi}^{\infty} |\varphi_{\epsilon,\delta}(x)|^{\frac{p}{2}} dx
\end{aligned}$$

shows that $\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} |\varphi'_{\epsilon,\delta}(x) - \varphi'_0(x)|^p dx = 0$ for any $p > 0$.

To show convergence of $\{\varphi''_{\epsilon,\delta}\}$ and $\{(\delta\varphi_{\epsilon,\delta}+\epsilon)\varphi'''_{\epsilon,\delta}\}$, we differentiate the equation

$$(\varphi'_{\epsilon,\delta})^2 = \frac{\varphi^2_{\epsilon,\delta}(1-\varphi_{\epsilon,\delta})}{\delta\varphi_{\epsilon,\delta}+\epsilon} = F(\varphi_{\epsilon,\delta})$$

on both sides with respect to x to obtain

$$\begin{aligned}
2\varphi''_{\epsilon,\delta} = F'(\varphi_{\epsilon,\delta}) &= \frac{-\varphi_{\epsilon,\delta}}{\delta(\delta\varphi_{\epsilon,\delta}+\epsilon)^2}[2(\delta\varphi_{\epsilon,\delta}+\epsilon)^2 - (\delta+\epsilon)(\delta\varphi_{\epsilon,\delta}+\epsilon) - (\epsilon^2+\epsilon\delta)] \\
&= -\frac{1}{\delta^2}[2(\delta\varphi_{\epsilon,\delta}+\epsilon) - (\delta+3\epsilon) + \frac{\epsilon^2(\epsilon+\delta)}{(\delta\varphi_{\epsilon,\delta}+\epsilon)^2}],
\end{aligned}$$

and

$$2\varphi'''_{\epsilon,\delta} = F''(\varphi_{\epsilon,\delta})\varphi'_{\epsilon,\delta} = \frac{2}{\delta}\left(1 - \frac{\epsilon^2(\epsilon+\delta)}{(\delta\varphi_{\epsilon,\delta}+\epsilon)^3}\right)\frac{(\operatorname{sign} x)\varphi_{\epsilon,\delta}\sqrt{1-\varphi_{\epsilon,\delta}}}{\sqrt{\delta\varphi_{\epsilon,\delta}+\epsilon}}.$$

Then it follows that

$$|\varphi''_{\epsilon,\delta}(x)| \le \frac{\varphi_{\epsilon,\delta}(x)}{\delta} + \frac{\delta+\epsilon}{\delta}\frac{\varphi_{\epsilon,\delta}(x)}{\delta\varphi_{\epsilon,\delta}(x)+\epsilon} \le \frac{2\delta+\epsilon}{\delta^2}, \tag{6.8}$$

$$|\varphi_{\epsilon,\delta}(x)\varphi'''_{\epsilon,\delta}(x)| \le \frac{3\delta+\epsilon}{\delta^2\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)} \quad \text{and} \quad |\epsilon\varphi'''_{\epsilon,\delta}(x)| \le \frac{2\epsilon+\delta}{\delta\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)} \tag{6.9}$$

for any $x \in \mathbb{R}$. Because $\frac{\varphi_{\epsilon,\delta}}{\delta\varphi_{\epsilon,\delta}+\epsilon} = \frac{\sin\theta+\sin\theta_0}{\delta(\sin\theta+1)} = \frac{-1}{\sqrt{\delta}}\frac{d\theta}{dx}$, where $\theta = \Theta_{\epsilon,\delta,0}(x)$ is the same as defined in (6.6), and it has been shown in Theorem 6.2 that $\frac{d\theta(x)}{dx} \to 0$ for any x with $|x| > \sqrt{\delta}\,\pi$ as $\epsilon \to 0$, it follows that the limit

$$\lim_{\epsilon\to 0}\frac{\varphi_{\epsilon,\delta}(x)}{\delta\varphi_{\epsilon,\delta}(x)+\epsilon} = 0. \tag{6.10}$$

exists for any $|x| > \sqrt{\delta}\,\pi$. On the other hand, it follows from (6.6) and uniform convergence of the functions $\theta = \Theta_{\epsilon,\delta,0}(x)$ on the real axis that for any fixed x_0 with $x_0 > \sqrt{\delta}\,\pi$, there is an $\epsilon_0 > 0$, such that whenever $x > x_0$ and $0 < \epsilon < \epsilon_0$, the following expression

$$\frac{\partial}{\partial\epsilon}\left(\frac{\varphi_{\epsilon,\delta}(x)}{\delta\varphi_{\epsilon,\delta}(x)+\epsilon}\right) = \frac{\epsilon\frac{\partial\varphi_{\epsilon,\delta}(x)}{\partial\epsilon} - \varphi(x)}{(\delta\varphi_{\epsilon,\delta}(x)+\epsilon)^2}$$
$$= \frac{\varphi(x)}{(\delta\varphi_{\epsilon,\delta}(x)+\epsilon)^2}\left(\frac{x+\sqrt{\delta}(\theta-\frac{\pi}{2})}{2\sqrt{\delta}}\,\frac{\cos\theta}{\sin\theta+1} - 1\right)$$
$$= \frac{\varphi(x)\cot(\frac{\pi}{4}+\frac{\theta}{2})}{(\delta\varphi_{\epsilon,\delta}(x)+\epsilon)^2}\left(\frac{x+\sqrt{\delta}(\theta-\frac{\pi}{2})}{2\sqrt{\delta}} - \tan(\frac{\pi}{4}+\frac{\theta}{2})\right) \geq 0,$$

holds, which means $\frac{\varphi_{\epsilon,\delta}(x)}{\delta\varphi_{\epsilon,\delta}(x)+\epsilon} \leq \frac{\varphi_{\epsilon_0,\delta}(x)}{\delta\varphi_{\epsilon_0,\delta}(x)+\epsilon_0}$ if $0 < \epsilon < \epsilon_0$ and $x > x_0$. Then applying (6.8), (6.10), the convergence of $\varphi_{\epsilon,\delta}$ in L^p and Lebesgue's Dominated Convergence Theorem to the integral

$$\int_{-\infty}^{\infty} |\varphi''_{\epsilon,\delta}(x) - \varphi''_0(x)|^p dx = 2\int_0^{\sqrt{\delta}\,\pi} |\varphi''_{\epsilon,\delta}(x) - \varphi''_0(x)|^p dx + 2\int_{\sqrt{\delta}\,\pi}^{\infty} |\varphi''_{\epsilon,\delta}(x)|^p dx,$$

one concludes that $\lim\limits_{\epsilon\to 0}\int_{-\infty}^{\infty} |\varphi''_{\epsilon,\delta}(x) - \varphi''_0(x)|^p dx = 0$. In a similar way, using (6.9) and Lebesgue's Dominated Convergence Theorem, one may also show that for any $p > 0$, the limits

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} |\varphi_{\epsilon,\delta}(x)\varphi'''_{\epsilon,\delta}(x) - \varphi_0(x)\varphi'''_0(x)|^p dx = 0 \quad \text{and} \quad \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} |\epsilon\varphi'''_{\epsilon,\delta}(x)| dx = 0$$

exist. Therefore, the compacton φ_0 belongs to the Sobolev space $W^{2,p}$ for any p with $p \geq 1$. Because $\Phi_{\epsilon,\delta,0}(z)$ converge to $\Phi_0(z)$ uniformly on any compact set of $\mathbb{C} \setminus \{z \in \mathbb{R};\ |z| \geq \sqrt{\delta}\,\pi\}$, $\Phi_{\epsilon,\delta,1}(z)$ converge to $\Phi_1(z)$ uniformly on any compact set of $\mathbb{C}\setminus\{z \in \mathbb{C};\ |\Re z| \leq \sqrt{\delta}\,\pi\}$ and $\Phi_0(x) = \Phi_1(x) = \varphi_0(x)$ for any $x \in \mathbb{R}$, for any positive integer n, the n-th order derivatives $\varphi^{(n)}_{\epsilon,\delta}(x)$ converge to $\varphi^{(n)}_0(x)$ everywhere except for the points $x = \pm\sqrt{\delta}\,\pi$. Furthermore, $\varphi_0(\pm\sqrt{\delta}\,\pi) = \varphi'_0(\pm\sqrt{\delta}\,\pi) = 0$. In consequence, we come to the conclusion that the compactons φ_0 and ϕ_0 given in (3.9) satisfy Equations (4.1) and (3.7) everywhere, respectively. □

7. Convergence to peakons.

In Section 5, we have shown two different extensions $t = \Xi_1(z)$ and $t = \Xi_2(z)$ for the solution $t = \Xi(x)$ of Equation (5.4) to the manifolds $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, such that $\Phi_{\delta,\rho,i} = \frac{\rho-\delta}{2\delta}\cosh\Xi_i - \frac{\rho+\delta}{2\delta}$ is an analytic extension of the solitary wave solution

$\varphi = \frac{\rho-\delta}{2\delta}\cosh\Xi - \frac{\rho+\delta}{2\delta}$ of Equation (5.1) to the manifold $\mathcal{Y}_i$, and continuous up to the boundary of $\mathcal{Y}_i$ for $i = 1, 2$. To indicate above functions or manifolds depending on the parameters δ and ρ and to show their convergence in this section, we denote $\Xi_{\delta,\rho,i} = \Xi_i$ for $i = 1, 2$,

$$\mathcal{Y}_{\delta,\rho,1} = \mathcal{Y}_1 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{iy;\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \leq y \leq \big((2n+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\},$$
$$\mathcal{Y}_{\delta,\rho,2} = \mathcal{Y}_2 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{x + i\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,\big)\pi;\ x \leq 0\}$$

and $\varphi_{\delta,\rho} = \varphi$. The following two theorems present results on convergence of the functions $\Phi_{\delta,\rho,1}$ and $\Phi_{\delta,\rho,2}$ to the limiting functions Φ_1 and Φ_2, respectively, whose restrictions to the real axis are identical and called a peakon.

Theorem 7.1. *When $\rho \to \delta$, the sequence of the functions $\{\Phi_{\delta,\rho,1}\}$ converges to the function Φ_1 defined as*

$$\Phi_1(z) = \begin{cases} -\,e^{\frac{-z}{\sqrt{\delta}}}, & \text{if } \Re z > 0 \\ -\,e^{\frac{z}{\sqrt{\delta}}}, & \text{if } \Re z < 0, \end{cases}$$

and for any $z \in \{iy;\ y \in \mathbb{R}\}$, $\lim_{\rho\to\delta} \Phi^l_{\delta,\rho,1}(z) = -e^{\frac{z}{\sqrt{\delta}}}$ and $\lim_{\rho\to\delta} \Phi^r_{\delta,\rho,1}(z) = -e^{\frac{-z}{\sqrt{\delta}}}$, where $\Phi^l_{\delta,\rho,1}(z_0) = \lim_{\substack{z\to z_0\\ Re z<0}} \Phi_{\delta,\rho,1}(z)$ and $\Phi^r_{\delta,\rho,1}(z_0) = \lim_{\substack{z\to z_0\\ Re z>0}} \Phi_{\delta,\rho,1}(z)$ for any $z_0 \in \{iy;\ y \in \mathbb{R}\}$. Therefore, Φ_1 is a holomorphic function on the left-half plane $\{z \in \mathbb{C};\ \Re z < 0\}$ and the right-half plane $\{z \in \mathbb{C};\ \Re z > 0\}$, having the imaginary axis as its natural boundary.

Since the proof of Theorem 7.1 follows from that of Theorem 7.2 as a direct consequence, we will show it after verifying Theorem 7.2.

Theorem 7.2. *When $\rho \to \delta$, the sequence of the functions $\{\Phi_{\delta,\rho,2}\}$ converges to the function Φ_2 defined as*

$$\Phi_2(z) = \begin{cases} -\,e^{\frac{-z}{\sqrt{\delta}}}, & \text{if } \Im z \neq 0, \text{ or } \Im z = 0 \text{ and } \Re z > 0 \\ -\,e^{\frac{z}{\sqrt{\delta}}}, & \text{if } \Im z = 0 \text{ and } \Re z \leq 0. \end{cases}$$

Therefore, Φ_2 is a holomorphic function on the open set $\mathbb{C} \setminus \{z \in \mathbb{C};\ \Re z \leq 0, \Im z = 0\}$.

Proof. The method to be used here is to show that the functions $(\rho - \delta)e^t$ are uniformly bounded on the half plane $\{z \in \mathbb{R};\ \Re z \geq -N\}$ for any constant $N > 0$, where $t = \Xi_{\delta,\rho,2}(z)$, so that as ρ is close to δ, $\{(\rho - \delta)e^t\}$ is a normal family convergent on the compact sets to be defined presently.

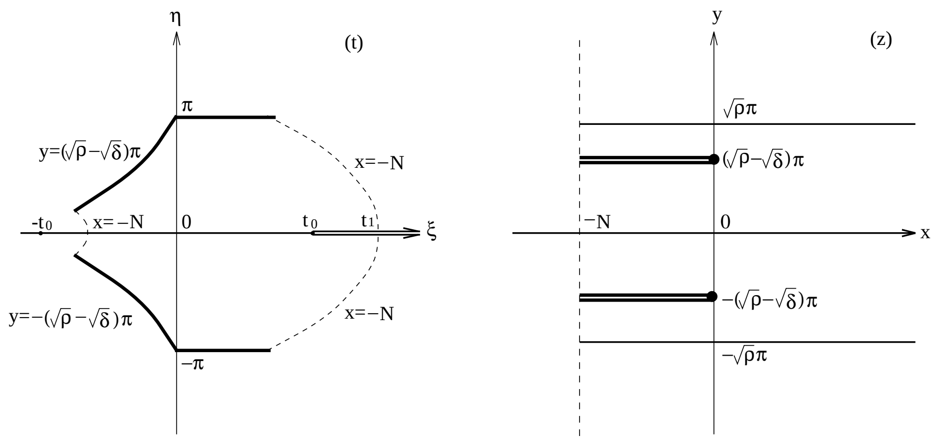

Fig. 15. A sketch of the image of the half plane $\{z \in \mathbb{C};\ \Re z \geq -N\}$

It follows from Lemma 5.1 and Theorem 5.4 that $\Xi_{\delta,\rho,2}$ maps the set $\overline{\mathcal{Y}_{\delta,\rho,2}} \setminus \{z \in \mathbb{C};\ \Re z < -N\}$ to the region bounded by the streamlines $\pm(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ and the equipotential $-N = x(\xi, \eta)$ of the single-valued branch Δ_0 as shown in Figure 15. Then $\sup_{\Re z \geq -N} |\Im\big(\Xi_{\delta,\rho,2}(z)\big)| \leq \pi$ and $t_0 < \sup_{\Re z \geq -N} |\Re\big(\Xi_{\delta,\rho,2}(z)\big)| \leq t_1$, where t_1 is the intersection point of the equipotential $-N = x(\xi, \eta)$ and the positive real axis, *i.e.* $t_1 = \Xi_{\delta,\rho,2}(-N + i\sqrt{\rho}\,\pi)$. It follows from (5.4) that

$$-N + i\sqrt{\rho}\,\pi + \sqrt{\delta}\,t_1 = -\sqrt{\rho}\,\log\Big(\frac{\tanh\frac{t_0}{2} - \tanh\frac{t_1}{2}}{\tanh\frac{t_0}{2} + \tanh\frac{t_1}{2}}\Big),$$

or

$$\tanh\frac{t_1}{2} = \sqrt{\frac{\delta}{\rho}}\coth\frac{\sqrt{\delta}\,t_1 - N}{2\sqrt{\rho}}.$$

Since the inequality $\tanh\frac{t_1}{2} \geq \tanh\big(\sqrt{\frac{\delta}{\rho}}\,\frac{t_1}{2}\big)$ holds when $0 < \delta < \rho$,

$$\tanh\Big(\sqrt{\frac{\delta}{\rho}}\,\frac{t_1}{2}\Big) \leq \sqrt{\frac{\delta}{\rho}}\coth\Big(\sqrt{\frac{\delta}{\rho}}\frac{t_1}{2} - \frac{N}{2\sqrt{\rho}}\Big) = \sqrt{\frac{\delta}{\rho}}\,\frac{1 - \tanh\big(\sqrt{\frac{\delta}{\rho}}\frac{t_1}{2}\big)\tanh\frac{N}{2\sqrt{\rho}}}{\tanh\big(\sqrt{\frac{\delta}{\rho}}\frac{t_1}{2}\big) - \tanh\frac{N}{2\sqrt{\rho}}},$$

which leads to the estimate

$$\tanh\Big(\sqrt{\frac{\delta}{\rho}}\frac{t_1}{2}\Big) \leq \frac{(\sqrt{\rho} - \sqrt{\delta}\,)\tanh\frac{N}{2\sqrt{\rho}} + \sqrt{4\sqrt{\delta\rho} + (\sqrt{\rho} - \sqrt{\delta}\,)^2\tanh^2\frac{N}{2\sqrt{\rho}}}}{2\sqrt{\rho}},$$

and thus

$$(\rho - \delta)e^{t_1} \leq \frac{(\sqrt{\rho} + \sqrt{\delta}\,)\left(\sqrt{\rho} + \sqrt{\delta} + \sqrt{4\sqrt{\delta\rho} + (\sqrt{\rho} - \sqrt{\delta}\,)^2\tanh^2\frac{N}{2\sqrt{\rho}}}\right)^{\sqrt{\frac{\rho}{\delta}}}}{(\sqrt{\rho} - \sqrt{\delta}\,)^{\sqrt{\frac{\rho}{\delta}}-1}\big(1 - \tanh\frac{N}{2\sqrt{\rho}}\big)^{\sqrt{\frac{\rho}{\delta}}}}.$$

Therefore, for any fixed constant ρ_0 with $\rho_0 > \delta$, the inequality $(\rho - \delta)|e^t| = (\rho - \delta)e^{\Re(\Xi_{\delta,\rho,2})} \le (\rho-\delta)e^{t_1} < \infty$ holds for any ρ with $\delta < \rho \le \rho_0$ on the set $\overline{\mathcal{Y}_{\delta,\rho,2}} \setminus \{z \in \mathbb{C};\ \Re z < -N\}$.

Let N_0 be any positive integer and let ν be any constant with $0 < \nu < \sqrt{\delta}$. Then there is a $\rho_0 > \delta$ such that for any integer k with $|k| \le N_0+1$ and any $\rho \in (\delta, \rho_0)$, the inequalities

$$(2k-1)\sqrt{\rho} < 2k\sqrt{\delta} - \nu < 2k\sqrt{\rho} - (\sqrt{\rho} - \sqrt{\delta}\,),$$

and

$$2k\sqrt{\rho} + (\sqrt{\rho} - \sqrt{\delta}\,) < 2k\sqrt{\delta} + \nu < (2k+1)\sqrt{\rho}$$

hold. Hence, for each $\rho \in (\delta, \rho_0)$, the function $(\rho-\delta)e^{\Xi_{\delta,\rho,2}}$ is analytic in the set

$$M = \{z \in \mathbb{C};\ \Re z \ge -N_0, |\Im z| \le (2N_0+1)\sqrt{\delta}\,\pi\} \setminus E,$$

where $E = \bigcup_{k=-N_0}^{N_0} \{z \in \mathbb{C};\ \Re z \le 0, (2k\sqrt{\delta} - \nu)\pi \le \Im z \le (2k\sqrt{\delta} + \nu)\pi\}$, and thus $\{(\rho - \delta)e^{\Xi_{\delta,\rho,2}}\}$ is a normal family in M. Let $\{(\rho_n - \delta)e^{\Xi_{\delta,\rho_n,2}}\}$ be a subsequence uniformly convergent to a holomorphic function, denoted by $g(z)$, in any compact set of M as $n \to \infty$, where $\rho_n > \rho_{n+1}$ and $\lim_{n\to\infty} \rho_n = \delta$. Since for $t = \Xi_{\delta,\rho_n,2}(z)$, it follows from (5.4) that $\tanh\frac{t}{2} = \tanh\frac{t_0}{2}\tanh\frac{z+\sqrt{\delta}\,t}{2\sqrt{\rho}}$, or

$$(\rho-\delta)e^t = \frac{(\rho-\delta)\big((\sqrt{\rho}+\sqrt{\delta}\,)e^{\frac{z+\sqrt{\delta}\,t}{\sqrt{\rho}}} + \sqrt{\rho} - \sqrt{\delta}\,\big)}{(\sqrt{\rho}-\sqrt{\delta}\,)e^{\frac{z+\sqrt{\delta}\,t}{\sqrt{\rho}}} + \sqrt{\rho} + \sqrt{\delta}},$$

taking the limit on both sides of the above equation, one obtains the relation

$$g(z) = \frac{4\delta g(z)}{g(z) + 4\delta e^{-\frac{z}{\sqrt{\delta}}}}. \tag{7.1}$$

It follows from Lemma 5.1 that for any fixed $x_1 > 0$, when $\Re z \ge x_1$,

$$\Re(\Xi_{\delta,\rho,2}(z)) \ge \Re(\Xi_{\delta,\rho,2}(x_1)) = \xi_1$$

and

$$\tanh\frac{\xi_1}{2} = \tanh\frac{t_0}{2}\tanh\frac{x_1+\sqrt{\delta}\,\xi_1}{2\sqrt{\rho}} \ge \frac{\delta}{\rho}\tanh\big(\frac{\xi_1}{2} + \frac{x_1}{2\sqrt{\delta}}\big), \tag{7.2}$$

which leads to the estimate

$$(\rho-\delta)e^{\xi_1} \ge \frac{(\delta+\rho)\tanh\frac{x_1}{2\sqrt{\delta}} + \sqrt{4\delta\rho\tanh^2\frac{x_1}{2\sqrt{\delta}} + (\rho-\delta)^2}}{\tanh\frac{x_1}{2\sqrt{\delta}} + 1}.$$

Therefore,

$$(\rho-\delta)\left|e^{\Xi_{\delta,\rho,2}(z)}\right| \geq (\rho-\delta)e^{\Xi_{\delta,\rho,2}(x_1)} \geq \frac{4\delta\tanh\frac{x_1}{2\sqrt{\delta}}}{\tanh\frac{x_1}{2\sqrt{\delta}}+1},$$

and $g(z)\neq 0$ when $\Re z \geq x_1$, which combined with (7.1) yields $g(z)=4\delta(1-e^{-\frac{z}{\sqrt{\delta}}})$ for any $z\in M$. Then

$$\lim_{n\to\infty}\Phi_{\delta,\rho_n,2}(z)=\lim_{n\to\infty}\left(\frac{\rho_n-\delta}{2\delta}\cosh\Xi_{\delta,\rho_n,2}(z)-\frac{\rho_n+\delta}{2\delta}\right)=\frac{g(z)}{4\delta}-1=-e^{-\frac{z}{\sqrt{\delta}}}=\Phi_2(z).$$

Because any convergent subsequence of the normal family $\{\Phi_{\delta,\rho,2}\}$ has the same limit as $\rho\to\delta$, the sequence $\{\Phi_{\delta,\rho,2}\}$ itself is convergent to $\Phi_2(z)$ in M. In addition, since m and ν are chosen arbitrarily,

$$\mathcal{M}=\mathbb{C}\setminus\bigcup_{k=-\infty}^{\infty}\{z\in\mathbb{C};\ \Re z\leq 0, \Im z=2k\sqrt{\delta}\pi\}$$

is the union of such sets as M defined above, and the sequence $\{\Phi_{\delta,\rho,2}\}$ is convergent to Φ_2 in $\mathcal{M}$.

To show the limit $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(z)=\Phi_2(z)$ exists for $z=x+i2k\sqrt{\delta}\,\pi$ and $x\leq 0$, one may take implicit differentiation with respect to ρ on both sides of the equation

$$\frac{\tanh\frac{t_0}{2}-\tanh\frac{t}{2}}{\tanh\frac{t_0}{2}+\tanh\frac{t}{2}}=e^{-\frac{z+\sqrt{\delta}t}{\sqrt{\rho}}} \tag{7.3}$$

with $t=\Xi_{\delta,\rho,2}(z)$ to obtain

$$(\rho-\delta)\frac{\partial t}{\partial\rho}=-\tanh\frac{t}{2}-\frac{z+\sqrt{\delta}\,t}{2\sqrt{\delta}}\big(\tanh^2\frac{t_0}{2}-\tanh^2\frac{t}{2}\big), \tag{7.4}$$

and then

$$\frac{\partial\Phi_{\delta,\rho,2}}{\partial\rho}=\frac{\rho-\delta}{2\delta}\frac{\partial t}{\partial\rho}\sinh t+\frac{\cosh t-1}{2\delta}=\frac{z+\sqrt{\delta}\,t}{2\sqrt{\delta}\,\rho}\frac{\sinh t\,\Phi_{\delta,\rho,2}}{\cosh t+1}. \tag{7.5}$$

When $z=x\leq 0$, $-t_0<t=\Xi_{\delta,\rho,2}(x)\leq 0$, and $\Phi_{\delta,\rho,2}(x)<0$. Hence, (7.5) shows that $\frac{\partial}{\partial\rho}\Phi_{\delta,\rho,2}(x)\leq 0$ which implies that both limits $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x)$ and $\lim_{\rho\to\delta}(\rho-\delta)e^{-\Xi_{\delta,\rho,2}(x)}$ exist. Taking the limit on both side of the equality

$$(\rho-\delta)e^{-t}=\frac{(\rho-\delta)\big(\sqrt{\rho}-\sqrt{\delta}+(\sqrt{\rho}+\sqrt{\delta})e^{-\frac{x+\sqrt{\delta}\,t}{\sqrt{\rho}}}\big)}{\sqrt{\rho}+\sqrt{\delta}+(\sqrt{\rho}-\sqrt{\delta})e^{-\frac{x+\sqrt{\delta}\,t}{\sqrt{\rho}}}}$$

obtained from (7.3) leads to the evaluations $\lim_{\rho\to\delta}(\rho-\delta)e^{-\Xi_{\delta,\rho,2}(x)} = 4\delta(1-e^{\frac{x}{\sqrt{\delta}}})$ and $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x) = -e^{\frac{x}{\sqrt{\delta}}}$.

If $z = x + 2k\sqrt{\delta}\,\pi i$ with $k > 0$ and $\delta < \rho < \rho_0$, then $(2k-1)\sqrt{\rho} < 2k\sqrt{\delta} < (2k-1)\sqrt{\rho} + \sqrt{\delta}$ and the function $\Xi_{\delta,\rho,2}$ maps the line $\{x + 2k\sqrt{\delta}\,\pi i;\ x \in \mathbb{R}\}$ to the right half plane with $-\pi < \Im(\Xi_{\delta,\rho,2}) = \eta < 0$ and $\frac{\sqrt{\delta}\,\Re(\Xi_{\delta,\rho,2})+x}{\sqrt{\rho}} \ge 0$. Then similar to the derivation of (7.1), one may show that the sequence $\{(\rho-\delta)e^{\Xi_{\delta,\rho,2}(z)}\}$ is bounded on the line $\{z \in \mathbb{C};\ \Im z = 2k\sqrt{\delta}\,\pi i,\ \Re z < 0\}$, and thus $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x + 2k\sqrt{\delta}\,\pi i) = -e^{-\frac{x}{\sqrt{\delta}}}$. Because $\overline{\Phi_{\delta,\rho,2}(x + 2k\sqrt{\delta}\,\pi i)} = \Phi_{\delta,\rho,2}(x - 2k\sqrt{\delta}\,\pi i)$, $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x - 2k\sqrt{\delta}\,\pi i) = -e^{-\frac{x}{\sqrt{\delta}}}$. □

Proof of Theorem 7.1. It follows from the definition of the function $\Xi_{\delta,\rho,1}(z)$ that

$$\Xi_{\delta,\rho,1}(z) = \Xi_{\delta,\rho,2}(z) \quad \text{and} \quad \lim_{\substack{z\to z_0\\ \Re z>0}} \Xi_{\delta,\rho,1}(z) = \Xi_{\delta,\rho,2}(z_0)$$

for any z and z_0 with $\Re z > 0$ and $\Re z_0 = 0$. Hence, $\lim_{\rho\to\delta}\Phi_{\delta,\rho,1}(z) = \lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(z) = e^{-\frac{z}{\sqrt{\delta}}}$ for $\Re z > 0$ and $\lim_{\rho\to\delta}\Phi_{\delta,\rho,1}(z) = \lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(z) = e^{-\frac{z}{\sqrt{\delta}}}$ for any z with $\Re z = 0$. On the other hand, $\Xi_{\delta,\rho,1}(z) = -\overline{\Xi_{\delta,\rho,1}(-\overline{z})}$ and then $\Phi_{\delta,\rho,1}(z) = \overline{\Phi_{\delta\rho,1}(-\overline{z})}$. Therefore, one may obtain the rest of results in Theorem 7.1 symmetrically. □

Remark. Although there are also other different analytic extensions of the solitary wave solution $\varphi_{\delta,\rho}$ besides $\Phi_{\delta,\rho,1}$ and $\Phi_{\delta,\rho,2}$, all of them are identical and analytic on the strip $\{z \in \mathbb{C};\ |\Im z| < (\sqrt{\rho} - \sqrt{\delta})\pi\}$, having branch points $z = \pm(\sqrt{\rho} - \sqrt{\delta})\pi i$. It is these two singularities approaching the real axis as $\rho \to \delta$ and $\rho > \delta$, which causes the formation of the singularity $z = 0$ of the peakon $\varphi_p(x) = -e^{-\frac{|x|}{\sqrt{\delta}}}$ as the limit of solitary wave solutions $\{\varphi_{\delta,\rho}\}$. On the other hand, since $\{\Phi_{\delta,\rho,1}(z)\}$ is a normal family of analytic functions, converging to $\Phi_1(z)$ on any compact set contained in the open set $\mathbb{C}\setminus\{z = iy;\ y \in \mathbb{R}\}$, the n-th order derivatives $\Phi^{(n)}_{\delta,\rho,1}(z)$ also converge to $\Phi^{(n)}_1(z)$ everywhere in $\mathbb{C}\setminus\{z = iy;\ y \in \mathbb{R}\}$ for any positive integer n.

Next, we show that the sequence $\{\varphi_{\delta,\rho}\}$ is uniformly convergent to φ_p on the real axis, and $\{\varphi_{\delta,\rho}\}$, $\{\varphi'_{\delta,\rho}\}$ and $\{(\delta\varphi_{\delta,\rho} + \rho)\varphi''_{\delta,\rho}\}$ are L^q-convergent to φ_p, φ'_p and $\delta(\varphi_p + 1)\varphi''_p$, respectively, for any $q \ge 1$, where $(\varphi_p + 1)\varphi''_p$ is defined as the difference of the two distributions $\frac{[(\varphi_p+1)^2]''}{2}$ and $(\varphi'_p)^2$ such that $(\varphi_p + 1)^2 \in H^2$, $(\varphi'_p)^2 \in H^1$ and

$$\big(\varphi_p(x) + 1\big)\varphi''_p(x) = \frac{e^{\frac{-|x|}{\sqrt{\delta}}}\big(e^{\frac{-|x|}{\sqrt{\delta}}} - 1\big)}{\delta}$$

almost everywhere. These results help understand that the limiting equation

$$\varphi' - \delta\varphi''' + 3\varphi\varphi' - \delta(2\varphi'\varphi'' + \varphi\varphi''') = 0 \tag{7.6}$$

of the equation

$$\varphi' - \rho\varphi''' + 3\varphi\varphi' - \delta(2\varphi'\varphi'' + \varphi\varphi''') = 0 \tag{7.7}$$

has the peakon solution φ_p as the limit of solitary solutions $\varphi_{\rho,\delta}$ satisfying Equation (7.7) when $\rho \to \delta$ and $\rho > \delta > 0$.

For the solution $t = \Xi_{\delta,\rho}(x) = \Xi_{\delta,\rho,i}(x)$ of Equation (7.3), where $i = 1, 2$ and $t, x \in \mathbb{R}$, it satisfies the equivalent equation $\tanh \frac{t}{2} = \tanh \frac{t_0}{2} \tanh \frac{x+\sqrt{\delta}\, t}{2\sqrt{\rho}}$, which leads to the estimates

$$\tanh \frac{\sqrt{\delta}\,|t|}{2\sqrt{\rho}} \geq \frac{\delta}{\rho} \tanh \frac{|x + \sqrt{\delta}\, t|}{2\sqrt{\rho}},$$

and

$$\begin{aligned} &(\rho-\delta)\cosh\frac{x+\sqrt{\delta}\,t}{\sqrt{\rho}} \\ \geq\ & \frac{(\rho-\delta)\left[2\rho\tanh\frac{|x|}{2\sqrt{\rho}} + \sqrt{4\delta\rho\tanh^2\frac{|x|}{2\sqrt{\rho}} + (\rho-\delta)^2} - (\rho-\delta)\right]\left(1+\tanh\frac{|x|}{2\sqrt{\rho}}\right)}{\left[2\rho\tanh\frac{|x|}{2\sqrt{\rho}} - \sqrt{4\delta\rho\tanh^2\frac{|x|}{2\sqrt{\rho}} + (\rho-\delta)^2} + (\rho-\delta)\right]\left(1-\tanh\frac{|x|}{2\sqrt{\rho}}\right)} \\ \geq\ & \frac{\delta\tanh\frac{|x|}{2\sqrt{\rho}}}{1-\tanh\frac{|x|}{2\sqrt{\rho}}}. \end{aligned} \tag{7.8}$$

Then using the identity

$$\Phi_{\delta,\rho,j}(z) = \frac{-2\rho}{(\rho-\delta)\cosh\frac{z+\sqrt{\delta}\,t}{\sqrt{\rho}} + \rho + \delta} \tag{7.9}$$

for $j = 1, 2$, and (7.8) yields the inequality

$$|\varphi_{\delta,\rho}(x)| \leq \frac{2\rho}{\frac{\delta\tanh\frac{|x|}{2\sqrt{\rho}}}{1-\tanh\frac{|x|}{2\sqrt{\rho}}} + 2\delta} \leq \frac{2\rho}{\delta}\left(1 - \tanh\frac{|x|}{2\sqrt{\rho}}\right),$$

which implies that the functions $\varphi_{\delta,\rho}(x)$ decay exponentially to zero at infinity. Then for any $\epsilon > 0$ and for a fixed $\rho_0 > \delta$, there is an $N > 0$, when $|x| > N$ and $\delta < \rho < \rho_0$,

$$|\varphi_{\delta,\rho}(x)| \leq \epsilon. \tag{7.10}$$

Since

$$\frac{d}{dx}\frac{\varphi_{\delta,\rho}}{\varphi_p} = \frac{\varphi_{\delta,\rho}(\rho-\delta)\,\mathrm{sign}\,x}{\sqrt{\delta}\,\varphi_p\sqrt{\delta\varphi_{\delta,\rho}+\rho}(\sqrt{\delta\varphi_{\delta,\rho}+\rho}+\sqrt{\delta\varphi_{\delta,\rho}+\delta})}\begin{cases} >0, & x>0 \\ <0, & x<0 \end{cases}$$

and $\frac{\varphi_{\delta,\rho}(0)}{\varphi_p(0)} = 1$, one has

$$0 \le \frac{\varphi_{\delta,\rho}(x)}{\varphi_p(x)} - 1 \le \frac{\varphi_{\delta,\rho}(N)-\varphi_p(N)}{\varphi_p(N)}$$

for any x with $|x| \le N$. Because $\varphi_{\delta,\rho}(N) \to \varphi_p(N)$ as $\rho \to \delta$ and $|\varphi_p(x)| \le 1$, the inequality $0 \le \varphi_p(x) - \varphi_{\delta,\rho}(x) \le \frac{\varphi_{\delta,\rho}(N)-\varphi_p(N)}{\varphi_p(N)} < \epsilon$ holds when ρ is sufficiently close to δ, which combined with (7.10) shows the uniform convergence of the sequence $\{\varphi_{\delta,\rho}\}$. Since (7.5) shows that $\frac{\partial\varphi_{\delta,\rho}}{\partial\rho} < 0$, *i.e.*

$$0 \ge \varphi_p \ge \varphi_{\delta,\rho_1} \ge \varphi_{\delta,\rho_2} \tag{7.11}$$

if $\delta < \rho_1 < \rho_2$, and using the integral transformation $x = \varphi^{-1}_{\delta,\rho}(y)$ yields the estimate

$$\int_{-\infty}^{\infty} |\varphi_{\delta,\rho}(x)|^q dx = 2\int_0^{-1} \frac{|y|^{q-1}\sqrt{\delta y+\rho}}{\sqrt{1+y}}dy < \infty$$

for any $q > 0$, together with the other estimates $|\varphi'_{\delta,\rho}| \le \frac{|\varphi_{\delta,\rho}|}{\sqrt{\delta}}$,

$$|(\delta\varphi_{\delta,\rho}+\rho)\varphi''_{\delta,\rho}| = \left|\frac{\varphi^2_{\delta,\rho}(\rho-\delta)}{2(\delta\varphi_{\delta,\rho}+\rho)} + \varphi_{\delta,\rho}(1+\varphi_{\delta,\rho})\right| \le 3|\varphi_{\delta,\rho}|$$

and the Lebesgue's Dominated Convergence Theorem, one concludes that the following limits

$$\lim_{\rho\to\delta}\int_{-\infty}^{\infty} |\varphi_p(x) - \varphi_{\delta,\rho}(x)|^q dx = 0, \quad \lim_{\rho\to\delta}\int_{-\infty}^{\infty} |\varphi'_p(x) - \varphi'_{\delta,\rho}(x)|^q dx = 0,$$

and

$$\lim_{\rho\to\delta}\int_{-\infty}^{\infty} |\delta(\varphi_p(x)+1)\varphi''_p(x) - (\delta\varphi_{\delta,\rho}(x)+\rho)\varphi''_{\delta,\rho}(x)|^q dx = 0.$$

exist. Now, we summarize the above results in the following theorem.

Theorem 7.3. *Let q be any constant with $q \geq 1$. Then as $\rho \to \delta$, the sequences of functions $\{\varphi_{\delta,\rho}\}$, $\{\varphi'_{\delta,\rho}\}$ and $\{(\delta\varphi_{\delta,\rho} + \rho)\varphi''_{\delta,\rho}\}$ converge to the peakon φ_p, its derivative φ'_p and $\delta(\varphi_p + 1)\varphi''_p$ in the Banach space L^q, respectively. Therefore, $\varphi_p \in W^{1,q}$ for any $q \geq 1$. In addition, $\{\varphi_{\delta,\rho}\}$ converges to φ_p uniformly on the real axis and for any positive integer n, the derivatives $\varphi^{(n)}_{\delta,\rho}$ converge to $\varphi^{(n)}_p$ uniformly on any compact set contained in $(-\infty, 0) \cup (0, \infty)$, and*

$$\begin{aligned}\varphi_p(0) + 1 &= \lim_{x \to 0^-} \left(\varphi'_p(x) + 3\varphi_p(x)\varphi'_p(x) - 2\delta\varphi'_p(x)\varphi''_p(x)\right) \\ &= \lim_{x \to 0^+} \left(\varphi'_p(x) + 3\varphi_p(x)\varphi'_p(x) - 2\delta\varphi'_p(x)\varphi''_p(x)\right) = 0.\end{aligned}$$

In consequence, the peakon solutions φ_p and $\phi_{\boldsymbol{p}} = \frac{\beta - \nu\alpha}{2} - [\frac{3}{2}(\beta + c\nu) - \frac{\nu}{2}(\alpha + c)]\varphi_p$ satisfy Equations (7.6) and (3.7) on the intervals $(-\infty, 0) \cup (0, \infty)$, having limits at $x = 0$, respectively.

8. Convergence to solitary wave solutions of the KdV equation.

In this section, we examine behavior of solitary wave solutions of Equation (4.1) or (5.1) as functions defined in the complex plane when nonlinear dispersion terms are vanishing or the parameter $\delta \to 0$ in these equations. We shall demonstrate that these functions converge to solitary wave solutions of the KdV equation on the real axis, but not necessarily on the entire complex plane, due to the different definition of their branch lines.

Theorem 8.1. *Let $\Phi_{\epsilon,\delta,0}(z)$ be the solitary wave solution of Equation (4.1) as defined in Theorem 6.1, and for any $z_0 \in \{z \in \mathbb{C};\ |\Im z| = \sqrt{\epsilon}\,\pi\}$, let*

$$\Phi^+_{\epsilon,\delta,0}(z_0) = \lim_{\substack{z \to z_0 \\ |\Im z| < \sqrt{\epsilon}\,\pi}} \Phi_{\epsilon,\delta,0}(z) \quad \textit{and} \quad \Phi^-_{\epsilon,\delta,0}(z_0) = \lim_{\substack{z \to z_0 \\ |\Im z| > \sqrt{\epsilon}\,\pi}} \Phi_{\epsilon,\delta,0}(z).$$

Then the following limit

$$\lim_{\delta \to 0} \Phi_{\epsilon,\delta,0}(z) = \begin{cases} \dfrac{1}{\cosh^2 \frac{z}{2\sqrt{\epsilon}}}, & |\Im z| < \sqrt{\epsilon}\,\pi \\ \infty, & |\Im z| > \sqrt{\epsilon}\,\pi, \end{cases}$$

exists, and the limits

$$\lim_{\delta \to 0} \Phi^+_{\epsilon,\delta,0}(z) = \frac{1}{\cosh^2 \frac{z}{2\sqrt{\epsilon}}} \quad \textit{and} \quad \lim_{\delta \to 0} \Phi^-_{\epsilon,\delta,0}(z) = \infty$$

hold for any z with $|z| = \sqrt{\epsilon}\,\pi$.

Proof. Let $\nu > 0$ be any fixed constant. It follows from Lemma 4.1 that

$$\inf_{|\Im z| \geq (\sqrt{\epsilon} + \nu)\pi} |\Im(\Theta_{\epsilon,\delta,0}(z))| = |\Im(\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi + i(\sqrt{\epsilon} + \nu)\pi))| = \eta_\nu,$$

and $\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi + i(\sqrt{\epsilon}+\nu)\pi) = -\frac{\pi}{2} - i\eta_\nu$, such that

$$\frac{\cos\theta_0 \sinh\eta_\nu}{\sin\theta_0\cosh\eta_\nu - 1} = \tan\frac{(\sqrt{\epsilon}+\nu)\pi - \sqrt{\delta}\,\eta_\nu}{\sqrt{\epsilon}}$$

with $0 \le (\sqrt{\epsilon}+\nu)\pi - \sqrt{\delta}\,\eta_\nu \le \sqrt{\epsilon}\,\pi$, or equivalently, $\frac{\nu\pi}{\sqrt{\delta}} \le \eta_\nu \le \frac{(\nu+\sqrt{\epsilon})\pi}{\sqrt{\delta}}$, which implies that $\lim\limits_{\delta\to 0}\eta_\nu = \infty$. Since $\Phi_{\epsilon,\delta,0} = \frac{\delta+\epsilon}{2\delta}(\sin\Theta_{\epsilon,\delta,0} + \sin\theta_0)$,

$$|\Phi_{\epsilon,\delta,0}(z)| = \frac{\delta+\epsilon}{2\delta}\sqrt{(\cosh\eta + \cos(\xi-\theta_0))(\cosh\eta - \cos(\xi+\theta_0))} \ge \frac{\delta+\epsilon}{2\delta}(\cosh\eta - 1),$$

where $\eta = \Im(\Theta_{\epsilon,\delta,0}(z))$ and $\xi = \Re(\Theta_{\epsilon,\delta,0}(z))$ for any $z \in \mathbb{C}$. When $|\Im z| \ge (\sqrt{\epsilon}+\nu)\pi$, $|\Phi_{\epsilon,\delta,0}(z)| \ge \frac{\delta+\epsilon}{2\delta}(\cosh\eta_\nu - 1) \to \infty$ as $\delta \to 0$. Because $\nu > 0$ is chosen arbitrarily, the limit $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,0}(z) = \infty$ holds for any z with $|\Im z| > \sqrt{\epsilon}\,\pi$.

To show $\lim\limits_{\delta\to 0}\Phi^-_{\epsilon,\delta,0}(z) = \infty$ for any z with $|z| = \sqrt{\epsilon}\,\pi$, one uses the streamline (4.8)

$$\sin\xi = -\sin\theta_0\cosh\eta - \cos\theta_0\sinh\eta\cot\frac{\sqrt{\delta}\,\eta}{\sqrt{\epsilon}}.$$

Since $|\eta| \le \eta_\delta$, where η_δ satisfies the equation

$$\sqrt{\epsilon}\,\pi = \sqrt{\delta}\,\eta_\delta + 2\sqrt{\epsilon}\tan^{-1}\Big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta_\delta}{2}\Big)$$

and has the limit $\lim\limits_{\delta\to 0}\eta_\delta = \eta_0$ with $\frac{\eta_0}{2}\tanh\frac{\eta_0}{2} = 1$, $|\Im(\Theta_{\epsilon,\delta,0}(z))|$ are uniformly bounded for all z with $|\Im z| = \sqrt{\epsilon}\,\pi$, which combined with Equation (4.7) leads to the result $\lim\limits_{\delta\to 0}(x + \sqrt{\delta}\,(\xi - \frac{\pi}{2})) = 0$. Then taking the limit on both sides of the equality

$$\Phi_{\epsilon,\delta,l}(z) = \frac{2\epsilon}{(\delta+\epsilon)\cosh\frac{z+\sqrt{\delta}\,(\Theta_{\epsilon,\delta,l}(z)-\frac{\pi}{2})}{\sqrt{\epsilon}} - \delta + \epsilon}, \quad l = 0, 1 \tag{8.1}$$

for $l = 0$ and as $\delta \to 0$, one obtains

$$\lim_{\delta\to 0}|\Phi_{\epsilon,\delta,0}(z)| = \lim_{\delta\to 0}\frac{2\epsilon}{\left|-(\delta+\epsilon)\cosh\frac{x+\sqrt{\delta}\,(\xi-\frac{\pi}{2})+\sqrt{\delta}\,\eta i}{\sqrt{\epsilon}} - \delta + \epsilon\right|} = \infty.$$

The limits $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,0}(z) = \frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$ for $|\Im z| < \sqrt{\epsilon}\,\pi$ and $\lim\limits_{\delta\to 0}\Phi^+_{\epsilon,\delta,0}(z) = \frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$ are direct consequences of the identities $\Phi_{\epsilon,\delta,0}(z) = \Phi_{\epsilon,\delta,1}(z)$ for any z with $|\Im z| < \sqrt{\epsilon}\,\pi$ and $\Phi^+_{\epsilon,\delta,0}(z) = \Phi_{\epsilon,\delta,1}(z)$ for any z with $|\Im z| = \sqrt{\epsilon}\,\pi$, as well as the limit $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,1}(z) = \frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$ for any $z \in \mathbb{C}$, which will be verified in the next theorem. □

Theorem 8.2. *Let $\Phi_{\epsilon,\delta,1}(z)$ be the solitary wave solution as defined in Theorem 6.2, satisfying Equation (4.1) on the manifold $Y_{\epsilon,\delta,1}$. Then*

$$\lim_{\delta\to 0}\Phi_{\epsilon,\delta,1}(z)=\frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$$

for any $z\in\mathbb{C}$.

Proof. Let μ be any fixed constant with $0<\mu<\frac{\sqrt{\epsilon}\,\pi}{2}$, and let

$$\mathcal{M}_\mu=\mathbb{C}\setminus\bigcup_{k=-\infty}^{\infty}\{z\in\mathbb{C};\ |\Im z-(2k+1)\sqrt{\epsilon}\,\pi|<\mu, |\Re z|<\mu\}.$$

Since $\Theta_{\epsilon,\delta,1}$ is a periodic function with the period $T=2\sqrt{\epsilon}\,\pi i$, $\Theta_{\epsilon,\delta,1}$ maps $\mathcal{M}_\mu$ to the set as shown in Figure 9, which is bounded by the streamlines $y_1=\sqrt{\epsilon}\,\pi-\mu=y(\xi,\eta)$ and $2\sqrt{\epsilon}\,\pi-y_1=\sqrt{\epsilon}\,\pi+\mu=y(\xi,\eta)$, and the equipotentials $x_1=\mu=x(\xi,\eta)$ and $-x_1=-\mu=x(\xi,\eta)$ of the single-valued branch Σ_0. Because

$$\sup_{z\in\mathcal{M}_\mu}|\Im\big(\Theta_{\epsilon,\delta,1}(z)\big)|=|\Im\big(\Theta_{\epsilon,\delta,1}(i(\sqrt{\epsilon}\,\pi-\mu))\big)|=\eta_\mu,$$

and $\Theta_{\epsilon,\delta,1}\big(i(\sqrt{\epsilon}\,\pi-\mu)\big)=\frac{\pi}{2}-i\eta_\mu$, it follows from (4.5) that

$$\sqrt{\epsilon}\,\pi-\mu=\sqrt{\delta}\,\eta_\mu+2\sqrt{\epsilon}\,\tan^{-1}\Big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta_\mu}{2}\Big),$$

or $\sqrt{\frac{\delta}{\epsilon}}\tan\frac{\sqrt{\epsilon}\,\pi-\mu-\sqrt{\delta}\,\eta_\mu}{2\sqrt{\epsilon}}=\tanh\frac{\eta_\mu}{2}$ with $0<\sqrt{\delta}\,\eta_\mu<\sqrt{\epsilon}\,\pi-\mu$, which implies $\lim_{\delta\to 0}\eta_\mu=0$. In addition, $|\Re\big(\Theta_{\epsilon,\delta,1}(z)\big)|\le\frac{3\pi}{2}$. Therefore, taking the limit on both sides of Equality (8.1) for $l=1$ as $\delta\to 0$, one obtains

$$\lim_{\delta\to 0}\Phi_{\epsilon,\delta,1}(z)=\frac{2\epsilon}{\epsilon(\cosh\frac{z}{\sqrt{\epsilon}}+1)}=\frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}.$$

for any $z\in\mathcal{M}_\mu$. Since $\mu>0$ is chosen arbitrarily small, the above limit holds for any $z\in\mathbb{C}\setminus\{(2k+1)\sqrt{\epsilon}\,\pi i;\ k=0,\pm1,\pm2,\cdots\}$. To show $\lim_{\delta\to 0}\Phi_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)=\infty$, one uses the identities $\Re\big(\Theta_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)\big)=\frac{\pi}{2}$, $|\Im\big(\Theta_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)\big)|=\eta_\epsilon$, and $\sqrt{\frac{\delta}{\epsilon}}\cot\sqrt{\frac{\delta}{\epsilon}}\frac{\eta_\epsilon}{2}=\tanh\frac{\eta_\epsilon}{2}$ with $0\le\sqrt{\delta}\,\eta_\epsilon\le\sqrt{\epsilon}\,\pi$. It follows that $\lim_{\delta\to 0}\eta_\epsilon=\eta_0$ exists and $\frac{\eta_0}{2}\tanh\frac{\eta_0}{2}=1$. Then taking the limit on both sides of Equality (8.1) for $l=1$ yields

$$\lim_{\delta\to 0}|\Phi_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)|=\lim_{\delta\to 0}\frac{2\epsilon}{\Big|(\delta+\epsilon)\cosh\frac{(2k+1)\sqrt{\epsilon}\,\pi i\pm\sqrt{\delta}\,\eta_\epsilon i}{\sqrt{\epsilon}}-\delta+\epsilon\Big|}=\infty$$

for any integer k. ◻

Theorem 8.3. *Let $\Phi_{\delta,\rho,1}$ be the solitary wave solution as defined in Section 5.2, satisfying Equation (5.1) on the manifold $\mathcal{Y}_{\delta,\rho,1}$. Then*

$$\lim_{\delta\to 0}\Phi_{\delta,\rho,1}(z)=\frac{-1}{\cosh^2\frac{z}{2\sqrt{\rho}}}$$

for any $z\in\mathbb{C}$.

Proof. Let ν be any fixed constant with $0<\nu<\sqrt{\rho}\,\pi$, and let

$$\mathcal{M}_\nu=\mathbb{C}\setminus\bigcup_{k=-\infty}^{\infty}\{z\in\mathbb{C};\ |\Im z-(2k+1)\sqrt{\rho}\,\pi|<\nu, |\Re z|<\nu\}.$$

Then $\Xi_{\delta,\rho,1}$ maps $\mathcal{M}_\nu$ to the region bounded by the streamlines $y_0=\sqrt{\rho}\,\pi-\nu=y(\xi,\eta)$ and $y_1=\sqrt{\rho}\,\pi+\nu=y(\xi,\eta)$, and the equipotentials $x_1=\nu=x(\xi,\eta)$ and $-x_1=-\nu=x(\xi,\eta)$ of the single-valued branch Δ_0, as illustrated in Figure 12. It follows from Lemma 5.1 and (5.3) that

$$\sup_{z\in\mathcal{M}_\nu}|\Re\big(\Xi_{\delta,\rho,1}(z)\big)|=\Xi_{\delta,\rho,1}(\nu+\sqrt{\rho}\,\pi i)=\xi_\nu,$$

and

$$-\nu-\sqrt{\rho}\,\pi i=\sqrt{\delta}\,\xi_\nu+\sqrt{\rho}\,\log\frac{\tanh\frac{t_0}{2}-\tanh\frac{\xi_\nu}{2}}{\tanh\frac{t_0}{2}+\tanh\frac{\xi_\nu}{2}},$$

or $\tanh\frac{\xi_\nu}{2}=\tanh\frac{t_0}{2}\coth\frac{\sqrt{\delta}\,\xi_\nu+\nu}{2\sqrt{\rho}}$, which leads to the estimate

$$\begin{aligned}e^{\xi_\nu}&\le\frac{1}{\big(1-\sqrt{\frac{\delta}{\rho}}\,\big)^{\sqrt{\frac{\rho}{\delta}}}}\left(1+\frac{\sqrt{\frac{\delta}{\rho}}+\sqrt{4\sqrt{\frac{\delta}{\rho}}+\big(1-\sqrt{\frac{\delta}{\rho}}\,\big)^2\tanh^2\frac{\nu}{2\sqrt{\rho}}}-\tanh\frac{\nu}{2\sqrt{\rho}}}{1+\tanh\frac{\nu}{2\sqrt{\rho}}}\right)^{\sqrt{\frac{\rho}{\delta}}}\\&\xrightarrow{\delta\to 0}e^{\frac{2}{\tanh\frac{\nu}{2\sqrt{\rho}}}}.\end{aligned}$$

Therefore, there is a $\delta_0>0$ such that

$$\sup_{z\in\mathcal{M}_\nu}|\Xi_{\delta,\rho,1}(z)|\le\sup_{0<\delta\le\delta_0}(\xi_\nu+\pi)<\infty.$$

Then taking the limit on both sides of Equality (7.9) for $j=1$ leads to the convergence

$$\lim_{\delta\to 0}\Phi_{\delta,\rho,1}(z)=\lim_{\delta\to 0}\frac{-2\rho}{(\rho-\delta)\cosh\frac{z+\sqrt{\delta}\,\Xi_{\delta,\rho,1}(z)}{\sqrt{\rho}}+\rho+\delta}=\frac{-1}{\cosh^2\frac{z}{2\sqrt{\rho}}}.$$

for any $z \in \mathcal{M}_\nu$. Since $\nu > 0$ is arbitrary, the above limit exists for any $z \in \mathbb{C} \setminus \{z = (2k+1)\sqrt{\rho}\,\pi i;\ k = 0, \pm 1, \pm 2, \cdots\}$.

When $z = (2k+1)\sqrt{\rho}\,\pi i$ for some integer k, $|\Xi_{\delta,\rho,1}(z)| = |\Re(\Xi_{\delta,\rho,1}(z))| = \xi_\delta$, where ξ_δ is the intersection point of the equipotential $0 = x(\xi, \eta)$ of Δ_0 and the real axis. It follows from (5.3) that $\tanh \frac{\xi_\delta}{2} = \sqrt{\frac{\delta}{\rho}} \coth \sqrt{\frac{\delta}{\rho}} \frac{\xi_\delta}{2}$, and thus $\lim\limits_{\delta \to 0} \xi_\delta = \xi_0$ exists with $\frac{\xi_0}{2} \tanh \frac{\xi_0}{2} = 1$. Taking the limit on both sides of Equality (7.9) for $j = 1$ and $z = (2k+1)\sqrt{\rho}\,\pi i$ again, one obtains

$$\lim_{\delta \to 0} |\Phi_{\delta,\rho,1}(z)| = \lim_{\delta \to 0} \frac{2\rho}{\left|(\rho - \delta) \cosh \frac{(2k+1)\sqrt{\rho}\,\pi i \pm \sqrt{\delta}\,\xi_\delta}{\sqrt{\rho}} + \rho + \delta\right|} = \infty.$$

□

Remark. The nonlinear integrable Equation (1.1) has provided an interesting example to show how singularities of solitary wave solutions change their nature when the nonlinear dispersion terms vanish. Both functions $\Phi_{\epsilon,\delta,1}$ and $\Phi_{\delta,\rho,1}$ have movable branch points of order three. When $\delta \to 0$, each pair of the singularities $\pm\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi$, for $n = 0, \pm 1, \pm 2, \ldots$, of the function $\Phi_{\epsilon,\delta,1}$ becomes closer and closer, and eventually collides with each other to form a movable pole of a solitary wave solution to the KdV equation. The same situation has also happened to each pair of singularities $i[(2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,]\pi$ of the function $\Phi_{\delta,\rho,1}$ as $\delta \to 0$. Whereas the branch points $i[(2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,]\pi$ and the branch lines $\{z = x + i[(2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,]\pi;\ x \le 0\}$ of the function $\Phi_{\delta,\rho,2}$ collide such that the limiting function becomes infinity on the line $\{z = x + i(2n+1)\sqrt{\rho}\,\pi;\ x \le 0\}$, for $n = 0, \pm 1, \pm 2, \ldots$, as δ reaches zero.

Theorem 8.4. *Let $\Phi_{\delta,\rho,2}$ be the solitary wave solution satisfying Equation (5.1) on the manifold $\mathcal{Y}_{\delta,\rho,2}$ as defined in Section 7. Then*

$$\lim_{\delta \to 0} \Phi_{\delta,\rho,2}(z) = \begin{cases} \dfrac{-1}{\cosh^2 \frac{z}{2\sqrt{\rho}}}, & \text{if } \Im z \ne (2n+1)\sqrt{\rho}\,\pi, \text{ or } \Im z = (2n+1)\sqrt{\rho}\,\pi \text{ and } \Re z > 0 \\ \infty, & \text{if } \Im z = (2n+1)\sqrt{\rho}\,\pi \text{ and } \Re z \le 0 \end{cases}$$

for integers $n = 0, \pm 1, \pm 2 \cdots$.

Proof. It follows from the definition of $\Phi_{\delta,\rho,1}$ and $\Phi_{\delta,\rho,2}$ in Section 4 that $\Phi_{\delta,\rho,1}(z) = \Phi_{\delta,\rho,2}(z)$ for any

$$z \in \mathbb{C} \setminus \bigcup_{k=-\infty}^{\infty} \{z \in \mathbb{C};\ \Re z < 0, \big((2k+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi < \Im z < \big((2k+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\}.$$

Therefore, for any z with $\Re z \geq 0$, or $z \neq (2k+1)\sqrt{\rho}\,\pi i$ and $\Re z \leq 0$, whenever δ is sufficiently small, the equality $\Phi_{\delta,\rho,1}(z) = \Phi_{\delta,\rho,2}(z)$ holds. Hence, $\lim\limits_{\delta\to 0}\Phi_{\delta,\rho,2}(z) = \lim\limits_{\delta\to 0}\Phi_{\delta,\rho,1}(z) = -\operatorname{sech}^2\frac{z}{2\sqrt{\rho}}$.

If $z = x + (2n+1)\sqrt{\rho}\,\pi i$ for some integer n and $x < 0$, then $\Xi_{\delta,\rho,2}(z) = \xi_x > 0$ and $\tanh\frac{\xi_x}{2} = \tanh\frac{t_0}{2}\coth\frac{\sqrt{\delta}\,\xi_x + x}{2\sqrt{\rho}}$. It follows that $\xi_x \xrightarrow{\delta\to 0} \infty$, and $\sqrt{\delta}\,\xi_x + x \xrightarrow{\delta\to 0} 0$. Taking the limit on both sides of Equality (7.9) for $j = 2$ and $z = x + (2n+1)\sqrt{\rho}\,\pi i$, one obtains

$$\lim_{\delta\to 0}|\Phi_{\delta,\rho,2}(z)| = \lim_{\delta\to 0}\frac{2\rho}{\left|(\rho-\delta)\cosh\frac{(2n+1)\sqrt{\rho}\,\pi i + \sqrt{\delta}\,\xi_x + x}{\sqrt{\rho}} + \rho + \delta\right|} = \infty.$$

□

Remark. It is worth observing how the mass of solitary wave solutions changes as $\delta \to 0$, when their definition is restricted to the real axis . Solutions of Equations (4.1) and (5.1) demonstrate different properties. For solitary wave solutions $\varphi_{\epsilon,\delta}$ of Equation (4.1), their mass

$$\int_{-\infty}^{\infty}\varphi_{\epsilon,\delta}(x)dx = 2\int_0^1\sqrt{\frac{\delta y + \epsilon}{1-y}}\,dy < \infty$$

decreases as $\delta \to 0$. Whereas the mass

$$\int_{-\infty}^{\infty}|\varphi_{\delta,\rho}(x)|dx = 2\int_{-1}^0\sqrt{\frac{\delta y + \rho}{1+y}}\,dy < \infty$$

of solitary wave solutions $\varphi_{\delta,\rho}$ of Equation (5.1) increases when $\delta \to 0$.

As a matter of fact, the computation on partial derivatives of $\varphi_{\epsilon,\delta}$ and $\varphi_{\delta,\rho}$ with respect to δ provides even more accurate estimates as follows,

$$\frac{\partial\varphi_{\epsilon,\delta}(x)}{\partial\delta} = \frac{(\delta+\epsilon)(\frac{\pi}{2}-\theta)\cos\theta(\sin\theta + \sin\theta_0)}{4\delta^2(1+\sin\theta)}$$

and

$$\frac{\partial\varphi_{\delta,\rho}(x)}{\partial\delta} = \frac{\rho\, t\sinh t}{4\delta^2}\big(\tanh^2\frac{t_0}{2} - \tanh^2\frac{t}{2}\big),$$

where $\theta = \Theta_{\epsilon,\delta,0}(x) = \Theta_{\epsilon,\delta,1}(x)$, $t = \Xi_{\delta,\rho,1}(x) = \Xi_{\delta,\rho,2}(x)$ for any $x \in \mathbb{R}$. Since $-\frac{\pi}{2} < -\theta_0 \leq \theta \leq \frac{\pi}{2}$ when $x \leq 0$ and $\frac{\pi}{2} \leq \theta \leq \pi + \theta_0 < \frac{3\pi}{2}$ if $x \geq 0$, $\frac{\partial\varphi_{\epsilon,\delta}(x)}{\partial\delta} \geq 0$ for any $x \in \mathbb{R}$. While $|t| \leq t_0$ for all $x \in \mathbb{R}$, and thus $\frac{\partial\varphi_{\delta,\rho}(x)}{\partial\delta} \geq 0$. In consequence, for any $x \in \mathbb{R}$, the following inequalities

$$\operatorname{sech}^2\frac{x}{2\sqrt{\epsilon}} \leq \varphi_{\epsilon,\delta_1}(x) \leq \varphi_{\epsilon,\delta_2}(x)$$

and

$$-\operatorname{sech}^2 \frac{x}{2\sqrt{\rho}} \le \varphi_{\delta_1,\rho}(x) \le \varphi_{\delta_2,\rho}(x) < 0$$

hold if $\delta_1 < \delta_2$. Therefore, the solitary wave solutions of the KdV equation attains the minimum mass among those of Equation (4.1); whereas it attains the maximum mass among solitary wave solutions of Equation (4.2).

9. Conclusion. Before we complete this paper, it is worth comparing solitary wave solutions of Equation (1.1) with those of the perturbed evolution equation

$$u_t + \epsilon u_{xxx} + (u^2)_x + \delta(u^2)_{xxx} = 0 \tag{9.1}$$

whose travelling wave solutions, including the compacton

$$u(x-\lambda t) = \begin{cases} \frac{4\lambda}{3}\cos^2\frac{x-\lambda t}{4\sqrt{\delta}}, & |x-\lambda t| \le 2\sqrt{\delta}\,\pi \\ 0, & |x-\lambda t| > 2\sqrt{\delta}\,\pi \end{cases}$$

have been studied by Rosenau and Hyman [20] when $\epsilon = 0$ and $\delta = 1$. If substituting a travelling wave solution $u(x,t) = \phi(x-ct)$ into Equation (9.1) for some constant $c > 0$, one obtains the equation

$$-c\phi' + \epsilon\phi''' + (\phi^2)' + \delta(\phi^2)''' = 0 \tag{9.2}$$

which is a counterpart of Equation (4.1) and has an analytic solitary wave solution decaying exponentially to zero at infinity, where parameters ϵ and δ are positive numbers. Equation (9.2) may be studied in a way similar to that we have used to deal with Equation (3.7), *i.e.* reducing Equation (9.2) by integration to obtain the equation

$$(\phi')^2 = \frac{\delta\phi^2(y_2-\phi)(\phi-y_1)}{(2\delta\phi+\epsilon)^2}, \tag{9.3}$$

and then applying the transformation $\phi(x) = \frac{y_2-y_1}{2}(\sin\theta(x)+\sin\theta_0)$ to (9.3) and integrating the resulting equation to derive the expression

$$-x = 2\sqrt{\delta}\left(\theta - \frac{\pi}{2}\right) + \sqrt{\frac{\epsilon}{c}}\log\frac{\tan\frac{\theta}{2}+\tan\frac{\theta_0}{2}}{1+\tan\frac{\theta_0}{2}\tan\frac{\theta}{2}}, \tag{9.4}$$

which implicitly determines θ as a function of x, where

$$y_1 = \frac{2c\delta - \epsilon - \sqrt{(2c\delta-\epsilon)^2 + 9\delta c\epsilon}}{3\delta}, \qquad y_2 = \frac{2c\delta - \epsilon + \sqrt{(2c\delta-\epsilon)^2 + 9\delta c\epsilon}}{3\delta},$$

and $\sin\theta_0 = \frac{y_2+y_1}{y_2-y_1}$ and $|\theta_0| < \pi/2$. Although Equations (9.4) and (4.3) look quite similar, there are still differences between corresponding solitary wave solutions in these two systems. If extending the solution $\theta(x)$ of (9.4) to the complex plane, one may find that the extension has singularities as movable branch points of order two, taking the form

$$z_n^{\pm} = \pm\sqrt{\delta}\left((4k+1)\pi + 2\tilde{\theta} - \sqrt{\frac{\epsilon}{\delta c}}\log\frac{\sin\frac{\tilde{\theta}-\theta_0}{2}}{\cos\frac{\tilde{\theta}+\theta_0}{2}}\right) + \sqrt{\frac{\epsilon}{c}}\,(2n+1)\pi i$$

for some integers k and n, where $\sin\tilde{\theta} = \sin\theta_0 + \frac{\sqrt{\epsilon}}{2\sqrt{\delta c}}\cos\theta_0$ and $0 < \tilde{\theta} < \pi/2$, such that the solitary wave solution ϕ of Equation (9.2), as a function extended to the complex plane, has a Frobenius series expansion near each branch point z_n, taking the form

$$\phi(z) = -\frac{\epsilon}{2\delta} + \sum_{k=1}^{\infty} a_k (z-z_n)^{\frac{k}{2}} = -\frac{\epsilon}{2\delta} + \frac{\sqrt{\epsilon}\,\sqrt[4]{\epsilon(\epsilon+4\delta c)}}{2\delta\sqrt[4]{3\delta}}(z-z_n)^{\frac{1}{2}} + \cdots.$$

In contrast, any extension of the solution $\theta(x)$ to Equation (4.3), and in consequence, the corresponding solitary wave solution Φ of (4.1) have branch points of order three, taking simpler expressions as $(2k+1)\sqrt{\delta}\,\pi + \sqrt{\epsilon}\,(2n+1)\pi i$ for some integers k and n. Near each movable branch point z_0, the Frobenius series of the function Φ takes the form

$$\Phi(z) = -\frac{\epsilon}{\delta} + \sum_{k=1}^{\infty} b_k (z-z_0)^{\frac{2k}{3}} = -\frac{\epsilon}{\delta} + \frac{1}{\delta}\left(\frac{9\epsilon^2(\epsilon+\delta)}{4\delta}\right)^{\frac{1}{3}}(z-z_0)^{\frac{2}{3}} + \cdots.$$

Another difference is that the derivative of solitary wave solution of Equation (9.2) has more zeros than that of the solitary wave solution to Equation (4.1). It follows from (9.3), (9.4) and the transformation $\phi = \frac{y_2-y_1}{2}(\sin\theta + \sin\theta_0)$ that when $\phi(x) = y_2$, $\phi'(x) = 0$, $\theta(x) = 2k\pi + \pi/2$ and $x = 4k\pi\sqrt{\delta} + 2n\sqrt{\frac{\epsilon}{c}}i$; whereas if $\phi(x) = y_1$, then $\phi'(x) = 0$, $\sin\theta(x) = -1$ and $x = 2\sqrt{\delta}\,(2k+1)\pi + \sqrt{\frac{\epsilon}{c}}(2n+1)\pi i$ for some integers k and n, which are located on the same horizontal lines as branch points $z_n^{\pm}$ of the function ϕ. Nevertheless, solitary wave solutions to either Equation (9.2) or Equation (4.1) converge to the compacton of the corresponding limiting equation as $\epsilon \to 0$, and they converge to the solitary wave solution of the corresponding limiting KdV equation as $\delta \to 0$. Included in our further studies are also singularities of solitary wave solutions to both Equation (5.1) and its counterpart equation

$$\phi' - \rho\phi''' + (\phi^2)' - \delta(\phi^2)''' = 0 \qquad (9.5)$$

whose solitary wave solution exists when $\rho > 4\delta > 0$ and converges to the peakon $\phi_p = -2e^{\frac{-|x|}{2\sqrt{\delta}}}$ as $\rho \to 4\delta$. Extensions of solitary wave solutions of Equation (9.5) to the complex plane have branch points of order two, having different structure from that of branch points of solitary wave solutions to Equation (5.1). Our inquiry is to understand how different nonlinear dispersion terms in these evolution equations influence behavior of their solitary wave solutions and other travelling wave solutions. As we have also mentioned in *Case II* of Section 3 that when $a \in (-\infty, -\beta - c\nu) \cup (\frac{\beta-\nu\alpha}{2}, \infty)$, System (3.7) has a cuspon at the fixed point $(a, 0, 0)$. That means there exists a function of the form $a + \phi_a(x)$ such that $\phi_a(x)$ is an even, continuous function, approaching zero at both $-\infty$ and ∞, and infinitely differentiable everywhere except at $x = 0$ at which the graph of ϕ_a has a cusp, *i.e.* $\lim_{|x|\to 0} |\phi'_a(x)| = \infty$. In addition, $\phi_a(x)$ satisfies Equation (3.11) everywhere except at $x = 0$ and the trajectory $(\phi_a(x), \phi'_a(x))$ is an orbit in the stable manifold of System (3.11) at the origin when $x > 0$, while the trajectory $(\phi_a(x), \phi'_a(x))$ is an orbit in the unstable manifold of System (3.11) at the origin for $x < 0$. The function ϕ_a may be obtained implicitly in the following way. If $a \in (\frac{\beta-\nu\alpha}{2}, \infty)$, one may adopt a procedure similar to that used in the beginning of Section 4.2 to reduce (3.11) to (5.1). Since $0 < \rho < \delta$ when $a > \frac{\beta-\nu\alpha}{2}$, one needs to use the transformation $\varphi = \frac{\delta-\rho}{2\delta}\cosh t - \frac{\rho+\delta}{2\delta}$ to Equation (5.1) to derive the expression (5.3) with the parameter t_0 defined as $\cosh t_0 = \frac{\delta+\rho}{\delta-\rho}$ and the derivative expressed by $\frac{dx}{dt} = \frac{\sqrt{\delta}\,(1-\cosh t)}{\cosh t - \cosh t_0}$. Therefore, Equation (5.3) still determines $t = t(x)$ as a continuous function of x defined on the real axis implicitly, but having a singularity at $x = 0$. In consequence, $\phi_a(x) = [3a + \nu(\alpha + c)]\big(\frac{\delta-\rho}{2\delta}\cosh t(x) - \frac{\rho+\delta}{2\delta}\big)$ is the function we have described above. In a similar way, one may obtain the continuous function ϕ_a implicitly for $a < -\beta - c\nu$ in *Case II* or for $a \neq -\beta - c\nu$ in *Case IV* as discussed in Section 3, such that ϕ_a also has the character of possessing a singularity at $x = 0$ in the form of a cusp in its graph and thus called a cuspon. Since these cuspons also have Frobenius series expansion of the form $\sum_{k=0}^{\infty} a_k x^{2k/3}$ near their singularity $x = 0$ and ϕ_a exponentially to zero at infinity, they are weak solutions of Equation (3.7) in the sense as defined in Definition 3.1. The singularities we have discussed above are caused by the nonlinear dispersion terms in Equation (1.1), a different feature from travelling wave solutions of the KdV equation and worth being studied further.

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