

THE REGULARITY AND SINGULARITY OF SOLUTIONS OF CERTAIN ELLIPTIC PROBLEMS ON POLYGONAL DOMAINS

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ABSTRACT. The regularity and singularity of variational solutions of problems $-\Delta u = f$ in Ω ; $\frac{\partial u}{\partial \nu} - T \frac{\partial^2 u}{\partial \tau^2} = g$ on Γ_1 ; $u = 0$ on Γ_2 ; $\frac{\partial u}{\partial \nu} = 0$ on Γ_3 with suitable compatibility conditions at vertices of Γ_1 for bounded polygonal domains $\Omega \subset R^2$ are studied by combining Grisvard's (cf. Grisvard[4]) results with perturbation theory and the method of continuity. The variational solutions are proved to be in $H^2(\Omega) \times H^2(\Gamma_1)$ for certain geometric polygonal domains. The singular decomposition forms of the variational solutions are given implicitly and explicitly for other geometric polygonal domains.

1. Motivation.

We here consider the following problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega & (1.1a) \\ \frac{\partial u}{\partial \nu} - T \frac{\partial^2 u}{\partial \tau^2} = g & \text{on } \Gamma_1 & (1.1b) \\ u = 0 & \text{on } \Gamma_2 & (1.1c) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_3 & (1.1d) \end{cases}$$

where $\Omega \subset R^2$ is a bounded open polygonal domain, $\Gamma_1, \Gamma_2, \Gamma_3$ are a part of $\partial\Omega$ and each of them consists of several sides of the polygon Ω , $mes(\Gamma_1) \neq 0$, $T > 0$ is a possibly different constant on different sides of Ω , ν is the external normal direction of boundary $\partial\Omega$ and τ is the tangential direction of boundary $\partial\Omega$ taken in counterclockwise direction. This kind of problem arises in the study of boundary control theory (see Littman[7][8]) where we need the compactness properties of the solution operator of equations (1.1a)-(1.1d). Physically this

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kind of problem is related to the deformation of a membrane with strings attached to several sides of the polygon Ω . In other words, we need to know the regularity and singularity properties of solutions u of equations (1.1a)-(1.1d) near those corners.

In [4], Grisvard used Fredholm Properties systematically to study the behavior of solutions of Laplace's equation with Dirichlet, Neumann or mixed boundary conditions near the corners of a polygonal domain. It is well-known that the variational solutions of Laplace's equation with Dirichlet, Neumann or mixed boundary conditions for $f \in L^2(\Omega)$ belong to $H^2(\Omega)$ when the polygonal domain Ω satisfies certain geometrical conditions. Otherwise the variational solution may have singularities near those corners independently of the smoothness of f . The decomposition for the singular part is described in Grisvard[4]. The same results for general elliptic equations were also obtained by Banasiak & Roach[2]. In [9] Mghazli investigated the regularity and singularities of solutions of the Laplace equation with Dirichlet-Robin boundary conditions where one actually has $u \in H^{1/2}(\partial\Omega)$ whenever variational solutions exist.

The problems (1.1a)-(1.1d) are quite different from the problem mentioned above because of the term $T \frac{\partial^2 u}{\partial \tau^2}$ on Γ_1 where trace imbedding theorems could not guarantee $u|_{\Gamma_1} \in H^2(\Gamma_1)$ when $u \in H^2(\Omega)$. It is easy to see that if the variational solutions of (1.1a)-(1.1d) with proper compatibility conditions at vertices are in $H^2(\Omega)$, then we must have $u|_{\Gamma_1} \in H^2(\Gamma_1)$. However $u \in H^2(\Omega)$ turns to be very difficult to establish. Can we expect that variational solutions u belong to $H^2(\Omega)$ and $u|_{\Gamma_1} \in H^2(\Gamma_1)$ when $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$? The answer is yes under certain conditions. Can we exclude singularities near the corners adjacent to Γ_1 due to the stronger effect of the term $T \frac{\partial^2 u}{\partial \tau^2}$ acting on Γ_1 ? The answer is negative as we shall see that the variational solutions could possibly have singularities at each corner of Ω .

The outline of this paper is as follows. In §2 we investigate the existence and interior regularity of the variational solutions of (1.1a)-(1.1d) with compatibility conditions at the corners. A density property needed for reaching our goal will be studied in §3. In §4 a priori estimates for solutions of (1.1a)-(1.1d) with compatibility conditions at the corners are obtained by using the Nirenberg-Gagliardo interpolation inequality. Finally in §5, the results of regularity are first proved by perturbation theory, the method of continuity and Grisvard's results. Then an extension theorem for (1.1a)-(1.1d) is derived and a decomposition for the singular part of the variational solutions is obtained.

2. Existence.

We first introduce some notation to be used throughout this paper. Let m be the number of corners of Ω . Let A_j , $j = 1, \dots, m$, denote the vertices of the domain Ω numbered in counterclockwise order and e_j , $j = 1, \dots, m$, denote sides of Ω such that $e_j = (A_{j-1}, A_j)$. We assume $A_{m+j} \equiv A_j$ and $e_{m+j} \equiv e_j$ modulo m . Let ω_j , $j = 1, \dots, m$, denote interior angle of A_j . Let τ_j , $j = 1, \dots, m$, denotes the tangential direction (counterclockwise) along the side e_j . Let ν_j , $j = 1, \dots, m$, denotes the external normal directions on e_j . We divide the set $\{1, \dots, m\}$ into three subsets D , N and S such that $D = \{j; e_j \subset \Gamma_2\}$, $N = \{j; e_j \subset \Gamma_3\}$ and $S = \{j; e_j \subset \Gamma_1\}$. We consider the existence of the variational solutions of equations (1.1a)-(1.1d) first. We need to assume the following compatibility conditions at the corners $A_j \in \bar{\Gamma}_1$ for a smooth function u .

$$\text{I): } u(A_j) = 0, \quad \forall A_j \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2. \quad (2.1)$$

$$\text{II): } \text{If } (j, j+1) \in S \times N, \text{ then either } u(A_j) = 0 \text{ or } \frac{\partial u}{\partial \tau_j}(A_j) = 0 \text{ but only } \frac{\partial u}{\partial \tau_j}(A_j) = 0 \text{ if } \omega_j = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}. \text{ If } (j, j+1) \in N \times S, \text{ then either } u(A_j) = 0 \text{ or } \frac{\partial u}{\partial \tau_{j+1}}(A_j) = 0 \text{ but only } \frac{\partial u}{\partial \tau_{j+1}}(A_j) = 0 \text{ if } \omega_j = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}. \quad (2.2)$$

III): Either $T_j \frac{\partial u}{\partial \tau_j}(A_j) - T_{j+1} \frac{\partial u}{\partial \tau_{j+1}}(A_j) = 0$ or $u(A_j) = 0$ for $(j, j+1) \in S \times S$ where T_j for $j \in S$ denotes the constant T on e_j . (2.3)

We also introduce some notation to be used throughout the paper.

$$\begin{aligned} L^2(\Gamma_1) &= \{v; \quad v|_{e_j} \in L^2(e_j) \quad \forall j \in S\}. \\ H^1(\Gamma_1) &= \{v; \quad v|_{e_j} \in H^1(e_j) \quad \forall j \in S\}. \\ \mathcal{V}_1 &= \{(v, p) \in H^1(\Omega) \times H^1(\Gamma_1); \quad v = 0 \text{ on } \Gamma_2, \quad p = v|_{\Gamma_1}, p(A_j) = 0 \\ &\quad \text{if the compatibility conditions require } v(A_j) = 0 \text{ in (2.1) - (2.3)}\}. \end{aligned}$$

The associated inner product on $L^2(\Gamma_1)$, $H^1(\Gamma_1)$ is naturally inherited from the inner products on $L^2(e_j)$, $H^1(e_j)$ respectively. The associated inner product on the test Hilbert space \mathcal{V}_1 is then inherited from the inner products on $H^1(\Omega)$ and $H^1(\Gamma_1)$. We say $v \in \mathcal{V}_1$ if $(v, v|_{\Gamma_1}) \in \mathcal{V}_1$. We consider the following variational problem: Find $u \in \mathcal{V}_1$ such that

$$\iint_{\Omega} \nabla u \cdot \nabla v dx dy + \int_{\Gamma_1} T \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial \tau} ds = \iint_{\Omega} f v dx dy + \int_{\Gamma_1} g v ds \quad \forall v \in \mathcal{V}_1. \quad (2.4)$$

To prove the existence and “uniqueness” of the variational solutions u of (2.4), one has to show that the left side of (2.4) defines an equivalent inner product on \mathcal{V}_1 or on $\mathcal{V}_1/\mathcal{P}^0$ which denotes the quotient space of \mathcal{V}_1 modulo constants. We need

Lemma 2.1. *Let $F_0 \dots F_{k_0}$ be the bounded linear functionals on the space $W^{k,p}(\Omega)$, $1 \leq p \leq \infty$. If $q \in \mathcal{P}^{k-1}$ where \mathcal{P}^k is the polynomial space of degree k of two variables such that we have $q = 0$ whenever $F_1(q) = F_2(q) = \dots = F_{k_0}(q) = 0$, then*

$$\|v\|_{W^{k,p}(\Omega)} \leq C[\|v\|_{W^{k,p}(\Omega)} + \sum_{i=0}^{k_0} |F_i(v)|] \quad (2.5)$$

where $|v|_{W^{k,p}(\Omega)}$ is the seminorm of the space $W^{k,p}(\Omega)$.

Proof. Suppose that (2.5) is not true. Then there is a sequence $v_n \in W^{k,p}(\Omega)$ such that

$$\|v_n\|_{W^{k,p}(\Omega)} = 1 \quad \forall n, \quad \text{but} \quad \lim_{n \rightarrow \infty} [|v_n|_{W^{k,p}(\Omega)} + \sum_{i=0}^{k_0} |F_i(v_n)|] = 0. \quad (2.6)$$

Therefore there is a subsequence, which we still denote by v_n , such that $\lim_{n \rightarrow \infty} v_n = \hat{v}$ weakly in $W^{k,p}(\Omega)$ since $W^{k,p}(\Omega)$ is weakly compact. One has $\lim_{n \rightarrow \infty} v_n = \hat{v}$ in $W^{k-1,p}(\Omega)$ by Rellich's theorem (cf Adams[1]). From (2.6) one easily has $|\hat{v}|_{W^{k,p}(\Omega)} + \sum_{i=0}^{k_0} |F_i(\hat{v})| = 0$. Therefore $\hat{v} = 0$ by assumptions. Then it follows that v_n converges to 0 in $W^{k,p}(\Omega)$ strongly since $\lim_{n \rightarrow \infty} |v_n|_{W^{k,p}(\Omega)} = 0$. Contradiction to (2.6). Q.E.D.

Theorem 2.1. *For any $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$, there is a variational solution $u \in \mathcal{V}_1$ such that (2.4) holds and*

$$\|u\|_{\mathcal{V}_1} \leq C[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)} + \|u\|_{L^1(\Gamma_1)}]. \quad (2.7)$$

Moreover if $\text{mes}(\Gamma_2) \neq 0$ or if either $u|_{e_j}(A_j) = 0$ or $u|_{e_j}(A_{j-1}) = 0$ for some $j \in S$ ($u|_{e_j}(A_j) = \lim_{A \in e_j, A \rightarrow A_j} u|_{e_j}(A)$, same for $u|_{e_j}(A_{j-1})$), then the term $\|u\|_{L^1(\Gamma_1)}$ on the right side of (2.7) can be dropped.

Proof. Consider the functional $F(v) = \int_{e_j} v ds$ for some $j \in S$. From lemma 2.1 we have $\|v\|_{H^1(\Omega)} \leq C[\|\nabla v\|_{L^2(\Omega)} + |F(v)|]$. It follows from Poincaré's inequality and the trace imbedding theorem (cf. Adams[1]) that $\iint_{\Omega} |\nabla u|^2 dx dy + \int_{\Gamma_1} T |\frac{\partial u}{\partial \tau}|^2 d\tau \geq C\|u\|_{\mathcal{V}_1}^2$ if $\text{mes}(\Gamma_2) \neq 0$ or if $u(A_j) = 0$ for some $j \in S$ and otherwise $\iint_{\Omega} |\nabla u|^2 dx dy + \int_{\Gamma_1} T |\frac{\partial u}{\partial \tau}|^2 ds \geq C\|u\|_{\mathcal{V}_1/\mathcal{P}^0}^2$. From the Lax-Milgram theorem (cf. Gilbarg & Trudinger[3]), we conclude that theorem 2.1 holds. Q.E.D.

Remark 2.1: From the proof of theorem 2.1 we also know that all of the variational solutions of (2.4) are the same up to a constant and would have the same regularity and singularity properties.

The following theorem tells us about the local regularity of the variational solutions of (2.4).

Theorem 2.2. *Let $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$ and u be a variational solution of (2.4). Then $u \in H_{loc}^2(\Omega \cup e_k) \times H_{loc}^2(e_j)$, $\forall j \in S$ and all k . Furthermore u satisfies (1.1a)-(1.1d).*

Proof. It is well known that (1.1a), (1.1c) and (1.1d) hold as well as $u \in H_{loc}^2(\Omega \cup e_j) \forall j \in D \cup N$. It is sufficient to check the regularity of u near e_j , $\forall j \in S$. Without loss of generality, we assume that e_j where $j \in S$ is fixed and lies on the x-axis. Let $x_0 \in e_j$. Consider the upper semiball $B_R^+(x_0) \subset\subset (\Omega \cup e_j)$ with center x_0 and radius R . For any $R' < R$, let $e'_j = B_{R'}(x_0) \cap e_j$ and $\Delta^h(u) = \frac{u(x+h,0) - u(x,0)}{h}$ where $0 < h < R - R'$. Then by (2.4) one has

$$\begin{aligned} & \left| \iint_{B_{R'}^+(x_0)} [\Delta^h(\nabla u)] \cdot \nabla v dx dy + \int_{e'_j} T \Delta^h \left(\frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial x} dx \right| \\ &= \left| \iint_{B_{R'}^+(x_0)} \nabla u \cdot [\Delta^h(\nabla v)] dx dy + \int_{e'_j} T \frac{\partial u}{\partial x} \Delta^h \left(\frac{\partial v}{\partial x} \right) dx \right| \\ &\leq C[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)}][\|v\|_{H^1(B_{R'}^+(x_0))} + \|v\|_{H^1(e'_j)}] \quad \forall v \in C_0^\infty(B_{R'}(x_0)). \end{aligned}$$

Thus

$$\|\Delta^h(\nabla u)\|_{L^2(B_{R'}^+(x_0))} + \left\| \Delta^h \frac{\partial u}{\partial x} \right\|_{L^2(e'_j)} \leq C[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)}]. \quad (2.8)$$

From (2.8) and lemma 7.24 in Gilbarg & Trudinger[3], one has $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y} \in L^2(B_{R'}^+(x_0))$ and $u|_{e_j} \in H^2(e'_j)$. Thus $\frac{\partial^2 u}{\partial y^2} \in L^2(B_{R'}^+(x_0))$ by (1.1a).

Q.E.D.

The above theorem tells us that we just need to look at the regularity of the variational solutions of (1.1a)-(1.1d) with compatibility conditions (2.1)-(2.3). For those corners which are not related to Γ_1 (i.e. those corners not next to the string part), one can use polar coordinates near the corners and the method of separation of variables to investigate regularity(cf. Grisvard[4]). This technique would not work for the corners related to Γ_1 because the operator $\frac{\partial^2}{\partial \theta^2}$, $\theta \in (0, \omega_j)$ for some j or $j + 1 \in S$, with boundary conditions at $\theta = 0, \omega_j$ is no longer self-adjoint.

3. Density Property.

In this section we focus on a density property of the Hilbert space

$$\mathcal{V}_2 = \{(v, p) \in H^2(\Omega) \times H^2(\Gamma_1); \quad v = 0 \quad \text{on } \Gamma_2, \quad p = v|_{\Gamma_1}, \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \quad v \text{ satisfies (2.1)-(2.3)}\}$$

with the associated norm $\|\cdot\|_{\mathcal{V}_2}^2 = \|\cdot\|_{H^2(\Omega)}^2 + \|\cdot\|_{H^2(\Gamma_1)}^2$. We sometimes find it convenient by slight abuse of notation to say $v \in \mathcal{V}_2$ if $(v, v|_{\Gamma_1}) \in \mathcal{V}_2$. Grisvard[4] proved that $H^k(\Omega) \cap \mathcal{V}_2$ is dense in \mathcal{V}_2 when $\Gamma_2 \cup \Gamma_3 = \partial\Omega$. However that $H^k(\Omega) \cap \mathcal{V}_2$ is dense in \mathcal{V}_2 in general is not obvious. If one proves that $H^3(\Omega) \cap \mathcal{V}_2$ is dense in \mathcal{V}_2 , then $H^k(\Omega) \cap \mathcal{V}_2$ is obviously dense in \mathcal{V}_2 . Here we have

Theorem 3.1. *For any $k \geq 3$, $H^k(\Omega) \cap \mathcal{V}_2$ is dense in \mathcal{V}_2 .*

Proof. Let $\mathbf{B} = \{\frac{\partial^k}{\partial \nu_j^k}\}_{k=0,1; 1 \leq j \leq m}$ be the trace operators and $\mathbf{B}(\mathcal{V}_2)$ be the image of the trace operator \mathbf{B} . From Grisvard's argument (cf. Grisvard[4] theorem 1.6.2) and the Hahn-Banach theorem, it is sufficient to show that $\mathbf{B}(H^k(\Omega) \cap \mathcal{V}_2)$ is dense in $\mathbf{B}(\mathcal{V}_2)$. Obviously $\mathbf{B}(\mathcal{V}_2)$ is a subspace of $\prod_{j \in S} [H^2(e_j) \times H^{1/2}(e_j)] \times \prod_{j \in D \cup N} [H^{3/2}(e_j) \times H^{1/2}(e_j)]$. Let $\prod_{j=1}^m (g_j, h_j) \in \mathbf{B}(\mathcal{V}_2)$. Then $\prod_{j=1}^m (g_j, h_j)$ satisfies $(g_j, h_j) \in H^2(e_j) \times H^{1/2}(e_j)$ if $j \in S$ and $(g_j, h_j) \in H^{3/2}(e_j) \times H^{1/2}(e_j)$

if $j \in D \cup N$ and

$$g_j = 0 \quad \text{on } e_j \text{ if } j \in D \quad (3.1)$$

$$h_j = 0 \quad \text{on } e_j \text{ if } j \in N \quad (3.2)$$

$$g_j(A_j) = g_{j+1}(A_j) \quad j = 1, 2, \dots, m \quad (3.3)$$

$$g'_j(A_j) \equiv [-\cos\omega_j g'_{j+1} + \sin\omega_j h_{j+1}](A_j) \quad j = 1, 2, \dots, m \quad (3.4)$$

$$h_j(A_j) \equiv -[\sin\omega_j g'_{j+1} + \cos\omega_j h_{j+1}](A_j) \quad j = 1, 2, \dots, m. \quad (3.5)$$

The symbol $'$ denotes the tangential derivative along e_j taken counterclockwise and the symbol \equiv means that when at most two of $\{g'_j(A_j), h_j(A_j), g'_{j+1}(A_j), h_{j+1}(A_j)\}$ for the A_j are not well defined, we can redefine their values at A_j such that (3.4) and (3.5) for the fixed j become identities. Checking $\mathbf{B}(H^k(\Omega) \cap \mathcal{V}_2)$ carefully, one has that $\mathbf{B}(H^k(\Omega) \cap \mathcal{V}_2)$ is a subspace of $\Pi_j H^{k-1/2}(e_j) \times H^{k-3/2}(e_j)$ and $\Pi_{j=1}^m(g_j, h_j)$ satisfies the following equation as well as (3.1)-(3.5).

$$[-\cos\omega_j \hat{g}''_j - \sin\omega_j \hat{h}'_j](A_j) \equiv [-\cos\omega_j \hat{g}''_{j+1} + \sin\omega_j \hat{h}'_{j+1}](A_j). \quad (3.6)$$

Therefore we just need to check if there are smooth functions (\hat{g}_j, \hat{h}_j) and $(\hat{g}_{j+1}, \hat{h}_{j+1})$ satisfying (3.1)-(3.6) locally to approximate (g_j, h_j) and (g_{j+1}, h_{j+1}) in corresponding trace spaces for each corner A_j .

The cases $(j, j+1) \in D \times D \cup D \times N \cup N \times D \cup N \times N$ were done by Grisvard [4] (Theorem 1.6.2). We here discuss the case $j \in S$.

Case 1: $j+1 \in D$. Then $g'_{j+1} = 0$ on e_{j+1} . If $\omega_j = \pi$, one has $g_j(A_j) = g'_j(A_j) = 0$ and $h_j(A_j) = h_{j+1}(A_j)$ by (3.1)-(3.5). If $\omega_j \neq \pi$, one has $g_j(A_j) = 0$, $h_j(A_j) = -\cot\omega_j g'_j(A_j)$ and $h_{j+1}(A_j) = \csc\omega_j g'_j(A_j)$ by (3.1)-(3.5) and (2.1). Thus one can choose smooth functions $\hat{g}_j, \hat{h}_j, \hat{h}_{j+1}$ with the above compatibility conditions at A_j and

$$\hat{g}_j^{(i_1)}(A_j) = \hat{g}_{j+1}^{(i_1)}(A_j) = \hat{h}_j^{(i_2)}(A_j) = \hat{h}_{j+1}^{(i_2)}(A_j) = 0 \quad (3.7)$$

where $i_1 \geq 2, i_2 \geq 1$ to approximate g_j, h_j, h_{j+1} in $H^2(e_j), H^{1/2}(e_j), H^{1/2}(e_{j+1})$ respectively.

Case 2: $j+1 \in N$. Then $h_{j+1} = 0$. If $\omega_j = \pi/2$ or $3\pi/2$, one has $g'_j(A_j) = 0$ and $h_j(A_j) = -\sin \omega_j g'_{j+1}(A_j)$. If $\omega_j \neq \pi/2$ or $3\pi/2$, one has $g_{j+1}(A_j) = -\sec \omega_j g'_j(A_j), h_j(A_j) = \tan \omega_j g'_j(A_j)$ by (3.4)-(3.5). g_j and g_{j+1} also satisfy the compatibility condition (2.2) and (3.3). Therefore one can choose smooth functions $\hat{g}_j, \hat{h}_j, \hat{g}_{j+1}$ which satisfy (3.7) and the above compatibility condition at A_j to approximate g_j, h_j, g_{j+1} in $H^2(e_j), H^{1/2}(e_j), H^{3/2}(e_{j+1})$ respectively.

Case 3: $j+1 \in S$. If $\omega_j = \pi$, then $g'_j(A_j) = g'_{j+1}(A_j)$ ($= 0$ if the first equality is required and $T_j \neq T_{j+1}$ in (2.3)) and $h_j(A_j) = h_{j+1}(A_j)$. Otherwise one has $h_j(A_j) = -\cot \omega_j g'_j(A_j) - \csc \omega_j g'_{j+1}(A_j), h_{j+1}(A_j) = \csc \omega_j g'_j(A_j) + \cot \omega_j g'_{j+1}(A_j)$ by (3.4)-(3.5). g_j and g_{j+1} also satisfy the compatibility condition (2.3) and (3.3). Once again, one can have smooth approximations of $g_j, h_j, g_{j+1}, h_{j+1}$ with the above compatibility conditions at A_j and (3.7) in $H^2(e_j), H^{1/2}(e_j), H^2(e_{j+1}), H^{1/2}(e_{j+1})$ respectively. Q.E.D.

4. A Priori Estimate.

Lemma 4.1. *Assume that Ω is a bounded open polygonal domain, $\beta \geq 0$ and $t \in (0, 1]$. If $v \in \mathcal{V}_2$, then there is $C(\Omega)$ independent of β and t such that*

$$\begin{aligned} & 2 \iint_{\Omega} \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 dx dy + \int_{\Gamma_1} \frac{T}{2} \left(\frac{\partial^2 v}{\partial \tau^2} \right)^2 ds \\ & \leq 2 \iint_{\Omega} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} dx dy + \int_{\Gamma_1} \frac{t}{T} \left(\frac{\partial v}{\partial \nu} - \frac{T}{t} \frac{\partial^2 v}{\partial \tau^2} + \frac{\beta}{t} v \right)^2 ds + C(\Omega) \int_{\Gamma_1} \left(\frac{\partial v}{\partial \tau} \right)^2 ds. \end{aligned} \quad (4.1)$$

Proof. By theorem 3.1 it is sufficient to prove that lemma 4.1 holds for $v \in H^4(\Omega) \cap \mathcal{V}_2$. Using integration by parts with respect to y first and then x , one

has

$$\iint_{\Omega} \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 dx dy = \iint_{\Omega} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} dx dy - \int_{\partial \Omega} \frac{\partial v}{\partial x} \frac{\partial}{\partial \tau} \left(\frac{\partial v}{\partial y} \right) ds. \quad (4.2)$$

Using integration by parts with respect to x first and then y , one has

$$\iint_{\Omega} \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 dx dy = \iint_{\Omega} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} dx dy + \int_{\partial \Omega} \frac{\partial v}{\partial y} \frac{\partial}{\partial \tau} \left(\frac{\partial v}{\partial x} \right) ds. \quad (4.3)$$

(4.2) and (4.3) give

$$\iint_{\Omega} \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 dx dy = \iint_{\Omega} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} dx dy + \frac{1}{2} \int_{\partial \Omega} \left[\frac{\partial v}{\partial y} \frac{\partial}{\partial \tau} \left(\frac{\partial v}{\partial x} \right) - \frac{\partial v}{\partial x} \frac{\partial}{\partial \tau} \left(\frac{\partial v}{\partial y} \right) \right] ds. \quad (4.4)$$

One has the following relationships of coordinate substitution.

$$\begin{cases} \frac{\partial v}{\partial x} = \tau^1 \frac{\partial v}{\partial \tau} + \tau^2 \frac{\partial v}{\partial \nu} \\ \frac{\partial v}{\partial y} = \tau^1 \frac{\partial v}{\partial \tau} - \tau^2 \frac{\partial v}{\partial \nu} \end{cases} \quad (4.5)$$

where $\tau = (\tau^1, \tau^2)$. Using (4.5), one has

$$\begin{aligned} \int_{\partial \Omega} \left[\frac{\partial v}{\partial y} \frac{\partial}{\partial \tau} \left(\frac{\partial v}{\partial x} \right) - \frac{\partial v}{\partial x} \frac{\partial}{\partial \tau} \left(\frac{\partial v}{\partial y} \right) \right] ds &= \sum_{j=1}^m \int_{e_j} \left[\frac{\partial v}{\partial y} \frac{\partial}{\partial \tau_j} \left(\frac{\partial v}{\partial x} \right) - \frac{\partial v}{\partial x} \frac{\partial}{\partial \tau_j} \left(\frac{\partial v}{\partial y} \right) \right] ds \\ &= \sum_{j=1}^m \int_{e_j} \left[\frac{\partial^2 v}{\partial \nu_j \partial \tau_j} \frac{\partial v}{\partial \tau_j} - \frac{\partial^2 v}{\partial \tau_j^2} \frac{\partial v}{\partial \nu_j} \right] ds = -2 \sum_{j=1}^m \int_{e_j} \frac{\partial^2 v}{\partial \tau_j^2} \frac{\partial v}{\partial \nu_j} ds + \sum_{j=1}^m \frac{\partial v}{\partial \tau_j} \frac{\partial v}{\partial \nu_j} \Big|_{A_{j-1}}. \end{aligned} \quad (4.6)$$

Consider the term $\sum_{j=1}^m \frac{\partial v}{\partial \tau_j} \frac{\partial v}{\partial \nu_j} \Big|_{A_{j-1}}$ in (4.6). If $j \in D \cup N$, then $\int_{e_j} \frac{\partial^2 v}{\partial \tau_j^2} \frac{\partial v}{\partial \nu_j} ds = \frac{\partial v}{\partial \tau_j} \frac{\partial v}{\partial \nu_j} \Big|_{A_{j-1}} = 0$. Let $j \in S$.

Case 1: $j+1 \in D$. If $\omega_j = \pi$, then $\frac{\partial v}{\partial \tau_j}(A_j) = \frac{\partial v}{\partial \tau_{j+1}}(A_j) = 0$. Otherwise one has $\frac{\partial v}{\partial \nu_j}(A_j) = -\cot \omega_j \frac{\partial v}{\partial \tau_j}(A_j)$ by (3.4) and (3.5).

Case 2: $j+1 \in N$. If $\omega_j = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, then $\frac{\partial v}{\partial \tau_j}(A_j) = \pm \frac{\partial v}{\partial \nu_{j+1}}(A_j) = 0$. Otherwise one has $\frac{\partial v}{\partial \nu_j}(A_j) = \tan \omega_j \frac{\partial v}{\partial \tau_j}(A_j)$ by (3.4) and (3.5).

Case 3: $j + 1 \in S$. If $\omega_j = \pi$, then $\frac{\partial v}{\partial \tau_{j+1}} \frac{\partial v}{\partial \nu_{j+1}}(A_j) - \frac{\partial v}{\partial \tau_j} \frac{\partial v}{\partial \nu_j}(A_j) = 0$. Otherwise one has $\frac{\partial v}{\partial \nu_{j+1}}(A_j) = \cot \omega_{j+1} \frac{\partial v}{\partial \tau_{j+1}}(A_j) + \csc \omega_j \frac{\partial v}{\partial \tau_j}(A_j)$ and $\frac{\partial v}{\partial \nu_j}(A_j) = -[\cot \omega_j \frac{\partial v}{\partial \tau_j} + \csc \omega_j \frac{\partial v}{\partial \tau_{j+1}}](A_j)$.

No matter which case it is, one has the following inequality if one applies a special case of the Nirenberg-Gagliardo interpolation inequality (cf. Adams[1], Lemma 5.18 or Henry[5]) to $v'(s)$ on e_j and Young's inequality

$$\begin{aligned} \left| \sum_{j=1}^m \frac{\partial v}{\partial \tau_j} \frac{\partial v}{\partial \nu_j} \Big|_{A_{j-1}} \right| &\leq C(\Omega) \sum_{j \in S} \left[\left| \frac{\partial v}{\partial \tau_j}(A_j) \right|^2 + \left| \frac{\partial v}{\partial \tau_j}(A_{j-1}) \right|^2 \right] \\ &\leq C(\Omega) \sum_{j \in S} \left\| \frac{\partial v}{\partial \tau_j} \right\|_{H^1(e_j)} \cdot \left\| \frac{\partial v}{\partial \tau_j} \right\|_{L^2(e_j)} \\ &\leq C(\Omega, \epsilon) \left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Gamma_1)}^2 + \epsilon \int_{\Gamma_1} \left(\frac{\partial^2 v}{\partial \tau^2} \right)^2 ds \end{aligned} \quad (4.7)$$

where $\epsilon > 0$ can be arbitrary small. We also have

$$\begin{aligned} \left| \int_{\Gamma_1} \frac{2\beta}{T} \frac{\partial v}{\partial \nu} v ds \right| &\leq \int_{\Gamma_1} \frac{t}{T} \left(\frac{\partial v}{\partial \nu} \right)^2 ds + \int_{\Gamma_1} \frac{\beta^2}{tT} v^2 ds, \\ - \int_{\Gamma_1} \frac{2\beta}{t} \frac{\partial^2 v}{\partial \tau^2} v ds &= \int_{\Gamma_1} \frac{2\beta}{t} \left(\frac{\partial v}{\partial \tau} \right)^2 ds \geq 0. \end{aligned} \quad (4.8)$$

If we choose ϵ sufficiently small such that $\epsilon \int_{\Gamma_1} \left(\frac{\partial^2 v}{\partial \tau^2} \right)^2 ds \leq \int_{\Gamma_1} \frac{T}{2} \left(\frac{\partial^2 v}{\partial \tau^2} \right)^2 ds$, then (4.4) and (4.6)-(4.8) provide (4.1). Q.E.D.

The term $\left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Omega)}$ on the right side of (4.1) can not be dropped in general except if $\beta \geq \beta_0 > 0$ or if Ω has special geometry. We now have

Theorem 4.1. *Assume that Ω is a polygonal domain, $\beta \geq 0$ and $t \in (0, 1]$. If $v \in \mathcal{V}_2$, then there is constant $C(\Omega)$ independent of t such that*

$$\begin{aligned} &\|v\|_{H^2(\Omega)} + \|v\|_{H^2(\Gamma_1)} \\ &\leq C(\Omega) \left[\|\Delta v\|_{L^2(\Omega)} + \left\| \frac{\partial v}{\partial \nu} - \frac{T}{t} \frac{\partial^2 v}{\partial \tau^2} + \frac{\beta}{t} v \right\|_{L^2(\Gamma_1)} + \|v\|_{L^1(\Gamma_1)} \right]. \end{aligned} \quad (4.9)$$

Moreover if $\text{mes}(\Gamma_2) \neq 0$ or if either $u(A_j) = 0$ or $u(A_{j-1}) = 0$ for some $j \in S$ or if $\beta > 0$, then the term $\|v\|_{L^1(\Gamma_1)}$ can be dropped from the right side of (4.9).

Proof. From lemma 4.1 we have

$$\begin{aligned} & |v|_{H^2(\Omega)}^2 + |v|_{H^2(\Gamma_1)}^2 \\ & \leq C[\|\frac{\partial^2 v}{\partial x^2}\|_{L^2(\Omega)}^2 + \|\frac{\partial^2 v}{\partial y^2}\|_{L^2(\Omega)}^2 + 2\|\frac{\partial^2 v}{\partial x \partial y}\|_{L^2(\Omega)}^2 + \int_{\Gamma_1} \frac{T}{2} (\frac{\partial^2 v}{\partial \tau^2})^2 ds] \\ & \leq C(\Omega)[\|\Delta v\|_{L^2(\Omega)}^2 + \|\frac{\partial v}{\partial \nu} - \frac{T}{t} \frac{\partial^2 v}{\partial \tau^2} + \frac{\beta}{t} v\|_{L^2(\Gamma_1)}^2 + \|\frac{\partial v}{\partial \tau}\|_{L^2(\Gamma_1)}^2]. \end{aligned} \quad (4.10)$$

The term $\|\frac{\partial v}{\partial \tau}\|_{L^2(\Gamma_1)}^2$ in (4.10) can be absorbed in $\|v\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Gamma_1)}^2$ which is equivalent to $\|v\|_{\mathcal{V}_1}$. Thus (4.9) follows from (4.10) and inequality similar to (2.7). Q.E.D.

Remark 4.1: Theorem 4.1 holds even if T is a piecewise smooth function on Γ_1 such that $0 < T_0 \leq T \leq T_1 < +\infty$ on Γ_1 where T_0 and T_1 are constants.

Corollary 4.1. *Assume that Ω is a polygonal domain and that $u \in \mathcal{V}_2$ is a variational solution of (2.4). Then*

$$\|u\|_{H^2(\Omega)} + \|u\|_{H^2(\Gamma_1)} \leq C(\Omega)[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)} + \|u\|_{L^1(\Gamma_1)}]. \quad (4.11)$$

Moreover if $\text{mes}(\Gamma_2) \neq 0$ or if either $u(A_j) = 0$ or $u(A_{j-1}) = 0$ for some $j \in S$, then the term $\|u\|_{L^1(\Gamma_1)}$ can be dropped from the right side of (4.11).

5. The Continuity Method and Singularity Analysis.

In this section we always assume that Ω is a bounded open polygonal domain. We first quote the method of continuity.

Lemma 5.1 (cf. Theorem 5.2 of Gilbarg and Trudinger[3]). *Let \mathcal{B}_1 and \mathcal{B}_2 be Banach spaces. $\mathcal{L}_t : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is bounded linear operator for every $t \in [a, b]$.*

If there is a constant C independent of t such that

$$\|v\|_{\mathcal{B}_1} \leq C \|\mathcal{L}_t v\|_{\mathcal{B}_2} \quad \forall t \in [a, b], \quad \forall v \in \mathcal{B}_1 \quad (5.1)$$

and \mathcal{L}_t is uniformly continuous in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ with respect to parameter t on $[a, b]$, then \mathcal{L}_b maps \mathcal{B}_1 onto \mathcal{B}_2 if and only if \mathcal{L}_a maps \mathcal{B}_1 onto \mathcal{B}_2 .

As an application of lemma 5.1, we consider $\mathcal{L}_t : \mathcal{V}_2 \rightarrow L^2(\Omega) \times L^2(\Gamma_1)$ defined by $\mathcal{L}_t(v) = (-\Delta v, t \frac{\partial v}{\partial \nu} - T \frac{\partial^2 v}{\partial \tau^2} + \beta v)$, $\forall t \in [0, 1]$ for some $\beta > 0$. From theorem 4.1 and remark 2.1 we know that \mathcal{L}_1 maps \mathcal{V}_2 onto $L^2(\Omega) \times L^2(\Gamma_1)$ if and only if \mathcal{L}_{t_0} maps \mathcal{V}_2 onto $L^2(\Omega) \times L^2(\Gamma_1)$ for some small parameter t_0 . However using perturbation theory, we claim

Theorem 5.1. *Let $\beta > 0$. If the following problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ -T \frac{\partial^2 u}{\partial \tau^2} + \beta u = g & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_3 \end{cases} \quad (5.2)$$

with compatibility conditions (2.1)-(2.3) has unique solution $u \in \mathcal{V}_2$ for any $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$, then the variational solutions of (1.1a)-(1.1d) with (2.1)-(2.3) belong to \mathcal{V}_2 for any $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$.

Proof. From the assumption we know that \mathcal{L}_0 maps \mathcal{V}_2 1-1 and onto $L^2(\Omega) \times L^2(\Gamma_1)$. So \mathcal{L}_0^{-1} maps $L^2(\Omega) \times L^2(\Gamma_1)$ onto \mathcal{V}_2 . Moreover $\|\mathcal{L}_0^{-1}(f, g)\|_{\mathcal{V}_2} \leq C(\Omega)[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)}]$. Thus $\|\mathcal{L}_0^{-1}(\mathcal{L}_0 - \mathcal{L}_t)\|_{\mathcal{L}(\mathcal{V}_2, \mathcal{V}_2)} \leq t \|\mathcal{L}_0^{-1} \circ (0, \frac{\partial}{\partial \nu})\|_{\mathcal{L}(\mathcal{V}_2, \mathcal{V}_2)} \leq t \cdot C(\Omega)$. Therefore we know that there exists a $t_0 > 0$ such that $\mathcal{L}_{t_0}^{-1} = [I - \mathcal{L}_0^{-1}(\mathcal{L}_0 - \mathcal{L}_{t_0})]^{-1} \mathcal{L}_0^{-1}$ exists. Q.E.D.

It is easy to see that the regularity of the solution of (5.2) can be reduced to the regularity of the solution of (5.2) with homogeneous Dirichlet boundary condition on Γ_1 instead of the second equation of (5.2). Using polar coordinates,

the Fredholm alternative theorem and the fact that $\frac{\partial^2}{\partial \theta^2}$ with Dirichlet, Neumann or mixed boundary conditions is a self-adjoint operator and has compact resolvent, Grisvard gave a complete study on the regularity and singularities of the solution of (5.2). Using theorem 5.1 and the results of theorem 2.3.7 in Grisvard[4], we have the following regularity result.

Corollary 5.1. *If $0 < \omega_j \leq \pi$ when $(j, j+1) \in (D \cup S) \times (D \cup S)$ or $N \times N$ and $0 < \omega_j \leq \frac{\pi}{2}$ when $(j, j+1) \in (D \cup S) \times N$ or $N \times (D \cup S)$ for any $j = 1, \dots, m$, then variational solutions of (1.1a)-(1.1d) with compatibility conditions (2.1) and (2.2) belong to \mathcal{V}_2 .*

Can that the variational solutions of equations (1.1a)-(1.1d) with compatibility conditions (2.1)-(2.3) belong to \mathcal{V}_2 but the solution of equations (5.2) with compatibility conditions (2.1)-(2.3) not belong to \mathcal{V}_2 ? We found from the proof of theorem 5.1 that it is possible that $\|u\|_{\mathcal{V}_2}$ is uniformly bounded by $\|\mathcal{L}_t\|_{L^2(\Omega) \times L^2(\Gamma_1)}$ for small parameter t and thus the proof of theorem 5.1 is reversible. For the rest of this section we assume that Γ_1 contains no two consecutive sides $e_j \subset \Gamma_1$ and $e_{j+1} \subset \Gamma_1$ for some j . We first have an extension theorem.

Theorem 5.2. *If $\beta > 0$ is a constant, then there is $t_0 > 0$ (depended on the geometry of Ω) such that for any $0 < t < t_0$ and $g \in L^2(\Gamma_1)$, there exists a $G \in \mathcal{V}_2$ (see notation at the beginning of §3) such that $t \frac{\partial G}{\partial \nu} - T \frac{\partial^2 G}{\partial \tau^2} + \beta G = g$ on Γ_1 . Moreover*

$$\|G\|_{\mathcal{V}_2} \leq C(\Omega) \|g\|_{L^2(\Gamma_1)} \quad (5.3)$$

where $C(\Omega)$ is independent of parameter t .

Proof. Consider one-dimensional auxiliary problem

$$\begin{cases} -Tu'' + tc(\xi)u' + \beta u = f & \xi \in (0, x_0) \\ u(0) = u(x_0) = 0 \quad \text{or} \quad u(0) = u'(x_0) = 0 \quad \text{or} \quad u'(0) = u'(x_0) = 0 \end{cases} \quad (5.4)$$

where $c(\xi) \in C^1[0, x_0]$. When $T\beta\alpha^{-2} > t$ where $\alpha = \|T^{-1}c(x)\|_{L^\infty(0, x_0)}$, (5.4) then allows a unique solution $u \in H^2(0, x_0)$ by Garding's inequality (cf. Gilbarg & Trudinger[3]) $\int_0^{x_0} [T|u'|^2 + tc(x)u'u]dx \geq \frac{T}{2} \int_0^{x_0} |u'|^2 dx - T^{-1}\alpha^2 t \int_0^{x_0} |u|^2 dx$. Moreover $\|u\|_{H^2(0, x_0)} \leq C\|f\|_{L^2(0, x_0)}$ where C is independent of the parameter t .

Using the above result, we shall construct $\{\hat{g}_j\}_{j=1}^m \in H^{3/2}(\partial\Omega)$, $\{\hat{h}_j\}_{j=1}^m \in H^{1/2}(\partial\Omega)$ from the given g such that $\{\hat{g}_j\}_{j=1}^m$, $\{\hat{h}_j\}_{j=1}^m$ satisfy (3.1)-(3.5) where $\hat{h}_j = c(\xi)\hat{g}'_j$, $\forall j \in S$ and where \hat{g}_j , $j \in S$ is a solution of (5.4) with respect to $f = g|_{e_j}$, $x_0 = |e_j|$, and $c(\xi)$ to be determined later with corresponding boundary conditions due to compatibility conditions (2.1) and (2.2). Then there is a $G \in \mathcal{V}_2$ such that $G = \hat{g}_j$, $\frac{\partial G}{\partial \nu} = \hat{h}_j$ on e_j , $j = 1, \dots, m$ by theorem 1.4.6, Grisvard[4] and the definition of \mathcal{V}_2 and $\|G\|_{\mathcal{V}_2} \leq C(\Omega) [\sum_{j \in D \cup N} (\|\hat{g}_j\|_{H^{3/2}(e_j)} + \|\hat{h}_j\|_{H^{1/2}(e_j)}) + \sum_{j \in S} (\|\hat{g}_j\|_{H^2(e_j)} + \|\hat{h}_j\|_{H^{1/2}(e_j)})]$.

Without loss of generality, we just need to consider (3.1)-(3.5) on $e_j = (A_{j-1}, A_j)$ for $j \in S$ fixed. There are four possible combinations for the neighborhood sides of e_j . But no matter what combination it is, we can always take $c(\xi)$ as a linear function depended on only the geometry of domain Ω as follows.

Case 1: $(j-1, j+1) \in D \times D$. Let $c(0) = \cot \omega_{j-1}$ or 0 if $\omega_{j-1} = \pi$ and $c(|e_j|) = \cot \omega_j$ or 0 if $\omega_j = \pi$. Let \hat{g}_j be a solution of (5.1) with $\hat{g}_j(A_{j-1}) = \hat{g}_j(A_j) = 0$. We let $\hat{h}_j = c(\xi)\hat{g}'_j(\xi)$ on e_j . Then we take h_{j-1} , h_{j+1} as arbitrary smooth functions such that $h_{j-1}(A_{j-1}) = -\csc \omega_{j-1}\hat{g}'_j(A_{j-1})$ or 0 if $\omega_{j-1} = \pi$ and $h_{j+1}(A_j) = \csc \omega_j\hat{g}'_j(A_j)$ or 0 if $\omega_j = \pi$.

Case 2: $(j-1, j+1) \in D \times N$. Let $c(0) = \cot \omega_{j-1}$ or 0 if $\omega_{j-1} = \pi$ and $c(|e_j|) = \tan \omega_j$ or 0 if $\omega_j = \frac{\pi}{2}, \frac{3\pi}{2}$. Let \hat{g}_j be a solution of (5.1) with \hat{g}_j satisfying compatibility conditions (2.1)-(2.2) at points A_{j-1} and A_j . We take \hat{h}_j as in 1) and \hat{g}_{j+1} as arbitrary smooth function such that $\hat{g}_{j+1}(A_j) = \hat{g}_j(A_j)$ and $\hat{g}'_{j+1}(A_j) = -\sec \omega_j \hat{g}'_j(A_j)$ or 0 if $\omega_j = \frac{\pi}{2}, \frac{3\pi}{2}$. Similarly one can construct $\hat{g}_{j-1}, \hat{g}_j, \hat{h}_j$ and \hat{h}_{j+1} when $(j-1, j+1) \in N \times D$.

Case 3: $(j-1, j+1) \in N \times N$. Let $c(0) = -\tan \omega_{j-1}$ or 0 if $\omega_{j-1} = \frac{\pi}{2}, \frac{3\pi}{2}$ and $c(|e_j|) = \tan \omega_j$ or 0 if $\omega_j = \frac{\pi}{2}, \frac{3\pi}{2}$. Let \hat{g}_j be a solution of (5.1) with \hat{g}'_j satisfying compatibility conditions (2.1)-(2.2) at points A_{j-1} and A_j . We take \hat{g}_{j+1} as in 2) and \hat{g}_{j-1} as arbitrary smooth function such that $\hat{g}_{j-1}(A_{j-1}) = \hat{g}_j(A_{j-1})$ and $\hat{g}'_{j-1}(A_{j-1}) = -\cos \omega_j \hat{g}'_j(A_j)$. Q.E.D.

Remark 5.1: theorem 5.2 still holds when Γ_1 contains more than two consecutive sides and $\omega_j = \pi, \forall (j, j+1) \in S \times S$.

Theorem 5.3. *Let $\beta > 0$ be a constant. If the variational solutions of equations (1.1a)-(1.1d) with compatibility conditions (2.1)-(2.2) belong to \mathcal{V}_2 for any $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$, then the variational solution of equations (5.2) with compatibility conditions (2.1)-(2.2) belongs to \mathcal{V}_2 .*

Proof. From the assumptions made and the method of continuity we know that \mathcal{L}_t maps \mathcal{V}_2 onto $L^2(\Omega) \times L^2(\Gamma_1)$ for each fixed small $t > 0$. By theorem 5.2 we know there is $G \in \mathcal{V}_2$ such that theorem 5.2 holds. Let $u(t)$ denote the solution of equation $\mathcal{L}_t u = (f, g)$. Replacing $u(t)$ by $u(t) - G$, we reduce the boundary part $t \frac{\partial u}{\partial \nu} - T \frac{\partial^2 u}{\partial \tau^2} + \beta u = g$ to the case $g = 0$. By theorem 4.1 we have $\|u(t)\|_{\mathcal{V}_2} \leq \|G\|_{\mathcal{V}_2} + \|u(t) - G\|_{\mathcal{V}_2} \leq C(\Omega)[\|g\|_{L^2(\Gamma_1)} + \|f - \Delta G\|_{L^2(\Omega)}] \leq C(\Omega)[\|g\|_{L^2(\Gamma_1)} + \|f\|_{L^2(\Omega)}]$ where $C(\Omega)$ is independent of the parameter t . Letting $t \rightarrow 0$, we conclude that the theorem 5.3 holds. Q.E.D.

Finally we study the singularities of the variational solutions of (2.4) for other general polygonal domain. It is easy to check that $\mathcal{L}_t(\mathcal{V}_2)$ is a closed subspace of $L^2(\Omega) \times L^2(\Gamma_1)$ for any $t \in [0, 1]$. Let \mathcal{Z}_t denote the orthogonal subspace of $\mathcal{L}_t(\mathcal{V}_2)$ for every fixed $t \in [0, 1]$ in $L^2(\Omega) \times L^2(\Gamma_1)$.

Lemma 5.2. $\dim(\mathcal{Z}_1) = \dim(\mathcal{Z}_0) < \infty$.

Proof. From the proof of theorem 5.1 and 5.2, one has that \mathcal{Z}_t , $t \in [0, 1]$ are the same and therefore have the same dimension and that $\dim(\mathcal{Z}_0) < \infty$ follows from the results of theorem 2.3.7 of Grisvard[4]. Q.E.D.

If the geometry of the polygonal domain does not satisfy the conditions in corollary 5.1, then $\dim(\mathcal{Z}_1) > 0$. Therefore \mathcal{L}_1 can not map \mathcal{V}_2 onto $L^2(\Omega) \times L^2(\Gamma_1)$ but it does map \mathcal{V}_1 onto $L^2(\Omega) \times L^2(\Gamma_1)$. We can choose a subspace \mathcal{X} of \mathcal{V}_1 such that \mathcal{L}_1 maps $\mathcal{X} \cup \mathcal{V}_2$ onto $L^2(\Omega) \times L^2(\Gamma_1)$.

Theorem 5.4. *There exist an integer $\sigma(\Omega)$ and linearly independent functions u_1, \dots, u_σ such that for any given $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$, there are unique constant c_1, \dots, c_σ such that*

$$u - \sum_{i=1}^{\sigma} c_i u_i \in \mathcal{V}_2 \quad (5.5)$$

where u is a variational solution of (2.4) with respect to (f, g) .

Proof. Let $\sigma = \dim(\mathcal{Z}_0) < \infty$ by lemma 5.2. Letting $(f_1, g_1), \dots, (f_\sigma, g_\sigma)$ be a basis of \mathcal{Z}_0 , one takes u_1, \dots, u_σ as variational solutions of (2.4) with respect to $(f_1, g_1), \dots, (f_\sigma, g_\sigma)$ individually. Then u_1, \dots, u_σ are linearly independent in \mathcal{V}_1 . Thus theorem 5.3 follows from the projection theorem of Hilbert space.

Q.E.D.

We now try to find an explicit form of such linearly independent functions u_1, \dots, u_σ such that (5.5) holds. Let r_j be the distance from (x, y) in the neighborhood of A_j to A_j . $\eta_j(r_j) \in \mathcal{D}(\bar{\Omega})$, $j = 1, \dots, m$ are smooth truncation functions such that $\eta_j \equiv 1$ in the very small neighborhood of A_j with $\text{supp}(\eta_j) \cap \text{supp}(\eta_i) = \emptyset$ whenever $i \neq j$. With θ running counterclockwise, consider

$$u_{j,k} = \eta_j(r_j) r_j^{\lambda_{j,k}} \varphi_{j,k}(\theta)$$

where $\varphi_{j,k}(\theta)$ and $\lambda_{j,k}$ are given as follows

$$\begin{aligned} \varphi_{j,k}(\theta) &= \sqrt{\frac{2}{\omega_j}} \sin(\theta \lambda_{j,k}), & \lambda_{j,k} &= \frac{k\pi}{\omega_j} \quad \text{if } (j, j+1) \in (D \cup S) \times (D \cup S), \\ \varphi_{j,k}(\theta) &= \sqrt{\frac{2}{\omega_j}} \sin(\theta \lambda_{j,k}), & \lambda_{j,k} &= \frac{(k - \frac{1}{2})\pi}{\omega_j} \quad \text{if } (j, j+1) \in N \times (D \cup S), \\ \varphi_{j,k}(\theta) &= \sqrt{\frac{2}{\omega_j}} \sin[(\omega_j - \theta) \lambda_{j,k}], & \lambda_{j,k} &= \frac{(k - \frac{1}{2})\pi}{\omega_j} \quad \text{if } (j, j+1) \in (D \cup S) \times N, \\ & \begin{cases} \varphi_{j,k}(\theta) = \sqrt{\frac{2}{\omega_j}} \cos(\theta \lambda_{j,k}), & \lambda_{j,k} = \frac{(k-1)\pi}{\omega_j}, & k \geq 2, \\ \varphi_{j,1}(\theta) = \sqrt{\frac{1}{\omega_j}}, & \lambda_{j,1} = 0 & \text{if } (j, j+1) \in N \times N. \end{cases} \end{aligned}$$

Direct calculation shows that $u_{j,k}$ where $j = 1, \dots, m$ and k is a positive integer belong \mathcal{V}_1 and are the variational solutions of (1.1a)-(1.1d) with compatibility conditions (2.1)-(2.2) with respect to certain $f = f_{j,k} \in \mathcal{D}(\bar{\Omega})$ and $g = g_{j,k} \in L^2(\Gamma_1)$. One also has that $(f_{j,k}, g_{j,k})$ are linearly independent and each $(f_{j,k}, g_{j,k})$ are not orthogonal to \mathcal{Z}_0 when $\lambda_{j,k} < 1$ (cf. §2.4, Grisvard[4]). Using the similar argument as in the proof of theorem 2.4.3 of Grisvard[4] and theorem 5.4, we have

Corollary 5.2. *If $0 < \omega_j < 2\pi$ for $j = 1, \dots, m$, then there exist unique numbers*

$c_{j,k}$ such that

$$u - \sum_j \left(\sum_{0 < \lambda_{j,k} < 1} c_{j,k} u_{j,k} \right) \in \mathcal{V}_2 \quad (5.6)$$

for every variational solution u of (2.4) with respect to $(f, g) \in L^2(\Omega) \times L^2(\Gamma_1)$.

Remark 5.2: Under the assumptions of corollary 5.2, it follows from (5.6) that we have that the variational solutions of (2.4) belong to $H^\sigma(\Omega)$ for every $\sigma < 1 + \inf\{\lambda_{j,k}; 0 < \lambda_{j,k} < 1\}$.

Remark 5.3: It seems likely that the same technique works for more general elliptic equation of second order in Ω and the curved polygons.

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