APPROXIMATION DYNAMICS AND THE STABILITY OF INVARIANT SETS

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ABSTRACT. We introduce the concept of a weakly, normally hyperbolic set for a system of ordinary differential equations. This concept includes the notion of a hyperbolic flow, as well as that of a normally hyperbolic invariant manifold. Moreover, it has the property that it is closed under finite set products. Consequently, the theory presented here can be used for the study of perturbations of the dynamics of coupled systems of weakly, normally hyperbolic sets. Our main objective is to show that under a small C^1-perturbation, a weakly, normally hyperbolic set K is preserved under a homeomorphism, where the image K^Y is a compact invariant set, with a related hyperbolic structure, for the perturbed equation. In addition, the homeomorphism is close to the identity in $C^{0,1}$ and the perturbed dynamics on K^Y are close to the original dynamics on K.

1. INTRODUCTION

The need to find new theories and new algorithms for the study of the long-time dynamical behavior of very high dimensional systems of differential equations is the basic goal of the relatively new theory of Approximation Dynamics. What lies behind this theory is that, in order to study the dynamics of a given system (S) of differential equations, one oftentimes needs to study a nearby system formed by small perturbation of the system (S). For example, in Sell (1993) it is shown that, under reasonable conditions, every approximate inertial manifold for the Navier-Stokes equations is an actual inertial manifold for a small perturbation of these equations. This fact, which is in principle valid for other systems of differential equations, offers a convenient framework for the study of the long-time dynamics

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of these equations. In particular, the basic issue one faces in the theory of Approximation Dynamics is how well one can approximate the long-time dynamics of the solutions of a given differential equation

$$x' = X(x) \tag{1.1}$$

by those of an approximate equation

$$y' = X(y) + Y(y), \tag{1.2}$$

where Y is small in a suitable norm.

While the theory of Approximation Dynamics did originate as an outgrowth of the theory of inertial manifolds and approximate inertial manifolds for partial differential equations, the approximation issues described above offer challenging dynamical problems for finite dimensional ordinary differential equations as well. In addition, by using the theory of inertial manifolds, one is able to derive good information comparing the long-time dynamics of infinite dimensional and finite dimensional problems, see Pliss and Sell (1993), for example. Furthermore, the issues of Approximation Dynamics are fundamental issues one faces when trying to study the long-time dynamics of very high dimensional systems of ordinary differential equations, such as the standard spatial discretizations of partial differential equations.

In Pliss and Sell (1991) it is shown that whenever equation (1.1) has a **hyperbolic attractor** $\mathcal{K}$ with a Lipschitz property, and Y is a sufficiently small C^1-perturbation, then equation (1.2) has a hyperbolic attractor $\mathcal{K}^Y$ which is homeomorphic to $\mathcal{K}$ and close to $\mathcal{K}$. Furthermore, the dynamics on $\mathcal{K}^Y$ are *close to* the dynamics on $\mathcal{K}$. In simpler terms, the long-time dynamics, as represented in certain attractors of equation (1.1), are preserved under small C^1-perturbations.

In this paper, our major objective is to extend the results of Pliss and Sell (1991) to the study of more complicated dynamical features, such as the Poincaré-Melnikov scenario describing bifurcations of homoclinic orbits, see Meyer and Sell (1989), and Bishop, et al (1990). It is known that this scenario underlies much of the complicated dynamical behavior one observes in various chaotic systems. Moreover, it is suspected that this scenario is a common occurrence in the route to chaos.

The principal dynamical objects which we study in this paper are compact, invariant sets $\mathcal{K}$ with suitable hyperbolic structures. Later we will give precise definitions of a **weakly hyperbolic** set and a **weakly, normally hyperbolic** set. It is important to note that these concepts generalize and include the notion of an hyperbolic flow, see Smale (1967), Arnold (1983), and Pilyugin (1992), as well as that of a normally hyperbolic manifold, see Sacker (1969), Fenichel (1971), and Hirsch, Pugh and Shub (1977). Also see Pliss and Sell (1991).

We will assume that both equations (1.1) and (1.2) are given on the Euclidean space R^n, where X and Y are C^1-functions from R^n to R^n. Let $\mathcal{K}$ be a given compact, invariant set for equation (1.1), and let $\|\cdot\|$ denote the Euclidean norm on R^n. In addition we will assume that for an appropriate $\delta > 0$ the perturbation term Y satisfies $\|Y\|_{C^1(\Omega)} \le \delta$, where

$$\|Y\|_{C^1(\Omega)} \stackrel{\text{def}}{=} \sup_{x\in\Omega} \max\left(\|Y(x)\|, \|DY(x)\|_{\text{op}}\right),$$

$\Omega = N(\mathcal{K}, \beta_0)$ is a given β_0-neighborhood of $\mathcal{K}$ in R^n, and $\beta_0 > 0$ is small and fixed. We let $DY = \frac{\partial Y}{\partial y}$ denote the Jacobian of a function Y, and for any bounded linear operator T, the operator norm $\|T\|$ is defined by

$$\|T\| = \|T\|_{op} \stackrel{\text{def}}{=} \sup\{\|Tv\| : \|v\| \leq 1\}.$$

We will let $x(t, x_0) = S_1(t)x_0$ and $y(t, y_0) = S_2(t)y_0$ denote the maximally defined solutions of equations (1.1) and (1.2) that satisfy $S_1(0)x_0 = x_0$ and $S_2(0)y_0 = y_0$, respectively. Recall that a set $\mathcal{K}$ is invariant for (1.1) provided that one has $S_1(t)\mathcal{K} = \mathcal{K}$, for all $t \in R$. The following statement is the main result proven in this paper.

Theorem A. *Let $\mathcal{K}$ be a given weakly, normally hyperbolic set for equation (1.1) that satisfies the Lipschitz property, and let $\Omega = N(\mathcal{K}, \beta_0)$ denote a fixed β_0-neighborhood of $\mathcal{K}$. Then for every $\epsilon > 0$ there exist δ_i, $i = 1, 2, 3$, such that $0 < \delta_3 \leq \delta_2 \leq \delta_1$ and the following three properties hold:*

1. *If $\|Y\|_{C^1(\Omega)} \leq \delta_1$, then there is a continuous mapping $h : \mathcal{K} \to R^n$ such that the image $\mathcal{K}^Y = h(\mathcal{K})$ is a compact, invariant set for equation (1.2), and $\|h(x) - x\| \leq 2\epsilon$ for all $x \in \mathcal{K}$.*
2. *If, in addition, $\|Y\|_{C^1(\Omega)} \leq \delta_2$, then $\mathcal{K}^Y$ is a weakly, normally hyperbolic set for equation (1.2).*
3. *Moreover, if $\|Y\|_{C^1(\Omega)} \leq \delta_3$, then the mapping h is a homeomorphism.*

In addition to the above result, the associated dynamics $S_2(t)$ on $\mathcal{K}^Y$ is essentially a faithful representation of the unperturbed dynamics $S_1(t)$ on $\mathcal{K}$, in a sense we now make precise. Let $\mathcal{K}$ be a compact, invariant set for equation (1.1). One says that a continuous mapping $h : \mathcal{K} \to R^n$ is a homomorphism, if there is an induced flow $S^Y(t)$ on $\mathcal{K}$ such that

$$S_2(t)h(x_0) = h(S^Y(t)x_0), \qquad \text{for all } x_0 \in \mathcal{K},\ t \geq 0. \tag{1.5}$$

If in addition, the mapping $h : \mathcal{K} \to R^n$ is one-to-one, i.e., h is a homeomorphism, then one says that h is an isomorphism. The flow $S^Y(t)$ satisfying equation (1.5) on the given compact, invariant set $\mathcal{K}$ is induced by the perturbation Y in equation (1.2). We refer to $S^Y(t)$ as the **shadow flow** on $\mathcal{K}$. We now have the following result:

Shadow Theorem B. *Let the hypotheses and conclusions of Theorem 1.1 be satisfied. Then for every Y with $\|Y\|_{C^1(\Omega)} \leq \delta_1$, there is a shadow flow $S^Y(t)$ on $\mathcal{K}$ satisfying equation (1.5).*

Remarks. (1) The commutivity relationship (1.5) contains some valuable dynamical information. For example, if $\mathcal{K}$ is a stable periodic orbit, then its homeomorphic image $\mathcal{K}^Y$ is also a cycle, and the perturbed flow $S^Y(t)$ differs from the unperturbed dynamics $S_1(t)$ by either a change in the speed, the period and/or the phase along the cycle $\mathcal{K}$. When h is a homeomorphism, then equation (1.5) is a statement of the lower semicontinuity with respect to the perturbation term Y of the weakly, normal hyperbolicity of the set $\mathcal{K}$. Generally speaking, lower semicontinuity theorems occur much less frequently in the literature than their upper semicontinuous counterparts, see Hale and Raugel (1989, 1990), for example.

(2) There are two special cases in which the conclusions of the Theorem A are known: (i) when $\mathcal{K}$ is a normally hyperbolic, compact invariant manifold and (ii) when $\mathcal{K}$ is the phase space of an hyperbolic flow, see the references cited above. The theorem proved here offers both a generalization and a unification of the theories of these two cases.

(3) As we will see below, the concept of a weakly, normally hyperbolic set introduced here is closed under finite set products.[1] Except for the case of finite products of compact, normally invariant manifolds, the earlier theories did not apply to set products, whereas Theorems A and B do. Because of this feature, one can now use the theory presented here in the study of the dynamical coupling between weakly, normally hyperbolic sets.

In Section 2 we give formal definitions of the various hyperbolicity concepts used in this paper, and we derive some basic properties associated with vector fields containing compact invariant sets with these hyperbolic structures. In Section 3 we give the proof of the existence of the continuous mapping $h : \mathcal{K} \to R^n$ satisfying both Part (1) of Theorem A and the Shadow Theorem B. In Sections 4 and 5 we will give proofs of Parts (2) and (3) of Theorem A. Finally in Section 6, we present some applications of our theory, including an introduction of issues of Approximation Dynamics to the Bubnov-Galerkin method as it is used in numerical analysis.

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2. Weak Hyperbolicity

Let equations (1.1) and (1.2) be given as in Section 1, and let $\mathcal{K}$ be a given compact, invariant set for equation (1.1). Since the vector field X is a C^1-function, the Jacobian $A(x) = \frac{\partial X(x)}{\partial x}$ satisfies

$$X(x+z) - X(x) = A(x)z + F(x,z), \tag{2.1}$$

where $F = F(x,z)$ is a continuous function which is C^1 in the z-variable with

$$F(x,0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial z}(x,0) = 0, \tag{2.1a}$$

for all $x \in R^n$. Consequently, one has

$$|F(x,z)| = o(|z|), \qquad \text{as } |z| \to 0. \tag{2.1b}$$

For each $x_0 \in \mathcal{K}$, we let $\Phi(t, x_0)$ denote the fundamental operator solution of the linear system

$$\partial_t u = A(x(t,x_0))\, u \tag{2.2}$$

that satisfies $\Phi(0, x_0) = I$, where I is the identity operator on R^n. Note that Φ satisfies the co-cycle identity

$$\Phi(s+t, x_0) = \Phi(t, x(s,x_0))\Phi(s,x_0) \qquad x_0 \in \mathcal{K},\ s,t \in R. \tag{2.2a}$$

[1]Hyperbolic flows, for example, do not have this closure property.

Definition 2.1. A compact, invariant set $\mathcal{K}$ is said to be **weakly hyperbolic** if there exist six real numbers $a \geq 1$, λ_0, $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, where $\lambda_1 < 0 < \lambda_4$, $\lambda_0 \geq \lambda_3$, and complementary linear spaces $U^s(x_0)$, $U^o(x_0)$, $U^u(x_0)$, defined for all $x_0 \in \mathcal{K}$, and such that dim $U^u(x_0) = m$, dim $U^o(x_0) = k$, dim $U^s(x_0) = n - m - k$, for all $x_0 \in \mathcal{K}$, and

(1) the invariance condition

$$\Phi(t, x_0)U^i(x_0) = U^i(x(t, x_0)), \qquad \text{for } i = s, o, u, \tag{2.2b}$$

holds, for all $x_0 \in \mathcal{K}$ and $-\infty < t < +\infty$;

(2) if $v \in U^s(x_0)$, then one has

$$|\Phi(t, x_0)v| \leq a|v|e^{\lambda_1 t}, \qquad \text{for } t \geq 0; \tag{2.2c}$$

(3) if $v \in U^o(x_0)$, then one has

$$\begin{cases} |\Phi(t, x_0)v| \leq a|v|e^{\lambda_2 t}, & \text{for } t \leq 0, \\ |\Phi(t, x_0)v| \leq a|v|e^{\lambda_3 t}, & \text{for } t \geq 0; \end{cases} \tag{2.2d}$$

(4) if $v \in U^u(x_0)$, then one has

$$|\Phi(t, x_0)v| \leq a|v|e^{\lambda_4 t}, \qquad \text{for } t \leq 0; \tag{2.2e}$$

(5) for all $v \in R^n$ and all $x_0 \in \mathcal{K}$ one has

$$|\Phi(t, x_0)v| \leq a|v|e^{\lambda_0 t}, \qquad \text{for } t \geq 0. \tag{2.2f}$$

Remarks. (1) The **index** of a weakly hyperbolic set $\mathcal{K}$ is defined to be index$(\mathcal{K}) = k + m$. This quantity does arise in some applications.

(2) The concept of a weakly hyperbolic set can be reformulated in terms of a pair of exponential dichotomies by using the shifted linear skew product flows $\Phi_\lambda(t, x_0) = e^{-\lambda t}\Phi(t, x_0)$ and $\Phi_\mu(t, x_0) = e^{-\mu t}\Phi(t, x_0)$, for appropriate choices of λ and μ with $\lambda < 0 < \mu$, see Sacker and Sell (1978, 1980), Sell (1978), and Coppel (1978), for more details.

(3) In our main applications, one has $\lambda_2 \leq 0 \leq \lambda_3$, but we will not add this relationship in our definition. Since $\mathcal{K}$ is invariant, it follows from (2.2a), (2.2b), and (2.2d) that for any $v \in U^o(x_0)$ one has

$$a^{-1}e^{\lambda_2 t}|v| \leq |\Phi(t, x_0)v|, \qquad \text{for } t \geq 0. \tag{2.3a}$$

Since the identity $\Phi(-\tau, x(\tau, x_0)) = \Phi^{-1}(\tau, x_0)$ follows from the cocycle property (2.2a), inequality (2.2e) is equivalent to

$$|\Phi^{-1}(\tau, x_0)v| \leq a|v|e^{-\lambda_4 \tau}, \qquad \text{for } \tau \geq 0, \tag{2.3b}$$

provided that $v \in U^u(x(\tau, x_0))$.

The six parameters a and λ_i, for $0 \le i \le 4$, are the **characteristics** of the compact, invariant set $\mathcal{K}$. Many of the quantities described in the sequel will depend on these characteristics.

The linear spaces $U^s(x_0)$, $U^o(x_0)$, and $U^u(x_0)$ are referred to as the **stable**, **neutral**, and **unstable** linear subspaces for $\Phi(t, x_0)$. It is well known that if $\mathcal{K}$ is weakly hyperbolic, then there exists an $\alpha > 0$ such that the angles satisfy

$$\begin{aligned} &\angle(U^s(x_0), U^o(x_0) \oplus U^u(x_0)) > \alpha, \\ &\angle(U^o(x_0), U^s(x_0) \oplus U^u(x_0)) > \alpha, \\ &\angle(U^u(x_0), U^s(x_0) \oplus U^o(x_0)) > \alpha, \end{aligned} \tag{2.4}$$

for all $x_0 \in \mathcal{K}$, see Pliss (1977), or Sacker and Sell (1974, 1976ab). For $x \in \mathcal{K}$ and $i = s, o, u$, let $P^i(x)$ denote the projections of R^n onto $U^s(x)$, $U^o(x)$, $U^u(x)$ that satisfy $\mathcal{R}(P^i(x)) = U^i(x)$, where $\mathcal{R}$ denotes the range of the projection, and $P^s(x) + P^o(x) + P^u(x) = I$. One should note that the invariance condition (2.2b) is equivalent to the condition that

$$P^i(x(t, x_0))\Phi(t, x_0) = \Phi(t, x_0)P^i(x_0), \qquad \text{for } i = s, o, u, \tag{2.4a}$$

for all $x_0 \in \mathcal{K}$ and $-\infty < t < \infty$. The projector Q^o, which is defined by

$$Q^o(x) = I - P^o(x), \qquad \text{for } x \in \mathcal{K}, \tag{2.4b}$$

also satisfies (2.4a). Note that $Q^o(x) = P^s(x) + P^u(x)$, for all $x \in \mathcal{K}$. Also note that for all $x \in \mathcal{K}$ one has $P^i(x)P^j(x) = 0$, if $i \ne j$. Furthermore, it follows from relations (2.2c)-(2.2f) and (2.4a) that

$$|P^i(x_0)| \le a, \qquad \text{for } x_0 \in \mathcal{K} \text{ and } i = s, o, u. \tag{2.4c}$$

Definition 2.2. The invariant set $\mathcal{K}$ is said to be **weakly, normally hyperbolic** if it is weakly hyperbolic and there exists an $r > 0$ such that for each $x_0 \in \mathcal{K}$ there exists a k-dimensional, locally invariant disk $\mathcal{D}(x_0) = \mathcal{D}_r(x_0) \subset \mathcal{K}$ with the center at the point x_0 and radius r such that if $x \in \mathcal{D}(x_0)$ then the neutral linear space $U^o(x)$ is tangent to the disk $\mathcal{D}(x_0)$ at the point x. A weakly, normally hyperbolic invariant set is said to satisfy a **Lipschitz property** if $U^o(x_0)$ is locally Lipschitz continuous in $x_0 \in \mathcal{K}$, and if for each $x_0 \in \mathcal{K}$, the spaces $U^s(x)$ and $U^u(x)$ are Lipschitz continuous functions of $x \in \mathcal{D}(x_0)$. The Lipschitz property can be reformulated in terms of the projectors P^i, $i = s, o, u$. In particular, this property is equivalent to saying that (i) there is an $L > 0$ such that one has

$$|P^o(x_1) - P^o(x_2)| \le L|x_1 - x_2|, \qquad \text{for all } x_1,\, x_2 \in \mathcal{K} \text{ with } |x_1 - x_2| \le r, \tag{2.4d}$$

and (ii) for $i = s, u$, one has

$$|P^i(x_1) - P^i(x_2)| \le L|x_1 - x_2|, \quad \text{for all } x_1,\, x_2 \in \mathcal{D}(x_0) \text{ and all } x_0 \in \mathcal{K}. \tag{2.4e}$$

If $\mathcal{K}$ is a weakly, normally hyperbolic set with $m = \dim U^u(x_0) = 0$, for all $x_0 \in \mathcal{K}$, i.e., with $\operatorname{index}(\mathcal{K}) = k$, then $\mathcal{K}$ is referred to as an **hyperbolic attractor**, see Pliss and Sell (1991).

Let $\mathcal{K}$ be a weakly, normally hyperbolic invariant set for (1.1) that satisfies the Lipschitz property. For $x_0 \in \mathcal{K}$, let $\mathcal{D}(x_0)$ be the disk given in Definition 2.2. Let us move the origin to the point x_0 and rotate the coordinate axes, to obtain a new coordinate system (u, v) where u is a k-dimensional vector and v is an $(n-k)$-dimensional vector. We assume that the coordinate axes have been fixed so that the space $v = 0$ coincides with the neutral space $U^o(x_0)$. The disk $\mathcal{D}(x_0)$ can then be represented in the form

$$\mathcal{D}(x_0) = \{v = f(u) : |u| \leq r\} \tag{2.4f}$$

in terms of a suitable function f. Because of the Lipschitz property for $\mathcal{K}$, the tangent space to the curve $v = f(u)$ is Lipschitz continuous, which implies that the function f, itself, is of class $C^{1,1}$. Let the point $x_1 \in \mathcal{D}(x_0)$ have coordinates $u = u_1$, $v = f(u_1)$, where $|u_1| \leq r$. Let $\mathcal{D}(x_1)$ be the disk in $\mathcal{K}$ centered at x_1, If the radius r is small enough, then the disk $\mathcal{D}(x_1)$ can be represented as a graph

$$\mathcal{D}(x_1) = \{v = g(u) : |u - u_1| \leq r\},$$

where g is also of class $C^{1,1}$ Let U be the subset of R^n given by

$$U = \{(u, 0) : |u| \leq r \text{ and } |u - u_1| \leq r\}.$$

We then have the following result, which establishes the uniqueness of the disks $\mathcal{D}(x)$.

Lemma 2.3. *For all $u \in U$, one has $f(u) = g(u)$. In particular, for each $x_0 \in \mathcal{K}$, the disk $\mathcal{D}(x_0)$ is uniquely determined.*

Proof. Let $(u, 0) \in U$. Since U is a convex set, the line segment

$$\{((1-s)u_1 + su, 0) : 0 \leq s \leq 1\}$$

belongs to U. Define $w = w(s)$ by

$$w(s) = f((1-s)u_1 + su) - g((1-s)u_1 + su), \qquad 0 \leq s \leq 1.$$

One then has

$$\frac{dw}{ds} = \left(\frac{\partial f}{\partial u} - \frac{\partial g}{\partial u}\right)(u - u_1)$$

and

$$\frac{d|w|}{ds} \leq \left|\frac{dw}{ds}\right| = \left|\frac{\partial f}{\partial u} - \frac{\partial g}{\partial u}\right| |u - u_1|.$$

From inequalities (2.4d) and (2.4e) one obtains

$$\left|\frac{\partial f}{\partial u} - \frac{\partial g}{\partial u}\right| \leq L|f((1-s)u_1 + su) - g((1-s)u_1 + su)| = L|w|,$$

perhaps with a larger value of L. Consequently, one has

$$\frac{d|w|}{ds} \leq L|u - u_1| \cdot |w|.$$

The Gronwall inequality then implies that

$$|w(s)| \le |w(0)|e^{L|u-u_1|s}, \qquad \text{for } 0 \le s \le 1.$$

Since $x_1 \in \mathcal{D}(x_0)$, one has $w(0) = f(u_1) - g(u_1) = 0$, and consequently $w(1) = 0$, or $f(u) = g(u)$. $\square$

For $x_0 \in \mathcal{K}$ we define the sets $\mathcal{S}_1(x_0), \mathcal{S}_2(x_0), \dots, \mathcal{S}_i(x_0)$ by

$$\mathcal{S}_1(x_0) = \bigcup_{x \in \mathcal{D}(x_0)} \mathcal{D}(x), \qquad \mathcal{S}_i(x_0) = \bigcup_{x \in \mathcal{S}_{i-1}(x_0)} \mathcal{D}(x), \quad \text{for } i \ge 2,$$

and

$$\mathcal{S}(x_0) = \bigcup_{i=1}^{\infty} \mathcal{S}_i(x_0).$$

Note that for each $x_0 \in \mathcal{K}$, $\mathcal{S}(x_0)$ is an invariant set for equation (1.1), and $\mathcal{S}(x_0) \subset \mathcal{K}$. Furthermore, one has $x_1 \in \mathcal{S}(x_0)$ if and only if $\mathcal{S}(x_1) = \mathcal{S}(x_0)$, because of Lemma 2.3. This implies that if $\mathcal{S}(x_0) \cap \mathcal{S}(x_1)$ is nonempty, then $\mathcal{S}(x_0) = \mathcal{S}(x_1)$. The set $\mathcal{S}(x_0)$ is referred to as the **leaf** of $\mathcal{K}$ through x_0. We will refer to k as the **dimension** of the disk $\mathcal{D}(x_0)$ and of the leaf $\mathcal{S}(x_0)$. Note that $\mathcal{K}$ is the disjoint union of its leaves, and that the leaves all have the same dimension k.

Let us return to the representation of the disk $\mathcal{D}(x_0)$ given by (2.4f). Since the space $v = 0$ coincides with the neutral space $U^o(x_0)$, it follows that the derivative $D_u f = \frac{\partial f}{\partial u}$ satisfies $D_u f(0) = 0$. Furthermore, because of the Lipschitz property for $x \to U^o(x)$, it follows that $D_u f(u)$ is differentiable almost everywhere and the second derivative $D_u^2 f = \frac{\partial^2 f}{\partial u^2}$ satisfies $|D^2 f(u)| \le \hat{L}$, for $|u| \le r$. (The constant $\hat{L}$ depends on the Lipschitz coefficient for the mapping $x \to U^o(x)$ and is independent of the base point x_0.) Consequently, by using a larger value for L, if necessary, one has the validity of both

$$|f(u_1) - f(u_2)| \le Lr|u_1 - u_2|, \qquad \text{for } |u_1|, |u_2| \le r, \tag{2.7}$$

and inequalities (2.4d) and (2.4e). As a consequence of this and the angle condition (2.4), we have the following result, which treats the radius r of the disk $\mathcal{D}(x_0)$ as a parameter.

Lemma 2.4. *There exists a bounded function $k_0 = k_0(r) > 0$ such that for all $x_0 \in \mathcal{K}$ and all x_1, $x_2 \in \mathcal{D}_r(x_0)$, one has*

$$\begin{aligned} &|P^o(x_0)(x_1 - x_0)| - k_0|P^o(x_0)(x_1 - x_0)|^2 \\ &\qquad \le |x_1 - x_0| \le |P^o(x_0)(x_1 - x_0)| + k_0|P^o(x_0)(x_1 - x_0)|^2. \end{aligned}$$

In addition, one obtains

$$\begin{aligned} |P^s(x_0)(x_1 - x_2)| &\le a^2 k_0 \max(|x_1 - x_0|, |x_2 - x_0|)|x_1 - x_2| \\ |P^u(x_0)(x_1 - x_2)| &\le a^2 k_0 \max(|x_1 - x_0|, |x_2 - x_0|)|x_1 - x_2|. \end{aligned} \tag{2.7b}$$

In particular, for r sufficiently small one has

$$|P^o(x_0)(x_1 - x_0)| \ge \frac{3}{4}|x_1 - x_0|. \tag{2.7c}$$

By combining Lemma 2.4 with the Definition 2.1 on weak hyperbolicity and (2.4c), we obtain the following result:

Lemma 2.5. *For all $x_0 \in \mathcal{K}$, all $x_1 \in \mathcal{D}(x_0)$, and all $t \geq 0$, the following are valid:*

$$|\Phi(t,x_0)P^o(x_0)(x_1-x_0)| \leq a^2|x_1-x_0|e^{\lambda_3 t},$$
$$|\Phi(t,x_0)P^s(x_0)(x_1-x_0)| \leq a^3k_0r|x_1-x_0|e^{\lambda_1 t},$$
$$|\Phi(t,x_0)P^u(x_0)(x_1-x_0)| \leq a^3k_0r|x_1-x_0|e^{\lambda_0 t},$$
$$|\Phi(t,x_0)(x_1-x_0)| \leq a^2|x_1-x_0|e^{\lambda_3 t} + 2a^3k_0r|x_1-x_0|e^{\lambda_0 t},$$
$$|\Phi(t,x_0)P^o(x_0)(x_1-x_0)| \geq \frac{3}{4a}|x_1-x_0|e^{\lambda_2 t},$$
$$|\Phi(t,x_0)(x_1-x_0)| \geq \left(\frac{3}{4a}e^{\lambda_2 t} - 2a^2k_0re^{\lambda_0 t}\right)|x_1-x_0|.$$

While the inequalities in this lemma are valid for all $t \geq 0$, we will be using them when t is restricted to a finite interval $0 \leq t \leq 2T$, where T is described below.

Remarks. (1) It follows directly from the definitions that if $\mathcal{K}_i$ is a weakly, normally hyperbolic set for an ordinary differential equation $x_i' = X_i(x_i)$ on R^{n_i}, for $i = 1, \cdots, k$, then the product set

$$\mathcal{K} = \Pi_{i=1}^k \mathcal{K}_i$$

is a weakly, normally hyperbolic set for the product system $x' = X(x)$, on R^n, where $n = n_1 + \cdots + n_k$, $x = (x_1, \cdots, x_k)$, and $X(x) = (X_1(x_1), \cdots, X_k(x_k))$. Moreover, if each $\mathcal{K}_i$, for $i = 1, \cdots, k$, has the Lipschitz property, then so does $\mathcal{K}$.

(2) A weakly, normally hyperbolic set $\mathcal{K}$ is said to be the **phase space of a hyperbolic flow**, and the flow $S_1(t)$ restricted to $\mathcal{K}$ is said to be **hyperbolic**, provided that (i) the flow $S_1(t)$ has no fixed points on $\mathcal{K}$, and (ii) the leaves have dimension 1, see Arnold (1983). Because of this dimension requirement for the leaves, it is immediate that, while the product of two hyperbolic sets is not a hyperbolic set, it is a weakly, normally hyperbolic set.

3. Existence Property

In this section we will assume that the hypotheses of Theorem 1.1 are satisfied and we will show that if $\|Y\|_{C^1(\Omega)} \leq \delta$, for a sufficiently small δ, then the system (1.2) has a compact, invariant set $\mathcal{K}^Y$ in a small neighborhood of the set $\mathcal{K}$. We also establish some special properties of $\mathcal{K}^Y$, and derive the Shadow Theorem B.

Local Coordinates: Let $\mathcal{S}$ be a given leaf of the compact, invariant set $\mathcal{K}$. Recall that $N(A, \beta_0)$ is the β_0-neighborhood of the set A. Due to the Lipschitz property of $\mathcal{K}$, it follows from the Tubular Neighborhood Theorem, see for example, Guillemin and Pollack (1974), that if the radius r of the disks $\mathcal{D}(x_0)$ and the number β_0 are sufficiently small, then one can construct a new (local) coordinate system in the vicinity of each point $x_0 \in \mathcal{K}$ as follows:

(1) Let $y \in N(\mathcal{D}(x_0), \beta_0)$, where $x_0 \in \mathcal{K}$. Then there is one and only one point $v \in \mathcal{D}(x_0)$ such that $y - v \in U^s(v) \oplus U^u(v)$. Furthermore, the mapping $\psi : y \to v \overset{\text{def}}{=} \psi(y) = \psi(x_0, y)$ is of class $C^{1,1}$ on $N(\mathcal{D}(x_0), \beta_0)$ with $\psi(x_0, x_1) = x_1$, for all $x_1 \in \mathcal{D}(x_0)$.

(2) Moreover, the Jacobian of $\psi(y)$ satisfies

$$D\psi(y) = P^o(v) = P^o(\psi(y)), \tag{3.0}$$

where $P^o(v)$ is the projection defined in the paragraph containing (2.4a).

(3) The mapping ψ can be written in the form

$$\psi(y) = \psi(x_0, y) = y - \phi(x_0, y) = y - \phi(y),$$

and one then has

$$D\phi(y) = Q^o(v) = Q^o(\psi(y)),$$

where $Q^o(v) = P^s(v) + P^u(v)$, see (2.4b). The mapping ϕ has the property that for all $v \in \mathcal{K}$, one has

$$\phi(v + n) = n, \qquad \text{for } n \in \mathcal{R}(Q^o(v)). \tag{3.0c}$$

(4) We will denote the new (nonlinear) coordinates of the point y by

$$y = v + s + u = v + n = (v, s, u), \tag{3.0a}$$

where $v \in \mathcal{S} = \mathcal{S}(x_0)$, $s \in U^s(v)$, $u \in U^u(v)$, and $n = s + u$. By (3.0c) one then has $\phi(y) = n = s + u$.

In the sequel we fix r and β_0 so that $r \le \beta_0$ and the representation given by (3.0a) holds. This new coordinate system for the point y depends on the base point $x_0 \in \mathcal{K}$. It can happen, of course, that one can choose different base points associated with a given $y \in N(\mathcal{K}, \beta_0)$. In this case one will obtain different representations of y in terms of the coordinates (v, s, u). However, if two disks $\mathcal{D}(x_0)$ and $\mathcal{D}(x_1)$ have a nontrivial intersection and $y \in N(\mathcal{D}(x_0), \beta_0) \cap N(\mathcal{D}(x_1), \beta_0)$, then $\psi(x_0, y) = \psi(x_1, y)$, i.e., the coordinate representations agree.

Let $y = y(t) = y(t, y_0)$ be a solution of the perturbed equation (1.2), where $y_0 = v_0 + n_0$, $v_0 = x_0 \in \mathcal{K}$, and $n_0 \in \mathcal{R}(Q^o(v_0))$. Assume that one has $y(t) \in N(\mathcal{D}(x(t, x_0)), \beta_0)$ for t in some interval, say $0 \le t \le 2T$. Then the local coordinate representation $y(t) = v(t) + n(t)$ is well-defined for all t, $0 \le t \le 2T$. Furthermore, the chain rule and (3.0) imply that the time-derivatives satisfy $v' = P^o(v)y'$ and $n' = Q^o(v)y'$. This leads to the system of equations

$$\begin{aligned} v' &= P^o(v)[X(v + n) + Y(v + n)] \\ n' &= Q^o(v)[X(v + n) + Y(v + n)], \end{aligned} \tag{3.1}$$

where v and n satisfy the constraints

$$v \in \mathcal{K} \qquad \text{and} \qquad P^o(v)n = 0. \tag{3.1a}$$

Since $P^o(v)$ and $Q^o(v) = I - P^o(v)$ are Lipschitz continuous in v, the system (3.1) has a unique solution $(v(t), n(t))$ for every initial condition (v_0, n_0).

Standard Notation: On several occasions in the following argument we will be studying two solutions $y_i(t)$ of the perturbed equation (1.2) where $y_i(t) \in N(\mathcal{K}, \beta_0)$, for $0 \le t \le 2T$. We will write this in the form

$$y_i = y_i(t) = v_i(t) + n_i(t), \qquad \text{for } 0 \le t \le 2T, \tag{3.1b}$$

where $v_i(t) \in \mathcal{K}$ and $P^o(v_i(t))n_i(t) = 0$, for $i = 0, 1$, and $0 \le t \le 2T$.

With the characteristics a, λ_0, λ_1, λ_2, λ_3, and λ_4 of the compact, invariant set $\mathcal{K}$ given as in Definition 2.1, we seek a real number $\tau > 0$ such that

$$\text{(3.2)} \qquad \begin{cases} 4ae^{\lambda_1\tau} & < 1, \\ 18a^2e^{(\lambda_1-\lambda_2)\tau} & < 1, \\ 16a^2e^{-\lambda_4\tau} & < 1, \\ 3a^2e^{(\lambda_3-\lambda_4)\tau} & < 1. \end{cases}$$

Note that each of the exponents in the inequalities (3.2) is negative. Consequently, there does exist a time $T > 0$ such that, for all $\tau \ge T$, these four inequalities are satisfied. We fix one such T for the sequel. We will use the fact that (3.2) is valid for all τ with $T \le \tau \le 2T$.

By replacing the radius r of the disks $\mathcal{D}_r(x_0)$ with a smaller value, if necessary, there is no loss in generality in assuming that

$$3a^3k_0re^{\lambda_0 t} \le ae^{\lambda_3 t} \qquad \text{and} \qquad 2a^3k_0re^{\lambda_0 t} \le \frac{1}{4a}e^{\lambda_2 t},$$

for $0 \le t \le 2T$. As a result, Lemma 2.5 implies that for $x_1 \in \mathcal{D}(x_0)$, one has

$$\text{(3.2c)} \quad \frac{1}{2a}|x_1 - x_0|e^{\lambda_2 t} \le |\Phi(t, x_0)(x_1 - x_0)| \le 2a|x_1 - x_0|e^{\lambda_3 t}, \qquad \text{for } 0 \le t \le 2T.$$

In the sequel we will use the class Σ consisting of all positive, real-valued functions $\beta = \beta(\sigma)$, defined for $0 < \sigma < \sigma_0$, where $\sigma_0 = \sigma_0(\beta) > 0$, and satisfying

$$\beta(\sigma) \to 0, \qquad \text{as } \sigma \to 0.$$

For example, if some real-valued function $\xi(\epsilon)$ is of order $o(\epsilon)$, as $\epsilon \to 0$, then one can write this in the form $|\xi(\epsilon)| = \epsilon\beta(\epsilon)$, where $\beta \in \Sigma$. We will let $\beta, \beta_1, \beta_2, \cdots$ denote various elements of Σ.

Let $C(\mathcal{K}, R^n)$ denote the Banach space of continuous functions $f : \mathcal{K} \to R^n$ with the sup-norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathcal{K}\}.$$

Next we define two function classes which are subsets of $C(\mathcal{K}, R^n)$: $\mathcal{F} = \mathcal{F}(\epsilon, \ell)$ and $\mathcal{G} = \mathcal{G}(\epsilon, \ell)$, where the parameters $\epsilon > 0$ and $\ell > 0$ will be chosen later. We will require in the sequel that $\epsilon \le \min(r, \beta_0/3)$ A vector-valued function f is said to belong to $\mathcal{F}(\epsilon, \ell)$, if $f \in C(\mathcal{K}, R^n)$ and, for each $v \in \mathcal{K}$, one has $f(v) \in U^s(v)$ with $|f(v)| \le \epsilon$, and the restriction of f to each disk $\mathcal{D}(x)$ in $\mathcal{K}$ is Lipschitz continuous with Lipschitz coefficient ℓ. Similarly, a vector-valued function g is said to belong to $\mathcal{G}(\epsilon, \ell)$, if $g \in C(\mathcal{K}, R^n)$ and, for each $v \in \mathcal{K}$, one has $g(v) \in U^u(v)$ with $|g(v)| \le \epsilon$, and the restriction of g to each disk $\mathcal{D}(x)$ in $\mathcal{K}$ is Lipschitz continuous with Lipschitz coefficient ℓ. Since the vector bundles

$$\text{(3.3a)} \quad \mathcal{U}^s = \{(x, v) : x \in \mathcal{K},\ v \in U^s(x)\} \quad \text{and} \quad \mathcal{U}^u = \{(x, v) : x \in \mathcal{K},\ v \in U^u(x)\}$$

are closed subsets of $\mathcal{K} \times R^n$, see Sacker and Sell (1974, 1976ab), it follows that for every $\epsilon > 0$ and $\ell > 0$ the spaces $\mathcal{F}(\epsilon, \ell)$ and $\mathcal{G}(\epsilon, \ell)$ are closed sets in $C(\mathcal{K}, R^n)$.

Consequently, the product space $\mathcal{F}(\epsilon,\ell)\times\mathcal{G}(\epsilon,\ell)$ is a complete metric space with the metric

$$d(w_1,w_2)=\|w_1-w_2\|_\infty=\|f_1-f_2\|_\infty+\|g_1-g_2\|_\infty,$$

where $w_i=(f_i,g_i)\in\mathcal{F}\times\mathcal{G}$, for $i=1,2$. We define the sets

$$\text{(3.4)}\qquad \mathcal{U}^s_\epsilon=\{(x,v)\in\mathcal{U}^s : |v|\le\epsilon\}\qquad\text{and}\qquad \mathcal{U}^u_\epsilon=\{(x,v)\in\mathcal{U}^u : |v|\le\epsilon\}.$$

Let $\epsilon\le\min(r,\beta_0/3)$. For any pair $w=(f,g)\in\mathcal{F}\times\mathcal{G}$, where $\mathcal{F}=\mathcal{F}(\epsilon,\ell)$ and $\mathcal{G}=\mathcal{G}(\epsilon,\ell)$, we define $h:\mathcal{K}\to R^n$ by

$$y_0=h(x_0)=x_0+f(x_0)+g(x_0),\qquad\text{for } x_0\in\mathcal{K},$$

where $s_0=f(x_0)$ and $u_0=g(x_0)$. Since $|y_0-x_0|\le 2\epsilon<\beta_0$, the solution $y(t,y_0)$ of the perturbed equation (1.2) will remain in $N(x(t,x_0),\beta_0)$ on some interval $I=I(y_0)$, where $0\in I$. Furthermore, one has

$$\lim_{(\epsilon,\delta)\to(0,0)} y(t,y_0)=x(t,x_0),$$

where the limits are uniform for $x_0\in\mathcal{K}$ and uniform on compact sets in R. As a result, there is an ϵ_0, with $0<\epsilon_0\le\min(r,\beta_0/3)$, and a $\delta_0>0$ such that if $(f,g)\in\mathcal{F}\times\mathcal{G}$, for any ϵ satisfying $0<\epsilon\le\epsilon_0$ and any $\ell>0$, and if $\|Y\|_{C^1(\Omega)}\le\delta_0$, then one has $y(t,y_0)\in N(\mathcal{D}(x(t,x_0)),\beta_0)$, for $0\le t\le 2T$. As a result, there is a unique local coordinate representation with $v(0,y_0)=x_0$ and

$$\text{(3.4b)}\qquad y(t,y_0)=v(t,y_0)+s(t,y_0)+u(t,y_0),\qquad\text{for } 0\le t\le 2T.$$

where $v(t,y_0)\in\mathcal{S}(x_0)$, $s(t,y_0)\in U^s(v(t,y_0))$, and $u(t,y_0)\in U^u(v(t,y_0))$, see (3.0a). Furthermore these coordinates are jointly continuous in (x_0,t), for $x_0\in\mathcal{K}$ and $0\le t\le 2T$.

From the general theory of differential equations on the continuous dependence of solutions on parameters, the mapping $(x_0,y_0,t)\to v(t,y_0)$ of the set

$$\{(x_0,y_0): x_0\in\mathcal{K},\ y_0\in R^n,\ \|y_0-x_0\|\le 2\epsilon\}\times[0,2T]$$

into $\mathcal{K}$ is jointly continuous in x_0, y_0, and t. Furthermore, for every $x_0\in\mathcal{K}$, $t\in[0,2T]$, and $(f,g)\in\mathcal{F}\times\mathcal{G}$, there is a point $v_1=v_1(t,x_0)\in\mathcal{S}(x_0)$ such that if $y_1=v_1+f(v_1)+g(v_1)$, then $v(t,y_1)=x_0$. This implies that, by replacing ϵ_0 and δ_0 with smaller values, if necessary, then for any $w=(f,g)\in\mathcal{F}\times\mathcal{G}$ and for each $t\in[0,2T]$, one has

$$\text{(3.4c)}\qquad \{v(t,h(x_0)): x_0\in\mathcal{K}\}=\mathcal{K}.$$

The mapping $v(t,y_0)=\psi(y(t,y_0))=\psi(x_0,y(t,y_0))$, which is defined by the local coordinate representation and which is valid on some interval $0\le t\le t_1$, where $0<t_1\le 2T$, admits a well-defined extension

$$v(t,y_0)=\psi^e(y(t,y_0))=\psi(x(t,x_0),y(t,y_0)),$$

which is now valid for $0 \le t \le 2T$. Furthermore, if t_i satisfies $0 \le t_i \le 2T$, for $i = 1, 2$, and $0 \le t_1 + t_2 \le 2T$, then one has

$$v(t_1, y(t_2, y_0)) = v(t_1 + t_2, y_0) = v(t_2, y(t_1, y_0)), \tag{3.4d}$$

since one has $S_2(t_1)S_2(t_2) = S_2(t_1 + t_2) = S_2(t_2)S_2(t_1)$ and

$$\begin{aligned} v(t_1, y(t_2, y_0)) &= \psi^e(S_2(t_1)y(t_2, y_0)) = \psi^e(S_2(t_1)S_2(t_2)y_0) \\ &= \psi^e(S_2(t_1 + t_2)y_0) = v(t_1 + t_2, y_0) \\ &= \psi^e(S_2(t_2)S_2(t_1)y_0) = v(t_2, y(t_1, y_0)). \end{aligned}$$

The next lemma is a basic result we need for the study of the stability of invariant sets.

Lemma 3.0. *Let $\mathcal{K}$ be a weakly, normally hyperbolic set with the Lipschitz property for equation (1.1), and let $T > 0$ be given as above. Then there exist nonnegative constants C_0, C_1, and C_2, which depend on the characteristics of $\mathcal{K}$, but which do not depend on the Lipschitz coefficient of Y, such that whenever $0 < \epsilon \le \epsilon_0$, $0 < \delta \le \delta_0$, and $\|Y\|_{C^1(\Omega)} \le \delta$, then the following hold:*

(1) *For any $x_0 \in \mathcal{K}$ and $y_0 = x_0 + n_0$, where $n_0 \in U^s(x_0) + U^u(x_0)$ and $|n_0| \le 2\epsilon$, one has*

$$\left.\begin{cases} |y(t, y_0) - x(t, x_0)| \\ |v(t, y_0) - x(t, x_0)| \end{cases}\right. \le C_0(\epsilon + \delta), \qquad \text{for } 0 \le t \le 2T. \tag{3.5c}$$

and

$$z(t) = \Phi(x_0, t)z(0) + H_3(t) \qquad \text{for } 0 \le t \le 2T, \tag{3.5d}$$

where $z(t) = y(t, y_0) - x(t, x_0)$, $y(t) = y(t, y_0) = v(t) + n(t)$ satisfies (3.1) and (3.1a), and $H_3(t)$ satisfies

$$|H_3(t)| \le (\epsilon + \delta)\beta_1(\epsilon + \delta) + C_2\delta, \qquad \text{for } 0 \le t \le 2T, \tag{3.5g}$$

for some $\beta_1 \in \Sigma$.

(2) *Moreover, if for $i = 1, 2$ one has $y_i \in N(\mathcal{D}(x_0), \beta_0)$, for some $x_0 \in \mathcal{K}$, and if $|y_i - \psi(x_0, y_i)| \le 2\epsilon$ and $y(t, y_i) \in N(\mathcal{D}(x(t, x_0)), \beta_0)$, for $0 \le t \le 2T$, then one has*

$$\left.\begin{cases} |y(t, y_1) - y(t, y_2)| \\ |v(t, y_1) - v(t, y_2)| \end{cases}\right. \le C_1|y_1 - y_2|, \qquad \text{for } 0 \le t \le 2T, \tag{3.5e}$$

and

$$w(t) = \Phi(x_2, t)(y_1 - y_2) + H_4(t) \qquad \text{for } 0 \le t \le 2T, \tag{3.5f}$$

where $w(t) = y(t, y_1) - y(t, y_2)$ and $H_4(t)$ satisfies $|H_4(t)| \le \beta_2(\epsilon + \delta)|y_1 - y_2|$, for some $\beta_2 \in \Sigma$.

Proof. First note that $z(t) = y(t, y_0) - x(t, x_0)$ is a solution of the differential equation

$$\partial_t z = X(x(t, x_0) + z) - X(x(t, x_0)) + Y(y(t, y_0)). \tag{3.6}$$

Now from (2.1), equation (3.6) can be rewritten in the form

$$\partial_t z = B(x(t, x_0))z + F(x(t, x_0), z) + Y(y(t, y_0)), \tag{3.6a}$$

where $B = DX$ is the Jacobian of X. From inequalities (2.1b) and (3.5c) we obtain

$$|F(x(t, x_0), z(t))| \leq \beta_3(\epsilon + \delta)|z(t)|, \qquad \text{for } t \in [0, 2T], \tag{3.6b}$$

for some $\beta_3 \in \Sigma$. By using the Variation of Constants Formula on (3.6a) together with $z(0) = n_0$, we find that $z(t)$ satisfies (3.5d), where

$$H_3(t) = \int_0^t \Phi(t, x_0)\Phi^{-1}(s, x_0)\left(F(x(s, x_0), z(s)) + Y(y(s, y_0))\right) ds.$$

From (2.2f) we obtain $|\Phi(t, x_0)n_0| \leq ae^{\lambda_0 2T}2\epsilon$. The first inequality in (3.5c) and the bound for $|H_3(t)|$ now follow from a direct application of the Gronwall inequality. The second inequality in (3.5c) follows from the Lipschitz continuity of ψ and the facts that $v(t, y_0) = \psi(y(t, y_0))$ and $x(t, x_0) = \psi(x(t, x_0))$.

For Item (2) we note that $w(t) = y(t, y_1) - y(t, y_2)$ is a solution of the differential equation

$$\partial_t w = X(y(t, y_2) + w) - X(y(t, y_2)) + Y(y(t, y_2) + w) - Y(y(t, y_2)). \tag{3.6d}$$

Now equation (3.6d) can be rewritten in the form

$$\partial_t w = C(y(t, y_2))w + G(y(t, y_2), w), \tag{3.6e}$$

where $C = D(X + Y)$ is the Jacobian of $(X + Y)$ and

$$|G(y(t, y_2), w(t))| \leq \beta_4(\epsilon + \delta)|w(t)|, \qquad \text{for } t \in [0, 2T], \tag{3.6f}$$

for some $\beta_4 \in \Sigma$. Let $x_2 = \psi(x_0, y_2)$. One then has $|y(t, y_2) - x(t, x_2)| \leq C_0(\epsilon + \delta)$, by (3.5c), and equation (3.6e) can be written in the form

$$\partial_t w = B(x(t, x_2))w + H_5(t),$$

where $H_5(t) = (C(y(t, y_2)) - B(x(t, x_2)))w(t) + G(y(t, y_2), w(t))$. By using the Variation of Constants Formula on the last equation we obtain (3.5f), where

$$H_4(t) = \int_0^t \Phi(t, x_2)\Phi^{-1}(s, x_2)H_5(s)\, ds.$$

Since $\|Y\|_{C^1(\Omega)} \leq \delta$, it follows from inequalities (3.5c) and (3.6f) that $|H_5(t)| \leq \beta_5(\epsilon + \delta)|w(t)|$, for $0 \leq t \leq 2T$. As argued in the last paragraph, inequalities (3.5e) and the bound on $|H_4(t)|$ follow from an application of the Gronwall inequality. ☐

For each pair $w = (f, g) \in \mathcal{F} \times \mathcal{G}$, we define a new function $\bar{f}_\tau$ by

$$\bar{f}_\tau(v(\tau, y_0)) = P^s(v(\tau, y_0))(y(\tau, y_0) - v(\tau, y_0)), \qquad \text{for } T \le \tau \le 2T, \tag{3.6i}$$

where $y_0 = x_0 + f(x_0) + g(x_0)$. Note that for each $x_0 \in \mathcal{K}$, it follows from (3.0a) that

$$\bar{f}_\tau(v(\tau, y_0)) = s(\tau, y_0) \in U^s(v(\tau, y_0)), \qquad \text{for } T \le \tau \le 2T,$$

since $U^s(x)$ is the range of the projection $P^s(x)$. Equation (3.6i) gives the value of $\bar{f}_\tau$ at the point $v(\tau, y_0) \in \mathcal{K}$. Due to (3.4c), we see that $\bar{f}_\tau$ is defined everywhere on $\mathcal{K}$, and the mapping $(x_0, \tau) \to \bar{f}_\tau(x_0)$ is a continuous mapping of $\mathcal{K} \times [T, 2T]$ into R^n. We will show later that $\bar{f}_\tau \in \mathcal{F}(\epsilon, \ell)$, provided that ϵ and δ are sufficiently small.

Lemma 3.1. *There is an ϵ_4, with $0 < \epsilon_4 \le \epsilon_0$, such that for all ϵ with $0 < \epsilon \le \epsilon_4$, there is a $\delta_4 = \delta_4(\epsilon)$, with $0 < \delta_4 \le \delta_0$, such that if $\|Y\|_{C^1(\Omega)} \le \delta = \delta_4$, and if $f \in \mathcal{F}(\epsilon, \ell)$ and $g \in \mathcal{G}(\epsilon, \ell)$ for any $\ell > 0$, then one has*

$$|\bar{f}_\tau(x)| \le \frac{3}{4}\epsilon, \qquad \textit{for all } x \in \mathcal{K} \textit{ and } T \le \tau \le 2T. \tag{3.7}$$

Proof. Due to (3.4c), it suffices to verify (3.7) when $x = v(\tau, y_0)$ and $y_0 = x_0 + f(x_0) + g(x_0)$, for some $x_0 \in \mathcal{K}$ and $T \le \tau \le 2T$. Let $z(t) = y(t, y_0) - x(t, x_0)$, $s_0 = f(x_0)$, and $u_0 = g(x_0)$. From (3.5d) in Lemma 3.0 one has

$$z(t) = \Phi(t, x_0)z(0) + H_3(t), \qquad \text{for } 0 \le t \le 2T, \tag{3.7a}$$

where H_3 satisfies (3.5g). Also one has $z(0) = y_0 - x_0 = s_0 + u_0$. From the definition of $\bar{f}_\tau$ in (3.6i) we have

$$|\bar{f}_\tau(v(\tau, y_0))| = |P^s(v(\tau, y_0))\,(z(\tau) + x(\tau, x_0) - v(\tau, y_0))\,|,$$

and from (2.8) we obtain

$$|\bar{f}_\tau(v(\tau, y_0))| \le I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= |P^s(v(\tau, y_0))\Phi(\tau, x_0)s_0| \\ I_2 &= |P^s(v(\tau, y_0))\Phi(\tau, x_0)u_0| \\ I_3 &= |P^s(v(\tau, y_0))H_3(\tau)| \\ I_4 &= |P^s(v(\tau, y_0))(x(\tau, x_0) - v(\tau, y_0))|. \end{aligned}$$

From (2.4a) we obtain $P^s(x(\tau, x_0))\Phi(\tau, x_0)s_0 = \Phi(\tau, x_0)s_0$. By using the continuity of P^s, (2.2c), (3.2), (3.5c), and the fact that $|s_0| \le \epsilon$, we find that there is a $\beta_2 \in \Sigma$ such that

$$\begin{aligned} I_1 &\le |[P^s(v(\tau, y_0)) - P^s(x(\tau, x_0))]\,\Phi(\tau, x_0)s_0| + |\Phi(\tau, x_0)s_0| \\ &\le (4\beta_2(\epsilon + \delta) + 1)|\Phi(\tau, x_0)s_0| \le \beta_2(\epsilon + \delta)\epsilon + \frac{1}{4}\epsilon. \end{aligned}$$

Similarly (2.4a) implies that $P^s(x(\tau,x_0))\Phi(\tau,x_0)u_0 = 0$, since $u_0 \in U^u(x_0)$. Therefore from (3.5c) we find that there is a $\beta_3 \in \Sigma$ such that

$$I_2 = |(P^s(v(\tau,y_0)) - P^s(x(\tau,x_0)))\,\Phi(\tau,x_0)u_0| \le \beta_3(\epsilon+\delta)|u_0| \le \beta_3(\epsilon+\delta)\epsilon.$$

From (3.5g) and (2.4c) one has

$$I_3 \le a((\epsilon+\delta)\beta_1(\epsilon+\delta) + C_2\delta).$$

Lastly, since $v(\tau,y_0) \in \mathcal{D}(x(\tau,x_0))$, it follows from (2.7b) and (3.5c) that

$$I_4 \le a^2k_0|x(\tau,x_0) - v(\tau,y_0)|^2 \le C_0^2a^2k_0(\epsilon+\delta)^2.$$

From these estimates on I_1, I_2, I_3, and I_4, we find that

$$|\bar{f}_\tau(v(\tau,y_0))| \le (\epsilon+\delta)\beta_5(\epsilon+\delta) + \frac{1}{4}\epsilon + aC_2\delta,$$

for some $\beta_5 \in \Sigma$. Next choose $\epsilon_4 > 0$ so that $\beta_5(2\epsilon_4) \le \frac{1}{4}$ and $\epsilon_4 \le \epsilon_0$. Then with

$$\delta_4 = \delta_4(\epsilon) = \min\left(\delta_0, \epsilon_4, \frac{1}{4}\left(aC_2 + \frac{1}{4}\right)^{-1}\epsilon\right), \qquad \text{for } 0 < \epsilon \le \epsilon_4,$$

the inequality (3.7) holds whenever $x \in \mathcal{K}$ is of the form $x = v(\tau,y_0)$. □

In the following result, we argue that $\bar{f}_\tau$ is (locally) Lipschitz continuous on each disk $\mathcal{D}(x_0) \subset \mathcal{K}$.

Lemma 3.2. *There is a number ℓ_1, with $0 < \ell_1 \le 1$ and which depends on T and the characteristics of the compact invariant set $\mathcal{K}$ such that the following is valid: There is an ϵ_5 with $0 < \epsilon_5 \le \epsilon_4$ such that for all ϵ with $0 < \epsilon \le \epsilon_5$, there exist $\delta_5 = \delta_5(\epsilon)$ with $0 < \delta_5 \le \delta_4$, where ϵ_4 and δ_4 are given in Lemma 3.1, and there is an $\ell_5 = \ell_5(\epsilon)$ with $0 < \ell_5(\epsilon) \le \ell_1$, such that if $\|Y\|_{C^1(\Omega)} \le \delta = \delta_5$ and if $f \in \mathcal{F} = \mathcal{F}(\epsilon,\ell)$ and $g \in \mathcal{G} = \mathcal{G}(\epsilon,\ell)$ with $\ell \le \ell_1$ and $\epsilon \le \epsilon_5$, then the restriction of $\bar{f}_\tau$ to the disk $\mathcal{D}(x_0)$ is Lipschitz continuous with Lipschitz coefficient ℓ_5, for every $x_0 \in \mathcal{K}$. Moreover, one has $\ell_5(\epsilon) \to 0$, as $\epsilon \to 0$.*

Proof. Let $(f,g) \in \mathcal{F}\times\mathcal{G}$ be given. We assume that ϵ and δ satisfy $0 < \epsilon \le \epsilon_4$ and $0 < \delta \le \delta_4$ so that Lemma 3.1 holds. Let $x_0 \in \mathcal{K}$ and $x_1 \in \mathcal{D}(x_0)$ be given. Let $s_i = f(x_i)$, $u_i = g(x_i)$, $y_{i0} = x_i + s_i + u_i$, and $n_{i0} = s_i + u_i$, for $i = 0,1$. Define $y_i = y_i(t) = y(t,y_{i0})$, and let $v_i = v_i(t)$ and $n_i = n_i(t)$, for $i = 0,1$, be given by (3.1b). Let τ satisfy $T \le \tau \le 2T$.

Due to (3.4c), it will suffice to show that, under the hypotheses stated in this lemma, one has

$$\frac{|\bar{f}_\tau(v_1(\tau)) - \bar{f}_\tau(v_0(\tau))|}{|v_1(\tau) - v_0(\tau)|} \le \ell_5, \tag{3.8}$$

for every $x_0 \in \mathcal{K}$ and all $x_1 \in \mathcal{D}(x_0)$. Now set $z = z(t) = y_1 - y_0$ and $w = w(t) = v_1 - v_2$. From Lemma 3.0, one obtains

$$z(t) = \Phi(t,x_0)z(0) + H(t), \qquad \text{for } 0 \le t \le 2T, \tag{3.8a}$$

where

$$|H(t)| \leq \beta_3(\epsilon+\delta)|y_{10}-y_{00}|, \qquad \text{for } 0 \leq t \leq 2T,$$

and some $\beta_3 \in \Sigma$. Since $f \in \mathcal{F}(\epsilon, \ell)$ and $g \in \mathcal{G}(\epsilon, \ell)$, we find that

$$|y_{10}-y_{00}| \leq |x_1-x_0| + |s_1-s_0| + |u_1-u_0| \leq (1+2\ell)|x_1-x_0|.$$

Therefore one has

$$|H(\tau)| \leq (1+2\ell)\beta_3(\epsilon+\delta)|x_1-x_0|, \qquad \text{for } T \leq \tau \leq 2T. \tag{3.8d}$$

Next we estimate the term $\Phi(t,x_0)z(0)$ in (3.8a). Note that one has

$$\Phi(t,x_0)z(0) = \Phi(t,x_0)(x_1-x_0) + \Phi(t,x_0)(n_{10}-n_{00}),$$

where $n_{i0} = s_i + u_i$, for $i = 0,1$. Also one has $n_{i0} \in \mathcal{R}(Q^o(x_i))$ and $|n_{i0}| \leq 2\epsilon$, for $i = 0,1$, and $|n_{10}-n_{00}| \leq 2\ell|x_1-x_0|$. From this fact and inequality (2.2f), one then obtains

$$|\Phi(t,x_0)(n_{10}-n_{00})| \leq a|n_{10}-n_{00}|e^{\lambda_0 t} \leq 2a\ell|x_1-x_0|e^{\lambda_0 t}, \qquad \text{for } 0 \leq t \leq 2T.$$

As a result, (3.2c) implies that

$$|\Phi(t,x_0)z(0)| \leq 2a|x_1-x_0|e^{\lambda_3 t} + 2a\ell|x_1-x_0|e^{\lambda_0 t}, \qquad \text{for } 0 \leq t \leq 2T.$$

If ℓ satisfies $\ell \leq e^{(\lambda_3-\lambda_0)T} \leq 1$, one obtains

$$|\Phi(t,x_0)z(0)| \leq 4a|x_1-x_0|e^{\lambda_3 t}, \qquad \text{for } 0 \leq t \leq 2T.$$

Similarly, (3.2c) implies that

$$|\Phi(t,x_0)z(0)| \geq \frac{1}{2a}|x_1-x_0|e^{\lambda_2 t} - 2a\ell|x_1-x_0|e^{\lambda_0 t}, \qquad \text{for } 0 \leq t \leq 2T.$$

As a result, if ℓ satisfies $\ell \leq \frac{1}{4a^2}e^{(\lambda_2-\lambda_0)T}$, then one obtains

$$|\Phi(t,x_0)z(0)| \geq \frac{1}{4a}|x_1-x_0|e^{\lambda_2 t}, \qquad \text{for } 0 \leq t \leq 2T.$$

Assume now that ℓ satisfies

$$\ell \leq \ell_1 \stackrel{\text{def}}{=} \min\left(2C_0^{-1}, e^{(\lambda_3-\lambda_0)T}, \frac{1}{4a^2}e^{(\lambda_2-\lambda_0)T}\right) \leq 1, \tag{3.8h}$$

where C_0 is given by Lemma 3.0. By putting together the above inequalities with (3.8a) and (3.8d), we find that

$$|z(\tau)| \leq \left(4ae^{\lambda_3\tau} + 3(\beta_3(\epsilon+\delta)+C_1\delta)\right)|x_1-x_0|.$$

Similarly, one obtains

$$|z(\tau)| \geq \left(\frac{1}{4a}e^{\lambda_2\tau} - 3(\beta_3(\epsilon+\delta) + C_1\delta)\right)|x_1 - x_0|.$$

Next choose ϵ_0 so that $0 < \epsilon_0 \leq \epsilon_4$, and

$$3(\beta_3(2\epsilon_0) + C_1\epsilon_0) \leq \frac{1}{8a}e^{\lambda_2 T} = \min\left(4ae^{\lambda_3 T}, \frac{1}{8a}e^{\lambda_2 T}\right).$$

Then for ϵ and δ with $0 < \epsilon \leq \epsilon_0$ and $0 < \delta \leq \epsilon_0$, one has

$$\text{(3.8m)} \qquad \frac{1}{8a}e^{\lambda_2\tau}|x_1 - x_0| \leq |z(\tau)| \leq 8ae^{\lambda_3\tau}|x_1 - x_0|, \qquad \text{for } T \leq \tau \leq 2T.$$

Let us return to the system (3.1). It follows from (3.1b) and (3.5d) that the solutions $v(t)$ and $n(t)$ are Lipschitz continuous functions of the initial data (v_0, n_0). Now the solution $n(t, v_0, n_0)$ satisfies

$$\lim_{(\epsilon,\delta)\to(0,0)} n(t, v_0, n_0) = 0, \qquad \text{for } 0 \leq t \leq 2T,$$

while the initial data satisfies

$$\lim_{(\epsilon,\delta)\to(0,0)} (v_0, n_0) = (x_0, 0).$$

Since $\mathcal{K}$ is compact, these limits are uniform for $x_0 \in \mathcal{K}$. This implies that there is a $\beta_4 \in \Sigma$ with the property that

$$\text{(3.8p)} \quad |n_1(t) - n_0(t)| \leq \beta_4(\epsilon+\delta)(|y_{10} - y_{00}| + |n_{10} - n_{00}|), \qquad \text{for } 0 \leq t \leq 2T,$$

and

$$\text{(3.8q)} \qquad |n_i(t)| \leq \beta_4(\epsilon+\delta), \qquad \text{for } i = 0, 1, \text{ and } 0 \leq t \leq 2T,$$

for all $x_0 \in \mathcal{K}$ and all $x_1 \in \mathcal{D}(x_0)$. Since $|y_{10} - y_{00}| \leq (1 + 2\ell)|x_1 - x_0|$ and $|n_{10} - n_{00}| \leq 2\ell|x_1 - x_0|$, we see that for $\ell \leq 1$, one has

$$\text{(3.8r)} \qquad |n_1(t) - n_0(t)| \leq 5\beta_4(\epsilon+\delta)|x_1 - x_0|, \qquad \text{for } 0 \leq t \leq 2T.$$

Since $w(\tau) = v_1(\tau) - v_2(\tau) = z(\tau) - n_1(\tau) + n_0(\tau)$, it follows from (3.8m) that

$$\text{(3.8s)} \qquad \frac{1}{16a}e^{\lambda_2\tau}|x_1 - x_0| \leq |w(\tau)| \leq 16ae^{\lambda_3\tau}|x_1 - x_0|,$$

provided that

$$\text{(3.8t)} \qquad 5\beta_4(\epsilon+\delta) \leq \min\left(8ae^{\lambda_3 T}, \frac{1}{16a}e^{\lambda_2 T}\right).$$

Let us now turn to the terms that appear in the numerator of (3.8). From the definition of $\bar{f}_\tau$ in (3.6i) one obtains

$$\bar{f}_\tau(v_1(\tau)) - \bar{f}_\tau(v_0(\tau)) = P^s(v_1(\tau))n_1(\tau) - P^s(v_0(\tau))n_0(\tau) \pm P^s(v_1(\tau))n_0(\tau).$$

From the Lipschitz continuity of P^s on the disk $\mathcal{D}(x(\tau, x_0))$, (2.4c), (3.8q), (3.8r), and (3.8s), one then finds that there is a constant $L > 0$ such that

$$\begin{aligned} |\bar{f}_\tau(v_1(\tau)) - \bar{f}_\tau(v_0(\tau))| &\le |P^s(v_1(\tau))|\,|(n_1(\tau) - n_0(\tau))| \\ &\qquad + |P^s(v_1(\tau)) - P^s(v_0(\tau))|\,|n_0(\tau)| \\ &\le 5a\beta_4(\epsilon+\delta)|x_1 - x_0| + L|v_1(\tau) - v_2(\tau)|\beta_4(\epsilon+\delta) \\ &\le (5a + 16Lae^{\lambda_3\tau})\beta_4(\epsilon+\delta)|x_1 - x_0|. \end{aligned}$$

As a result, (3.8s) implies that

$$\frac{|\bar{f}_\tau(v_1(\tau)) - \bar{f}_\tau(v_0(\tau))|}{|v_1(\tau) - v_0(\tau)|} \le 16a(5a + 16Lae^{\lambda_3\tau})e^{-\lambda_2\tau}\beta_4(\epsilon+\delta).$$

Finally we choose ϵ_5 and so that $0 < \epsilon_5 \le \epsilon_0 \le \epsilon_4$, inequality (3.8t) is valid for $\epsilon = \delta = \epsilon_5$, and

$$\sup_{T \le \tau \le 2T} 16a(5a + 16Lae^{\lambda_3\tau})e^{-\lambda_2\tau}\beta_4(2\epsilon_5) \le \ell_1,$$

where ℓ_1 is given by (3.8h). Set $\delta_5(\epsilon) = \min(\epsilon, \delta_4(\epsilon))$ and

$$\ell_5(\epsilon) \stackrel{\text{def}}{=} \sup_{T \le \tau \le 2T} 16a(5a + 16Lae^{\lambda_3\tau})e^{-\lambda_2\tau}\beta_4(\epsilon + \delta_5(\epsilon)) \le \ell_1, \qquad \text{for } 0 < \epsilon \le \epsilon_5.$$

This then completes the proof of the lemma. ☐

Next we will derive a contracting property for the mapping $(f, g) \to \bar{f}_\tau$.

Lemma 3.3. *There is an ϵ_6 with $0 < \epsilon_6 \le \epsilon_5$ and for every ϵ with $0 < \epsilon \le \epsilon_6$ there is a $\delta_6 = \delta_6(\epsilon) \le \delta_5(\epsilon)$, where ϵ_5 and δ_5 are given in Lemma 3.2, such that if $\|Y\|_{C^1(\Omega)} \le \delta_6$, then for all $w_i = (f_i, g_i) \in \mathcal{F}(\epsilon, \ell) \times \mathcal{G}(\epsilon, \ell)$, where $\ell \le 1$, one has*

$$|\bar{f}_{\tau,1}(x) - \bar{f}_{\tau,0}(x)| \le \frac{1}{3} d(w_0, w_1), \qquad \textit{for all } x \in \mathcal{K},$$

where $\bar{f}_{\tau,i}$ be given by (3.6i), for $i = 0, 1$.

Proof. Let $x_0 \in \mathcal{K}$ be given, and let $s_i = f_i(x_0)$ and $u_i = g_i(x_0)$, for $i = 0, 1$. Let $y_{i0} = x_0 + s_i + u_i$ and let $y_i(t) = y(t, y_{i0})$, $v_i(t) = v(t, y_{i0})$, and $n_i(t)$ satisfy (3.1b), for $i = 0, 1$. From equation (3.6i) one finds that

$$\bar{f}_{\tau,i}(v_i(\tau)) = P^s(v_i(\tau))n_i(\tau), \qquad \text{for } i = 0, 1 \text{ and } T \le \tau \le 2T.$$

As argued in the proof of Lemma 3.2, the functions $v_i(t)$ and $n_i(t)$ are Lipschitz continuous functions of the initial data $(y_i(0), n_i(0))$. In particular, inequalities (3.5e) and (3.8p) are valid. From the construction of these solutions one has

$$|y_1(0) - y_0(0)| = |n_1(0) - n_0(0)| \le |f_1(x_0) - f_0(x_0)| + |g_1(x_0) - g_0(x_0)| \le d(w_1, w_0).$$

From the Lipschitz continuity of P^s on the disk $\mathcal{D}(v_i(\tau))$ and inequalities (2.4c) and (0.8g), one then finds that there is a constant $C_1 > 0$ such that

$$\begin{aligned} |\bar{f}_{\tau,1}(v_1(\tau)) - \bar{f}_{\tau,0}(v_0(\tau))| &\le |P^s(v_1(\tau))(n_1(\tau) - n_0(\tau))| \\ &\quad + |(P^s(v_1(\tau)) - P^s(v_0(\tau)))n_0(\tau)| \\ &\le 2a\beta_1(\epsilon + \delta)d(w_1, w_0) + L\epsilon|v_1(\tau) - v_0(\tau)| \\ &\le (2a\beta_1(\epsilon + \delta) + C_1\epsilon\beta_1(\epsilon + \delta))d(w_1, w_0). \end{aligned}$$

The conclusion then follows by choosing ϵ_6 so that

$$0 < \epsilon_6 \le \epsilon_5 \quad \text{and} \quad (2a + C_1\epsilon_6)\beta_1(2\epsilon_6) \le 1/3.$$

Lastly, set $\delta_6(\epsilon) = \min(\epsilon, \delta_5(\epsilon))$, for $0 < \epsilon \le \epsilon_6$, to complete the proof. □

Let $(f, g) \in \mathcal{F} \times \mathcal{G}$. We now seek to define a new function $\bar{g}_\tau$, which is a companion to the function $\bar{f}_\tau$ given by (3.6i). Among other things, we want $\bar{g}_\tau(x_0)$ to be in $U^u(x_0)$, for every $x_0 \in \mathcal{K}$. Let $x_0 \in \mathcal{K}$ be given, and define $y_0 = y_0(V) = x_0 + f(x_0) + V$, where $V \in U^u(x_0)$ will be treated as a parameter. Consider the equation

$$g(v(\tau, y_0(V))) = P^u(v(\tau, y_0(V)))(y(\tau, y_0(V)) - v(\tau, y_0(V))). \tag{3.10d}$$

Our objective is to show that if ϵ and δ are sufficiently small, then equation (3.10d) has a unique solution $V \in U^u(x_0)$. In this case we will denote this solution by $V = \bar{g}_\tau(x_0)$. Before proving this property, it is convenient to write equation (3.10d) in the abbreviated form

$$g = P^u(v)(y - v),$$

where $y = y(\tau, y_0(V))$, $v = v(\tau, y_0(V))$, and $g = g(v)$. Now with $x = x(\tau, x_0)$, equation (3.10d) takes on the equivalent form

$$g = P^u(x)(y - x) - P^u(x)(v - x) + (P^u(v) - P^u(x))(y - v). \tag{3.11b}$$

Notice that each of the terms y, v, g, and $P^u(v)$ are Lipschitz continuous functions of the parameter V, while the term x does not depend on V.

Lemma 3.4. *There is an $\epsilon_7 > 0$ such that $\epsilon_7 \le \epsilon_6$ and for all ϵ with $0 < \epsilon \le \epsilon_7$, there is a δ_7 with $0 < \delta_7 \le \delta_6$, where ϵ_6 and δ_6 are given in Lemma 3.3, such that if $\|Y\|_{C^1(\Omega)} \le \delta_7$, and if $f \in \mathcal{F}(\epsilon, \ell)$ and $g \in \mathcal{G}(\epsilon, \ell)$ for any ℓ with $0 < \ell \le \ell_1$, where ℓ_1 is given by (3.8h), then for each $x_0 \in \mathcal{K}$ there is a unique solution $V = \bar{g}_\tau(x_0)$*

of (3.10d), where $V \in U^u(x_0)$, *with* $|V| \le \frac{3}{4}\epsilon$. *Moreover,* $\bar{g}_\tau(x_0)$ *is continuous for* $x_0 \in \mathcal{K}$ *and* $T \le \tau \le 2T$.

Proof. Let $z = z(t) = y(t, y_0) - x(t, x_0)$, where $y_0 = x_0 + f(x_0) + V$ and $V \in U^u(x_0)$. As argued in Lemma 3.0, one has

$$z(\tau) = \Phi(\tau, x_0)z(0) + H(\tau), \qquad \text{for } T \le \tau \le 2T, \tag{3.11c}$$

where $z(0) = f(x_0) + V$ and $H(\tau) = H_3(\tau)$ satisfies (3.5g). By applying $P^u(x)$ to (3.11c), one obtains the term $P^u(x)(y - x)$ in (3.11b), and by using the invariance property (2.4a), one obtains

$$P^u(x)(y - v) = \Phi(\tau, x_0)V + P^u(x)H(\tau), \qquad \text{for } T \le \tau \le 2T.$$

As a result, (3.11b) can be rewritten in the form

$$V = G(V) = G(V, x_0), \tag{3.11f}$$

where $x_0 \in \mathcal{K}$, $V \in U^u(x_0)$, and

$$\begin{aligned} G(V, x_0) = {} & \Phi^{-1}(\tau, x_0)P^u(x)(-H(\tau) + (v - x)) \\ & + \Phi^{-1}(\tau, x_0)(-(P^u(v) - P^u(x))(y - v) + g). \end{aligned} \tag{3.11g}$$

Notice that G is a continuous function of $(x_0, V) \in \mathcal{U}^u_\epsilon$, where $\mathcal{U}^u_\epsilon$ is given by (3.4).

The remainder of the argument is to show that $G(V, x_0)$, the right side of equation (3.11f), is a contraction in the V- variable under the conditions stated in this lemma. In particular, we will now show that for small ϵ and δ, the following two properties are valid:

(1) For any $(x_0, V) \in \mathcal{U}^u_\epsilon$ one has $|G(V, x_0)| \le \frac{3}{4}\epsilon$.
(2) There is a k, $0 \le k < 1$ such that for any $(x_0, V_i) \in \mathcal{U}^u_\epsilon$, for $i = 1, 2$, one has $|G(V_1, x_0) - G(V_2, x_0)| \le k|V_1 - V_2|$.

Since one has $g(v) \in U^u(v)$, it follows that $P^u(v)g(v) = g(v)$, and therefore

$$g(v) = P^u(x)g(v) + (P^u(v) - P^u(x))g(v). \tag{3.11k}$$

By applying $\Phi^{-1}(\tau, x_0)$ to equation (3.11k) and using the continuity of P^u and inequalities (2.3b), (3.2), and (3.5c), we find that there exists a $\beta_0 \in \Sigma$ such that

$$\begin{aligned} |\Phi^{-1}(\tau, x_0)g(v)| &\le a|g(v)|e^{-\lambda_4 \tau} + |\Phi^{-1}(\tau, x_0)(P^u(v) - P^u(x))|\,|g(v)| \\ &\le \left(\frac{1}{4} + \beta_0(\epsilon + \delta)\right)\epsilon. \end{aligned}$$

Since $|V| \le \epsilon$, it follows from (2.3b), (2.7b), (3.2), and (3.5c) that

$$|\Phi^{-1}(\tau, x_0)P^u(x)(v - x)| \le ae^{-\lambda_4 \tau}|P^u(x)(v - x)| \le \frac{1}{4}a^2 k_0|v - x|^2 \le \frac{1}{4}a^2 k_0 C_0^2(\epsilon + \delta)^2.$$

From the continuity of P^u and Lemma 3.0 one finds that there is a $\beta_1 \in \Sigma$ such that

$$|\Phi^{-1}(\tau, x_0)(P^u(v) - P^u(x))(y - v)| \leq \beta_1(\epsilon + \delta)(\epsilon + \delta)$$

From (3.5g), one finds that there is a $\beta_2 \in \Sigma$ such that

$$|\Phi^{-1}(\tau, x_0)P^u(x)H(\tau)| \leq \beta_2(\epsilon + \delta)(\epsilon + \delta).$$

Putting the last four inequalities together, one obtains

$$|G(V, x_0)| \leq \frac{1}{4}\epsilon + \beta_3(\epsilon + \delta)(\epsilon + \delta),$$

for some $\beta_3 \in \Sigma$. Now choose $\bar{\epsilon}$ so that $0 < \bar{\epsilon} \leq \epsilon_6$, where ϵ_6 is given in Lemma 3.3, so that $\beta_3(2\bar{\epsilon}) \leq \frac{1}{4}$, and set $\bar{\delta}(\epsilon) = \min(\epsilon, \delta_6(\epsilon))$, where δ_6 is given in Lemma 3.3. Then with $\delta \leq \bar{\delta}$ one has $|G(V, x_0)| \leq \frac{3}{4}\epsilon$.

In order to prove the Lipschitz property for $G(V)$, we let $(x_0, V_i) \in \mathfrak{U}^u_\epsilon$, for $i = 1, 2$. Next we define $x = x(\tau, x_0)$, $y_i = y(\tau, y_0(V_i))$, $v_i = v(\tau, y_0(V_i))$, and $g_i = g(v_i)$, for $i = 1, 2$. In order to estimate $|G(V_1, x_0) - G(V_2, x_0)|$, we note that (3.11b) implies that

$$\begin{aligned} g_1 - g_2 = {} & P^u(x)(y_1 - y_2) - P^u(x)(v_1 - v_2) \\ & (P^u(v_1) - P^u(x))(n_1 - n_2) + (P^u(v_1) - P^u(v_2))(y_2 - v_2). \end{aligned} \tag{3.11r}$$

Now (3.5f) in Lemma 3.0 implies that

$$y_1 - y_2 = \Phi(\tau, x_0)(V_1 - V_2) + H_4(\tau),$$

where $|H_4(\tau)| \leq \beta_2(\epsilon + \delta)|V_1 - V_2|$. From the invariance property (2.4a), one then obtains

$$P^u(x)(y_1 - y_2) = \Phi(\tau, x_0)(V_1 - V_2) + P^u(x)H_4(\tau).$$

By combining this with (3.11r), one then obtains

$$\begin{aligned} g_1 - g_2 &= G(V_1, x_0) - G(V_2, x_0) \\ &= \Phi^{-1}(\tau, x_0)(P^u(x)H_4(\tau) + P^u(x)(v_1 - v_2) + (g_1 - g_2)) \\ &\quad - \Phi^{-1}(\tau, x_0)((P^u(v_1) - P^u(x))(n_1 - n_2) \\ &\quad + (P^u(v_1) - P^u(v_2))(y_2 - v_2)). \end{aligned} \tag{3.11u}$$

The bound on H_4 implies that

$$|\Phi^{-1}(\tau, x_0)P^u(x)H_4(\tau)| \leq \beta_3(\epsilon + \delta)|V_1 - V_2|,$$

for some $\beta_3 \in \Sigma$. Since $v_1, v_2 \in \mathcal{D}(x)$, inequalities (2.3b), (2.7b), (3.2), and (3.5e) imply that

$$|\Phi^{-1}(\tau, x_0)P^u(x)(v_1 - v_2)| \leq \frac{1}{16}ak_0rC_1|V_1 - V_2|.$$

By choosing a smaller value for r, if necessary, there is no loss in generality in assuming that $ak_o rC_1 \leq 4$, in which case one has

$$|\Phi^{-1}(\tau, x_0)P^u(x)(v_1 - v_2)| \leq \frac{1}{4}|V_1 - V_2|.$$

Since $P^u(v_i)g_i = g_i$, for $i = 1, 2$, one has

$$g_1 - g_2 = P^u(x)(g_1 - g_2) + (P^u(v_1) - P^u(x))(g_1 - g_2) + (P^u(v_1) - P^u(v_2))g_2.$$

From inequalities (2.3b), (2.4c), (3.2), (3.5c), the facts that $\ell \leq 1$, $g \in \mathcal{G}$ and that P^u is Lipschitz continuous on the disk $\mathcal{D}(x)$, one finds that there is a constant $C_2 > 0$ such that

$$|\Phi^{-1}(\tau, x_0)(g_1 - g_2)| \leq \frac{1}{4}|V_1 - V_2| + C_2(\epsilon + \delta)|V_1 - V_2|.$$

Similarly one shows that the two remaining terms on the right side of (3.11u) are bounded by $C_3(\epsilon + \delta)|V_1 - V_2|$, for some constant $C_3 > 0$. As a result one has

$$|G(V_1, x_0) - G(V_2, x_0)| \leq \left(\frac{1}{2} + \beta_7(\epsilon + \delta)\right)|V_1 - V_2|,$$

for some $\beta_7 \in \Sigma$. Finally we let ϵ_7 be chosen so that $0 < \epsilon_7 \leq \bar{\epsilon}$ and $\beta_7(2\epsilon_7) \leq \frac{1}{4}$. Then set

$$\delta_7(\epsilon) = \min(\epsilon, \bar{\delta}(\epsilon)), \qquad \text{for } 0 < \epsilon \leq \epsilon_7.$$

One then finds that

$$|G(V_1, x_0) - G(V_2, x_0)| \leq \frac{3}{4}|V_1 - V_2|, \tag{3.12k}$$

for all $(x_0, V_i) \in U^u_\epsilon$ for $i = 1, 2$.

Finally the function $\bar{g}_\tau(x_0)$ is a continuous function of $x_0 \in \mathcal{K}$, because: (1) the set $\mathcal{U}^u_\epsilon = \{(x, V) \in \mathcal{U}^u : |V| \leq \epsilon\}$ is compact; (2) the mapping $(x, V) \to G(V, x)$ given by equation (3.11g) is continuous on $\mathcal{U}^u_\epsilon$; and (3) the value $\bar{g}_\tau(x_0)$ is the unique fixed point of a contraction mapping. Similarly, one shows that $\bar{g}_\tau(x_0)$ is jointly continuous in x_0 and τ. This completes the proof of the lemma. □

Now equation (3.11f) assumes the form

$$V = G(V, x_1), \qquad \text{for } x_1 \in \mathcal{D}(x_0),\ V \in U^u(x_1). \tag{3.13}$$

Our next objective is to study the dependence of the new function $\bar{g}_\tau(x_1) = V = V(x_1)$ of the equation (3.13) on x_1, as x_1 varies over some disk $\mathcal{D}(x_0)$ in $\mathcal{K}$. It is convenient to make a change of variables in equation (3.13). Let us restrict the vector bundle $\mathcal{U}^u$ to

$$\mathcal{U}^u(\mathcal{K} \cap N(x_0, \beta_0)) = \{(x, v) \in \mathcal{U}^u : x \in \mathcal{K} \text{ and } |x - x_0| \leq \beta_0\},$$

see equation (3.3a). As argued in Sacker and Sell (1974, 1976ab), there is a mapping $M = M(x)$, which is defined for $x \in \mathcal{K} \cap N(x_0, \beta_0)$ such that, for small β_0, one has

(1) $M(x) : U^u(x_0) \to U^u(x)$ is a linear isomorphism of $U^u(x_0)$ onto $U^u(x)$, for $x \in \mathcal{K} \cap N(x_0, \beta_0)$.

(2) The mapping $x \to M(x)$ is continuous, and $M(x_0) = I$.

Furthermore, one can show that $M(x)$ can be chosen to be an isometry, i.e., one has

$$|M(x)v| = |v|, \qquad \text{for all } x \in \mathcal{K} \cap N(x_0, \beta_0) \text{ and } v \in U^u(x_0). \tag{3.13b}$$

As a result, equation (3.13) can be written in the form

$$M(x_1)W = G(M(x_1)W, x_1),$$

where $x_1 \in \mathcal{D}(x_0)$, $V = M(x_1)W$, and $W \in U^u(x_0)$. Equivalently, one can write

$$W = \hat{G}(W, x_1), \qquad x_1 \in \mathcal{D}(x_0) \text{ and } W \in U^u(x_0), \tag{3.13d}$$

where $\hat{G}(W, x_1) = M^{-1}(x_1)G(M(x_1)W, x_1)$. Consequently, $W \in U^u(x_0)$ is a fixed point of $W = \hat{G}(W, x_1)$ if and only if $V = M(x_1)W \in U^u(x_1)$ is a fixed point of $V = G(V, x_1)$.

We will use a property of contraction operators, which says states that if

$$|\hat{G}(W_1, x_1) - \hat{G}(W_2, x_1)| \le k|W_1 - W_2|, \tag{3.13e}$$

for all $x_1 \in \mathcal{D}(x_0)$, and all $W_i \in U^u(x_0)$ with $|W_i| \le \epsilon$ (for $i = 1, 2$), where $0 \le k < 1$, and if

$$|\hat{G}(W, x_1) - \hat{G}(W, x_0)| \le L|x_1 - x_0|,$$

for all $x_0 \in \mathcal{K}$, $x_1 \in \mathcal{D}(x_0)$, and $W \in U^u(x_0)$ with $|W| \le \epsilon$, then the fixed point $W = W(x_1)$ satisfies

$$|W(x_1) - W(x_0)| \le L(1-k)^{-1}|x_1 - x_0|, \qquad \text{for all } x_0 \in \mathcal{K} \text{ and } x_1 \in \mathcal{D}(x_0).$$

(In our case, one has $M(x_1)W(x_1) = \bar{g}_\tau(x_1)$, for all $x_1 \in \mathcal{D}(x_0)$.) Since the isometry property (3.13b) and inequality (3.12k) imply that

$$\begin{aligned} |\hat{G}(W_1, x) - \hat{G}(W_2, x)| &= |M^{-1}(x)G(M(x)W_1, x) - M^{-1}(x)G(M(x)W_2, x)| \\ &= |G(M(x)W_1, x) - G(M(x)W_2, x)| \\ &\le \frac{3}{4}|M(x)W_1 - M(x)W_2| = \frac{3}{4}|W_1 - W_2|, \end{aligned}$$

for $x \in \mathcal{D}(x_0)$, we see that one can choose $k = 3/4$ in equation (3.13e). We next prove the following result, which establishes the local Lipschitz continuity of $\bar{g}_\tau$ on each disk $\mathcal{D}(x_0)$, for $x_0 \in \mathcal{K}$.

Lemma 3.5. *Let ℓ_1 be given by Lemma 3.2. Then there is an ϵ_8 with $0 < \epsilon_8 \le \epsilon_7$ such that for all ϵ with $0 < \epsilon \le \epsilon_8$, there exists a $\delta_8 = \delta_8(\epsilon)$ with $0 < \delta_8 \le \delta_7$, where ϵ_7 and δ_7 are given in Lemma 3.4, and there exists an $\ell_8 = \ell_8(\epsilon) > 0$ with $\ell_8(\epsilon) \le \ell_1$, such that if $\|Y\|_{C^1(\Omega)} \le \delta_8$ and if $f \in \mathcal{F} = \mathcal{F}(\epsilon, \ell)$ and $g \in \mathcal{G} = \mathcal{G}(\epsilon, \ell)$ with $\ell \le \ell_1$ and $\epsilon \le \epsilon_8$, then the restriction of $\bar{g}_\tau$ to the disk $\mathcal{D}(x_0)$ is Lipschitz continuous with Lipschitz coefficient ℓ_8, for every $x_0 \in \mathcal{K}$. Moreover, one has $\ell_8(\epsilon) \to 0$, as $\epsilon \to 0$.*

Proof. First we will show that there is a $\beta \in \Sigma$ with the property that

$$|\bar{g}_\tau(x_1) - \bar{g}_\tau(x_0)| \le 4\beta(\epsilon)|x_1 - x_0|, \qquad \text{for all } x_0 \in \mathcal{K} \text{ and } x_1 \in \mathcal{D}(x_0). \tag{3.14}$$

Indeed, let $(f, g) \in \mathcal{F} \times \mathcal{G}$ be given and let $x_0 \in \mathcal{K}$ and $x_1 \in \mathcal{D}(x_0)$. Define $s_i = f(x_i)$ and $u_i = \bar{g}_\tau(x_i)$, and set $y_{i0} = x_i + s_i + V$, for $i = 1, 2$, where $V = M(x_i)W$ is a parameter with $V \in U^u(x_i)$ (or $W \in U^u(x_0)$) and $|V| = |W| \leq \epsilon$. Let $y_i = y_i(t) = y(t, y_{i0})$ and $v_i = v_i(t) = v(t, y_{i0})$. One then has

$$\lim_{(\epsilon,\delta)\to(0,0)} y_i(t) = x(t, x_i) \qquad \text{and} \qquad \lim_{(\epsilon,\delta)\to(0,0)} v_i(t) = x(t, x_i),$$

for $i = 0, 1$. Also the initial conditions satisfy

$$\lim_{(\epsilon,\delta)\to(0,0)} y_{i0} = x_i, \qquad \text{for } i = 0, 1.$$

From equation (3.11g) and the continuity of the terms in this equation, we find that

$$\lim_{(\epsilon,\delta)\to(0,0)} \hat{G}(W, x_i) = 0, \qquad \text{for } i = 0, 1,$$

where all these limits are uniform for $x_0 \in \mathcal{K}$ and $x_1 \in \mathcal{D}(x_0)$. Consequently, there is a $\beta_1 \in \Sigma$ such that

$$|\hat{G}(W, x_1) - \hat{G}(W, x_0)| \leq \beta_1(\epsilon + \delta)|x_1 - x_0|,$$

for all $x_0 \in \mathcal{K}$, $x_1 \in \mathcal{D}(x_0)$, and $W \in U^u(x_0)$ with $|W| \leq \epsilon$. Next choose ϵ_8 so that $0 < \epsilon_8 \leq \epsilon_7$ and $4\beta_1(2\epsilon_8) \leq \ell_1$, and set $\delta_8(\epsilon) = \min(\epsilon, \delta_7(\epsilon))$. Define $\beta \in \Sigma$ by

$$\beta(\epsilon) = \beta_1(\epsilon + \delta_7(\epsilon)), \qquad \text{for } 0 < \epsilon \leq \epsilon_8.$$

Now $\bar{g}_\tau(x_i) = M(x_i)W_i$ is a fixed point of (3.13), and W_i is a fixed point of (3.13d), for $i = 0, 1$. As a result, one has

$$\begin{aligned} |\bar{g}_\tau(x_1) - \bar{g}_\tau(x_0)| &= |\hat{G}(W_1, x_1) - \hat{G}(W_0, x_0) \pm \hat{G}(W_0, x_1)| \\ &\leq |\hat{G}(W_1, x_1) - \hat{G}(W_0, x_1)| + |\hat{G}(W_0, x_1) - \hat{G}(W_0, x_0)| \\ &\leq \frac{3}{4}|\bar{g}_\tau(x_1) - \bar{g}_\tau(x_0)| + \beta(\epsilon)|x_1 - x_2|, \end{aligned}$$

which implies inequality (3.14). With ℓ_8 defined by $\ell_8(\epsilon) = \min(\ell_1, 4\beta(\epsilon))$, this completes the proof of the lemma. □

In the following result we derive the contracting property for the mapping $(f, g) \to \bar{g}_\tau$.

Lemma 3.6. *There is an ϵ_9 with $0 < \epsilon_9 \leq \epsilon_8$ and for every ϵ with $0 < \epsilon \leq \epsilon_9$ there is a $\delta_9 = \delta_9(\epsilon) \leq \delta_8(\epsilon)$, where ϵ_8 and δ_8 are given in Lemma 3.5, such that if $\|Y\|_{C^1(\Omega)} \leq \delta_9$, then for all $w_i = (f_i, g_i) \in \mathcal{F}(\epsilon, \ell) \times \mathcal{G}(\epsilon, \ell)$, where $\ell \leq \ell_1$, and $\bar{g}_{\tau,i}$ is given by (3.13), for $i = 0, 1$. One then has*

$$|\bar{g}_{\tau,1}(x) - \bar{g}_{\tau,0}(x)| \leq \frac{2}{3} d(w_0, w_1) \qquad \textit{for all } x \in \mathcal{K}. \tag{3.15}$$

Proof. Let $x_0 \in \mathcal{K}$ and let $w_i = (f_i, g_i) \in \mathcal{F} \times \mathcal{G}$ be given, where $i = 1, 2$, $\mathcal{F} = \mathcal{F}(\epsilon, \ell)$ and $\mathcal{G} = \mathcal{G}(\epsilon, \ell)$. Set $y_{i0} = x_0 + f_i(x_0) + \bar{g}_{\tau,i}(x_0)$, for $i = 1, 2$. Let $x = x(\tau, x_0)$, $y_i = y(\tau, y_{i0})$, and $v_i = v(\tau, y_{i0})$, for $i = 1, 2$, and set $z(t) = y(t, y_{10}) - y(t, y_{20})$ Then one has $g_i(v_i) = P^u(v_i)(y_i - v_i)$, for $i = 1, 2$. By using equation (3.11b), one obtains

$$\begin{aligned} g_1(v_1) - g_2(v_2) = P^u(x)(y_1 - y_2) &- P^u(x)(v_1 - v_2) \\ &+ (P^u(v_1) - P^u(v_2))(y_2 - v_2) \\ &+ (P^u(v_1) - P^u(x))(y_1 - y_2 + v_2 - v_1), \end{aligned}$$

which we will write in the form

$$\begin{aligned} P^u(x)z(\tau) = P^u(x)(v_1 - v_2) &+ g_1(v_1) - g_2(v_2) \\ &- (P^u(v_1) - P^u(v_2))(y_2 - v_2) \\ &- (P^u(v_1) - P^u(x))(y_1 - y_2 + v_2 - v_1). \end{aligned} \tag{3.15b}$$

As argued in Lemma 3.0, the function $z(t)$ satisfies equation (3.5f), where $z(0) = f_1(x_0) - f_2(x_0) + \bar{g}_{\tau,1}(x_0) - \bar{g}_{\tau,2}(x_0)$ and $H_4(t)$ satisfies

$$|H_4(t)| \le \beta_2(\epsilon + \delta)|y_{10} - y_{20}|, \qquad \text{for } 0 \le t \le 2T \tag{3.15c}$$

As a result, (3.15b) becomes

$$P^u(x)(\Phi(\tau, x_0)z(0) + H(\tau)) = H_2(\tau),$$

where $H_2(\tau)$ is the right side of (3.15b). Because of the invariance property (2.4a) and the fact that

$$P^u(x_0)z(0) = \bar{g}_{\tau,1}(x_0) - \bar{g}_{\tau,2}(x_0),$$

one obtains

$$\bar{g}_{\tau,1}(x_0) - \bar{g}_{\tau,2}(x_0) = -\Phi^{-1}(\tau, x_0)H(\tau) + \Phi^{-1}(\tau, x_0)H_2(\tau).$$

From (3.15c) we see that there is a constant $C_1 > 0$ such that

$$|\Phi^{-1}(\tau, x_0)H(\tau)| \le C_1\beta_1(\epsilon + \delta)|y_{10} - y_{20}|. \tag{3.15g}$$

We will now show that $|\Phi^{-1}(\tau, x_0)H_2(\tau)|$ satisfies an estimate similar to (3.15g). First by (2.3b), (2.7b), and Lemma 3.0, there is a constant $C_2 > 0$ such that one has

$$|\Phi^{-1}(\tau, x_0)P^u(x)(v_1 - v_2)| \le a|P^u(x)(v_1 - v_2)|e^{-\lambda_4\tau} \le C_2(\epsilon + \delta)|y_{10} - y_{20}|.$$

Due to the continuity of P^u and Lemma 3.0, there is a $\beta_2 \in \Sigma$ such that

$$|\Phi^{-1}(\tau, x_0)(P^u(v_1) - P^u(x))(y_1 - y_2)| \le \beta_2(\epsilon + \delta)|y_{10} - y_{20}|,$$

with similar estimates valid for

$$|\Phi^{-1}(\tau, x_0)(P^u(v_1) - P^u(x))(v_1 - v_2)| \quad \text{and} \quad |\Phi^{-1}(\tau, x_0)(P^u(v_1) - P^u(v_2))(y_2 - v_2)|.$$

Using the fact that $P^u(v_i)g_j(v_i) = g_j(v_i)$, for $i, j = 1, 2$, one obtains

$$g_1(v_1) - g_2(v_2) = G_1 + G_2 + G_3 + G_4 + G_5,$$

where

$$G_1 = P^u(x)(g_1(v_1) - g_1(v_2)), \qquad G_2 = (P^u(v_1) - P^u(x))(g_1(v_1) - g_1(v_2)),$$

and

$$G_3 = (P^u(v_1) - P^u(v_2))g_1(v_2),$$

as well as,

$$G_4 = P^u(x)(g_1(v_2) - g_2(v_2)), \qquad G_5 = (P^u(v_2) - P^u(x))(g_1(v_2) - g_2(v_2)).$$

It then follows from the Lipschitz continuity of P^u on the disk $\mathcal{D}(x(\tau, x_0))$, (2.3b), (2.7b), and Lemma 3.0 that there is a constant $C_3 > 0$ such that, when $\ell \leq \ell_1$, the five terms $|\Phi^{-1}(\tau, x_0)G_i|$, for $i = 1, \cdots, 5$, are bounded respectively by

$$2a^2e^{-\lambda_4\tau}|y_{10} - y_{20}|, \quad C_3(\epsilon + \delta)|y_{10} - y_{20}|, \quad C_3\epsilon|y_{10} - y_{20}|,$$

and

$$a^2e^{-\lambda_4\tau}\|g_1 - g_2\|_\infty, \quad C_3(\epsilon + \delta)|y_{10} - y_{20}|.$$

Since $|y_{10} - y_{20}| \leq \|f_1 - f_2\|_\infty + \|\bar{g}_{\tau,1} - \bar{g}_{\tau,2}\|_\infty$, one finds that there is a $\beta_3 \in \Sigma$ such that

$$\begin{aligned}|\bar{g}_{\tau,1}(x_0) - \bar{g}_{\tau,2}(x_0)| \leq (2a^2e^{-\lambda_4\tau} + \beta_3(\epsilon + \delta))\|\bar{g}_{\tau,1} - \bar{g}_{\tau,2}\|_\infty \\ + (3a^2e^{-\lambda_4\tau} + \beta_3(\epsilon + \delta))\|w_1 - w_2\|_\infty.\end{aligned}$$

From (3.2) we obtain

$$\|\bar{g}_{\tau,1} - \bar{g}_{\tau,2}\|_\infty \leq (\frac{1}{8} + \beta_3(\epsilon + \delta))\|\bar{g}_{\tau,1} - \bar{g}_{\tau,2}\|_\infty + (\frac{3}{16} + \beta_3(\epsilon + \delta))\|w_1 - w_2\|_\infty.$$

Now choose ϵ_9 so that $0 < \epsilon_9 \leq \epsilon_8$ and $\beta_3(2\epsilon_9) \leq \frac{1}{5}$ and define δ_9 by $\delta_9(\epsilon) = \min(\epsilon, \delta_8(\epsilon))$. For $0 < \epsilon \leq \epsilon_9$, one then has

$$\|\bar{g}_{\tau,1} - \bar{g}_{\tau,2}\|_\infty \leq \left(\frac{1}{8} + \frac{1}{5}\right)\|\bar{g}_{\tau,1} - \bar{g}_{\tau,2}\|_\infty + \left(\frac{3}{16} + \frac{1}{5}\right)\|w_1 - w_2\|_\infty,$$

which implies (3.15). □

The objective of the next result is to give a more precise formulation of Part (1) of Theorem A given in Section 1. Among other things, we show the existence of a continuous mapping $h : \mathcal{K} \to R^n$, where the image $\mathcal{K}^Y = h(\mathcal{K})$ is a compact invariant set for the perturbed equation (1.2).

Theorem 3.7. *For each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ and an $\ell = \ell(\epsilon) \le \ell_1$, where $\delta, \ell \in \Sigma$ such that if $\|Y\|_{C^1(\Omega)} \le \delta$, then there is a continuous mapping $h : \mathcal{K} \to R^n$ such that the following properties hold:*

(1) *the image $\mathcal{K}^Y = h(\mathcal{K})$ is a compact invariant set for the perturbed equation (1.2);*
(2) *for each $x \in \mathcal{K}$, one has $(h(x) - x) \in U^s(x) \oplus U^u(x)$;*
(3) *the restriction of h to any disk $\mathcal{D}(x_0)$, where $x_0 \in \mathcal{K}$, is Lipschitz continuous with*

$$|h(x_1) - x_1 - h(x_2) + x_2| \le 2\ell |x_1 - x_2|, \tag{3.16}$$

for all x_1, $x_2 \in \mathcal{D}(x_0)$;
(4) *and one has $|h(x) - x| \le 2\epsilon$, for all $x \in \mathcal{K}$.*

Proof of Theorem 3.7 and the Shadow Theorem B. Let T be given by (3.2), and for $T \le \tau \le 2T$, let A_τ be the mapping on $\mathcal{F} \times \mathcal{G} = \mathcal{F}(\epsilon, \ell_1) \times \mathcal{G}(\epsilon, \ell_1)$ defined by

$$A_\tau : w = (f, g) \to w_\tau = (\bar{f}_\tau, \bar{g}_\tau), \tag{3.16a}$$

where ℓ_1 is given by (3.8h), $\bar{f}_\tau$ is given by equation (3.6i), and $V = \bar{g}_\tau$ is given by equation (3.11f).

Let ϵ_9 and δ_9 be given by Lemma 3.6, and let ℓ_1, ℓ_5, and ℓ_8 be given by Lemmas 3.2 and 3.5. Define $\ell_9 = \ell_9(\epsilon)$ by

$$\ell_9(\epsilon) = \max(\ell_5(\epsilon), \ell_8(\epsilon)), \qquad \text{for } 0 < \epsilon \le \epsilon_9. \tag{3.16b}$$

From the construction of ℓ_5 and ℓ_8 on has $\ell_9(\epsilon) \le \ell_1 \le 1$, for $0 < \epsilon \le \epsilon_9$, and $\ell_9 \in \Sigma$. Let ϵ be fixed with $0 < \epsilon \le \epsilon_9$, and set $\delta = \delta_9(\epsilon)$ and $\ell = \ell_9(\epsilon)$.

Assume that $\|Y\|_{C^1(\Omega)} \le \delta$. From Lemmas 3.1, 3.2, 3.4, and 3.5, we see that A_τ maps $\mathcal{F} \times \mathcal{G}$ into itself, for each τ with $T \le \tau \le 2T$. Also Lemmas 3.3 and 3.6 imply that A_τ is a strict contraction on $\mathcal{F} \times \mathcal{G}$. Since $\mathcal{F} \times \mathcal{G}$ is a complete metric space, the mapping A_τ has a unique fixed point, which we will denote by

$$w_\tau = (f_\tau, g_\tau) \in \mathcal{F} \times \mathcal{G}, \qquad \text{for each } \tau \text{ with } T \le \tau \le 2T.$$

For the fixed point $w_\tau = (f_\tau, g_\tau)$ define $h_\tau : \mathcal{K} \to R^n$ by

$$h_\tau(x) = x + f_\tau(x) + g_\tau(x), \qquad \text{for } x \in \mathcal{K} \text{ and } T \le \tau \le 2T. \tag{3.16d}$$

Let $w_T = (f_T, g_T)$ be the fixed point of A_T and let $h = h_T$ be given by equation (3.16d). For $0 \le t \le 2T$, define $S^Y(t)$ by

$$S^Y(t)x_0 = v(t, h(x_0)), \qquad \text{for } x_0 \in \mathcal{K}. \tag{3.16e}$$

Since $w_T = (f_T, g_T)$ is a fixed point of A_T, one has

$$f_T(v(T, h(x)) = P^s(v(T, h(x)))(y(T, h_(x)) - v(T, h(x))) = s(T, h(x))$$
$$g_T(v(T, h(x)) = P^u(v(T, h(x)))(y(T, h(x)) - v(T, h(x))) = u(T, h(x)),$$

for $x \in \mathcal{K}$. This can be rewritten in the form

$$y(T, h(x)) = h(v(T, h(x)), \qquad \text{for } x \in \mathcal{K},$$

or equivalently,

$$S_2(T)h(x) = h(S^Y(T)x), \qquad \text{for } x \in \mathcal{K}. \tag{3.16j}$$

For $0 \le t \le 2T$ we define h_t by

$$h_t(S^Y(t)x_0) = S_2(t)h(x_0), \qquad \text{for } x_0 \in \mathcal{K}. \tag{3.16m}$$

Because of (3.4c), we see that h_t is well-defined for all $x_0 \in \mathcal{K}$. Also from (3.4b) we see that one has the local coordinate representation

$$h_t(v(t, h(x_0))) = y(t, h(x_0)) = v(t, h(x_0)) + s(t, h(x_0)) + u(t, h(x_0)).$$

Now define (f_t, g_t) by

$$f_t(v(t, h(x_0))) = s(t, h(x_0)) \qquad \text{and} \qquad g_t(v(t, h(x_0))) = u(t, h(x_0)). \tag{3.16o}$$

From (3.16j), (3.16m), and the commutivity relations $S_2(T)S_2(t) = S_2(t)S_2(T)$ and $S^Y(T)S^Y(t) = S^Y(t)S^Y(T)$, see (3.4d), we then obtain

$$\begin{aligned} S_2(T)h_t(S^Y(t)x_0) &= S_2(T)S_2(t)h(x_0) = S_2(t)S_2(T)h(x_0) \\ &= S_2(t)h(S^Y(T)x_0) = h_t(S^Y(t)S^Y(T)x_0) \\ &= h_t(S^Y(T)S^Y(t)x_0), \end{aligned} \tag{3.16n}$$

for all $x_0 \in \mathcal{K}$. Now (3.16n) implies that $w_t = (f_t, g_t)$, where (f_t, g_t) is given by (3.16o), is a fixed point of A_T.

We claim that there is an ϵ_{10}, with $0 < \epsilon_{10} \le \epsilon_9$, such that, for $0 < \epsilon \le \epsilon_{10}$, there is a $t_0 > 0$ where $w_t \in \mathcal{F} \times \mathcal{G}$, for $0 \le t \le t_0$. Indeed, from Lemmas 3.1 and 3.4, and since w_τ is in the range of A_τ, one has $|f_0(x)| \le \frac{3}{4}\epsilon$ and $|g_0(x)| \le \frac{3}{4}\epsilon$, for all $x \in \mathcal{K}$. By continuity, there is a $t_1 > 0$ such that $|f_t(x)| \le \epsilon$ and $|g_t(x)| \le \epsilon$, for all $x \in \mathcal{K}$ and $0 \le t \le t_1$. Next choose ϵ_{10} so that $0 < \epsilon_{10} \le \epsilon_9$ and where $\ell_j(\epsilon_{10}) < \ell_1$ and ℓ_j are given by Lemmas 3.2 and 3.5, for $j = 5, 8$. By continuity, it follows from Lemmas 3.2 and 3.5 that there is a $t_2 > 0$ such that both f_t and g_t are Lipschitz continuous on each disk $\mathcal{D}(x_0)$ with Lipschitz coefficient ℓ_1, for $0 \le t \le t_2$. By setting $t_0 = \min(t_1, t_2)$, we conclude that $w_t \in \mathcal{F} \times \mathcal{G}$, for $0 \le t \le t_0$.

Since the fixed point of A_T is unique, we have $h_t = h$, for all t, with $0 \le t \le t_0$. By iteration of this argument, we conclude that $h_t = h$, for all $t \ge 0$. This implies that

$$S_2(t)h(x_0) = h(S^Y(t)x_0), \qquad \text{for } x_0 \in \mathcal{K} \text{ and } t \ge 0. \tag{3.17}$$

Furthermore, $S^Y(t) : \mathcal{K} \to \mathcal{K}$ is a semiflow in the sense that $S^Y(t)x_0$ is jointly continuous in $(x_0, t) \in \mathcal{K} \times [0, \infty)$; $S^Y(0)x_0 = x_0$; and the semigroup property

$$S^Y(t)S^Y(s) = S^Y(t+s), \qquad \text{for } s, t \ge 0,$$

is valid. In addition, one has $S^Y(t)\mathcal{K} = \mathcal{K}$, for all $t \geq 0$, by (3.4c). This completes the proof of Theorem 3.7 and the Shadow Theorem B. $\square$

One can readily verify that $v(t) = v(t, h(x_0))$, see equation (3.16e), is the solution of the differential equation

$$v' = P^o(v)[X(v + h(v)) + Y(v + h(v))], \tag{3.17b}$$

with $v(0) = x_0 \in \mathcal{K}$.

Proof of Theorem A, Part (1). This now follows directly from Theorem 3.7 by setting $\delta_1 = \delta_9(\epsilon_9)$. $\square$

4. Perturbation of Hyperbolic Structures

As in Section 3, we assume that the hypotheses of Theorem A are satisfied. We will make extensive use of the notation and concepts developed above. The first step is to show that the dynamics $S_2(t)$ on $\mathcal{K}^Y$ are weakly hyperbolic, see Definition 2.1. We begin by fixing ϵ_9 and δ_9 as in Lemma 3.6. Also let ℓ_1 and ℓ_9 be given by (3.8h) and (3.16b), and let $\delta_1 = \delta_9(\epsilon_9)$ be given as in the proof of Theorem A, Part (1). In this section, we restrict the parameters to satisfy $0 < \epsilon \leq \epsilon_9$, $\delta \leq \delta_9(\epsilon)$, and $\ell \leq \ell_9(\epsilon)$.

Theorem 4.1. *Let $\mathcal{K}$ be a weakly, normally hyperbolic set for the given equation (1.1) that satisfies the Lipschitz property, and let a and λ_i, for $i = 0, \cdots, 4$, denote the characteristics of $\mathcal{K}$. Then there is a $\sigma_0 > 0$ such that for each σ with $0 < \sigma \leq \sigma_0$, there exists a $\delta = \delta(\sigma) > 0$ such that if $\|Y\|_{C^1(\Omega)} \leq \delta$, then the set $\mathcal{K}^Y$, which is an invariant set of the system (1.2), is weakly hyperbolic with characteristics $a + \sigma$, $\lambda_0 + \sigma$, $\lambda_1 + \sigma < \lambda_2 - \sigma < \lambda_3 + \sigma < \lambda_4 - \sigma$.*

For each $y \in \mathcal{K}^Y$, let $U^s_Y(y)$, $U^o_Y(y)$, and $U^u_Y(y)$ denote the associated stable, neutral, and unstable linear spaces. Then for $y_0 = h(x_0)$, the spaces $U^s_Y(y(t, y_0))$, $U^o_Y(y(t, y_0))$, and $U^u_Y(y(t, y_0))$ are uniformly close (in the sense of angular measure) to the spaces $U^s(v(t, y_0))$, $U^o(v(t, y_0))$, and $U^u(v(t, y_0))$, respectively, for all $t \in R$.

The proof of this theorem uses the same argument as in Pliss (1977; Theorem 1,3, Chap. 4, p. 257). Since this argument is not available in English, we present the key ideas here for the convenience of the reader. (A different proof is given in Pliss and Sell (1997).) Let us first consider the system

$$\partial_t x = Q_\tau(t)x = Q(\tau + t)x, \qquad \tau \in R, \tag{VP1}$$

where $Q = Q(t)$ is a piecewise continuous matrix-valued function with $\|Q\|_\infty = \sup\{|Q(t)| : t \in R\} \leq H$. Let $\Phi(t, Q_\tau)$ denote the fundamental solution operator of (VP1) with $\Phi(0, Q_\tau) = I$. Assume that (1) is weakly hyperbolic with characteristics $a \geq 1$, λ_0, $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, $\lambda_1 < 0 < \lambda_4$, and $\lambda_0 \geq \lambda_3$. In particular, there exist linear spaces $U^s(t)$, $U^o(t)$, and $U^u(t)$ such that $\dim U^u = m$, $\dim U^o = k$, and $\dim U^s = n - m - k$, and the inequalities (2.2c)-(2.2f) hold, for all $\tau \in R$, where $\Phi(t, x_0)$ is replaced by $\Phi(t, Q_\tau)$. The following lemma now applies to equation (VP1).

Lemma 4.2. *Let equation (VP1) be weakly hyperbolic in the sense described above. For each $\sigma > 0$ there exists a $\Delta = \Delta(H, a, \lambda_i, \sigma) > 0$ such that if $\|\bar{Q} - Q\|_\infty \le \Delta$, then the system*

$$\partial_t x = \bar{Q}_\tau(t)x = \bar{Q}(\tau + t)x, \qquad \tau \in R \tag{VP6}$$

is weakly hyperbolic with characteristics $a+\sigma$, $\lambda_0+\sigma$, $\lambda_1+\sigma$, $\lambda_2-\sigma$, $\lambda_3+\sigma$, $\lambda_4-\sigma$, and spaces the $U_y^u(t)$, $U_y^o(t)$, $U_y^s(t)$, corresponding to equation (VP6), are σ-close to $U_y^s(t)$, $U_y^o(t)$, and $U_y^u(t)$.

As noted in Section 2, the weak hyperbolicity is equivalent to a pair of exponential dichotomies for the shifted equation $\partial_t y = (Q(t) + \lambda I)y$, where

$$\lambda = -\frac{1}{2}(\lambda_1 + \min(0, \lambda_2)) \quad \text{and} \quad \lambda = -\frac{1}{2}(\lambda_4 + \max(\lambda_3, 0)).$$

Consequently, a proof of this lemma, in English, can be found in Coppel (1978, Section 4), and we will omit the details.

Proof of the Theorem 4.1. We consider the systems (1.1) and (1.2) in a neighborhood Ω of $\mathcal{K}$. Note that there is an $H > 0$ such that

$$\|DX(x)\|_{\text{op}} \le H, \qquad \text{for all } x \in \Omega. \tag{P1}$$

Because of the continuity of the mapping $x \to (X(x), DX(x))$, for every η with $0 < \eta \le \frac{1}{2}$, there exists a $\delta_0 > 0$ such that if $x_1, x_2 \in \mathcal{K}$ and $|x_1 - x_2| \le \delta_0$, then $|X(x_1) - X(x_2)| \le \eta$ and $\|DX(x_1) - DX(x_2)\|_{\text{op}} \le \eta$. Furthermore, since the spaces $U^s(x_0)$, $U^o(x_0)$, $U^u(x_0)$ vary continuously in x_0, by choosing the value of δ_0 smaller, if necessary, we can assume that for each pair $x_1, x_2 \in \mathcal{K}$ with $|x_1 - x_2| \le \delta_0$, there exists a matrix $A = A(x_1, x_2)$ such that $AU^i(x_1) = U^i(x_2)$ for $i = s, o, u$, and $|A - I| \le \eta$ and $|A^{-1} - I| \le \eta$.

Let T be a number such that $T > 1$ and

$$6a^2 e^{-\frac{\sigma}{2}T} \le 1, \tag{P7}$$

where σ is small. We will now use Theorem 3.7. Let $\epsilon > 0$ be given and assume that $\delta > 0$ satisfies $0 < \delta \le \delta_1$ and let Y satisfy $\|Y\|_{C^1(\Omega)} \le \delta$. Then one has $\mathcal{K}^Y \subset N_{2\epsilon}(\mathcal{K})$. By choosing a smaller value of δ, if necessary, we can assume that if $\|Y\|_{C^1(\Omega)} \le \delta$ and if $\psi(t)$ is a solution of equation (1.2) that satisfies $|x_0 - \psi(t_0)| \le 2\epsilon$, where $(t_0, x_0) \in R \times \mathcal{K}$, then one has

$$|S_1(t - t_0)x_0 - \psi(t)| \le \frac{1}{2}\delta_0, \qquad \text{for } |t - t_0| \le T. \tag{P8}$$

Now let $\psi(t)$ be any solution of (1.2) with $\psi(0) \in \mathcal{K}^Y$. Our goal is to prove, perhaps with a smaller value of δ, that the system

$$\partial_t y = \bar{Q}_\tau(t)y = \bar{Q}(\tau + t)y, \qquad \tau \in R,$$

where $\bar{Q}(t) = DX(\psi(t)) + DY(\psi(t))$, is weakly hyperbolic with good characteristics. We will use Lemma 4.2 by constructing a matrix-valued function $Q(t)$, where $\|Q - \bar{Q}\|_\infty$ is small and such that the system (VP1) is weakly hyperbolic, with good characteristics.

Let $x_0 \in \mathcal{K}$ satisfy $|x_0 - \psi(0)| \leq 2\epsilon$. Let $t_m = mT$, and pick $x_m \in \mathcal{K}$ so that $|x_m - \psi(t_m)| \leq 2\epsilon$, for all $m \in Z$. Next we define the piecewise continuous function $Q^1(t)$, for $t \in R$, by

$$\text{(P11)} \qquad Q^1(t) = DX(S_1(t - t_m)x_m), \qquad \text{for } t_m < t \leq t_{m+1} \text{ and } m \in Z.$$

We will let $\Phi(t, Q^1_\tau)$ denote the fundamental solution operator of the system

$$\partial_t y = Q^1_\tau(t)y = Q^1(\tau + t)y, \qquad \tau \in R.$$

Since the solution $S_1(t - t_m)x_m$ is in $\mathcal{K}$, for all $t \in R$, and for each $m \in Z$, one has $\|Q^1\|_\infty \leq H$, by inequality (P1). Also inequality (P8) implies that

$$|S_1(t - t_m)x_m - \psi(t)| \leq \frac{1}{2}\delta_0, \qquad \text{for } t_m \leq t \leq t_{m+1} \text{ and } m \in Z.$$

Furthermore, if ϵ and δ are sufficiently small, then one has $|x_{m+1} - S_1(T)x_m| \leq \delta_0$, for all $m \in Z$. Consequently, the matrix $A_m = A(S_1(T)x_m, x_{m+1})$ satisfies

$$\text{(P15)} \qquad A_m U^i(S_1(T)x_m) = U^i(x_{m+1}), \qquad \text{for } i = s, o, u,$$

and

$$\text{(P16)} \qquad |A_m - I| \leq \eta, \qquad \text{and} \qquad |A_m^{-1} - I| \leq \eta.$$

The next step is to construct a continuously differentiable matrix-valued function $B = B(t)$ with good properties. In particular, we construct B as a cubic spline on the interval $(t_{m+1} - 1, t_{m+1})$ with $B(t_{m+1} - 1) = I$, $B(t_{m+1}) = A_m$, and

$$\partial_t B(t_{m+1} - 1) = \partial_t B(t_{m+1}) = 0,$$

for $m \in Z$. Thus one has

$$B(t) - I = 2(t - t_{m+1} + 1)^2(-t + t_{m+1} + 1/2)(A_m - I),$$

for $t_{m+1} - 1 \leq t \leq t_{m+1}$ and $m \in Z$. For $t_m < t < t_{m+1} - 1$, we define $B(t) = I$. The function $B(t)$ is a C^1-function on each interval $t_m < t < t_{m+1}$, with a discontinuity at t_m when $A_{m-1} \neq I$. By using (P16), one readily obtains

$$\text{(P16c)} \qquad |B(t) - I| \leq \eta \quad \text{and} \quad |\partial_t B(t)| \leq 5\eta, \qquad \text{for } t_m < t \leq t_{m+1}.$$

Since $0 < \eta \leq \frac{1}{2}$, the inverse $B^{-1}(t)$ exists and one has $|B^{-1}(t) - I| \leq 2\eta$, for $t_m < t \leq t_{m+1}$.

Next we define $Q(t)$, for $t \in R$, by

$$Q(t) = \partial_t B(t)\, B^{-1}(t) + B(t)Q^1(t)B^{-1}(t), \qquad \text{for } t_m < t \leq t_{m+1} \text{ and } m \in Z.$$

The functions $B(t)$ and $Q(t)$ have been chosen so that $\Psi(t) \stackrel{\text{def}}{=} B(t)\Phi(t, Q^1)$ satisfies the differential equation $\partial_t \Psi(t) = Q(t)\Psi(t)$. In other words, the identity

$$\Phi(t, Q) = B(t)\Phi(t, Q^1), \qquad t \in R,$$

is valid. More generally, one has

$$\text{(P17)} \qquad \Phi(t, Q_{t_m}) = B(t_m + t)\Phi(t, Q^1_{t_m}) = B_{t_m}(t)\Phi(t, Q^1_{t_m}),$$

for all $t \in R$ and $m \in Z$. One can readily show that there is a constant $K = K(H) > 0$ such that

$$\text{(P18)} \qquad \|Q - \bar{Q}\|_\infty \leq K\eta.$$

Now let $v_m \in U^s(x_m)$, for some $m \in Z$. It then follows from inequality (2.2c) that $|\Phi(t, Q^1_{t_m})v_m| \leq ae^{\lambda_1 t}|v_m|$, for $0 \leq t \leq T$. As a result, (P16c) and (P17) imply that

$$|\Phi(t, Q_{t_m})v_m| \leq a(1+\eta)e^{\lambda_1 t}|v_m|, \qquad \text{for } 0 \leq t \leq T.$$

Since (P15) and (P17) imply that

$$\Phi(T, Q_{t_m})v_m = B(t_m + T)\Phi(T, Q^1_{t_m})v_m = A_m\Phi(T, Q^1_{t_m})v_m \in U^s(x_{m+1}),$$

it follows from the cocycle identity (2.2a) and inequality (P7) that

$$\begin{aligned}\Phi(T+t, Q_{t_m})v_m = |\Phi(t, Q_{t_{m+1}})\Phi(T, Q_{t_m})v_m| &\leq ae^{\lambda_1 t}|\Phi(T, Q_{t_m})v_m| \\ &\leq a^2(1+\eta)e^{\lambda_1(T+t)}|v_m| \leq a(1+\eta)e^{(\lambda_1+\frac{\sigma}{2})(T+t)}|v_m|,\end{aligned}$$

for $0 \leq t \leq T$. By induction, one then obtains

$$\text{(P25)} \qquad |\Phi(kT+t, Q_{t_m})v_m| \leq a(1+\eta)e^{(\lambda_1+\frac{\sigma}{2})(kT+t)}|v_m|,$$

for $0 \leq t \leq T$ and $k = 0, 1, \cdots$. Similarly, if $v_m \in U^o(x_m)$, then one has

$$|\Phi(kT+t, Q_{t_m})v_m| \leq a(1+\eta)e^{(\lambda_3+\frac{\sigma}{2})(kT+t)}|v_m|,$$

for $0 \leq t \leq T$ and $k = 0, 1, \cdots$. For $t \leq 0$, one uses B^{-1} in place of B. In particular, for $v_m \in U^u(x_m)$, one obtains

$$|\Phi(kT+t, Q_{t_m})v_m| \leq a(1+2\eta)e^{(\lambda_4-\frac{\sigma}{2})(kT+t)}|v_m|,$$

for $-T \leq t \leq 0$ and $k = 0, -1, -2, \cdots$, and for $v_m \in U^o(x_m)$ one has

$$|\Phi(kT+t, Q_{t_m})v_m| \leq a(1+2\eta)e^{(\lambda_3+\frac{\sigma}{2})(kT+t)}|v_m|,$$

for $-T \le t \le 0$ and $k = 0, -1, -2 \cdots$.

We see then that, for η satisfying $2a\eta \le \frac{\sigma}{2}$, the solution operator $\Phi(t, Q_\tau)$ satisfies inequalities (2.2c)-(2.2f) with characteristics $a + \frac{\sigma}{2}$, $\lambda_0 + \frac{\sigma}{2}$, $\lambda_1 + \frac{\sigma}{2}$, $\lambda_2 - \frac{\sigma}{2}$, $\lambda_3 + \frac{\sigma}{2}$, and $\lambda_4 - \frac{\sigma}{2}$, provided that $\tau = t_m$ with $m \in Z$. In particular, one has $U^i(Q_{t_m}) \stackrel{\text{def}}{=} U^i(x_m)$, for $i = s, o, u$ and $m \in Z$.

What happens when $\tau \ne t_m$? Let us examine this for $t_m < \tau < t_{m+1}$, where $m \in Z$ is now fixed, but arbitrary. First we define

$$U^i(Q_\tau) \stackrel{\text{def}}{=} \Phi(\tau - t_m, Q_{t_m}) U^i(Q_{t_m}) = B(\tau)\Phi(\tau - t_m, Q^1_{t_m}) U^i(x_m), \quad \text{for } i = s, o, u.$$

Let us now examine the stable manifold with $v_\tau \in U^s(Q_\tau)$. Set $w_\tau = B(\tau)^{-1} v_\tau$. One then has $w_\tau \in U^s(S_1(\tau)x_m)$ and $|w_\tau| \le (1 + 2\eta)|v_\tau|$. Furthermore, one has

$$|\Phi(t - \tau, Q^1_\tau) w_\tau| \le a e^{\lambda_1 (t-\tau)} |w_\tau|, \qquad \text{for } \tau \le t \le t_{m+1}.$$

Hence one finds that

$$\text{(P30)} \quad |\Phi(t - \tau, Q_\tau) v_\tau| \le |B(t)\Phi(t - \tau, Q^1_\tau) w_\tau| \le a(1 + \eta)(1 + 2\eta) e^{\lambda_1 (t-\tau)} |v_\tau|,$$

for $\tau \le t \le t_{m+1}$. Let $v_{m+1} \stackrel{\text{def}}{=} \Phi(t_{m+1} - \tau, Q_\tau) v_\tau$. One then has $v_{m+1} \in U^s(Q_{t_{m+1}})$, and since $0 < \eta \le \frac{1}{2}$, one finds that

$$|v_{m+1}| = |\Phi(t_{m+1} - \tau, Q_\tau) v_\tau| \le 3a e^{\lambda_1 (t_{m+1} - \tau)} |v_\tau|.$$

It then follows from (P7) and (P25) that

$$\text{(P31)} \qquad |\Phi(t - \tau, Q_\tau) v_\tau| \le a(1 + \eta) e^{(\lambda_1 + \frac{\sigma}{2})(t-\tau)} |v_\tau|, \qquad \text{for } t_{m+1} \le t < \infty.$$

The arguments for the remaining inequalities in (2.2c)-(2.2f), with $v_\tau \in U^i(Q_\tau)$, for $i = o, u$, are similar, and we omit the details. Consequently, it follows from inequalities (P30) and (P31) that if η is sufficiently small, say $\eta \le \min\left(1, \frac{\sigma}{10a}\right)$, then equation (VP1), with this choice of Q, is weakly hyperbolic with characteristics $a + \frac{\sigma}{2}$, $\lambda_0 + \frac{\sigma}{2}$, $\lambda_1 + \frac{\sigma}{2}$, $\lambda_2 - \frac{\sigma}{2}$, $\lambda_3 + \frac{\sigma}{2}$, and $\lambda_4 - \frac{\sigma}{2}$. Finally Theorem 4.1 follows from Lemma 4.2 and inequality (P18) by choosing η sufficiently small. This, in turn, is made possible by using Theorem 3.7, perhaps with a smaller value of δ. □

The following result is a more detailed formulation of Theorem A, Part (2).

Theorem 4.3. *There is a δ_2 with $0 < \delta_2 \le \delta_1$ such that if $\|Y\|_{C^1(\Omega)} \le \delta_2$, then for every $x_0 \in \mathcal{K}$ the space $U^o_Y(y_0)$ is tangent to the set $h(\mathcal{D}(x_0))$ at the point y_0, where $y_0 = h(x_0)$. In particular, the set $\mathcal{K}^Y$ is weakly, normally hyperbolic.*

Proof. Assume on the contrary that the space $U^o_Y(y_0)$ is not tangent to $h(\mathcal{D}(x_0))$ at the point y_0. This then implies that there exist an $\bar{\alpha} > 0$ and a sequence of points $x_i \in \mathcal{D}(x_0)$, for $i = 1, 2, \ldots$, such that $x_i \to x_0$, as $i \to \infty$, and for each i, the angle satisfies

$$\text{(4.3)} \qquad \angle(U^o_Y(y_0), y_i - y_0) \ge \bar{\alpha}, \qquad \text{for all } i \ge 1.$$

where $y_i = h(x_i)$. Let $P_Y^s(y)$, $P_Y^o(y)$, and $P_Y^u(y)$, denote the projections of R^n onto $U_Y^s(y)$, $U_Y^o(y)$, and $U_Y^u(y)$, respectively, where $P_Y^s(y) + P_Y^o(y) + P_Y^u(y) = I$, for all $y \in \mathcal{K}^Y$. Let $Q_Y^o(y) = I - P_Y^o(y)$, so that $Q_Y^o(y) = P_Y^s(y) + P_Y^u(y)$, for all $y \in \mathcal{K}^Y$. By taking a subsequence, if necessary, we can assume that, either one has either

$$|P_Y^u(y_0)(y_i - y_0)| \geq |P_Y^s(y_0)(y_i - y_0)|, \tag{4.3a}$$

or

$$|P_Y^s(y_0)(y_i - y_0)| \geq |P_Y^u(y_0)(y_i - y_0)|, \tag{4.3b}$$

for all i. We will consider only the case (4.3a). The argument for the case (4.3b) is analogous.

From the Theorem 4.1 it follows that inequality (2.4) is valid for all $y \in \mathcal{K}^Y$, where $U_Y^i(y)$ replaces $U^i(x)$, for $i = s, o, u$, and $\frac{\alpha}{2}$ replaces α, provided that δ is sufficiently small. In particular, we set $\bar{\delta}$, where $0 < \bar{\delta} \leq \min(\delta_1, \delta(\sigma_0))$, where $\delta(\sigma_0)$ is given by Theorem 4.1, so that if $\|Y\|_{C^1(\Omega)} \leq \bar{\delta}$, then for each $y \in \mathcal{K}^Y$ the inequality

$$\angle(U_Y^o(y), U_Y^s(y) \oplus U_Y^u(y)) \geq \frac{\alpha}{2}, \qquad \text{for } y \in \mathcal{K}^Y,$$

is true. Next define the two cone bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ by

$$\mathcal{V}_1 = \{(y, v) \in \mathcal{K}^Y \times R^n : |Q_Y^o(y)v| \leq 2|v|\},$$

and for $C > 0$,

$$\mathcal{V}_2(C) = \{(y, v) \in \mathcal{K}^Y \times R^n : C|P_Y^o(y)v| \leq |Q_Y^o(y)v|\}.$$

If one has $Q_Y^o(y)v = v$, i.e., if $v \in U_Y^s(y) \oplus U_Y^u(y)$, then $(y, v) \in \mathcal{V}_1$. Also if one has $P_Y^o(y)v = 0$, i.e., if $v \in U_Y^s(y) \oplus U_Y^u(y)$, then one has $(y, v) \in \mathcal{V}_2(C)$, for any $C > 0$. Consequently for large C_1, which we now fix so that $C_1 \geq 18$, one has $\mathcal{V}_2 \subset \mathcal{V}_1$, where $\mathcal{V}_2 = \mathcal{V}_2(C_1)$. To put it another way, the implication

$$|Q_Y^o(y)v| \geq C_1|P_Y^n(y)v| \quad \Longrightarrow \quad 2|v| \geq |Q_Y^o(y)v| \tag{4.3f}$$

is valid, and $C_1 \geq 18$.

By using Theorem 4.1, inequalities (4.3) and (4.3a), and the fact that $y_i \to y_0$, as $i \to \infty$, we conclude that there is a $T > 0$ and an $i_0 \geq 1$ such that

$$|Q_Y^o(y(T, y_0))(y(T, y_i) - y(T, y_0))| \geq C_1|P_Y^o(y(T, y_0))(y(T, y_i) - y(T, y_0))| \tag{4.3g}$$

for $i \geq i_0$. By combining (4.3f) and (4.3g), we find that

$$|y(T, y_i) - y(T, y_0)| \geq \frac{1}{2}|Q_Y^o(y(T, y_0))(y(T, y_i) - y(T, y_0))|, \qquad \text{for } i \geq i_0. \tag{4.3h}$$

From inequality (3.5c) in Lemma 3.0, there is an $i_1 \geq i_0$ such that

$$|y(t, y_i) - y(t, y_0)| \leq \epsilon, \quad \text{and} \quad |v(t, x_i) - v(t, x_0)| \leq \epsilon,$$

for $i \geq i_1$, and $0 \leq t \leq T$. Let $\bar{x}_i = v(T, y_i)$, and $\bar{y}_i = y(T, y_i)$, for $i \geq 1$, and set $\bar{x}_0 = v(T, y_0)$ and $\bar{y}_0 = y(T, y_0)$. Then $\bar{x}_i \in \mathcal{D}(\bar{x}_0)$ and one has $\bar{x}_i \to \bar{x}_0$, and $\bar{y}_i \to \bar{y}_0$, as $i \to \infty$, where $\bar{y}_i = h(\bar{x}_i)$. From inequality (3.16), we obtain

$$|\bar{y}_i - \bar{y}_0| = |h(\bar{x}_i) - h(\bar{x}_0)| \leq 2|\bar{x}_i - \bar{x}_0| \tag{4.3k}$$

since $\ell \leq \ell_1 \leq 1$.

By projecting the equality

$$\bar{y}_i - \bar{y}_0 = (\bar{y}_i - \bar{x}_i) - (\bar{y}_0 - \bar{x}_0) + (\bar{x}_i - \bar{x}_0).$$

into $U^o(\bar{x}_0)$ we obtain:

$$P^o(\bar{x}_0)(\bar{y}_i - \bar{y}_0) = P^o(\bar{x}_0)(\bar{y}_i - \bar{x}_i) - P^o(\bar{x}_0)(\bar{y}_0 - \bar{x}_0) + P^o(\bar{x}_0)(\bar{x}_i - \bar{x}_0). \tag{4.3n}$$

Since $(\bar{y}_0 - \bar{x}_0) \in U^s(\bar{x}_0) \oplus U^u(\bar{x}_0)$, one has

$$P^o(\bar{x}_0)(\bar{y}_0 - \bar{x}_0) = 0. \tag{4.3o}$$

Also the term $P^o(\bar{x}_0)(\bar{y}_i - \bar{x}_i)$ assumes the form

$$P^o(\bar{x}_0)(\bar{y}_i - \bar{x}_i) = P^o(\bar{x}_i)(\bar{y}_i - \bar{x}_i) + (P^o(\bar{x}_0) - P^o(\bar{x}_i))(\bar{y}_i - \bar{x}_i). \tag{4.3p}$$

Since $(\bar{y}_i - \bar{x}_i) \in U^s(\bar{x}_i) \oplus U^u(\bar{x}_i)$, one has $P^o(\bar{x}_i)(\bar{y}_i - \bar{x}_i) = 0$. From inequality (2.4d) we obtain

$$|(P^o(\bar{x}_i) - P^o(\bar{x}_0)(\bar{y}_i - \bar{x}_i)| \leq L|\bar{x}_i - \bar{x}_0|\,|\bar{y}_i - \bar{x}_i|.$$

By combining this inequality with (4.3p) and using Theorem 3.7 and the relation $|\bar{y}_i - \bar{x}_i| = |h(\bar{x}_i) - \bar{x}_i| \leq 2\epsilon$, one obtains

$$|P^o(\bar{x}_0)(\bar{y}_i - \bar{x}_i)| \leq 2L\epsilon|\bar{x}_i - \bar{x}_0|. \tag{4.3r}$$

Since the neutral space $U^o(\bar{x}_0)$ is tangent to the disk $\mathcal{D}(\bar{x}_0)$ at the point $\bar{x}_0$, there is a sequence ρ_i with $\rho_i \to 0$, as $i \to \infty$, and such that

$$|P^o(\bar{x}_0)(\bar{x}_i - \bar{x}_0)| \geq |\bar{x}_i - \bar{x}_0| - \rho_i|\bar{x}_i - \bar{x}_0|. \tag{4.3s}$$

By combining (4.3o), (4.3r), and (4.3s) with equation (4.3n) it follows that

$$|P^o(\bar{x}_0)(\bar{y}_i - \bar{y}_0)| \geq (1 - \rho_i - 2L\epsilon)|\bar{x}_i - \bar{x}_0|. \tag{4.3t}$$

Let us next estimate the norm $|P^o_Y(\bar{y}_0)(\bar{y}_i - \bar{y}_0)|$. First note that

$$P^o_Y(\bar{y}_0)(\bar{y}_i - \bar{y}_0) = P^o(\bar{x}_0)(\bar{y}_i - \bar{y}_0) + (P^o_Y(\bar{y}_0) - P^o(\bar{x}_0)(\bar{y}_i - \bar{y}_0). \tag{4.3u}$$

Since one has

$$\lim_{(\epsilon,\delta)\to(0,0)} P^o_Y(y_0) = P^o(x_0)$$

uniformly for $x_0 \in \mathcal{K}$, there is a $\beta_1 \in \Sigma$ such that

$$|(P_Y^o(\bar{y}_0) - P^o(\bar{x}_0))(\bar{y}_i - \bar{y}_0)| \leq \beta_1(\epsilon + \delta)|\bar{y}_i - \bar{y}_0|.$$

By combining this inequality with (4.3t) and (4.3u) we find that

$$|P_Y^o(\bar{y}_0)(\bar{y}_i - \bar{y}_0)| \geq (1 - \rho_i - 2L\epsilon)|\bar{x}_i - \bar{x}_0| - \beta_1(\epsilon + \delta)|\bar{y}_i - \bar{y}_0|. \tag{4.3x}$$

Now inequalities (4.3g) and (4.3h) imply that

$$|\bar{y}_i - \bar{y}_0| \geq \frac{C_1}{2}|P_Y^o(\bar{y}_0)(\bar{y}_i - \bar{y}_0)|.$$

By combining this with (4.3x) we conclude that

$$|\bar{y}_i - \bar{y}_0| \geq \frac{C_1(1 - \rho_i - 2L\epsilon)}{2}|\bar{x}_i - \bar{x}_0| - \frac{C_1\beta_2(\epsilon + \delta)}{2}|\bar{y}_i - \bar{y}_0|,$$

which implies that

$$|\bar{y}_i - \bar{y}_0| \geq \frac{C_1(1 - \rho_i - 2L\epsilon)}{2(1 + \frac{C_1\beta_2(\epsilon+\delta)}{2})}|\bar{x}_i - \bar{x}_0|.$$

Now choose ϵ_{10} so that $0 < \epsilon_{10} \leq \epsilon_9$, $L\epsilon_{10} \leq \frac{1}{12}$, and $\beta_1(2\epsilon_{10}) \leq \frac{2}{3C_1}$. Set $\delta_{10}(\epsilon) = \min(\epsilon, \delta_9(\epsilon), \bar{\delta})$, for $0 < \epsilon \leq \epsilon_{10}$, and choose $i_2 \geq i_1$ where $\rho_i \leq \frac{1}{6}$, for $i \geq i_2$. One can then show that

$$|\bar{y}_i - \bar{y}_0| \geq \frac{5}{2}|\bar{x}_i - \bar{x}_0|,$$

which contradicts inequality (4.3k). By setting $\delta_2 = \delta_{10}(\epsilon_{10})$, we thereby complete the proof of Theorem 4.3. □

5. The One-to-One Property

The continuous mapping $h : \mathcal{K} \to \mathcal{K}^Y$, which was constructed in the proof of Theoren 3.7 and chosen to satisfy the identity (4.3t), for all $t \geq 0$, is sometimes referred to as a **lifting** of (the dynamics $S^Y(t)$ on) $\mathcal{K}$ to (the dynamics $S_2(t)$ on) $\mathcal{K}^Y$. Why the term *lifting*? Since h may not be one-to-one, various orbits of $S^Y(t)$ on $\mathcal{K}$ may be pinched together (by h) and mapped onto a single orbit of $S_2(t)$ on $\mathcal{K}^Y$, see Sacker and Sell (1977). The identity (4.3t) insures that the any orbit of $S_2(t)$ on $\mathcal{K}^Y$ must be the image of full orbits of $S^Y(t)$ on $\mathcal{K}$. In this sense, the dynamics $S^Y(t)$ on $\mathcal{K}$ has greater complexity[2] than that of $S_2(t)$ on $\mathcal{K}^Y$.

Because of the inequality (4.3s), we see that if $\ell < \frac{1}{2}$, a condition which is necessarily satisfied for small ϵ because one has $\ell = \ell(\epsilon) \in \Sigma$, then the restriction of h to any disk $\mathcal{D}(x_0)$ in $\mathcal{K}$ is one-to-one. The object of this section is to show that for small ϵ, δ, and ℓ the mapping h is globally one-to-one.

As in Sections 3 and 4, we assume that the hypotheses of Theorem A are satisfied. We will make extensive use of the notation and concepts developed above. We begin by fixing ϵ_{10}, δ_{10}, and δ_2 as in the proof of Theorem 4.2. Let L be fixed so that inequalities (2.4d), (2.4e), and (2.7) are valid. In this section, we restrict the parameters to satisfy $0 < \epsilon \leq \epsilon_{10}$, $\delta \leq \delta_{10}$, and $\ell \leq \ell_9(\epsilon)$.

[2] A good llustration of this feature is that of an almost automorphic extension of an almost periodic minimal set, see Bochner (1962, 1975), Johnson (1980), and Shen and Yi (1993), for example.

Lemma 5.1. *There is a $\rho > 0$ and $T_0 > 0$ with the property that if $x_0, x_1 \in \mathcal{K}$ satisfy $x_1 \neq x_0$ and*

$$|P^o(x_0)(x_1 - x_0)| \le \frac{1}{2}|x_1 - x_0|, \tag{5.1}$$

then the following statements are valid:

(1) *If one has $|x(t, x_1) - x(t, x_0)| \le \rho$, for $0 \le t \le T_0$, and*

$$|P^u(x_0)(x_1 - x_0)| \ge |P^s(x_0)(x_1 - x_0)|, \tag{5.1a}$$

then the following two inequalities are valid for $\tau = T_0$:

$$|x(\tau, x_1) - x(\tau, x_0)| \ge \frac{12}{\sin \frac{\alpha}{2}}|x_1 - x_0| \tag{5.1b}$$

and

$$\angle(U^n(x(\tau, x_0)), (x(\tau, x_1) - x(\tau, x_0))) \ge \frac{3\alpha}{4}. \tag{5.1c}$$

(2) *If instead, one has $|x(t, x_1) - x(t, x_0)| \le \rho$, for $-T_0 \le t \le 0$, and*

$$|P^s(x_0)(x_1 - x_0)| \ge |P^u(x_0)(x_1 - x_0)|, \tag{5.1d}$$

then inequalities (5.1b) and (5.1c) are valid for $\tau = -T_0$.

Proof. First we fix $\rho > 0$ so that $|x_1 - x_0| \le \rho$. Now inequalities (2.7c) and (5.1) imply that

$$|Q^o(x_0)(x_1 - x_0)| \ge |x_1 - x_0| - |P^o(x_0)(x_1 - x_0)| \ge \frac{1}{2}|x_1 - x_0|,$$

and that $x_1 \notin \mathcal{D}(x_0)$. Therefore $|Q^o(x_0)(x_1 - x_0)| > 0$. The basic idea of the proof is that, for ρ sufficiently small, the norm $|x(t, x_1) - x(t, x_0)|$ behaves very much like the norm of the linear problem $|\Phi(t, x_0)(x_1 - x_0)|$. Since $|Q^o(x_0)(x_1 - x_0)| > 0$, either (5.1a) or (5.1d) is valid and the left side of at least one of these inequalities is nonzero. The existence of a $T_0 > 0$ and a sufficiently small $\rho > 0$ satisfying the conditions above, now follows from the weak hyperbolicity of $\mathcal{K}$. □

By replacing the radius r of the disks $\mathcal{D}(x_0)$ with a smaller value, if necessary, there is no loss in generality in assuming that

$$r \le \min\left\{\frac{\ln 2}{L}, \frac{1}{2L}\sin\frac{\alpha}{2}, \frac{\rho}{2}\right\}, \tag{5.1f}$$

where L is given by inequalitites (2.4d), (2.4e), and (2.7), and α satisfies (2.4) in addition to

$$\angle(U^n(x_0), U^n(x_1)) \le \frac{\alpha}{4}, \qquad \text{for } x_0 \in \mathcal{K} \text{ and } x_1 \in \mathcal{D}(x_0). \tag{5.2}$$

It follows from the continuity of solutions of (1.1), that there exists a $\mu \in (0, \frac{1}{3})$, such that $N(x(t, x_0), \mu r) \subset x(t, N(x_0, r))$ for all $-T_0 \le t \le T_0$ and all $x_0 \in \mathcal{K}$.

Assume for the moment that the following lemma is valid. We will then use this lemma to prove Part (3) of Theorem A.

Lemma 5.2. *There exists a δ_3, with $0 < \delta_3 \le \delta_2$, such that if $\|Y\|_{C^1(\Omega)} \le \delta_3$, and if there are two points x_0 and x_1 in $\mathcal{K}$ with $h(x_0) = h(x_1)$, then there exists two points x_0^1 and x_1^1 in $\mathcal{K}$ that satisfy $h(x_0^1) = h(x_1^1)$ and*

$$|x_1^1 - x_0^1| \ge 2|x_1 - x_0|. \tag{5.2a}$$

Proof of Theorem A, Part (3). We apply Lemma 5.2 repeatedly to obtain sequences x_0^m and x_1^m in $\mathcal{K}$ with $h(x_0^m) = h(x_1^m)$ and

$$|x_1^m - x_0^m| \ge 2^m |x_1 - x_0|, \qquad \text{for } m = 1, 2, \cdots.$$

Since $\mathcal{K}$ is a bounded set, the last inequality implies that $x_0 = x_1$. Hence the mapping $h : \mathcal{K} \to \mathcal{K}^Y$ is one-to-one. $\square$

It remains only to verify Lemma 5.2.

Proof of Lemma 5.2. From Lemma 3.0, there is a constant $C_1 > 0$ such that

$$|x(t, x_0) - y(t, y_0)| \le C_1(\epsilon + \delta), \qquad \text{for } x_0 \in \mathcal{K} \text{ and } y_0 \text{ with } |x_0 - y_0| \le 2\epsilon.$$

Now choose ϵ_{11} so that

$$\epsilon_{11} \le \min\left(\frac{1}{4L}, \frac{\mu r}{6C_1}, \epsilon_{10}\right) \tag{5.2d}$$

and set $\delta_{11}(\epsilon) = \min(\epsilon, \delta_{10}(\epsilon))$, for $0 < \epsilon \le \epsilon_{11}$. Set $\delta_3 = \delta_{11}(\epsilon_{11})$. One then has $\delta_3 \le \delta_2$. Also, for $\epsilon \le \epsilon_{11}$ and $\delta \le \delta_{11}(\epsilon)$, one has

$$|x(t, x_0) - y(t, y_0)| \le \frac{\mu r}{3} \qquad \text{for } -T_0 \le t \le T_0, \tag{5.3b}$$

and for all $x_0 \in \mathcal{K}$ and y_0 with $|x_0 - y_0| \le 2\epsilon$.

Let $x_0 \in \mathcal{K}$ and $x_1 \in \mathcal{K}$ be two points with $x_0 \ne x_1$ and $h(x_0) = h(x_1) = y_0$. Since Part (1) of Theorem A implies that $|y_0 - x_0| \le 2\epsilon$ and $|y_0 - x_1| \le 2\epsilon$, one has $|x_0 - x_1| \le 4\epsilon$. We claim that inequality (5.1) is satisfied. In order to prove this, let us first estimate the norm $|P^o(x_0)(x_1 - x_0)|$. From the definition of h it follows that $h(x_0) - x_0 = y_0 - x_0 \in U^s(x_0) \oplus U^u(x_0)$ and $y_0 - x_1 \in U^s(x_1) \oplus U^u(x_1)$. Therefore one has $P^o(x_0)(y_0 - x_0) = P^o(x_1)(y_0 - x_1) = 0$. Consequently, we have

$$P^o(x_0)(x_1 - x_0) = P^o(x_0)(y_0 - x_0) + (P^o(x_1) - P^o(x_0))(y_0 - x_1).$$

Since P_o is Lipschitz continuous <u>on</u> $\mathcal{K}$, it follows from inequality (2.4d) that one has

$$|P^o(x_0)(x_1 - x_0)| \le L|x_1 - x_0|\,|x_1 - y_0|.$$

Since $|x_1 - y_0| \le 2\epsilon$, it follows from (5.2d) that

$$|P^o(x_0)(x_1 - x_0)| \le 2L\epsilon|x_1 - x_0| \le \frac{1}{2}|x_1 - x_0|.$$

Let us return to the local coordinates given by equation (3.0a). Since $\mu < \frac{1}{3}$, it follows from inequality (5.3b) that the solution $y(t, y_0)$ of the perturbed equation (1.2) satisfies

$$y(t, y_0) \in \mathcal{D}(x(t, x_i)), \qquad \text{for } i = 0, 1 \text{ and } -T_0 \le t \le T_0.$$

Consequently, the representation given by equation (3.0a) will depend on the base point x_i, for $i = 0, 1$. As a result, we will write $y(t, y_0)$ in the form

$$y(t, y_0) = v_0(t, y_0) + n_0(t, y_0) = v_1(t, y_0) + n_1(t, y_0), \qquad \text{for } -T_0 \le t \le T_0.$$

where the $i = 0$ decomposition depends on the base point x_0, and the $i = 1$ decomposition depends on x_1. Because of equation (4.3t) one has

$$h(v_0(t, y_0)) = h(v_1(t, y_0)) = y(t, y_0), \qquad \text{for } -T_0 \le t \le T_0.$$

For $x \in \mathcal{K}$, let $d(x)$ be the k-dimensional disk with the center at x, radius μr and such that $d(x) \subset \mathcal{D}(x)$. Using the facts that the mapping h is continuous, $v_i(t, y_0) \in \mathcal{D}(x(t, x_i))$, for $i = 0, 1$, and $-T_0 \le t \le T_0$, together with (5.3b), one obtains

$$v_i(t, y_0) \in d(x(t, x_i)), \qquad \text{for } i = 0, 1 \text{ and } -T_0 \le t \le T_0. \tag{5.3i}$$

Let us now turn to Part(1) of the lemma, where inequality (5.1a) is satisfied. For $i = 0, 1$ we define $x_i = x(T_0, x_i)$ and $x_i^1 = v_i(T_0, y_0)$. From (5.3i) it follows that $x_i^1 \in d(x_i)$, and therefore $x_i \in d(x_i^1) \subset \mathcal{D}(x_i^1)$, for $i = 0, 1$. Let us transform the origin into the point x_0^1 and rotate the coordinate axis to obtain the new coordinate system (ξ, η) where ξ is a k-dimensional vector and η is an $(n-k)$-dimensional vector. We assume that in the new coordinate system the space $\eta = 0$ coincides with the linear space $U^n(x_0^1)$. Furthermore, as argued in the paragraph preceeding Lemma 2.3, the two disks $\mathcal{D}(x_0^1)$ and $\mathcal{D}(x_1^1)$ can be represented in the form

$$\mathcal{D}(x_i^1) = \{\eta = f_i(\xi) : |\xi| \le r\}, \qquad i = 0, 1,$$

where f_i is an $(n-k)$-dimensional vector-valued function of class $C^{1,1}$, for $i = 0, 1$, and $f_0(0) = 0$. In this coordinate system the points x_i have representations $(\xi_i, f_i(\xi_i))$, for $i = 0, 1$. Let $\bar{x}_0 = (\xi_1, f_0(\xi_1))$. Consider the three vectors $z_0 = x_0 - \bar{x}_0$, $z_1 = x_1 - x_0$, and $z_2 = x_1 - \bar{x}_0$. It is clear that $z_2 = z_0 + z_1$.

Let P be projection of R^n onto the space $\{(0, \eta)\}$ with null space $\{(\xi, 0)\}$. We then have $Pz_2 = Pz_0 + Pz_1$ and

$$|Pz_2| \ge |Pz_1| - |Pz_0|. \tag{5.4a}$$

From (5.1c) it follows that $\angle(U^o(x_0), z_1) \ge \frac{3\alpha}{4}$. By using the additivity of the angles and (5.2) we find that $\angle(U^o(x_0), U^o(x_0^1)) \le \frac{\alpha}{4}$, and consequently, $\angle(U^o(x_0^1), z_1) \ge \frac{\alpha}{2}$. From this inequality it follows that

$$|Pz_1| \ge |z_1| \sin \frac{\alpha}{2}. \tag{5.4b}$$

Since $Pz_0 = (0, f_0(\xi_0) - f_0(\xi_1))$, it follows from inequality (2.7) that we get

$$|Pz_0| = |f_0(\xi_0) - f_0(\xi_1)| \leq Lr|\xi_0 - \xi_1| \leq Lr|z_1|.$$

From this inequality, (5.4a), and (5.4b), it follows that

$$|Pz_2| \geq \left(\sin\frac{\alpha}{2} - Lr\right)|z_1|,$$

and consequently, (5.1f) implies that

$$|Pz_2| \geq \frac{1}{2}\sin\frac{\alpha}{2}|z_1|$$

which can be rewritten in the form

$$|f_1(\xi_1) - f_0(\xi_1)| \geq \frac{1}{2}\sin\frac{\alpha}{2}|x(T_0, x_1) - x(T_0, x_0)|.$$

By using (5.1b), we then find that

(5.4g) $$|f_1(\xi_1) - f_0(\xi_1)| \geq 6|x_1 - x_0|.$$

Let $w(s) = f_1((1-s)\xi_1) - f_0((1-s)\xi_1), 0 \leq s \leq 1$. Then by the argument of Lemma 2.3, one has

$$\left|\frac{d|w|}{ds}\right| \leq L|w(s)||\xi_1| \leq Lr|w(s)|,$$

since $|\xi_1| \leq r$. By using (5.1f) we then have

$$\frac{d|w|}{ds} \geq -\ln 2\,|w|,$$

and the Gronwall inequality implies that $2|w(1)| \geq |w(0)|$, or equivalently,

$$|f_1(0) - f_0(0)| \geq \frac{1}{2}|f_1(\xi_1) - f_0(\xi_1)|.$$

Combining this with (5.4g) and using the fact that $f_0(0) = 0$, we obtain

(5.4k) $$|f_1(0)| \geq 3|x_1 - x_0|.$$

In the (ξ, η) coordinates one has $x_0^1 = (0, 0)$, and for some ξ_2 with $|\xi_2| \leq \frac{r}{3}$, one has $x_1^1 = (\xi_2, f_1(\xi_2))$. Therefore, one has

(5.4m) $$|x_1^1 - x_0^1|^2 = |\xi_2|^2 + |f_1(\xi_2)|^2.$$

Since $|\xi_2| \leq \frac{r}{3}$, inequality (2.7) implies that

$$|f_1(\xi_2)| \geq |f_1(0)| - |f_1(\xi_2) - f_1(0)| \geq |f_1(0)| - \frac{Lr}{3}|\xi_2|.$$

As a result, equation (5.4m) implies that

$$|x_1^1 - x_0^1|^2 \geq |\xi_2|^2 + (|f_1(0)| - \frac{Lr}{3}|\xi_2|)^2.$$

By expanding and using the Young inequality $2ab \leq \frac{9}{5}a^2 + \frac{5}{9}b^2$, we get

$$|x_1^1 - x_0^1|^2 \geq (1 - \frac{4}{45}L^2r^2)|\xi_2|^2 + \frac{4}{9}|f_1(0)|^2,$$

which together with (5.1f) and (5.4k), implies that

$$|x_1^1 - x_0^1| \geq \frac{2}{3}|f_1(0)| \geq 2|x_1 - x_0|.$$

This completes the proof of Part (1) of the lemma. The argument for Part (2) is similar. In this case the new points x_i^1 assume the form $x_i^1 = v_i(-T_0, y_0)$, for $i = 0, 1$. □

6. Conclusions

In this section we discuss some applications of the theory given above, and we include an introduction of issues of Approximation Dynamics to the Bubnov-Galerkin method as it is used in numerical analysis.

Applications: As noted above, the theory developed here is a generalization of the theory of hyperbolic attractors, see Pliss and Sell (1991). The examples contained in our earlier paper are examples of weakly, normally hyperbolic sets, as well. The newer theory also applies to such problems as small quasiperiodic forcings of conservative systems with homoclinic orbits, see Meyer and Sell (1989). For example, it is shown in the latter reference that, for small positive values of η, and for suitable functions f and g, the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_1 - x_1^3 + \eta f(x, \theta, \eta) \\ \theta' &= \omega + \eta g(x, \theta, \eta), \end{aligned}$$

where $x = (x_1, x_2) \in R^2$, θ, $\omega \in T^k$, and T^k is the the k-dimensional torus, has a compact invariant set $\mathcal{K}$ (a **Poincaré-Melnikov invariant** set) in $R^2 \times T^2$ and that $\mathcal{K}$ satisfies the defining properties for a weakly, normally hyperbolic set with the Lipschitz property. Moreover, one has $\operatorname{index}(\mathcal{K}) = k + 1$. Consequently, Theorems A and B are applicable to the perturbed problem

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_1 - x_1^3 + \eta f(x, \theta, \eta) + Y(x, \theta, \eta) \\ \theta' &= \omega + \eta g(x, \theta, \eta) + \Theta(x, \theta, \eta). \end{aligned}$$

Coupled Systems of Weakly, Normally Hyperbolic Sets: As noted above, the concept of a weakly, normally hyperbolic set is closed under finite set products. For example, if $\mathcal{K}_1$ is a Poincaré-Melnikov invariant set for $x_1' = X_1(x_1)$ and $\mathcal{K}_2 = T^p$ is a normally hyperbolic torus arising from a quasi periodic solution of $x_2' = X_2(x_2)$, then $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ is a weakly, normally hyperbolic set for the product system $x' = X(x)$, where $x = (x_1, x_2)$ and $X = (X_1, X_2)$. The perturbed equation then allows for dynamical coupling between the two sets $\mathcal{K}_1$ and $\mathcal{K}_2$.

Bubnov-Galerkin Approximations: The use of these approximations is a standard methodology arising in the numerical study of solutions of partial differential equations. These approximations offer one method used for converting a system of partial differential equations (PDEs) into a finite dimensional system of ordinary differential equations (ODEs), and they are directly related to the spatial discretization of the PDEs. It is convenient to make note of some of the dynamical features of the infinite dimensional problem arising in the Bubnov-Galerkin scheme. For this purpose, we present a brief discussion of theory of nonlinear evolutionary equations as they are used in the study of systems of reaction diffusion equations, see Hale (1988), Henry (1981, 1985), Pazy (1983), Sell and You (1996), and Temam (1988) for more details.

The nonlinear evolutionary equation of interest here is given by

$$\partial_t u + Au = F(u), \qquad \text{for } u \in H, \tag{6.0}$$

where H is a Hilbert space.[3] We will assume here that A is a positive definite, selfadjoint linear operator on H with a compact inverse A^{-1}. The spectral theory then implies that A has a countable sequence of eigenvalues

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \le \mu_{n+1} \le \cdots \le \mu_m \le \cdots ,$$

and corresponding eigenfunctions $e_1, e_2, \cdots$ form an orthonormal basis for H.

We will assume that the nonlinear term $F : H \to H$ is a continuously Gateaux differentiable function and that there are constants K_0 and K_1 such that F satisfies[4]

$$\|F(u)\| \le K_0, \qquad \text{for all } u \in H \tag{6.1c}$$

and

$$\|F(u_1) - F(u_2)\| \le K_1 \|u_1 - u_2\|, \qquad \text{for all } u_1,\, u_2 \in H. \tag{6.1d}$$

Let R denote the orthogonal projection of H onto the m dimensional space $\text{Span}(e_1, \cdots, e_m) \sim R^m$. Then the Bubnov-Galerkin approximation (of order m) is the ODE on R^m given by

$$\partial_t r + Ar = RF(r), \qquad \text{for } r \in R^m. \tag{6.2}$$

(Note that R satisfies the commutivity relationship $RA = AR$.) We will treat the dimension m as fixed and finite, but very large. Next we let P and Q denote the orthogonal projections of R^m onto the spaces $\text{Span}(e_1, \cdots, e_n) \sim R^n$ and $\text{Span}(e_{n+1}, \cdots, e_m)$, respectively. One then has $R = P + Q$ and $r = p + q$, where $p = Pr$ and $q = Qr$. The ODE (6.2) can be written in the equivalent form

$$\begin{aligned} \partial_t p + Ap &= PF(p+q) \\ \partial_t q + Aq &= QF(p+q). \end{aligned} \tag{6.3}$$

Note that the Bubnov-Galerkin approximation of order n is the ODE

$$\partial_t p + Ap = PF(p), \qquad \text{for } p \in R^n, \tag{6.3a}$$

which is formed by setting $q = 0$ in the p-equation in (6.3) and ignoring the q-equation.

Our next objective is to compare the longtime dynamics of the solutions of the two ODEs (6.3) and (6.3a). The key to this comparison is to view equation (6.3a) as an approximation to equation (6.3). Even though, these two equations are defined on different spaces: R^n and R^m, where $m >> n$, equation (6.3a) can be lifted to the larger space R^m, by using the system

$$\begin{aligned} p_t + Ap &= PF(p+q) \\ q_t + Aq &= QF(q) - QF(0). \end{aligned} \tag{6.4}$$

[3] Typically the space H is a Sobolev space, such as $H_0^1(\Omega)$.

[4] These inequalities are normally too restrictive for typical PDE problems. However by using a standard ODE idea, one can modify the PDE "near infinity" so that these inequalities are valid, see Foias, Sell, and Temam (1988), Mallet-Paret and Sell (1988), and Pliss and Sell (1993), for example.

An alternate version replaces the q-equation in (6.4) with $q_t + Aq = 0$. In either case, the hyperplane $q = 0$ is an invariant set for equation (6.4). This is a triangular system in the sense used in Sacker and Sell (1976b). The perturbation term is $Y = \pm[QF(Pw) - QF(0)]$.

Equation (6.3) can be viewed as the given equation (1.1) and equation (6.4) is the perturbed equation (1.2), or vice versa. From the point-of-view of applying the theory presented above, it does not matter which one of these two equations plays the role of the given equation (1.1). However, there are other mathematical issues which arise in the Bubnov-Galerkin scheme, and these issues result in differences in the resulting theories.

If we consider $\mathcal{K}$ to be a weakly, normally hyperbolic set for the fine grid model given by equation (6.3), and we ask whether equation (6.4) has a related compact, invariant set $\mathcal{K}^Y$, as given in Theorem A, then we wish to know whether *essentially* the same dynamics as in $\mathcal{K}$ can be found by studying the coarse grid model given by equation (6.3a). A related question is to determine the dimension n in order that the conclusions of Theorem A hold.

On the other hand, if we consider $\mathcal{K}$ to be a weakly, normally hyperbolic set for the coarse grid model given by equation (6.4), then our interest in the set $\mathcal{K}^Y$ in R^m is motivated by the question of whether the coarse grid model gives any useful information about the fine grid structures. This then becomes an issue of subgrid scale modeling. For example, Theorem A establishes the existence of a mapping $h : \mathcal{K}_0 \to R^m$, where $\mathcal{K}_0$ is defined below and $\mathcal{K}^Y = h(\mathcal{K}_0)$. When Theorem A is applicable, the q-component of h is given by $Qh(x_0)$, for $x_0 \in \mathcal{K}_0$. We see then that the q-component is a slave variable in this problem. By replacing q with $Qh(x_0)$, one achieves a desired subgrid scale modeling.[5]

We next prove the following result, which gives a necessary condition for equation (6.4) to be a good model for (6.3a).

Lemma 6.1. *Let $\mathcal{K}$ be a weakly, normally hyperbolic set for equation (6.3a) with characteristics a and λ_i, for $0 \le i \le 4$. Then $\mathcal{K}_0 \overset{\text{def}}{=} \{(p,q) : p \in \mathcal{K} \text{ and } q = 0\}$ is a compact, invariant set for (6.4). If in addition, one has*

$$\mu_{n+1} > K_1 - \lambda_1 > 0, \tag{6.5}$$

then $\mathcal{K}_0$ is a weakly, normally hyperbolic set for equation (6.4), with the same characteristics and the same index as for $\mathcal{K}$.

Proof. First note that $\mathcal{K}_0$ is an compact, invariant set for equation (6.4). By using the Variation of Constants Formula on the q-equation in (6.4), one obtains

$$q(t) = e^{-AQt} q_0 + \int_0^t e^{-AQ(t-s)} [QDF(q) - QF(0)]\, ds, \qquad \text{for } t \ge 0.$$

By using inequality (6.1d) and some properties of selfadjoint operators, one finds that

$$\|q(t)\| \le e^{-\mu_{n+1} t} \|q_0\| + K_1 \int_0^t e^{-\mu_{n+1}(t-s)} \|q(s)\|\, ds, \qquad \text{for } t \ge 0.$$

[5] A similar situation arises in the study of inertial manifolds, see Foias, Sell, and Temam (1988), and Mallet-Paret and Sell (1988).

By using the Gronwall inequality and inequality (6.5), one obtains

$$\|q(t)\| \leq e^{-(\mu_{n+1}-K_1)t}\|q_0\| \leq \|q_0\|e^{\lambda_1 t}, \qquad \text{for } t \geq 0.$$

Since $a \geq 1$, this completes the proof. □

In order to apply Theorems A and B in this setting, one needs to estimate the C^1-norm of the perturbation term Y. Due to the assumptions on the nonlinearity F, one has $\|Y\|_{C^1} \leq \max(2K_0, K_1)$, where K_0 and K_1 are given by (6.1c) and (6.1d). If these numbers are small, then Theorems A and B apply immediately. Unfortunately, in most problems of interest, and especially in problems where one lets $m \to \infty$ and where equation (6.2) is replaced by the infinite dimensional problem (6.0), these numbers need not be small. One needs to derive an extension of Theorems A and B to cover this situation. This extension, which takes advantage of the fact that the nonlinear term $F(u)$ is suitably dominated by the linear term Au, is developed in Pliss and Sell (1997).

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