High order finite-difference approximations of the wave equation with absorbing boundary conditions: a stability analysis

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Abstract

This paper deals with the stability of finite difference approximations of initial value problems for the wave equation with absorbing boundary conditions. The stability of a family of high order variational numerical schemes is studied by energy techniques. Dirichlet, sponge and first order paraxial absorbing boundary conditions are treated. The variational form of the schemes as well as the use of the image principle are essential for the discrete energy estimates. With these estimates conditional stability is shown to be equivalent to the positivity of the kinetic energy operator. The main result of the paper is to show how to couple the discretization of the equation in the interior to the discretization of the boundary condition for a particular class of schemes.

Key Words. Hyperbolic Equations, Finite Difference Schemes, Energy Estimates, Stability

AMS(MOS) subject classifications. 65M06, 65M12

1 Introduction

The stability of a numerical approximation is an essential question. Indeed for linear problems, under the assumption of consistency it guarantees convergence. Therefore stability analysis has been extensively studied (cf [15, 17, 14]). For equations in unbounded domains with constant coefficients the analysis is relatively easy. In this case the well known Von Neumann analysis (cf [15]) gives a necessary stability condition. But stability analysis is difficult for initial boundary value problems with varying or even non smooth coefficients (cf [17]). The stability theory of Kreiss (cf [14]) is available but it is very cumbersome when applicable. Another tool to tackle stability analysis of initial boundary value problems of hyperbolic equations is energy or apriori estimates. Recently Ha-Duong and Joly (cf [10]) have used this technique to show the well-posedness of initial boundary value problems for the wave equation. These estimates have also been used to prove existence results (cf [7, 9, 8]) and to study stability properties of numerical approximation of parabolic and hyperbolic equations (cf [16, 18]).
But these results did not address high order approximations of initial boundary value problems and in particular did not deal with the coupling of high order interior scheme with the boundary

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conditions. In applications, however it is desirable to use high order schemes to minimize the numerical dispersion (cf [1, 19, 20]). Such schemes make the treatment of boundary conditions complicated and the stability analysis difficult.
We propose in this paper to investigate the stability properties of high order variational finite difference schemes for the wave equation with varying coefficients and Dirichlet or absorbing boundary conditions. A class of high order schemes is introduced and analysed. These schemes enjoy the property of implicitly giving a discrete weak formulation. This characteristic feature is very important in dealing with differential absorbing boundary conditions. In fact the use of symmetry in the discrete Green's formula given by these schemes turns out to be essential for the stability analysis. Variational finite difference schemes are based on an approximation of the first order derivative. An approximation of the second order derivatives is obtained by composing this approximation with its adjoint.
The paper is organized as follows. In section 2 the equivalence theorem is recalled in a form well suited for our problem. Section 3 is devoted to the detailed study of stability for the wave equation with Dirichlet boundary conditions. In this section a number of lemmas concerning energy estimates are used. Their proofs is in the appendix. They are used throughout the paper. Section 4 deals with the sponge boundary condition and section 5 with the first order absorbing boundary condition. In each of the preceeding sections the continuous energy estimate is recalled and then the discrete problem and the stability analysis are treated. Section 6 ends the paper with discussions, conclusion and open questions.

2 The Equivalence Theorem

Consider two Hilbert spaces X and Y provided respectively with the norms $||\ ||_X$ and $||\ ||_Y$. Given $y \in Y$ we want to solve the linear equation:

$$Find\ \ x \in X \ \ / \ \ Lx = y \tag{1}$$

where $L \in \mathcal{L}(X,Y)$ is a linear operator from X to Y.
L is assumed to have a continuous inverse $L^{-1} \in \mathcal{L}(Y,X)$. Let (X_n) be a dense sequence of subspaces of X and (Y_n) be a dense sequence of subspaces of Y, so:

$$\overline{\bigcup_{n=1}^{+\infty} X_n} = X \qquad\qquad \overline{\bigcup_{n=1}^{+\infty} Y_n} = Y \tag{2}$$

The projection operator from X onto X_n (resp. Y onto Y_n) is denoted by P_X^n (resp. P_Y^n). Consider a sequence of operators $L_n \in \mathcal{L}(X_n, Y_n)$ and the problem. Given $y_n = P_Y^n y$

$$Find\ \ x_n \in X_n \ \ / \ \ L_n x_n = y_n \tag{3}$$

The sequence of operators L_n is consistent with the operator L if and only if (cf [13]):

$$\forall \varepsilon > 0, \quad \exists N \geq 0, \quad \forall n \geq N \quad \forall x \in X \quad ||P_Y^n L x - L_n P_X^n x||_Y \ \leq\ \varepsilon ||x||_X \tag{4}$$

We are going to show that the sequence of solutions (x_n) of problem (3) converges to the solution x of (1) under the hypothesis of stability for the operators L_n.
The sequence of operators (L_n) is stable if and only if :

i) - $S_n = L_n^{-1}$ exists for all integer n

ii) - $\exists K > 0\ \ , \forall n \geq 0\ \ ||S_n|| = ||L_n^{-1}||_{\mathcal{L}(Y,X)} \leq K$

Remark: The constant K is *independent* of n.
Let us prove first that stability implies convergence. We have:

$$\begin{aligned} ||P_X^n x - x_n||_X &= ||L_n^{-1}(L_n P_X^n x - L_n x_n)||_X \\ &\leq ||L_n^{-1}||_{\mathcal{L}(Y,X)} ||L_n P_X^n x - L_n x_n||_Y \\ &\leq K(||L_n P_X^n x - P_Y^n Lx||_Y + ||P_Y^n Lx - L_n x_n||_Y) \end{aligned}$$

But $P_Y^n Lx = P_Y^n y = y_n = L_n x_n$, so:

$$||P_X^n x - x_n||_X \leq K||L_n P_X^n x - P_Y^n Lx||_Y$$

by consistency we have:

$$\forall \varepsilon > 0, \ \exists N \geq 0, \ \forall n \geq N \ \ ||P_Y^n Lx - L_n P_X^n x||_Y \leq \frac{\varepsilon}{2K}$$

Furthermore since (X_n) is dense in X we know that:

$$\forall \varepsilon > 0, \ \exists N \geq 0, \ \forall n \geq N \ \forall x \in X \ \ ||P_X^n x - x||_X \leq \frac{\varepsilon}{2}$$

The convergence of (x_n) to x is easily proven, since:

$$||x - x_n||_X \leq ||x - P_X^n x||_X + ||P_X^n x - x_n||_X \leq \varepsilon$$

Let us prove now that convergence imply stability. This is an application of Banach-Steinhaus theorem (cf [5]):

Theorem 1 *Let (T_n) be a sequence of bounded linear operator from Y to X such that:*
$\forall v \in Y \quad \exists\, C > 0 \quad \sup_n ||T_n v||_Y \leq C$ *then* $\exists\, K > 0 \ / \ ||T_n||_{\mathcal{L}(Y,X)} \leq K$

In our case we have $T_n = S_n o P_Y^n$. Obviously $T_n \in \mathcal{L}(Y, X)$ and for all $y \in Y$ we have:

$$T_n y = S_n(P_Y^n y) = S_n y_n = L_n^{-1} y_n = x_n$$

Assuming that the sequence (x_n) is convergent in X, it is bounded; so

$$\forall y \in Y \quad \exists C > 0 \quad \exists N \geq 0, \quad \forall n \geq N \quad ||T_n y||_X \leq C$$

By Banach-Steinhaus theorem, $\exists \ K > 0 \ \forall n \geq 1 \ \ ||T_n||_{\mathcal{L}(Y,X)} \leq K$
Since P_Y^n is a projection we have:

$$||S_n|| = \sup_{y_n \in Y_n} \frac{||S_n y_n||_X}{||y_n||_Y} = \sup_{y_n \in Y_n} \frac{||S_n P_Y^n y_n||_X}{||y_n||_Y} = \sup_{y_n \in Y_n} \frac{||T_n y_n||_X}{||y_n||_Y} \leq \sup_{y \in Y} \frac{||T_n y||_X}{||y||_Y} \leq K$$

Therefore the sequence of operators $(S_n) = (L_n^{-1})$ is uniformly bounded which implies that the scheme is stable. The operator S_n is called the solution operator.

The stability analysis rests on the uniform boundedness of the solution operator S_n. This uniform boundedness depends on the space X and Y used as well as the norm chosen on these spaces. A convenient choice will turn out to be the energy spaces and energy norms as defined below.

3 Dirichlet Boundary Conditions

In this section we tackle the stability analysis of high order variational schemes for the simplest mixed boundary value problem, the Dirichlet problem for the wave equation.

3.1 Classical results on the continuous problem

Consider the one dimensional wave equation in the interval $\Omega =]0,1[$. The medium where the wave propagates is defined by its bulk modulus $K(x)$ and its density $\rho(x)$. We assume that the coefficients K and ρ are such that:

$$\left\{ \begin{array}{lll} K \in L^2(\Omega) & \exists\ K_m, K_M > 0 & K_m \leq K(x) \leq K_M \\ \rho \in L^2(\Omega) & \exists\ \rho_m, \rho_M > 0 & \rho_m \leq \rho(x) \leq \rho_M \end{array} \right. \tag{5}$$

Given initial conditions $u_0 \in H^1_0(\Omega)$ and $v_0 \in L^2(\Omega)$, we look for $u \in C^0(0,T;H^1_0(\Omega)) \bigcap C^1(0,T;L^2(\Omega))$ (cf [8]) solution of:

$$\left\{ \begin{array}{l} \dfrac{1}{K(x)}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho(x)}\dfrac{\partial u}{\partial x}\right) = 0 \qquad x \in \Omega \qquad t > 0 \\ u(0,t) = u(1,t) = 0 \qquad t > 0 \\ u(x,0) = u_0(x) \qquad \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \qquad x \in \Omega \end{array} \right. \tag{6}$$

The solution operator S defined by :

$$\begin{array}{rcl} S : H^1_0(\Omega) \times L^2(\Omega) & \longrightarrow & C^0(0,T;H^1_0(\Omega)) \bigcap C^1(0,T;L^2(\Omega)) \\ (u_0, v_0) & \longmapsto & u(x,t) \quad \textit{solution of } (6) \end{array}$$

is linear continuous. The linearity of S is obvious from the linearity of the wave equation and the continuity comes from the following a priori estimate, usually called energy estimate:

Lemma 1 *The quantity* $E(t) = \displaystyle\int_0^1 \frac{1}{K(x)}(\frac{\partial u}{\partial t})^2 + \frac{1}{\rho(x)}(\frac{\partial u}{\partial x})^2 \, dx$ *verifies:* $\dfrac{dE}{dt}(t) = 0$

Proof: Multiplication by u_t and integration by parts in space gives the result with the use of boundary conditions. (cf. [8])

With the assumption (5) the norm:

$$||u||^2_{1,\Omega} = \int_0^1 \frac{1}{\rho}|\nabla u|^2 \, dx$$

defines a norm equivalent to the usual norm on $H^1_0(\Omega)$ and the norm

$$||u||^2_{0,K} = \int_0^1 \frac{u^2}{K} \, dx$$

defines a norm equivalent to the usual norm on $L^2(\Omega)$.
The energy $E(t)$ can be written as $E(t) = \frac{1}{2}(||u_t||^2_{0,K} + ||u||^2_{1,\Omega})$.
Let $W = C^0(0,T;H^1_0(\Omega)) \bigcap C^1(0,T;L^2(\Omega))$ be provided with the following norm:

$$||u||^2_W \;=\; \sup_{0\le t\le T}\left(||u||^2_{1,\Omega} + ||\frac{\partial u}{\partial t}||^2_{0,K}\right) \;=\; \sup_{0\le t\le T} E(t)$$

The energy estimate shows that $E(t) = E(0)$, so

$$||u||^2_W \;=\; \sup_{0\le t\le T} E(t) = E(0) \le (||u_0||^2_{1,\Omega} + ||v_0||^2_{0,K})$$

So

$$||S(u_0,v_0)||^2_W \;\le\; ||(u_0,v_0)||^2_{H^1_0(\Omega)\times L^2(\Omega)}$$

Thus the continuity of S. Incidently note that S is a contraction, which means that the system is dissipating. This estimate justifies the choice of spaces and norms above. The main goal of this paper is to extend this "dissipation" result to the numerical approximation of the wave equation with boundary conditions.

3.2 The Discrete Problem

Numerical wave propagation suffers from dispersion (cf [20, 1, 19]). This phenomenon consists in making the waves' velocity frequency dependent. Therefore a travelling pulse is smeared as it propagates. To control that numerical artefact two solutions are commonly chosen: decrease the mesh size or increase the accuracy of the discrete operator. The second solution is generally chosen because it does not increase the number of mesh points and therefore is computationnally cheap. To discretize the wave equation with high order variational schemes, the following functional spaces are needed:

$$L^2_o \;=\; \left\{ u \in L^2(\Omega) \;/\; u = \sum_{j=J_0}^{J_1} u_j \; 1_{](j-1/2)\Delta x,(j+1/2)\Delta x]}(x) \right\}$$

$$L^2_* \;=\; \left\{ u \in L^2(\Omega) \;/\; u = \sum_{j=J_0}^{J_1-1} u_{j+1/2} \; 1_{]j\Delta x,(j+1)\Delta x]}(x) \right\}$$

equipped respectively with the norms :

$$||u||_{L^2_o} = ||u||_o \;=\; \left(\sum_{j=J_0}^{J_1} u_j^2 \Delta x\right)^{1/2} \qquad ||u||_{L^2_*} = ||u||_* \;=\; \left(\sum_{j=J_0}^{J_1-1} u_{j+1/2}^2 \Delta x\right)^{1/2}$$

The finite difference operator A_L is defined by:

$$\begin{array}{llll} A_L \;: & L^2_o & \longrightarrow & L^2_* \\ & (u_j)_{j=J_0}^{J_1} & \longmapsto & (A_L u)_{j=J_0}^{J_1-1} \;/\; A_L u_{j+\frac{1}{2}} = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x}[u_{j+l} - u_{j-l+1}] \end{array}$$

Its adjoint A_L^* for the scalar products above is defined by:

$$\begin{array}{cccc} A_L^* & : & L_*^2 & \longrightarrow & L_o^2 \\ & & (u_{j+\frac{1}{2}})_{j=J_0}^{J_1-1} & \longmapsto & (A_L^* u)_{j=J_0}^{J_1} \ / \ A_L^* u_j = -\sum_{l=1}^{L} \frac{\beta_l}{\Delta x}[u_{j+l-\frac{1}{2}} - u_{j-l+\frac{1}{2}}] \end{array}$$

The operators A_L map a function defined on the "integer" grid (L_o^2) to a function defined on the "shifted" (half-integer) grid (L_*^2). The coefficients $(\beta_l)_{l=1..L}$, given in appendix 1, are chosen so that A_L is an approximation of $\frac{\partial}{\partial x}$ of order $2L$. Composing A_L and its adjoint provides an approximation of the second derivative in the space L_o^2 (the integer grid).

Now let $H_L^1 = L_o^2$ be equipped with the semi-norm $||u||_{H_L^1}^2 \ = \ ||\frac{1}{\sqrt{\rho}} A_L u||_*^2 = \left(\sum_{j=J_0}^{J_1} \frac{(A_L u_{j+1/2})^2}{\rho_{j+1/2}} \Delta x \right)$.

This semi-norm is actually a norm (see proof in appendix 2). Let's also define the followimg norm on (L_o^2): $||u||_{0,K}^2 \ = \ \left(\sum_{j=J_0}^{J_1} \frac{u_j^2}{K_j} \Delta x \right)$

The fully discrete problem is the following given $u_0, v_0 \in L_o^2$ find (u_j^n) solution of:

$$\left\{ \begin{array}{l} \dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{K_j \Delta t^2} + A_L^* \left(\dfrac{1}{\rho} A_L \right) u_j^n \ = \ 0 \\ u_{J_0}^n \ = \ u_{J_1}^n \ = 0 \\ u_j^0 \ = \ u_0(j\Delta x) - \Delta t v_0(j\Delta x) \\ u_j^1 \ = \ u_0(j\Delta x) + \Delta t v_0(j\Delta x) \end{array} \right. \tag{7}$$

where J_0 corresponds to $x = 0$ and J_1 to $x = 1$. The scheme (7) is consistent with equation (6) since its truncation error is $O(\Delta t^2 + \Delta x^{2L})$. So according to the equivalence theorem, the stability of the scheme is equivalent to showing that the sequence of operators $S_{(\Delta t, \Delta x)}$ defined by:

$$\begin{array}{rcl} S_{(\Delta t, \Delta x)} : H_L^1 \times L_o^2 & \longrightarrow & W_{\Delta t, \Delta x} \\ (u_0, v_0) & \longmapsto & u_j^n \ \textit{solution of} \ (7) \end{array}$$

is uniformly bounded where $W_{(\Delta t, \Delta x)} = \left\{ u \in L_o^2 \times L_{o,T}^2 \right\}$ is equipped with the following norm :

$$||u||_{W_{(\Delta t, \Delta x)}}^2 \ = \ \sup_{0 \le n \le N} \left(||\frac{u^{n+1} + u^n}{2}||_{H_L^1}^2 + ||\frac{u^{n+1} - u^n}{\Delta t}||_{0,K}^2 \right)$$

and L_o^2 is defined by

$$L_{o,T}^2 \ = \ \left\{ u \in L^2(]0, T[) \ / \ u = \sum_{j=N_0}^{N_1} u^n \ 1_{](n-1/2)\Delta t, (n+1/2)\Delta t]}(t) \right\}$$

The key to the uniform boundedness of the operators $S_{(\Delta t, \Delta x)}$ is a discrete energy estimate. This estimate given in the following proposition is based on a discrete Green's formula. This is the justification for the use of the variational finite difference . As shown below symmetry or skew-symmetry simplifies greatly the boundary terms in the discrete Green's formula.

Lemma 2 *(Discrete Green's formula)*
$\forall\, u \in L^2_0 \qquad \forall\, v \in L^2_*$ we have:

$$(A_L u, v)_* - (u, A_L^* v)_o \;=\; B_r(u,v) - B_l(u,v)$$

where B_r (resp B_l) is the right (resp left) boundary term defined by:

$$B_r(u,v) = \sum_{l=1}^{L} a_l u_{J_1} v_{J_1+l-\frac{1}{2}} + \sum_{l=1}^{L}\sum_{j=1}^{l-1} a_l (u_{J_1+j} v_{J_1+j-l+\frac{1}{2}} + u_{J_1-j} v_{J_1-j+l-\frac{1}{2}})$$

$$B_l(u,v) = \sum_{l=1}^{L} a_l u_{J_0} v_{J_0-l+\frac{1}{2}} + \sum_{l=1}^{L}\sum_{j=1}^{l-1} a_l (u_{J_0+j} v_{J_0+j-l+\frac{1}{2}} + u_{J_0-j} v_{J_0-j+l-\frac{1}{2}})$$

Proof: See appendix 3
To apply the operator $A_L^*(\frac{1}{\rho}A_L)$ on all the points $(x_j)_{j=J_0..J_1}$ the field u must be extended outside the interval $[0,1]$. To do so, set:

$$\left\{ \begin{array}{lcll} u_{J_0-l} & = & -u_{J_0+l} & \qquad l = 1...2L-2 \\ & & & \\ u_{J_1+l} & = & -u_{J_1-l} & \qquad l = 1...2L-2 \end{array} \right. \tag{8}$$

Proposition 1 *(Conservation of Energy)*
Let the discrete energy $E^{n+\frac{1}{2}}$ be the sum of the potential energy $E_p^{n+\frac{1}{2}}$ and kinetic energy $E_c^{n+\frac{1}{2}}$ defined by:

$$E_c^{n+\frac{1}{2}} \;=\; \frac{1}{2}\left(T\frac{u^{n+1}-u^n}{\Delta t}\,,\,\frac{u^{n+1}-u^n}{\Delta t}\right)_{0,K} \qquad \text{with} \qquad T = I - \frac{\Delta t^2}{4} K A_L^* \left(\frac{1}{\rho}A_L\right)$$

$$E_p^{n+\frac{1}{2}} \;=\; \frac{1}{2}\left|\left|\frac{u^{n+1}+u^n}{2}\right|\right|_{H^1_L}$$

We have

$$\frac{E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}}{\Delta t} \;=\; 0$$

which expresses the conservation of the discrete energy.

Proof:
Multiply (7) by $\dfrac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}$ and integrate in space, that is sum over j. The detail of the calculation is given in appendix 4. □

The stability of the scheme comes from the properties of the kinetic energy operator $T = I - \frac{\Delta t^2}{4} K A_L^* \left(\frac{1}{\rho}A_L\right)$.

Lemma 3 *The kinetic energy operator $T = I - \frac{\Delta t^2}{4} K A_L^* \left(\frac{1}{\rho}A_L\right)$ is a linear continuous operator from L^2_o into L^2_o. It is positive definite iff*

$$\left|\frac{C_\# \Delta t}{\Delta x}\right| < \left(\sum_{l=1}^{L} |\beta_l|\right)^{-1} \tag{9}$$

with $C_{\#}^2 = \left(\sum_{l=1}^{L} |\beta_l|\right)^{-1} \sup_j \left(\sum_{l=1}^{L} |\beta_l| \frac{K_i}{2\rho_{i+l-1/2}} + \frac{K_i}{2\rho_{i-l+1/2}}\right)$

<u>Proof:</u>

The linearity of T is obvious from the linearity of A_L. The positivity of T is equivalent to:

$$\forall u \in L_o^2 \quad u \neq 0 \qquad (Tu\ ,\ u)_o > 0$$

which is equivalent to :

$$I \ = \ \frac{\Delta t^2}{4}\left(A_L^*\left(\frac{1}{\rho}A_L\right)u\ ,\ u\right)_o \ < \ ||u||_{0,K}^2$$

because of (8) (see appendix 4) $(A_L u, v)_* = (u, A_L^* v)_o$, so:

$$\begin{aligned}
I \quad &= \quad \frac{\Delta t^2}{4}\Delta x \sum_{j=J_0}^{J_1-1} \frac{1}{\rho_{j+1/2}}\left(\sum_{l=1}^{L} a_l(u_{j+l} - u_{j-l+1})\right)^2 \\
&\leq \quad \frac{\Delta t^2}{4}\Delta x \sum_{j=J_0}^{J_1-1} \frac{1}{\rho_{j+1/2}}\left(\sum_{l=1}^{L}|a_l|\right)\left(\sum_{l=1}^{L}|a_l|(u_{j+l} - u_{j-l+1})^2\right) \\
&\leq \quad \frac{\Delta t^2}{2}\Delta x \sum_{j=J_0}^{J_1-1} \frac{1}{\rho_{j+1/2}}\left(\sum_{l=1}^{L}|a_l|\right)\left(\sum_{l=1}^{L}|a_l|(u_{j+l}^2 + u_{j-l+1}^2)\right) \\
&\leq \quad \frac{\Delta t^2}{2}\left(\sum_{l=1}^{L}|a_l|\right)\Delta x \sum_{l=1}^{L}|a_l|\left(\sum_{j=J_0}^{J_1-1}\frac{u_{j+l}^2}{\rho_{j+1/2}} \ + \ \frac{u_{j-l+1}^2}{\rho_{j+1/2}}\right) \\
&\leq \quad \frac{\Delta t^2}{2}\left(\sum_{l=1}^{L}|a_l|\right)\Delta x\left(\sum_{l=1}^{L}|a_l|\sum_{j=J_0}^{J_1-1}\left(\frac{K_j}{\rho_{j-l+1/2}} \ + \ \frac{K_j}{\rho_{j+l-1/2}}\right)\right)\frac{u_j^2}{K_i} \\
&\leq \quad \frac{\Delta t^2}{2}\left(\sum_{l=1}^{L}|a_l|\right)\sup_j\left(\sum_{l=1}^{L}|a_l|(\frac{K_j}{\rho_{j-l+1/2}} \ + \ \frac{K_j}{\rho_{j+l-1/2}})\right)||u||_{0,K}^2
\end{aligned}$$

By definition of $a_l \ = \ \dfrac{\beta_l}{\Delta x}$ so with:

$$C_{\#}^2 \ = \ \left(\sum_{l=1}^{L}|\beta_l|\right)^{-1}\sup_j\left(\sum_{l=1}^{L}|\beta_l|(\frac{K_j}{2\rho_{j-l+1/2}} \ + \ \frac{K_j}{2\rho_{j+l-1/2}})\right)$$

we have

$$I \ \leq \ \left(\frac{C_{\#}\Delta t}{\Delta x}\right)^2\left(\sum_{l=1}^{L}|\beta_l|\right)^2 ||u||_{0,K}^2$$

So T is positive definite if and only if:

$$\frac{C_{\#}\Delta t}{\Delta x} \ < \ \left(\sum_{l=1}^{L}|\beta_l|\right)^{-1} \tag{10}$$

Furthermore since $(\frac{1}{\rho}A_L u, A_L u)_* \geq 0$ we have $(Tu,u) \leq ||u||_{0,K}$. With $\Delta t/\Delta x$ verifying (10), there exists a constant $\alpha = 1 - \left(\frac{C_{\#}\Delta t}{\Delta x}\right)^2 \left(\sum_{l=1}^{L} |\beta_l|\right)^2 > 0$ such that,

$$\alpha ||u||_{0,K}^2 \leq (Tu,u)_{0,K} \leq ||u||_{0,K}^2$$

In other words, the kinetic energy is a positive definite bilinear form on $L_o^2 \times L_o^2$. Therefore:

$$\frac{\alpha}{2}\left|\left|\frac{u^{n+1}-u^n}{\Delta t}\right|\right|_o^2 \leq E_c^{n+1/2} \leq \frac{1}{2}\left|\left|\frac{u^{n+1}-u^n}{\Delta t}\right|\right|_{0,K}^2$$

so the energy $E^{n+\frac{1}{2}}$ satisfies:

$$\frac{\alpha}{2}\left|\left|\frac{u^{n+1}-u^n}{\Delta t}\right|\right|_{0,K}^2 + \frac{1}{2}\left|\left|\frac{u^{n+1}+u^n}{2}\right|\right|_{H_L^1}^2 \leq E^{n+1/2} \leq \frac{1}{2}\left|\left|\frac{u^{n+1}-u^n}{\Delta t}\right|\right|_o^2 + \frac{1}{2}\left|\left|\frac{u^{n+1}+u^n}{2}\right|\right|_{H_L^1}^2$$

Therefore

$$E^{n+1/2} \geq \frac{\alpha}{2}\left(\left|\left|\frac{u^{n+1}-u^n}{\Delta t}\right|\right|_{0,K}^2 + \left|\left|\frac{u^{n+1}+u^n}{2}\right|\right|_{H_L^1}^2\right)$$

$$E^{n+1/2} \leq \frac{1}{2}\left(\left|\left|\frac{u^{n+1}-u^n}{\Delta t}\right|\right|_{0,K}^2 + \left|\left|\frac{u^{n+1}+u^n}{2}\right|\right|_{H_L^1}^2\right)$$

Taking the maximum over $n = 1..N$ we have:

$$\alpha\, ||u||_{W(\Delta t,\Delta x)}^2 \leq \sup_{n=1..N} E^{n+1/2} \leq ||u||_{W(\Delta t,\Delta x)}^2$$

The energy result of proposition 1 shows that $\sup_{n=1..N} E^{n+1/2} = E^{1/2}$. But:

$$\begin{aligned} E^{1/2} &= \frac{1}{2}\left(\left|\left|T\frac{u^1-u^0}{\Delta t}\right|\right|_{0,K}^2 + \left|\left|\frac{u^1+u^0}{2}\right|\right|_{H_L^1}^2\right) \\ &\leq \frac{1}{2}\left(\left|\left|\frac{u^1-u^0}{\Delta t}\right|\right|_{0,K}^2 + \left|\left|\frac{u^1+u^0}{2}\right|\right|_{H_L^1}^2\right) \\ &\leq ||v_o||_{0,K}^2 + ||u_o||_{H_L^1}^2 \;=\; ||(u_o,v_o)||_{H_L^1\times L_o^2}^2 \end{aligned}$$

Therefore finally since $\alpha > 0$:

$$||u||_{W(\Delta t,\Delta x)}^2 \;=\; ||S_{(\Delta t,\Delta x)}(u_o,v_o)||_{W(\Delta t,\Delta x)}^2 \;\leq\; \frac{1}{\alpha}||(u_o,v_o)||_{H_L^1\times L_o^2}^2$$

This last inequality shows that under the condition (9) the sequence of operators $S_{(\Delta t,\Delta x)}$ is uniformly bounded independently of Δt and Δx. Therefore the scheme is stable.

Remark:

If we choose the second order scheme ($L = 1$) and a homogeneous medium (that K and ρ constant) the condition of positivity of the kinetic energy operator is the classical Courant Friedrichs Lewy condition (cf [7]):

$$\frac{C\Delta t}{\Delta x} < 1 \qquad \text{where} \qquad C = \sqrt{\frac{K}{\rho}} \tag{11}$$

This condition shows that the discrete kinetic energy defines a norm equivalent to the L^2_o norm. When this condition is violated the discrete equation (7) still enjoys a conservation of energy result, but the kinetic energy is not positive, which implies imaginary solutions. This is easily seen since the dispersion relation in that case is given by:

$$\sin(\frac{\omega\Delta t}{2}) = \frac{C\Delta t}{\Delta x}\sin(\frac{k\Delta x}{2}) \tag{12}$$

When condition (11) is not satisfied equation (12) admits complex ω solutions, therefore the plane wave solution $exp(i(kj\Delta x - \omega n\Delta t))$ is unbounded as $n\Delta t \longrightarrow +\infty$

4 "Sponge" Boundary Conditions

Typically in applications wave propagation problems are in infinite or semi-infinite domains. Boundary conditions have to be applied to the finite computational domain to simulate unbounded propagation. Many methods have been studied to reach that goal ([11]). They fall essentially into two classes. The first consists in changing the equation on the boundary of the domain. The wave equation is replaced by another differential equation such that waves coming from the interior of the domain undergo little or no reflexion. The second method consists in adding a layer to the domain where the energy is dissipated ([12]). We consider first this second method and study its stability properties.

4.1 The Continuous Problem

Consider the domain $\Omega =]0, 1+\varepsilon[$. Given initial conditions $u_0 \in H^1_0(\Omega)$ and $v_0 \in L^2(\Omega)$, we look for $u \in C^0(0,T; H^1_0(\Omega)) \bigcap C^1(0,T; L^2(\Omega))$ solution of:

$$\left\{ \begin{array}{rclll} \frac{1}{K(x)}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial u}{\partial x}\right) + \alpha(x)\frac{\partial u}{\partial t} & = & 0 & x \in \Omega & t > 0 \\ u(0,t) \;=\; u(1+\varepsilon, t) & = & 0 & t > 0 & \\ u(x,0) \;=\; u_0(x) \qquad \frac{\partial u}{\partial t}(x,0) & = & v_0(x) & x \in \Omega & \end{array} \right. \tag{13}$$

with $\alpha(x) = 0 \;\; for \;\; x \in]0,1[$ and $\alpha(x) > 0 \;\; for \;\; x \in]1, 1+\varepsilon[$. The following energy estimate holds:

Lemma 4 *Let* $E(t) = \frac{1}{2}\int_0^{1+\varepsilon}\left(\frac{u_t^2}{K(x)} + \frac{u_x^2}{\rho(x)}\right)dx$ *be the energy in* Ω*, then we have* $\frac{dE}{dt} \le 0.$

Proof:
Multiplication of the first equation of (13) by u_t and integration by parts gives :

$$\frac{dE}{dt} = -\int_1^{1+\varepsilon}\alpha(x)u_t^2(x,t)dx \le 0$$

whence the lemma. Using the same argument as before, the solution operator S:

$$\begin{array}{rcl} S : H^1_0(\Omega)\times L^2(\Omega) & \longrightarrow & C^0(0,T;H^1_0(\Omega))\bigcap C^1(0,T;L^2(\Omega)) \\ (u_0, v_0) & \longmapsto & u(x,t) \;\; solution \;\; of \; (13) \end{array}$$

is linear continuous.

4.2 The discrete problem

We discretize (13) as follows:

$$\left\{\begin{array}{l} \dfrac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{K_j\Delta t^2}+A_L^*\left(\dfrac{1}{\rho}A_L\right)u_j^n+\alpha_j\dfrac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} \;=\; 0 \\ u_{J_0}^n \;=\; u_{J_1}^n \;=\; 0 \\ u_j^0 \;=\; u_0(j\Delta x)-\Delta t v_0(j\Delta x) \\ u_j^1 \;=\; u_0(j\Delta x)+\Delta t v_0(j\Delta x) \end{array}\right. \tag{14}$$

where J_0 corresponds to $x = 0$ and J_1 to $x = 1+\varepsilon$. The solution of (14) satisfies the following energy estimate:

Lemma 5 *Let $E^{n+1/2}$ be defined as in Lemma 2, then* $\dfrac{E^{n+1/2}-E^{n-1/2}}{\Delta t} \leq 0$

<u>Proof:</u>
Proceed in a fashion similar to the Dirichlet case. First symetrize the field at the end of the interval $x = 0$ and $x = 1+\varepsilon$. Then multiply the first equation of (14) by $\dfrac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}$ and integrate in space. We easily get:

$$\frac{E^{n+1/2}-E^{n-1/2}}{\Delta t} = -\frac{1}{2}\left(\alpha\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t},\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}\right)_o \;\leq\; 0$$

whence the lemma.

<u>Remark:</u>
The discretization of the term $\alpha\dfrac{\partial u}{\partial t}$ with a non centered formula like $\dfrac{u_j^{n+1}-u_j^n}{2\Delta t}$ does not allow us to conclude that the energy is decreasing.
This energy estimate and the same reasonning as in the Dirichlet case shows the stability of the scheme under the same stability condition.

5 Differential Absorbing Boundary Conditions

There is a large variety of such absorbing boundary conditions. We consider the first order paraxial approximation (cf [6, 2, 3, 11]).

5.1 The Continuous Problem

Let $\Omega =]0,1[$, given initial conditions $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$, we look for $u \in C^0(0,T;H_0^1(\Omega))\bigcap C^1(0,T;L^2($
solution of:

$$
\begin{cases}
\dfrac{1}{K(x)}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho(x)}\dfrac{\partial u}{\partial x}\right) = 0 \qquad x \in \Omega \qquad t > 0 \\
u(0,t) \;=\; 0 \qquad t > 0 \\
\dfrac{1}{c(1)}\dfrac{\partial u}{\partial t}(1,t) + \dfrac{\partial u}{\partial x}(1,t) = 0 \qquad t > 0 \\
u(x,0) \;=\; u_0(x) \qquad \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \qquad x \in \Omega
\end{cases}
\tag{15}
$$

where $c(1) = K(1)/\rho(1)$ is the velocity at $x = 1$.
The solution of (15) satisfies the following energy estimate:

Lemma 6 *Let the energy $E(t)$ be defined by $E(t) = \int_0^1 \frac{u_t^2}{K(x)} + \frac{u_x^2}{\rho(x)}\, dx$. The function $E(t)$ is a decreasing function of time, which means that equation (15) is dissipative.*

Proof:
Multiply the first equation of (15) by u_t and integrate over space. After integration by parts we have

$$
\frac{dE}{dt}(t) \;=\; \frac{\partial u}{\partial x}(1,t)u_t(1,t) \qquad\qquad t > 0 \tag{16}
$$

The boundary condition in $x = 1$ gives:

$$
\frac{dE}{dt}(t) \;=\; -\frac{1}{c(1)}u_t^2(1,t) \;\leq\; 0 \qquad\qquad t > 0
$$

5.2 The Discrete problem

Equation (15) is discretized as follows:

$$
\begin{cases}
\dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{K_j \Delta t^2} + A_L^*\left(\dfrac{1}{\rho}A_L\right)u_j^n \;=\; 0 \\
\dfrac{u_{J_1}^{n+1} - u_{J_1}^{n-1}}{2C_{J_1}\Delta t} + \displaystyle\sum_{l=1}^{L}\gamma_l\left(A_L u^n_{J_1-l+1/2} + A_L u^n_{J_1+l-1/2}\right) \;=\; 0 \\
u^n_{J_0} \;=\; 0 \\
u_j^0 \;=\; u_0(j\Delta x) - \Delta t v_0(j\Delta x) \\
u_j^1 \;=\; u_0(j\Delta x) + \Delta t v_0(j\Delta x)
\end{cases}
\tag{17}
$$

The coefficients γ_l must satisfy $2\sum_{l=1}^{L}\gamma_l = 1$ in order to have a first order approximation of the derivative at $x = 1$. Since a linear combination of the sum of symmetric terms with respect to $x = 1$ is used we actually have a second order approximation of the derivative at $x = 1$. This seems contradictory to the high accuracy of the scheme in the interior. However the first order

absorbing boundary condition is poorly accurate for waves impinging upon the boundary with large angles (greater that 15^o). So second order accuracy is acceptable is that context.
To accomodate the Dirichlet condition in $x = 0$ we extend the field to the left of $x = 0$ (of indice J_0) by skew symmetry:

$$u^n_{J_0-l} = -u^n_{J_0+l} \qquad \forall n \geq 0 \quad l = 1..2L-2$$

and extend the field in $x = 1$ (of indice J_1) by skew symmetry:

$$u^n_{J_1-l} = -u^n_{J_1+l} \qquad \forall n \geq 0 \quad l = 1..2L-2$$

We have the following lemma:

Lemma 7 *(Energy Dissipation)*
Let the discrete energy be defined by:

$$E^{n+1/2} = \frac{1}{2}\left[|| \frac{u^{n+1}-u^n}{K\Delta t} ||_{0,K} + \left(\frac{1}{\rho} A_L u^{n+1}, A_L u^n \right)_* \right]$$

Then we have

$$\frac{E^{n+1/2} - E^{n-1/2}}{\Delta t} \leq 0$$

<u>Proof:</u>

To compute the discrete energy, multiply the first equation of (17) by $\dfrac{u^{n+1}_j - u^{n-1}_j}{2\Delta t}$ and perform a discrete integration in space.

$$0 = \left(\frac{u^{n+1} - 2u^n + u^{n-1}}{K\Delta t^2}, \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o + \left(A^*_L(\frac{1}{\rho} A_L u^n), \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o$$

$$= \frac{1}{2\Delta t} \left\{ \left(\frac{u^{n+1} - u^n}{K\Delta t}, \frac{u^{n+1} - u^n}{\Delta t} \right)_o - \left(\frac{u^n - u^{n-1}}{K\Delta t}, \frac{u^n - u^{n-1}}{\Delta t} \right)_o \right.$$

$$\left. + \left(\frac{1}{\rho} A_L u^n, A_L \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_* - B_r \left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}, \frac{1}{\rho} A_L u^n \right) + B_l \left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}, \frac{1}{\rho} A_L u^n \right) \right\}$$

As in the Dirichlet case, the left boundary term vanishes:

$$B_l \left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}, \frac{1}{\rho} A_L u^n \right) = 0$$

so with:

$$E^{n+1/2} = \frac{1}{2} \left(\frac{u^{n+1} - u^n}{K\Delta t}, \frac{u^{n+1} - u^n}{\Delta t} \right)_o + \left(\frac{1}{\rho} A_L u^{n+1}, A_L u^n \right)_*$$

we have :

$$\frac{E^{n+1/2} - E^{n-1/2}}{\Delta t} = B_r \left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}, \frac{1}{\rho} A_L u^n \right)$$

This relation is the discrete equivalent of (16). We have extended the field u by skew symmetry in $j = J_1$. Therefore $A_L u$ is symmetric with repect to J_1 so:

$$A_L u_{J_1+j-\frac{1}{2}} = A_L u_{J_1-j+\frac{1}{2}} \quad j = 1..2L$$

Therefore the right boundary term B_r simplifies to (see Lemma 3):

$$B_r\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t},\frac{A_L u^n}{\rho}\right) = \sum_{l=1}^{L} a_l \frac{u_{J_1}^{n+1}-u_{J_1}^{n-1}}{2\Delta t}\frac{A_L u^n_{J_1-l+\frac{1}{2}}}{\rho_{J_1-l+\frac{1}{2}}}$$

The boundary condition in (17) implies:

$$\begin{aligned}\frac{u_{J_1}^{n+1}-u_{J_1}^{n-1}}{2\Delta t} &= -C_{J_1}\sum_{l=1}^{L}\gamma_l\left(A_L u^n_{J_1-l+\frac{1}{2}}+A_L u^n_{J_1+l-\frac{1}{2}}\right)\\ &= -C_{J_1}\sum_{l=1}^{L}2\gamma_l A_L u^n_{J_1-l+\frac{1}{2}}\end{aligned}$$

Now since $\sum_{l=1}^{L}\beta_l>0$ (see appendix 1) we choose $\gamma_l=\dfrac{\beta_l}{2\sum_{l=1}^{L}\beta_l}$ so $\sum_{l=1}^{L}\gamma_l=\dfrac{1}{2}$ and we have (see appendix 3):

$$\begin{aligned}B_r\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t},\frac{1}{\rho}A_L u^n\right) &= \frac{u_{J_1}^{n+1}-u_{J_1}^{n-1}}{2\Delta t}\sum_{l=1}^{L}\frac{\beta_l}{\Delta x}\frac{1}{\rho_{J_1-l+\frac{1}{2}}}A_L u^n_{J_1-l+\frac{1}{2}}\\ &= -\frac{C_{J_1}}{\Delta x\sum_{l=1}^{L}\beta_l}\sum_{m=1}^{L}\beta_m A_L u^n_{J_1-m+\frac{1}{2}}\sum_{l=1}^{L}\frac{\beta_l}{\rho_{J_1-l+\frac{1}{2}}}A_L u^n_{J_1-l+\frac{1}{2}}\\ &= -\frac{C_{J_1}}{\Delta x\sum_{l=1}^{L}\beta_l}\sum_{m=1}^{L}\sum_{l=1}^{L}\frac{\beta_m\beta_l}{\rho_{J_1-l+\frac{1}{2}}}A_L u^n_{J_1-m+\frac{1}{2}}A_L u^n_{J_1-l+\frac{1}{2}}\end{aligned}$$

with $s_k=\beta_k A_L u^n_{J_1-k+\frac{1}{2}}$, we can write :

$$B_r\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t},\frac{1}{\rho}A_L u^n\right) = -\frac{C_{J_1}}{\Delta x\sum_{l=1}^{L}\beta_l}\sum_{l=1}^{L}\sum_{m=1}^{L}\frac{s_l s_m}{\rho_{J_1+m-\frac{1}{2}}}$$

Let M be the matrix defined by $m_{i,j}=\rho^{-1}_{J_1-j+\frac{1}{2}}$. We note that the rank of M is 1, since all the columns are equal. Therefore it has only one non zero eigenvalue. This eigenvalue is easily found if we consider the transposed matrix of M. With $e=(1,1,...,1)^T$ we have $M^T e=\left(\sum_{l=1}^{L}\frac{1}{\rho^{-1}_{J_1+m-\frac{1}{2}}}\right)e$, therefore the unique non zero eigenvalue is positive. Finally we can conclude

$$B_r\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t},\frac{1}{\rho}A_L u^n\right)\leq 0$$

whence

$$\frac{E^{n+1/2} - E^{n-1/2}}{\Delta t} \leq 0$$

The same reasoning of section 1 shows that the scheme is stable under the condition of positivity for the kinetic energy.

6 Conclusion and Discussion

The stability of high order variational finite difference schemes for initial boundary value problems for the wave equation with varying coefficients has been ivestigated. Energy estimates were used to show the stability of the schemes. These estimates are based on a discrete Green's formula for variational schemes. These schemes based on an approximation of the first order derivative give simple boundary terms in the Green's formula by use of symmetry. In the three boundary conditions treated (Dirichlet, Sponge and first order paraxial equation) these schemes satisfy a conservation or dissipation of energy result.In the case of the first order paraxial equation the coupling between the interior scheme and the boundary condition scheme was done by appropriatly choosing a linear combinaison of the discrete operators. The variational characteristic of the schemes introduced in this paper was used here since the same discrete operators were used on the boundary.
Energy conservation was the key to the stability analysis of the scheme since it provided an estimation of the energy of the solution as a function of the energy of the initial data. The stabilty result was then obtained by choosing the time and space steps such that the kinetic energy operator was positive definite. This is a generalization of the classical Courant-Friedrichs-Lewy condition to initial boundary value problems.
This approach can be generalized to higher dimensions for the three conditions treated. Generalization to higher order boundary condition is not straight forward because of the corner problem (cf [4]). In this case one needs to think about the coupling of the corner condition with the boundary condition and interior schemes.
Finally essential use of the image principle (cf [21]) (that is symmetry) was made in this paper. While this is acceptable for acoustic wave at a free surface, the extension of this work to elastic waves requires another method for defining the field outside the free surface. The method of images does not work in this case because of the coupling of pressure and shear waves at the boundary.

7 Acknowledgments

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8 Appendix

8.1 Appendix 1

For the scheme to be consistent β_l verify $\displaystyle\sum_{l=1}^{L} \frac{\beta_l}{2l-1} = 1$ and to approximate the first derivative to order $2L$

$$\sum_{l=1}^{L} (2l-1)^{2p-1} \beta_l = 0 \qquad p = 1,..,L$$

Solving this linear system in β_l gives:

$$\beta_l = \frac{(-1)^{l+1}}{2l-1} \frac{\prod_{m \neq l} (2m-1)^2}{\mid \prod_{m \neq l} (2m-1)^2 - (2l-1)^2 \mid}$$

8.2 Appendix 2

We want to prove that the semi norm defined on H^1_L is a norm. We only need to prove that $||u||_{H^1_L} = 0 \implies u = 0$. We have

$$||u||_{H^1_L} = 0 \iff ||A_L u||_* = 0$$

$$\iff \sum_{j=0}^{J-1} (A_L u)^2_{j+1/2} = 0$$

$$\implies \sum_{l=1}^{L} \beta_l (u_{j+l} - u_{j-l+1}) = 0 \qquad j = 0,..,J-1$$

We can write this linear system of equations in matrix form. With the vector $\vec{U}$ and the matrix B defined by

$$\vec{U}^T = (u_{1-L}, ..., u_1, ..., u_J, ..., u_{J-L+1})$$

$$B = \begin{pmatrix} -\beta_L & -\beta_{L-1} & \cdots & \beta_{L-1} & \beta_L & 0 & \cdots & 0 \\ 0 & -\beta_L & -\beta_{L-1} & \cdots & \beta_{L-1} & \beta_L & 0 \cdots & 0 \\ . \\ . \\ . \\ . \\ 0 & \cdots & 0 & -\beta_L & -\beta_{L-1} & \cdots & \beta_{L-1} & \beta_L \end{pmatrix}$$

we can write the previous equations as

$$B.\vec{U} = 0$$

B is a $J+2L-1$ by J matrix. If we consider $\vec{U'}$ composed of the J first components of $\vec{U}$ and the matrix B' composed of the first J line of B, we find $\vec{U'}$ solution of

$$B'.\vec{U'} = 0$$

It is clear from the definition of the matrix B' that the only eigenvalue is $\lambda = \beta_L$ and that its multiplicity is J. The eigenvector associated to λ is $u_\lambda = (1, 1, ..., 1, 1)$. Therefore the only solution is

$$u_{1-L} = \cdots = u_0 = \cdots = u_{J-L}$$

since by definition $u_0 = 0$, we have

$$u_{1-L} = \cdots = u_{J-L} = 0$$

Now to prove that $u_{J-L+1} \cdots u_{J+L-1}$ are also zero, we proceed the same, by considering the sub system composed of the last J point of $\vec{U}$.

8.3 Appendix 3

Let $u \in L_0^2$ and $v \in L_*^2$, we have:

$$(Au, v)_* = \sum_{j=J_0}^{J_1-1} \sum_{l=1}^{L} a_l (u_{j+l} - u_{j-l+1}) v_{j+\frac{1}{2}} \, \Delta x$$

$$= \sum_{j=J_0}^{J_1-1} \sum_{l=1}^{L} a_l u_{j+l} v_{j+\frac{1}{2}} \, \Delta x - \sum_{j=J_0}^{J_1-1} \sum_{l=1}^{L} a_l u_{j-l+1} v_{j+\frac{1}{2}} \, \Delta x$$

$$= \sum_{l=1}^{L} a_l \sum_{j=J_0}^{J_1-1} u_{j+l} v_{j+\frac{1}{2}} \, \Delta x - \sum_{l=1}^{L} a_l \sum_{j=J_0}^{J_1-1} u_{j-l+1} v_{j+\frac{1}{2}} \, \Delta x$$

$$= \sum_{l=1}^{L} a_l \sum_{j=J_0+l}^{J_1+l-1} u_j v_{j-l+\frac{1}{2}} \, \Delta x - \sum_{l=1}^{L} a_l \sum_{j=J_0-l+1}^{J_1-l} u_j v_{j+l-\frac{1}{2}} \, \Delta x$$

$$= \sum_{l=1}^{L} a_l \sum_{j=J_0}^{J_1} u_j v_{j-l+\frac{1}{2}} \, \Delta x - \sum_{l=1}^{L} a_l \sum_{j=J_0}^{J_0+l-1} u_j v_{j-l+\frac{1}{2}} \, \Delta x + \sum_{l=1}^{L} a_l \sum_{j=J_1+1}^{J_1+l-1} u_j v_{j-l+\frac{1}{2}} \, \Delta x$$

$$- \left(\sum_{l=1}^{L} a_l \sum_{j=J_0}^{J_1} u_j v_{j+l-\frac{1}{2}} \, \Delta x + \sum_{l=1}^{L} a_l \sum_{j=J_0-l+1}^{J_0-1} u_j v_{j+l-\frac{1}{2}} \, \Delta x - \sum_{l=1}^{L} a_l \sum_{j=J_1-l+1}^{J_1} u_j v_{j+l-\frac{1}{2}} \, \Delta x \right)$$

$$= \sum_{l=1}^{L} a_l \sum_{j=J_0}^{J_1} u_j (v_{j-l+\frac{1}{2}} - v_{j+l-\frac{1}{2}}) \, \Delta x - \sum_{l=1}^{L} a_l (\sum_{j=J_0}^{J_0+l-1} u_j v_{j-l+\frac{1}{2}} + \sum_{j=J_0-l+1}^{J_0-1} u_j v_{j+l-\frac{1}{2}}) \, \Delta x$$

$$+ \sum_{l=1}^{L} a_l (\sum_{j=J_1+1}^{J_1+l-1} u_j v_{j-l+\frac{1}{2}} + \sum_{j=J_1-l+1}^{J_1} u_j v_{j+l-\frac{1}{2}}) \, \Delta x$$

Therefore

$$(A_L u, v)_* = (u, A_L^* v)_0 + B_r(u, v) - B_l(u, v)$$

with

$$B_r(u,v) = \sum_{l=1}^{L} a_l (\sum_{j=J_1+1}^{J_1+l-1} u_j v_{j-l+\frac{1}{2}} + \sum_{j=J_1-l+1}^{J_1} u_j v_{j+l-\frac{1}{2}}) \, \Delta x$$

$$B_l(u,v) = \sum_{l=1}^{L} a_l (\sum_{j=J_0}^{J_0+l-1} u_j v_{j-l+\frac{1}{2}} + \sum_{j=J_0-l+1}^{J_0-1} u_j v_{j+l-\frac{1}{2}}) \, \Delta x$$

We can write the boundary terms B_l and B_r by a change of variable in the summation. We get after some algebraic manipulation:

$$B_l(u,v) = \sum_{l=1}^{L} a_l u_{J_0} v_{J_0-l+\frac{1}{2}} + \sum_{l=1}^{L} \sum_{j=1}^{l-1} a_l (u_{J_0+j} v_{J_0+j-l+\frac{1}{2}} + u_{J_0-j} v_{J_0-j+l-\frac{1}{2}})$$

$$B_r(u,v) = \sum_{l=1}^{L} a_l u_{J_1} v_{J_1+l-\frac{1}{2}} + \sum_{l=1}^{L} \sum_{j=1}^{l-1} a_l (u_{J_1+j} v_{J_1+j-l+\frac{1}{2}} + u_{J_1-j} v_{J_1-j+l-\frac{1}{2}})$$

This form of the boundary terms makes it easy to see how the symmetrization or skew-symmetrization of the field u simplifies these terms.

8.4 Appendix 4

Multiplying (7) by $(u_j^{n+1} - u_j^{n-1})/2\Delta t$ and summing over j that is integrating in space we have

$$\begin{aligned} 0 &= \left(\frac{1}{K} \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + A_L^*(\frac{1}{\rho} A_L u^n) \, , \, \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o \\ &= \left(\frac{1}{K} \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \, , \, \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o + \left(A_L^*(\frac{1}{\rho} A_L u^n) \, , \, \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o \end{aligned}$$

Using Lemma 2 (the discrete Green's formula) gives:

$$\begin{aligned} \left(A_L^*(\frac{1}{\rho} A_L u^n) \, , \, \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o &= \left(\frac{1}{\rho} A_L u^n \, , \, A_L \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_* \\ &\quad - B_r(\frac{u^{n+1} - u^{n-1}}{2\Delta t} \, , \, \frac{1}{\rho} A_L u^n) + B_l(\frac{u^{n+1} - u^{n-1}}{2\Delta t} \, , \, \frac{1}{\rho} A_L u^n) \end{aligned}$$

Since we have extended the field u by skew symmetry at the boundaries with the relation (8) we have:

$$B_r(\frac{u^{n+1} - u^{n-1}}{2\Delta t} \, , \, \frac{1}{\rho} A_L u^n) = B_l(\frac{u^{n+1} - u^{n-1}}{2\Delta t} \, , \, \frac{1}{\rho} A_L u^n) = 0$$

Therefore

$$\left(A_L^*(\frac{1}{\rho} A_L u^n) \, , \, \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_o = \left(\frac{1}{\rho} A_L u^n \, , \, A_L \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)_*$$

So

$$\begin{aligned} 0 \;=\;& \frac{1}{2\Delta t}\left(\frac{1}{K}\frac{u^{n+1}-u^n}{\Delta t}\,,\,\frac{u^{n+1}-u^n}{\Delta t}\right)_o - \frac{1}{2\Delta t}\left(\frac{1}{K}\frac{u^{n}-u^{n-1}}{\Delta t}\,,\,\frac{u^{n}-u^{n-1}}{\Delta t}\right)_o \\ &+ \frac{1}{2\Delta t}\left(\frac{1}{\rho}A_L u^{n+1}\,,\,A_L u^n\right)_* - \frac{1}{2\Delta t}\left(\frac{1}{\rho}A_L u^{n}\,,\,A_L u^{n-1}\right)_* \end{aligned}$$

Using the following identities

$$\begin{aligned} \left(\frac{1}{\rho}A_L u^{n+1}\,,\,A_L u^n\right)_* \;=\;& \left(\frac{1}{\rho}A_L \frac{u^{n+1}+u^n}{2}\,,\,A_L\frac{u^{n+1}+u^n}{2}\right)_* \\ &- \left(\frac{1}{\rho}A_L \frac{u^{n+1}-u^n}{2}\,,\,A_L\frac{u^{n+1}-u^n}{2}\right)_* \\ \left(\frac{1}{\rho}A_L u^{n}\,,\,A_L u^{n-1}\right)_* \;=\;& \left(\frac{1}{\rho}A_L \frac{u^{n}+u^{n-1}}{2}\,,\,A_L\frac{u^{n}+u^{n-1}}{2}\right)_* \\ &- \left(\frac{1}{\rho}A_L \frac{u^{n}-u^{n-1}}{2}\,,\,A_L\frac{u^{n}-u^{n-1}}{2}\right)_* \end{aligned}$$

we have

$$\begin{aligned} 0 \;=\;& \frac{1}{2\Delta t}\Bigg\{\left(\frac{1}{K}\frac{u^{n+1}-u^n}{\Delta t}\,,\,\frac{u^{n+1}-u^n}{\Delta t}\right)_o - \left(\frac{1}{K}\frac{u^{n}-u^{n-1}}{\Delta t}\,,\,\frac{u^{n}-u^{n-1}}{\Delta t}\right)_o \\ &+ \left(\frac{1}{\rho}A_L \frac{u^{n+1}+u^n}{2}\,,\,A_L\frac{u^{n+1}+u^n}{2}\right)_* - \frac{\Delta t^2}{4}\left(\frac{1}{\rho}A_L \frac{u^{n+1}-u^n}{\Delta t}\,,\,A_L\frac{u^{n+1}-u^n}{\Delta t}\right)_* \\ &- \left(\frac{1}{\rho}A_L \frac{u^{n}+u^{n-1}}{2}\,,\,A_L\frac{u^{n}+u^{n-1}}{2}\right)_* + \frac{\Delta t^2}{4}\left(\frac{1}{\rho}A_L \frac{u^{n}-u^{n-1}}{\Delta t}\,,\,A_L\frac{u^{n}-u^{n-1}}{\Delta t}\right)_*\Bigg\} \end{aligned}$$

With

$$\begin{aligned} E_c^{n+1/2} \;=\;& \frac{1}{2}\left(I - \frac{\Delta t^2}{4}KA_L^*\left(\frac{1}{\rho}A_L\right)\frac{u^{n+1}-u^n}{\Delta t}\,,\,\frac{u^{n+1}-u^n}{\Delta t}\right)_{0,K} \\ E_p^{n+1/2} \;=\;& \frac{1}{2}\|\frac{u^{n+1}+u^n}{2}\|_{H_L^1} \\ E^{n+1/2} \;=\;& E_c^{n+1/2} + E_p^{n+1/2} \end{aligned}$$

we find that

$$\frac{E^{n+1/2}-E^{n+1/2}}{\Delta t} = 0$$

which expresses the conservation of the discrete energy.

References

[1] R.M Alford, K.R Kelly, D.M Boore *Accuracy of finite difference modeling of the acoustic wave equation.* Geophysics, Vol 39, 834-842, 1974.

[2] A. Bamberger, B. Engquist, L. Halpern, P. Joly, *Parabolic wave equation approximations in heterogeneous media.* SIAM Journal on Applied Mathematics, 129-154, 1988.

[3] A. Bamberger, B. Engquist, L. Halpern, P. Joly, *Higher order paraxial wave equation approximation in heterogeneous media.* SIAM Journal on Applied Mathematics, 1988.

[4] A. Bamberger, P. Joly, J.E Robrets, *Second order absorbing boundary conditions for the wave equation: A solution to the corner problem.*, SIAM J. Numer. Anal., 27(2), 323-352, April 1990.

[5] H. Brezis *Analyse Functionnelle : Theorie et Applications*, Masson, 1983.

[6] R. Clayton and B. Engquist, *Absorbing boundary conditions for acoustic and elastic wave equations.* Bull Seis Soc Amer, Vol 67:6, 1529-1540, 1977.

[7] R. Courant, K. Friedrichs and H. Lewy *On the Partial Difference Equations of Mathematical Physics.* Mathematische Annalen 100, 32-74, 1928. *English Translation in IBM Journal March 1967.*

[8] R. Dautray and J.L Lions *Mathematical analysis and Numerical Methods for Science and Technology*, Vol 5, Springer-Verlag, 1988.

[9] K. Friedrichs *Symmetric hyperbolic linear differential equations.* Comm Pure and Applied Math, Vol 7, No 2, 345-392, 1954.

[10] T. Ha-Duong and P. Joly *On the stability analysis of boundary conditions for the wave equation by energy methods. Part I: the homogeneous case*, Math. Comp, Vol 62, Number 206, 539-563, 1994.

[11] R.L Higdon *Initial boundary value problems for linear hyperbolic systems.* SIAM Review, 28, 177-217, 1986.

[12] M. Israeli and S. Orszag, *Approximation of radiation boundary conditions.* Journal of Computational Physics, Vol 41, 1981.

[13] L.V Kantorovich, *Functional Analysis and Applied Mathematics*, Armed Services Technical Information Agency, 1952.

[14] H.O. Kreiss *Initial boundary value problems for hyperbolic systems*, Communications on Pure and Applied Mathematics, Vol 23, 277-298, 1970.

[15] P.D Lax and R.D Richmyer *Survey of the Stability of Linear Finite Difference Equations.* Communications on Pure and Applied Mathematics, Vol IX, 267-293, 1956.

[16] M. Less, *Energy inequalities for the solution of differential equations.* Trans. Amer. Math. Soc, 58-73, January 1960.

[17] R.D Richmyer and K.W Morton *Difference Methods for Initial Value Problems.* Interscience Publishers, 1967.

[18] A.A Samarskii *A priori estimates for difference equations.* Zh. vych. mat., No 6, 972-1000, 1961.

[19] A. Sei, *A Family of schemes for the computation of elastic waves.* SIAM Journal on Scientific Computing, Vol 16, Number 4, pp 898-916.

[20] R. Vichnevetski and J. Bowles, *Fourier analysis of numerical approximations of hyperbolic equations*, Philadelphia, SIAM, 1982.

[21] A.N Tikhonov and A.A Samarskii *Partial differential equations of mathematical physics.* San Francisco. Holden-Day, 1964.