# OPERATOR PENCIL AND HOMOGENIZATION IN THE PROBLEM OF VIBRATION OF FLUID IN A VESSEL WITH A FINE NET ON THE SURFACE

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*Dedicated to Olga Arsenievna Oleinik*

**Abstract:** In this paper we study low vibrations of heavy viscous incompressible fluid in a vessel with a fine net on the surface near equilibrium level. We consider normal vibrations, i.e. the dependence of the velocity field on the time variable has the exponential form. For simplicity we consider a model scalar case. Under this assumption we obtain the spectral problem for the Laplace operator. Using the Krein scheme and homogenization method, we obtain estimates for deviation of the eigenelements of this problem and of the limit problem.

**1. Introduction.** Low vibrations of heavy viscous incompressible fluid in a vessel with a net on the surface near equilibrium level are described by linearized Navier-Stokes equations.

We consider normal vibrations, i.e. the dependence of the velocity field on the time variable has the exponential form $e^{-\lambda t}$. For simplicity we consider a model scalar case.

Under this assumption we obtain the spectral problem for the Laplace operator, which we will study on the base of the Krein scheme.

**2. Some questions of functional analysis.**

**Definition 2.1.** *Two Hilbert spaces $E$ and $F$ form a* ***Hilbert pair****, if $F$ is a linear dense subset of $E$, and if there exists an $a > 0$ such that for all $x \in F$*

$$\|x\|_E \leq a\|x\|_F. \tag{2.1}$$

The scalar product $(x, y)_E$ with a fixed $y \in E$ is a linear bounded functional on $F (x \in F)$.

Using the Riesz theorem one can deduce that there exists a unique $z \in F$ such that

$$(x, y)_E = (x, z)_F \quad \forall\, x \in F,$$

and moreover,

$$\|z\|_F \leq a\|y\|_E. \tag{2.2}$$

Thus a map $Vy = z$ is defined.

**Lemma 2.1.** *The operator $V$ is linear, selfadjoint, and positive.*

**Proof:** It is easy to see that the operator $V$ is linear and

$$\frac{1}{a}\|Vy\|_E \leq \|Vy\|_F \leq a\|y\|_E. \tag{2.3}$$

Using the definition of $V$, i.e.

$$(x, y)_E = (x, Vy)_F, \tag{2.4}$$

and taking $x = Vy_1 (y_1 \in E)$ we obtain the selfadjointness of the operator $V$

In fact

$$(Vy, y)_E = (Vy_1, Vy)_F = \overline{(Vy, Vy_1)_F} = \overline{(Vy, y_1)_E} = (y_1, Vy)_E.$$

Using (2.4) we obtain the positiveness of $V$.

If $y_1 = y$, then

$$(Vy, y)_E = (Vy, Vy)_F \geq 0.$$

If $Vy = 0$, then (2.4) gives us the equality $(x, y)_E = 0$ for all $x \in F$. Since $F$ is dense in $E$, one obtains that $y = 0$. That is $(Vy, y) > 0$ if $y \neq 0$. The lemma is proved.

If $F$ is compactly embedded in $E$, then $V$ is the compact operator.

Now we can consider the inverse operator $A = V^{-1}$.

The operator $A$ is unbounded, selfadjoint, and positive. Its domain of definition $D(A)$ is everywhere dense in $E$ and coincides with the range of values of the operator $V$, i.e. $D(A) = R(V)$ .

The formula (2.4) gives us

$$(x, Az)_E = (x, z)_F \tag{2.5}$$

for all $x \in F, z \in D(A)$.

If $x \in D(A)$, then we can rewrite (2.5) as follows

$$(A^{\frac{1}{2}}x, A^{\frac{1}{2}}z)_E = (x,z)_F \quad x, z \in D(A). \tag{2.6}$$

If we take $z = x$, then we obtain

$$\|A^{\frac{1}{2}}x\|_E = \|x\|_F. \tag{2.7}$$

**Lemma 2.2.** *The domain of definition of the operator $A^{\frac{1}{2}}$ coincides with $F$.*

**Proof:** It is known that the domain of definition of the operator
$A^{\frac{1}{2}}$ is a completion of $D(A)$ with respect to the norm $\|A^{\frac{1}{2}}x\|$.

Therefore, in our case $D(A^{\frac{1}{2}})$ is the completion of $D(A)$ with respect to the norm of the space $F$ (see (2.7)) and consequently it is a subset of $F$.

Otherwise there exists $u_0 \in F, u_0 \neq 0$, such that $(u_0, z)_F = 0$ for all $z \in D(A^{\frac{1}{2}})$. Since the range of values $R(A) = E$ we obtain, using (2.5), that

$$0 = (u_0, z)_F = (u_0, Az)_E$$

and consequently $u_0 \equiv 0$. We arrive to a contradiction. The lemma is proved.

Using properties of functions of the operator $A$ (see, for example [2]) one can conclude that $A^{\frac{1}{2}}$ is a positive selfadjoint operator.

If $F$ is compactly embedded in $E$, then the inverse operator $A^{-\frac{1}{2}}$ is compact.

**Definition 2.2:** *The operator $A$ is a* ***generating*** *operator of the Hilbert pair $(F, E)$.*

It is easy to show (see [2]) that for each Hilbert pair the operator $A$ is unique.

**Example 1:** Let $\Omega$ be a domain such that its boundary $\partial\Omega$ consists of two parts $S$ and $\Gamma$.

Consider the space $H^1(\Omega, \Gamma)$ formed by the functions from $H^1(\Omega)$, whose traces vanish on $\Gamma$, and the Hilbert pair $\Big(H^1(\Omega,\Gamma), L_2(\Omega)\Big)$.

From (2.5) we obtain the following identity

$$\int_\Omega u(x)Av(x)dx = \int_\Omega \nabla u \nabla v dx \tag{2.8}$$

for all $u \in H^1(\Omega,\Gamma),\ \ v \in D(A)$.

If $v(x) \in C^2(\Omega)$, then from (8) we deduce

$$\int_\Omega uAvdx = -\int_\Omega u\Delta v dx + \int_{\partial\Omega} u\frac{\partial v}{\partial n}ds = -\int_\Omega u\Delta v dx + \int_\Gamma u\frac{\partial v}{\partial n}ds.$$

Now we can conclude that

$$Av := -\Delta v \quad \text{in} \quad \Omega,$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on} \quad S.$$

Thus the operator $V$, the inverse operator for $A$, gives us a generalized solution of the problem:

$$\begin{cases} -\Delta v = f \quad \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \Gamma, \\ \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad S. \end{cases} \tag{2.9}$$

If $(F, E)$ is a Hilbert pair and $A$ is its generating operator, then we define $E^\alpha$ as the following Hilbert spaces:

if $\alpha > 0$, then $E^\alpha$ is the domain of definition of the operator $A^\alpha$ with the scalar product

$$(x, y)_{E^\alpha} := (A^\alpha x, A^\alpha y)_E \quad (x, y \in D(A^\alpha)),$$

if $\alpha \leq 0$, then $E^\alpha$ is the completion of the space $E$ with respect to the norm

$$\|x\|_{E^\alpha} := \|A^\alpha x\|_E.$$

**Definition 2.3.** *The family of spaces $E^\alpha$ is called a **Hilbert scale** of spaces.*

According to the Definition 2.3 we can note that the space $F$ is the space $E^{\frac{1}{2}}$.

It is easy to see that if $\alpha > 1$, then the restriction $A\big|_{E^\alpha}$ is a one-to-one continuous mapping $E^\alpha$ onto $E^{\alpha-1}$; further, if $\alpha \leq 1$, then the continuous extension of $A$ to $E^\alpha$ is a one-to-one continuous mapping $E^\alpha$ onto $E^{\alpha-1}$.

In an analogous way, the operator $A^{\frac{1}{2}}$ maps $E^{\frac{1}{2}} = F$ onto $E$, and $E$ onto $E^{-\frac{1}{2}}$.

If we fix $x \in E$, then $l_x(y) := (x, y)_E$ is a linear bounded functional on $E^{\frac{1}{2}} = F$.

Actually,

$$|l_x(y)| = |(x, y)_E| = |(A^{-\frac{1}{2}} x, A^{-\frac{1}{2}} y)_E| \leq \|A^{-\frac{1}{2}} x\|_E \, \|A^{-\frac{1}{2}} y\|_E = \|A^{-\frac{1}{2}} x\|_E \, \|y\|_F.$$

Therefore,

$$\|l_x\|_{F^*} \leq \|A^{-\frac{1}{2}} x\|_E.$$

**Lemma 2.3.** $E^{-\frac{1}{2}} = (E^{\frac{1}{2}})^*$.

**Proof:** We have shown that $E$ is embedded in $F^*$. We obtain from here that one can identify $E^{-\frac{1}{2}}$ with a subspace of $F^*$.

Now we will show that this subspace coincides with the whole space $F^*$.

Otherwise, there exists $y_0 \in F, y_0 \neq 0$ such that $l_x(y_0) = (x, y_0)_E = 0$ for all $x \in E$. Hence $y_0 = 0$. It is a contradiction. The lemma is proved.

Suppose that $\gamma : F \to G$ is a linear bounded operator such that

$$\|\gamma u\|_G \le b\|u\|_F \quad (u \in F). \tag{2.10}$$

Here $F$ and $G$ are Hilbert spaces.

From the formula (2.10) we conclude that $ker\,\gamma$ is a subspace in $F$, i.e. $F = N \oplus M$, where $N := ker\,\gamma$, and $M$ is the orthogonal complement of $N$ in $F$.

Denote by $G_+$ the range of values $R(\gamma)$ of the operator $\gamma$.

The operator $\gamma$ is a one-to-one correspondence between $M$ and $G_+$.

Therefore we can consider $G_+$ as a Hilbert space with the following scalar product

$$(\varphi, \psi)_{G_+} := (u, v)_F, \tag{2.11}$$

where $u, v \in M, \gamma u = \varphi, \gamma v = \psi$.

Then $\|\varphi\|_{G_+} = \|u\|_F \quad (\gamma u = \varphi,\ u \in M)$.

If $u \in F$, and $\gamma u = \varphi$, then there exists a $v \in M$ such that $\gamma v = \varphi$, i.e. $\gamma(u - v) = 0$, and, consiquently, $u - v \in\ ker\,\gamma \equiv N$.

Furthermore,

$$\|u\|_F^2 = \|v + (u - v)\|_F^2 = \|v\|_F^2 + \|u - v\|_F^2.$$

From here

$$\|\varphi\|_{G_+}^2 = \|v\|_F^2 \le \|u\|_F^2$$

for all $u, \gamma u = \varphi$. In other words, we can write

$$\|\varphi\|_{G+} = \min_{\gamma u = \varphi} \|u\|_F. \tag{2.12}$$

Now we suppose that $G_+$ is densely embedded in $G$. From (2.10) we obtain that for $\varphi \in G_+, \quad u \in M, \quad \varphi = \gamma u$ the following estimate takes place

$$\|\varphi\|_G = \|\gamma u\|_G \le b\|u\|_F = b\|\varphi\|_{G_+}.$$

It means that the spaces $G$ and $G_+$ form a Hilbert pair, and we can construct a Hilbert scale $G^\alpha$ such that $G_+ = G^{\frac{1}{2}}$; moreover, we can identify the space $G^{-\frac{1}{2}}$ with $(G_+)^*$.

Denote by $T$ the operator, conjugated to the operator $\gamma$ in the sense of scalar product in $G$. Since $\gamma : M \to G^{\frac{1}{2}}$ is an isometric operator, we obtain that $T : G^{-\frac{1}{2}} \to M$ is an isometry too (see [2]).

The definition of $T$ gives us for $\psi \in G^{-\frac{1}{2}}$

$$(T\psi, v)_F = (\psi, \gamma v)_G \tag{2.13}$$

for all $v \in M$.

Denote by $\partial$ the inverse operator for $T$, i.e. $\partial := T^{-1}$.

**Lemma 2.4.** *The operator $C := \gamma T$ is isometric bounded positive and selfadjoint in $G$. If the embedding $G_+ \hookrightarrow G$ is compact, then $C$ is compact.*

**Proof:** It is easy to see that $C : G^{-\frac{1}{2}} \to G^{\frac{1}{2}}$ is an isometric map.

Using properties of the scale $G^\alpha$ and (10), we obtain

$$\|\varphi\|_{G^{-\frac{1}{2}}} \leq b\|\varphi\|_G \quad (\varphi \in G).$$

We have

$$\|\gamma T\varphi\|_G \leq b\|T\varphi\|_{G^{\frac{1}{2}}} = b\|\varphi\|_{G^{-\frac{1}{2}}} \leq b^2\|\varphi\|_G^2, \quad \varphi \in G.$$

We will prove the selfadjointness of $C$. The formula (2.13) with $v = T\varphi, \varphi, \psi \in G$ gives us

$$(C\varphi, \psi)_G = (\gamma T\varphi, \psi)_G = \overline{(\psi, \gamma T\varphi)_G} = \overline{(T\psi, T\varphi)_F} = (T\varphi, T\psi)_F = (\varphi, \gamma T\psi)_G = (\varphi, C\psi)_G.$$

Now we will show the positiveness of $C$. In fact,

$$(C\varphi, \varphi)_G = (T\varphi, T\varphi)_F = \|T\varphi\|_F^2 \geq 0.$$

If $C\varphi = 0$, i.e. $\gamma T\varphi = 0$, then from (2.11) and (2.13) with $\gamma v = \psi$ we have

$$0 = (\gamma T\varphi, \psi)_{G_+} = (T\varphi, v)_F = (\varphi, \gamma v)_G = (\varphi, \psi)_G$$

for all $\psi \in G_+$. Since $G_+$ is densely embedded in $G$, we obtain that $\varphi = 0$. The lemma is proved.

**Corollary.** *The operator $C^{-1} = \partial\gamma^{-1}$ is the generating operator of the Hilbert pair $(G_+, G)$.*

**Example 2.** Suppose that $\partial\Omega = \overline{S} \cup \Gamma$. Denote by $H^1_\Gamma(\Omega)$ the set of functions from $H^1(\Omega)$ with $\int\limits_\Gamma \gamma u ds = 0$ and the norm

$$\|u\|_F := \Big(\int\limits_\Omega \|\nabla u\|^2 dx\Big)^{\frac{1}{2}}$$

(the correctness see [3]). Let $F$ be $H^1_\Gamma(\Omega)$, the space $G$ be $L_{2,\Gamma} := L_2(\Gamma) \ominus \{1\}$ and the operator $\gamma_\Gamma$ be a composition of the trace operator and the restriction operator from $\partial\Omega$ on $\Gamma$. It is easy to see that the subspace $N$ is the space $H^1(\Omega, \Gamma)$.

**Lemma 2.5.** *The orthogonal supplement $M$ of $N$ is the space of harmonic functions in $\Omega$ from $H^1(\Omega)$ with the normal derivative, vanishing on $S$. We denote it by $H^1_{h,S}(\Omega)$.*

**Proof:** For finding the orthogonal supplement of $N$ we use the Green formula.

For $q \in N$ and for smooth $p \in N$ we have

$$0 = \int_{\Omega} \nabla q \nabla p dx = - \int_{\Omega} q \Delta p dx + \int_{\partial\Omega} \gamma q \frac{\partial p}{\partial n} ds = - \int_{\Omega} q \Delta p dx + \int_{\Gamma} \gamma q \frac{\partial p}{\partial n} ds.$$

Assuming that $q$ is a compactly supported function in $\Omega$, we find that $\Delta p = 0$.

Since the set of traces of continuous functions in $\overline{\Omega}$, vanishing on $\Gamma$, is dense in $L_2(S)$, we obtain that $\frac{\partial p}{\partial n} = 0$ on $S$.

Thus,

$$H^1_\Gamma(\Omega) = H^1(\Omega, \Gamma) \oplus H^1_{h,S}(\Omega). \tag{2.14}$$

The lemma is proved.

It can be seen that the operator $\gamma_\Gamma : H^1_{h,S}(\Omega) \to H^{\frac{1}{2}}_\Gamma$ is an isometric map. The space $H^{\frac{1}{2}}_\Gamma$ with the following norm

$$\|\varphi\|_{H^{\frac{1}{2}}} = \min_{\gamma q = \varphi} \|q\|_{H^1_\Gamma(\Omega)} \quad (q \in H^1_\Gamma(\Omega)). \tag{2.15}$$

is dense in the space $L_{2,\Gamma}$.

This fact we can reformulate as follows : for all $\varphi \in H^{\frac{1}{2}}_\Gamma$ such that $\int_\Gamma \varphi d\Gamma = 0$, there exists a unique solution $p \in H^1_\Gamma(\Omega)$ of the problem

$$\begin{cases} \Delta p = 0 \quad \text{in} \quad \Omega, \\ p = \varphi \quad \text{on} \quad \Gamma, \\ \frac{\partial p}{\partial n} = 0 \quad \text{on} \quad S. \end{cases} \tag{2.16}$$

Moreover,

$$\|p\|_{H^1_\Gamma(\Omega)} = \|\varphi\|_{H^{\frac{1}{2}}_\Gamma}. \tag{2.17}$$

**Lemma 2.6.** *The problem* (2.16) *is solvable for all* $\varphi \in H^{\frac{1}{2}}(\Gamma) := H^{\frac{1}{2}}_\Gamma + \boldsymbol{R}$.

**Proof:** Note that for $\varphi \equiv 1$ the problem has the solution $p \equiv 1$. Consiquently, the problem is solvable for all $\varphi \in H^{\frac{1}{2}}(\Gamma)$.

### 2.1 The Green formula.

We suppose that the space $F$ is densely embedded in $E$, the operator $\gamma : F \to G_+$ is bounded, $G_+$ is a dense subset of $G$. Let $N = ker\, \gamma$ be dense in $E$.

We suppose that $L$ is the generating operator of the Hilbert pair $(N, E)$. The operator $L^{-1}$ gives the solution of the problem

$$\begin{cases} Lu = f, \\ \gamma u = 0 \end{cases}$$

for any $f \in E$.

We define the operator $L$ to be zero on the ortogonal supplement $M$ of $N$ in $F$.

Now we construct the operator $\partial$. On the space $M$ we define $\partial$ as above and on $N$ as follows:

We consider the inequality

$$|(Lu, v)_E| \leq \|Lu\|_E \ \|v\|_E \leq a\|Lu\|_E \ \|v\|_F = a\|Lu\|_E \ \|\gamma v\|_{G^{\frac{1}{2}}},$$

which takes place for $u \in N, v \in M$.

The form $(Lu, v)_E$ can be considered as a linear bounded functional on $G^{\frac{1}{2}}$.

The Riesz theorem says that there exists an element $(-\partial u) \in G^{-\frac{1}{2}}$ such that

$$(Lu, v)_E = -(\partial u, \gamma v)_G \quad (u \in N, \ v \in F);$$

moreover,

$$\|\partial u\|_{G^{-\frac{1}{2}}} \leq a\|Lu\|_E.$$

Then the operator $\partial$ can be extended as a linear operator on $N \oplus M$.

**Lemma 2.7.** *The following Green formula holds:*

$$(Lu, v)_E = (u, v)_F - (\partial u, \gamma v)_G \tag{2.18}$$

*for all* $v \in F, u \in D(L)$.

**Proof:** We suppose that $u = u_1 + u_2, v = v_1 + v_2$, where $u_1, v_1 \in N$ and $u_2, v_2 \in M$.

Using the definition of the operators $L$ and $\partial$, we obtain

$$(Lu, v)_E = (L(u_1 + u_2), v_1 + v_2)_E = (Lu_1, v_1 + v_2)_E =$$

$$= (Lu_1, v_1)_E + (Lu_1, v_2)_E = (u_1, v_1)_F - (\partial u_1, \partial u_2)_G.$$

Moreover,

$$(u, v)_F = (u_1, v_1)_F + (u_2, v_2)_F.$$

Finaly, (2.13) and the equality $\partial = T^{-1}$ give us

$$(\partial u, \gamma v)_G = (\partial u_1, \gamma v_2)_G + (\partial u_2, \gamma v_2)_G = (\partial u_1, \gamma v_2)_G + (u_2, v_2)_F.$$

From these equalities we deduce the formula (2.18). The lemma is proved.

### 2.2 The operator pencil.

Let $E$ be a separable Hilbert space. We consider the following spectral problem:

$$\varphi = \lambda A\varphi + \lambda^{-1} B\varphi \quad (\lambda \neq 0), \tag{2.19}$$

where $A : E \to E$ is a compact positive operator and $B : E \to E$ is a compact nonnegative operator.

By virtue of the properties of the operators $A$ and $B$, the spectrum of the operator pencil $L(\lambda) := I - \lambda A - \lambda^{-1} B$ in $\mathbf{C}\backslash\{0\}$ consists of isolated points, which are the eigenvalues. These points of the spectrum can have only 0 and $+\infty$ as the points of concentration.(see [2]).

**Theorem 2.1.** *The spectrum of the problem* (2.19) *consists of the countable set of the eigenvalues, which have finite multiplicities and are disposed in the right-hand half-plane. Only* 0 *and* $+\infty$ *can be the points of concentration. The nonreal eigenvalues are disposed symmetrically about the real axis, and they have a finite number of adjoint elements in the segment*

$$Re\,\lambda \geq (2\|A\|)^{-1}, \quad |\lambda| \leq 2\|B\|. \tag{2.20}$$

*If the following condition holds*

$$4\|A\|\,\|B\| < 1, \tag{2.21}$$

*then all eigenvalues are real and there are no adjoint elements.*

**Proof:** If $\lambda_0 \neq 0$ is an eigenvalue, then, by virtue of the compactness of the operators $A$ and $B$, the eigenspace, which corresponds to $\lambda_0$, is finite-dimensional. Moreover, the resolvent $L^{-1}(\lambda) = (I - \lambda A - \lambda^{-1} B)^{-1}$ has a pole in $\lambda_0$, and the root subspace, which corresponds to $\lambda_0$, is finite-dimensional.

Let $\lambda_0$ be an eigenvalue and $\varphi_0$ be the corresponding eigenelement:

$$\varphi_0 = \lambda_0 A\varphi_0 + \lambda_0^{-1} B\varphi_0. \tag{2.22}$$

From (2.22) we obtain

$$\lambda_0^2 (A\varphi_0, \varphi_0) - \lambda_0 (\varphi_0, \varphi_0) + (B\varphi_0, \varphi_0) = 0 \tag{2.23}$$

and

$$\lambda_0 = \frac{(\varphi_0, \varphi_0) \pm \sqrt{(\varphi_0, \varphi_0)^2 - 4(A\varphi_0, \varphi_0)\,(B\varphi_0, \varphi_0)}}{2(A\varphi_0, \varphi_0)}. \tag{2.24}$$

It is easy to see that $Re\,\lambda_0 > 0$. Further, we suppose, that $Im\,\lambda_0 \neq 0$. It means that

$$(\varphi_0, \varphi_0)^2 < 4(A\varphi_0, \varphi_0)\,(B\varphi_0, \varphi_0) \Longleftrightarrow \frac{(\varphi_0, \varphi_0)}{2(A\varphi_0, \varphi_0)} < \frac{2(B\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)}. \tag{2.25}$$

From (2.24) we deduce that

$$\|\lambda_0\|^2 = \frac{(B\varphi_0, \varphi_0)}{(A\varphi_0, \varphi_0)} = \frac{2(B\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)} \cdot \frac{(\varphi_0, \varphi_0)}{2(A\varphi_0, \varphi_0)}.$$

Using this equality and (2.25), we obtain

$$\left(\frac{(\varphi_0, \varphi_0)}{2(A\varphi_0, \varphi_0)}\right)^2 < \|\lambda_0\|^2 < \left(\frac{2(B\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)}\right)^2. \tag{2.26}$$

It is easy to note that by virtue of (2.21), the right-hand side of (2.26) is greater then the left-hand one. It means that there are no nonreal eigenvalues.

Using (2.24) and (2.26), we conclude that all nonreal eigenvalues are disposed on the segment (2.20).

Note that if $\lambda_0$ is a nonreal eigenvalue of the operator pencil $L(\lambda)$, then $\overline{\lambda_0}$ is an eigenvalue of this operator pencil too.

In fact, because of selfadjointness of the operator pencil $L(\lambda)$, the operator $L(\lambda_0)$ is reversible if and only if the operator $(L(\lambda_0))^* = L(\overline{\lambda_0})$ is reversible.

Now we will consider the question about the adjoint elements for the element $\varphi_0$. We suppose that $Im\,\lambda_0 = 0$. The first adjoined element for $\varphi_0$ is the solution of the equation

$$(I - \lambda_0 A - \lambda_0^{-1} B)\varphi_1 + (-A + \lambda_0^{-2} B)\varphi_0 = 0.$$

Using the selfadjointness of the operator $L(\lambda_0)$ the equation (2.22), we obtain that

$$((-A + \lambda_0^{-2} B)\varphi_0, \varphi_0) = 0.$$

Considering this equation and the equation (2.23), we find

$$2\lambda_0(A\varphi_0, \varphi_0) = (\varphi_0, \varphi_0), \quad 2\lambda_0^{-1}(B\varphi_0, \varphi_0) = (\varphi_0, \varphi_0).$$

If (2.21) is valid, then these equalities are false, and hence there are no adjoint elements. The theorem is proved.

Now we consider the square operator pencil $M(\lambda) = \lambda L(\lambda)$.

**Definition 2.4.** *The set of $\lambda$, for which there exists an element $\varphi \neq 0$ such that*

$$(M(\lambda)\varphi, \varphi) = 0, \tag{2.27}$$

*is called the **root domain** of the pencil.*

Denote by $p_\pm(\varphi)$ the roots of the equation (2.27), i.e.

$$p_\pm(\varphi) := \frac{(\varphi,\varphi) \pm \sqrt{(\varphi,\varphi)^2 - 4(A\varphi,\varphi)(B\varphi,\varphi)}}{2(A\varphi,\varphi)}. \tag{2.28}$$

If the condition (2.21) takes place, then $p_\pm(\varphi)$ are positive and different. By virtue of homogeneity of the functionals $p_\pm(\varphi)$, one can consider them on the unit sphere $S$ of the space $E$.

We consider the following partition of the root domain of the pencil $M(\lambda)$ in two root zones: the zone $\Delta_+\big(\Delta_-\big)$ consists of all values of the functional $p_+(\varphi)\big(p_-(\varphi)\big)$ on the sphere $S$.

**Lemma 2.8.** *The following equalities take place:*

$$\inf_{\varphi\in S} \Delta_- = 0, \quad \sup_{\varphi\in S} \Delta_+ = +\infty.$$

**Proof:** In fact, if $\|\varphi_0\| = 1$ and $B\varphi_0 = 0$, i.e. $\varphi_0 \in ker\, B \neq \{0\}$ then $p_-(\varphi_0) = 0$. If $B > 0$, then we take the eigenvalues $\lambda_n(B)$ and the sequence of the orthonormal eigenelements $\varphi_n(B)$ of the operator $B$ and write:

$$p_-(\varphi_n(B)) = \frac{1 - \sqrt{1 - 4(A\varphi_n,\varphi_n)\lambda_n(B)}}{2(A\varphi_n,\varphi_n)} =$$

$$= \frac{2\lambda_n(B)}{1 + \sqrt{1 - 4\lambda_n(B)(A\varphi_n,\varphi_n)}} \sim \lambda_n(B) \to 0 \quad \text{as} \quad n \to \infty.$$

Further, we have

$$p_+(\varphi_n(A)) = \frac{1 + \sqrt{1 - 4\lambda_n(A)(B\varphi_n,\varphi_n)}}{2\lambda_n(A)} \sim [\lambda_n(A)]^{-1} \to +\infty \quad \text{as} \quad n \to \infty;$$

here $\{\lambda_n(A)\}_{n=1}^\infty$ are the eigenvalues of the operator $A$, and $\{\varphi_n(A)\}_{n=1}^\infty$ are its eigenelements. The lemma is proved.

Suppose that

$$\alpha_- := \sup_{\varphi\in S} \Delta_-, \quad \alpha_+ := \inf_{\varphi\in S} \Delta_+$$

.

**Lemma 2.9.** *The following inequality*

$$\alpha_+ = \inf_{\varphi\in S} \Delta_+ \; > \; \alpha_- = \sup_{\varphi\in S} \Delta_-$$

*is valid.*

**Proof:** It is easy to see that $p_-(\varphi) \le r_-$ and $p_+(\varphi) \ge r_+$, where

$$r_\pm = \frac{1 \pm \sqrt{1 - 4\|A\| \, \|B\|}}{2\|A\|}.$$

Therefore,

$$\alpha_+ - \alpha_- \ge r_+ - r_- = \frac{\sqrt{1 - 4\|A\| \, \|B\|}}{\|A\|} > 0.$$

The lemma is proved.

Using properties of the operator pencil $L(\lambda)$, one can prove the following theorem (see [2]):

**Theorem 2.2.** *If the condition* (2.21) *takes place, then the spectrum of the operator pencil $L(\lambda)$ is contained in the zones $\Delta_-$ and $\Delta_+$, and has two branches: $\{\lambda_n^-\}_{n=1}^\infty$, $\lambda_n^- \to 0$ as $n \to \infty$, and $\{\lambda_n^+\}_{n=1}^\infty$, $\lambda_n^+ \to +\infty$ as $n \to \infty$. Furthermore, the eigenelements $\{\varphi_n^-\}_{n=1}^\infty$ and $\{\varphi_n^+\}_{n=1}^\infty$ of the first and the second branches , respectively, are the Riesz bases in the spaces $E_1$ and $E$, respectively, where $E = E_0 \oplus E_1, E_0 := ker\, B$.*

Using variational principle for operator pencils, we obtain the following proposition involving the Fisher-Courant-Weyl principle and the Poincaré-Ritz principle (see [2]).

**Lemma 2.10.** *For the eigenvalues $\{\lambda_n^-\}_{n=1}^\infty$ the following variational principles hold:*

$$\lambda_n^- = \min_{dim\ N^\perp = n-1} \ \max_{0\ne\varphi\in N} \ p_-(\varphi), \tag{2.29}$$

$$\lambda_n^- = \max_{dim\ M = n} \ \min_{0\ne\varphi\in M} \ p_-(\varphi); \tag{2.30}$$

*here $M$ is an $n$ - dimensional subspace of $E$ and $N$ is an $n-1$ - codimensional subspace of $E$.*

**Lemma 2.11.** *The following estimates are valid:*

$$\lambda_n(B) \le \lambda_n^- \le \frac{\lambda_n(B)}{1 - 2\lambda_n(B)\|A\|} \quad (n = 1, 2, \cdots), \tag{2.31}$$

$$\frac{1}{\lambda_n(A)} - 2\|B\| \le \lambda_n^+ \le \frac{1}{\lambda_n(A)} \quad (n = 1, 2, \cdots). \tag{2.32}$$

**Proof:** Now we consider the change of variables $\mu = \frac{1}{\lambda}$ in the pencil $M(\lambda)$. The eigenvalues $\lambda_n^-$ become $\mu_n^+ = \frac{1}{\lambda_n^-}$, for which the following variational principles take place:

$$\mu_n^+ = \frac{1}{\lambda_n^-} = \max_{dim\ N^\perp = n-1} \min_{0 \neq \varphi \in N} \frac{1}{p_-(\varphi)}, \tag{2.33}$$

$$\mu_n^+ = \frac{1}{\lambda_n^-} = \min_{dim\ M = n} \max_{0 \neq \varphi \in M} \frac{1}{p_-(\varphi)}. \tag{2.34}$$

Keeping in mind the following inequality

$$0 \leq \frac{(\varphi,\varphi)}{(B\varphi,\varphi)} - \frac{1}{p_-(\varphi)} = \frac{(\varphi,\varphi)}{(B\varphi,\varphi)} - \frac{(\varphi,\varphi) + \sqrt{(\varphi,\varphi)^2 - 4(A\varphi,\varphi)\,(B\varphi,\varphi)}}{2(B\varphi,\varphi)} =$$

$$= \frac{(\varphi,\varphi) - \sqrt{(\varphi,\varphi)^2 - 4(A\varphi,\varphi)\,(B\varphi,\varphi)}}{2(B\varphi,\varphi)} =$$

$$= \frac{2(A\varphi,\varphi)}{(\varphi,\varphi) + \sqrt{(\varphi,\varphi)^2 - 4(A\varphi,\varphi)\,(B\varphi,\varphi)}} \leq 2\frac{(A\varphi,\varphi)}{(\varphi,\varphi)} \leq 2\|A\|,$$

we deduce

$$\frac{(\varphi,\varphi)}{(B\varphi,\varphi)} - 2\|A\| \leq \frac{1}{p_-(\varphi)} \leq \frac{(\varphi,\varphi)}{(B\varphi,\varphi)}. \tag{2.35}$$

Using (2.33), the relations

$$\max_{dim\ N^\perp = n-1} \min_{0 \neq \varphi \in N} \frac{(\varphi,\varphi)}{(B\varphi,\varphi)} = \frac{1}{\min\limits_{dim\ N^\perp = n-1} \max\limits_{0 \neq \varphi \in N} \frac{(B\varphi,\varphi)}{(\varphi,\varphi)}} = \frac{1}{\lambda_n(B)},$$

and inequality (2.35), we conclude that

$$\frac{1}{\lambda_n(B)} - 2\|A\| \leq \frac{1}{\lambda_n^-} \leq \frac{1}{\lambda_n(B)}. \tag{2.36}$$

Estimates (2.31) follow from (2.36).

A similar reasoning with the formal changes $\lambda_n^- \mapsto \frac{1}{\lambda_n^+}$, $A \mapsto B$, $B \mapsto A$ gives us estimate (2.32). The lemma is proved.

**Corollary.** *The following asymptotics for* $\lambda_n^-$ *and* $\lambda_n^+$ *take place*

$$\lambda_n^- = \lambda_n(B)\Big(1 + o(1)\Big) \quad \text{as} \quad n \to +\infty, \tag{2.37}$$

$$\lambda_n^+ = \frac{1}{\lambda_n(A)} + O(1) \quad \text{as} \quad n \to +\infty. \tag{2.38}$$

## 3. Setting of the problem.

Let $\Omega$ be a domain in $R^3_- = R^2 \times R_-$ $(n=3), \partial\Omega = \Gamma_1 \cup \Gamma_2$. The part $\Gamma_1$ is contained in the plane $\{x_3 = 0\}$.

We suppose that $Q^\varepsilon$ is a semiball $\{\xi \in R^3 \parallel \xi_n < 0,\ \xi_1^2 + \xi_2^2 + \xi_3^2 < \varepsilon^2\}$, $Q_0^\varepsilon = \sum\limits_k (Q^\varepsilon + k), k = (k_1, k_2, 0),\ \ k_1, k_2 \in Z$. We also suppose that $Q_\varepsilon = \delta Q_0^\varepsilon (\delta(\varepsilon) \to 0$ as $\varepsilon \to 0), \gamma_\varepsilon = \partial\Omega \cap \overline{Q_\varepsilon}, \Gamma_\varepsilon = \Gamma_1 \backslash \gamma_\varepsilon$.

We will study the following spectral problem with two spectral parameters:

$$\begin{cases} \Delta u^\varepsilon = -\lambda_\varepsilon \rho_\varepsilon(x) u^\varepsilon & \text{in} \quad \Omega, \\ u^\varepsilon = 0 \quad \text{on} \quad \Gamma_2 \cup \gamma_\varepsilon, \\ \lambda_\varepsilon \frac{\partial u^\varepsilon}{\partial x_3} - q u^\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \end{cases} \quad \text{where} \quad \rho_\varepsilon(x) = \begin{cases} 1 & \text{in} \quad \Omega \backslash Q_\varepsilon, \\ \frac{1}{(\varepsilon\delta)^m} & \text{in} \quad Q_\varepsilon \end{cases}, \ m < 2, \tag{3.1}$$

where $q \equiv const > 0$.

We suppose that $u^\varepsilon(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$ satisfies the integral identity

$$\int\limits_\Omega \nabla u^\varepsilon(x) \nabla v(x) dx - \int\limits_{\Gamma_\varepsilon} \frac{q}{\lambda_\varepsilon} v(x) ds = \int\limits_\Omega \lambda_\varepsilon \rho_\varepsilon(x) u^\varepsilon(x) v(x) dx \tag{3.2}$$

for all $v(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.

## 4. Auxiliary boundary value problems.

Let us introduce into consideration the following two boundary value problems.

The first problem is to find the solution of the Poisson equation in the domain $\Omega$ with rapidly alternating boundary conditions :

$$\begin{cases} \Delta w^\varepsilon(x) = -\rho_\varepsilon(x) f(x) & \text{in} \quad \Omega, \\ w^\varepsilon(x) = 0 \quad \text{on} \quad \Gamma_2 \cup \gamma_\varepsilon, \\ \frac{\partial w^\varepsilon(x)}{\partial x_3} = 0 \quad \text{on} \quad \Gamma_\varepsilon, \end{cases} \tag{4.1}$$

**Definition 4.1.** *A function $w^\varepsilon(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$ is a* ***generalized solution*** *of problem* (4.1) *if it satisfies the integral identity*

$$\int\limits_\Omega \nabla w^\varepsilon(x) \nabla v(x) dx = \int\limits_\Omega \rho_\varepsilon(x) f(x) v(x) dx \tag{4.2}$$

*for all $v(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.* Similarly to Example 1, we define the operator $A_\varepsilon$ as the operator of problem (4.1), i.e. $A_\varepsilon[w^\varepsilon] = f$, or

$$\int\limits_\Omega (\nabla w^\varepsilon, \nabla v) dx = \int\limits_\Omega A_\varepsilon w^\varepsilon v dx$$

for all $v \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$. Moreover,

$$\int\limits_\Omega (\nabla u, \nabla v)dx = \int\limits_\Omega A_\varepsilon^{\frac{1}{2}} u A_\varepsilon^{\frac{1}{2}} v dx$$

for all $u, v \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.

Note that the operator $A_\varepsilon$ has discrete positive spectrum $\{\lambda_\varepsilon^n(A_\varepsilon)\}_{n=1}^\infty$, $\lim\limits_{n\to\infty} \lambda_\varepsilon^n(A_\varepsilon) = +\infty$.

Now we consider the second auxiliary problem, which corresponds to the operator $T = \partial^{-1}$ in our scheme. The problem is to find the solution of the Laplace equation with nonzero rapidly alternating boundary conditions :

$$\begin{cases} \Delta s^\varepsilon(x) = 0 \quad \text{in} \quad \Omega, \\ s^\varepsilon(x) = 0 \quad \text{on} \quad \Gamma_2 \cup \gamma_\varepsilon, \\ \frac{\partial s^\varepsilon(x)}{\partial x_3} = \varphi \quad \text{on} \quad \Gamma_\varepsilon. \end{cases} \tag{4.3}$$

**Definition 4.2.** *A function $s^\varepsilon(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$ is a* ***generalized solution*** *of problem (4.3) if it satisfies the integral identity*

$$\int\limits_\Omega \nabla s^\varepsilon(x) \nabla v(x) dx = \int\limits_{\Gamma_\varepsilon} \varphi(x) v(x) ds \tag{4.4}$$

*for all $v(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.*

We define $T_\varepsilon$ as the operator of problem (4.3), i.e. $T_\varepsilon[\varphi] = s^\varepsilon$, and $\Gamma$ as the projector, $\Gamma \ : \ H^1(\Omega) \to L_2(\Gamma_1)$. Thus

$$\int\limits_\Omega (\nabla T\varphi, \nabla v)dx = \int\limits_{\Gamma_\varepsilon} \varphi \Gamma v ds$$

for all $v \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.

According to the scheme, one can consider the operator $C := \Gamma T_\varepsilon$. The restriction of this operator to the space $L_{2,\Gamma_1}$ is a bounded, selfadjoint, positive, compact, and isometric operator.

Supposing that $u^\varepsilon$ is the classical solution of the problem (3.1) we can consider the following two problems, which correspond to problems (4.1) and (4.3):

$$\int\limits_\Omega (\nabla w^\varepsilon, \nabla v)dx = \int\limits_\Omega \lambda_\varepsilon u^\varepsilon v dx, \tag{4.5}$$

$$\int\limits_\Omega (\nabla s^\varepsilon, \nabla v)dx = \int\limits_{\Gamma_\varepsilon} \frac{q}{\lambda} u^\varepsilon v ds \tag{4.6}$$

for all $v \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.

From (3.2), (4.5), and (4.6) we conclude that

$$u^\varepsilon = w^\varepsilon + s^\varepsilon. \tag{4.7}$$

Bearing in mind the definitions of the operators $A_\varepsilon, T_\varepsilon$, we obtain the following equations:

$$A_\varepsilon[w^\varepsilon] = \lambda_\varepsilon u^\varepsilon,$$

$$\frac{q}{\lambda_\varepsilon} T_\varepsilon \Gamma[u^\varepsilon] = s^\varepsilon.$$

The equality (4.7) allows us to exclude $u^\varepsilon$:

$$\begin{cases} \lambda_\varepsilon w^\varepsilon = A_\varepsilon w^\varepsilon - q T_\varepsilon \Gamma[w^\varepsilon + s^\varepsilon], \\ \lambda_\varepsilon s^\varepsilon = q T_\varepsilon \Gamma[w^\varepsilon + s^\varepsilon]. \end{cases} \tag{4.8}$$

**Definition 4.3.** *A function $u^\varepsilon(x) \in H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$ such that $u^\varepsilon(x) = w^\varepsilon(x) + s^\varepsilon(x)$, is a* ***generalized solution*** *of problem* (3.1), *if $w^\varepsilon(x)$ and $s^\varepsilon(x)$ are generalized solutions of problem* (4.8) *in the space $H^1(\Omega, \Gamma_2 \cup \gamma_\varepsilon)$.*

On the base of the results from [2] we can prove the following proposition.

**Lemma 4.1.** *Problem* (4.8) *has a unique solution.*

According to the section 2, we can represent $u^\varepsilon, w^\varepsilon$ and $s^\varepsilon$ as follows :

$$u^\varepsilon = A_\varepsilon^{-\frac{1}{2}} U^\varepsilon, \quad w^\varepsilon = A_\varepsilon^{-\frac{1}{2}} W^\varepsilon, \quad s^\varepsilon = A_\varepsilon^{-\frac{1}{2}} S^\varepsilon. \tag{4.9}$$

Substituting (4.9) in (4.8) and acting on the equality by the operator $A_\varepsilon^{\frac{1}{2}}$, we obtain

$$\begin{cases} \lambda_\varepsilon W^\varepsilon = A_\varepsilon W^\varepsilon - q B_\varepsilon[W^\varepsilon + S^\varepsilon], \\ \lambda_\varepsilon S^\varepsilon = q B_\varepsilon[W^\varepsilon + S^\varepsilon], \end{cases} \tag{4.10}$$

where $B_\varepsilon := A_\varepsilon^{\frac{1}{2}} T_\varepsilon \Gamma A_\varepsilon^{-\frac{1}{2}}$.

**Lemma 4.2.** *The operator $B_\varepsilon$ is a nonnegative, selfadjoint, and compact operator.*

**Proof:** Noting that the operator $A_\varepsilon^{-\frac{1}{2}}$ is bounded, one can prove that $P := \Gamma A_\varepsilon^{-\frac{1}{2}}$ is compact in respective spaces.

Using the definition of $T_\varepsilon$ we can write:

$$(A_\varepsilon^{\frac{1}{2}} T_\varepsilon \psi, \eta)_{L_2(\Omega)} = (\psi, \Gamma A_\varepsilon^{-\frac{1}{2}} \eta)_{L_{2,\Gamma_1}}. \tag{4.11}$$

It means that the operator $A_\varepsilon^{\frac{1}{2}} T_\varepsilon$ is the adjoint one for $P$ and, consequently, it is compact too.

Moreover, the operator $B_\varepsilon = (A_\varepsilon^{\frac{1}{2}} T_\varepsilon)(\Gamma A_\varepsilon^{-\frac{1}{2}}) = P^* P$ is selfadjoint, compact and nonnegative. The lemma is proved.

Keeping in mind (4.7), (4.9), and (4.10), we obtain the main spectral problem

$$U^\varepsilon = \lambda_\varepsilon A_\varepsilon^{-1} U^\varepsilon + \frac{q}{\lambda_\varepsilon} B_\varepsilon U^\varepsilon. \tag{4.12}$$

Note that the selfadjoint operator pencil, which corresponds to problem (4.12), has the following form

$$L(\lambda_\varepsilon) := I - \lambda_\varepsilon A_\varepsilon^{-1} - \frac{q}{\lambda_\varepsilon} B_\varepsilon. \tag{4.13}$$

According to the properties of the auxiliary problems, the operator $A_\varepsilon^{-1}$ is compact and positive, the operator $B_\varepsilon$ is compact and nonnegative. Consequently, (4.13) is a special case of the operator pencil from problem (2.19) with $A = A_\varepsilon^{-1}, B = qB_\varepsilon$.

On the base of the results of the section 2 we can describe the properties of problem (4.13).

• The spectrum of problem (4.13) is discrete, and it consists of a countable set of eigenvalues with finite multiplicities, which are disposed in the right-hand half-plane. Only 0 and $+\infty$ can be the points of concentration.

• If

$$1 \le 4q \|A_\varepsilon^{-1}\| \, \|B_\varepsilon\|,$$

then the branch $\{(\lambda_\varepsilon^k)^+\}_{k=1}^\infty$ of the real eigenvalues is situated on the interval $\Delta_+ = (2q\|B_\varepsilon\|, +\infty)$ with the point of concentration $\lambda = +\infty$. The corresponding eigenelements have no adjoint elements. The branch $\{(\lambda_\varepsilon^k)^-\}_{k=1}^\infty$ of the real eigenvalues is situated on the interval $\Delta_- = (0, (2\|A_\varepsilon^{-1}\|)^{-1})$ and $(\lambda_\varepsilon^k)^- \to 0$. The corresponding eigenelements have no adjoint elements. Finally, on the segment

$$\Delta_0 = \{\lambda : \ Re\,\lambda \ge (2\|A_\varepsilon^{-1}\|)^{-1}, \ \ \|\lambda\| \le 2q\|B_\varepsilon\| \}$$

a finite number of nonreal eigenvalues can be situated symmetrically about the real axis, as well as of real eigenvalues, the eigenelements of which possess adjoint elements. The eigenvalues from this segment are called intermediate.

• If

$$1 > 4q \|A_\varepsilon^{-1}\| \, \|B_\varepsilon\|,$$

then the segment $\Delta_0$ is empty, and the eigenvalues are situated in the intervals $(0, r_-)$, $(r_+, +\infty)$, where

$$r_\pm = \frac{1 \pm \sqrt{1 - 4q\|A_\varepsilon^{-1}\| \, \|B_\varepsilon\|}}{2\|A_\varepsilon^{-1}\|}, \quad 0 < r_- < r_+.$$

• The eigenvalues $(\lambda_\varepsilon^k)^+$ correspond to the normal vibrations of viscous fluid, called internal dissipative waves.

• The eigenvalues $(\lambda_\varepsilon^k)^-$ correspond to the normal vibrations of viscous fluid, called surface gravitational waves.

• The intermediate eigenvalues correspond to the normal vibrations of viscous fluid, called intermediate waves. For this type of vibrations, their dependence on the time variable is characterized by the following laws:

† for real $\lambda$

$$e^{-\lambda t}, \quad t^m e^{-\lambda t} \quad \text{with a certain integer} \quad m,$$

‡ for nonreal $\lambda$

$$e^{-(Re\,\lambda)t}\, e^{-(Im\,\lambda)t}.$$

• The following estimates and asymptotic formulae take place

$$\lambda_\varepsilon^k(A_\varepsilon) - 2q\|B_\varepsilon\| \le (\lambda_\varepsilon^k)^+ \le \lambda_\varepsilon^k(A_\varepsilon) \ \ (k = 1, 2, \cdots),$$

$$(\lambda_\varepsilon^k)^+ = \lambda_\varepsilon^k(A_\varepsilon)[1 + o(1)] \ \text{ as } \ k \to +\infty,$$

$$q\lambda_\varepsilon^k(B_\varepsilon) \le (\lambda_\varepsilon^k)^- \le \frac{q\lambda_\varepsilon^k(B_\varepsilon)}{1 - 2q\lambda_\varepsilon^k(B_\varepsilon)\|A_\varepsilon^{-1}\|}\,(k = 1, 2, \cdots),$$

$$(\lambda_\varepsilon^k)^- = q\lambda_\varepsilon^k(B_\varepsilon)[1 + o(1)] \ \text{ as } \ k \to +\infty.$$

## 5. Homogenization.

We suppose that $\lim\limits_{\varepsilon\to 0} \frac{\varepsilon}{\pi^2\delta} = p \in [0, +\infty]$.

In the framework of the theorems from [7] we obtain the following homogenized problems for (4.1) and (4.3), respectively, as $\varepsilon \to 0$:

$$\begin{cases} \Delta w^0(x) = -f(x) \quad \text{in} \quad \Omega; \\ w^0(x) = 0 \quad \text{on} \quad \Gamma_2, \\ \frac{\partial w^0(x)}{\partial x_3} + pw^0 = 0 \quad \text{on} \quad \Gamma_1 \quad (p < \infty); \\ w^0 = 0 \quad \text{on} \quad \partial\Omega \quad (p = \infty) \end{cases} \tag{5.1}$$

and

$$\begin{cases} \Delta s^0(x) = 0 \quad \text{in} \quad \Omega; \\ s^0(x) = 0 \quad \text{on} \quad \Gamma_2, \\ \frac{\partial s^0(x)}{\partial x_3} + ps^0 = \varphi \quad \text{on} \quad \Gamma_1 \quad (p < \infty); \\ s^0 = 0 \quad \text{on} \quad \partial\Omega \quad (p = \infty). \end{cases} \tag{5.2}$$

We define operators $\widehat{A}$ and $\widehat{T}$ as the operators of problem (5.1) and (5.2), respectively, i.e. $\widehat{A}[w^0] = f$; $\widehat{T}[\varphi] = s^0$ if $p < \infty$, and $\widehat{T} \equiv 0$ if $p = \infty$. On the base of the theorem from [1] (see also [6]), we prove the following theorem:

**Theorem 5.1.** *Let the operator $A_\varepsilon$ have eigenvalues $\lambda_\varepsilon^k(A_\varepsilon)$ such that $\lambda_\varepsilon^1(A_\varepsilon) \le \lambda_\varepsilon^2(A_\varepsilon) \le \cdots$, and let the operator $B_\varepsilon$ have eigenvalues $\lambda_\varepsilon^k(B_\varepsilon)$ such that $\lambda_\varepsilon^1(B_\varepsilon) \ge \lambda_\varepsilon^2(B_\varepsilon) \ge \cdots > 0$. ( Here the eigenvalues repeat with respect to their multiplicities.)*

*Then there exist constants $C_1, C_2, C_3$ and $C_4$, depending only on $k$, such that*

$$|\lambda_\varepsilon^k(A_\varepsilon) - \lambda_0^k(\widehat{A})| \le C_1\Big(\frac{\varepsilon^{\frac{1}{2}}}{\pi} + \frac{(\varepsilon\delta)^{3-m}}{\pi} + |\frac{\varepsilon}{\pi^2\delta} - p|\Big), \quad \text{if} \quad p < \infty \tag{5.3}$$

$$|\lambda_\varepsilon^k(A_\varepsilon) - \lambda_0^k(\widehat{A})| \le C_2\Big(\frac{\varepsilon^{\frac{1}{2}}}{\pi} + \frac{(\varepsilon\delta)^{3-m}}{\pi} + \big(\frac{\pi^2\delta}{\varepsilon}\big)\Big), \quad \text{if} \quad p = \infty \tag{5.4}$$

$$|\frac{1}{\lambda_\varepsilon^k(B_\varepsilon)} - \frac{1}{\lambda_0^k(B_\varepsilon)}| \le C_3\big(\frac{\varepsilon^{\frac{1}{2}}}{\pi} + \frac{(\varepsilon\delta)^{3-m}}{\pi} + |\frac{\varepsilon}{\pi^2\delta} - p|\big), \quad \text{if} \quad p < \infty \tag{5.5}$$

$$|\frac{1}{\lambda_\varepsilon^k(B_\varepsilon)} - \frac{1}{\lambda_0^k(B_\varepsilon)}| \le C_4\Big(\frac{\varepsilon^{\frac{1}{2}}}{\pi} + \frac{(\varepsilon\delta)^{3-m}}{\pi} + \big(\frac{\pi^2\delta}{\varepsilon}\big)\Big), \quad \text{if} \quad p = \infty \tag{5.6}$$

Thus we have the following result:

**Theorem 5.2.** *If $u^\varepsilon$ is the solution of the problem (3.1),then $u^\varepsilon \rightharpoonup u^0$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$, where*

$$\begin{cases} \Delta u^0 = -\lambda u^0 \quad \text{in} \quad \Omega; \\ u^0 = 0 \quad \text{on} \quad \Gamma_2, \\ \lambda\Big(\frac{\partial u^0}{\partial x_3} + pu^0\Big) - qu^0 = 0 \quad \text{on} \quad \Gamma_1, \quad (p < \infty), \\ u^0 = 0 \quad \text{on} \quad \partial\Omega, \quad (p = \infty). \end{cases} \tag{5.7}$$

**Definition 5.1.** *The operator pencil*

$$I - \lambda\widehat{A}^{-1} - \frac{q}{\lambda}\widehat{A}^{\frac{1}{2}}\widehat{T}\Gamma\widehat{A}^{-\frac{1}{2}} \quad (p < \infty), \tag{5.8}$$

*or*

$$I - \lambda\widehat{A}^{-1} \quad (p = \infty), \tag{5.9}$$

*is called a* ***homogenized operator pencil*** *for* (4.13).

In the same way the cases, where $m > 2$, $m = 2$, and $n = 2$ can be considered. The convergence of eigenvalues of problem (4.1) was proved in [4], see also [5]. An analogous problem of boundary homogenization was considered in [6].

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