ON VIRTUAL INERTIA EFFECTS DURING DIFFUSION OF A DISPERSED MEDIUM IN A SUSPENSION

BY

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G. Capriz and P. Giovine

1. Introduction

The equations of motion for a mixture of two immiscible fluids (or a suspension of solid particles in a fluid) are still a matter for discussion. Indeed the fundamental choice of an appropriate expression for the inertia terms is itself not trivial, because virtual mass effects have an important rôle during diffusion and are not easily modelled within the scheme of a continuum.

It is usually claimed that the additional inertia force f due to those effects is proportional to a relative acceleration; however, the volume fraction itself may vary in time and from place to place and these changes are not irrelevant for the correct evaluation of f. Some authors believe that f should be objective (see, for instance, [1]), but such requirement does not appear to be mandatory (an up to date discussion is promised in [2]).

It seems easier, and perhaps less controversial, to start with the proposal of an expression for the virtual kinetic energy, following suggestions from classical analyses of the motion of a sphere (or of a disc) in a fluid, and then to formulate an appropriate variational principle. If one chooses a variational principle of material type, however, one must, perhaps implicitly, assume a mechanism for the transport of the virtual kinetic energy (see, for instance, [4]); thus, in a sense, one begs at least part of the question. We prefer to adopt here a spatial variational principle, along lines followed within classical contexts in [3].

Because we want to concentrate attention on virtual mass effects, we examine only the most elementary type of two-phase flow that implies them: a suspension of incompressible particles, or bubbles, in an incompressible perfect fluid. After a brief justification of the choice of the expression for the total kinetic energy density, we introduce the variational principle. We give some details of the developments that lead to the balance equations of momentum. We check the consistency of those equations with Truesdell's 'metaphysical' principles for mixtures and we conclude with the study of a special problem of sedimentation.

The equations of motion we obtain seem consistent with those that have been derived within an extremely general context in the paper already quoted by Bedford and Drumheller [4]. In view of the simplicity of the situation envisaged here, the details of the derivation can be followed explicitly; a constant effort is also made to assure physical insight into the analytical developments.

2. The variational principle

In this section we propose a continuum model of a two-phase system where the particles of the dispersed phase are large enough so that virtual inertia effects cannot be disregarded. As is usual in theories of continua with microstructure, the idea is to obtain hints for an appropriate expression of the relevant densities from results of analyses of simple motions in classical continua.

One can imagine an element of our continuum as a spherical drop of an incompressible perfect fluid containing a concentric spherical inclusion (the dispersed phase), the latter moving with an instantaneous purely translational relative speed z with respect to the drop. If υ is the volume of the element, β the ratio of the volume of the inclusion to υ, v_1 the absolute speed of the mass centre of the fluid, $v_2 = v_1 + z$ the absolute speed of the inclusion , ρ_1 the density of the fluid and ρ_2 the density of the inclusion, then the total kinetic energy of the element turns out to be (see [5], Sect. 93) υ times the expression

$$\frac{1}{2}\left(\rho_1(1-\beta)v_1^2 + \rho_2\beta v_2^2 + \rho_1\psi(\beta)z^2\right) , \qquad (2.1)$$

where

$$\psi(\beta) = \frac{1}{2}\,\beta\,\frac{1+2\beta}{1-\beta} \quad . \tag{2.2}$$

The expression (2.1) will be taken below as the density of kinetic energy per unit volume of the mixture; the special choice (2.2) for the function ψ, presumably adequate only when β is sufficiently small, need not be made.

Remark. The model suggested above for an elementary drop of the mixture could be sharpened to include the effects of relative rotation of the inclusion. One could also, perhaps more appropriately in certain situations, attribute to the suspended particle an ellipsoidal shape, perhaps even the shape of a disc. But then the analysis would become much more complex, because the suspended phase would have to be modelled through a continuum with microstructure, where the unit vector of the axis of symmetry of the ellipsoid would be the microstructural variable. Also, the effects of the interference between micromotions in neighbouring elements could be brought to bear; then a non-local expression for the kinetic energy density would be required (a very interesting notion, in principle), or at least an expression involving the gradients of β and z; for instance, formulae (16) of Sect. 98 and (7) of Sect. 99 of [5] hint at expressions of the latter type. Finally the idea, (embodied in (2.2) and which conforms to the spirit of the paper) that the flow between the two spheres is that pertaining to a perfect fluid, could be forfeited in favour of a similar hypothesis but involving a viscous fluid; but then (as far as concerns formula (2.1)) only the dependence of ψ on β would change.

To reduce developments to essentials, we consider only the case when the mixture is contained in a fixed rigid vessel with impermeable walls; we call B the region of space delimited by the vessel and ∂B the boundary. Kinematic compatibility imposes the conditions

$$v_1 \cdot n = 0 \quad , \quad v_2 \cdot n = 0 \ , \quad \text{on} \quad \partial B \ , \tag{2.3}$$

if n is the unit exterior vector normal to ∂B .

Again to simplify matters radically, both phases are assumed to be incompressible; ρ_1 and ρ_2 are then constant. Also mass exchanges between phases are excluded. Thus conservation of mass requires that

$$\left.\begin{aligned} &\frac{\partial\,((1-\beta)\rho_1)}{\partial\tau} + \operatorname{div}\,(1-\beta)\rho_1 v_1) = 0 \\ &\frac{\partial(\beta\rho_2)}{\partial\tau} + \operatorname{div}\,(\beta\rho_2 v_2) = 0\ , \end{aligned}\right\}\quad \text{in } B;$$

or

$$\left.\begin{aligned} &\frac{-\partial\beta}{\partial\tau} + \operatorname{div}\,((1-\beta)v_1\ ) = 0\ , \\ &\frac{\partial\beta}{\partial\tau} + \operatorname{div}\,(\beta v_2) = 0\ , \end{aligned}\right\}\quad \text{in } B\ . \qquad (2.4)$$

As for the external body forces, they are taken here to be conservative with a potential energy ω per unit mass. The potential energy of internal actions (σ, per unit volume) is taken to depend at most on β .

A mechanical process of duration $\bar{\tau}$ in the mixture is portrayed through the assignment at each instant τ in $[0,\bar{\tau}]$ of the fields v_1 , v_2 and β over B , subject to (2.3) . Among all these virtual processes the natural process is distinguished because it satisfies a variational principle and the balance conditions (2.4). To state the principle the usual minor technical definitions are required.

Let $\{\hat{v}_1(x,\tau),\hat{v}_2(x,\tau),\ \hat{\beta}(x,\tau)\}$ be the natural process and $\{v_1(x,\tau,\varepsilon),v_2(x,\tau,\varepsilon),\ \beta(x,\tau,\varepsilon)\}$ a family of virtual processes depending smoothly on a parameter ε , for ε in a neighbourhood N_ε of the origin, and such that

$\{v_1(x,\tau,0),v_2(x,\tau,0),\beta(x,\tau,0)\} \equiv \{\hat{v}_1(x,\tau),\hat{v}_2(x,\tau),\hat{\beta}(x,\tau)\},\ \forall x \in B,\ \forall\tau \in [0,\bar{\tau}],$

and

$$\{v_1(x,0,\varepsilon), v_2(x,0,\varepsilon), \beta(x,0,\varepsilon)\} \equiv \{\hat{v}_1(x,0), \hat{v}_2(x,0), \hat{\beta}(x,0)\},$$

$$\{v_1(x,\bar{\tau},\varepsilon), v_2(x,\bar{\tau},\varepsilon), \beta(x,\bar{\tau},\varepsilon) \equiv \{\hat{v}_1(x,\bar{\tau}), \hat{v}_2(x,\tau), \hat{\beta}(x,\bar{\tau})\}, \quad \forall \varepsilon \in N_\varepsilon \quad, \forall x \in B \; .$$

The variation $\delta\Gamma$ of any quantity Γ defined on a process class is given by

$$\delta\Gamma \; : = \frac{\partial \Gamma}{\partial \varepsilon} \Big|_{\varepsilon=0} \quad . \tag{2.5}$$

The conditions imposed upon $v_1(x,\tau,\varepsilon), v_2(x,\tau,\varepsilon), \beta(x,\tau,\varepsilon)$ assure us that

$$\delta v_1 = 0 \;, \quad \delta v_2 = 0 \;, \; \delta\beta = 0, \text{ for } \quad \tau = 0, \; \tau = \bar{\tau} \;, \tag{2.6}$$

whereas

$$\delta v_1 \cdot n = \delta v_2 \cdot n = 0 \quad \text{in} \quad \partial B. \tag{2.7}$$

The variational principle asserts that, during the natural motion of the body, the equality

$$\delta \int_0^{\bar{\tau}} d\tau \int_B \Big(\frac{1}{2} (\rho_1(1-\beta)v_1^2 + \rho_2\beta v_2^2 + \rho_1\psi(\beta)z^2) - (\rho_1(1-\beta) + \rho_2 \beta)\omega +$$

$$+ \quad \sigma + \phi(\frac{\partial\beta}{\partial\tau} + \mathrm{div}(\beta v_2)) + \gamma(- \frac{\partial\beta}{\partial\tau} + \mathrm{div}\,((1-\beta)v_1))\Big) \, dB = 0$$

holds for all virtual processes; here ϕ and γ are Lagrange multipliers of the constraints imposed upon the natural process by mass balance.

3. The balance equations for momentum

In view of the restrictive hypothesis made on B, reflected in the relations (2.3), the transport theorem leads, for any Γ , to the equalities

$$\delta \int_0^{\bar{\tau}} d\tau \int_B \Gamma \, dB = \int_0^{\bar{\tau}} d\tau \int_B \delta\Gamma \, dB \;,$$

$$\int_0^{\bar\tau} dt \int_B \frac{\partial \Gamma}{\partial \tau} dB = [\int_B \Gamma \, dB]_0^{\bar\tau} ,$$

and these, plus repeated recourse to integration by parts in the usual manner, lead to the following consequences of (2.8)

$$\frac{1}{2} (\rho_2 v_2^2 - \rho_1 v_1^2) + \frac{1}{2} \rho_1 \frac{d\psi}{d\beta} z^2 + (\rho_1 - \rho_2)\omega - \frac{d\sigma}{d\beta} =$$

$$= \frac{\partial \phi}{\partial \tau} + v_2 \cdot \text{grad } \phi - \frac{\partial \gamma}{\partial \tau} - v_1 \cdot \text{grad } \gamma , \tag{3.1}$$

$$\rho_1 v_1 - \rho_1 \frac{\psi z}{(1-\beta)} = \text{grad } \gamma , \tag{3.2}$$

$$\rho_2 v_2 + \rho_1 \frac{\psi z}{\beta} = \text{grad } \phi .$$

The similarity of these consequences with those obtained from a variational principle in classic cases is obvious (see [3]); one can also proceed similarly to eliminate ϕ and γ by cross-differentiation and obtain

$$\text{grad}((\rho_1 - \rho_2)\omega - \frac{1}{2} \rho_1 \frac{d\psi}{d\beta} z^2 - \frac{d\sigma}{d\beta}) = \rho_2 a_2 - \rho_1 a_1 +$$

$$+ \frac{\rho_1}{\beta} (\frac{\partial(\psi z)}{\partial \tau} + (\text{grad } v_2)^T \psi z) + \frac{\rho_1}{1-\beta} (\frac{\partial(\psi z)}{\partial \tau} + (\text{grad } v_1)^T \psi z) +$$

$$+ \{\rho_1 \psi z \frac{\partial}{\partial \tau} (\frac{1}{\beta}) + (\text{grad } (\frac{\rho_1 \psi z}{\beta}))^T v_2 - v_2 \times \text{rot } (\frac{\rho_1 \psi z}{\beta})\} + \tag{3.3}$$

$$+ \{\rho_1 \psi z \frac{\partial}{\partial \tau} (\frac{1}{1-\beta}) + (\text{grad } (\frac{\rho_1 \psi z}{1-\beta}))^T v_1 - v_1 \times \text{rot } (\frac{\rho_1 \psi z}{1-\beta})\} ,$$

where the peculiar accelerations a_1 , a_2 have been introduced

$$a_i = \frac{\partial v_i}{\partial \tau} + \text{grad } \frac{v_i}{2} - v_i \times \text{rot } v_i \ , \ i = 1,2$$

and obvious consequences of (3.2) have been exploited, i.e.,

$$\text{rot}\ \rho_1 v_1 = \text{rot}\ \rho_1 \frac{\psi z}{(1-\beta)} \quad , \quad \text{rot}\ \rho_2 v_2 = -\ \text{rot}\ \rho_2 \frac{\psi z}{\beta} \quad . \qquad (3.4)$$

Now one can use (2.4) to show that the two terms between curly brackets in (3.4) are equal respectively to

$$\frac{1}{\beta}\ \text{div}\ (\rho_1 \psi z \otimes v_2) \quad \text{and} \quad \frac{1}{1-\beta} \text{div}\ (\rho_1 \psi z \otimes v_1)$$

with the conclusion that

$$\beta(\rho_1(1-\beta)(a_1 + \text{grad}(\omega + \frac{1}{2}\ \frac{d\psi}{d\beta}\ z^2) - \rho_1(\frac{\partial \psi z}{\partial \tau} + (\text{grad}\ v_1)^T \psi z) - \rho_1 \text{div}(\psi z \times v_1)) =$$

$$= (1-\beta)\ (\beta(\rho_2 a_2 + \rho_2\ \text{grad}\omega + \text{grad}\ \frac{d\sigma}{d\beta}\ ) + \rho_1(\frac{\partial \psi z}{\partial \tau} + (\text{grad} v_1)^T \psi z) + \rho_1 \psi\ \text{grad}\ \frac{z^2}{2}\ +$$

$$+ \rho_1\ \text{div}(\psi z \otimes v_2)) \ . \qquad (3.5)$$

If one calls $\beta(1-\beta)\ u$ the common value of right and left-hand side of (3.5), one discovers, by way of (2.4) and (3.4), that the vector $u + \beta\ \text{grad}\ \frac{d\sigma}{d\beta}$ is irrotational . Then, introducing a scalar π (the physical meaning of which will be clear in the sequel), such that

$$u + \beta\ \text{grad}\ \frac{d\sigma}{d\beta} = \text{grad}\ (\pi - \frac{1}{2}\ \rho_1 z^2(\psi + (1-\beta)\ \frac{d\psi}{d\beta}\ ))\ , \qquad (3.6)$$

one is led to the balance equations of momentum for the two constituents of the mixture

$$\begin{aligned} \rho_1(1-\beta)a_1 &= -\rho_1(1-\beta)\ \ \text{grad}\ \ \omega - (1-\beta)\ \text{grad}\ \pi - \text{div}\ (\rho_1 \psi z \otimes z) + f\ , \\ \rho_2 \beta a_2 &= -\rho_2 \beta\ \text{grad}\ \omega - \beta\ \text{grad}\ \pi - f\ , \end{aligned} \qquad (3.7)$$

where: π has the obvious meaning of a pressure, and the terms which involve it express the Archimedean buoyancy forces;

a term involves the Reynolds stress $\rho\psi z\otimes z$ due to relative motion; the interaction force f between the two phases has the expression:

$$f = (1-\beta)\ \mathrm{grad}\ (\frac{1}{2}\ \rho_1 z^2(\psi - \beta\frac{d\psi}{d\beta})) + \rho_1(\frac{\partial(\psi z)}{\partial\tau} + (\mathrm{grad}\ v_1)^T\ (\psi z)) +$$

$$+ \mathrm{div}\ (\rho_1\psi z\otimes v_2) + \beta(1-\beta)\ \mathrm{grad}\ \frac{d\sigma}{d\beta}\ . \qquad (3.8)$$

Thus our analysis leads to an expression for the interaction force which is not objective, contrary to what some authors would prefer it to be.

There are many other forms into which eqns (3.7), (3.8) can be put; one variant shows that they obey the 'metaphysical' principles set by Clifford Truesdell as the basis of any theory of mixtures [6]:

$$\begin{aligned} \rho_1(1-\beta)a_1 &= -\ \rho_1(1-\beta)\ \mathrm{grad}\ \omega + \mathrm{div}\ T_1 + h\ , \\ \rho_2\beta a_2 &= -\ \rho_2\beta\ \mathrm{grad}\ \omega + \mathrm{div}\ T_2 - h\ , \end{aligned} \qquad (3.9)$$

with

$$\begin{aligned} T_1 &= \rho_1\psi z\otimes v_1 - (1-\beta)(\pi - \frac{1}{2}\ \rho_1 z^2(\psi-\beta\frac{d\psi}{d\beta}))1\ , \\ T_2 &= -\rho_1\psi z\otimes v_2 - (\beta\pi + \frac{1}{2}\ \rho(1-\beta)z^2(\psi - \beta\frac{d\psi}{d\beta}))1\ , \end{aligned} \qquad (3.10)$$

and

$$h = f - \mathrm{div}\ (\rho\psi z\otimes v_2 + \frac{1}{2}\ (1-\beta)\rho_1 z^2(\psi - \beta\frac{d\psi}{d\rho})1) - \qquad (3.11)$$

$$-\ \pi\ \mathrm{grad}\ \beta\ .$$

Thus each constituent satisfies the balance equation of momentum of an ordinary continuum; besides, if one sums up term by term either (3.7) or (3.9), introduces mixture density ρ , velocity v and acceleration a

$$\rho = \rho_1(1-\beta) + \rho_2\beta \ ,$$
$$\rho v = \rho_1(1-\beta)v_1 + \rho_2\beta v_2 \ ,$$
$$a = \frac{\partial v}{\partial \tau} + (\text{grad } v)v \ ,$$

and remembers the kinematic relation ([6] , formula (5.16))

$$\rho a = \rho_1(1-\beta)a_1 + \rho_2\beta a_2 - \text{div}(\rho_1(1-\beta)v_1 \otimes v_1 + \rho_2\beta\, v_2 \otimes v_2 - \rho v \otimes v), \quad (3.13)$$

one is led to the balance equation for the mixture

$$\rho a = - \rho \text{ grad } \omega + \text{div } T, \quad (3.14)$$

where

$$T = \rho v \otimes v - \rho_1(1-\beta)v_1 \otimes v_1 - \rho_2\beta v_2 \otimes v_2 - \rho_1\, \psi\, z \otimes z - \pi\, 1. \quad (3.15)$$

Again here Truesdell's general requirement is satisfied.

4. A problem of sedimentation

As we have repeatedly declared, our aim was the derivation, from a single (and perhaps less disputable) principle, of an expression for the interaction force due to virtual mass effects. For a discussion of practical problems, however, our balance equations are far too special. In particular the absence of terms accounting for viscosity is unrealistic; of course, such terms could easily be added, but the process of adaptation of (3.7) to a more concrete form will not be pursued.

Here, just for the sake of showing what can be expected from our analysis, we treat a particular and, as already admitted, rather artificial sedimentation problem. We consider a mixture occupying the half-space $\zeta \geq 0$, under the effect of gravity ($\omega = - |g|\zeta$; g , acceleration due to gravity). We look for steady-state solutions of (3.7), (3.8) where: (i) all unknown functions depend only on ζ ; (ii) more specifically

$$v_1 = \alpha(\zeta)c \quad , \; v_2 = \gamma(\zeta)c \; , \tag{4.1}$$

(c, unit vector of ζ-axis); (iii) the mean velocity vanishes.

Conservation of mass (i.e., eqns (2.4)) requires

$$\begin{aligned} \alpha &= (1-\beta_0)\alpha_0(1-\beta)^{-1} \; , \\ \gamma &= \beta_0\gamma_0\beta^{-1} \; , \end{aligned} \tag{4.2}$$

if $\beta_0, \alpha_0, \gamma_0$ are values of β , α , γ at $\zeta = 0$, for instance. Hypothesis (iii) imposes the relation

$$\varepsilon(1-\beta_0)\alpha_0 = - \beta_0\gamma_0 \; , \tag{4.3}$$

if $\varepsilon = \rho_1/\rho_2$.

By introducing (4.2) in (3.7) we obtain a system of two ordinary differential equations of the first order in $\beta(\zeta)$ and $\pi(\zeta)$. Actually π can be easily eliminated; in fact one could refer to (3.5) and obtain directly a single equation in β .

We spare the reader the details of the algebraic manipulations and register below results valid for small β for the case when ψ is taken to be zero

$$\beta = \beta_0 \left(1 + \frac{2(1-\varepsilon)|g|\zeta}{\gamma_0^2} \right)^{-1/2}$$

and for the case when ψ is given by (2.2)

$$\beta = \beta_0\left(1 + \frac{4(1-\varepsilon)|g|\zeta}{(2+\varepsilon)\gamma_0^2} \right)^{-1/2} .$$

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