DIFFERENTIAL GAMES AND DIRECTIONAL DERIVATIVES OF VISCOSITY SOLUTIONS OF ISAACS' EQUATIONS II

BY

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DIFFERENTIAL GAMES AND DIRECTIONAL DERIVATIVES OF VISCOSITY SOLUTIONS OF ISAACS' EQUATIONS II

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Introduction

We present here a remark and a counterexample which complement the results of P.L. Lions and P.E. Souganidis [2]. In [2], we considered deterministic control and differential games problems and we studied the relations between two formulations of the associated Bellman and Isaacs equations. In the case of Lipschitz continuous functions, we compared the notion of viscosity solutions of general Hamilton-Jacobi equations (introduced by M.G. Crandall and P.L. Lions [1]) and a formulation due to A.I. Subbotin [3] concerning directional derivatives of the value functions. The equivalence of these notions was shown in [2] in the case of control or differential games problems under the Isaacs' condition: in fact, even in the general case of upper and lower value functions, we proved "one half" of the set of the inequalities to be checked. This combined with the Isaacs condition was enough for the value functions. Here we present a simple example showing that the other half is in general false and we give a positive answer in a very particular situation.

To simplify the presentation, we restrict ourselves to the case of infinite horizon problems without state constraints:

$$(1)\quad \inf_{y\in Y}\ \sup_{z\in Z}\ \{-f(x,y,z)\cdot DV - \ell(x,y,z)\} + V(x) = 0 \quad \text{in} \quad R^N$$

where Y,Z are fixed compact metric spaces and f,ℓ are bounded, uniformly continuous on $R^N \times Y \times Z$ and Lipschitz continuous in $x \in R^N$ uniformly in $(y,z) \in Y \times Z$. As it is well-known, (1) corresponds to the equation satisfied by the lower value of a differential game (see [2] and below for more details).

In [2] we proved that a Lipschitz continuous function V is a viscosity supersolution (see below for the definition) of (1) if and only if

$$(2)\quad \inf_{y\in Y}\ \sup_{(f,\ell)\in K(x,y)} \{\overline{\lim_{t\to 0_+}}\ \frac{V(x)-V(x+tf)}{t} - \ell\} + V(x) \geq 0 \text{ in } R^N$$

where for $x \in R^N$, $y \in Y$ $K(x,y) = \overline{co}\{(f(x,y,z), \ell(x,y,z)) / z \in Z\}$. On the other hand, it is easy to check (see also [2]) that if V satisfies

$$(3)\quad \inf_{y\in Y}\ \sup_{(f,\ell)\in K(x,y)} \{\underline{\lim}_{t\to 0_+}\ \frac{V(x)-V(x+tf)}{t} - \ell\} + V(x) \leq 0 \text{ in } R^N ,$$

then V is a viscosity subsolution of (1). The converse statement was left open. In Section 1 we show that a Lipschitz continuous viscosity subsolution of (1) does not necessarily satisfy (3). Recalling that lower values are always viscosity solutions of (1), this counterexample shows the limitation of the formulation of (1) by the pair of inequalities (2) - (3). Finally, in Section 2 we describe a particular situation where the converse statement holds. (Some other situations where the equivalence holds are indicated in [2].)

We conclude the introduction by recalling the definition of the viscosity solution. Let $C(\mathcal{O})$ $(C^{0,1}(\mathcal{O}))$ denote the set of continuous (Lipschitz continuous) functions defined on $\mathcal{O}$. We have:

<u>Definition</u>: Let $F \in C(R^N \times R \times R^N)$. $u \in C(R^N)$ is a viscosity supersolution (subsolution) of

$$F(x,u,Du) = 0 \quad \text{in} \quad R^N$$

if

$$F(x,u(x), D\phi(x)) \leq 0$$

$$(F(x,u(x),D\phi(x)) \geq 0\ )$$

for every smooth function ϕ in R^N and every local maximum (minimum) x of $u-\phi$ in R^N.

1. A counterexample

Let $\ell(x)$ be a bounded Lipschitz continuous function on R^N such that

$$\begin{cases} |\ \ell(x) - \ell(y)\ | < |x-y|, \text{ for all } x,y \in R^N \\ \text{and} \\ \ell(x) = -\ |x| \text{ in a neighborhood of } 0 . \end{cases}$$

We take $Y = Z = \{\xi \in R^N ,\ |\xi| < 1\}$ and we choose

$$f(x,y,z) = z, \quad \ell(x,y,z) = \ell(x) - (y,z)$$

Thus,

$$\sup_{z \in Z} \{-\ f \cdot p -\ \ell\} = |p-y| - \ell(x) \quad \text{for all } x,p \in R^N ;$$

and

$$\inf_{y \in Y} \ \sup_{z \in Z} \{\ -\ f \cdot p - \ell\} = (|p| - 1)^+ - \ell(x) .$$

Obviously, $V(x) = \ell(x)$ is a viscosity solution of (1). On the other hand, if we try to check (3) at $x=0$ we obtain

$$\sup_{(f,\ell) \in K(0,y)} \{ \lim_{t \to 0_+} \frac{V(0) - V(0 + tf)}{t} - \ell\} + V(0) =$$

$$= \sup_{z \in Z} \{|z| + (y,z)\} = 1 + |y|$$

The infimum over y yields 1. Hence, (3) does not hold at $x = 0$.

2. A particular situation.

In this section we prove that if $N = 1$ and $\ell(x,y,z) = \ell(x)$, then any viscosity subsolution $V \in C^{0,1}(R)$ of (1) satisfies (3). We will prove this claim following a general scheme of proof and indicating where the above restrictive assumptions are needed.

Let $P(Z)$ be the set of probability measures on Z and identify Z with the subspace of $P(Z)$ consisting of Dirac measures. For $\mu \in P(Z)$, $y \in Y$, $x \in R^N$ we set

$$f(x,y,\mu) = \int_Z f(x,y,z)\, d\mu(z)$$

$$\ell(x,y,\mu) = \int_Z \ell(x,y,z)\, d\mu(z) .$$

Let M and N be the set of measurable functions y_s and z_s from $[0,\infty)$ into Y and $P(Z)$ respectively. We consider the set Δ of strategies β which are maps from M into N such that for all $t \geqslant 0$

$$\beta(y^1)_s = \beta(y^2)_s \quad \text{on} \quad [0,t], \quad \text{if} \quad y_s^1 = y_s^2 \quad \text{on} \quad [0,t].$$

Observing that for all $x \in R^N$, $y \in Y$, $p \in R^N$

$$\sup_{z \in Z} \{ - f(x,y,z)\cdot p - \ell(x,y,z)\} = \sup_{\mu \in P(z)} \{ - f(x,y,\mu)\cdot p - \ell(x,y,\mu)\},$$

we deduce from [2] that if $V \in C(R^N)$ is viscosity subsolution of (1), then for all $h > 0$ and $x \in R^N$ we have

$$(4) \qquad V(x) \leqslant \inf_{\beta\in\Delta}\ \sup_{y\in M} \{ \textstyle\int_0^h e^{-s}\ell(x_s,y_s,\ \beta[y]_s)ds + V(x_h)e^{-h}\}$$

where x_t is the solution of

$$\frac{dx_t}{dt} = f(x_t\ ,\ y_t\ ,\ \beta[y]_t) \quad \text{on} \quad [0,\infty[\ ,\ x_0 = x .$$

To prove (3) we argue by contradiction. Indeed, we assume that there exist $x \in R^N$ and $\delta>0$ such that

$$\inf_{y \in Y}\ \sup_{(f,\ell)\in K(x,y)} \{ \lim_{t\to 0_+} \frac{V(x) - V(x+tf)}{t} - \ell\} + V(x) \geqslant \delta > 0 .$$

Hence, for every $y \in Y$, there exist $t_y > 0$, $m_y \in N^+$ (positive integers) $z(y) \in Z$ and $\theta_i(y) \in (0,1)$ such that

$$\sum_{i=1}^{m_y} \theta_i(y) = 1,$$

and

$$\frac{V(x) - V(x+tf_y)}{t} - \ell_y + V(x) \geqslant \frac{\delta}{2} > 0 \qquad \text{for} \quad 0 < t < t_y ,$$

where

$$f_y = \sum_i \theta_i(y)\ f(x,y,z_i(y))\ ,\quad \ell_y = \sum_i \theta_i(y)\ \ell(x,y,z_i(y))\ .$$

Here we used the fact that $V \in C^{0,1}(R^N)$ and that $\frac{V(x) - V(x+tf)}{t}$ is continuous in f uniformly for $t > 0$. If we set

$$f_{\overline{y},y} = \sum_{i=1}^{m_y} \theta_i(y)\ f(x,\overline{y},z_i(y))\ ,\quad \ell_{\overline{y},y} = \sum_{i=1}^{m_y} \theta_i(y)\,\ell(x,\overline{y},z_i(y))$$

the properties of f and ℓ imply that for every $y \in Y$ there exists $r_y > 0$ such that

$$\frac{1}{t}\{V(x) - V(x+t\ f_{\overline{y},y})\} - \ell_{\overline{y},y} + V(x) \geq \frac{\delta}{4} > 0$$

for $0 < t < t_y$ and $d(\overline{y},y) < r_y$, where d denotes the metric of Y . Using the compactness of Y we obtain the existence of $M \geq 1$, $\overline{t} > 0$, $y_1,\ldots,y_m \in Y$ and $r_1\ ,\ldots,r_M > 0$ such that $Y \subset \bigcup_{k=1}^{M} B(y_k,r_k)$ and

$$(5)\qquad \frac{1}{t}\{V(x) - V(x+t\overline{f}_y)\} - \overline{\ell}_y + V(x) \geq \frac{\delta}{4} \quad \text{for } 0 < t < \overline{t}$$

where for $y \in B(y_j,r_j) \setminus \bigcup_{k=1}^{j-1} B(y_k,r_k)$, $\overline{f}_y$, $\overline{\ell}_y$ are defined by

$$\overline{f}_y = \sum_{i=1}^{m_{y_j}} \theta_i(y_j)\quad f(x,y_j,\ z_i(y_j))$$

$$\overline{\ell}_y = \sum_{i=1}^{m_{y_j}} \theta_i(y_j)\ \ell(x,y_j,\ z_i(y_j))$$

Next, we introduce a map $\tilde{\beta}$ from Y into P (z) defined by

$$\tilde{\beta}(y) = \sum_{i=1}^{m_{y_j}} \theta_i(y_j)\ \delta_{z_i(y_j)} \quad \text{if}\quad y \quad B(y_j,r_j)\setminus \bigcup_{k=1}^{j-1} B(y_k,r_k)$$

This map obviously defines a strategy of $\beta \in \Delta$ by

$$\beta[y]_s = \tilde{\beta}(y_s) \quad \text{for} \quad s \geq 0 .$$

Moreover,

$$f(x,y_s , \beta[y]_s) = \overline{f}_{y_s} , \ell(x,y_s, \beta[y]_s) = \overline{\ell}_{y_s} .$$

Hence, for any $y \in M$, we have

$$\frac{1}{t} \{V(x) - V(x_t)e^{-t}\} - \frac{1}{t} \int_0^t e^{-s} \ell(x_s,y_s,\beta[y]_s)ds \geq - ct +$$

$$+ \frac{1}{t} \{V(x) - V(x_t) - \frac{1}{t} \int_0^t \ell(x_s,y_s, \beta[y]_s)ds + V(x) \geq$$

$$\geq - Ct + V(x) + \frac{1}{t} \{V(x) - V(x + \int_0^t f(x,y_s,\beta[y]_s)ds)\} +$$

$$- \frac{1}{t} \int_0^t \ell(x,y_s,\beta[y]_s)ds$$

$$\geq - Ct + V(x) + \frac{1}{t} \{V(x) - V(x + \int_0^t \overline{f}_{y_s} ds)\} - \frac{1}{t} \int_0^t \overline{\ell}_{y_s} ds ;$$

where C denotes various constants independent of t , y M.

Next observe that the suboptimality principle (4) yields that the left-hand side of the preceding inequality is nonpositive. We wish to reach a contradiction by showing that (5) implies that the right-hand side is nonnegative for t small. The counter example of Section 1 shows that this is not possible in general. However, in the particular situation where $\ell(x,y,z) = \ell(x)$ (hence $\ell_y = \ell(x)$ for all y) and N = 1, there exists $s_0 \in (0,t)$ such that

$$V(x) + \frac{1}{t} \{V(x) - V(x+ \int_0^t f_{y_s} ds)\} - \frac{1}{t} \int_0^t \ell_{y_s} ds =$$

$$= \frac{1}{t} \{V(x) - V(x+t \overline{f}_{y_{s_0}})\} + V (x) - \ell(x) \geq \frac{\delta}{4}$$

for $t < \overline{t}$. The contradiction proves our claim.

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