RECURSIVE RECOVERY OF A FAMILY OF MARKOV TRANSITION PROBABILITIES FROM BOUNDARY VALUE DATA

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RECURSIVE RECOVERY OF A FAMILY OF MARKOV TRANSITION PROBABILITIES FROM BOUNDARY VALUE DATA

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Abstract. Recent developments in optical imaging inspired the model of photon transport discussed below. (Infrared radiation is used to image relatively soft and homogeneous tissue.) The difficulty of solving Maxwell's equations, or even linear transport equations, led to this "diffuse tomographic" model. A recursive scheme for solving the two dimensional problem is sketched and the first recursive step is detailed.

1. Introduction. Nearly one century has passed since Röntgen took the first radiograph of his wife's hand. A concise history of the development of medical imaging can be found in [21]; more mathematical detail can be found in [1]. The word "tomography" refers to imaging an object by slices. X rays, for example, have high energy and travel straight through the body. Data analysis is linear and yields a scalar valued function. The oxymoron "diffuse tomography" refers to low energy imaging in which the paths of the radiant energy are not necessarily straight and are *unknown*. Data analysis in diffuse tomography is highly nonlinear and yields many parameters per pixel. Problems in diffuse tomography are highly nonlinear because the radiant energy is relatively low. Clinical applications such as neonatal imaging and annual mammograms are not amenable to high energy techniques which might overexpose the patient to harmful radiation. Experimentalists in the medical arena work with near infrared radiation. See [22, 23, 24, 25, 26] for cursory descriptions; [27, 28] contain more details. Because Maxwell's equations are difficult to solve for this problem many scientists and mathematicians turn to other models of photon transport.

The integral-differential equation governing linear transport theory appears more tractable than Maxwell's equations. Even so, the difficulty of handling the scattering term and the high number of scattering events led to wide adoption of a diffusion model. Although they do not account for the anisotropic scattering of photons in biological tissue, such models permit quick and accurate solution to the forward problem. See [6]

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for an overview and [7] for a finite element solution to the forward problem. Redheffer's work on network and transport theory, especially [4], was an early precursor of recent discrete models of photon transport. Inverse solutions for several isotropic lattice models were found during the late 80's [2, 3, 9, 10]. Work on the general lattice model commenced in the early 90's. The inverse solution for small systems in the plane was derived in [11]; range conditions upon the data were studied in [12, 17] for two and three dimensional systems respectively. This paper presents a solution to the inverse problem for the most general lattice model on square systems in the plane.

1.1. Description of the model. Consider an $n \times n$ array of pixels in $\mathbb{R}^2$ enclosing the object to be reconstructed. On each of the $4n$ outer edges there are two devices. One device shoots photons across the outside edge into the neighboring pixel; the other device detects photons as they leave the system. For each of the $4n$ outside edges we collect $4n$ pieces of data. Within the array, photons travel in four directions: *n*orth, *s*outh, *e*ast, and *w*est. They may change direction as long as they travel in one of the preferred directions. They do not interact and may be absorbed within a pixel. Photons move according to a Markov process. The probabilities with which a photon moves to a neighboring pixel depend upon its previous, as well as present, location. In this two step formulation the state space consists of locations. We may redefine the state space so that photons move according to a one step Markov process. In the new state space a single state consists of the photon's location and direction of travel.

There are three different types of these Markov states: incoming, outgoing, and hidden. The probabilities with which photons move from one state to another are referred to as transition probabilities. For example a photon which travels east into pixel 1,1 continues to travel east into pixel 1,2 with some probability, which we denote $e11e$. See figure 1. The same photon travels south into pixel 2,1 with probability $e11s$. These probabilities are the nonzero entries of the transition matrix, M. M is sparse and may be written as a block matrix with nontrivial subblocks which we refer to as P_{io}, P_{ih}, P_{ho}, and P_{hh}. P_{io}, for example, contains the probabilities with which photons in incoming states move directly to outgoing states. P_{ih} contains the probabilities with which photons in incoming states move to hidden states. P_{ho} and P_{hh} are the transition matrices for photons starting in hidden states traveling to outgoing and hidden states, respectively. P_{io} and P_{hh} are always square matrices. If we order the Markov states carefully, all four of these submatrices have a block structure.

1.2. Forward Problem. For a $n \times n$ system the forward map takes $16n^2$ transition probabilities to a $4n \times 4n$ data matrix Q. The domain of the forward map lies in the unit cube in $\mathbb{R}^{16n^2}$ and is defined by

$$\begin{aligned} eije + eijw + eijn + eijs &\leq 1 \\ wije + wijw + wijn + wijs &\leq 1 \\ nije + nijw + nijn + nijs &\leq 1 \\ sije + sijw + sijn + sijs &\leq 1 \end{aligned} \tag{1.1}$$

for $i, j = 1, 2, \ldots, n$. Furthermore, we assume that none of these transition probabilities

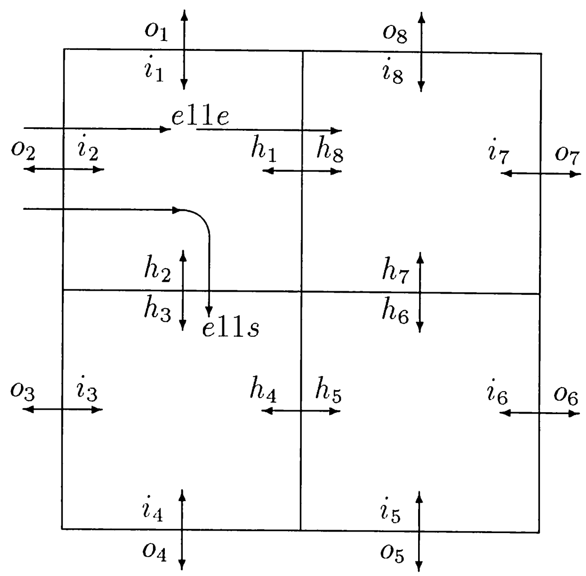

FIG. 1. 2×2 *base case. Incoming, hidden, and outgoing states are labeled with i's, h's, and o's respectively.*

is zero. $Q_{i,j}$ represents the probability that a photon which enters the system at source i exits the system at detector j. Q provides no time-of-flight information. Because Q is a transition matrix acceptable solutions also lie in the unit cube in $\mathbb{R}^{16n^2}$ and satisfy

$$0 \leq \sum_{\lambda=1}^{4n} Q[i,\lambda] \leq 1, \qquad i = 1, 2, \ldots, 4n \tag{1.2}$$

The forward map is given by the following matrix equation:

$$Q = P_{io} + P_{ih} \sum_{n=0}^{\infty} P_{hh}^{n} \, P_{ho} \tag{1.3}$$

Given a physical set of Markov transition probabilities, (equivalently the matrix M), the solution to the forward problem is

$$Q = P_{io} + P_{ih} \, (I - P_{hh})^{-1} \, P_{ho} \tag{1.4}$$

1.3. Inverse Problem. Our primary interest, however, is the inverse problem. Given Q, we wish to recover the transition probabilities. For a given object the transition probabilities give a discretized "image" of the object. In traditional imaging, we recover a single parameter per pixel. From this information a visual picture of the object is made. In diffuse tomography, however, we want to recover many parameters per pixel. From this information we could make several "pictures" of the object. In both classical and diffuse tomography, fine discretizations of the covering array are required to obtain detailed information about the object being imaged.

1.4. Useful facts. Less general diffuse tomographic models have been studied for relatively large systems; general models have been studied only for specific geometries. This section contains a brief summary of work pertinent to the rest of the paper.

1.4.1. Consistency Conditions. An understanding of range conditions on the forward map have typically preceded its inversion. Consistency conditions have been studied for the smallest nontrivial three dimensional system [17] and for the general problem in the plane [12].

A $n \times n$ system generates a $4n \times 4n$ data matrix which is subject to at least $8n(n-1)$ independent consistency conditions, where $n = 2^k$ and $k \in \mathbb{N}$. These conditions appear as rank deficient submatrices of Q. A full set of conditions can be found by considering only submatrices representing travel from one "side" of the system to the other "side". Such matrices are generically of rank n. These conditions can be derived using the Markovian nature of the system and the fact that a barrier of n hidden states separates the sides of the system.

Notation: Let $Q_{\mathbf{r},\mathbf{c}}$ denote the submatrix of Q taken from rows $\mathbf{r}$ and columns $\mathbf{c}$. Let $dQ_{\mathbf{r},\mathbf{c}}$ denote the determinant of this submatrix.

For example, the 8×8 data matrix for the 2×2 base case has four rank deficient submatrices. The submatrix representing travel from left to right, $Q_{[1,2,3,4],[5,6,7,8]}$, is generically of rank two. Similarly, the submatrices $Q_{[3,4,5,6],[1,2,7,8]}$, $Q_{[5,6,7,8],[1,2,3,4]}$, and $Q_{[1,2,7,8],[3,4,5,6]}$ are generically of rank two.

Consistency conditions have the unfortunate effect of reducing the amount of independent data. When working on an inverse problem we would like to have as much information as possible. At best, we may recover only as much information as we have independent data. In two dimensions, there are precisely as many data as unknowns. Consistency conditions amongst the data prevent inversion of the forward map.

1.4.2. Base Case. A recovery algorithm has been developed for the general model on a 2×2 array. In this base case the transition submatrices P_{io}, P_{ho}, P_{ih}, and P_{hh} are sparse 8×8 matrices. Each has 16 nonzero entries. P_{io} and P_{ho} have 4×4 blocks along their diagonals; P_{ih} and P_{hh} have identical off-diagonal block structures. See figure 2. Assuming that P_{ho} is invertible we write $A = P_{ho}^{-1}$. Several nonlinear changes of variables transform the governing equations from a highly nonlinear system to several *smaller*, *decoupled*, and *linear* systems. The transition probabilities can be recovered in terms of the data and the parameters in the matrix A. A set of ad hoc assumptions were applied to the 2×2 problem in order to close the underdetermined system of equations [11].

1.4.3. Graßmann - Plücker Embedding. Since the inverse problem involves linear systems, it is not surprising that Graßmannians and the Graßmann-Plücker embedding come into play. Graßmannians and the identities which embed them in projective space will be used in the following sections. The algebraic identities which embed $G(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are derived below. A cursory explanation of the embedding can be found in [11, 20]. For a more thorough exposition see [15, 16].

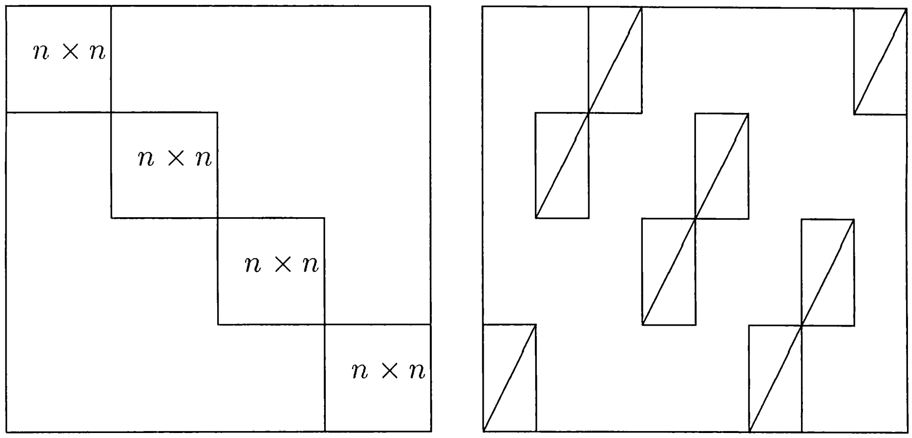

FIG. 2. *Block structures for a $n \times n$ system. The block structure of P_{io}, P_{ho}, and A are shown on the left. P_{ih} and P_{hh} have the structures of the arrays on the right. Each of the blocks in the rightmost array is $n \times n/2$.*

Let Λ be any rectangular matrix with k rows and n columns where $k < n - 1$ and $\Lambda = (a)_{ij}$. Let $I = (i_1, i_2, i_3, \ldots, i_{(k-1)})$ index $(k-1)$ distinct columns of Λ. Let $J = (j_1, j_2, j_3, \ldots, j_{(k+1)})$ index $(k+1)$ distinct columns of Λ. Consider the sum,

$$\sum_{\lambda=1}^{k+1}(-1)^{\lambda+1}\begin{vmatrix} a_{1,i_1} & a_{1,i_2} & \cdots & a_{1,i_{k-1}} & a_{1,j_\lambda} \\ \vdots & & & \vdots & \\ a_{k,i_1} & \cdots & \cdots & a_{k,i_{k-1}} & a_{k,j_\lambda} \end{vmatrix}$$

$$\begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_{\lambda-1}} & a_{1,j_{\lambda+1}} & \cdots & a_{1,j_{k+1}} \\ \vdots & & \vdots & \vdots & & \\ a_{k,j_1} & \cdots & a_{k,j_{\lambda-1}} & a_{k,j_{\lambda+1}} & \cdots & a_{k,j_{k+1}} \end{vmatrix} \tag{1.5}$$

To simplify 1.5, expand the first determinant along the last column as shown below

$$\begin{vmatrix} a_{1,i_1} & a_{1,i_2} & \cdots & a_{1,i_{k-1}} & a_{1,j_\lambda} \\ \vdots & & & & \vdots \\ a_{k,i_1} & & & a_{k,i_{k-1}} & a_{k,j_\lambda} \end{vmatrix} = \sum_{\mu=1}^{k} a_{\mu,j_\lambda} CF_\mu \tag{1.6}$$

where CF_μ is the cofactor of the matrix on the left-hand side of 1.6 about the $(\mu, k)_{th}$ entry. Then 1.5 becomes

$$\sum_{\lambda=1}^{k+1}(-1)^{\lambda+1}\sum_{\mu=1}^{k} a_{\mu,j_\lambda} CF_\mu \begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_{\lambda-1}} & a_{1,j_{\lambda+1}} & \cdots & a_{1,j_{k+1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k,j_1} & \cdots & a_{k,j_{\lambda-1}} & a_{k,j_{\lambda+1}} & \cdots & a_{k,j_{k+1}} \end{vmatrix}$$

$$= \sum_{\mu=1}^{k} CF_\mu \sum_{\lambda=1}^{k+1} \begin{vmatrix} 0 & \cdots & 0 & a_{\mu,j_\lambda} & 0 & \cdots & 0 \\ a_{1,j_1} & \cdots & a_{1,j_{\lambda-1}} & a_{1,j_\lambda} & a_{1,j_{\lambda+1}} & \cdots & a_{1,j_{k+1}} \\ \vdots & & \vdots & \vdots & & & \vdots \\ a_{k,j_1} & \cdots & a_{k,j_{\lambda-1}} & a_{k,j_\lambda} & \cdots & \cdots & a_{k,j_{k+1}} \end{vmatrix}$$

$$(1.7) \qquad = \sum_{\mu=1}^{k} CF_\mu \begin{vmatrix} a_{\mu,j_1} & a_{\mu,2} & \dots & a_{\mu,j_{k+1}} \\ a_{1,j_1} & a_{1,j_2} & \dots & a_{1,j_{k+1}} \\ \vdots & \vdots & & \vdots \\ a_{k,j_1} & a_{k,j_2} & \dots & a_{k,j_{k+1}} \end{vmatrix}$$

$$= \sum_{\mu=1}^{k} CF_\mu \cdot 0$$

$$= 0$$

Denote by π_I the determinant of the minor of Λ whose columns are indexed by the multi-index I. Then

$$(1.8) \qquad \sum_{\lambda=1}^{k+1} \pi_{(i_1,i_2,\dots,i_{k-1},j_\lambda)} \pi_{(j_1,j_2,\dots,j_{\lambda-1},j_{\lambda+1},\dots,j_{k+1})} = 0$$

Equation 1.8 defines the Graßmann relations.

2. Recursive Inversion Algorithm. The recusive algorithm developed in this section is the inverse counterpart of the ideas of invariant imbedding developed in [18, 19]. Rather than doubling the fundamental cell, the macroscopic system is split apart into subsystems, which are consequently split into more sub-subsystems. The process ends when the subsystems are sufficiently small that the microscopic level has been reached.

2.1. Writing the equations for general n. Only when $n = 2$ is the number of hidden states equal to the number of incoming and outgoing states. For a $n \times n$ system P_{io} is a $4n \times 4n$ matrix and P_{hh} is a $(4n^2 - 4n) \times (4n^2 - 4n)$ matrix. P_{ih} is a $4n \times (4n^2 - 4n)$ matrix and P_{ho} is a $(4n^2 - 4n) \times 4n$ matrix. See figure 3. For $n \geq 4$ the governing equations are so horribly large and nonlinear that MAPLE cannot even solve the *forward* problem analytically. (Inverting $(I - P_{hh})$ is too much for MAPLE.) In order to begin work on the inverse problem one must somehow cut this monstrosity down to size.

Even if MAPLE were able to handle the equations for any large $n \times n$ system the algorithm described in [11] is doomed to failure. It assumes that the transition submatrix P_{ho} is invertible. Unfortunately, P_{ho} is not square for large systems. We would like to preserve the "squareness" of the transition submatrices as well as reduce the complexity of the problem. A recursive approach allowing only one layer of hidden states at any recursive level achieves both goals. The algorithm described below decomposes the system into subsystems which are subsequently decomposed into subsystems of their own. A system is broken into subsystems by ignoring most of its hidden states. No matter how the original system is decomposed the new system may not violate the original system's range conditions.

2.2. 4×4 problem. In this section the recursive algorithm is developed in detail for a 4×4 system. Only one recursion is required and no assumptions are made to close

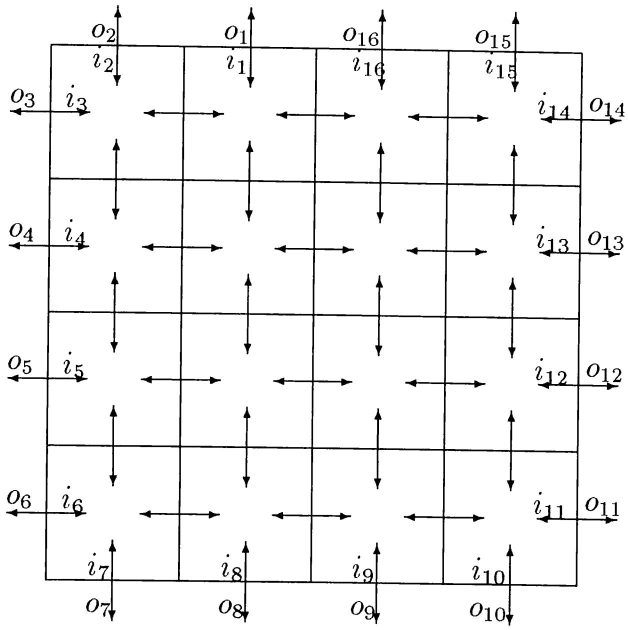

FIG. 3. *A 4×4 system. The incoming and outgoing states are labeled; all unlabeled states are hidden states. There are 16 incoming states, 16 outgoing states, and 48 hidden states.*

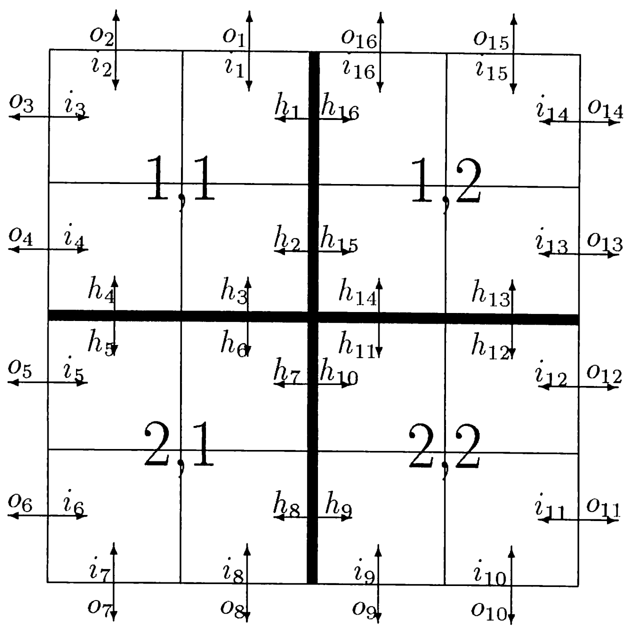

FIG. 4. *Decomposition of a 4×4 system into four 2×2 subsystems. Thick lines separate the subsystems. The "modified" 4×4 system disregards individual pixels. Only the subsystems are relevant at the first level of this recursive procedure.*

the resulting underdetermined systems of equations. The 16×16 data matrix is subject to $8 * 4(4-1) = 96$ independent consistency conditions, leaving only $256 - 96 = 160$ independent data. Since there are $16 * 16 = 256$ unknown transition probabilities a 96 parameter family of solutions to the 4×4 problem is found in sections 2.2.2 and 2.2.3.

2.2.1. Decomposing into subsystems. Consider a $n \times n$ system. Let b_h be a straight horizontal barrier separating the bottom of the system from the top. Let b_v be a vertical barrier. There are exactly $4n$ hidden states associated with b_h and b_v. (Each of the barriers is associated with two rank deficient submatrices of rank n. The vertical barrier is associated with a right-left as well as a left-right submatrix; the horizontal barrier is associated with a top-bottom as well as a bottom-top submatrix.) Recall that there are exactly $4n$ incoming and $4n$ outgoing states.

Consider the example in figure 4. The 4×4 array of pixels is decomposed into four subarrays labeled 11, 12, 21, and 22. There are 16 incoming states and 16 outgoing states. There are 16 relevant hidden states, those associated with the barriers. The incoming states which send photons into a subarray are adjacent only to hidden and outgoing states which send photons out of that subarray. Similarly, hidden states sending photons into one subarray are adjacent to hidden and outgoing states which send photons out of that subarray. Finally, hidden states are adjacent to outgoing states which send photons out of that subarray. As in the base case, it is assumed that photons can only travel from one state to adjacent states.

The shortest possible path between states in this modified system may require that the photon travel several steps in the original system. The most important thing to notice is that these modified transition probabilities are the *data* for the subarrays. These transitions matrices have the same block structure as their counterparts for the base case. See figure 2.

2.2.2. Solving for P_{ih}, P_{hh}, P_{io}, and P_{ho} in terms of A. We start by labeling the states for the 4×4 system as in figure 4, and each of the 2×2 subsystems as in figure 1. The transition matrices for the modified system are sparse block matrices. These matrices are all 16×16 and have the "same" block structures as their counterparts for the 2×2 base case. See figure 2. Examples of paths taken into account by the modified transition probabilities are shown in figure 5.

The governing matrix equation, 1.4, may be rewritten as the following:

$$(Q - P_{io})A(I - P_{hh}) - P_{ih} = \Theta \tag{2.1}$$

where Q is the data matrix, P_{io}, P_{ih}, P_{ho}, and P_{hh} are probability transition matrices for this modified system and $A = P_{ho}^{-1}$. The following changes of variables allow us to "remove" the nonlinearity from 2.1.

$$\begin{aligned} W &= AP_{hh} \\ X &= P_{io}A \\ Y &= P_{io}W - P_{ih} \end{aligned} \tag{2.2}$$

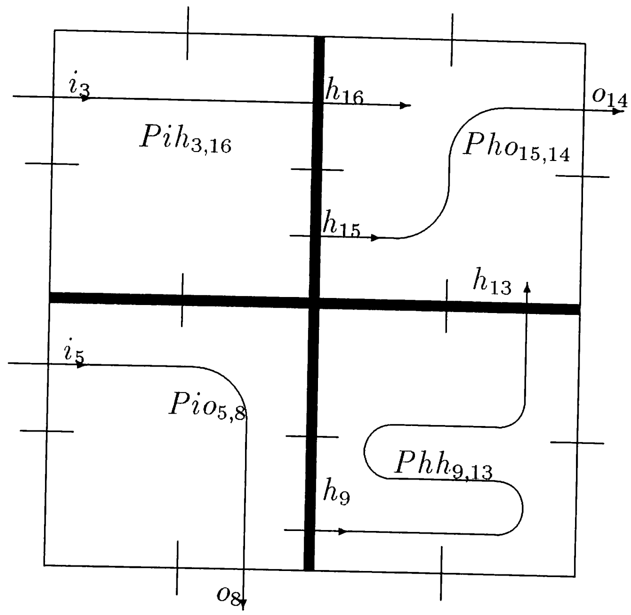

FIG. 5. *Examples of paths which are taken into account by transition probabilities for this modified system.*

Notice that X has the same zero structure as P_{ho} and P_{io} and that Y and W have identical zero structures as P_{ih} and P_{hh}. Equation 2.1 becomes linear in the unknowns A, W, X, and Y.

(2.3) $$Q(A - W) - (X - Y) = \Theta$$

As in the base case the columns in 2.3 come in groups; four of the columns correspond to the same matrix equation. The eleventh through fourteenth columns of 2.3 are written below.

$$
\begin{bmatrix}
Q_{1,9} & Q_{1,10} & Q_{1,11} & Q_{1,12} & Q_{1,13} & Q_{1,14} & Q_{1,15} & Q_{1,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{2,9} & Q_{2,10} & Q_{2,11} & Q_{2,12} & Q_{2,13} & Q_{2,14} & Q_{2,15} & Q_{2,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{3,9} & Q_{3,10} & Q_{3,11} & Q_{3,12} & Q_{3,13} & Q_{3,14} & Q_{3,15} & Q_{3,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{4,9} & Q_{4,10} & Q_{4,11} & Q_{4,12} & Q_{4,13} & Q_{4,14} & Q_{4,15} & Q_{4,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{5,9} & Q_{5,10} & Q_{5,11} & Q_{5,12} & Q_{5,13} & Q_{5,14} & Q_{5,15} & Q_{5,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{6,9} & Q_{6,10} & Q_{6,11} & Q_{6,12} & Q_{6,13} & Q_{6,14} & Q_{6,15} & Q_{6,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{7,9} & Q_{7,10} & Q_{7,11} & Q_{7,12} & Q_{7,13} & Q_{7,14} & Q_{7,15} & Q_{7,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{8,9} & Q_{8,10} & Q_{8,11} & Q_{8,12} & Q_{8,13} & Q_{8,14} & Q_{8,15} & Q_{8,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{9,9} & Q_{9,10} & Q_{9,11} & Q_{9,12} & Q_{9,13} & Q_{9,14} & Q_{9,15} & Q_{9,16} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{10,9} & Q_{10,10} & Q_{10,11} & Q_{10,12} & Q_{10,13} & Q_{10,14} & Q_{10,15} & Q_{10,16} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{11,9} & Q_{11,10} & Q_{11,11} & Q_{11,12} & Q_{11,13} & Q_{11,14} & Q_{11,15} & Q_{11,16} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
Q_{12,9} & Q_{12,10} & Q_{12,11} & Q_{12,12} & Q_{12,13} & Q_{12,14} & Q_{12,15} & Q_{12,16} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
Q_{13,9} & Q_{13,10} & Q_{13,11} & Q_{13,12} & Q_{13,13} & Q_{13,14} & Q_{13,15} & Q_{13,16} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
Q_{14,9} & Q_{14,10} & Q_{14,11} & Q_{14,12} & Q_{14,13} & Q_{14,14} & Q_{14,15} & Q_{14,16} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
Q_{15,9} & Q_{15,10} & Q_{15,11} & Q_{15,12} & Q_{15,13} & Q_{15,14} & Q_{15,15} & Q_{15,16} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Q_{16,9} & Q_{16,10} & Q_{16,11} & Q_{16,12} & Q_{16,13} & Q_{16,14} & Q_{16,15} & Q_{16,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
(2.4) \qquad
\begin{bmatrix}
A_{9,11} & A_{9,12} & -W_{9,13} & -W_{9,14} \\
A_{10,11} & A_{10,12} & -W_{10,13} & -W_{10,14} \\
A_{11,11} & A_{11,12} & -W_{11,13} & -W_{11,14} \\
A_{12,11} & A_{12,12} & -W_{12,13} & -W_{12,14} \\
-W_{13,11} & -W_{13,12} & A_{13,13} & A_{13,14} \\
-W_{14,11} & -W_{14,12} & A_{14,13} & A_{14,14} \\
-W_{15,11} & -W_{15,12} & A_{15,13} & A_{15,14} \\
-W_{16,11} & -W_{16,12} & A_{16,13} & A_{16,14} \\
X_{9,11} & X_{9,12} & Y_{9,13} & Y_{9,14} \\
X_{10,11} & X_{10,12} & Y_{10,13} & Y_{10,14} \\
X_{11,11} & X_{11,12} & Y_{11,13} & Y_{11,14} \\
X_{12,11} & X_{12,12} & Y_{12,13} & Y_{12,14} \\
Y_{13,11} & Y_{13,12} & X_{13,13} & X_{13,14} \\
Y_{14,11} & Y_{14,12} & X_{14,13} & X_{14,14} \\
Y_{15,11} & Y_{15,12} & X_{15,13} & X_{15,14} \\
Y_{16,11} & Y_{16,12} & X_{16,13} & X_{16,14}
\end{bmatrix}
= \Theta
$$

Since this is a homogenous linear system the solution is interesting only if the leftmost matrix is rank deficient. Fortunately, $Q_{[1,\ldots,8],[9,\ldots,16]}$ represents travel from the left half of the system to the right half; $Q_{[1,\ldots,8],[9,\ldots,16]}$ is generically of rank four. We can solve for the $W_{i,j}$'s, $X_{i,j}$'s, and $Y_{i,j}$'s in terms of the $A_{i,j}$'s. Solving the first two columns of 2.4 is equivalent to solving

$$\begin{bmatrix} Q_{5,13} & Q_{5,14} & Q_{5,15} & Q_{5,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{6,13} & Q_{6,14} & Q_{6,15} & Q_{6,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{7,13} & Q_{7,14} & Q_{7,15} & Q_{7,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{8,13} & Q_{8,14} & Q_{8,15} & Q_{8,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{9,13} & Q_{9,14} & Q_{9,15} & Q_{9,16} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{10,13} & Q_{10,14} & Q_{10,15} & Q_{10,16} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{11,13} & Q_{11,14} & Q_{11,15} & Q_{11,16} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ Q_{12,13} & Q_{12,14} & Q_{12,15} & Q_{12,16} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ Q_{13,13} & Q_{13,14} & Q_{13,15} & Q_{13,16} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ Q_{14,13} & Q_{14,14} & Q_{14,15} & Q_{14,16} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ Q_{15,13} & Q_{15,14} & Q_{15,15} & Q_{15,16} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ Q_{16,13} & Q_{16,14} & Q_{16,15} & Q_{16,16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -W_{13,11} & -W_{13,12} \\ -W_{14,11} & -W_{14,12} \\ -W_{15,11} & -W_{15,12} \\ -W_{16,11} & -W_{16,12} \\ X_{9,11} & X_{9,12} \\ X_{10,11} & X_{10,12} \\ X_{11,11} & X_{11,12} \\ X_{12,11} & X_{12,12} \\ Y_{13,11} & Y_{13,12} \\ Y_{14,11} & Y_{14,12} \\ Y_{15,11} & Y_{15,12} \\ Y_{16,11} & Y_{16,12} \end{bmatrix}$$

$$(2.5) \qquad = \begin{bmatrix} Q_{5,9} & Q_{5,10} & Q_{5,11} & Q_{5,12} \\ Q_{6,9} & Q_{6,10} & Q_{6,11} & Q_{6,12} \\ Q_{7,9} & Q_{7,10} & Q_{7,11} & Q_{7,12} \\ Q_{8,9} & Q_{8,10} & Q_{8,11} & Q_{8,12} \\ Q_{9,9} & Q_{9,10} & Q_{9,11} & Q_{9,12} \\ Q_{10,9} & Q_{10,10} & Q_{10,11} & Q_{10,12} \\ Q_{11,9} & Q_{11,10} & Q_{11,11} & Q_{11,12} \\ Q_{12,9} & Q_{12,10} & Q_{12,11} & Q_{12,12} \\ Q_{13,9} & Q_{13,10} & Q_{13,11} & Q_{13,12} \\ Q_{14,9} & Q_{14,10} & Q_{14,11} & Q_{14,12} \\ Q_{15,9} & Q_{15,10} & Q_{15,11} & Q_{15,12} \\ Q_{16,9} & Q_{16,10} & Q_{16,11} & Q_{16,12} \end{bmatrix} \begin{bmatrix} A_{9,11} & A_{9,12} \\ A_{10,11} & A_{10,12} \\ A_{11,11} & A_{11,12} \\ A_{12,11} & A_{12,12} \end{bmatrix}$$

Doing the same for other pairs of columns yields solutions for W, X, and Y in terms of A and exhausts the supply of equations given by equation 2.3. Since A is invertible we can return to our original variables, computing the entries of P_{ih}, P_{hh}, P_{io}, and P_{ho} in terms of the data and $A_{i,j}$'s. The forms of the solutions are similar among variables from the same transition matrix. Samples of solutions in terms of $A_{i,j}$'s for one variable from each matrix are listed below. The simplest solutions are for the entries of P_{ho}:

$$Pho_{4,3} = -\frac{dA_{[1,2,4],[1,2,3]}}{dA_{[1,2,3,4],[1,2,3,4]}} \tag{2.6}$$

The next simplest solutions are those for entries of P_{hh}.

$$\begin{aligned} &Phh_{3,15} = \\ &\quad -\Big(dA_{[1,2,3],[1,2,4]}\Big(\mathrm{dQ}_{[5,6,7,8],[1,2,3,13]}A_{13,15} + \mathrm{dQ}_{[5,6,7,8],[1,2,3,14]}A_{14,15} \\ &\qquad +\mathrm{dQ}_{[5,6,7,8],[1,2,3,15]}A_{15,15} + \mathrm{dQ}_{[5,6,7,8],[1,2,3,16]}A_{16,15}\Big) \\ &\quad + dA_{[1,2,4],[1,2,4]}\Big(\mathrm{dQ}_{[5,6,7,8],[1.2,4,13]}A_{13,15} + \mathrm{dQ}_{[5,6,7,8],[1,2,4,14]}A_{14,15} \\ &\qquad +\mathrm{dQ}_{[5,6,7,8],[1,2,4,15]}A_{15,15} + \mathrm{dQ}_{[5,6,7,8],[1,2,4,16]}A_{16,15}\Big) \\ &\quad + dA_{[1,3,4],[1,2,4]}\Big(\mathrm{dQ}_{[5,6,7,8],[1,3,4,13]}A_{13,15} + \mathrm{dQ}_{[5,6,7,8],[1,3,4,14]}A_{14,15} \\ &\qquad +\mathrm{dQ}_{[5,6,7,8],[1,3,4,15]}A_{15,15} + \mathrm{dQ}_{[5,6,7,8],[1,3,4,16]}A_{16,15}\Big) \\ &\quad + dA_{[2,3,4],[1,2,4]}\Big(\mathrm{dQ}_{[5,6,7,8],[2,3,4,13]}A_{13,15} + \mathrm{dQ}_{[5,6,7,8],[2,3,4,14]}A_{14,15} \\ &\qquad +\mathrm{dQ}_{[5,6,7,8],[2,3,4,15]}A_{15,15} + \mathrm{dQ}_{[5,6,7,8],[2,3,4,16]}A_{16,15}\Big)\Big) / \\ &\qquad\quad \mathrm{dQ}_{[5,6,7,8],[1,2,3,4]}dA_{[1,2,3,4],[1,2,3,4]} \end{aligned} \tag{2.7}$$

The solutions for the entries of P_{io} are a little bit longer:

$$\begin{aligned} &Pio_{5,6} = -\frac{1}{dA_{[5,6,7,8],[5,6,7,8]}} \\ &\quad\Big(dA_{[5,7,8],[6,7,8]}\Big(\mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,5]}A_{5,5}+ \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,6]}A_{6,5} + \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,7]}A_{7,5} + \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]}A_{8,5}\Big) / \mathrm{dQ}_{[13,14,15,16],[1,2,3,4]} - \\ &\quad dA_{[5,7,8],[5,7,8]}\Big(\mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,5]}A_{5,6}+ \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,6]}A_{6,6} + \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,7]}A_{7,6} + \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]}A_{8,6}\Big) / \mathrm{dQ}_{[13,14,15,16],[1,2,3,4]} - \\ &\quad dA_{[5,7,8],[5,6,7]}\Big(\mathrm{dQ}_{[5,13,14,15,16],[5,9,10,11,12]}A_{5,8}+ \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[6,9,10,11,12]}A_{6,8} + \mathrm{dQ}_{[5,13,14,15,16],[7,9,10,11,12]}A_{7,8} + \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]}A_{8,8}\Big) / \mathrm{dQ}_{[13,14,15,16],[9,10,11,12]} \\ &\quad dA_{[5,7,8],[5,6,8]}\Big(\mathrm{dQ}_{[5,13,14,15,16],[5,9,10,11,12]}A_{5,7}+ \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[6,9,10,11,12]}A_{6,7} + \mathrm{dQ}_{[5,13,14,15,16],[7,9,10,11,12]}A_{7,7} + \\ &\qquad \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]}A_{8,7}\Big) / \mathrm{dQ}_{[13,14,15,16],[9,10,11,12]}\Big) \end{aligned} \tag{2.8}$$

Solutions for the entries of P_{ih} are of the form:

$$
\begin{aligned}
Pih_{10,14} = & \frac{1}{\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}} \\
& \Big(\mathrm{dQ}_{[5,6,7,8,10],[9,10,11,12,13]}A_{13,14} + \mathrm{dQ}_{[5,6,7,8,10],[9,10,11,12,14]}A_{14,14} + \\
& \mathrm{dQ}_{[5,6,7,8,10],[9,10,11,12,15]}A_{15,14} + \mathrm{dQ}_{[5,6,7,8,10],[9,10,11,12,16]}A_{16,14}\Big) + \\
& \frac{1}{\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}dA_{[9,10,11,12],[9,10,11,12]}} \Big(\Big(\mathrm{dQ}_{[5,6,7,8],[9,11,12,13]}A_{13,14}+ \\
& \mathrm{dQ}_{[5,6,7,8],[9,11,12,14]}A_{14,14} + \mathrm{dQ}_{[5,6,7,8],[9,11,12,15]}A_{15,14} + \\
& \mathrm{dQ}_{[5,6,7,8],[9,11,12,16]}A_{16,14}\Big) \\
& \Big(dA_{[9,11,12],[9,10,11]}\Big(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,12} \\
& + \mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,12} + dQ_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,12} \\
& +\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,12}\Big) / \mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} \\
& - dA_{[9,11,12],[9,10,12]}\Big(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,11} \\
& + \mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,11} + \mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,11} \\
& +\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,11}\Big) / \mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} \\
& + dA_{[9,11,12],[9,11,12]}\Big(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,10} \\
& + \mathrm{dQ}_{[10,13.14,15,16],[5,6,7,8,10]}A_{10,10} + \mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,10} \\
& +\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,10}\Big) / \mathrm{dQ}_{[13,14,15,16],[5,6,7,8]} \\
& - dA_{[9,11,12],[10,11,12]}\Big(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,9} \\
& + \mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,9} + \mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,9} \\
& +\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,9}\Big) / \mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}\Big)
\end{aligned} \tag{2.9}
$$

$$+\left(\mathrm{dQ}_{[5,6,7,8],[9,10,12,13]}A_{13,14}+\mathrm{dQ}_{[5,6,7,8],[9,10,12,14]}A_{14,14}\right.$$
$$\left.+\mathrm{dQ}_{[5,6,7,8],[9,10,12,15]}A_{15,14}+\mathrm{dQ}_{[5,6,7,8],[9,10,12,16]}A_{16,14}\right)$$
$$\left(dA_{[9,10,12],[9,10,11]}\left(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,12}\right.\right.$$
$$+\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,12}+\mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,12}$$
$$\left.+\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,12}\right)/\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}$$
$$-dA_{[9,10,12],[9,10,12]}\left(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,11}\right.$$
$$+\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,11}+\mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,11}$$
$$\left.+\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,11}\right)/\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}$$
$$+dA_{[9,10,12],[9,11,12]}\left(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,10}\right.$$
$$+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,10}+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,10}$$
$$\left.+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,10}\right)/\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}$$
$$-dA_{[9,10,12],[10,11,12]}\left(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,9}\right.$$
$$+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,9}+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,9}$$
$$\left.\left.+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,9}\right)/\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}\right)$$
$$+\left(\mathrm{dQ}_{[5,6,7,8],[10,11,12,13]}A_{13,14}+\mathrm{dQ}_{[5,6,7,8],[10,11,12,14]}A_{14,14}\right.$$
$$\left.+\mathrm{dQ}_{[5,6,7,8],[10,11,12,15]}A_{15,14}+\mathrm{dQ}_{[5,6,7,8],[10,11,12,16]}A_{16,14}\right)$$

$$
\begin{aligned}
&\Big(dA_{[10,11,12],[9,10,11]}\Big(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,12}\\
&\quad+\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,12}+\mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,12}\\
&\qquad+\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,12}\Big)/\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\\
&-dA_{[10,11,12],[9,10,12]}\Big(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,11}\\
&\quad+\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,11}+\mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,11}\\
&\qquad+\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,11}\Big)/\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\\
&+dA_{[10,11,12],[9,11,12]}\Big(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,10}\\
&\quad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,10}+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,10}\\
&\qquad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,10}\Big)/\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}\\
&-dA_{[10,11,12],[10,11,12]}\Big(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,9}\\
&\quad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,9}+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,9}\\
&\qquad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,9}\Big)/\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}\Big)\\
&+\Big(\mathrm{dQ}_{[5,6,7,8],[9,10,11,13]}A_{13,14}+\mathrm{dQ}_{[5,6,7,8],[9,10,11,14]}A_{14,14}\\
&\qquad+\mathrm{dQ}_{[5,6,7,8],[9,10,11,15]}A_{15,14}+\mathrm{dQ}_{[5,6,7,8],[9,10,11,16]}A_{16,14}\Big)\\
&\Big(dA_{[9,10,11],[9,10,11]}\Big(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,12}\\
&\quad+\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,12}+\mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,12}\\
&\qquad+\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,12}\Big)/\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\\
&-dA_{[9,10,11],[9,10,12]}\Big(\mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}A_{9,11}\\
&\quad+\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}A_{10,11}+\mathrm{dQ}_{[5,6,7,8,10],[11,13,14,15,16]}A_{11,11}\\
&\qquad+\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}A_{12,11}\Big)/\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\\
&+dA_{[9,10,11],[9,11,12]}\Big(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,10}\\
&\quad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,10}+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,10}\\
&\qquad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,10}\Big)/\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}\\
&-dA_{[9,10,11],[10,11,12]}\Big(\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,10]}A_{10,9}\\
&\quad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,9]}A_{9,9}+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,11]}A_{11,9}\\
&\qquad+\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}A_{12,9}\Big)/\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}\Big)\Big)
\end{aligned}
$$

2.2.3. Eliminating $A_{i,j}$'s. Each of the four subsystems has an 8×8 data matrix which is subject to consistency conditions. The data matrix for the 1,1 subsystem can be written in terms of the entries in P_{io}, P_{ih}, P_{hh}, and P_{ho}.

$$
(2.10)\,\mathrm{Q11} = \begin{bmatrix}
Pio_{2,2} & Pio_{2,3} & Pio_{2,4} & Pih_{2,5} & Pih_{2,6} & Pih_{2,15} & Pih_{2,16} & Pio_{2,1} \\
Pio_{3,2} & Pio_{3,3} & Pio_{3,4} & Pih_{3,5} & Pih_{3,6} & Pih_{3,15} & Pih_{3,16} & Pio_{3,1} \\
Pio_{4,2} & Pio_{4,3} & Pio_{4,4} & Pih_{4,5} & Pih_{4,6} & Pih_{4,15} & Pih_{4,16} & Pio_{4,1} \\
Pho_{4,2} & Pho_{4,3} & Pho_{4,4} & Phh_{4,5} & Phh_{4,6} & Phh_{4,15} & Phh_{4,16} & Pho_{4,1} \\
Pho_{3,2} & Pho_{3,3} & Pho_{3,4} & Phh_{3,5} & Phh_{3,6} & Phh_{3,15} & Phh_{3,16} & Pho_{3,1} \\
Pho_{2,2} & Pho_{2,3} & Pho_{2,4} & Phh_{2,5} & Phh_{2,6} & Phh_{2,15} & Phh_{2,16} & Pho_{2,1} \\
Pho_{1,2} & Pho_{1,3} & Pho_{1,4} & Phh_{1,5} & Phh_{1,6} & Phh_{1,15} & Phh_{1,16} & Pho_{1,1} \\
Pio_{1,2} & Pio_{1,3} & Pio_{1,4} & Pih_{1,5} & Pih_{1,6} & Pih_{1,15} & Pih_{1,16} & Pio_{1,1}
\end{bmatrix}
$$

$Q11$ has four rank deficient submatrices. They are 4×4 submatrices of rank two (or less). Two constraints are required to force a generic vector in $\mathbb{R}^4$ to lie in a given two dimensional subspace. Four conditions are required, therefore, to force a generic 4×4 matrix to be of rank two. These consistency conditions upon $Q11$ may be expressed as the vanishing of 3×3 minors. Substituting the solutions for the modified transition probabilities into these minors forces highly nonlinear polynomials of the $A_{i,j}$s to be identically zero. These conditions will be studied in order of increasing complexity. (Clearly, the conditions which involve variables from P_{ih} are bound to be horrendous, so they are not considered until much later.) Eight of the conditions are identities of the form $A_{i,j} = 0$. The rest reduce (at a generic point) to four term linear equations.

Notation: Right-left, left-right, top-bottom, and bottom-top rank deficient submatrices are labeled as Qij_{rl}, Qij_{lr}, Qij_{tb}, and Qij_{bt} where $i, j = 1, 2$. Consider the submatrix representing travel from right to left across subsystem $1,1$:

$$
Q11_{rl} = \begin{bmatrix}
\mathrm{Q11}_{5,1} & \mathrm{Q11}_{5,2} & \mathrm{Q11}_{5,3} & \mathrm{Q11}_{5,4} \\
\mathrm{Q11}_{6,1} & \mathrm{Q11}_{6,2} & \mathrm{Q11}_{6,3} & \mathrm{Q11}_{6,4} \\
\mathrm{Q11}_{7,1} & \mathrm{Q11}_{7,2} & \mathrm{Q11}_{7,3} & \mathrm{Q11}_{7,4} \\
\mathrm{Q11}_{8,1} & \mathrm{Q11}_{8,2} & \mathrm{Q11}_{8,3} & \mathrm{Q11}_{8,4}
\end{bmatrix}
$$

$$
(2.11) \qquad = \begin{bmatrix}
Pho_{3,2} & Pho_{3,3} & Pho_{3,4} & Phh_{3,5} \\
Pho_{2,2} & Pho_{2,3} & Pho_{2,4} & Phh_{2,5} \\
Pho_{1,2} & Pho_{1,3} & Pho_{1,4} & Phh_{1,5} \\
Pio_{1,2} & Pio_{1,3} & Pio_{1,4} & Pih_{1,5}
\end{bmatrix}
$$

Identities. Since $Q11_{rl}$ is rank two, any of its 3×3 minors has a zero determinant. Hence,

$$(2.12) \qquad 0 = \begin{vmatrix} Q11_{5,1} & Q11_{5,2} & Q11_{5,3} \\ Q11_{6,1} & Q11_{6,2} & Q11_{6,3} \\ Q11_{7,1} & Q11_{7,2} & Q11_{7,3} \end{vmatrix} = \begin{vmatrix} Pho_{3,2} & Pho_{3,3} & Pho_{3,4} \\ Pho_{2,2} & Pho_{2,3} & Pho_{2,4} \\ Pho_{1,2} & Pho_{1,3} & Pho_{1,4} \end{vmatrix}$$

Recall that A is a 16×16 block matrix with 4×4 blocks on the diagonal and that $A = P_{ho}^{-1}$. Therefore,

$$(2.13) \qquad -A_{1,4} = \begin{vmatrix} Pho_{1,2} & Pho_{1,3} & Pho_{1,4} \\ Pho_{2,2} & Pho_{2,3} & Pho_{2,4} \\ Pho_{3,2} & Pho_{3,3} & Pho_{3,4} \end{vmatrix} / dPho_{[1,2,3,4],[1,2,3,4]} = 0$$

The same reasoning applies to $Q11_{bt}$ and shows that $A_{4,1} = 0$. This argument also applies to the rank-deficient submatrices $Q21_{rl}$, $Q21_{tb}$, $Q12_{lr}$, $Q12_{bt}$, $Q22_{tb}$, and $Q22_{lr}$ and yields the following identities:

$$(2.14) \qquad \begin{array}{cccc} A_{1,4} = 0, & A_{4,1} = 0, & A_{5,8} = 0, & A_{8,5} = 0 \\ A_{9,12} = 0, & A_{12,9} = 0, & A_{13,16} = 0, & A_{16,13} = 0 \end{array}$$

So really the upper left subblock of A looks like

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ 0 & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix}$$

For larger systems there are even more zero valued $A_{i,j}$s. In the first recursive step for the 8×8 problem A is a 32×32 block diagonal matrix with four 8×8 blocks along the diagonal. For exactly the same reason that the blocks of A in the 4×4 problem have zero valued corners the blocks of A for the 8×8 problem have three zero valued entries in each of their off diagonal corners. The upper left block has the zero structure:

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & 0 & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & A_{4,8} \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} & A_{5,8} \\ A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} & A_{6,8} \\ 0 & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & A_{7,8} \\ 0 & 0 & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & A_{8,7} & A_{8,8} \end{bmatrix}$$

In general, for a $n \times n$ where $n = 2^k, k \in \mathbb{N}$, the matrix A at the first level in this recursive algorithm has four $n \times n$ blocks and each of these blocks contains $\sum_{j=1}^{(k-1)} j = \frac{k(k-1)}{2}$ zeros in each of its off diagonal corners.

Easy Conditions. Notice that there are sixteen 3×3 minors of the matrix $Q11_{rl}$. Each rank deficient submatrix like $Q11_{rl}$ yields at most four independent consistency conditions. Since we already know that $A_{4,1} \equiv 0$, we can hope to get at most three more independent conditions by setting the 3×3 minors of $Q11_{rl}$ to zero. When the other fifteen 3×3 minors are first written down, they seem highly nonlinear, but upon closer inspection they proved to be quite simple. Graßmann relations may be used to simplify the equations. Although we need not consider all fifteen remaining minors, we do so for the $Q11_{rl}$ submatrix. (In later sections minors of other matrices will turn out to be so cumbersome that we only consider an independent set of minors.)

"Easy" Conditions before Graßmann

Eight of the minors factor very easily. The minor $dQ11_{rl[2,3,4],[1,2,3]}$ factors to become

$$\begin{aligned}
&\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} \\
&\quad \big(-dA_{[1,2,4],[1,2,3]} \; dA_{[1,3,4],[1,3,4]} \; dA_{[1,2,3],[2,3,4]} + \\
&\quad dA_{[1,2,4],[2,3,4]} \; dA_{[1,3,4],[1,3,4]} \; dA_{[1,2,3],[1,2,3]} - \\
&\quad dA_{[1,2,4],[1,3,4]} \; dA_{[1,3,4],[2,3,4]} \; dA_{[1,2,3],[1,2,3]} + \\
&\quad dA_{[1,2,4],[1,3,4]} \; dA_{[1,2,3],[2,3,4]} \; dA_{[1,3,4],[1,2,3]} - \\
&\quad dA_{[1,2,4],[2,3,4]} \; dA_{[1,2,3],[1,3,4]} \; dA_{[1,3,4],[1,2,3]} + \\
&\quad dA_{[1,2,4],[1,2,3]} \; dA_{[1,2,3],[1,3,4]} \; dA_{[1,3,4],[2,3,4]}\big) \\
&\quad \big(\mathrm{dQ}_{[1,13,14,15,16],[3,5,6,7,8]} A_{3,4} + \mathrm{dQ}_{[1,13,14,15,16],[4,5,6,7,8]} A_{4,4} + \\
&\qquad \mathrm{dQ}_{[1,13,14,15,16],[1,5,6,7,8]} A_{1,4} + A_{2,4} \mathrm{dQ}_{[1,13,14,15,16],[2,5,6,7,8]}\big)
\end{aligned} \tag{2.15}$$

The minors $dQ11_{rl[1,3,4],[1,2,3]}$ and $dQ11_{rl[1,2,4],[1,2,3]}$ take the same form, sharing $dQ11_{rl[2,3,4],[1,2,3]}$'s linear term. By permuting the rows and columns of A their cubic terms could be made identical to $dQ11_{rl[2,3,4],[1,2,3]}$'s. See [20] for more details.

"Easy" Conditions after Graßmann

Graßmann relations can be used to simplify the minors $dQ11_{rl[2,3,4],[1,2,3]}$, $dQ11_{rl[1,3,4],[1,2,3]}$, and $dQ11_{rl[1,2,4],[1,2,3]}$. The computation is carried out explicitly here only for 2.15. The cubic term in equation 2.15 may be rewritten as

$$\begin{aligned}(2.16)\; & dA_{[1,2,4],[1,3,4]}\left(dA_{[1,2,3],[2,3,4]}dA_{[1,3,4],[1,2,3]} - dA_{[1,3,4],[2,3,4]}dA_{[1,2,3],[1,2,3]}\right) - \\ & dA_{[1,2,4],[2,3,4]}\left(-dA_{[1,2,3],[1,2,3]}dA_{[1,3,4],[1,3,4]} + dA_{[1,2,3],[1,3,4]}dA_{[1,3,4],[1,2,3]}\right) + \\ & dA_{[1,2,4],[1,2,3]}\left(dA_{[1,2,3],[1,3,4]}dA_{[1,3,4],[2,3,4]} - dA_{[1,2,3],[2,3,4]}dA_{[1,3,4],[1,3,4]}\right)\end{aligned}$$

Using the matrix

$$(2.17)\qquad \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & 1 & 0 & 0 & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & 0 & 1 & 0 & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & 0 & 0 & 1 & 0 \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & 0 & 0 & 0 & 1 \end{bmatrix}$$

we may rewrite 2.16 in Graßmann notation as

$$\begin{aligned}(2.18)\qquad & \pi_{1,3,4,7}\;(\pi_{2,3,4,8}\;\pi_{1,2,3,6} - \pi_{2,3,4,6}\;\pi_{1,2,3,8}) + \\ & \pi_{2,3,4,7}\;(-\pi_{1,3,4,6}\;\pi_{1,2,3,8} + \pi_{1,3,4,8}\;\pi_{1,2,3,6}) + \\ & \pi_{1,2,3,7}\;(-\pi_{1,3,4,8}\;\pi_{2,3,4,6} + \pi_{1,3,4,6}\;\pi_{2,3,4,8})\end{aligned}$$

We may next make use of the Graßmann relations

$$\begin{aligned} 0 &= \pi_{2,3,4,8}\;\pi_{1,2,3,6} - \pi_{2,3,4,1}\;\pi_{8,2,3,6} + \pi_{2,3,4,2}\;\pi_{8,1,3,6} - \\ (2.19)\qquad & \qquad \pi_{2,3,4,3}\;\pi_{8,1,2,6} + \pi_{2,3,4,6}\;\pi_{8,1,2,3} \\ &= \pi_{2,3,4,8}\;\pi_{1,2,3,6} - \pi_{2,3,4,6}\;\pi_{1,2,3,8} - \pi_{2,3,6,8}\;\pi_{1,2,3,4}\;, \end{aligned}$$

$$\begin{aligned} 0 &= \pi_{1,3,4,8}\;\pi_{1,2,3,6} - \pi_{1,3,4,1}\;\pi_{8,2,3,6} + \pi_{1,3,4,2}\;\pi_{8,1,3,6} - \\ (2.20)\qquad & \qquad \pi_{1,3,4,3}\;\pi_{8,1,2,6} + \pi_{1,3,4,6}\;\pi_{8,1,2,3} \\ &= \pi_{1,3,4,8}\;\pi_{1,2,3,6} - \pi_{1,3,4,6}\;\pi_{1,2,3,8} - \pi_{1,3,6,8}\;\pi_{1,2,3,4}\;, \quad \text{and} \end{aligned}$$

$$\begin{aligned} 0 &= \pi_{1,3,4,8}\;\pi_{2,3,4,6} - \pi_{1,3,4,2}\;\pi_{8,3,4,6} + \pi_{1,3,4,3}\;\pi_{8,2,4,6} - \\ (2.21)\qquad & \qquad \pi_{1,3,4,4}\;\pi_{8,2,3,6} + \pi_{1,3,4,6}\;\pi_{8,2,3,4} \\ &= \pi_{1,3,4,8}\;\pi_{2,3,4,6} - \pi_{1,3,4,6}\;\pi_{2,3,4,8} + \pi_{3,4,6,8}\;\pi_{1,2,3,4} \end{aligned}$$

Using these relations, the expression in 2.18 may be simplified as

$$(2.22)\qquad \pi_{1,3,4,7}\ \pi_{2,3,6,8}\ \pi_{1,2,3,4}\ -\ \pi_{2,3,4,7}\ \pi_{1,3,6,8}\ \pi_{1,2,3,4}\ +\ \pi_{1,2,3,7}\ \pi_{3,4,6,8}\ \pi_{1,2,3,4}$$

Finally, we can make use of the Graßmann relation

$$(2.23)\qquad \pi_{1,3,4,7}\ \pi_{2,3,6,8}\ -\ \pi_{2,3,4,7}\ \pi_{1,3,6,8}\ +\ \pi_{1,2,3,7}\ \pi_{3,4,6,8}\ -\ \pi_{3,6,7,8}\ \pi_{1,2,3,4}\ =\ 0$$

to simplify 2.22. When equation 2.23 is used to simplify equation 2.22, the result is a product

$$(2.24)\qquad \pi_{3,6,7,8}\ \pi^2_{1,2,3,4}$$

which equals $A_{1,3}dA^2_{[1,2,3,4],[1,2,3,4]}$ in the original notation. $dQ11_{rl[2,3,4],[1,2,3]}$ can be expressed in terms of the $A_{i,j}$s as

$$(2.25)\qquad -\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}A_{1,3}\quad dA^2_{[1,2,3,4],[1,2,3,4]} \left(\mathrm{dQ}_{[1,13,14,15,16],[3,5,6,7,8]}A_{3,4} + \mathrm{dQ}_{[1,13,14,15,16],[4,5,6,7,8]}A_{4,4} + \mathrm{dQ}_{[1,13,14,15,16],[1,5,6,7,8]}A_{1,4} + A_{2,4}\mathrm{dQ}_{[1,13,14,15,16],[2,5,6,7,8]}\right)$$

The same manipulations show that

$$(2.26)\qquad 0 = dQ11_{rl[2,3,4],[1,2,3]} = -\frac{A_{1,3}}{A_{1,2}}dQ11_{rl[1,3,4],[1,2,3]} = \frac{A_{1,3}}{A_{1,1}}dQ11_{rl[1,2,4],[1,2,3]}$$

Generically $dA_{[1,2,3,4],[1,2,3,4]}$, $A_{1,1}$, $A_{1,2}$, $dQ_{[5,6,7,8],[13,14,15,16]}$, and $A_{1,3}$ are generically nonzero. Requiring $dQ11_{rl[2,3,4],[1,2,3]}$, $dQ11_{rl[1,3,4],[1,2,3]}$, and $dQ11_{rl[1,2,4],[1,2,3]}$ to be zero is equivalent to requiring that their common linear term is zero.

$$(2.27)\qquad \mathrm{dQ}_{[1,13,14,15,16],[3,5,6,7,8]}A_{3,4} + \mathrm{dQ}_{[1,13,14,15,16],[4,5,6,7,8]}A_{4,4} + \mathrm{dQ}_{[1,13,14,15,16],[1,5,6,7,8]}A_{1,4} + A_{2,4}\mathrm{dQ}_{[1,13,14,15,16],[2,5,6,7,8]} = 0$$

Three more minors of $Q11_{rl}$ factor easily. $dQ11_{rl[1,2,3],[1,2,4]}$ equals

$$
\begin{aligned}
&\Big(-A_{5,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,5]} - A_{6,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,6]}- \\
&\quad A_{8,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,8]} - A_{7,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,7]}\Big) \\
&\quad \Big(dA_{[1,3,4],[1,3,4]}dA_{[1,2,4],[2,3,4]}dA_{[2,3,4],[1,2,4]}+ \\
&\quad dA_{[1,3,4],[2,3,4]}dA_{[2,3,4],[1,3,4]}dA_{[1,2,4],[1,2,4]} - \\
&\quad dA_{[1,3,4],[1,3,4]}dA_{[2,3,4],[2,3,4]}dA_{[1,2,4],[1,2,4]} - \\
&\quad dA_{[1,3,4],[2,3,4]}dA_{[1,2,4],[1,3,4]}dA_{[2,3,4],[1,2,4]} + \\
&\quad dA_{[1,3,4],[1,2,4]}dA_{[1,2,4],[1,3,4]}dA_{[2,3,4],[2,3,4]} - \\
&\quad dA_{[1,3,4],[1,2,4]}dA_{[2,3,4],[1,3,4]}dA_{[1,2,4],[2,3,4]}\Big)
\end{aligned}
\tag{2.28}
$$

Just as before, the minors for $dQ11_{rl[1,2,3],[2,3,4]}$ and $dQ11_{rl[1,2,3],[1,3,4]}$ have the same form as 2.29 and Graßmann relations may be used to simplify them. Provided that $A_{2,4}$, $A_{3,4}$, and $A_{4,4}$ are nonzero they yield only one relevant term:

$$
\begin{aligned}
&\Big(A_{5,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,5]} + A_{6,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,6]}+ \\
&\quad A_{8,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,8]} + A_{7,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,7]}\Big) = 0
\end{aligned}
\tag{2.29}
$$

The same hold true for the other rank deficient submatrices, $Q11_{bt}$, $Q21_{rl}$, $Q21_{tb}$, $Q12_{lr}$, $Q12_{bt}$, $Q22_{tb}$, and $Q22_{lr}$. Each submatrix has several 3×3 minors which factor easily but these easily factorizable minors yield only two relevant equations per submatrix. Fortunately, these equations are *linear* in the unknowns! Recall that each of these submatrices has one 3×3 minor which yields one of the identities in 2.15. So we expect to find only one more independent relation per rank deficient submatrix. Fortunately, the remaining minors are easily cleaned up. They are sums of many terms, some of which are eqivalent to the identities just found (like 2.29) multiplied by some other term. When these identities are subtracted from one of the remaining minors, another relation amongst the $A_{i,j}$s appears. One example is given below:

$$
\begin{aligned}
&\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}dA_{[1,4],[2,4]} \\
&\quad \Big(A_{7,5}\mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,7]} + A_{6,5}\mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,6]}+ \\
&\quad A_{5,5}\mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,5]} + \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,8]}A_{8,5}\Big) = 0
\end{aligned}
\tag{2.30}
$$

Since $\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}$, $\mathrm{dQ}_{[13,14,15,16],[5,6,7,8]}$, and $dA_{[1,4],[2,4]}$ are generically nonzero the relevant term is that last one. In fact, all of the remaining 3×3 minors of $Q11_{rl}$ yield the same relevant term, as we might expect. Each of the other seven rank deficient submatrices that have been studied thus far yields exactly one more independent relation amongst the $A_{i,j}$s.

If the rank deficient submatrices $Q11_{rl}$, $Q11_{bt}$, $Q21_{rl}$, $Q21_{tb}$, $Q12_{lr}$, $Q12_{bt}$, $Q22_{tb}$, and $Q22_{lr}$ were all independent of each other then they would correspond to $8*4=32$ independent 3×3 minors. Unfortunately, this is not the case. The 24 nontrivial minors may be grouped in eight sets of three according to their unknowns. Graßmann relations may be used to show that all three equations per group are equivalent. One of the sets is shown below:

$$\begin{aligned} \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,7]}A_{7,5} + \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,6]}A_{6,5}+ \\ \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,5]}A_{5,5} + \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,8]}A_{8,5} &= 0 \end{aligned} \tag{2.31}$$

$$\begin{aligned} \mathrm{dQ}_{[13,14,15,16],[2,3,4,5]}A_{5,5} + \mathrm{dQ}_{[13,14,15,16],[2,3,4,6]}A_{6,5}+ \\ \mathrm{dQ}_{[13,14,15,16],[2,3,4,8]}A_{8,5} + \mathrm{dQ}_{[13,14,15,16],[2,3,4,7]}A_{7,5} &= 0 \end{aligned} \tag{2.32}$$

$$\begin{aligned} \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,7]}A_{7,5} + \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,8]}A_{8,5}+ \\ \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,6]}A_{6,5} + \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,5]}A_{5,5} &= 0 \end{aligned} \tag{2.33}$$

Notice that this equation does not take the identities 2.15 into account. When the identities are considered the Jacobian of the above system becomes:

$$\begin{bmatrix} \mathrm{dQ}_{[13,14,15,16],[2,3,4,5]} & \mathrm{dQ}_{[13,14,15,16],[2,3,4,6]} & \mathrm{dQ}_{[13,14,15,16],[2,3,4,7]} \\ \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,5]} & \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,6]} & \mathrm{dQ}_{[1,13,14,15,16],[1,2,3,4,7]} \\ \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,5]} & \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,6]} & \mathrm{dQ}_{[8,13,14,15,16],[1,2,3,4,7]} \end{bmatrix} \tag{2.34}$$

This Jacobian portends trouble: either the solution to the system 2.31, 2.32, and 2.33 is trivial or else these equations are not all independent. Graßmann relations may be used to show the latter. For the equations to be equivalent the rank of this matrix must be one. The rank is one if and only if every 2×2 minor is identically zero. Start with the upper left 2×2 minor

$$\begin{aligned} &dQ_{[13,14,15,16],[2,3,4,5]} \quad dQ_{[1,13,14,15,16],[1,2,3,4,6]} \quad - \\ &\qquad dQ_{[13,14,15,16],[2,3,4,6]} \quad dQ_{[1,13,14,15,16],[1,2,3,4,5]} \end{aligned} \tag{2.35}$$

In order to use Graßmann identities to show that 2.35 is identically zero, consider the matrix

$$\begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & Q_{1,4} & Q_{1,5} & Q_{1,6} & Q_{1,7} & 1 & 0 \\ Q_{8,1} & Q_{8,2} & Q_{8,3} & Q_{8,4} & Q_{8,5} & Q_{8,6} & Q_{8,7} & 0 & 1 \\ Q_{13,1} & Q_{13,2} & Q_{13,3} & Q_{13,4} & Q_{13,5} & Q_{13,6} & Q_{13,7} & 0 & 0 \\ Q_{14,1} & Q_{14,2} & Q_{14,3} & Q_{14,4} & Q_{14,5} & Q_{14,6} & Q_{14,7} & 0 & 0 \\ Q_{15,1} & Q_{15,2} & Q_{15,3} & Q_{15,4} & Q_{15,5} & Q_{15,6} & Q_{15,7} & 0 & 0 \\ Q_{16,1} & Q_{16,2} & Q_{16,3} & Q_{16,4} & Q_{16,5} & Q_{16,6} & Q_{16,7} & 0 & 0 \end{bmatrix} \tag{2.36}$$

In the Graßmann notation with respect to this matrix,

$$\pi_{2,3,4,5,8,9}\ \pi_{1,2,3,4,6,9} - \pi_{2,3,4,6,8,9}\ \pi_{1,2,3,4,5,9} \tag{2.37}$$

The Graßmann relation beginning with these terms is

$$\pi_{2,3,4,5,8,9}\ \pi_{1,2,3,4,6,9} - \pi_{2,3,4,6,8,9}\ \pi_{1,2,3,4,5,9} - \pi_{2,3,4,5,6,9}\ \pi_{1,2,3,4,8,9} = 0 \tag{2.38}$$

Hence, the upper left 2×2 minor in equation 2.35 equals

$$dQ_{[1,13,14,15,16],[2,3,4,5,6]} \quad dQ_{[13,14,15,16].[1,2,3,4]} \tag{2.39}$$

But the submatrix of Q from rows $[1, 8, 9, 10, 11, 12, 13, 14, 15, 16]$ and columns $[2, 3, 4, 5, 6, 7]$ is of rank four so $dQ_{[1,13,14,15,16],[2,3,4,5,6]} \equiv 0$. Then the expression in 2.39 is identically zero, forcing the minor in 2.35 to be identically zero. Graßmann identities plus consistency conditions can be used to show that each 2×2 minor of the Jacobian in 2.34 is identically zero. The same holds for each of the eight sets of three equations. Amongst the two dozen conditions found, only eight are independent. The author prefers to work with the relations whose coefficients are of lowest degree in the data and uses the following solutions to eliminate eight of the $A_{i,j}$s.

$$
\begin{aligned}
A_{5,5} &= -\frac{A_{6,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,6]} + A_{7,5}\mathrm{dQ}_{[13,14,15,16],[2,3,4,7]}}{\mathrm{dQ}_{[13,14,15,16],[2,3,4,5]}} \\
A_{12,12} &= -\frac{A_{10,12}\mathrm{dQ}_{[5,6,7,8],[10,13,14,15]} + A_{11,12}\mathrm{dQ}_{[5,6,7,8],[11,13,14,15]}}{\mathrm{dQ}_{[5,6,7,8],[12,13,14,15]}} \\
A_{13,13} &= -\frac{\mathrm{dQ}_{[5,6,7,8],[10,11,12,14]}A_{14,13} + A_{15,13}\mathrm{dQ}_{[5,6,7,8],[10,11,12,15]}}{\mathrm{dQ}_{[5,6,7,8],[10,11,12,13]}} \\
A_{8,8} &= -\frac{A_{7,8}\mathrm{dQ}_{[13,14,15,16],[7,9,10,11]} + A_{6,8}\mathrm{dQ}_{[13,14,15,16],[6,9,10,11]}}{\mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}} \\
A_{9,9} &= -\frac{A_{11,9}\mathrm{dQ}_{[13,14,15,16],[6,7,8,11]} + A_{10,9}\mathrm{dQ}_{[13,14,15,16],[6,7,8,10]}}{\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}} \\
A_{4,4} &= -\frac{A_{2,4}\mathrm{dQ}_{[13,14,15,16],[2,5,6,7]} + A_{3,4}\mathrm{dQ}_{[13,14,15,16],[3,5,6,7]}}{\mathrm{dQ}_{[13,14,15,16],[4,5,6,7]}} \\
A_{16,16} &= -\frac{A_{15,16}\mathrm{dQ}_{[5,6,7,8],[1,2,3,15]} + A_{14,16}\mathrm{dQ}_{[5,6,7,8],[1,2,3,14]}}{\mathrm{dQ}_{[5,6,7,8],[1,2,3,16]}} \\
A_{1,1} &= -\frac{\mathrm{dQ}_{[5,6,7,8],[2,14,15,16]}A_{2,1} + \mathrm{dQ}_{[5,6,7,8],[3,14,15,16]}A_{3,1}}{\mathrm{dQ}_{[5,6,7,8],[1,14,15,16]}}
\end{aligned}
\tag{2.40}
$$

Hard Conditions. We may now substitute the solutions in 2.41 and 2.15 back into the modified probabilities, (the data for the 2×2 subsystems which are the nonzero entries of the modified transition matrices P_{ih}, P_{io}, P_{hh}, and P_{ho}). The eight 4×4 rank deficient submatrices which were *not* used to find the solutions in 2.41 and 2.15 may now be used to eliminate more of the $A_{i,j}$s. As before eight 4×4 submatrices of rank two yield at most 32 independent conditions amongst the remaining 48 $A_{i,j}$s.

The submatrices which have not yet been used to eliminate $A_{i,j}$s are $Q11_{lr}$, $Q11_{tb}$, $Q12_{rl}$, $Q12_{bt}$, $Q21_{lr}$, $Q21_{bt}$, $Q22_{bt}$, and $Q22_{rl}$. Since the 3×3 minors of these equations cannot all be independent we need not bother simplifying all of them. These 3×3 minors are extremely cumbersome so the author generated a phantom data set and substituted its data into the minors in order to look for the simplest maximal spanning set of these minors. Some of these minors were much simpler than others. Although each rank deficient submatrix corresponds to four independent 3×3 minors, there may be dependencies between minors generated by different submatrices. As with the submatrices $Q11_{rl}$, $Q11_{bt}$, $Q12_{lr}$, $Q12_{tb}$, $Q21_{rl}$, $Q21_{tb}$, $Q22_{tb}$, and $Q22_{lr}$, which had only sixteen independent minors amongst them, the remaining eight submatrices correspond to only sixteen independent minors.

The author prefers the path of least resistance, choosing to simplify as few of the general minors as possible. Consider first the minors corresponding to rows [1,2,3] and [1,2,4] and columns [1,2,3] for the submatrices $Q12_{rl}$, $Q21_{lr}$, $Q22_{bt}$, and $Q11_{tb}$. This choice of equations is not unique and is made simply because these equations look simplest. After writing down the minors (with help from MAPLE) their denominators are eliminated. Appropriate terms in these equations must be collected. When a data set is substituted into the general equations the resulting equations have numerical coefficients and are referred to as numerical equations. The numerical equations have nearly as many terms as the general equations and many of their terms share the same coefficient. Once terms in the numerical equations with like coefficients are collected, the resulting equations have 1000 terms each. The arguments of like coefficients factor into a neat form (sometimes zero!). Collecting the general equations with respect to the minors of the data matrix, Q, yields 1000 term general equations. Each of these terms may be simplified individually. These terms are products of polynomials of minors of Q times expressions that are either of the same form as 2.16 or of the form

$$-dA_{[2,3,4],[1,3,4]}\ dA_{[1,2,3],[2,3,4]} + dA_{[1,2,3],[1,3,4]}\ dA_{[2,3,4],[2,3,4]} \tag{2.41}$$

Referring to the matrix 2.17, we can write 2.41 in Graßmann notation

$$\pi_{1,3,4,5}\ \pi_{2,3,4,8} - \pi_{1,3,4,8}\ \pi_{2,3,4,5} \tag{2.42}$$

which is the beginning of the Graßmann relation

$$\pi_{1,3,4,5}\ \pi_{2,3,4,8} - \pi_{1,3,4,8}\ \pi_{2,3,4,5} - \pi_{3,4,5,8}\ \pi_{1,2,3,4} = 0 \tag{2.43}$$

From 2.43 we can make the substitution

$$dA_{[2,3],[3,4]} dA_{[1,2,3,4],[1,2,3,4]} = -dA_{[2,3,4],[1,3,4]} dA_{[1,2.3],[2,3,4]} + dA_{[1,2,3],[1,3,4]} dA_{[2,3,4],[2,3,4]}$$

Once substitutions like this are made, the equation can be factored. The resulting equation is a product of several generically nonzero minors of Q and the square of one of the following minors:

$$dA_{[1,2,3,4],[1,2,3,4]},\ dA_{5,6,7,8],[5,6,7,8]},\ dA_{[9,10,11,12],[9,10,11,12]},\ and$$

$$dA_{[13,14,15,16],[13,14,15,16]}$$

This step reduces the degree of the equations in $A_{i,j}$s from thirteen to five and many of the terms in the equations are functions of minors of A. As long as the minors are written in the shorthand using the symbol dA, the identities in 2.15 are not recognized. So we must (have MAPLE) write out the minors and substitute the identities 2.15 into the equations. In both the numerical and general cases, the resulting equations have 256 terms, once they are collected with respect to the $A_{i,j}$s. In the numerical case, the equations factor to be the product of a sixteen term quadratic, a four term quadratic, and a four term linear sum. Generically, the relevant term is the linear one. Unfortunately, the author's general equations do not factor. The coefficients of each of the terms is a polynomial in minors of Q. The minors are expressed in the by-now familiar shorthand using the symbol dQ. As the Graßmann relations show, there may be many ways of writing a polynomial in minors of a matrix. If MAPLE were able to handle equations of arbitrary size then the easiest thing to do would be to rewrite the equations without the dQ notation and ask MAPLE to factor them. At present, that is not possible. So we must work.

Assuming that the general equations should factor just as their numerical counterparts do, we make good use of that knowledge. Consider the 64 combinations of variables which occur if the quadratic terms in one of the equations are expanded. The coefficient of any one of these combinations is the desired linear term. It is relatively easy to take the coefficient of each one of these 64 combinations. Fortunately, they are all equivalent. Although tedious, it is just as straightforward to show that these linear equations are equivalent as it is to show that the relations 2.31, 2.32, and 2.33 are equivalent. Once again, we choose to work with the equations which have the coefficients of lowest degree in the data. Two of the identities are shown below

$$
\begin{aligned}
0 = \; & \Big(\mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]}\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} - \\
& \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,8]}\mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]}\Big) A_{8,6} + \\
& \Big(\mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,7]}\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} - \\
& \quad \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,7]}\mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} - \\
& \quad \mathrm{dQ}_{[13,14,15,16],[1,2,3,4]}\mathrm{dQ}_{[5,6,13,14,15,16],[7,8,9,10,11,12]}\Big) A_{7,6} + \\
& \Big(-\mathrm{dQ}_{[13,14,15,16],[1,2,3,4]}\mathrm{dQ}_{[5,6,13,14,15,16],[5,8,9,10,11,12]} + \\
& \quad \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,5]}\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} - \\
& \quad \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,5]}\mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]}\Big) A_{5,6} + \\
& \Big(-\mathrm{dQ}_{[13,14,15,16],[1,2,3,4]}\mathrm{dQ}_{[5,6,13,14,15,16],[6,8,9,10,11,12]} - \\
& \quad \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,6]}\mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} + \\
& \quad \mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]}\mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,6]}\Big) A_{6,6}
\end{aligned}
\tag{2.44}
$$

and

$$
\begin{aligned}
(2.45)\qquad 0 = & \Big(-dQ_{[5,13,14,15,16],[1,2,3,4,8]}dQ_{[7,13,14,15,16],[8,9,10,11,12]}+ \\
& dQ_{[5,13,14,15,16],[8,9,10,11,12]}dQ_{[7,13,14,15,16],[1,2,3,4,8]}\Big)\, A_{8,6} + \\
& \Big(-dQ_{[5,13,14,15,16],[1,2,3,4,7]}dQ_{[7,13,14,15,16],[8,9,10,11,12]}+ \\
& \quad dQ_{[7,13,14,15,16],[1,2,3,4,7]}dQ_{[5,13,14,15,16],[8,9,10,11,12]} + \\
& \quad dQ_{[5,7,13,14,15,16],[7,8,9,10,11,12]}dQ_{[13,14,15,16],[1,2,3,4]}\Big)\, A_{7,6} + \\
& \Big(-dQ_{[5,13,14,15,16],[1,2,3,4,5]}dQ_{[7,13,14,15,16],[8,9,10,11,12]}+ \\
& \quad dQ_{[5,13,14,15,16],[8,9,10,11,12]}dQ_{[7,13,14,15,16],[1,2,3,4,5]} + \\
& \quad dQ_{[5,7,13,14,15,16],[5,8,9,10,11,12]}dQ_{[13,14,15,16],[1,2,3,4]}\Big)\, A_{5,6} + \\
& \Big(-\, dQ_{[7,13,14,15,16],[8,9,10,11,12]}dQ_{[5,13,14,15,16],[1,2,3,4,6]}+ \\
& \quad dQ_{[5,13,14,15,16],[8,9,10,11,12]}dQ_{[7,13,14,15,16],[1,2,3,4,6]} + \\
& \quad dQ_{[5,7,13,14,15,16],[6,8,9,10,11,12]}dQ_{[13,14,15,16],[1,2,3,4]}\Big)\, A_{6,6}
\end{aligned}
$$

These equations are independent and yield the following solutions for $A_{8,6}$ and $A_{5,6}$.

$$
\begin{aligned}
A_{8,6} = \Big(- \Big(&\mathrm{dQ}_{[6,7,13,14,15,16],[1,2,3,4,5,6]} \ \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,6]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[5,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[5,7,13,14,15,16],[5,8,9,10,11,12]} \ \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,6]} + \\
&\mathrm{dQ}_{[5,7,13,14,15,16],[6,8,9,10,11,12]} \ \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,5]} - \\
&\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[1,2,3,4,5,6]} - \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,5]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[6,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[6,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,5]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[1,2,3,4,5,6]} + \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[5,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,6]} + \\
&\mathrm{dQ}_{[5,6,7,13,14,15,16],[5,6,8,9,10,11,12]} \ \mathrm{dQ}_{[13,14,15,16],[1,2,3,4]}\Big) A_{6,6} - \\
&\Big(\mathrm{dQ}_{[6,7,13,14,15,16],[1,2,3,4,5,7]} \ \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} - \\
&\mathrm{dQ}_{[5,7,13,14,15,16],[5,8,9,10,11,12]} \ \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,7]} - \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,5]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[7,8,9,10,11,12]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,7]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[5,8,9,10,11,12]} + \\
&\mathrm{dQ}_{[5,7,13,14,15,16],[7,8,9,10,11,12]} \ \mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,5]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[1,2,3,4,5,7]} - \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[7,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,5]} - \\
&\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[1,2,3,4,5,7]} + \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[5,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,7]} + \\
&\mathrm{dQ}_{[13,14,15,16],[1,2,3,4]} \ \mathrm{dQ}_{[5,6,7,13,14,15,16],[5,7,8,9,10,11,12]}\Big) A_{7,6}\Big) / \\
&\Big(\mathrm{dQ}_{[6,7,13,14,15,16],[1,2,3,4,5,8]} \ \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[5,8,9,10,11,12]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[5,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[1,2,3,4,5,8]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[1,2,3,4,5,8]} + \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[5,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]}\Big)
\end{aligned}
$$

$$
\begin{aligned}
A_{5,6} = \Big(\Big(-&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[6,8,9,10,11,12]} + \\
&\mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[6,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[6,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]} - \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[1,2,3,4,6,8]} \ \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} + \\
&\mathrm{dQ}_{[5,7,13,14,15,16],[1,2,3,4,6,8]} \ \mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} - \\
&\mathrm{dQ}_{[5,6,13,14,15,16],[1,2,3,4,6,8]} \ \mathrm{dQ}_{[7,13,14,15,16],[8,9,10,11,12]}\Big) A_{6,6} + \\
&\Big(\mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[7,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[7,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[6,7,13,14,15,16],[7,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[5,6,13,14,15,16],[1,2,3,4,7,8]} \ \mathrm{dQ}_{[7,13,14,15,16],[8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[1,2,3,4,7,8]} \ \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} + \\
&\mathrm{dQ}_{[5,7,13,14,15,16],[1,2,3,4,7,8]} \ \mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]}\Big) A_{7,6}\Big) \, / \\
&\Big(\mathrm{dQ}_{[6,7,13,14,15,16],[1,2,3,4,5,8]} \ \mathrm{dQ}_{[5,13,14,15,16],[8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,7,13,14,15,16],[5,8,9,10,11,12]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[1,2,3,4,8]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[5,8,9,10,11,12]} - \\
&\mathrm{dQ}_{[6,13,14,15,16],[8,9,10,11,12]} \mathrm{dQ}_{[5,7,13,14,15,16],[1,2,3,4,5,8]} + \\
&\mathrm{dQ}_{[7,13,14,15,16],[8,9,10,11,12]} \ \mathrm{dQ}_{[5,6,13,14,15,16],[1,2,3,4,5,8]} + \\
&\mathrm{dQ}_{[6,7,13,14,15,16],[5,8,9,10,11,12]} \ \mathrm{dQ}_{[5,13,14,15,16],[1,2,3,4,8]}\Big)
\end{aligned}
\tag{2.46}
$$

Very Hard Conditions . The only rank deficient submatrices we have not yet accounted for are $Q22_{rl}$, $Q11_{lr}$, $Q21_{bt}$, and $Q12_{tb}$. The author chose to simplify the simplest minors generated by the phantom, those from columns [1,2,4] and both sets of rows [1,2,4] and [1,3,4]. The equations obtained by subtituting 2.47 and its counterparts into these remaining minors are polynomials in the remaining $A_{i,j}$s and with coefficients which are *large polynomials* in the minors of Q. In preliminary work with a phantom each of these equations was a quintic in the $A_{i,j}$s and became the product of a 32 term quartic and a linear term after factorization. Upon substituting the phantom's values for the $A_{i,j}$s into the minors, the relevant terms turned out to be the linear terms. Unfortunately, the general versions of these minors did not factor, presumably because the coefficients were written in the "dQ" notation. As with the hard conditions, the author assumed that the coefficient of any of the 32 terms in the quartic is the desired linear term.

Because their coefficients are so cumbersome, only a caricature of one of these identities is shown below

$$
\begin{aligned}
&\Big(c_5\ (c_4\, a_8 - b_2\, c_6 + b_6\, c_8)\, d_1 - \mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]} \\
&\qquad \Big(\mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}a_1 - a_4\, \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}\Big) \\
&\qquad\quad \Big(b_1\, \mathrm{dQ}_{[11,13,14,15,16],[6,7,8,9,11]} - b_3\, \mathrm{dQ}_{[10,13,14,15,16],[6,7,8,9,11]}\Big) \\
&-\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}a_6\ \Big(a_5\, \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]} \\
&\quad +\mathrm{dQ}_{[5,6,7,8,10],[10,13,14,15,16]}a_2\, \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}\Big) \\
&+\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}a_1\ \Big(a_7\, \mathrm{dQ}_{[10,13,14,15,16],[8,9,10,11,12]} \\
&\qquad -a_5\, \mathrm{dQ}_{[11,13,14,15,16],[8,9,10,11,12]}\Big) \qquad\qquad (2.47) \\
&+\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\ \Big(a_7\, \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]} \\
&\quad +\mathrm{dQ}_{[5,6,7,8,11],[10,13,14,15,16]}a_2\, \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}\Big)\ a9 \\
&-\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\ \Big(b_4\ \ \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]} + b_5\, \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}\Big)\ c_5\, a_4 \\
&+\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\ \Big(c_1\, a_2\, \mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}a_3 - \ \mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}c_2\, a_2\, a_4 \\
&\quad -\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}a_{10}\, a_3 + \mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}b_5\, a_1\, c_5 \\
&\quad +\mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}b_4\, c_5\, a_3 + \mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}a_1\, \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}a_{10} \\
&\quad +\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}c_3\, a_2\, a_6 - \mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}c_7\, a9\, a_2\Big)\Big)\ A[10,11] \\
+\ \ &\big(-c_9\ (c_4\, a_8 - b_2\, c_6 + b_6\, c_8)\, d_1 - \mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]} \\
&\qquad \Big(b_5\, \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]} + \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}a_2\Big) \\
&\qquad\quad \Big(-\mathrm{dQ}_{[5,6,7,8,11],[9,13,14,15,16]}a9 + \mathrm{dQ}_{[5,6,7,8,10],[9,13,14,15,16]}a_6\Big) \\
&+\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\ \Big(b_4\ \ \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]} + b_5\, \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}\Big)\ c_9\, a_4 \\
&-\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\ \Big(-\mathrm{dQ}\ _{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}a_{11}\, a_3 \\
&\quad +\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}a_1\, \mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}a_{11} - c_{10}\, a_2\, \mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}a_3 \\
&\quad +\mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}b_4\, c_9\, a_3 - c_{10}\, b_5\, \mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}a_1 \\
&\quad +\mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}b_5\, a_1\, c_9\Big)\Big)\ A[9,11]
\end{aligned}
$$

$$
\begin{aligned}
+\quad & \Big(\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\Big(-\mathrm{dQ}_{[11,13,14,15,16],[8,9,10,11,12]} \\
& \quad \Big(-b_1\,\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} + \mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}a_2\Big) \\
& - \Big(-b_3\,\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} + \mathrm{dQ}_{[5,6,7,8,11],[12,13,14,15,16]}a_2\Big)\,\mathrm{dQ}_{[10,13,14,15,16],[8,9,10,11,12]}\Big)\,a_3 \\
& +\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\Big(-\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}b_3\,\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} \\
& \quad +\mathrm{dQ}_{[5,6,7,8,11],[12,13,14,15,16]}b_5\;\;\mathrm{dQ}_{[13,14,15,16],[8,9,10,11]} \\
& +\mathrm{dQ}_{[5,6,7,8,11],[12,13,14,15,16]}\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}a_2\Big)\,a9 \\
& -\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\Big(-\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}b_1\,\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]} \\
& \quad +\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}b_5\;\;\mathrm{dQ}_{[13,14,15,16],[8,9,10,11]} \\
& \quad +\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}a_2\Big)\,a_6 \\
& -\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}\Big(b_4\;\;\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]} + b_5\,\mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}\Big)\,c_{11}\,a_4 \\
& +\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]} \\
& \quad \Big(\mathrm{dQ}_{[5,6,7,8,11],[12,13,14,15,16]}\mathrm{dQ}_{[5,6,7,8,10,11],[10,12,13,14,15,16]}\mathrm{dQ}_{[9,10,13,14,15,16],[5,6,7,8,9,12]} \\
& \quad +\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}\mathrm{dQ}_{[11,13,14,15,16],[5,6,7,8,12]}\mathrm{dQ}_{[5,6,7,8,9,10,11],[9,10,12,13,14,15,16]} \\
& \quad -\mathrm{dQ}_{[5,6,7,8,11],[12,13,14,15,16]}\mathrm{dQ}_{[10,13,14,15,16],[5,6,7,8,12]}\mathrm{dQ}_{[5,6,7,8,9,10,11],[9,10,12,13,14,15,16]} \\
& \quad +\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}\mathrm{dQ}_{[5,6,7,8,10,11],[9,12,13,14,15,16]}\mathrm{dQ}_{[9,11,13,14,15,16],[5,6,7,8,10,12]} \\
& \quad -\mathrm{dQ}_{[5,6,7,8,10],[12,13,14,15,16]}\mathrm{dQ}\;\;{}_{[5,6,7,8,10,11],[10,12,13,14,15,16]}\mathrm{dQ}_{[9,11,13,14,15,16],[5,6,7,8,9,12]} \\
& \quad -\mathrm{dQ}_{[5,6,7,8,11],[12,13,14,15,16]}\mathrm{dQ}_{[5,6,7,8,10,11],[9,12,13,14,15,16]}\mathrm{dQ}_{[9,10,13,14,15,16],[5,6,7,8,10,12]} \\
& \quad -\;\;\mathrm{dQ}_{[10,11,13,14,15,16],[5,6,7,8,10,12]}\mathrm{dQ}_{[5,6,7,8,10,11],[9,12,13,14,15,16]}\mathrm{dQ}_{[5,6,7,8,9],[12,13,14,15,16]} \\
& \quad +\mathrm{dQ}_{[10,11,13,14,15,16],[5,6,7,8,9,12]}\mathrm{dQ}_{[5,6,7,8,10,11],[10,12,13,14,15,16]}\;\;\mathrm{dQ}_{[5,6,7,8,9],[12,13,14,15,16]}\Big) \\
& \quad \Big(-\;\;\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]}a_1 + \mathrm{dQ}_{[13,14,15,16],[8,9,10,11]}a_3\Big) \\
& +\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]} \\
& \quad \Big(-a_4\,\mathrm{dQ}_{[13,14,15,16],[8,9,11,12]} + \mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}a_3\Big) \\
& \qquad \Big(b_1\,\mathrm{dQ}_{[11,13,14,15,16],[6,7,8,9,11]} - b_3\,\mathrm{dQ}_{[10,13,14,15,16],[6,7,8,9,11]}\Big) \\
& +c_8\,c_{11}\,d_1\,b_6 + c_4\,c_{11}\,a_8\,d_1 - c_6\,c_{11}\,d_1\,b_2 \\
& +c_{11}\;(b_4\,a_3 + a_1\,b_5)\;\;\mathrm{dQ}_{[13,14,15,16],[8,9,10,12]}\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]} \\
& +b_5\;(-a9\,c_7 - a_4\,c_2 + c_3\,a_6)\,\mathrm{dQ}_{[5,6,7,8],[13,14,15,16]}\;\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]} \\
& +\;\mathrm{dQ}_{[5,6,7,8],[9,10,11,12]}c_{12}\,b_5\,\mathrm{dQ}_{[13,14,15,16],[6,7,8,9]}a_1\Big)\;A[12,11]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \Big(-dQ_{[5,6,7,8],[9,10,11,12]}\,(-c_{14}\,a9 + c_{15}\,a_6) \\
& \quad \Big(b_5\, dQ_{[13,14,15,16],[8,9,10,11]} + dQ_{[13,14,15,16],[8,9,11,12]}a_2\Big) \\
& +dQ_{[5,6,7,8],[13,14,15,16]}dQ_{[5,6,7,8],[9,10,11,12]} \\
& \ \Big(-dQ_{[13,14,15,16],[8,9,11,12]}a_1 + dQ_{[13,14,15,16],[8,9,10,11]}a_3\Big) \\
& \quad \Big(b_1\, dQ_{[11,13,14,15,16],[6,7,8,9,11]} - b_3\, dQ_{[10,13,14,15,16],[6,7,8,9,11]}\Big) \\
& -c_6\, c_{13}\, d_1\, b_2 + dQ_{[5,6,7,8],[9,10,11,12]}dQ_{[13,14,15,16],[8,9,10,12]}b_5\, a_1\, c_{13} \\
& -dQ_{[5,6,7,8],[9,10,11,12]}\Big(b_4\ \ dQ_{[13,14,15,16],[8,9,11,12]} + b_5\, dQ_{[13,14,15,16],[8,9,10,11]}\Big)\, c_{13}\, a_4 \\
& +dQ_{[5,6,7,8],[9,10,11,12]}dQ_{[13,14,15,16],[8,9,10,12]}b_4\, a_3\, c_{13} \\
& +c_4\, c_{13}\, a_8\, d_1 + c_8\, c_{13}\, d_1\, b_6 + dQ_{[13,14,15,16],[6,7,8,9]}dQ_{[5,6,7,8],[9,10,11,12]} \\
& \Big(-dQ_{[5,6,7,8,10],[11,13,14,15,16]}dQ_{[5,6,7,8,10,11],[10,12,13,14,15,16]}dQ_{[9,11,13,14,15,16],[5,6,7,8,9,12]} \\
& \quad -\ dQ_{[10,11,13,14,15,16],[5,6,7,8,10,12]}dQ_{[5,6,7,8,9,11],[9,12,13,14,15,16]}dQ_{[5.6,7,8,10],[11,13,14,15,16]} \\
& \quad +dQ_{[10,11,13,14,15,16],[5,6,7,8,10,12]}dQ_{[5,6,7,8,9,10],[9,12,13,14,15,16]}\ dQ_{[5,6,7,8,11],[11,13,14,15,16]} \\
& \quad -dQ_{[5,6,7,8,9,10,11],[9,10,12,13,14,15,16]}dQ_{[10,13,14,15,16],[5,6,7,8,12]}dQ_{[5,6,7,8,11],[11,13,14,15,16]} \\
& \quad +dQ_{[5,6,7,8,10],[11,13,14,15,16]}dQ\ _{[11,13,14,15,16],[5,6,7,8,12]}dQ_{[5,6,7,8,9,10,11],[9,10,12,13,14,15,16]} \\
& \quad +dQ_{[10,11,13,14,15,16],[5,6,7,8,9,12]}dQ_{[5,6,7,8,9,11],[10,12,13,14,15,16]}dQ_{[5,6,7,8,10],[11,13,14,15,16]} \\
& \quad -\ dQ_{[5,6,7,8,9,10],[10,12,13,14,15,16]}dQ_{[5,6,7,8,11],[11,13,14,15,16]}dQ_{[10,11,13,14,15,16],[5,6,7,8,9,12]} \\
& \quad -dQ_{[9,10,13,14,15,16],[5,6,7,8,10,12]}dQ_{[5,6,7,8,10,11],[9,12,13,14,15,16]}\ dQ_{[5,6,7,8,11],[11,13,14,15,16]} \\
& \quad +dQ_{[9,10,13,14,15,16],[5,6,7,8,9,12]}dQ_{[5,6,7,8,10,11],[10,12,13,14,15,16]}dQ_{[5,6,7,8,11],[11,13,14,15,16]} \\
& \quad +dQ_{[5,6,7,8,10],[11,13,14,15,16]}\ dQ_{[5,6,7,8,10,11],[9,12,13,14,15,16]}dQ_{[9,11,13,14.15,16],[5,6,7,8,10,12]} \\
& \quad +dQ_{[9,10,11,13,14,15,16],[5,6,7,8,9,10,12]}dQ_{[5,6,7,8],[13,14,15,16]}dQ_{[5,6,7,8,10,11],[11,12,13,14,15,16]}\Big) \\
& \quad \Big(-dQ_{[13,14,15,16],[8,9,11,12]}a_1 + dQ_{[13,14,15,16],[8,9,10,11]}a_3\Big) \\
& -dQ_{[5,6,7,8],[9,10,11,12]}\Big(dQ\ _{[13,14,15,16],[6,7,8,9]}c_16 + c_2\, dQ_{[5,6,7,8],[13,14,15,16]}\Big) \\
& \quad (a_2\, a_3 + a_1\, b_5)\big)\, A[11,11]
\end{aligned}
$$

where each of the a_is represents a six term quadratic polynomial in minors of Q. The b_is represent ten term quadratics in minors of Q. The c_is represent two term quadratics in minors of Q.

The author has little doubt that equation 2.47 and its counterparts can be simplified significantly if enough time, energy, and computing power are devoted to the cause. This last set of minors can then be solved linearly for eight more of the $A_{i,j}$s in terms of the remaining $A_{i,j}$s, leaving a 32 parameter family of solutions for the modified 4×4 problem. Therefore, we have 32 parameter family data sets for each of the four 2×2 subsystems. The last step is to solve each of these subsystems as done in [11]. The solution for each of the 2×2 subsystems is a 16 parameter family of solutions in terms of the data. The process of solving all four subsystems introduces another $4 * 16 = 64$ parameters and yields the result promised at the beginning of section 2.2, a $96 = 4 * 16 + 32$ parameter family of solutions for the unknown transition probabilities for a 4×4 system.

2.3. $n \times n$ problem where $n = 2^k$, $k \in \mathbb{N}$. In the previous section only one recursion was required to solve the 4×4 problem. The author's vision of the algorithms for the 8×8 and $n \times n$ problems are sketched below.

The first step in tackling the 8×8 problem is to break up the 8×8 system into four 4×4 subsystems. See figure 6. Only 32 of the original $8^2 * 16 - 32 = 992$ hidden states are considered in this modified system. The modified transition probabilities are the probabilities with which a photon travels from one of the pertinent states to another such that its travel path lies entirely inside one of the subsystems. These modified transition probabilities comprise the data for the 4×4 subsystems. Furthermore, the same process for solving the governing equations 1.4 that was used in section 2.2.2 permits expression of the modified transition probabilities in terms of the entries of $A = P_{ho}^{-1}$. P_{ho} is a 32×32 block diagonal matrix with four 8×8 blocks along its diagonal. Since A has the same zero structure, we have a $4 * 8^2 = 256$ parameter solutions for the modified transition probabilities. There are many consistency conditions amongst the data for each of the 4×4 subsystems. These conditions should allow us to solve for all but 64 $A_{i,j}$s in terms of the remaining $A_{i,j}$s.

Notation: Let A, P_{ho}, P_{io}, P_{hh}, and P_{ih} denote the modified transition matrices at the first level of this recursive algorithm. At the next level of the recursive process each of the four 4×4 subsystems will have its own data matrix, Qi_1j_1, where we refer to the subsystems as systemi_1j_1, where $i_1, j_1 = 1, 2$. The transition matrices for the *modified* 4×4 systemi_1j_1 are referred to as Ai_1j_1, Pi_1j_{1ho}, Pi_1j_{1io}, Pi_1j_{1hh}, and Pi_1j_{1ih}. At the last recursive level of this recovery algorithm, each of the 4×4 modified systems will be broken into four 2×2 subsystems. The $(i_2, j_2)^{th}$ 2×2 subsystem of the $(i_1, j_1)^{th}$ 4×4 subsystem will be referred to as system$i_1j_{1_{i_2j_2}}$. The data matrices for these sub-subsystems will be referred to as $Qi_1j_{1_{i_2j_2}}$; the transition matrices as $Pi_1j_{1_{i_2j_2ho}}$, $Pi_1j_{1_{i_2j_2io}}$, $Pi_1j_{1_{i_2j_2hh}}$, and $Pi_1j_{1_{i_2j_2ih}}$.

Now that we have 64 parameter solutions for $Q11$, $Q12$, $Q21$, and $Q22$ we can implement the recovery algorithm for the 4×4 problem on each of the 4×4 subsystems as done in section 2.2. See figure 7. For subsystem$i_1 j_1$ we recover $Qi_1 j_{1_{i_2 j_2}}$ for each combination $i_2, j_2 = 1, 2$ in terms of $Qi_1 j_1$ and half of the nonzero entries of $Ai_1 j_1$. Since each $Ai_1 j_1$ is a 16×16 block diagonal matrix with four 4×4 blocks along the diagonal, we introduce $1/2 * 4 * 4 * 4^2 = 128$ additional parameters to our solutions for the data submatrices for the 2×2 sub-subsystems. The resulting data matrices, $Qi_1 j_{1_{i_2 j_2}}$, should be functions of $64 + 128 = 192$ parameters. Now, we must simpy implement the 2×2 recovery algorithm on each of the $4^2 = 16$ sub-subsystems. We may solve for each set of transition matrices $Pi_1 j_{1_{i_2 j_2 ho}}$, $Pi_1 j_{1_{i_2 j_2 io}}$, $Pi_1 j_{1_{i_2 j_2 hh}}$, and $Pi_1 j_{1_{i_2 j_2 ih}}$ in terms of $Ai_1 j_{1_{i_2 j_2}}$, introducing another $4^2 * 16 = 256$ parameters. The end result is a $256 + 128 + 64 = 448 = 64(4 + 2 + 1) = 8 * 8(8 - 1)$ parameter family of solutions for the transition probabilities in terms of the data matrix Q. Recall that the forward map is subject to at least $8n(n-1)$. For the 8×8 problem we can at best find a $8n(n-1)|_{n=8} = 448$ parameter family of solutions.

In general, the recovery algorithm for a $n \times n$ system where $n = 2^k$, $k \in \mathbb{N}$, requires $k - 1$ recusive levels before the 2×2 "base case" is reached. The author expects that at level i in the recursion, 2^{k+2+i} parameters will be introduced to the data sets for the 4^i $2^{k-i} \times 2^{k-i}$ subsystems. This would result in a

$$\sum_{i=1}^{k} 2^{k+2+i} = 2^{k+3} \sum_{i=0}^{k-1} 2^i = 2^3 2^k (2^k - 1) = 8n(n-1)$$

parameter family of solutions. Pseudocode for this algorithm is shown below:

```
solveasubsystem := proc(sysin)
m := edgesize/2 of sysin
if m = 1 then solve base case
elif log_2 m ∈ N then
    break up sysin into four m × m subsystems
    for each subsystem
        1. solve for (modified) transition probabilities in terms of data and A
        2. eliminate all but 8m parameters using consistency conditions
        3. call solveasubsystem with this subsystem as input
    fi;
else print('error - input system not of proper size');
fi;
```

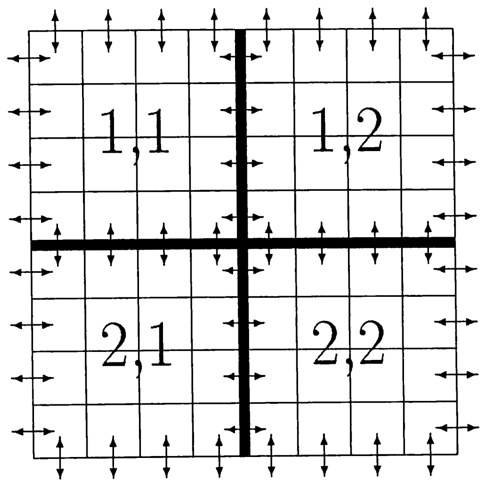

FIG. 6. *Decomposition of an 8×8 system into four 4×4 subsystems. The thick lines separate the subsystems. Only states which are considered when solving for the subsystems' data are denoted with arrows.*

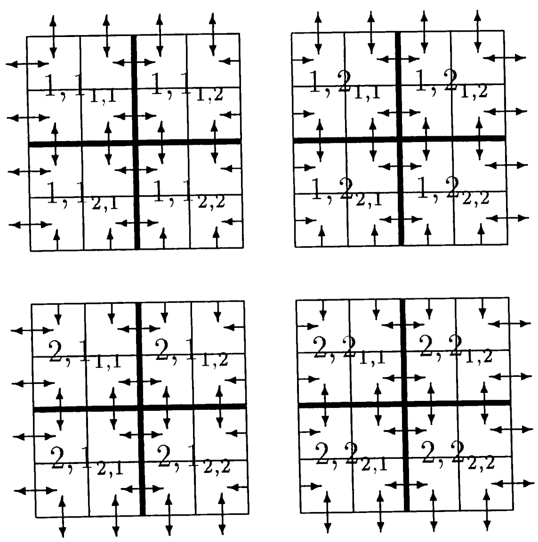

FIG. 7. *Decomposition of a 8×8 system into four 4×4 which are subsequently decomposed into 2×2 subsystems. The thick lines separate the subsystems.*

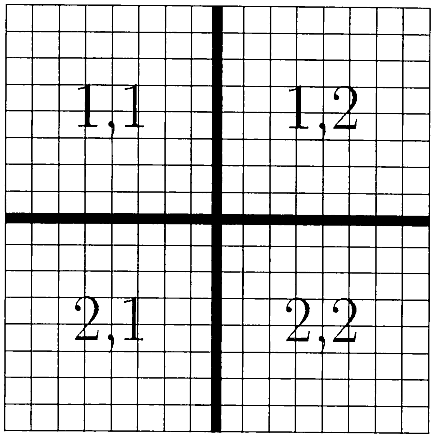

FIG. 8. *Decomposition of a* 16×16 *system into four* 8×8 *subsystems. The thick lines separate the subsystems.*

if $m = 2$ then
  solve each subsystem for its transition probabilities in terms of its data and 16 parameters

3. Conclusion. Diffuse tomography is still in its infancy, and there are many areas which should be explored. As yet unexplored areas which pique the author's interest include completion of a careful study of consistency conditions for the three dimensional model [17]. Understanding the consistency conditions is crucial because the amount and type of additional information required to close the resulting system of equations is directly tied to the number and type of conditions. In order to apply this recursive algorithm to larger systems a more clever/less computationally intensive way of using subsystems' consistency conditions to reduce the number of parameters must be found. The same sort of physical arguments used to prove the identities

$$\begin{array}{cccc} A_{1,4} = 0, & A_{4,1} = 0, & A_{5,8} = 0, & A_{8,5} = 0 \\ A_{9,12} = 0, & A_{12,9} = 0, & A_{13,16} = 0, & A_{16,13} = 0 \end{array} \tag{3.1}$$

must be found for the other consistency conditions. The very next item on the agenda is to implement the recursive recovery algorithm in three dimensions. The algorithm will be analogous to its two dimensional predecessor. Since clinical applications require that these algorithms must be stable, a careful stability study is crucial.

An extremely general Markovian model of photon transport was considered above. Neither time-of-flight information nor any physical information about photon transport through tissue were taken into account. A priori information about photon transport can and should be incorporated into this model. (The author doubts that clinicians would find a set of $36n^3$ Markov transition probabilities helpful diagnostic information.) The general model generates far more independent data than simpler models and the recovery algorithm makes full use of all independent data. This indicates (to the author, at least) that data generated by multiply scattered photons contains pertinent information.

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