Galois descent and the rational homotopy of the K(n)-local Picard space

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL

OF THE UNIVERSITY OF MINNESOTA

BY

Marshall David Smith

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

Doctor of Philosophy

Advisor: Craig Westerland

February, 2024

Acknowledgements

First, I would like to acknowledge my advisor, Craig Westerland for his immense patience and support during both my undergraduate and graduate tenures. His guidance and help have been invaluable over the years, and I am grateful to him for introducing me to the beautiful world of chromatic homotopy theory and for his ever-deep well of linguistic trivia.

Next, I would like to thank my committee of Sasha Voronov, Tyler Lawson, and Tian-Jun Li for reviewing this dissertation and providing useful feedback as well as for many interesting conversations, both math-related and otherwise.

I would also like to thank my fellow topology students in the department for their friendship and discussions.

Finally, I would like to thank my friends and family who have supported me through both the ups and downs of my academic career, and whose love and encouragement have been unwavering.

Dedication

To my family.

Abstract

Using a form of Galois descent, we construct a family of spectral sequences computing the homotopy groups of the Picard space $\mathcal{P}\mathrm{ic}_n$ whose 0^{th} homotopy group is the Picard group of the $K(n)$-local category. For all primes p and heights n, we compute the rank of $\pi_*\mathcal{P}\mathrm{ic}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to be zero for $* \geq 2$ and 1 for $* = 1$. Finally, using these methods, we describe the rank of $\pi_0\mathcal{P}\mathrm{ic} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in terms of a limit of module categories and discuss implications involving the algebraic Picard group.

Contents

Chapter 1

Introduction

The computation of the homotopy groups $\pi_i S^n$ of spheres is a key long-standing open problem in algebraic topology, having driven the development of many of the field's modern tools. One easing of this problem comes in the form of the stable homotopy groups $\pi_i^S = \varinjlim \pi_{i+n} S^n$, with the stable homotopy category of spectra being designed to tackle such questions in a more structured way. In 1951, through the use of spectral sequences, Serre computed in [Ser51] that $\pi_0^S \cong \mathbb{Z}$ and $\pi_{i>0}^S \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Further, he showed that for fixed i, π_i^S is finitely-generated. These spectral sequences, new at that time (having only been introduced in 1946 by Leray), are now ubiquitous in algebraic topology. Serre was, in effect, studying π_*^S via the Postnikov filtration, having filtration quotients π_n^S (see [BB19a, §1]). We will instead be interested in studying stable homotopy groups via the "chromatic" filtration.

Through the language of complex-oriented cohomology theories, formal groups, and Morava's extraordinary K-theories, these computations can be refined further to one prime and one "chromatic layer" $M_n X$ of a spectrum X at a time. The functor M_n and the Bousfield localization functor $L_{K(n)}$ with respect to the n^{th} Morava K-theory are mutually inverse equivalences between the monochromatic and $K(n)$-local symmetric monoidal categories of spectra [HS99b, Theorem 6.19]. In general, the $K(n)$-localizations of spectra are more amenable to a study via algebraic tools, and an understanding of the category $L_{K(n)}\mathbf{Sp}$ of $K(n)$-local spectra is of relevance, in particular the structure of its Picard group and the construction of these invertible objects.

An overview of chromatic homotopy theory in Chapters 2-7. We then discuss the $K(n)$-local Picard group and review existing results related to its structure and the construction of some elements in Chapters 8-9.

Then, in Chapter 10, we construct a family of descent spectral sequences computing the homotopy groups of the Picard space $\mathcal{P}\mathrm{ic}_n$ whose 0^{th} homotopy group is the Picard group of the $K(n)$-local category. We then compute the rank of $\pi_*\mathcal{P}\mathrm{ic}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to be zero for $* \geq 2$ and 1 for $* = 1$. Finally, using these methods, we describe the rank of Pic_n in terms of a limit of module categories and discuss implications involving the algebraic Picard group, $\mathrm{Pic}_n^{\mathrm{alg}}$.

Lastly, in Chapter 11 we discuss additional results and indicate possible directions for future research.

The content of Chapters 1-9 and Sections 11.4-11.6 represents a very slight revision of the author's (unpublished) master's expository paper.

Chapter 2

Morava K-theories

2.1 Description and basic properties

We begin with a description of Morava's extraordinary K-theories, often called the "fields" of homotopy theory, a term justified by (3) and (4) of Proposition 2.1.1.

Proposition 2.1.1 ([JW75], [Rav92]). *For each prime p and $n \geq 0$, there exist homology theories $K(n)_*$ (suppressing p from the notation from this point forward) which have the following properties:*

1. $K(0)_*(X) = H\mathbb{Q}_*(X)$ *and* $\widetilde{K(0)}_*(X) = 0$ *when* $\widetilde{H\mathbb{Z}}_*X$ *is torsion.*

2. $K(1)_*X$ *is one of the* $p-1$ *isomorphic summands of* $(KU/p)_*(X)$*, where* KU/p *is mod* p *complex* K*-theory.*

3. $K(0)_* = \mathbb{Q}$ *and for* $n > 0$*,* $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$*, with* $|v_n| = 2p^n - 2$*, which are graded fields, in that every graded module over* $K(n)_*$ *is free.*

4. We have a Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

5. $K(n) \wedge K(m) \simeq *$ *if* $n \neq m$*.*

6. Let X *be a finite* p*-local CW complex. Then,*

$$\widetilde{K(n)}_*(X) = 0 \implies \widetilde{K(n-1)}_*(X) = 0.$$

7. Let X *be a finite* p*-local CW complex. Then, for* $2p^n - 2 \geq \dim(X)$*,*

$$\widetilde{K(n)}_* X \cong K(n)_* \otimes \widetilde{H\mathbb{F}_p}_* X.$$

We will let $K(\infty)$ *be the spectrum* $H\mathbb{F}_p$*.*

Any module over $K(n)$, $0 \leq n \leq \infty$ splits as a wedge sum of suspension shifts of $K(n)$, thus furthering our analogy with the fields of algebra. More generally, any ring spectrum $K \in \mathbf{Sp}$ such that every K-module spectrum M is of the form $\bigvee_\alpha \Sigma^{i_\alpha} K$ will be called a *field*. The following classification result of Hopkins-Smith tells us that not only are the Morava K-theories fields with exceptionally useful properties, but in a sense, they are the only such theories we need to consider.

Theorem 2.1.2 (Hopkins-Smith [HS98])**.** *Any field object in* $\mathbf{Sp}_{(p)}$*, the category of* p*-local spectra, splits as a wedge of suspension shifts of the Morava* K*-theories.*

The Morava K-theories for $0 < n < \infty$ can be constructed by localizing and quotienting the Brown-Peterson spectrum BP, which is an irreducible summand of the p-local complex cobordism spectrum $MU_{(p)}$, with homotopy groups $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],\ |v_i| = 2p^i - 2$ to achieve the desired homotopy groups, a process which we describe briefly below. We'll discuss the spectrum MU a bit more in Chapter 3. See [HS98] for further details.

2.2 Construction

2.2.1 Quotients in homotopy

Let E be a ring spectrum. Then, given $x \in \pi_* E$, $E/(x)$ is the cofiber of multiplication by x map:

$$\Sigma^{|x|} E \xrightarrow{x\cdot} E \to E/(x).$$

More explicitly, this is the map $S^{|x|} \wedge E \xrightarrow{x \wedge \mathrm{id}} E \wedge E \xrightarrow{\mu} E$. The quotient $E/(x)$ is not necessarily a ring spectrum, but is in good cases, including all of the cases we need in this dissertation. Further, if x is not a zero-divisor, then quotienting has the expected result on homotopy:

Proposition 2.2.1. *Let E be a ring spectrum, and suppose $x \in \pi_* E$ is not a zero divisor. Then, $\pi_*(E/(x)) = E_*/(x)$.*

Proof. From the cofiber sequence $\Sigma^{|x|} E \to E \to E/(x)$, we get a long exact sequence in homotopy

$$\cdots \to \pi_{n+1}(E/(x)) \to \pi_{n-|x|} E \to \pi_n E \to \pi_n(E/(x)) \to \pi_{n-|x|-1} E \to \cdots$$

Because x is not a zero-divisor, the map $\pi_{n-|x|} E \to \pi_n E$ is injective. For the same reason, the image of $\pi_n(E/(x))$ in $\pi_{n-|x|-1} E$ is zero, and the long exact sequence splits into short exact sequences

$$0 \to \pi_{n-|x|} E \xrightarrow{x\cdot} \pi_n E \longrightarrow \pi_n(E/(x)) \to 0,$$

and we get $\pi_n(E/(x)) \cong (\pi_n E)/(x)$.

□

2.2.2 Localization with respect to a multiplicative subset

Given a multiplicative subset $S \subseteq \pi_* E$ (for E a ring spectrum), we can form a localization $S^{-1}E$. Since $S^{-1}\pi_* E$ is flat over $\pi_* E$, $S^{-1}\pi_* E \otimes_{\pi_* E} E_*(-)$ is a legitimate homology theory (the only axiom of concern is whether it converts cofiber sequences to exact sequences, and E_* being a homology theory and flatness assures this) and is thus represented by a spectrum $S^{-1}E$. As with quotients, this procedure does not always yield a ring spectrum, but will in all the cases we need. By construction, we always have $(S^{-1}E)_*(X) = S^{-1}(E_*(X))$.

Quotienting BP_* by the ideal $J_n := (p = v_0, v_1, \ldots, v_{n-1}, \widehat{v}_n, v_{n+1}, \ldots)$ and then localizing with respect to the prime ideal $S = (v_n) \subseteq \mathbb{F}_p[v_n] = BP_*/J_n$ yields $\mathbb{F}_p[v_n, v_n^{-1}]$, and the corresponding operations on the level of spectra give the Morava K-theory $K(n)$. Writing $I_n := (v_0, \ldots, v_{n-1})$, it is shown in [Wür77][5.1] that the quotients BP/I_n admit a ring structure, with homotopy commutative multiplication for $p > 2$, and by [Wür77][7.2], $K(n)$ can be given the same type of structure. (That is, a ring structure which is homotopy commutative for $p > 2$.)

Chapter 3

Formal group laws and complex-oriented cohomology theories

3.1 Formal group laws

Definition 3.1.1. Let R be a commutative ring with unit. A (one-dimensional, commutative) *formal group law over* R is a power series $f \in R[[x, y]]$ such that each of the following holds:

(1) $f(x, 0) = f(0, x) = x$ (identity),

(2) $f(x, y) = f(y, x)$ (commutativity), and

(3) $f(x, f(y, x)) = f(f(x, y), z)$ (associativity).

Example 3.1.2. The *additive formal group law* is given by $f(x, y) = x + y$ over any ring R.

Example 3.1.3. The *multiplicative formal group law* is given by

$$f(x,y) = x + y + xy = (1+x)(1+y) - 1.$$

By a homomorphism of formal group laws $f \to g$, we mean a power series H such that $g(H(x), H(y)) = H(f(x,y))$, and isomorphisms will correspond to such series H which are invertible in the sense that there is another power series G such that $G(H(x)) = H(G(x)) = x$. These are precisely those H with $H(0) = 0$ and $H'(0) \in R^\times$, and we'll use H^{-1} to denote this compositional inverse to H. We will use the notation

$$[n]_f(x) = \underbrace{x +_f x +_f \cdots +_f x}_{n \text{ terms}},$$

where $\alpha +_f \beta = f(\alpha, \beta)$. This is called the *$n$-series of f*. When $p = 0$ in R, we will be particularly interested in the p-series of f.

When in characteristic $p > 0$, the p-series of a non-zero formal group law can be written as

$$[p]_f(x) = ax^{p^h} + \text{ higher order terms.}$$

We will call h the *height* of f if a is a unit. Otherwise, we'll say that f has height *at least* h.

Letting $\mathrm{FGL}(R)$ denote the set of formal group laws over R, we see that a morphism $R \to S$ of commutative rings gives a map $\mathrm{FGL}(R) \to \mathrm{FGL}(S)$, where we replace the coefficients of a formal group law over R with their images in S. This functor from commutative rings to sets is corepresentable, meaning there is some commutative ring L such that we have a natural isomorphism $\mathrm{Hom}_{\mathrm{rings}}(L, R) \cong \mathrm{FGL}(R)$. We now construct L more explicitly.

Writing a power series $f \in R[[x,y]]$ as $f = \sum c_{i,j} x^i y^j$, we see that requiring f to be a formal group law is the same as imposing a collection of relations on the $c_{i,j}$. For instance, to satisfy $f(x,0) = f(0,x) = x$, we must have $c_{1,0} = c_{0,1} = 1$ for and $c_{i,0} = c_{0,i}$ for $i \neq 1$. The commutativity requirement forces the symmetry $c_{i,j} = c_{j,i}$. The relations forced by associativity are more complicated, but still polynomial in nature. So, giving a formal group law over R is equivalent to giving elements $c_{i,j}$ satisfying these polynomial relations. This leads to the following definition:

Definition 3.1.4. Let $L := \mathbb{Z}[c_{i,j}]/I$, where I is the ideal generated by the polynomial relations described above. L is known as the *Lazard ring*, and is the object whose existence is asserted above by the corepresentability of FGL$(-)$. Indeed, a map $L \to R$ is precisely the selection of elements $c_{i,j} \in R$ satisfying the relations necessary to form a formal group law over R. The *universal formal group law* $f \in L[[x,y]]$ is $\sum c_{i,j}x^iy^j$.

The formal group law $f = \sum c_{i,j}x^iy^j$ is universal in the sense that given a formal group law $g \in \mathrm{FGL}(R)$, there is a unique morphism $\varphi : L \to R$ such that $g = \varphi_* f$, that is,

$$g(x,y) = \sum \varphi(c_{i,j})x^iy^j.$$

Note that for fixed n, the coefficient of x^k in the n-series of g is an integer polynomial in the images of the $c_{i,j}$ which doesn't depend on g or R. So, φ carries the coefficients of $[n]_f(x)$ to the corresponding coefficients of $[n]_g(x)$.

We make L a graded ring by putting $|c_{i,j}| = 2(i+j-1)$, so that if x, y have degree -2, then $\sum c_{i,j}x^iy^j$ has degree -2 as well.

Theorem 3.1.5 (Lazard)**.** [Rav92, 3.2.3] *There is an isomorphism of graded rings* $L \cong \mathbb{Z}[x_1, x_2, \dots]$*, where* $|x_i| = 2i$*.*

The ring L contains a family of distinguished elements v_i, which we'll make use of later via the Landweber Exact Functor Theorem. They are defined as follows:

Definition 3.1.6. Let v_i denote the coefficient of x^{p^i} in the p-series for the universal formal group law on L. These v_i for $i > 0$ are the so-called *Araki generators* for BP_*, and satisfy $|v_i| = 2(p^i - 1)$.

3.2 Complex-oriented cohomology theories

On the surface, this discussion of formal group laws seems to be purely algebraic and have no obvious ties to homotopy theory. However, it is a theorem of Quillen ([Qui69]) that $L \cong MU^{-*} = MU_*$, where MU is the universal complex Thom spectrum. This spectrum is central to the Nilpotence Theorem:

Theorem 3.2.1 (Devinatz-Hopkins-Smith [DHS88])**.** *Let R be a ring spectrum. The kernel of the Hurewicz homomorphism*

$$\pi_* R \to MU_* R$$

consists of nilpotent elements.

An identical theorem holds for p-local ring spectra R and MU replaced with BP. Alternatively, we have the following theorem of a more "chromatic" flavor:

Theorem 3.2.2 (Hopkins-Smith [HS98])**.** *Let R be a p-local ring spectrum. Then, $\alpha \in \pi_* R$ is nilpotent if and only if image under the map*

$$\pi_* R \to K(i)_* R$$

is nilpotent for all $0 \leq i \leq \infty$. In particular, the intersection of the kernels of these maps consists of nilpotent elements.

In addition to its usefulness in the detection of nilpotent elements in stable homotopy, MU is the universal example of a "complex-oriented cohomology theory." (This statement is made exact in Example 3.2.10.) The study of such cohomology theories, especially via the theory of formal group laws and their heights, lies at the heart of the chromatic approach to stable homotopy theory. We now describe what it means for a cohomology theory to be "complex-oriented," along with some useful examples, and explain their close relationship with formal group laws.

Adjoint to the equivalence $S^1 \simeq K(\mathbb{Z},1) \xrightarrow{\sim} \Omega K(\mathbb{Z},2)$ is a map

$$\alpha : \Sigma S^1 \simeq S^2 \to K(\mathbb{Z},2) \simeq BU(1) \simeq \mathbb{CP}^\infty.$$

Alternatively, α can be viewed as the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$.

Definition 3.2.3. A multiplicative cohomology theory E is called complex-orientiable if any of the following equivalent conditions hold:

- The map $\alpha^* : E^2(\mathbb{CP}^\infty) \to E^2(S^2)$ is surjective.
- The map $\tilde{\alpha}^* : \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(S^2) \cong \pi_0 E$ is surjective
- The unit $1 \in \pi_0 E$ lies in $\mathrm{Im}(\tilde{\alpha}^*)$.

A *complex-orientation* is a choice of element c_1^E such that $\widetilde{\alpha}^*(c_1^E) = 1$.

Given a complex-oriented cohomology theory E, there is an isomorphism of cohomology rings

$$E^*(\mathbb{CP}^\infty) \cong E^*[[c_1^E]],$$

with $|c_1^E| = 2$ ([Lur10, Lec. 4]). We will prove this in the special case that E is even-periodic shortly. Now, consider the map

$$\mu : BU(1) \times BU(1) \to BU(1)$$

classifying the tensor product of line bundles. (This is the same as the multiplication on $\Omega K(\mathbb{Z},3)$ given by concatenation of loops, or alternatively, a twice-delooping of the addition map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, a description we use below.) This then induces a map

$$E^*[[c_1^E]] \cong E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = E^*[[c_1^E \otimes 1, 1 \otimes c_1^E]].$$

Writing $c_1^E \otimes 1 = x$ and $1 \otimes c_1^E = y$, the image of c_1^E under the above morphism is an element $F_E(x, y) \in E^*[[x, y]]$. The map

$$\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$$

is the map

$$K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \simeq K(\mathbb{Z} \times \mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$$

given (up to homotopy) by addition. That is, we have

$$[K(\mathbb{Z} \times \mathbb{Z}, n), K(\mathbb{Z}, n)] = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}),$$

and the homotopy class of μ corresponds to addition. As a result, this gives $\mathbb{CP}^\infty$ the structure of an H-space with a homotopy associative and homotopy commutative multiplication. So, F_E is a formal group law over E^*, and writing

$$F_E = \sum_{i,j>0} c_{i,j} x^i y^j,$$

we have $c_{i,j} \in E^{-2(i+j-1)}$ so that $F_E(x, y)$ is in degree 2.

Remark 3.2.4. The notation c_1^E is meant to parallel that of the first Chern class, and for good reason: By our construction using the classification map of the tensor product, we can describe the association of a formal group law F_E to a complex-oriented cohomology E theory via the relation

$$c_1^E(L_1 \otimes L_2) = F_E(c_1^E(L_1), c_1^E(L_2)) = c_1^E(L_1) +_{F_E} c_1^E(L_2)$$

for line bundles L_1, L_2, paralleling the usual relation. Here, we can consider complex line bundles L over any space X. Such an L is given as the pullback $\rho^*\mathcal{O}(1)$ of the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^\infty$ by a continuous map $\rho : X \to \mathbb{CP}^\infty$, and take

$$c_1(L) := \rho^*(c_1^E) \in E^2(X).$$

Furthermore, $E^*(BU(n)) \cong E^*[[c_1^E, \ldots, c_n^E]]$, and we can think of these c_i^E as the E-analogue of higher Chern classes.

Proposition 3.2.5. *Different choices of orientation c_1^E yield (strictly) isomorphic formal group laws.*

Proof. Let c_1, c_1' be complex orientations for E, with associated formal group laws F, F', respectively. Then, $c_1' \in E^*(\mathbb{CP}^\infty) = E^*[[c_1]]$, so that $c_1' = g(c_1)$ for some power series

$$g(x) = b_0 x + b_1 x^2 + \cdots .$$

(Reversing the roles of c_1 and c_1' shows that the function g has no constant term, as it is invertible.) Furthermore, as c_1' restricts to $1 \in E^2(S^2)$, we must have $b_0 = 1$. We then have for line bundles L_1, L_2,

$$\begin{aligned} c_1'(L_1 \otimes L_2) &= g(c_1(L_1 \otimes L_2)) \\ &= g\left(F(c_1(L_1), c_1(L_2))\right) \\ &= g\left(F(g^{-1}(c_1'(L_1)), g^{-1}(c_1'(L_2)))\right), \end{aligned}$$

so that $F'(x, y) = g\left(F(g^{-1}(x), g^{-1}(y))\right)$. □

Example 3.2.6. For a ring R, the Eilenberg-Mac Lane spectrum HR is complex-orientable, as restriction gives an isomorphism

$$R \simeq HR^2(\mathbb{CP}^\infty) \cong H^2(\mathbb{CP}^1, R) \simeq HR^2(S^0).$$

In this case, the associated formal group law is the additive one:

$$F_{HR}(x, y) = x + y.$$

Proof. Write

$$F_{HR}(x, y) = \sum_{i,j>0} c_{i,j} x^i y^j.$$

We have $|c_{i,j}| \in HR^{-2(i+j-1)}$, meaning that $c_{i,j} = 0$ unless $i + j = 1$. The requirement

$$F_{HR}(x, 0) = F_{HR}(0, x) = x$$

forces the remaining two coefficients to be $c_{1,0} = c_{0,1} = 1$ so that $F_{HR}(x, y) = x + y$. □

Remark 3.2.7. In the case $R = \mathbb{Z}$, we recover the classical relation for the first Chern class of a tensor product of line bundles:

$$c_1^{H\mathbb{Z}}(L_1 \otimes L_2) = c_1^{H\mathbb{Z}}(L_1) + c_1^{H\mathbb{Z}}(L_2) \in H^2(\mathbb{CP}^\infty, \mathbb{Z}).$$

Example 3.2.8. Let E be an even-periodic cohomology theory. That is, let $E^{2k+1}(S^0) = 0$ for $k \in \mathbb{Z}$ and let there be a unit $u \in E^2(S^0)$ inducing isomorphisms $E^2(-) \to E^0(-)$. Then E is complex-orientable.

Proof. Consider the Atiyah-Hirzebruch spectral sequence with

$$E_2^{p,q} = H^p(\mathbb{CP}^n; E^q(S^0)) \implies E^{p+q}(\mathbb{CP}^n).$$

For fixed q, we have

$$H^*(\mathbb{CP}^n; E^q(S^0)) \cong E^q(S^0)[x]/x^{n+1},$$

with $|x| = (2, 0)$. Since $E^q = 0$ for q odd, $H^p(\mathbb{CP}^n; E^q(S^0))$ is zero if either p or q is odd, meaning that there are no non-trivial differentials (d_2 has bi-degree $(2, -1)$) and the spectral sequence collapses with $E_2^{p,q} = E_\infty^{p,q}$. Since every object on the E_2-page is a free module and there are no non-trivial extensions of free modules by free modules, we must have

$$E^*(\mathbb{CP}^n) = E^*[x]/x^{n+1}.$$

The maps

$$\pi_{-*}E[x]/x^{n+1} \to \pi_{-*}E[x]/x^{m+1}$$

are surjective, and thus the inverse system $E^*(\mathbb{CP}^n)$ satisfies the Mittag-Leffler condition and we have

$$E^*(\mathbb{CP}^\infty) = E^*(\varinjlim_n \mathbb{CP}^n) \cong \varprojlim_n E^*(\mathbb{CP}^n) \cong E^*[[x]].$$

Furthermore, via this identification, the map $E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^1) = E^*(S^2)$ is the quotient by x^n and is therefore surjective, meaning that E is complex-orientable. The choice of generator x is the choice of complex orientation. □

The prototypical example of an even periodic cohomology theory is complex K-theory, KU, with unit $\beta^{-1} \in KU^2(S^0)$, where β is Bott element. Here, we have a canonical choice

$$c_1^{KU}(L) = \beta^{-1}([L] - 1) \in KU^2(X),$$

where 1 denotes the trivial rank 1 complex vector bundle on X, and

$$F_{KU}(x, y) = x + y + \beta xy,$$

which is isomorphic to the multiplicative formal group law.

Proof. We have for line bundles L, M,

$$\begin{aligned}
&c_1^{KU}(L) +_{F_{KU}} c_1^{KU}(M) \\
=&c_1^{KU}(L \otimes M) \\
=&\beta^{-1}([L \otimes M] - 1) \\
=&\beta^{-1}([L][M] - 1) \\
=&\beta^{-1}([L][M] + ([L] - 1) + ([M] - 1) - [L] - [M] + 1) \\
=&\beta^{-1}(([L] - 1) + ([M] - 1) + [L][M] - [L] - [M] + 1) \\
=&\beta^{-1}([L] - 1) + \beta^{-1}([M] - 1) + \beta^{-1}([L] - 1)([M] - 1) \\
=&\beta^{-1}([L] - 1) + \beta^{-1}([M] - 1) + (-1)^{|\beta^{-1}||[L]-1|}([L] - 1)\beta^{-1}([M] - 1) \\
=&\beta^{-1}([L] - 1) + \beta^{-1}([M] - 1) + ([L] - 1)\beta^{-1}([M] - 1) \\
=&\beta^{-1}([L] - 1) + \beta^{-1}([M] - 1) + \beta(\beta^{-1}([L] - 1)\beta^{-1}([M] - 1)) \\
=&c_1^{KU}(L) + c_1^{KU}([M]) + \beta(c_1^{KU}(L)c_1^{KU}(M))
\end{aligned}$$

□

Example 3.2.9. For $n \geq 1$, the Morava K-theories $K(n)$ are complex-oriented, with the associated formal group law (known as the *Honda formal group law*) being of height exactly n with p-series

$$[p](x) = v_n x^{p^n}.$$

(See [Wür91, 1.3].)

Example 3.2.10. The second space of MU is $MU_2 = MU(1) \cong \mathbb{CP}^\infty$. (Warning: $MU(n)$ is actually the $2n^{\text{th}}$ space of the spectrum MU, with the odd spaces being suspensions of the even ones.) The equivalence $\mathbb{CP}^\infty \to MU_2$ gives a degree -2 map from $\Sigma^\infty\mathbb{CP}^\infty \to MU$ and thus an element in $\widetilde{MU}^2(\mathbb{CP}^\infty)$, which Quillen showed in [Qui71] to be a complex orientation c_1^{MU} for MU. The formal group law coming from the spectrum MU is the universal one. That is, the (degree-reversing) map $L \to MU^*$ classifying the associated formal group law F_{MU} is an isomorphism and the induced map $\mathrm{FGL}(L) \to \mathrm{FGL}(MU^*)$ carries the universal formal group law to the formal group law to F_{MU}. Furthermore, the spectrum MU is universal amongst complex-oriented cohomology theories, in that given a complex-oriented cohomology theory E, there is a map $g : MU \to E$ satisfying $g_*(c_1^{MU}) = c_1^E$ and $g^*(F_{MU}) = F_E$. (See [Rav03, 4.1.13].)

Remark 3.2.11. In general, a formal group law over E^* is equivalent to a map $MU^* \to E^*$. The universality of MU implies for E a complex-oriented cohomology theory, this map lifts to a map of spectra.

Chapter 4

A geometric viewpoint: The moduli stack of formal groups

4.1 Construction of $\mathcal{M}_{FG}$

For a commutative ring R, a map $L \to R$ corresponds to a choice of formal group law over R. In this sense, the affine scheme $\operatorname{Spec} L$ parametrizes formal group laws. However, we have the notion of an isomorphism of formal groups, as defined in the previous chapter, and we would like to have a way to study formal group laws taking this into account. This will require the language of stacks.

One good way to understand the structure of a scheme Y is to study the category of quasi-coherent sheaves on Y. So, to get started, we consider an example of a quasi-coherent sheaf on $\operatorname{Spec} L$. As $\operatorname{Spec} L$ is affine, such a quasi-coherent sheaf is nothing more than an L-module, but thinking in these terms will be helpful later on.

For X a spectrum, $MU_*(X)$ is a (left) module over the (commutative) Lazard ring

$L = \pi_* MU$ via

$$S \simeq S \wedge S \to MU \wedge MU \wedge X \to MU \wedge X,$$

where the last map is given by multiplication on MU. So, we can think of $MU_*(X)$ as a quasi-coherent sheaf on the scheme $\operatorname{Spec} L$, which parametrizes formal group laws. Furthermore, this sheaf carries an action of the group scheme

$$G := \operatorname{Spec} \mathbb{Z}[b_1, \ldots],$$

which assigns to a commutative ring R the group

$$\{g \in R[[t]] \,|\, g(t) = t + b_1 t^2 + b_2 t^3 + \cdots\} = \operatorname{Hom}(\mathbb{Z}[b_1, \ldots], R),$$

this action being compatible with the action of G on $\operatorname{Spec} L$ by

$$(g \cdot f)(x, y) = gf(g^{-1}(x), g^{-1}(y)) \in \mathrm{FGL}(R),$$

for $g \in G(R)$, $f \in \mathrm{FGL}(R) = \operatorname{Spec} L(R)$.

Two formal group laws over R are called *strictly isomorphic* if they differ by a change of variables via an element of $G(R)$. The quotient stack $\mathcal{M}^s_{FG} := \operatorname{Spec} L/G$ parametrizes formal group laws and strict isomorphisms.

Notice that a $\mathbb{Z}$-grading on a ring A corresponds to a homomorphism $\varphi : A \to A[t^{\pm}]$, with associated decomposition

$$A = \bigoplus_{r \in \mathbb{Z}} \varphi^{-1}(t^r A).$$

Equivalently, we can regard φ as a map of affine schemes

$$\operatorname{Spec} A[t^{\pm}] = \operatorname{Spec}(\mathbb{Z}[t^{\pm}] \otimes_{\mathbb{Z}} A) = \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A \to \operatorname{Spec} A.$$

That is, a grading on A is equivalent to an action of the multiplicative group $\mathbb{G}_m$ on $\operatorname{Spec} A$.

The Lazard ring L comes with a equipped with a grading, and the corresponding action on the R-points of $\operatorname{Spec} L$ is given by:

$$(\lambda, f(x,y)) \mapsto \lambda f(\lambda^{-1}x, \lambda^{-1}y),$$

where $\lambda \in \mathbb{G}_m(R) = R^\times$, $f(x,y) \in \mathrm{FGL}(R)$. Similarly, the natural grading of $\mathbb{Z}[b_1, \dots]$ corresponds to an action of $\mathbb{G}_m$ on G.

Let G^+ denote the group scheme assigning to a ring R the group

$$G^+(R) = \{g \in R[[t]] \,|\, g = b_0 t + b_1 t^2 + b_2 t^3 + \cdots, b_0 \in R^\times\}$$

of power series with $g(0) = 0$, $g'(0) \in R^\times$.

Then, G^+ can be written as the semi-direct product $G^+ = \mathbb{G}_m \ltimes G$, so that G^+ acts on $\operatorname{Spec} L$ via this identification. Alternatively, we can write

$$G^+ = \operatorname{Spec} \mathbb{Z}[b_0^{\pm}, b_1, b_2, \dots].$$

Definition 4.1.1. The moduli stack of formal groups $\mathcal{M}_{FG}$ is the quotient stack $\operatorname{Spec} L/G^+$.

This stack parametrizes formal groups and isomorphisms between them. Specifically,

$$\mathcal{M}_{FG}(\operatorname{Spec} R)$$

is the groupoid of formal group laws over R and isomorphisms of formal group laws.

Returning to our quasi-coherent sheaf on $\operatorname{Spec} L$ corresponding to $MU_*(X)$, note that $MU_*(X)$ is a graded L-module. The action of G on $MU_*(X)$ is compatible with this action, meaning that the even graded part of $MU_*(X)$, $MU_{\text{even}}(X)$ is a representation of G^+, compatible with the action of G^+ on $\operatorname{Spec} L$. So, $MU_{\text{even}}(X)$ is a quasi-coherent sheaf on $\operatorname{Spec} L/G^+ = \mathcal{M}_{FG}$.

The stack $\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Q} = B\mathbb{G}_m$ parametrizes formal group laws over $\mathbb{Q}$-algebras, every such formal group law being strictly isomorphic to the additive formal group law via its logarithm, with automorphism group $\mathbb{G}_m$. Similarly, $\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ does so for $\mathbb{Z}_{(p)}$-algebras. For the rest of this chapter, we will fix a prime p and write $\mathcal{M}_{FG}$ in the place of $\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)}$.

4.2 Stratification of $\mathcal{M}_{FG}$ by height

In studying $\mathcal{M}_{FG}$, it is often useful to restrict our attention to strata corresponding to the heights of formal groups.

To that end, let v_n denote the coefficient of x^{p^n} in the p-series of a formal group law f. Then, following the definition of height given in Chapter 3, f is of height at least n if $v_i = 0$ for $i < n$. By taking f to be the universal formal group law over L, this gives us a way to identify elements $v_i \in L$. In the context of our stack $\mathcal{M}_{FG}$, this allows us to identify the closed substack

$$\mathcal{M}_{FG}^{\geq n} := \operatorname{Spec}(L_{(p)}/(v_0, \ldots, v_{n-1}))/G^+$$

as the moduli stack of formal group laws of height at least n. For $0 \leq n < \infty$, a formal group law is of height exactly n if v_n is invertible and $v_0, \ldots, v_{n-1}$ are all zero. So, the substack of formal group laws of height exactly n is

$$\mathcal{M}_{FG}^{n} := \mathcal{M}_{FG}^{\geq n} - \mathcal{M}_{FG}^{\geq n+1} = \operatorname{Spec}(L_{(p)}[v_n^{-1}]/(v_0, \ldots, v_{n-1}))/G^+.$$

The substack of formal groups of infinite height is

$$\mathcal{M}_{FG}^{\infty} = \operatorname{Spec}(L/(v_0, \ldots))/G^+.$$

The locally-closed substacks $\mathcal{M}_{FG}^{n}$, $0 \leq n \leq \infty$ form a stratification of the stack $\mathcal{M}_{FG}$.

The existence of a formal group law of height 0 over a $\mathbb{Z}_{(p)}$-algebra R forces $p \in R^\times$ so that R is in fact a $\mathbb{Q}$-algebra, meaning that $\mathcal{M}_{FG}^0 = B\mathbb{G}_m$.

Let $1 \leq n < \infty$. Every formal group law over a $\mathbb{Z}_{(p)}$-algebra R is strictly isomorphic to a p-typical one. So, isomorphism classes of formal group laws of height n over R correspond to maps

$$v_n^{-1}BP_*/(p, v_1, \ldots, v_{n-1}) = \mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots][v_n^{\pm}] \to R,$$

and following [Goe08, §6], this gives us an *fpqc* presentation

$$\operatorname{Spec} \mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots][v_n^{\pm}] \to \mathcal{M}_{FG}^n$$

of the stratum $\mathcal{M}_{FG}^n$. In the next chapter, we will see another description of $\mathcal{M}_{FG}^n$ involving the Morava stabilizer group.

Any formal group law of infinite height over a $\mathbb{Z}_{(p)}$-algebra R (which is then necessarily an $\mathbb{F}_p$-algebra) is isomorphic to the additive formal group law. So, $\mathcal{M}_{FG}^\infty = B\mathrm{Aut}(f)$, where f is additive formal group law over $\mathbb{F}_p$. The group $\mathrm{Aut}(f)$ is closely related to the group schemes $\operatorname{Spec} \mathcal{A}^\vee$, where $\mathcal{A}^\vee$ is the dual mod p Steenrod algebra.

Chapter 5

Lubin-Tate theory and Morava modules

5.1 The Landweber Exact Functor Theorem

As we've seen, complex-oriented cohomology theories give rise to formal group laws via their Chern classes. One might ask: Can we reverse this? That is, given a formal group law over a ring R given by a map $MU_* \to R$, can we produce a complex-oriented cohomology theory carrying this formal group law?

As we'll see, the Landweber Exact Functor Theorem provides a partial answer to this question.

Let f be a formal group law over a ring R. Begin by defining a functor

$$E_*(X) := MU_*(X) \otimes_{MU_*} R,$$

where X is a space and R is a left MU_*-module via the map classifying f. ($MU_*(X)$ is naturally a left MU_*-module, as described in Chapter 4. Make it into a right MU_*-module via the Koszul sign rule.)

To be a homology theory, E_* must satisfy the Eilenberg-Steenrod axioms (minus the dimension axiom). With the exception of converting fiber sequences to long exact sequences, these follow easily from the fact that MU_* is a homology theory. (For example, E_* is clearly homotopy-invariant.) It would suffice to require R to be flat over MU_*, but this is quite strict. For a less-restrictive requirement, we have the following theorem of Landweber:

Theorem 5.1.1 (Landweber Exact Functor Theorem). *Let $f : MU_* \to R$ be a formal group law over R. If for every prime p, the images of $p = v_0, v_1, \dots$ form a regular sequence in R, then*

$$E_*(-) = MU_*(-) \otimes_{MU_*} R$$

is a homology theory.

Via Brown Representability, this procedure produces a representing spectrum E.

A variant of this exists for Brown-Peterson homology, BP, where we fix a prime p and consider p-typical formal group laws given by maps from the ring

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots],$$

and we require that the sequence of the images of $p, v_1, \dots$ be regular for our fixed prime p.

Homology theories arising in this way are called *Landweber exact* homology theories. Via the Conner-Floyd isomorphism, complex K-theory is Landweber exact with

$$KU_*(X) \cong MU_*(X) \otimes_{MU_*} KU_*,$$

and this construction of the functor E_* from a formal group law on a ring R is sometimes called the *Conner-Floyd construction*. One notable non-example is the Morava K-theories for $n > 1$. (In $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$, $v_1 = 0$, and multiplication by 0 is not injective on $\mathbb{F}_p[v_n^{\pm}]/(p) = \mathbb{F}_p[v_n^{\pm}]$.)

In the language of Chapter 4, a formal group law over R determines a Landweber exact homology theory E if and only if the corresponding map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ is flat.

5.2 Deformations of formal groups and Lubin-Tate theory

A family of spectra produced in this way and of great use in the chromatic approach to stable homotopy theory is the collection of *Lubin-Tate theories* (sometimes called *Morava E-theories*). We now describe their construction, which will require a discussion of Lubin and Tate's theory of deformations of formal groups to complete local rings.

Fix a perfect field k of characteristic $p > 0$ and a formal group Γ of height n over k.

Definition 5.2.1. Let $(B, \mathfrak{m})$ be a complete local Noetherian ring. A *deformation* of Γ to B is a pair (G, i), where $G \in \mathrm{FGL}(B)$, $i : k \to B/\mathfrak{m}$ is a homomorphism, such that

$$i_*\Gamma = \varphi_* G$$

as formal group laws on $B/\mathfrak{m}$, where $\varphi : B \to B/\mathfrak{m}$ is the quotient map. Let $\mathrm{Def}_\Gamma(B)$ denote the category of deformations of Γ to B, where by a *morphism* of deformations

$$(G_1, i) \to (G_2, i),$$

we will mean an isomorphism $g : G_1 \to G_2$ of formal group laws such that $\varphi_* g \in \mathrm{Aut}(i_*\Gamma)$ is the identity. That is, we require

$$g(x) \equiv x \mod \mathfrak{m}.$$

Such a map g is called a $\star$-*isomorphism.* We take

$$\mathrm{Hom}((G_1, i_1), (G_2, i_2)) = \emptyset$$

for $i_1 \neq i_2$. This makes the category $\mathrm{Def}_\Gamma(B)$ into a groupoid, where $\pi_0 \mathrm{Def}_\Gamma(B)$ is the collection of $\star$-isomorphism classes of deformations of Γ to B.

Example 5.2.2. Let's examine at least one non-trivial example of a deformation of Γ. For simplicity, consider the case where $B/\mathfrak{m} = k$. Any complete local Noetherian ring with residue field k is uniquely a continuous $\mathbb{W}(k)$-algebra. Since Γ is a formal group law of height n over a characteristic p field, the map $\alpha : L \to k$ necessarily factors through $L_{(p)}/(p, v_1, \ldots, v_{n-1})$. So, it is convenient to consider the the local ring

$$E := \mathbb{W}(k)[[v_1, \ldots, v_{n-1}]],$$

which has maximal ideal $(p, v_1, \ldots, v_{n-1})$ and residue field k. Take any lift

$$\widetilde{\alpha} : L_{(p)} \to E$$

of α, such that $\widetilde{\alpha}(v_i) = v_i$ for $1 \leq i \leq n-1$. The resulting commutativity of the diagram

$$\begin{array}{ccc} L_{(p)} & \xrightarrow{\widetilde{\alpha}} & \mathbb{W}(k)[[v_1, \ldots, v_{n-1}]] \\ \downarrow & & \downarrow \\ L_{(p)}/(p, v_1, \ldots, v_{n-1}) & \xrightarrow{\alpha} & k \end{array}$$

implies that $\widetilde{\alpha}$ determines a deformation $F \in \mathrm{Def}_\Gamma(E)$.

Theorem 5.2.3 (Lubin-Tate)**.** *The functor $\pi_0\mathrm{Def}_\Gamma(-)$ takes values in sets and is corepresentable by the ring $E := \mathbb{W}(k)[[v_1, \ldots, v_{n-1}]]$. That is, up to $\star$-isomorphism, a deformation of Γ to a complete local Noetherian ring B is the the same a continuous $\mathbb{W}(k)$-algebra map*

$$\mathbb{W}(k)[[v_1, \ldots, v_{n-1}]] \to B.$$

Furthermore, the deformation F described above is the universal deformation of Γ in the sense that the $\star$-isomorphism class corresponding to a map

$$\psi : E \to B$$

is that of $\psi_* F$.

Notice that the sequence $p, v_1, \ldots, v_{n-1} \in E$ is trivially regular. As a result, by the Landweber Exact Functor Theorem, we can produce a $2(p^n - 1)$-periodic homology theory $\mathcal{E}(n)$ defined by

$$\mathcal{E}(n)(X) := MU_*(X) \otimes_{MU_*} E[v_n^{\pm}],$$

with $|v_i| = 2(p^i - 1)$, as before.

It is more common to work with a 2-periodic version of $\mathcal{E}(n)$, defined as follows:

Take $E(k, \Gamma)_* = \mathbb{W}(k)[[v_1, \ldots, v_{n-1}]][u^{\pm}]$ with $|u| = -2$ (and $|v_i| = 2(p^i - 1)$). This element u allows us to write

$$E(k, \Gamma)_* = \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]][u^{\pm}],$$

where $u_i = v_i u^{p^i - 1}$, so that $|u_i| = 0$ for all i. As with $\mathcal{E}(n)$, this defines a Landweber exact cohomology theory (where the MU_*-module structure on the ring $E(k, \Gamma)$ is given by the map classifying the universal deformation $\widetilde{\Gamma}$ of Γ), and we'll denote the representing spectrum $E(k, \Gamma)$. Having coefficients in $E(k, \Gamma)_*$, $\widetilde{\Gamma}$ is naturally a degree -2 formal group law. The unit u allows us to equivalently consider a degree 0 formal group law $u\widetilde{\Gamma}(u^{-1}x, u^{-1}y)$.

When Γ is taken to be the Honda formal group law Γ_n of height n over $k = \mathbb{F}_{p^n}$ (having p-series $[p](x) = x^{p^n}$), we'll write E_n instead, and call this spectrum *Morava E-theory.*

In this case, following [BB19a, §3.1], the universal deformation $u\widetilde{\Gamma_n}(u^{-1}x, u^{-1}y)$ of Γ_n is a p-typical formal group law classified by a map $BP_* \to E(k, \Gamma_n)$ determined by

$$v_i \mapsto \begin{cases} u_i u^{1-p^i} & 1 \le i \le n-1 \\ u^{1-p^n} & i = n \\ 0 & i > n. \end{cases}$$

5.3 The Morava stabilizer group

Definition 5.3.1. Let $\mathbb{S}_n = \mathrm{Aut}(\Gamma_n)$ be the automorphism group of the Honda formal group law Γ_n. That is, elements of $\mathbb{S}_n$ are invertible power series $f \in \mathbb{F}_{p^n}[[x]]$ such that

$$\Gamma_n(f(x), f(y)) = f(\Gamma_n(x, y)).$$

This group is acted on by the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, and the semi-direct product

$$\mathbb{G}_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma_n)$$

is called the *Morava stabilizer group*.

The semi-direct summand $\mathbb{S}_n$ is the group of units $\mathcal{O}_n^\times$ of

$$\mathcal{O}_n = \mathbb{W}(\mathbb{F}_{p^n})\langle S\rangle/(S^n = p, S\omega = \omega^\sigma S),$$

where $\omega \in \mathbb{W}(\mathbb{F}_{p^n})$ and σ is a lift of the Frobenius $x \mapsto x^p$ on $\mathbb{F}_{p^n}$ to $\mathbb{W}(\mathbb{F}_{p^n})$. Letting $F_i = \{x \in \mathbb{S}_n \,|\, x \equiv 1 \mod (S)^i\}$ gives a filtration by normal subgroups

$$\mathbb{S}_n = F_0 \supset F_1 \supset \cdots,$$

and the canonical map

$$\mathbb{S}_n \to \varprojlim_i \mathbb{S}_n/F_i$$

is an isomorphism, meaning that $\mathbb{S}_n$ (and similarly $\mathbb{G}_n$) has the structure of a profinite topological group ([Hen17]).

Remark 5.3.2. Let $\overline{\Gamma_n}$ denote the Honda formal group law of height n over $\mathbb{F}_p$ (again with p-series $[p](x) = x^{p^n}$) considered as a formal group law over $\overline{\mathbb{F}_p}$. Then, by [Laz55, Thm. IV], as $\overline{\mathbb{F}_p}$ is algebraically closed and characteristic p, $\overline{\Gamma_n}$ is the unique height n formal group law over $\overline{\mathbb{F}_p}$ up to isomorphism.

The map $\mathrm{Spec}\,\overline{\mathbb{F}_p} \to \mathcal{M}_{FG}^n$ classifying $\overline{\Gamma_n}$ is faithfully flat. That is, given a formal group law $\mathrm{Spec}\,R \to \mathcal{M}_{FG}^n$ of height n, we can form a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{\mathbb{F}_p} & \longrightarrow & \mathcal{M}_{FG}^n, \end{array}$$

where R' is faithfully flat over R. (As a result of Theorem 1 of [Lur10, Lec. 14], we can write the ring R' as the direct limit of finite étale extensions of $R \otimes \overline{\mathbb{F}_p}$, meaning that R' is faithfully flat over R.)

Furthermore, writing

$$\overline{\mathbb{G}_n} := \operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \ltimes \operatorname{Aut}(\overline{\Gamma_n})$$

for the automorphism group of the pair $(\overline{\mathbb{F}_p}, \overline{\Gamma_n})$, we can identify

$$\mathcal{M}_{FG}^n = \operatorname{Spec} \overline{\mathbb{F}_p}/\overline{\mathbb{G}_n}.$$

See [Lur10, Lec. 19] for a more thorough discussion.

This group $\overline{\mathbb{G}_n}$ is also sometimes called the Morava stabilizer group, but we will reserve that terminology for $\mathbb{G}_n$. Note that $\overline{\mathbb{G}_n}$ is a pro-finite group, and as such can be regarded as an affine group scheme. Concretely, following [Goe08, §2] we can identify the automorphism group $\operatorname{Aut}(\overline{\Gamma_n})$ as the affine $\overline{\mathbb{F}_p}$-scheme:

$$\operatorname{Aut}(\overline{\Gamma_n}) = \operatorname{Spec}(\overline{\mathbb{F}_p} \otimes_L \otimes W \otimes_L \overline{\mathbb{F}_p}),$$

where $W = \mathbb{Z}[b_0^{\pm}, b_1, b_2, \dots]$, so that $\operatorname{Spec} W = G^+$. This description works for any field k in place of $\overline{\mathbb{F}_p}$ and any map $L \to k$. In the case at hand, there is an isomorphism of Hopf algebras

$$\overline{\mathbb{F}_p} \otimes_L \otimes W \otimes_L \overline{\mathbb{F}_p} = \overline{\mathbb{F}_p}[b_0^{\pm}, b_1, \dots]/(b_i^{p^n} - b_i), \quad i \geq 0,$$

which follows from the computation of the Morava stabilizer algebra carried out in [Rav03, §6.1].

For a L/K a finite Galois extension, we can identify $\mathrm{Gal}(L/K)$ as the L-group scheme

$$\mathrm{Gal}(L/K) = \mathrm{Spec}(L \otimes_K L).$$

As such, $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ is an affine $\overline{\mathbb{F}_p}$-group scheme via

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \varprojlim \mathrm{Spec}(\mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}) = \mathrm{Spec}(\varinjlim(\mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})) = \mathrm{Spec}(\overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}).$$

In order to understand how these strata glue together, we want to understand infinitesimal neighborhoods in $\mathcal{M}_{FG}$ of a point in $\mathcal{M}_{FG}$, which correspond to deformations of a formal group law over a field k to infinitesimal thickenings of k.

Using the Lubin-Tate theory of lifts of formal group laws, one can define an action of $\mathbb{S}_n$ and $\mathbb{G}_n$ on $(E_n)_*$ (see [DH95] and [DH04]) for details).

Theorem 5.3.3 (Goerss-Hopkins-Miller)**.** *The action of $\mathbb{G}_n$ on $(E_n)_*$ lifts to a coherent action of $\mathbb{G}_n$ on the spectrum E_n by E_∞ maps. Furthermore, the space* $\mathrm{Map}_{E_\infty}(E_n, E_n)$ *of E_∞ self-maps of E_n has contractible path-components and*

$$\pi_0(\mathrm{Map}_{E_\infty}(E_n, E_n)) \cong \mathbb{G}_n.$$

For finite group G acting on a spectrum X, we can define the homotopy fixed point spectrum

$$X^{hG} = F(\Sigma^\infty_+ EG, X)^G,$$

where $F(-,-)$ is the function spectrum. There is an associated "Homotopy Fixed Point Spectral Sequence"

$$E_2^{p,q} = H^{-p}(G, \pi_q(X)) \implies \pi_{p+q}(X^{hG}).$$

So, for $H \leq \mathbb{G}_n$ finite, we get a fixed point spectrum E_n^{hH} (by replacing E_n with an equivalent spectrum on which H literally acts). If we are willing to discretize $\mathbb{G}_n$, this idea can be extended to arbitrary subgroups $H \subseteq \mathbb{G}_n$, but it turns out that we should care more about *continuous* group cohomology of $\mathbb{G}_n$; the $K(n)$-local E_n-based Adams Spectral Sequence is of the form

$$H^*_{\text{cont}}(\mathbb{G}_n; (E_n)_*) \implies \pi_* L_{K(n)} S^0,$$

where $L_{K(n)}S^0$ denotes the $K(n)$-local sphere spectrum (see Chatper 6 for more details on localization). In the case of finite (or equivalently, closed and discrete, as $\mathbb{G}_n$ is compact Hausdorff) H, there is no distinction between continuous and ordinary group cohomology, and E_n^{hH} can thus also be regarded as a "continuous" homotopy fixed point spectrum.

Theorem 5.3.4 (Devinatz-Hopkins [DH04]). *For all closed subgroups $H \leq \mathbb{G}_n$, there exists a (continuous) homotopy fixed point spectrum (abusively denoted) E_n^{hH} with an associated Homotopy Fixed Point Spectral Sequence*

$$E_2^{p,q} = H^{-p}_{cont}(H; \pi_q E_n) \implies \pi_{p+q}(E_n^{hH}).$$

In the case that $H = \mathbb{G}_n$, this Homotopy Fixed Point Spectral Sequence corresponds to the $K(n)$-local E_n-based Adams Spectral Sequence mentioned above, and $E_n^{h\mathbb{G}_n} = L_{K(n)}S^0$.

Remark 5.3.5. The usual construction for homotopy fixed points (for H a discrete group) requires that the action of H be a literal action, and not just a homotopy coherent action. As such, the spectra E_n^{hH} constructed in [DH04] are not literally homotopy fixed point spectra in the usual sense (at least when H is not finite). However, they have all of the desired functoriality properties, and agree with the usual homotopy fixed point construction in the case where H is finite, and are therefore the "correct" notion of homotopy fixed point spectra in this situation.

5.4 Movava modules

In practice, a completed version of Morava E-theory is seen to be a more natural choice of covariant version of E_n^* than the usual $(E_n)_*(X) = \pi_*(E_n \wedge X)$.

Definition 5.4.1. For X a spectrum, let

$$(E_n)_*^\vee(X) := \pi_* L_{K(n)}(E_n \wedge X)$$

We will call $(E_n)_*^\vee(X)$ the *height n Morava module* of X.

A description of the Morava module of a spectrum X will be of particular use in determining whether X is invertible. Furthermore, by [DH04], the Morava module of the fixed point spectrum E_n^{hH} for closed subgroup $H \leq \mathbb{G}_n$ can be computed as

$$(E_n)_*^\vee(E_n^{hH}) = \mathrm{Maps}_{\mathrm{cont}}(\mathbb{G}_n/H, (E_n)_*).$$

Remark 5.4.2. In [HMS94], the authors define the Morava module of X to be

$$\mathcal{K}_{n,*}(X) := \varprojlim_{(i_0,\ldots,i_{n-1})} [E_n/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})]_*(X).$$

Each of these quotients can be realized by smashing with an appropriate generalized Moore spectrum, as is described explicitly in [HMS94, §7]. Alternatively, we can construct quotients described in Chapter 2, and as each sequence $p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}$ is regular, we have

$$[E_n/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})]_* = (E_n)_*/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$$

In nice cases (see Remark 9.2.2), these two definitions coincide. We will continue to use the term "Morava module of X" to mean in the sense of Definition 5.4.1.

Chapter 6

Bousfield localization

The stable homotopy category of spectra is difficult to study in its entirety. In this chapter, we describe the technique of Bousfield Localization, which is often employed to restrict to smaller, more well-behaved subcategories.

6.1 Preliminaries on E-local and E-acyclic spectra

Definition 6.1.1. A morphism $f : X \to Y$ is called an E-*equivalence* if $f \wedge \mathrm{id} : X \wedge E \to Y \wedge E$ is an equivalence, that is, if $E_*(f)$ is an isomorphism.

Definition 6.1.2. Given a spectrum E, a spectrum X is called E-*acyclic* if $E_*X = 0$ (equivalently, if $E \wedge X \simeq *$). A spectrum X is called E-*local* if for every E-acyclic spectrum A, $[A, X]_* = 0$.

Notice that an E-equivalence $A \to B$ is a map whose homotopy fiber F is E-acyclic (look at the long exact sequence in E-homology), meaning that asking for a spectrum X to be E-acyclic is the same as requiring that every E-equivalence $A \to B$ gives an isomorphism $[B, X]_* \xrightarrow{\sim} [A, X]_*$.

Example 6.1.3. If E is a ring spectrum and M is a module over E (with the action of E on M denoted ρ), then M is E-local. To see this, let A be an E-acyclic spectrum and take any map $f : A \to M$. Then, factoring f as

$$S \wedge A \xrightarrow{\eta \wedge 1} E \wedge A \xrightarrow{1 \wedge f} E \wedge M \xrightarrow{\rho} M$$

shows that $f = 0$, because $E \wedge A \simeq *$.

E-local spectra satisfy many useful properties, some of which are as follows:

Proposition 6.1.4 ([Rav92] 7.1.2)**.**

1. *The homotopy inverse limit of E-local spectra is also E-local.*
2. *Given a cofiber sequence*
   $$X \to Y \to Z,$$
   if any two of X, Y, Z are E-local, then so is the third.
3. *An E-equivalence of E-local spectra is an equivalence.*
4. *If $X \vee Y$ is E-local, then so are X and Y.*
5. *If X is E-local, then it is $E \vee F$-local for any F.*

Proof.

1. Let $\{X_i\}_{i \in I}$ be an inverse system of E-local spectra and let A be E-acyclic. Letting $F(-,-)$ denote the mapping spectrum, we have
   $$F(A, \operatorname{holim} X_i) \simeq \operatorname{holim}\,(F(A, X_i)),$$
   There is then a Milnor exact sequence
   $$0 \to \varprojlim{}^1 \pi_{*+1} F(A, X_i) \to \pi_* \operatorname{holim} F(A, X_i) \to \varprojlim \pi_* F(A, X_i) \to 0.$$

As each X_i is E-local, $\pi_* F(A, X_i) = [A, X_i]_* = 0$, so that both $\varprojlim^1 \pi_{*+1} F(A, X_i)$ and $\varprojlim \pi_* F(A, X_i)$ vanish, and we have

$$[A, \operatorname{holim} X_i]_* = \pi_* F(X, \operatorname{holim} X_i) = \pi_* \operatorname{holim} (F(A, X_i)) = 0,$$

so that holim X_i is E-local.

2. Let A be an E-acyclic spectrum. From the cofiber sequence

$$X \to Y \to Z,$$

we get a long exact sequence

$$\cdots \to [A, X]_* \to [A, Y]_* \to [A, Z]_* \to [A, X]_{*-1} \to \cdots.$$

By hypothesis, two out of every three consecutive terms is zero, meaning that the third must be also.

3. Let $f : X \to Y$ be an E_*-equivalence of E-local spectra. Then the homotopy fiber F of f is E-acyclic. Further, by (2), F is E-local. Being both E-acyclic and E-local means that $F \simeq *$, so that $f : X \to Y$ is an equivalence.

4. Let A be E-acyclic. Then,

$$0 = [A, X \vee Y]_* = [A, X]_* \oplus [A, Y]_*,$$

so that both $[A, X]_*$ and $[A, Y]_*$ are zero.

5. Any $E \vee F$-acyclic spectrum is E-acyclic and the result follows immediately.

□

So, if we restrict our attention to E-local spectra, E_*-homology is sufficient for detecting equivalences. If we already know that a map $A \to B$ is between E-local spectra, this is great. Otherwise, we might hope find E-local replacements for A and B and work with those instead.

6.2 The Bousfield lattice and localization

In [Bou79], Bousfield showed that given a homology theory E and spectrum X we can functorially replace X with an E-local spectrum. More specifically, he showed that there is a homotopy cofiber sequence

$$G_E(X) \to X \to L_E(X),$$

functorial in X, where $G_E(X)$ is E-acyclic and $L_E(X)$ is E-local. As the homotopy fiber of $X \to L_E(X)$ is E-acyclic, this means that $X \to L_E(X)$ is an E-equivalence. So, the functor L_E provides a way of replacing a spectrum with an E-local one whose difference is invisible to the eyes of E. The functor L_E is called a *Bousfield localization functor*. Up to homotopy, there is only one choice spectrum $L_E(X)$ which is both E-local and E-equivalent to X. On the categorical level, the functor L_E localizes the category of spectra at the collection of morphisms f for which $f \wedge \mathrm{Id}_E$ is an equivalence.

Definition 6.2.1. Let $\langle E \rangle$ denote the class of E-acyclic spectra. Then, two homology theories E and F give the same localization functor if and only if $\langle E \rangle = \langle F \rangle$, which we'll call *Bousfield equivalence*. The collection (which is a set) of Bousfield equivalence classes is partially ordered by reverse inclusion, and is called the *Bousfield lattice*.

Remark 6.2.2. By [Kra08, 4.9.1], the Bousfield localization functor L_E can be constructed as a Verdier quotient $\mathbf{Sp} \to \mathbf{Sp}/\langle E \rangle$ followed by a fully faithful right adjoint $\mathbf{Sp}/\langle E \rangle \to \mathbf{Sp}$.

6.3 Examples

Many previously understood constructions can be phrased in terms of Bousfield localization:

Example 6.3.1. Let $M\mathbb{Z}/p\mathbb{Z}$ be the mod-p Moore spectrum. That is,

$$M\mathbb{Z}/p\mathbb{Z} = S^0/p = \mathrm{hocofib}(S^0 \xrightarrow{p} S^0).$$

The functor $L_{M\mathbb{Z}/p\mathbb{Z}}$ is called *p-completion*, and for connective spectra,

$$L_{M\mathbb{Z}/p\mathbb{Z}}X = \varprojlim((S^0/p^n) \wedge X) = X_p^\wedge.$$

For a not-necessarily connective X, we will also denote $X_p^\wedge := L_{M\mathbb{Z}/p\mathbb{Z}}X$.

Example 6.3.2. Let $H\mathbb{F}_p$ denote the mod-p Eilenberg-Mac Lane spectrum. For a connective spectrum X, localization $L_{H\mathbb{F}_p}X$ coincides with the p-completion $L_{M\mathbb{Z}/p\mathbb{Z}}X$ of the previous example (see [BB19b, 2.6]). So, this gives an alternative notion of p-completion of a spectrum. Throughout this dissertation, "p-completion" will continue to mean localization with respect to the Moore spectrum $M\mathbb{Z}/p\mathbb{Z}$.

Example 6.3.3. $L_{M\mathbb{Z}_{(p)}}$ is p-localization, where $M\mathbb{Z}_{(p)}$ is the Moore spectrum for the p-local integers.

Example 6.3.4. Localization L_{S^0} does nothing, i.e. $L_{S^0}X \simeq X$: A spectrum A is S^0-acyclic if and only if $S^0 \wedge A \simeq A \simeq *$, so $[A, X]_* = 0$ for all X, meaning that any spectrum X is already S^0-local.

Example 6.3.5. On the other end of the Bousfield lattice is the localization $L_{\mathrm{pt.}}$ with respect to pt. $= *$: Everything is pt.-acyclic, including the sphere spectrum. So, to be pt.-local is to force $\pi_*X = 0$.

Example 6.3.6. Localization with respect to Johnson-Wilson theory

$$E(n) := v_n^{-1}BP/(v_{n+1}, v_{n+2}, \dots)$$

can be described quite nicely:

$$L_{E(n)}X \simeq X \wedge L_{E(n)}S^0.$$

That is, $E(n)$-localization is the same thing as smashing with the $E(n)$-local sphere. Such a localization is called *smashing.*

For any E, $X \in \langle E \rangle \iff L_E X \simeq *$, so that when L_E is a smashing localization, we have

$$\langle E \rangle = \langle L_E S^0 \rangle.$$

Remark 6.3.7. The Bousfield classes of $E(n)$, E_n and $K(0) \vee \cdots \vee K(n)$ are the same, meaning that $L_{E(n)} = L_{E_n} = L_{K(0)\vee\cdots\vee K(n)}$. See [Rav84, 2.1] for the equivalence

$$\langle E(n) \rangle = \langle K(0) \rangle \vee \cdots \vee \langle K(n) \rangle.$$

The equivalence of these with $\langle E_n \rangle$ follows from [Hov93, 1.12]. In [Bak00, 3.4], it is shown that $\langle E(n) \rangle = \langle \widehat{E(n)} \rangle$, where $\widehat{E(n)} = E(n)^{\wedge}_{I_n}$ is completed Johnson-Wilson theory.

We finish with a useful fact, which we will use later to build the "chromatic fracture square."

Proposition 6.3.8 ([Law19] 9.26)**.** *Suppose $L_K L_E \simeq *$. Then, for any spectrum X, there is a homotopy pullback diagram*

$$\begin{array}{ccc} L_{E\vee K} X & \longrightarrow & L_E X \\ \downarrow & & \downarrow \\ L_K X & \longrightarrow & L_E L_K X. \end{array}$$

Proof. Let P be the homotopy pullback of the diagram

$$\begin{array}{ccc} & & L_E X \\ & & \downarrow \\ L_K X & \longrightarrow & L_E L_K X. \end{array}$$

Note that each object in the above diagram is either E-local or K-local, so by Proposition 6.1.4 (5), they are $E \vee K$-local. Note that this means that P is also $E \vee K$-local: Looking at the Mayer-Vietoris sequence for associated to the homotopy pullback, letting A be $E \vee K$-acyclic, we have a long exact sequence

$$\cdots \to [A, L_E L_K X]_{*+1} \to [A, P]_* \to [A, L_K X]_* \oplus [A, L_E X]_* \to \cdots,$$

and by hypothesis, the outside terms vanish, so $[A, P]_* = 0$.

So, it suffices to show that the map $X \to P$ is an $E \vee K$ equivalence, as we will then have $L_{E\vee K} X \simeq L_{E\vee K} P \simeq P$. Next, note that any homotopy pullback square is simultaneously a homotopy pushout square. So, as the smash product commutes with homotopy colimits, it necessarily commutes with taking homotopy pullbacks. So,

$$\begin{array}{ccc} P \wedge (E \vee K) & \longrightarrow & L_E X \wedge (E \vee K) \\ \downarrow & & \downarrow \\ L_K X \wedge (E \vee K) & \longrightarrow & L_E L_K X \wedge (E \vee K) \end{array}$$

is a homotopy pullback square and we are reduced to showing that

$$\begin{array}{ccc} X \wedge (E \vee K) & \longrightarrow & L_E X \wedge (E \vee K) \\ \downarrow & & \downarrow \\ L_K X \wedge (E \vee K) & \longrightarrow & L_E L_K X \wedge (E \vee K) \end{array}$$

is also, so that $X \wedge (E \vee K) \to P \wedge (E \vee K)$ is an equivalence.

The requirement that $L_K L_E \simeq *$ means that $L_E X$ is K-acyclic, as is $L_E L_K X$, so splitting the wedge summand factors apart, we are reduced to showing that

$$\begin{array}{ccc} X \wedge E & \xrightarrow{\sim} & L_E X \wedge E \\ \downarrow & & \downarrow \\ L_K X \wedge E & \xrightarrow{\sim} & L_E L_K X \wedge E \end{array}$$

and

$$\begin{array}{ccc} X \wedge K & \longrightarrow & * \\ \sim\downarrow & & \downarrow \\ L_K X \wedge K & \longrightarrow & * \end{array}$$

are homotopy pullback squares, which is clear.

□

For the rest of this dissertation, we'll be mostly concerned with localization at Morava K-theory and Morava E-theory.

Chapter 7

The chromatic tower and chromatic convergence theorem

7.1 Chromatic convergence and monochromatic layers

Fix a prime p and let $L_n = L_{E_n} = L_{K(0)\vee\cdots\vee K(n)}$.

Definition 7.1.1. The *chromatic tower* for a p-local spectrum X is the system

$$X \longrightarrow \cdots \longrightarrow L_2X \longrightarrow L_1X \longrightarrow L_0X.$$

Theorem 7.1.2 (Chromatic convergence)**.** *For X a p-local finite CW complex, $X \simeq \varprojlim L_nX$.*

Definition 7.1.3. The homotopy fibers $M_nX = \mathrm{hofib}(L_nX \to L_{n-1}X)$ are called the *monochromatic layers* of X.

The monochromatic layers M_nX are the "quotients" of our filtration, and they themselves decompose into periodic spectra with periods multiples of $2p^n - 2$ (the same $2p^n - 2$ as

$|v_n|$ in $K(n)_*$). Explicitly, we can write

$$\mathrm{hocolim}_\alpha F_\alpha \xrightarrow{\sim} M_n X,$$

where $F_\alpha \simeq \Sigma^{(2p^n-2)p^{e(\alpha)}} F_\alpha$ for some $e(\alpha) \geq 0$. This follows as a result of [HS99b, 7.10(c)] and the Periodicity Theorem of [HS98].

The functors $L_{K(n)}$ and M_n restrict to an adjunction on $L_n\mathbf{Sp}$, and give a symmetric monoidal equivalence

$$M_n : \mathbf{Sp}_{K(n)} \rightleftarrows \mathfrak{M}_n : L_{K(n)},$$

where $\mathfrak{M}_n$ is the essential image of the functor M_n. So, instead of working with the fibers M_nX of the filtration, we can equivalently work with the localizations $L_{K(n)}X$.

7.2 The chromatic fracture square

Proposition 7.2.1. *$L_{K(n)}L_{n-1}X \simeq *$ for any X.*

Proof. We wish to show that $L_{n-1}X$ is $K(n)$-acyclic. As L_{n-1} is a smashing localization, we have $K(n) \wedge L_{n-1}X \simeq K(n) \wedge L_{n-1}S^0 \wedge X$, and it suffices to show that $K(n) \wedge L_{n-1}S^0 \simeq *$. Now,

$$K(n) \wedge L_{n-1}S^0 \simeq L_{n-1}K(n),$$

so this is equivalent to verifying that $K(n)$ is E_{n-1}-acyclic. We have,

$$\langle L_{n-1}S^0 \rangle = \langle E_{n-1} \rangle = \langle K(0) \vee \cdots K(n-1) \rangle.$$

So,

$$L_{n-1}K(n) \simeq * \iff K(n) \wedge \bigvee_{i=0}^{n-1} K(i) \simeq *.$$

But, $K(n) \wedge K(i) \simeq *$ for $n \neq i$ by Proposition 2.1.1 (5), so the right hand side is indeed trivial and $L_{K(n)}L_{n-1}S^0 \simeq *$, as desired.

□

Corollary 7.2.2. Combining Propositions 6.3.8 and 7.2.1 we have the *chromatic fracture square*:

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X. \end{array}$$

Here, the left vertical map is the same as the map within the chromatic tower. So, using the chromatic fracture square and chromatic convergence, one can hope to study a spectrum X (p-locally) by studying its localizations $L_{K(n)}X$, and through this lens, the importance of understanding the $K(n)$-local category of spectra is immediate.

Remark 7.2.3. Being a homotopy pullback square, the homotopy fibers of the vertical maps in the above diagram are equivalent. As $L_{K(n)}X \simeq L_nL_{K(n)}X$, this gives

$$M_nX \simeq M_nL_{K(n)}X.$$

Remark 7.2.4. Let X be a complex-oriented cohomology theory with formal group law of height exactly m. Then, $L_nX \simeq * \iff m > n$. (See [Lur10][Lec. 29].) In this sense, L_n acts on complex-oriented cohomology theories like restriction to the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n}$. Similarly, $L_{K(n)}$ acts like completion along the locally-closed substack $\mathcal{M}_{\mathrm{FG}}^{n}$ ([Lur10][Lec. 22]).

Through this guise, the chromatic fracture square can be interpreted as saying that given a sheaf on $\mathcal{M}_{\mathrm{FG}}^{\leq n-1}$ and one on the completion $\widehat{\mathcal{M}_{\mathrm{FG}}^{n}}$ agreeing on a formal neighborhood $\widehat{\mathcal{M}_{\mathrm{FG}}^{n}} \cap \mathcal{M}_{\mathrm{FG}}^{\leq n-1}$ of $\mathcal{M}_{\mathrm{FG}}^{n}$ in $\mathcal{M}_{\mathrm{FG}}^{\leq n-1}$, we can glue these to get a sheaf on $\mathcal{M}_{\mathrm{FG}}^{\leq n}$.

7.3 The Sullivan arithmetic square

This chromatic fracture square should bring to mind the usual "arithmetic fracture square"

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_p \mathbb{Z}_p\right). \end{array}$$

In more direct analogy, we have the fracture square for spectra:

Proposition 7.3.1. *For any spectrum X, there is a homotopy pullback square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p L_p X \\ \downarrow & & \downarrow \\ L_{\mathbb{Q}} X & \longrightarrow & L_{\mathbb{Q}} \left(\prod_p L_p X\right), \end{array}$$

where $L_{\mathbb{Q}}X$ denotes rationalization $L_{\mathbb{Q}}X \cong H\mathbb{Q} \wedge X$ and $L_pX = X_p^{\wedge} = L_{M\mathbb{Z}/p\mathbb{Z}}X$ is p-completion.

Proof. This follows from Proposition 6.3.8 by taking $K = \bigvee_p M\mathbb{Z}/p\mathbb{Z}$ and $E = H\mathbb{Q}$. The equivalence $\prod_p L_pX \simeq L_{M(\bigoplus_p \mathbb{Z}/p\mathbb{Z})}X$ is shown in [Bou79, 2.6], and

$$M\left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right) \simeq \bigvee_p M\mathbb{Z}/p\mathbb{Z}$$

by definition of Moore spectra. The condition that $L_K L_E X \simeq 0$ is equivalent to requiring that $K \wedge E \wedge X \simeq *$, which holds because $K \wedge E \simeq \bigvee_p M(\mathbb{Z}/p\mathbb{Z}) \wedge H\mathbb{Q} \simeq *$. Finally, by Serre, there is an equivalence $M\mathbb{Q} \simeq H\mathbb{Q}$, and by [Rav92, 7.2.5], there is a Bousfield equivalence

$$\langle S^0 \rangle = \langle M\mathbb{Q} \rangle \vee \bigvee_p M\mathbb{Z}/p\mathbb{Z},$$

so that $L_{E \vee K} X \simeq L_{S^0} X \simeq X$. ∎

Remark 7.3.2. This idea of building X from its mod p-localizations and its rationalization is originally due to Sullivan [SR05] (in the case where X is a nilpotent space), and as such the pullback square of Proposition 7.3.1 is often called the "Sullivan arithmetic square."

7.4 The chromatic filtration on homotopy groups

In the introduction, we promised a new filtration to replace the Postnikov filtration of $\pi_* S$with quotient groups $\pi_n S$. The filtration above, however, is a filtration on the level of *spectra*, rather than on their homotopy groups. An alternative chromatic filtration on the homotopy groups, however, does exist.

Definition 7.4.1. The *chromatic filtration on* $\pi_* S^0$ is the descending filtration

$$\pi_* S^0 \supseteq F_0(\pi_* S^0) \supseteq F_1(\pi_* S^0) \supseteq \cdots ,$$

where $F_n(\pi_* S^0) = \ker(\pi_* S^0 \to \pi_* L_n S^0)$.

This construction works equally well with any other spectrum in place of S^0.

Chapter 8

The $K(n)$-local Picard group

Definition 8.0.1. Given a symmetric monoidal category $(\mathbf{C}, \otimes, I)$, we call an object $X \in \mathbf{C}$ *invertible* if there is some object $Y \in \mathbf{C}$ such that $X \otimes Y \cong I$. Should the collection of isomorphism classes of invertible elements form a set (for instance, when $\mathbf{C}$ is essentially small), then we can define the *Picard group of the category* $\mathbf{C}$ to be the collection of isomorphism classes of such elements with group operation $\otimes$ and identity element I, and we'll denote it $\mathrm{Pic}(\mathbf{C})$.

8.1 Invertibility in Sp

Example 8.1.1. The only invertible objects in the stable homotopy category $\mathbf{Sp} = (\mathbf{Sp}, \wedge, S^0)$ are the spheres, so we have $\mathrm{Pic}(\mathbf{Sp}) \cong \mathbb{Z}$, with cyclic generator S^1.

Proof. By the Künneth isomorphism for homology with coefficients in a field k, for $Z \wedge Z' \simeq S^0$,

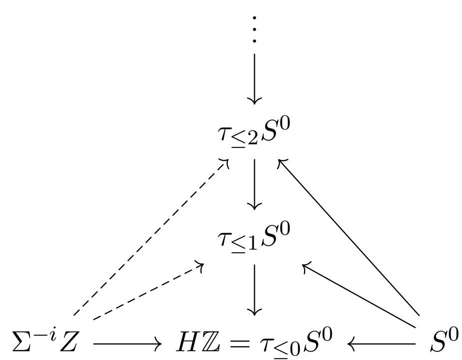
$$Hk_*Z \otimes_k Hk_*Z' \cong Hk_*S^0 = k,$$

so that $Hk_*Z = H\mathbb{Z}_*Z \otimes_{\mathbb{Z}} k = k$ generated in some degree i. Since this is true for all fields k, we must have $H\mathbb{Z}_*Z \simeq \mathbb{Z}$. (First, as Z is invertible, it is strongly dualizable. By [MLC$^+$96, XVI.7.4] this means that Z is the retract of a finite spectrum F. In particular, the homology groups $H\mathbb{Z}_*Z$ are finitely generated, being a direct summand of $H\mathbb{Z}_*F$. Taking $k = \mathbb{Q}$ shows that for some fixed i, we have

$$H\mathbb{Z}_*Z = \begin{cases} \mathbb{Z} \oplus T_i & * = i \\ T_j & j \neq i, \end{cases}$$

where the T_i consist entirely of torsion. So, for taking $k = \mathbb{F}_\ell$ for any prime ℓ, we see that T_i and T_j, $j \neq i$ have no ℓ-torsion. As ℓ was arbitrary, these torsion groups are all 0.)

Now, consider the Postnikov tower for S^0.



The 0$^{\text{th}}$ truncation of the Postnikov tower of $\Sigma^{-i}Z$ is $\tau_{\leq 0}\Sigma^{-i}Z = H\mathbb{Z}$, giving the bottom map. We have

$$\operatorname{hocofib}(\tau_{\leq m}S^0 \to \tau_{\leq m-1}S^0) = \Sigma^{m+1}H\pi_m S^0.$$

So, obstructions to lifting to a map $\Sigma^{-i}Z \to \tau_{\leq m}S^0$ live in

$$\begin{aligned}[\Sigma^{-i}Z, \Sigma^{m+1}H\pi_m S^0]_0 &= H^{i+m+1}(Z; \pi_m S^0) \\ &\cong \operatorname{Hom}(H_{i+m+1}(Z;\mathbb{Z}), \pi_m S^0) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{i+m}(Z;\mathbb{Z}), \pi_m S^0)\end{aligned}$$

by the Universal Coefficients Theorem. For $m > 1$,

$$H_{i+m+1}(Z;\mathbb{Z}) = H_{i+m}(Z;\mathbb{Z}) = 0,$$

so that the obstructions vanish and we get a map $\Sigma^{-i}Z \to S^0$, which is evidently an isomorphism on $H\mathbb{Z}_*(-)$. Similarly, we get a map $\Sigma^i Z' \to S^0$, which after smashing with Z gives a map $S^0 \to \Sigma^{-i}Z$ which is also an isomorphism on $H\mathbb{Z}_*(-)$, inverse to the map induced by $\Sigma^{-i}Z \to S^0$. In total, this gives a composite

$$S^0 \to \Sigma^{-i}Z \to S^0,$$

which is an isomorphism on $H\mathbb{Z}_*S^0$. As S^0 is connective, by the Hurewicz theorem, this composite is a homotopy equivalence $\pi_*S^0 \xrightarrow{\sim} \pi_*S^0$, so that we get a splitting

$$\Sigma^{-i}Z \simeq S^0 \vee A$$

for some A.

Identically, we can write

$$\Sigma^i Z' \simeq S^0 \vee A'$$

for some A'. In all, this gives

$$S^0 \simeq (S^0 \vee A) \wedge (S^0 \vee A') \simeq S^0 \vee A \vee A' \vee A \wedge A',$$

so that $A \simeq A' \simeq *$ and

$$Z \simeq S^i$$

□

8.2 Invertibility in $L_{K(n)}\mathbf{Sp}$

Definition 8.2.1. In [HMS94], Hopkins, Mahowald, and Sadofsky show that for $\mathbf{C} = L_{K(n)}\mathbf{Sp}$ (with monoidal product $X \otimes Y = L_{K(n)}(X \wedge Y)$ and unit $L_{K(n)}S^0$), the collection of isomorphism classes of invertible elements is indeed a set, so we may therefore define the $K(n)$-*local Picard group*, $\mathrm{Pic}_n := \mathrm{Pic}(L_{K(n)}\mathbf{Sp})$. Furthermore, for $X \in \mathrm{Pic}_n$, finding its inverse is straightforward: It is given by the function spectrum $F(X, L_{K(n)}S^0)$, i.e., the $K(n)$-local Spanier-Whitehead dual of X.

8.3 The algebraic Picard group

Definition 8.3.1. By *Morava module*, we'll mean a complete $(E_n)_*$-module M with a continuous action of $\mathbb{G}_n$ which is compatible with the action of $(E_n)_*$ in the sense that

$$g(e(m)) = g(e)g(m)$$

for $g \in \mathbb{G}_n$, $e \in (E_n)_*$, and $m \in M$. This is compatible with Definition 5.4.1 in that for a spectrum X, $(E_n)^\vee_* X$ is a Morava module.

Definition 8.3.2. By *algebraic Picard group*, we'll mean the Picard group of the category of Morava modules, and we'll denote it $\mathrm{Pic}_n^{\mathrm{alg}}$. A Morava module M is in $\mathrm{Pic}_n^{\mathrm{alg}}$ if and only if it is free of rank 1 over $(E_n)_*$. There is a natural map $\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}}$ given by $X \mapsto (E_n)^\vee_* X$. An element in $\kappa_n := \ker(\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}})$ is called *exotic*.

Theorem 8.3.3 (Pstrągowski)**.** [Pst18] *For $2p - 2 > n^2 + n$, the map*

$$\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}}$$

is an isomorphism.

Remark 8.3.4. Injectivity of the map $\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}}$ when $2p - 2 > n^2$ and $(p-1) \nmid n$ follows from [HMS94][7.5]. For the same range, $2p - 2 > n^2 + n$, as Theorem 8.3.3, Hovey and Sadofsky [HS99a] show that the Picard group of the $E(n)$-local category is isomorphic to $\mathbb{Z}$, with generator $L_n S^1$.

As E_n is even-periodic, $\mathrm{Pic}_n^{\mathrm{alg}}$ is endowed with a $\mathbb{Z}/2\mathbb{Z}$ grading, and we define $\mathrm{Pic}_n^{\mathrm{alg},0}$ to be the index 2 subgroup of $\mathrm{Pic}_n^{\mathrm{alg}}$ concentrated in even degrees. Hy [HMS94, 8.4], this subgroup can be described via continuous group cohomology as

$$\mathrm{Pic}_n^{\mathrm{alg},\,0} = H^1_{\mathrm{cont}}(\mathbb{G}_n, (E_n)_0^\times).$$

Chapter 9

Existing results

9.1 Preliminaries on Thom spectra and orientations

Critical to an understanding of known constructions of invertible $K(n)$-local spectra is the theory of (reduced) Thom spectra. We review the pertinent material now.

Definition 9.1.1. Let $f : X \to BU$ be a complex vector bundle over a space X. For X compact, the image is contained in some finite $BU(n)$. In this case, we define the *Thom spectrum* $M(f)$ as the desuspension

$$\Sigma^{-\dim f}\Sigma^{\infty}\mathrm{Th}(f)$$

of the reduced suspension spectrum of the Thom space of f by the dimension of the bundle. For non-compact X, define $M(f)$ as

$$\varinjlim_{\substack{C \subseteq X \\ \text{compact}}} M(f|_C),$$

taking the limit of the Thom spectra over the compact subspaces of X.

Definition 9.1.2. For a vector bundle of dimension n, the Thom space has a spherical cell in dimension n. Let $\overline{Th}(f)$ be the cofiber of the inclusion $S^n \hookrightarrow \mathrm{Th}(f)$. This will be called the *reduced Thom space*. Similarly, by desuspending by the dimension of the bundle in the construction of the Thom spectrum, we guarantee a unique spherical 0-cell S^0 in $M(f)$. The *reduced Thom spectrum* X^f will be the cofiber

$$S^0 \hookrightarrow M(f) \to X^f.$$

We wish to extend this construction to give Thom spectra Mf associated to a spherical fibration $f : X \to BG = BGL_1S^0$. Classical ($H\mathbb{Z}-$)orientability of such a fibration gives us Thom isomorphisms

$$H\mathbb{Z}_*Mf \xrightarrow{\sim} H\mathbb{Z}_*\Sigma^\infty_+X,$$

and

$$H\mathbb{Z}^*\Sigma^\infty_+X \xrightarrow{\sim} H\mathbb{Z}^*Mf$$

If we instead seek such an equivalence on R-(co)homology for some ring spectrum R, we can modify the requirement of orientability to the notion of an *R-orientation*, which we discuss in what follows.

The theory of R-orientations of Thom spectra for an E_∞ ring spectrum R was developed in [May77]. We review that material, as well as an extension to A_∞ spectra from the point of view of [ABG$^+$08].

Definition 9.1.3. Let A be an associative ring spectrum. We then define GL_1A, the *space of units of A* to be the pullback in the diagram of unpointed spaces

$$\begin{array}{ccc} GL_1A & \longrightarrow & \Omega^\infty A \\ \downarrow & & \downarrow \\ (\pi_0A)^\times & \longrightarrow & \pi_0A. \end{array}$$

Here (for A a sequential spectrum), the map $\Omega^\infty \to \pi_0A$ is determined by taking the natural maps

$$\Omega^n A_n \to \pi_n A_n \to \varinjlim \pi_n A_n = \pi_0 A$$

and applying the universal property of $\Omega^\infty A = \varinjlim \Omega^n A_n$. If A is an A_∞ ring spectrum, then we have a delooping

$$GL_1 A \simeq \Omega BGL_1 A.$$

If A further happens to be an E_∞ ring spectrum, then we get an infinite delooping

$$GL_1 A \simeq \Omega^\infty gl_1 A.$$

Let R be an E_∞ ring spectrum, and let b be a spectrum with a map $f : b \to bgl_1 R = \Sigma gl_1 R$. Now, let p be the homotopy pullback of the diagram

$$b \longrightarrow bgl_1 R \longleftarrow egl_1 R \simeq *.$$

We can then form the diagram

$$\begin{array}{ccc} gl_1 R & \xrightarrow{=} & gl_1 R \\ \downarrow & & \downarrow \\ p & \longrightarrow & egl_1 R \\ \downarrow & & \downarrow \\ b & \xrightarrow{f} & bgl_1 R. \end{array}$$

Given an E_∞ map $R \to A$ gives through composition and functoriality

$$\begin{array}{ccc} gl_1 R & \longrightarrow & gl_1 A \\ \downarrow & & \downarrow \\ p & \longrightarrow & egl_1 A \simeq * \\ \downarrow & & \downarrow \\ b & \longrightarrow & bgl_1 A. \end{array}$$

Theorem 9.1.4. [ABG+08, 3.2] *The functors $\Sigma^\infty_+ \Omega^\infty$ and gl_1 participate in an adjunction*

$$\Sigma^\infty_+ \Omega^\infty : \mathrm{ho}((-1) - \textit{connected spectra}) \rightleftarrows \mathrm{ho}(E_\infty \textit{ ring spectra}) : gl_1.$$

Definition 9.1.5. The *Thom spectrum* Mf associated to the map $f : b \to bgl_1 R$ is the homotopy pushout in the following diagram of E_∞ spectra:

$$\begin{array}{ccc} \Sigma^\infty_+ \Omega^\infty gl_1 R & \longrightarrow & R \\ \downarrow & & \downarrow \\ \Sigma^\infty_+ \Omega^\infty p & \longrightarrow & Mf. \end{array}$$

Here, the map

$$\Sigma^\infty_+ \Omega^\infty gl_1 R \longrightarrow R$$

is the counit of the adjunction in Theorem 9.1.4

The spectrum underlying Mf is the derived smash product

$$Mf = \Sigma^\infty_+ P \wedge^L_{\Sigma^\infty_+ GL_1 R} R,$$

where $P = \Omega^\infty p$.

Remark 9.1.6. For R an A_∞ ring spectrum (but not necessarily E_∞) we can form a Thom spectrum associated to a map of spaces $f : B \to BGL_1 R$. In this case, we can still form a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & EGL_1 R \\ \downarrow & & \downarrow \\ B & \longrightarrow & BGL_1 R, \end{array}$$

and we define $Mf = \Sigma^\infty_+ P \wedge^L_{\Sigma^\infty_+ GL_1 R} R$, just as in the E_∞ case.

Using this construction, we can form a Thom spectrum for any map $B \to BGL_1 R$ (where R is A_∞). Specializing to $R = S^0$, we arrive at the standard case of the Thom spectrum associated to a spherical fibration. We will primarily be concerned with (a p-completion of) this case.

In the classical case of the Thom space of a vector bundle $f : U \to V$ of rank n, there is a Thom isomorphism

$$H^*(V; \mathbb{Z}/2\mathbb{Z}) \to \widetilde{H}^{*+n}(\mathrm{Th}(f); \mathbb{Z}/2\mathbb{Z}),$$

where $\mathrm{Th}(f)$ is the Thom space of f, where the isomorphism is given by cupping with a Thom class $c \in \widetilde{H}^n(\mathrm{Th}(f); \mathbb{Z}/2\mathbb{Z})$.

The space G_n of homotopy equivalences of S^{n-1} is an associative H-space under composition, and its delooping BG_n is the classifying space for spherical fibrations with fiber S^{n-1} (see [Sta63]). Furthermore, G_n admits an inclusion $O_n \hookrightarrow G_n$. This allows us to consider such the vector bundle $f : U \to V$ as a spherical fibration $V \to BG_n$ with fiber S^{n-1}, and an orientation of such a map upgrades this to an isomorphism on *integral* homology, and working stably, this gives the integral Thom isomorphism for a spherical fibration $X \to BG = BGL_1S^0$ previously discussed (with $G = \varinjlim G_n$, the maps $G_n \to G_{n+1}$ given by suspension). We now generalize this to define R-orientations for a cohomology theory R, which will give us an R-(co)homological Thom isomorphisms, with classical orientability corresponding to $H\mathbb{Z}$-orientability.

Definition 9.1.7. Let R be an A_∞ ring spectrum and $f : B \to BGL_1R$ a map of spaces. For $x \in B$, let Mf_x be the Thom spectrum associated to

$$\{x\} \hookrightarrow B \xrightarrow{f} BGL_1R.$$

A map $Mf \to R$ (or, equivalently, an element of $R^0(Mf)$) is an *orientation* if and only if

$$Mf_x \to Mf \to R$$

is a weak equivalence for every $x \in B$.

By [ABG$^+$08, 2.20], if f factors as

$$f : B \xrightarrow{g} BGL_1S^0 \xrightarrow{BGL_1\eta} BGL_1R,$$

then the set of lifts

$$\begin{array}{ccccc} P & \longrightarrow & B(S^0, R) & \longrightarrow & EGL_1R \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ B & \xrightarrow[g]{} & BGL_1S^0 & \longrightarrow & BGL_1R \end{array}$$

is in bijection with the set of orientations $u : Mf \to R$, where $B(S^0, R)$ is defined so that the right hand square is a homotopy pullback square. In this situation, we will also call an orientation $u : Mf \to R$ an *R-orientation* for g.

Remark 9.1.8. When R is E-local, the analogous statement holds if we replace $BGL_1(S^0)$ with $BGL_1(L_E S^0)$ and consider lifts $B \to B(L_E S^0, R)$, where $B(L_E S^0, R)$ is defined in the obvious way. See [May77, §3] or [Wes17, §4] for this.

Given a map $u : Mf \to R$, we can form the composite

$$\rho(u) : Mg \wedge R \simeq Mf \xrightarrow{\mathrm{Th}(\Delta)} \Sigma^\infty_+ B \wedge Mf \xrightarrow{1 \wedge u} \Sigma^\infty_+ B \wedge R,$$

where $\mathrm{Th}(\Delta)$ is the Thom diagonal.

Theorem 9.1.9 (Thom isomorphism, homological version, [ABG$^+$08])**.** *Let $f : B \to BGL_1 R$ be a map let $u : Mf \to R$ be an orientation. Then, the map*

$$\rho(u) : Mf \to \Sigma^\infty_+ B \wedge R$$

is a weak equivalence. In particular, we get an isomorphism

$$R_*(Mg) \cong R_*(\Sigma^\infty_+ B)$$

Theorem 9.1.10 (Thom isomorphism, cohomological version)**.** *Let $u : Mf \to R$ be an orientation, with $f = (BGL_1 \eta) \circ g$, as before. Then, cupping with u gives an isomorphism*

$$R^*(\Sigma^\infty_+ B) \cong R^*(Mg).$$

9.2 Results of Hopkins-Mahowald-Sadofsky

When discussing the Picard group of the $K(n)$-local category, there are three main things we would like to do:

1. Find computable invariants to determine whether a given spectrum lies in Pic_n

2. Compute the group structure of Pic_n.

3. Construct explicit elements of Pic_n.

It is usually helpful to restrict ourselves to looking at a specific height and a specific prime at a time. Hopkins-Mahowald-Sadofsky ([HMS94]) provide an answer to (1) at all heights and all primes:

Theorem 9.2.1 (Hopkins-Mahowald-Sadofsky)**.** *The following are equivalent:*

1. $L_{K(n)}(Z) \in \mathrm{Pic}_n$*.*

2. $\dim_{K(n)_*} K(n)_*(Z) = 1$*.*

3. $(E_n)^\vee_*(Z)$ *is a free* E_**-module of rank 1.*

Proof. (1)$\Rightarrow$ (2): The localization map $Z \to L_{K(n)}Z$ is a $K(n)_*$-equivalence, so it suffices to suppose $L_{K(n)}Z = Z \in \mathrm{Pic}_n$. Suppose that $Z \wedge Z' = L_{K(n)}S^0$. Then, by the Künneth isomorphism for Morava K-theory,

$$K(n)_*(Z \wedge Z') = K(n)_*Z \otimes_{K(n)_*} K(n)_*Z' = K(n)_*(L_{K(n)}S^0) = K(n)_*S^0 = \mathbb{F}_p[v_n^{\pm}].$$

As $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$ is a graded field, each $K(n)_*Z$ and $K(n)_*Z'$ are free modules over $K(n)_*$ whose ranks multiply to 1. Hence, $\dim_{K(n)_*} K(n)_*Z = 1$.

(2)$\Rightarrow$(1): Suppose $\dim_{K(n)_*} K(n)_*(Z) = 1$ and let $Z' := F(Z, L_{K(n)}S^0)$.

<u>Claim:</u> The evaluation map

$$Z \wedge Z' \to L_{K(n)}S^0$$

is a $K(n)$-equivalence.

<u>Proof of claim:</u> Let $\mathcal{C}_n$ denote the collection of all spectra X for which

$$q_X : Z \wedge F(Z, L_{K(n)}X) \to L_{K(n)}X$$

is a $K(n)$-equivalence. Here, q_X is adjoint to the identity in

$$[F(Z, L_{K(n)}X), F(Z, L_{K(n)}X)].$$

Then, $\mathcal{C}_n$ is closed under taking coproducts and cofibers. Furthermore, for $X = K(n)$,

$$q_{K(n)} : Z \wedge F(Z, K(n)) \to K(n)$$

is a $K(n)$-equivalence. To see this, note that

$$\pi_* F(Z, K(n)) = K(n)^{-*}(Z) = \mathrm{Hom}_{K(n)_*}(K(n)_{-*}(Z), K(n)_*).$$

Also, adjoint to the map

$$K(n) \wedge Z \wedge F(Z, K(n)) \stackrel{1 \wedge q_{K(n)}}{\longrightarrow} K(n) \wedge K(n) \stackrel{\mu}{\longrightarrow} K(n)$$

is a map

$$K(n) \wedge F(Z, K(n)) \to F(Z, K(n))$$

which makes $F(Z, K(n))$ into a $K(n)$-module, and thus a wedge of suspensions of $K(n)$. So, if $K(n)_* Z = K(n)_{*+m}$, then $\pi_* F(Z, K(n)) = K(n)_{*-m}$, so that

$$F(Z, K(n)) \simeq \Sigma^m K(n),$$

and

$$K(n)_*(F(Z, K(n))) = K(n)_{*-m} K(n),$$

and the equivalence follows.

Now, for X finite and type n (i.e., $K(n)_* X \neq 0$ and $K(i)_* X = 0$ for $i < n$), it is a fact, shown by Hopkins and Ravenel in unpublished work (and which follows non-trivially from [Rav92, §8.3]) that $L_{K(n)}X$ possesses a finite filtration wherein each cofiber is a wedge of $K(n)$'s, so $X \in \mathcal{C}_n$. But, for finite spectra, $L_{K(n)}$ is a smashing localization, so that our map

$$Z \wedge F(Z, L_{K(n)}X) \to L_{K(n)}X.$$

is equivalent to

$$Z \wedge Z' \wedge X = Z \wedge F(Z, L_{K(n)}S^0) \wedge X \to L_{K(n)}S^0 \wedge X.$$

As X is type n, $K(n)_*X \neq 0$, so that this is a $K(n)$-equivalence if an only if

$$Z \wedge Z' \to L_{K(n)}S^0$$

is, meaning that $S^0 \in \mathcal{C}_n$ and we're done.

(3)⇒(2): Suppose $(E_n)_*^\vee(Z) = (E_n)_*^\vee(S^k)$. Then, $(E_n)_*^\vee(Z)$ is pro-free, so that by [HS99b, 8.4],

$$K(n)_*(X) = (E_n)_*^\vee(X)/I_n,$$

where $I_n = (p, v_1, \ldots, v_{n-1})$. Thus, reducing modulo I_n gives

$$K(n)_*Z = K(n)_*S^k = K(n)_{*-k}.$$

For (2)⇒(3), see [HMS94, §7]. Alternatively, for a proof of (1)⇔(3), see [Dev17, 5.3]. □

Remark 9.2.2. In [HMS94], in place of (3), the authors include the condition that $\mathcal{K}_{n,*}$ be a free $(E_n)_*$-module of rank 1. Following [HS99b, 8.4], there is a Milnor exact sequence

$$0 \to \varprojlim_I{}^1 (E_n/I)_{*+1}X \to (E_n)_*^\vee X \to \underbrace{\varprojlim_I (E_n/I)_*X}_{=\mathcal{K}_{n,*}(X)} \to 0,$$

where I ranges over ideals of $(E_n)_*$ of the form $(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$. In [BF15, 6.2], it is shown that the $\varprojlim^1$-term vanishes if $(E_n)_*^\vee X$ is pro-free or if X is strongly dualizable in $L_{K(n)}\mathbf{Sp}$. In particular, when $X \in \mathrm{Pic}_n$, $(E_n)_*^\vee X \cong \mathcal{K}_{n,*}$ as E_*-modules.

9.3 Calculation of Pic_1 at odd primes following [HMS94]

For $p \neq 2$, the p-adic units, $\mathbb{Z}_p^\times$, are topologically cyclic. Let g be any topological generator, for example $(1+p)\zeta$, where ζ is a primitive $(p-1)^{\text{st}}$ root of unity. Then, specializing Definition 5.4.1 to the case $n = 1$ (and identifying $(E_n)_*^\vee(X) \cong \mathcal{K}_{n,*}(X)$ per Remark 9.2.2), we have for $X \in \mathrm{Pic}_1$,

$$(E_1)_*^\vee(X) = \varprojlim [E_1/(p^j)]_*(X) = \varprojlim (KU_p^\wedge)_*(X \wedge M(p^j)),$$

where $KU_p^\wedge$ is complex K-theory completed at p and $M(p^j)$ is the $\mathbb{Z}/p^j\mathbb{Z}$ Moore spectrum. This identification allows us to act on $(E_1)_*^\vee(X)$ by Adams operations ψ^a for any $a \in \mathbb{Z}_p^\times$. Alternatively, this action of $\mathbb{Z}_p^\times$ can be described via the action the Morava stabilizer group $\mathbb{G}_1 = \mathbb{S}_1 \cong \mathbb{Z}_p^\times$ on $E_1 \simeq KU_p^\wedge$.

Theorem 9.3.1. Pic_1 *sits in an exact sequence*

$$0 \to M \to \mathrm{Pic}_1 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

By Theorem 9.2.1, for $X \in \mathrm{Pic}_1$, $K(1)_*X$ is generated by a single element over $K(1)_*$. The map $\mathrm{Pic}_1 \to \mathbb{Z}/2\mathbb{Z}$ takes X to the dimension of that generator mod 2, and M is the kernel of that map. Furthermore, we have an isomorphism $ev : M \to \mathbb{Z}_p^\times$ taking X to the eigenvalue of ψ^g on $(E_1)_0^\vee(X) \cong \mathbb{Z}_p$.

We begin by outlining the proof of the equivalence $M \cong \mathbb{Z}_p^\times$.

Proposition 9.3.2. *The map $ev : M \to \mathbb{Z}_p^\times$ is a homomorphism.*

Proof. By Theorem 9.2.1, for $X \in M$, $(E_1)_*^\vee(X) = (E_1)_*^\vee(S^k)$ for some $k \equiv 0 \mod 2$. By Bott Periodicity, we can take $k = 0$. For $X, Y \in M$, we then have a Künneth isomorphism of $(KU_p^\wedge)_*$-modules

$$(E_1)_*^\vee(X \wedge Y) \cong (E_1)_*^\vee(X) \otimes_{(KU_p^\wedge)_*} (E_1)_*^\vee(Y),$$

which extends to an isomorphism of modules over the Adams operations ψ^a by multiplicativity of the Adams operations. □

Proposition 9.3.3. *$ev : M \to \mathbb{Z}_p^\times$ is injective.*

Proof. (Sketch) Suppose $X \in \ker ev$, i.e., that $ev(X) = 1$. We then have a diagram whose top row is a fiber sequence:

$$\begin{array}{ccccc} X & \xrightarrow{\eta \wedge 1} & (KU_p^\wedge \wedge X)_p^\wedge & \xrightarrow{[\psi^g - 1] \wedge 1} & (KU_p^\wedge \wedge X)_p^\wedge \\ & \nwarrow^{\tilde{s}} & \uparrow s & \nearrow_{0} & \\ & & S^0, & & \end{array}$$

where s is a generator of $\pi_0(KU_p^\wedge \wedge X) = (KU_p^\wedge)_0 X$. The fact that the top row is a fiber sequence follows from [Bou79] Theorem 4.3 and Proposition 2.11, and the full proof is detailed in Lemma 2.3 of [HMS94]. As the composition $[\psi^g - 1] \wedge 1 \circ s$ is null-homotopic (by our assumption on X), we can lift s to a map $\tilde{s} : S^0 \to X$. Further, as X is $K(1)$-local, this gives us a map

$$f : L_{K(1)} S^0 \to X.$$

Now, s is injective on $K(1)$-homology, meaning that $K(1)_*(f)$ is an injective homomorphism between objects both isomorphic to $K(1)_*$. So, f is a $K(1)_*$-equivalence of $K(1)$-local spectra and is thus a homotopy equivalence, meaning that $X \simeq L_{K(1)} S^0$ and ev is injective. □

Proposition 9.3.4. *ev is surjective.*

Proof. This is Corollary 2.6 of [HMS94], whose proof we omit. □

Corollary 9.3.5. We identify $M \cong \mathbb{Z}_p^\times$, and thus have an extension

$$0 \to \mathbb{Z}_p^\times \to \mathrm{Pic}_1 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Proposition 9.3.6. *This extension is not split, so*

$$\mathrm{Pic}_1 \cong \mathbb{Z}_p \oplus \mathbb{Z}/q\mathbb{Z},$$

where $q = 2p - 2$.

Proof. For $\mu \in \mathbb{Z}_p^\times$, let X_μ be the fiber of $KU_p^\wedge \xrightarrow{\psi^g - \mu} KU_p^\wedge$. Then, by Lemma 2.5 of [HMS94],

$$X_{g^n} \simeq L_{K(1)} S^{2n}.$$

It follows from the proof of [HMS94, 2.6] that $X_\mu \in M$ for all $\mu \in \mathbb{Z}_p^\times$ and that ψ^g acts by multiplication by μ^{-1} on $E_0^\vee(X_\mu)$, so that $\mu \mapsto X_{\mu^{-1}}$ determines an isomorphism $\mathbb{Z}_p^\times \to M$ inverse to ev.

Now, suppose a section $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}_1$ exists. The image of $1 \in \mathbb{Z}/2\mathbb{Z}$ under this section must be of the form $S^{-1} \wedge X$ for some $X = X_\mu \in M$ and be of order 2. So, we find that after $K(n)$-localizing,

$$S^0 \simeq S^{-2} \wedge X_{\mu^2}.$$

So,

$$X_{\mu^2} \simeq S^2 \simeq X_g \implies \mu^2 = g.$$

But, $g = (1+p)\xi$ has no square root in $\mathbb{Z}_p$, meaning that there is no section $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}_1$ and the extension is not split. Furthermore, we for p odd, $\mathbb{Z}_p$ splits as

$$\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p.$$

So we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}_p^\times) &\cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/(p-1)\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}_p)\\ &\cong (\mathbb{Z}/(p-1)\mathbb{Z})\,[2] \oplus \mathbb{Z}_p[2] \cong \mathbb{Z}/2\mathbb{Z} \oplus 0,\end{aligned}$$

so that $\mathbb{Z}_p \oplus \mathbb{Z}/q\mathbb{Z}$ is the only non-split extension.

□

In particular, there is an element of order $q = 2p - 2$ in Pic_1. As in the proof of the previous proposition, any such element must be of the form

$$Z \simeq S^{-1} \wedge X_\mu,$$

with (again, omitting the localizations)

$$S^0 \simeq Z^{\wedge q} \simeq S^{-q} \wedge X_{\mu^q},$$

so that $\mu^q = g^{p-1}$. This μ^q is then a generator of the summand $\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ of units congruent to 1 modulo p.

The existence of such a root $\mu \in \mathbb{Z}_p$ is guaranteed by Hensel's lemma: Let

$$f(x) = x^q - g^{p-1} \in \mathbb{Z}_p[x].$$

Then, $f(1) \equiv 1 - \xi^{p-1} \equiv 0 \mod p\mathbb{Z}_p$, and $f'(1) = q \not\equiv 0 \mod p\mathbb{Z}_p$ (because $p \neq 2$). Hensel's lemma then gives a unique root of $f(x)$ in $\mathbb{Z}_p$ congruent to $1 \mod p\mathbb{Z}_p$.

Explicitly, letting $Z = X_\mu$ for this μ, we have an isomorphism

$$\mathbb{Z}_p \oplus \mathbb{Z}/q\mathbb{Z} \to \mathrm{Pic}_1$$

given by

$$(\lambda, n) \mapsto X_{(1+p)^\lambda} \wedge Z^n.$$

This construction, however, is not terribly geometric. We use the theory of Thom spectra of complex vector bundles to construct particular elements of Pic_1. For now, by *Thom spectrum*, we will mean in the sense of Definition 9.1.2.

Let Σ_p denote the symmetric group on p letters and $\xi = \rho - [p]$ be the virtual complex vector bundle over $B\Sigma_p$, where ρ is the permutation bundle (with $\rho : B\Sigma_p \to BU(p)$ being the delooping of the inclusion $\Sigma_p \hookrightarrow U(p)$) and $[p]$ is the trivial bundle of dimension p. Then,

$$(\widetilde{KU_p^\wedge})^0((B\Sigma_p)_{(p)}) = \mathbb{Z}_p$$

is topologically cyclic with topological generator ξ. A sketch of much of this computation occurs below. Alternatively, this follows from the computation of the (unreduced) Morava E-theory of $B\Sigma_k$ for arbitrary k found in [Str98, 3.2].

Remark 9.3.7. We will want to define Thom spectra $(B\Sigma_p)^{\lambda\xi}$ for $\lambda \in \mathbb{Z}_p$. For $\lambda \in \mathbb{Z}$ an ordinary integer, the classical Definition 9.1.2 will suffice. However, for arbitrary λ, we need a bit of a modification. Using Definition 9.1.5, and writing $S := S^0$ for the sphere spectrum, we can form a Thom spectrum X^f if we have a map

$$X := (B\Sigma_p)_{(p)} \xrightarrow{f} BGL_1(S_p^\wedge).$$

Alternatively, we can regard this map as an element of $\widetilde{BGL_1(S_p^\wedge)}^0(X)$. The product $\lambda \cdot \xi$ naturally lives as an element

$$\lambda \cdot \xi \in (\widetilde{KU_p^\wedge})^0((B(\Sigma_p)_{(p)}) \cong \mathbb{Z}_p.$$

A delooping BJ of the J-homomorphism gives us a map $KU \to BGL_1(S)$, and p-completing then gives

$$KU_p^\wedge \to BGL_1(S_p^\wedge),$$

so that we get

$$(\widetilde{KU}_p^\wedge)^0(X) \xrightarrow{BJ_p^\wedge} BGL_1(S_p^\wedge)^0(X)$$

By $X^{\lambda\xi}$, we will mean the Thom spectrum associated to the composite $BJ_p^\wedge \circ (\lambda\xi)$.

Definition 9.3.8. For $\lambda \in \mathbb{Z}_p$, let

$$R_\lambda = ([B\Sigma_p]^{\lambda\xi})_{(p)},$$

where the subscript (p) denotes p-localization.

Atiyah [Ati61] showed that the complex K-theory of the classifying space BG of a finite group G is given by the completion $\hat{R}(G)$ of the representation ring at the augmentation ideal. In [Kuh87a, Kuh87b], Kuhn uses these results to show that for G a finite group with abelian Sylow p-subgroup,

$$\dim_{K(1)_*} K(1)_*(BG) = \#\{\text{conjugacy classes of elements of } G \text{ of order a power of } p\}.$$

In the case $G = \Sigma_p$, G contains $\mathbb{Z}/p\mathbb{Z}$ as a Sylow p-subgroup, and has a single conjugacy class of order p elements (as well as a single conjugacy class of order $p^0 = 1$ corresponding to the use here of unreduced $K(1)$-homology). Furthermore, it follows as a result of Atiyah's work that for $n = 1$, $K(1)_*(BG)$ is concentrated in even degrees. So, $L_{K(1)}\Sigma^\infty_+ B\Sigma_p \in M$ and by the Thom isomorphism for complex K-theory and Proposition 2.1.1(2), we also have $L_{K(1)}R_\lambda \in M$.

As it turns out, these spectra are sufficient to generate all of Pic_1 at odd primes:

Theorem 9.3.9 (Hopkins-Mahowald-Sadofsky)**.**

$$\mathrm{Pic}_1 = \{L_{K(1)}(R_\lambda \wedge S^i) \,|\, \lambda \in \mathbb{Z}_p, 0 \le i < q\}.$$

Proof. (Sketch.) The proof of this theorem involves some rather grotesque manipulations with quotients of CW-skeleta of R_λ involving the p-adic expansion of $\lambda \in \mathbb{Z}_p$. We lay out a few of the ideas here, and direct the curious reader to [HMS94] for the full details.

Define

$$R^m_\alpha := ((B\Sigma_p)_{(p)})^m / ((B\Sigma_p)_{(p)})^{q\alpha+q-2},$$

where superscript n denotes taking the n-skeleton.

By [HMS94, 2.9], we can write R_λ as

$$R_\lambda = \varinjlim \left(R^q_{a_{-1}} \hookrightarrow \Sigma^{-a_0 q} R^{a_0 q+2q}_{a_0} \hookrightarrow \Sigma^{-a_1 q} R^{a_1 q+3q}_{a_1} \hookrightarrow \Sigma^{-a_1 q} R^{a_2 q+4q}_{a_2} \hookrightarrow \cdots \right),$$

where $\lambda \in \mathbb{Z}_p$ is written as $\lambda = \sum_{i=0}^{\infty} \lambda_i p^i$, with $0 \le \lambda_i \le p-1$ and $a_m = \sum_{i=0}^{m}$ is the m^{th} partial sum so that $\lambda = \lim_{m\to\infty} a_m$. Write $a_{-1} = 0$.

For $\lambda \in \mathbb{Z}$, this sequence is constant by [HMS94, 2.9], and we have

$$R_\lambda = \Sigma^{-\lambda q}[(B\Sigma_p)_{(p)}/(B\Sigma_p)^{\lambda q}_{(p)}],$$

and after $K(1)$-localization,

$$L_{K(1)}R_\lambda \simeq L_{K(1)}S^{-q\lambda}$$

This last relation can be checked by noting that

$$L_{K(1)}R_\lambda \in M,$$

and

$$ev(R_\lambda) = ev(S^{-q\lambda}).$$

(Recall that by Proposition 9.3.3, ev is injective.) Taking inverse limits (modulo p^i for all all i), it follows that for arbitrary $\lambda \in \mathbb{Z}_p$, we have

$$L_{K(1)}R_\lambda \simeq X_{g^{\lambda(1-p)}}.$$

For g a topological generator of $\mathbb{Z}_p^\times$, g^{1-p} is a topological generator of the summand $\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, and as a result, the collection of all $g^{\lambda(1-p)}$ for $\lambda \in \mathbb{Z}_p$ is precisely this $\mathbb{Z}_p$-summand, and the result follows. $\square$

Remark 9.3.10. $p = 2$ case. When $p = 2$, $\mathbb{Z}_2^\times$ is not topologically cyclic, and

$$\mathbb{Z}_2^\times \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z},$$

which differs from the odd prime case. Furthermore, localizations $L_{K(n)}S^{2m}$ of even-dimensional spheres have square roots $L_{K(n)}(Z \wedge Z) \simeq L_{K(n)}S^{2m}$ which are not topologically close to localizations of spheres. As a result, the analog of Theorem 9.3.9 does not hold when $p = 2$. We instead have

$$\mathrm{Pic}_1 \cong \mathbb{Z}_2^\times \oplus \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

In a similar fashion to the $p > 2$ case, [HMS94] constructs a $(\mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z})$-family of spectra in Pic_1 via quotients of skeleta of $\mathbb{RP}^\infty = B\Sigma_2$, though spheres and elements constructed in this way do not account for the whole of Pic_1, and examples of more "interesting" elements of Pic_1 are given in [HMS94, §5].

9.4 Summary of results at low heights

- $\underline{n = 1, p = 2}$:

  By [HMS94],
  $$\mathrm{Pic}_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$
  Following [GHMR14, 2.10],
  $$\mathrm{Pic}_1^{\mathrm{alg}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$
  and $\kappa_1 \cong \mathbb{Z}/2\mathbb{Z}$.

- $\underline{n = 1,\ p \geq 3}$:

  By Theorem 8.3.3 and Proposition 9.3.6, we have
  $$\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}} \cong \mathbb{Z}_p \oplus \mathbb{Z}/(2p-2)\mathbb{Z},$$
  and $\kappa_1 = 0$.

- $\underline{n = 2, p = 2}$

  By [BBG$^+$22, Theorem 12.29],
  $$\kappa_2 = (\mathbb{Z}/8\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})^3.$$

- $\underline{n = 2, p = 3}$:

  [GHMR14] compute

$$\mathrm{Pic}_2 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z},$$

$$\mathrm{Pic}_2^{\mathrm{alg}} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}/16\mathbb{Z},$$

  and $\kappa_2 = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

- $\underline{n = 2, p \geq 5}$:

  In this range,

$$\mathrm{Pic}_2 \cong \mathrm{Pic}_2^{\mathrm{alg}} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}/(2p^2 - 2)\mathbb{Z}.$$

  The computation of Pic_n is due to Hopkins (see [Lad13]). Alternatively, $\mathrm{Pic}_n^{\mathrm{alg}}$ is computed in [Kar10] and [GHMR14], and agrees with the result at $p = 3$. Equivalence of Pic_n and $\mathrm{Pic}_n^{\mathrm{alg}}$ follows from [Pst18], as we have

$$2p - 2 \geq 8 > n^2 + n = 6.$$

9.5 Some additional important examples

- Following [GHMR05] and [Wes17]: Let p be an odd prime. Recall that the Morava stablilizer group $\mathbb{G}_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma_n)$ acts on the spectrum E_n compatibly with its action on the homotopy groups $(E_n)_*$, where again, Γ_n denotes the Honda formal group law. Per Devinatz and Hopkins [DH04], we can define continuous homotopy fixed point spectra E_n^{hH} with respect to closed subgroups $H \leq \mathbb{G}_n$. For $H = \mathbb{G}_n$, we have $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$.

Recall from Chapter 5 that the non-Galois part $\mathrm{Aut}(\Gamma_n)$ of $\mathbb{G}_n$, sometimes referred to as the *small Morava stabilizer group* is the group of units $\mathbb{S}_n = \mathcal{O}_n^\times$ of the left $\mathbb{W}(\mathbb{F}_{p^n})$-module

$$\mathcal{O}_n = \mathbb{W}(\mathbb{F}_{p^n})\langle S\rangle/(S^n = p, S\omega = \omega^\sigma S).$$

This $\mathcal{O}_n$ is a rank n module over $\mathbb{W}(\mathbb{F}_{p^n})$. Thus, right multiplication gives a homomorphism

$$\mathbb{S}_n \to \mathrm{GL}_n(\mathbb{W}(\mathbb{F}_{p^n})),$$

and taking the determinant of the action defined by left multiplication gives a map

$$\det : \mathrm{Aut}(\Gamma_n) \to \mathbb{W}(\mathbb{F}_{p^n})^\times.$$

This image of det is known to lie in $\mathbb{Z}_p^\times$ (see [Rav76, 2.9]). We can extend this to a map

$$\det{}_\pm : \mathbb{G}_n \to \mathbb{Z}_p^\times$$

by sending the Frobenius map in $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ to $(-1)^{n-1}$. Let $S\mathbb{G}_n^\pm = \ker \det_\pm$. The group $\mathbb{Z}_p^\times = \mathbb{G}_n/S\mathbb{G}_n^\pm$ acts on the homotopy fixed point spectrum $E_n^{hS\mathbb{G}_n^\pm}$. Write the action as $\psi^k \in \mathrm{Aut}(E_n^{hS\mathbb{G}_n^\pm})$ for $k \in \mathbb{Z}_p^\times$ in analogy with the usual Adams operations.

As p is odd, $\mathbb{Z}_p^\times$ is topologically cyclic. Letting g be the topological generator, set for $\gamma \in \mathbb{Z}_p^\times$

$$F_\gamma := \mathrm{hofib}(\psi^g - \gamma : E_n^{hS\mathbb{G}_n^\pm} \to E_n^{hS\mathbb{G}_n^\pm}).$$

F_γ is an invertible $K(n)$-local spectrum and thus defines an element of Pic_n. Further, the association $\gamma \mapsto F_\gamma$ gives a homomorphism $\mathbb{Z}_p^\times \to \mathrm{Pic}_n$.

Example 9.5.1. When $\gamma = 1$, we get the spectrum

$$F_1 = \left(E_n^{hS\mathbb{G}_n^\pm}\right)^{h\mathbb{Z}_p^\times} \simeq E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0,$$

the $K(n)$-local sphere spectrum.

Example 9.5.2. When $\gamma = g$, the resulting spectrum F_g is known as the *determinantal sphere spectrum*, $S\langle\det\rangle$.

- For primes $p > 2$, Westerland [Wes17] constructs an invertible spectrum $Z \in \mathrm{Pic}_n$ as a summand of the $K(n)$-localization of the suspension spectrum

$$L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p, n),$$

where $K(\mathbb{Z}/p, n)$ denotes the Eilenberg-Mac Lane space having a single non-trivial homotopy group $\mathbb{Z}/p$ in degree n. We give an overview of this construction.

Let ζ be a primitive $(p-1)^{\text{st}}$ root of unity. The group $\mu_{p-1} = \langle\zeta\rangle$ of $(p-1)^{\text{st}}$ roots of unity is isomorphic to $\mathbb{F}_p^\times$ and thus acts on $\mathbb{Z}/p$ by multiplication. Delooping this action n times gives an action of μ_{p-1} on $K(\mathbb{Z}/p, n)$. Denote the action of ζ^k given in this way by ψ^{ζ^k}.

We can define another action of μ_{p-1} on $L_{K(n)}K(\mathbb{Z}/p\mathbb{Z}, n)$. $K(n)$-local spectra are p-complete, and any p-complete spectrum X admits an action of $\mathbb{Z}_p^\times$ such that $\alpha \in \mathbb{Z}_p^\times$ acts on π_*X by multiplication. Under the identification

$$\mu_{p-1} \subseteq \mathbb{Z}_p^\times = \mu_{p-1} \oplus (1 + p\mathbb{Z}_p),$$

this gives another action of μ_{p-1} on $L_{K(n)}K(\mathbb{Z}/p, n)$. Denote the action of ζ^k defined in this way by ζ^k.

One then can define an endomorphism e on $L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p, n)$ by

$$e := \frac{1}{p-1}\sum_{k=0}^{p-2} \zeta^{-k}\psi^{\zeta^k}.$$

This e acts as a homotopy idempotent, that is, $\pi_*(e^2) = \pi_*(e)$, and as a result, we get a splitting

$$L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p\mathbb{Z}, n) \simeq Z \vee Z'.$$

To see this, write $X := L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p\mathbb{Z}, n)$ and let

$$e^{-1}X = \mathrm{hocolim}(X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \longrightarrow \cdots).$$

Then, the natural map

$$X \longrightarrow e^{-1}X \vee (1-e)^{-1}X$$

is an equivalence. e acts as the identity on $e^{-1}X$ and as 0 on $(1-e)^{-1}X$. We take $Z := e^{-1}X$.

By [Wes17, 3.7], $\dim_{K(n)_*} K(n)_*Z = 1$, so that by Theorem 9.2.1, $Z \in \mathrm{Pic}_n$. This Z satisfies

$$L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}_p, n+1),$$

and we get a splitting

$$L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p\mathbb{Z}, n) \simeq \bigvee_{k=0}^{p-2} L_{K(n)}(Z^{\wedge k}).$$

Note: By Proposition 6.1.4(4), there is no need to re-localize the smash powers $Z^{\wedge k}$ in this splitting.

- A number of other interesting examples are listed in [BB19a, 6.16].

9.6 Construction of invertible spectra via determinantal K-theory

In [Wes17], using maps from the determinantal sphere $S\langle \det \rangle$, Westerland defines the notion of an n-oriented $K(n)$-local ring spectrum in analogy with the definition of a complex orientation given in Chapter 3, and constructs the universal multiplicative n-oriented spectrum, R_n. He then constructs invertible spectra as the homotopy fibers of endomorphisms of R_n, much in the same way as the first example of the previous section.

We present now the construction of these R_n and describe their use in constructing invertible spectra.

We continue to let $p > 2$. Earlier, we defined for $f : S^0 \to E$ the localization $f^{-1}E$. We now extend this notion to define localization with respect to maps $A \to X$ for $A \in \mathrm{Pic}_n$.

Definition 9.6.1. Let $A \in \mathrm{Pic}_n$ and $X \in L_{K(n)}\mathbf{Sp}$ be a ring spectrum. Then, for a map $f : A \to X$, define

$$f^{-1}X$$

as the $K(n)$-localization of the homotopy colimit of

$$X \xrightarrow{m_f} A^{-1} \otimes X \xrightarrow{1\otimes m_f} A^{-1} \otimes (A^{-1} \otimes X) \xrightarrow{1\otimes 1\otimes m_f} \cdots .$$

Here, $- \otimes - = L_{K(n)}(- \wedge -)$, and m_f is the map

$$X \simeq A^{-1} \otimes A \otimes X \xrightarrow{1\otimes f\otimes 1} A^{-1} \otimes X \otimes X \to A^{-1} \otimes X,$$

where the last map is given by the $K(n)$-localization of the multiplication $X \wedge X \to X$.

Let Z be the spectrum defined earlier as $e^{-1}L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p\mathbb{Z}, n)$ and let

$$i : Z \to L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p\mathbb{Z}, n)$$

be the inclusion of this summand. From the exact sequence

$$\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{\bmod p} \mathbb{Z}/p\mathbb{Z},$$

we get a Bockstein map $\mathbb{Z}/p\mathbb{Z} \to B\mathbb{Z}_p$, giving a map

$$K(\mathbb{Z}/p\mathbb{Z}, n) \to K(\mathbb{Z}_p, n+1).$$

Taking suspension spectra and $K(n)$-localizing results in a map

$$\beta : L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}/p\mathbb{Z}, n) \to L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}_p, n+1).$$

Define the *determinantal K-theory spectrum* R_n to be the localization

$$\alpha^{-1} L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}_p, n+1),$$

where $\alpha = \beta \circ i : Z \to L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}_p, n+1)$.

Remark 9.6.2. The "K-theory" part of the name *determinantal K-theory* comes from the analogy with equivalence

$$\beta^{-1} \Sigma_+^\infty K(\mathbb{Z}, 2) \to KU.$$

due to Snaith, where $\beta : S^2 \to K(\mathbb{Z}, 2)$ is the Bott class.

Westerland shows that, for $\gamma \in (\pi_0 R_n)^\times$, the homotopy fiber

$$F_\gamma := \mathrm{hofib}(\psi^g - \gamma)$$

belongs to Pic_n. Here, the maps ψ^g and γ are defined similarly to how ζ^k and ψ^{ζ^k} were defined in the case of $L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}/p\mathbb{Z}, n)$ earlier.

In particular, ψ^g is defined as localization of the natural action of the topological generator $g = (1+p)\zeta \in \mathbb{Z}_p^\times$ (for ζ a primitive $(p-1)^{\mathrm{st}}$ root of unity) on $L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}_p, n+1)$ via its action on $\mathbb{Z}_p$ by multiplication, and $\gamma : R_n \to R_n$ is the composite

$$R_n \simeq S^0 \wedge R_n \to R_n \wedge R_n \to R_n,$$

with the last map being the multiplication map for the ring spectrum R_n. When $\gamma = 1$ (i.e., $\gamma = \eta : S^0 \to R_n$), we get the $K(n)$-local sphere spectrum

$$F_\gamma = L_{K(n)} S^0.$$

The determinantal sphere spectrum $S\langle \det \rangle$ can be produced in this manner:

$$S\langle \det \rangle = F_g.$$

Theorem 9.6.3. [Wes17, 3.17] *The association $\gamma \mapsto F_\gamma$ determines a group homomorphism*

$$(\pi_0 R_n)^\times \to \mathrm{Pic}_n.$$

The similarity between these results and the results given in the first example in the previous section hint at a relationship between R_n and the homotopy fixed point spectrum $E_n^{hS\mathbb{G}_n^\pm}$, and it can be shown (see [Wes17, 3.25]) that there is a weak equivalence of E_∞-spectra

$$R_n \to E_n^{hS\mathbb{G}_n^\pm}$$

which is $\mathbb{Z}_p^\times$-equivariant.

Remark 9.6.4. As $S\mathbb{G}_n^\pm = \ker(\det_\pm : \mathbb{G}_n \to \mathbb{Z}_p^\times)$, this gives a reason for the "determinantal" part of "determinantal K-theory." Additionally, we can present R_n as a localization

$$\rho_n^{-1} L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}_p, n+1),$$

where $\rho_n : S\langle \det \rangle \to L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}_p, n+1)$ is a map from the determinantal sphere spectrum.

Chapter 10

A descent spectral sequence computing Pic_n and applications

In this chapter, we construct a family of descent spectral sequences which for $t \geq 1$ take the form

$$H^s(\mathbb{G}_n/U, \pi_t\,\mathcal{P}\mathrm{ic}(E_n^{hU})) \implies \pi_{t-s}\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}),$$

computing the homotopy groups of the Picard space $\mathcal{P}\mathrm{ic}_n := \mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp})$. In particular, taking $t - s = 0$, these spectral sequences compute the $K(n)$-local Picard group

$$\mathrm{Pic}_n = \pi_0\mathcal{P}\mathrm{ic}_n.$$

Further, making use of Davis' discrete $\mathbb{G}_n$-spectra $F_n := \operatorname*{hocolim}_{\substack{U \trianglelefteq \mathbb{G}_n \\ \text{open}}} E_n^{hU}$ (see [Dav06]), we investigate the colimit of these spectral sequences, which we then use to compute the rational homotopy groups of $\mathcal{P}\mathrm{ic}_n$.

Throughout, we'll fix a prime p and will let $U \trianglelefteq \mathbb{G}_n$ range over open (and therefore finite-index) normal subgroups of the Morava stabilizer group $\mathbb{G}_n$.

10.1 Additional Background

The spectral sequences we construct which compute $\pi_*\mathcal{P}\mathrm{ic}_n$ take the form of Bousfield-Kan Spectral Sequences, and in order to make computations using them, we will make heavy use of the Lyndon-Hochschild-Serre Spectral Sequence. We now provide a review of both.

We begin by introducing the idea of the totalization of a cosimplicial object, which is central to the construction of the Bousfield-Kan Spectral Sequence. For the original treatment of both the totalization functor and the Boufield-Kan Spectral Sequence, see [BK72, Part II]. See also [Bou03] and [McC01, 8^{bis}].

10.1.1 Totalization of a cosimplicial object

Let $\mathbf{\Delta}$ denote the finite ordinal category whose objects are finite totally ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ and morphisms are order-preserving maps.

Definition 10.1.1. Given a category $\mathcal{C}$, a *cosimpicial object in* $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$.

In order to define the totalization of cosimplicial object, we will need to utilize the cosimplicial space $\Delta^\bullet \in \mathrm{Fun}(\Delta, \mathrm{Top})$ of standard n-simplices $\Delta^n \in \mathrm{Top}$ for $n \geq 1$.

Totalization of a cosimplicial object should be though of as dual to the construction of the geometric realization of a simplicial object, and will be a key ingredient in the Bousfield-Kan Spectral Sequence used in Section 10.2.

Definition 10.1.2. Let $\mathcal{C}$ be pointed bicomplete (that is, all small limits and all small colimits exits) simplicial model category. Give the functor category $\mathrm{Fun}(\mathbf{\Delta}^\bullet, \mathcal{C})$ the Reedy model structure and let $X^\bullet$ be fibrant. Let

$$\mathrm{Tot}_s X^\bullet = \mathrm{Hom}(\mathrm{sk}_s \mathbf{\Delta}^\bullet, X^\bullet) \in \mathcal{C},$$

where $\mathrm{sk}_s\mathbf{\Delta}^\bullet$ is the cosimplicial space which termwise is the s-skeleton of $\mathbf{\Delta}^\bullet$. Then, there is a tower of fibrations

$$\cdots \to \mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet) \to \cdots \to \mathrm{Tot}_0(X^\bullet) \cong X^0,$$

with the *totalization* of $X^\bullet$ being the limit

$$\mathrm{Tot}(X^\bullet) \cong \varprojlim_n \mathrm{Tot}_n(X^\bullet).$$

10.1.2 Some useful spectral sequences

Such a tower of fibrations leads to an exact couple based on the long exact sequences of homotopy groups, and we therefore get a spectral sequence whose E_1-term is the homotopy groups of the fibers $\mathrm{Fib}_s X^\bullet = \mathrm{Fiber}(\mathrm{Tot}_s(X^\bullet) \to \mathrm{Tot}_{s-1}(X^\bullet))$:

Theorem 10.1.3 (Bousfield-Kan Spectral Sequence)**.** [BK72]

Given a fibrant, pointed, cosimplicial space $X^\bullet$, there is a spectral sequence associated to the tower of fibrations $\{\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)\}$ with

$$\begin{aligned} E_1^{s,t}(X^\bullet) &\cong \pi_{t-s}(\mathrm{Fib}_s X^\bullet), \qquad t \geq s \geq 0 \quad \textit{and differential} \\ d_r : &E_r^{s,t} \to E_r^{s+r,t+r-1}, \end{aligned}$$

converging under favorable conditions to $\pi_*(\mathrm{Tot}(X^\bullet))$.

Under ideal conditions, the Bousfield-Kan Spectral Sequence admits a multiplicative structure:

Theorem 10.1.4. [BK73] *When $X^\bullet$ and $Y^\bullet$ are cosimplicial simplicial pointed sets, there is a multiplicative pairing*

$$E_r^{s,t}(X^\bullet) \otimes E_r^{s',t'}(Y^\bullet) \to E_r^{s+s',t+t'}(X^\bullet \otimes Y^\bullet).$$

Theorem 10.1.5 (Lyndon-Hochschild-Serre Spectral Sequence)**.** *Let G be a group with $N \trianglelefteq G$ and let M be a G-module. Then, there is a spectral sequence*

$$H^s(G/N, H^t(N, M)) \implies H^{s+t}(G, M).$$

If G is a profinite group and $N \trianglelefteq G$ is closed, then we further have a spectral sequence

$$H^s_{\text{cont}}(G/N, H^t_{\text{cont}}(N, M)) \implies H^{s+t}_{\text{cont}}(G, M),$$

where H^*_{cont} denotes continuous cohomology.

10.1.3 Galois extensions of commutative ring spectra

Definition 10.1.6. [Rog08, 4.1.3] Let K be a spectrum, $A \to B$ be a map of K-local commutative ring spectra and G be a finite discrete group acting continuously on B on the left through A-algebra maps, such that the canonical maps

$$A \to B^{hG} = \text{Map}(EG_+, B)^G$$

and

$$B \otimes_A B \to \text{Map}(G_+, B)$$

(formed in the K-local category) are weak equivalences. We will then say that $A \to B$ is a *K-local G-Galois extension.* Further, we will call the extension *faithful* if for N an A-module, $N \wedge_A B \simeq * \implies N \simeq *$

10.2 Construction of the spectral sequence

10.2.1 Descendability

For a traditional finite Galois extension of fields L/K with Galois group $G = \text{Gal}(L/K)$, we have *Galois descent.* That is, the functor

$$\text{Vect}_K \xrightarrow{-\otimes_K L} \text{Vect}_{L,G}$$

from vector spaces over K to vector spaces over L with semilinear G-action is an equivalence of categories, with weak inverse functor

$$\mathrm{Vect}_{L,G} \xrightarrow{(-)^G} \mathrm{Vect}_K,$$

where $(-)^G$ denote the G-invariants. In order to produce the promised family of spectral sequences computing $\pi_*\mathcal{P}\mathrm{ic}_n$, we will need to extend this notion of Galois descent to the broader context of Galois extensions of commutative ring spectra. A good reference is [Mat16].

Definition 10.2.1. [Mat16, 2.1] A *stable homotopy theory* is a presentable, symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ (in the sense of [Lur17, Ch. 1]), where $\otimes$ commutes with all colimits.

Remark 10.2.2. Under this definition, the category $\mathbf{Sp}$ of spectra as well as the Bousfield localizations $L_E\,\mathbf{Sp}$ are stable homotopy theories (see [Mat16, Ex. 2.28]).

Remark 10.2.3. Following [Lur17, Ch. 3], given a symmetric monoidal ∞-category $\mathcal{C}$, there is a natural ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects. In the case $\mathcal{C} = \mathbf{Sp}$, this is the category of E_∞ ring spectra (see [Mat16, Def. 2.18]). Further, following [Lur17, Ch. 4], given an object $A \in \mathrm{CAlg}(\mathcal{C})$, there is a natural ∞-category $\mathrm{Mod}_{\mathcal{C}}(A)$ of A-modules in $\mathcal{C}$. $\mathrm{Mod}_{\mathcal{C}}(A)$ is itself a stable homotopy theory (see [Mat16, Def. 2.19]).

Remark 10.2.4. It is a consequence of the Goerss-Hopkins-Miller Theorem [GH05] that $E_n \in \mathrm{CAlg}(\mathbf{Sp})$ (and therefore $E_n \in \mathrm{CAlg}(L_{K(n)}\mathbf{Sp})$ as well). Further, by construction (see [DH04, Def. 1.5]), for $U \subseteq \mathbb{G}_n$ closed, $E_n^{hU} \in \mathrm{CAlg}(\mathbf{Sp})$ (and therefore $E_n^{hU} \in \mathrm{CAlg}(L_{K(n)}\mathbf{Sp})$).

Definition 10.2.5. [Mat16, 3.19] Given $A \in \mathrm{CAlg}(\mathcal{C})$, we say that A *admits descent* or is *descendable* if the thick $\otimes$-ideal generated by A is all of $\mathcal{C}$.

Proposition 10.2.6. [Mat16, 3.22] *Let $\mathcal{C}$ be a stable homotopy theory. Let $A \in \mathrm{CAlg}(\mathcal{C})$ admit descent. Then the adjunction*

$$\mathcal{C} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A)$$

given by tensoring with A and forgetting, is comonadic. In particular, the natural functor

$$\mathcal{C} \to \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(A) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \overset{\to}{\underset{\to}{\to}} \cdots\right)$$

is an equivalence.

Descent for E_n^{hU}

Theorem 10.2.7. *Let $U \subseteq \mathbb{G}_n$ be an open normal subgroup. Then E_n^{hU} admits descent in $L_{K(n)}\mathbf{Sp}$.*

Proof. By [Mat16, 4.18], E_n admits descent over $L_n S^0$. By [Mat16, 3.21], we can further localize to find that $L_{K(n)}E_n \simeq E_n$ is descendable over $L_{K(n)}S^0$. Finally, by [Mat16, 3.24], since the composite

$$L_{K(n)}S^0 \to E_n^{hU} \to E_n$$

admits descent, so does

$$L_{K(n)}S^0 \to E_n^{hU}.$$

□

Corollary 10.2.8. The adjunction

$$L_{K(n)}\mathbf{Sp} \rightleftarrows \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})$$

is comonadic and therefore there is an equivalence of categories

$$L_{K(n)}\mathbf{Sp} \simeq \mathrm{Tot}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}) \rightrightarrows \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU} \otimes E_n^{hU}) \overset{\to}{\underset{\to}{\to}} \cdots).$$

Proof. [Mat16, 3.22] □

Proposition 10.2.9. [MS16, 2.2.3] $\mathcal{P}\mathrm{ic}$ *commutes with limits and filtered colimits. In particular,* $\mathcal{P}\mathrm{ic}$ *commutes with the totalization functor* Tot*.*

Corollary 10.2.10. There is an equivalence

$$\mathcal{P}\mathrm{ic}_n = \mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}) \to \mathrm{Tot}\left(\mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{hU}\right)\right) \rightrightarrows \cdots\right).$$

In the case of a "global" (that is, S^0-local) finite Galois extension, we have the following result:

Proposition 10.2.11. [MS16, 3.3.1] *Let G be a finite group and $A \to B$ be a faithful G-Galois extension of E_∞-rings. Then there is a natural equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Mod}\,A \xrightarrow{\sim} (\mathrm{Mod}\,B)^{hG}.$$

We can extend this to our context:

Proposition 10.2.12. *Let $U \subseteq \mathbb{G}_n$ be an open normal subgroup. Write $A = L_{K(n)}S^0$ and $B = E_n^{hU}$ and $G = \mathbb{G}_n/U$. There is an equivalence*

$$\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\,A \xrightarrow{\sim} \left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\,B\right)^{hG},$$

where $\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\,A$ denotes the category of $K(n)$-local A-modules and similarly for $\mathrm{Mod}_{L_{K(n)}}\,B$.

Applying $\mathcal{P}\mathrm{ic}$*, we find*

$$\mathcal{P}\mathrm{ic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\,A) \to \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\,B)^{hG}.$$

Proof. By Theorem 10.2.7, B admits descent in $\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}} A$. So, by [Mat16, 3.22] there's an equivalence

$$\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}} A \xrightarrow{\sim} \mathrm{Tot}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(\otimes_A^{\bullet+1} B\right)\right),$$

where $- \otimes_A - = L_{K(n)}(- \wedge_A -)$. By [Rog08, 5.44, 5.49(b)], $L_{K(n)}S^0 \to E_n^{hU}$ is a finite faithful $K(n)$-local $\mathbb{G}_n/U$-Galois extension, so we have an equivalence

$$B \otimes_A B \simeq \mathrm{Map}(G_+, B),$$

where $\mathrm{Map}(-,-)$ denotes the mapping spectrum constructed in the $K(n)$-local category, so that we level-wise have $\otimes_A^{s+1} B \simeq \mathrm{Map}(G_+^{\times s}, B)$. This means that

$$\begin{aligned}\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}} A &\simeq \mathrm{Tot}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(\mathrm{Map}(G_+^{\times\bullet}, B)\right)\right)\\ &\simeq \mathrm{Tot}\left(\mathrm{Map}(G_+^{\times\bullet}, \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}} B)\right)\\ &= \left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}} B\right)^{hG}.\end{aligned}$$

□

Theorem 10.2.13. *For a fixed open normal subgroup $U \subseteq \mathbb{G}_n$, we have a spectral sequence*

$$E_{2,U}^{s,t} = H^s\left(\mathbb{G}_n/U; \pi_t \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{hU}\right))\right) \implies \pi_{t-s}\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}).$$

Proof. By Corollary 10.2.10, we have an equivalence

$$\mathcal{P}\mathrm{ic}_n = \mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}) \xrightarrow{\sim} \mathrm{Tot}\left(\mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{hU}\right)\right) \rightrightarrows \cdots\right).$$

By Proposition 10.2.12,

$$\begin{aligned}\left(\mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}E_n^{hU}\right)\right)^{h\mathbb{G}_n/U} &\cong \mathcal{P}\mathrm{ic}\left(\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{hU}\right)\right)^{h\mathbb{G}_n/U}\right)\\ &\cong \mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(\left(E_n^{hU}\right)^{h\mathbb{G}_n/U}\right)\right)\\ &\cong \mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{h\mathbb{G}_n}\right)\right)\\ &\cong \mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(L_{K(n)}S^0\right)\right)\\ &= \mathcal{P}\mathrm{ic}_n\end{aligned}$$

Finally, the claimed spectral sequence is the associated Homotopy Fixed Point Spectral Sequence (Bousfield-Kan Spectral Sequence) associated to the equivalence

$$\mathcal{P}\mathrm{ic}_n \simeq \left(\mathcal{P}\mathrm{ic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}E_n^{hU}\right)\right)^{h\mathbb{G}_n/U}.$$

□

10.2.2 Direct limits of spectral sequences

Theorem 10.2.13 establishes a family of spectral sequences indexed by open normal subgroups $U \trianglelefteq \mathbb{G}_n$ all with the same target. It is natural to ask whether we can then take a direct limit of these spectral sequences to get a new one also converging to $\pi_*\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp})$.

Given $U \subseteq V \subseteq \mathbb{G}_n$, we have a resulting quotient map $\varphi_{U,V} : \mathbb{G}_n/U \twoheadrightarrow \mathbb{G}_n/V$. Further, on homotopy fixed points, inclusion $U \hookrightarrow V$ induces a map

$$E_n^{hV} \to E_n^{hU},$$

and we therefore have an induction map

$$\mathrm{Ind}_V^U : \mathrm{Mod}_{E_n^{hV}}(L_{K(n)}\mathbf{Sp}) \to \mathrm{Mod}_{E_n^{hU}}(L_{K(n)}\mathbf{Sp})$$

given by

$$X \mapsto X \otimes_{E_n^{hV}} E_n^{hU},$$

so that we have a composite

$$H^*(\mathbb{G}_n/V, M_V) \overset{\varphi_{U,V}^*}{\to} H^*(\mathbb{G}_n/U, M_V) \overset{(\mathrm{Ind}_V^U)_*}{\to} H^*(\mathbb{G}_n/U, M_U).$$

In our case, we are considering

$$\mathrm{Ind} : M_V = \pi_* \mathrm{Pic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hV})) \to M_U = \pi_* \mathrm{Pic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})).$$

Note that in general, the direct limit of spectral sequences is indeed a spectral sequence, as filtered direct limits preserve exactness, so we can take the direct limit of our spectral sequences along these $(\mathrm{Ind}_V^U)_* \circ \varphi_{U,V}^*$ to get a new spectral sequence with E_2-term

$$E_2^{s,t} = \varinjlim_U E_{2,U}^{s,t} = \varinjlim_U H^s(\mathbb{G}_n/U, \pi_t \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}))).$$

Denote the abutment of this spectral sequence as P_{t-s}. This abutment need not be the direct limit of the abutments of the spectral sequences with E_2-terms $E_{2,U}^{s,t}$. Two possible issues may arise, quoted here from [Les95, §4]:

1. *Non-detection: It could be that an element of the direct limit of the abutments moves into higher and higher filtration as it moves through the direct limit of spectral sequences, in which case it would not be detected in the direct limit spectral sequence.*

2. *Fake cycles: If an element supports longer and longer differentials as it goes through the direct limit, it will be an infinite cycle in the direct limit spectral sequence even though it does not represent a class in the abutment at any finite stage.*

Showing that these hold is in general difficult, and is for the moment an open question in our case. (See [Mit97, Section 3.1.3] for a more thorough discussion.)

However, rationally, we can do better: By [Ser79, Prop. 8, Cor. 3],

$$E_2^{s,t} \otimes \mathbb{Q} \cong 0$$

for $s \geq 1$. So, by [Mit97, Proposition 3.3],

$$E_2^{s,t} \otimes \mathbb{Q} \implies \pi_{t-s}\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}) \otimes \mathbb{Q}.$$

We would now like to describe $E_2^{s,t}$ and P_{t-s} more conveniently.

Proposition 10.2.14. [Ser79, Proposition 8] *Let (G_i) be a projective system of profinite groups, and let (A_i) be an inductive system of discrete G_i-modules. Then one has*

$$H^q_{\mathrm{cont}}(\varprojlim G_i, \varinjlim A_i) = \varinjlim H^q_{\mathrm{cont}}(G_i, A_i)$$

for each $q \geq 0$.

Theorem 10.2.15. *There is a spectral sequence*

$$H^s_{\mathrm{cont}}\left(\mathbb{G}_n; \left(\pi_t\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^\delta\right) \implies P_{t-s},$$

where δ means to take the discrete topology. Furthermore,

$$P_{t-s} \cong \pi_{t-s}\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)^{h\mathbb{G}_n}$$

and

$$P_{t-s} \otimes \mathbb{Q} \cong \pi_{t-s}\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}) \otimes \mathbb{Q}.$$

Proof. By Proposition 10.2.14, we can identify

$$\varinjlim E_{2,U}^{s,t} = H^s_{\mathrm{cont}}\left(\mathbb{G}_n; \left(\pi_t\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^\delta\right).$$

We earlier defined P_{t-s} to be the abutment of the colimit spectral sequence with E_2-term being $\varinjlim E_{2,U}^{s,t}$. To see that the abutment P_{t-s} is identified as claimed, note that for a fixed V, there is a map of Homotopy Fixed Point Spectral Sequences which on E_2-terms is

$$
\begin{aligned}
E_{2,V}^{s,t} =& H^s_{\mathrm{cont}}\left(\mathbb{G}_n/V; \pi_t\mathcal{P}\mathrm{ic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{hU}\right)\right)\right) \\
&\longrightarrow H^s_{\mathrm{cont}}\left(\mathbb{G}_n; \left(\pi_t\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^{\delta}\right),
\end{aligned}
$$

the latter being the Homotopy Fixed Point Spectral Sequence for the discrete $\mathbb{G}_n$-space

$$\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}),$$

converging to

$$\pi_{t-s}\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)^{h\mathbb{G}_n}.$$

This map is induced by the natural maps

$$\mathbb{G}_n \to \mathbb{G}_n/V$$

and

$$\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hV}) \to \varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}).$$

As a result, in the limit, we have a map of spectral sequences which is an isomorphism on E_2-terms, so that we have an isomorphism of abutments

$$P_{t-s} \cong \pi_{t-s}\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)^{h\mathbb{G}_n}.$$

Finally, the identification of $P_{t-s}\otimes\mathbb{Q}$ follows from the above discussion on the rationalized spectral sequence

$$E_2^{s,t}\otimes\mathbb{Q} \implies \pi_{t-s}\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp})\otimes\mathbb{Q}.$$

□

When $t-s=0$, the right hand side is $\pi_0\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp})\otimes\mathbb{Q} = \mathrm{Pic}_n\otimes\mathbb{Q}$. For the rest of the chapter, our goal will be to use this spectral sequence to examine the torsion-free rank of Pic_n. As $\mathcal{P}\mathrm{ic}$ takes values in spaces, we'll be interested in the case $s = t \geq 0$. Further, for $t \geq 1$, we can identify

$$\pi_t\,\mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right) = \pi_{t-1}\mathrm{GL}_1(\varinjlim_U E_n^{hU}) = \pi_{t-1}\mathrm{GL}_1(F_n),$$

as by [ABG+13] §2.4 Proposition 2.9, $\mathrm{GL}_1(\mathcal{C}, \otimes, \mathbb{1}) = \mathrm{GL}_1(\mathbb{1})$ for a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$. Finally, we have

$$\pi_*\mathrm{GL}_1(F_n) = \begin{cases} (\pi_0 F_n)^\times; & * = 0 \\ \pi_* F_n; & * \geq 1, \end{cases}$$

so that our inquiry now turns to computing the torsion-free part of $\pi_* F_n$.

10.3 Computation of $\pi_* F_n \otimes \mathbb{Q}$

10.3.1 Higher homotopy is torsion

Proposition 10.3.1. *$\pi_* F_n$ is torsion for $* > 0$.*

Proof. Recall that $\mathbb{G}_n \cong \mathcal{O}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_n)$ where $\mathcal{O} = \mathbb{W}(\mathbb{F}_{p^n})\langle S\rangle/(S^n = p, Sw = w^\sigma S)$, where $\mathbb{W}(\mathbb{F}_{p^n})$ is the ring of Witt vectors over $\mathbb{F}_{p^n}$ and σ is the Frobenius. Fix an open normal subgroup $U \subseteq \mathbb{G}_n$. Then, since the collection of subgroups $\{(1+p^i\mathbb{Z}_p)^\times\}_{i\geq 2}$ form a fundamental system of neighborhoods of $1 \in (1+p^2\mathbb{Z}_p)^\times \subseteq Z(\mathbb{G}_n)$, there exists some i such that $Z_i = (1+p^i\mathbb{Z}_p)^\times \subseteq Z(\mathbb{G}_n) \cap U$. (Here, we use the notation $(-)^\times$ to indicate that the group structure is multiplication.) Since this Z_i is then normal in U, we have a Lyndon-Hochschild-Serre Spectral Sequence:

$$H^s_{\mathrm{cont}}(U/Z_i; H^t_{\mathrm{cont}}(Z_i; \pi_r E)) \implies H^{s+t}_{\mathrm{cont}}(U; \pi_r E)$$

for a fixed r.

This, in turn, is the E_2-page of the Homotopy Fixed Point Spectral Sequence

$$H^{s+t}_{\text{cont}}(U; \pi_r E) \implies \pi_{r-s-t} E_n^{hU}.$$

Next, we find that the cohomology group $H^t_{\text{cont}}(Z_i; \pi_r E)$ is torsion if $r \neq 0$:

The cohomological dimension of Z_i is 1, so that the only potentially non-zero groups are

$$H^0_{\text{cont}}(Z_i; \pi_r E) \text{ and } H^1_{\text{cont}}(Z_i; \pi_r E)$$

By [Hea15] [1.3.1], for $g \in Z_i$, $g_* u_j = u_j$ and $g_* u^\ell = g^\ell \cdot u^\ell$ and the action is $\mathbb{W}(\mathbb{F}_{p^n})$-linear, so that

$$H^0_{\text{cont}}(Z_i; \pi_* E) = (\pi_* E)^{Z_i} = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]] = E_0.$$

Next,

$$H^1_{\text{cont}}(Z_i; \pi_* E) = (\pi_* E)_{Z_i} = \frac{\pi_* E}{\gamma x = x},$$

where $\gamma = 1 + p^i$ is a topological generator of $\mathbb{Z}_i$. We have $\gamma|_{E_0} = \text{id}$ and in non-zero even degrees, we have the relation $\gamma^m \cdot u^m = u^m$, and since u^m is not a zero divisor, this gives $\gamma^m = 1$. In summary,

$$H^1_{\text{cont}}(Z_i; \pi_* E) = (\pi_* E)_{Z_i} = \begin{cases} E_0 & \text{if } * = 0 \\ E_{2m}/((1+p^i)^m - 1) & \text{if } * = 2m > 0 \\ 0 & \text{else.} \end{cases}$$

So, for $* > 0$, we conclude that $H^t_{\text{cont}}(Z_i; \pi_r E)$ is indeed torsion. Finally, it is of note that $\varepsilon_{m,i} := \text{val}_p((1+p^i)^m - 1) \geq i$, where val_p denotes p-adic valuation. (We'll leverage this fact in §10.5 to compute $\pi_1 F_1$.)

Without loss of generality, assume that $\text{cd}(U/Z_i) < \infty$ (if not, replace U with a finite index open subgroup of U so that $\text{cd}(U) = \text{vcd}(U)$, which can be chosen to be normal in $\mathbb{G}_n$ by [Wil98, Lemma 0.3.2]). Then for a fixed $r \neq 0$, $H^s_{\text{cont}}(U/Z_i; H^t(Z_i; \pi_r E))$ is a finite sum of torsion modules and is therefore torsion, and as a result, so is $H^{s+t}_{\text{cont}}(U; \pi_r E)$.

Next, via the Homotopy Fixed Point Spectral Sequence, we see that the associated graded of $\pi_d E^{hU}$ is a subquotient of

$$\bigoplus_{d=r-s-t} H^{s+t}_{\text{cont}}(U; \pi_r E).$$

So, if $d = r - s - t > 0$, then $r > 0$ and $\pi_d E^{hU}$ is torsion, since $- \otimes \mathbb{Q}$ and $\varinjlim$ commute,

$$\pi_d(F_n) = \pi_d(\underrightarrow{\text{hocolim}}\, E^{hU_i}) = \varinjlim \pi_d E^{hU}$$

is also torsion. □

10.3.2 Computation of $\pi_0 F_n \otimes \mathbb{Q}$

While the group $\pi_0 F_n$ does not appear as a coefficient group for any value of t in Theorem 10.2.15 (only its group of units does), we include a computation of its rationalization for completeness, and an application of this computation is presented in Section 11.2.

In order to understand $\pi_0 F_n = \varinjlim \pi_0 E^{hU}$, we would like to first understand the fixed points $(\pi_0 E_n)^U$ for $U \subseteq \mathbb{S}_n$. The action of $\mathbb{S}_n$ on $\pi_0 E_n$ is provided by Devinatz and Hopkins in [DH95]. However, it is in a bit of a roundabout way:

They construct first a graded $\mathbb{W}(\mathbb{F}_{p^n})$-algebra $\mathbb{W}(\mathbb{F}_{p^n})[[w_1, \ldots, w_{n-1}]][w, w^{-1}]$ (with $|w_i| = 0$ and $|w| = -2$) on which $\mathbb{S}_n$ acts and then show that there is an injective $\mathbb{S}_n$-equivariant graded $\mathbb{W}(\mathbb{F}_{p^n})$-algebra homomorphism

$$i : \pi_* E_n \to \mathbb{W}(\mathbb{F}_{p^n})\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}],$$

where $\mathbb{W}(\mathbb{F}_{p^n})\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$ denotes the divided power envelope of the $\mathbb{W}(\mathbb{F}_{p^n})$-algebra $\mathbb{W}(\mathbb{F}_{p^n})[[w_1, \ldots, w_{n-1}]]$. Finally, they describe the action of $\mathbb{S}_n$ on

$$\mathbb{W}(\mathbb{F}_{p^n})[[w_1, \ldots, w_{n-1}]][w, w^{-1}],$$

noting that it extends to an action on $\mathbb{W}(\mathbb{F}_{p^n})\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}]$, the restriction which to $\mathbb{W}(\mathbb{F}_{p^n})\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$ is by P.D. $\mathbb{W}(\mathbb{F}_{p^n})$-algebra maps.

Letting

$$i_0 : \pi_0 E_n \to \mathbb{W}(\mathbb{F}_{p^n})\langle\langle w_1, \dots, w_{n-1}\rangle\rangle$$

denote the restriction $i|_{\pi_0 E_n}$, we then conclude that for $U \subseteq \mathbb{S}_n$,

$$(\pi_0 E_n)^U = i_0^{-1}\left((i_0(\pi_0 E_n))^U\right)$$

Further, $\mathbb{G}_n$ acts trivially on $\mathbb{W}(\mathbb{F}_{p^n}) \subseteq \pi_0 E_n$, so that

$$\mathbb{W}(\mathbb{F}_{p^n}) \subseteq (\pi_0 E_n)^U = i_0^{-1}\left((i_0(\pi_0 E_n))^U\right).$$

We'll show that this containment is in fact equality.

Theorem 10.3.2. [DH95, 3.3] *The right action of* $\mathbb{S}_n$ *on* $\mathbb{W}(\mathbb{F}_{p^n})[[w_1, \dots, w_{n-1}]][w, w^{-1}]$ *is* $\mathbb{W}(\mathbb{F}_{p^n})$*-linear on the coordinates* $ww_1, \dots, ww_{n-1}, w$ *and for*

$$g = a_0 + a_1 S + \cdots + a_{n-1}S^{n-1} \in U \subseteq \mathbb{S}_n,$$

we have

$$g(w) = a_0 w + \sum_{j=1}^{n-1} a_{n-j}^{\sigma^j} ww_j$$

and

$$g(ww_i) = pa_i w + pa_{i+1}^{\sigma^{n-1}} ww_{n-1} + \cdots + pa_{n-1}^{\sigma^{i+1}} ww_{i+1} + a_0^{\sigma^i} ww_i + \cdots + a_{i-1}^{\sigma} ww_1,$$

where σ *denotes the Frobenius.*

Proposition 10.3.3. *Let*

$$g = a_0 + a_1 S + \cdots + a_{n-1}S^{n-1} \in U \subseteq \mathbb{S}_n$$

and

$$X = \sum \frac{x_{i_1,\dots,i_{n-1}}}{(i_1 + \cdots + i_{n-1})!} w_1^{i_1} \cdots w_{n-1}^{i_{n-1}} \in i_0(\pi_0 E_n),$$

where $x_{i_1,\dots,i_{n-1}} \in \mathbb{W}(\mathbb{F}_{p^n})$*. Then, if* $g(X) = X$*, we have*

$$\sum_{(i_1,\dots,i_{n-1}) \neq (0,\dots,0)} \left(\frac{p}{a_0}\right)^{i_1+\cdots+i_{n-1}} \frac{x_{i_1,\dots,i_{n-1}}}{(i_1 + \cdots + i_{n-1})!} a_1^{i_1} \cdots a_{n-1}^{i_{n-1}} = 0.$$

Proof. We first prove this at heights 1, 2, and 3 as it is illuminating to see these smaller cases worked out in detail and then include the general case at arbitrary height n.

We want to figure out for $g \in \mathbb{G}_n$ which elements of $\pi_0 E_n$ are fixed by g. It suffices to consider $g \in U \subseteq \mathbb{S}_n$.

Height 1:

Let $g = a_0$, with $a_0 \in \mathbb{W}(\mathbb{F}_p)^\times = \mathbb{Z}_p^\times$. Then, a generic element of $\pi_0 E_1$ is of the form x for $x \in \mathbb{W}(\mathbb{F}_p)$, and since $\mathbb{S}_1$ acts by $\mathbb{W}(\mathbb{F}_p)$-algebra homomorphisms, $g(x) = x \cdot g(1) = x$ so that x is fixed and g fixes all of $\pi_0 E_1$. So,

$$(\pi_0 E_1)^U = \pi_0 E_1$$

for all $U \subseteq \mathbb{S}_1 = \mathbb{Z}_p^\times$.

Height 2:

We do this via the action of $\mathbb{S}_n$ on $\mathbb{W}(\mathbb{F}_{p^n})\langle\langle w_1, \ldots, w_n \rangle\rangle[w, w^{-1}]$ through $\mathbb{W}(\mathbb{F}_{p^n})$-algebra maps as described in [DH95].

We want

$$g(x_0 + x_1 w_1 + \frac{x_2}{2!} w_1^2 + \cdots) = x_0 + x_1 w_1 + \frac{x_2}{2!} w_1^2 + \cdots$$

for $g = a_0 + a_1 S$, where a_0 is invertible and $x_i \in \mathbb{W}(\mathbb{F}_{p^2})$. Expansion gives

$$g(x_0 + x_1 w_1 + \cdots) = g(x_0) + g(x_1 w_1 + \cdots) = x_0 + g(x_1 w_1 + \cdots),$$

so that we are left with

$$x_1 g(w_1) + \frac{x_2}{2!} g(w_1)^2 + \cdots = x_1 w_1 + \frac{x_2}{2!} w_1^2 + \cdots .$$

Here,

$$\begin{aligned} g(ww_1) &= pa_1 w + a_0^\sigma ww_1 \\ g(w) &= a_0 w + a_1^\sigma ww_1, \end{aligned}$$

so that

$$\begin{aligned} g(w_1) &= \frac{pa_1 w + a_0^\sigma ww_1}{a_0 w + a_1^\sigma ww_1} \\ &= \frac{p\frac{a_1}{a_0} + \frac{a_0^\sigma}{a_0} w_1}{1 + \frac{a_1^\sigma}{a_0} w_1} \\ &= \left(p\frac{a_1}{a_0} + \frac{a_0^\sigma}{a_0} w_1\right)\left(1 - \left(\frac{a_1^\sigma}{a_0} w_1\right) + \left(\frac{a_1^\sigma}{a_0} w_1\right)^2 - \left(\frac{a_1^\sigma}{a_0} w_1\right)^3 + \cdots\right). \end{aligned}$$

So, collecting all terms of degree zero in w_1, we find

$$\sum_{i=1}^{\infty} \frac{x_i}{i!} \left(p\frac{a_1}{a_0}\right)^i = 0.$$

Height 3:

Set

$$X = \sum \frac{x_{i,j}}{(i+j)!} w_1^i w_2^j,$$

with $x_{i,j} \in \mathbb{W}(\mathbb{F}_{p^n})$.

We have

$$\begin{aligned} g(ww_1) &= pa_1 w + pa_2^{\sigma^2} ww_2 + a_0^\sigma ww_1, \\ g(ww_2) &= pa_2 w + a_0^{\sigma^2} ww_2 + a_1^\sigma ww_1, \quad \text{and} \\ g(w) &= a_0 w + a_2^\sigma ww_1 + a_1^{\sigma^2} ww_2, \end{aligned}$$

So,

$$
\begin{aligned}
g(w_1) =& \frac{pa_1 w + pa_2^{\sigma^2} ww_2 + a_0^{\sigma} ww_1}{a_0 w + a_2^{\sigma} ww_1 + a_1^{\sigma^2} ww_2} \\
=& \frac{pa_1 + pa_2^{\sigma^2} w_2 + a_0^{\sigma} w_1}{a_0 + a_2^{\sigma} w_1 + a_1^{\sigma^2} w_2} \\
=& \frac{p\frac{a_1}{a_0} + p\frac{a_2^{\sigma^2}}{a_0} w_2 + \frac{a_0^{\sigma}}{a_0} w_1}{1 + \frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2} \\
=& \left(p\frac{a_1}{a_0} + p\frac{a_2^{\sigma^2}}{a_0} w_2 + \frac{a_0^{\sigma}}{a_0} w_1 \right) \\
& \times \left(1 - \left(\frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2 \right) + \left(\frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2 \right)^2 - \cdots \right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
g(w_2) =& \frac{pa_2 w + a_0^{\sigma^2} ww_2 + a_1^{\sigma} ww_1}{a_0 w + a_2^{\sigma} ww_1 + a_1^{\sigma^2} ww_2} \\
=& \frac{pa_2 + a_0^{\sigma^2} w_2 + a_1^{\sigma} w_1}{a_0 + a_2^{\sigma} w_1 + a_1^{\sigma^2} w_2} \\
=& \frac{p\frac{a_2}{a_0} + \frac{a_0^{\sigma^2}}{a_0} w_2 + \frac{a_1^{\sigma}}{a_0} w_1}{1 + \frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2} \\
=& \left(p\frac{a_2}{a_0} + \frac{a_0^{\sigma^2}}{a_0} w_2 + \frac{a_1^{\sigma}}{a_0} w_1 \right) \\
& \times \left(1 - \left(\frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2 \right) + \left(\frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2 \right)^2 - \cdots \right).
\end{aligned}
$$

Let $g(X) = x_{0,0} + \frac{x_{1,0}}{1!} g(w_1) + \frac{x_{0,1}}{1!} g(w_2) + \cdots = X$.

Then, we find that

$$
\begin{aligned}
g(w_1^i w_2^j) =& \left(p\frac{a_1}{a_0} + p\frac{a_2^{\sigma^2}}{a_0} w_2 + \frac{a_0^{\sigma}}{a_0} w_1 \right)^i \left(p\frac{a_2}{a_0} + \frac{a_0^{\sigma^2}}{a_0} w_2 + \frac{a_1^{\sigma}}{a_0} w_1 \right)^j \\
& \times \left(1 - \left(\frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2 \right) + \left(\frac{a_2^{\sigma}}{a_0} w_1 + \frac{a_1^{\sigma^2}}{a_0} w_2 \right)^2 - \cdots \right)^{i+j}.
\end{aligned}
$$

Equating the degree zero terms (as a power series in w_1 and w_2), we find

$$\sum_{(i.j)\neq(0,0)} \frac{x_{i,j}}{(i+j)!}\left(p\frac{a_1}{a_0}\right)^i \left(p\frac{a_2}{a_0}\right)^j = 0.$$

Generally, for arbitrary height n, we have

$$\begin{aligned} g(w_i) &= \frac{g(ww_i)}{g(w)} \\ &= \frac{pa_i w + pa_{i+1}^{\sigma^{n-1}} ww_{n-1} + \cdots + pa_{n-1}^{\sigma^{i+1}} ww_{i+1} + a_0^{\sigma^i} ww_i + \cdots + a_{i-1}^{\sigma} ww_1}{a_0 w + \sum_{j=1}^{n-1} a_{n-j}^{\sigma^j} ww_j} \\ &= \frac{1}{a_0}\cdot\frac{pa_i + pa_{i+1}^{\sigma^{n-1}} w_{n-1} + \cdots + pa_{n-1}^{\sigma^{i+1}} w_{i+1} + a_0^{\sigma^i} w_i + \cdots + a_{i-1}^{\sigma} w_1}{1 + \sum_{j=1}^{n-1} \frac{a_{n-j}^{\sigma^j}}{a_0} w_j} \\ &= \frac{1}{a_0}\cdot\Big(pa_i + pa_{i+1}^{\sigma^{n-1}} w_{n-1} + \cdots + pa_{n-1}^{\sigma^{i+1}} w_{i+1} + a_0^{\sigma^i} w_i + \cdots + a_{i-1}^{\sigma} w_1\Big) \\ &\quad\times\left(1 - \left(\sum_{j=1}^{n-1}\frac{a_{n-j}^{\sigma^j}}{a_0} w_j\right) + \left(\sum_{j=1}^{n-1}\frac{a_{n-j}^{\sigma^j}}{a_0} w_j\right)^2 - \cdots\right). \end{aligned}$$

So that degree zero term (as a power series in the w_i) of

$$g(x_1^{i_1}\cdots x_{n-1}^{i_{n-1}})$$

is

$$\left(\frac{pa_1}{a_0}\right)^{i_1}\cdots\left(\frac{pa_{n-1}}{a_0}\right)^{i_{n-1}}$$

So, we find for

$$g = a_0 + a_1 S + \cdots + a_{n-1}S^{n-1} \in \mathbb{S}_n$$

and

$$X = \sum \frac{x_{i_1,\ldots,i_{n-1}}}{(i_1+\cdots+i_{n-1})!} w_1^{i_1}\cdots w_{n-1}^{i_{n-1}} \in i_0(\pi_0 E_n),$$

that setting $g(X) = x_{0,\ldots,0} + \frac{x_{1,0,\ldots,0}}{1!} g(w_1) + \cdots = X$ and equating the degree zero terms that

$$\sum_{(i_1,\ldots,i_{n-1})\neq(0,\ldots,0)} \left(\frac{p}{a_0}\right)^{i_1+\cdots+i_{n-1}} \frac{x_{i_1,\ldots,i_{n-1}}}{(i_1+\cdots+i_{n-1})!} a_1^{i_1}\cdots a_{n-1}^{i_{n-1}} = 0.$$

$\square$

Proposition 10.3.4. *Let $U \subseteq \mathbb{S}_n \subseteq \mathbb{G}_n$ be an open subgroup. Then*

$$(\pi_0 E_n)^U = \mathbb{W}(\mathbb{F}_{p^n}).$$

Proof. We begin by illustrating at height 2 and then prove the general case. Let $n = 2$. Fix an open subgroup $U \subseteq \mathbb{S}_n$. From the above, we find that for $g = a_0 + a_1 S \in U$ and $x = x_0 + \frac{x_1}{1!} w_1 + \frac{x_2}{2!} w_1^2 + \cdots \in i_0(\pi_0 E_2)$ that if $x \in i_0 (\pi_0 E_2)^U$, then

$$\sum_{i \geq 1} \frac{x_i}{i!} \left(\frac{p a_1}{a_0} \right)^i = 0.$$

Furthermore, for sufficiently large j, $1 + p^{j-1} S \in U$ so that letting $a_0 = 1$ and $a_1 = p^{j-1}$, we have

$$\sum_{i \geq 1} \frac{x_i}{i!} p^{ij} = 0.$$

Now, suppose that $i_0(\pi_0 E_2)^U$ properly contains $\mathbb{W}(\mathbb{F}_{p^2})$. Then, there is some minimal $I_0 \geq 1$ such that $x_{I_0} \neq 0$, so that

$$\sum_{i \geq I_0} \frac{x_i}{i!} p^{ij} = 0.$$

For $j >> 1$,

$$\begin{aligned} \mathrm{val}_p(i!) &= \left\lfloor \frac{i}{p} \right\rfloor + \left\lfloor \frac{i}{p^2} \right\rfloor + \left\lfloor \frac{i}{p^3} \right\rfloor + \cdots \\ &\leq i/(p-1) \\ &< ij = \mathrm{val}_p(p^{ij}) \end{aligned}$$

for all $i \geq 1$. In particular, $\mathrm{val}_p(i!) \leq i - 1$.

As a result, we can regard this as an equality in $\mathbb{W}(\mathbb{F}_{p^2})$.

Since this holds in $\mathbb{W}(\mathbb{F}_{p^2})$, it also holds in $\mathbb{W}(\mathbb{F}_{p^2})/p^r$ for all $r \geq 1$ and for $j >> 1$, so that

$$\sum_{\substack{i\geq I_0,\\ ij<r}} \frac{x_i}{i!}p^{ij} = \frac{x_{I_0}}{I_0!}p^{I_0 j} + \frac{x_{I_0+1}}{(I_0+1)!}p^{(I_0+1)j} + \cdots + \frac{x_s}{s!}p^{sj} + \cdots \equiv 0 \mod p^r,$$

Letting

$$I_0 j - \mathrm{val}_p(I_0!) \leq I_0 j \leq r < sj - s + 1 \leq \mathrm{val}_p\left(\frac{p^{sj}}{s!}\right)$$

for all $s > I_0$ (which is possible for large enough j), this gives

$$\frac{x_{I_0}}{I_0!}p^{I_0 j} \equiv 0 \mod p^r.$$

Since for large j, $\frac{x_{I_0}}{I_0!}p^{I_0 j} \in \mathbb{W}(\mathbb{F}_{p^2})$, we can rewrite this as

$$\frac{x_{I_0}}{I_0!}p^{I_0 j} = \eta \cdot x_{I_0} p^{I_0 j - \mathrm{val}_p(I_0!)} \equiv 0 \mod p^r$$

for some unit $\eta \in \mathbb{W}(\mathbb{F}_{p^2})^\times$. In particular,

$$x_{I_0} p^{I_0 j} \equiv 0 \mod p^r$$

and therefore

$$x_{I_0} \equiv 0 \mod p^{r - I_0 j}.$$

Finally, letting j grow, and selecting

$$r = (I_0+1)j - I_0 - 1,$$

we find that

$$x_{I_0} \equiv 0 \mod p^{j - I_0 - 1}.$$

for arbitrarily large j, so that $x_{I_0} = 0$ and

$$(i_0(\pi_0 E_2))^U = \mathbb{W}(\mathbb{F}_{p^2}),$$

and as i_0 is a $\mathbb{W}(\mathbb{F}_{p^2})$-linear, $\mathbb{S}_n$-equivariant inclusion,

$$(\pi_0 E_2)^U = \mathbb{W}(\mathbb{F}_{p^2}).$$

For the general case, let

$$g = a_0 + a_1 S + \cdots + a_{n-1} S^{n-1} \in U$$

and

$$X = \sum \frac{x_{i_1,\ldots,i_{n-1}}}{(i_1 + \cdots + i_{n-1})!} w_1^{i_1} \cdots w_{n-1}^{i_{n-1}} \in i_0(\pi_0 E_n)$$

with $g(X) = X$, where again $x_{i_1,\ldots,i_{n-1}} \in \mathbb{W}(\mathbb{F}_{p^n})$. For $j >> 0$, we can let $a_0 = 1$ and for $i > 1$, $a_i = p^{j-1}$. Then, from Proposition 10.3.3, we have

$$\begin{aligned} &\sum_{(i_1,\ldots,i_{n-1}) \neq (0,\ldots,0)} p^{i_1+\cdots+i_{n-1}} \frac{x_{i_1,\ldots,i_{n-1}}}{(i_1+\cdots+i_{n-1})!} a_1^{i_1} \cdots a_{n-1}^{i_{n-1}} \\ = &\sum_{(i_1,\ldots,i_{n-1}) \neq (0,\ldots,0)} p^{(i_1+\cdots+i_{n-1})j} \frac{x_{i_1,\ldots,i_{n-1}}}{(i_1+\cdots+i_{n-1})!} = 0. \end{aligned}$$

Assuming that there is some $X \notin \mathbb{W}(\mathbb{F}_{p^n})$ in $i_0\left((\pi_0 E_n)\right)^U$, let $I_0 = i_1 + \cdots + i_{n-1} \neq 0$ be minimal such that $x_{i_1+\cdots+i_{n-1}} \neq 0$. Then, we find that

$$\sum_{i_1+\cdots+i_{n-1} \geq I_0} p^{(i_1+\cdots+i_{n-1})j} \frac{x_{i_1,\ldots,i_{n-1}}}{(i_1+\cdots+i_{n-1})!} = 0.$$

As before, for j large enough, this can be regarded as an equation in $\mathbb{W}(\mathbb{F}_{p^n})$ rather than in $\mathrm{Frac}(\mathbb{W}(\mathbb{F}_{p^n}))$. and taking residues modulo p^r for $r = (I_0+1)j - I_0 - 1$, we find

$$\sum_{i_1+\cdots+i_{n-1} = I_0} \eta_{i_1,\ldots,i_{n-1}} \cdot x_{i_1,\ldots,i_{n-1}} \equiv 0 \mod p^{j-I_0-1}$$

for some units $\eta_{i_1,\ldots,i_{n-1}} \in \mathbb{W}(\mathbb{F}_{p^n}))^{\times}$. Letting j grow,

$$\sum_{i_1+\cdots+i_{n-1} = I_0} \eta_{i_1,\ldots,i_{n-1}} x_{i_1,\ldots,i_{n-1}} = 0.$$

For $n \geq 3$, this alone is insufficient to force each $x_{i_1,\ldots,i_{n-1}} = 0$. However, we have significant flexibility with our choice of the a_i. Let $\ell_k \in \mathbb{W}(\mathbb{F}_{p^n})$ be arbitrary for $k = 1, \ldots, n-1$. Repeating the above with $a_0 = 1$, $a_k = \ell_k p^{j-1}$ for $k = 1, \ldots n-1$, we find

$$\sum_{i_1+\cdots+i_{n-1} = I_0} \ell_1^{i_1} \ell_2^{i_2} \cdots \ell_{n-1}^{i_{n-1}} \left(\eta_{i_1,\ldots,i_{n-1}} x_{i_1,\ldots,i_{n-1}}\right) = 0. \tag{10.1}$$

Letting $V = \text{Span}_K\{x_{i_1,\ldots,i_{n-1}}\}_{i_1+\cdots i_{n-1}=I_0}$ where $K = \text{Frac}(\mathbb{W}(\mathbb{F}_{p^n}))$, we find that the vectors with entries $\prod \ell_k^{i_k}$ corresponding to each partition

$$i_1 + \cdots + i_k + \cdots + i_{n-1} = I_0$$

(extracted as the coefficients in Equation 10.1) span the $\binom{n-2+I_0}{n-2}$-dimensional dual space V^*.

(As an example, let $n = 4, I_0 = 2$. Then, the collection of $x_{i_1,\ldots,i_{n-1}}$ with $i_1+\cdots+i_{n-1} = I_0$ ordered lexicographically are

$$x_{0,0,2}, x_{0,1,1}, x_{0,2,0}, x_{1,0,1}, x_{1,1,0}, x_{2,0,0},$$

with corresponding vectors of the form

$$(\ell_3^2, \ell_2\ell_3, \ell_2^2, \ell_1\ell_3, \ell_1\ell_2, \ell_1^2).)$$

Since the $\binom{n-2+I_0}{n-2}$ monomials of the form $\prod \ell_k^{i_k}$ are K-linear independent, they indeed span V^*, so that letting the ℓ_k vary and noting that

$$\sum_{i_1+\cdots+i_{n-1}=I_0} \ell_1^{i_1}\ell_2^{i_2}\cdots\ell_{n-1}^{i_{n-1}}\eta_{i_1,\ldots,i_{n-1}}x_{i_1,\ldots,i_{n-1}} = 0,$$

for every choice of the ℓ_k, we find that each $x_{i_1,\ldots,i_{n-1}}$ with $i_1 + \cdots + i_{n-1} = I_0$ is zero and $X \in \mathbb{W}(\mathbb{F}_{p^n})$.

So, $x_{i_1,\ldots,i_{n-1}} = 0$ for all $i_1 + \cdots + i_{n-1} = I_0$ and therefore $X \in \mathbb{W}(\mathbb{F}_{p^n})$ and

$$(i_0(\pi_0 E_n))^U = \mathbb{W}(\mathbb{F}_{p^n})$$

so that

$$(\pi_0 E_n)^U = \mathbb{W}(\mathbb{F}_{p^n}).$$

□

Theorem 10.3.5. $\pi_0 F_n \otimes \mathbb{Q} = \mathbb{W}(\mathbb{F}_{p^n}) \otimes \mathbb{Q} \cong \mathbb{Q}_p^n$.

Proof. Taking the direct limit of Homotopy Fixed Point Spectral Sequences, we have

$$\varinjlim_U H^s_{\text{cont}}(U;\pi_r E_n) \implies \varinjlim_U \pi_{r-s} E_n^{hU} = \pi_{r-s} F_n.$$

Since the U's with finite cohomological dimension form a cofinal family, we can without loss of generality let each U have $\text{cd}(U) = n^2 = \text{vcd}(\mathbb{G}_n)$. As such, we need not worry about non-detection of a class in the direct limit nor fake cycles (as discussed in Subsection 10.2.2) and the direct limit of the spectral sequences does indeed converge to $\pi_{r-s}F_n$.

Since $-\otimes\mathbb{Q}$ is exact, we also have

$$\varinjlim_U H^s_{\text{cont}}(U;\pi_r E_n)\otimes\mathbb{Q} \implies \varinjlim_U \pi_{r-s} E_n^{hU}\otimes\mathbb{Q} = \pi_{r-s} F_n\otimes\mathbb{Q}.$$

By [Ser79, Prop. 8, Cor. 3], $H^s_{\text{cont}}(U;\pi_r E_n)$ is torsion for $s \geq 1$, meaning that the differential $d_a \otimes \mathbb{Q} = 0$ for all a so that we have

$$\varinjlim_U H^0_{\text{cont}}(U;\pi_0 E_n)\otimes\mathbb{Q} = \varinjlim_U \mathbb{W}(\mathbb{F}_{p^n})\otimes\mathbb{Q} = \mathbb{W}(\mathbb{F}_{p^n})\otimes\mathbb{Q} \implies \pi_0 F_n\otimes\mathbb{Q}.$$

So, $\pi_0 F_n \otimes \mathbb{Q} = \mathbb{W}(\mathbb{F}_{p^n})\otimes\mathbb{Q}$, which is rank n over $\mathbb{Q}_p$.

□

We can also compute the units in $\pi_0 F_n$:

Theorem 10.3.6. $\pi_0\text{GL}_1(F_n) = (\pi_0 F_n)^\times \cong \mathbb{W}(\mathbb{F}_{p^n})^\times$.

Proof. Again, taking the subgroups $U \subseteq \mathbb{S}_n \subseteq \mathbb{G}_n$ to be of finite cohomological dimension, we have a spectral sequence of algebras

$$\varinjlim_U H^s_{\text{cont}}(U;\pi_t E_n) \implies \pi_{t-s} F_n.$$

When $s = t = 0$, fixing U, we have

$$E_2^{s,t} = H^0_{\text{cont}}(U;\pi_0 E_n) = (\pi_0 E_n)^U = \mathbb{W}(\mathbb{F}_{p^n})$$

by Proposition 10.3.4.

This gives

$$E_2^0 = \bigoplus_{s=0}^{n^2} E_2^{s,s} = \underbrace{\mathbb{W}(\mathbb{F}_{p^n})}_{E_2^{0,0}} \oplus \bigoplus_{s=1}^{n^2} E_2^{s,s},$$

Furthermore, $E_r^{0,0}$ accepts no non-zero differentials. Also, for $x, y \in E_r^{0,0}$, we have via the Leibniz formula that

$$d_r(xy) = d_r(x) \cdot y + (-1)^{(0-0)} x d_r(y).$$

So, $\ker(d_r^{0,0})$ is closed under multiplication and contains 1 and therefore all of $\mathbb{Z}$. (seen by letting $x = y = 1$).

Further, d_r is continuous: We have

$$d_r = \varprojlim_I d_{r,I},$$

where $d_{r,I}$ denotes the differential of the homotopy fixed point spectral sequence

$$E_{2,I} = H^s(U; \pi_t E_{n,I}) \implies \pi_{t-s} E_{n,I}^{hU}.$$

Here, $\{E_{n,I}\}_I$ is the inverse system of discrete $\mathbb{G}_n$-spectra $E_{n,I} = E_n \wedge M_I$, where M_I is the generalized Moore spectrum satisfying $BP_* M_I = BP_*/I$ for $I = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$, so that $\pi_* E_{n,I} = (\pi_* E_n)/I$. Then $\varprojlim_I E_{n,I} \cong E_n$, and the spectral sequence $E_r^{s,t}$ is the inverse limit of the spectral sequences $E_{r,I}^{s,t}$. (See [BD10] [4.6, Proof of 8.25].) Each $d_{r,I}$ is continuous because each $E_{r,I}^{s,t}$ is finite. This gives the continuity of d_r.

So, $\ker(d_r^{0,0})$ is closed, and contains all of $\mathbb{Z}_p$ since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Further, if $z^m = 1$, then

$$0 = d_r(1) = d_r(z^m) = m z^{m-1} d_r(z) \implies d_r(z) = 0,$$

so that $\ker(d_r^{0,0})$ contains all roots of unity in $\mathbb{W}(\mathbb{F}_{p^n})$. As a result,

$$E_{r+1}^{0,0} = \ker(d_r^{0,0}) \cong \mathbb{W}(\mathbb{F}_{p^n}),$$

and we conclude that

$$E_\infty^0 = \mathbb{W}(\mathbb{F}_{p^n}) \oplus N,$$

where N is nilpotent and contained in positive filtration degree.

$$\mathbb{W}(\mathbb{F}_{p^n}) \oplus N \cong \bigoplus_{i=0}^{n^2} \Phi_i/\Phi_{i+1},$$

where $\pi_0 E_n^{hU} = \Phi_0 \supseteq \Phi_1 \supseteq \cdots \supseteq \Phi_{n^2} \supseteq \Phi_{n^2+1} = 0$ is the filtration on $\pi_0 E_n^{hU}$ with Φ_i representing elements coming from cohomological degree $\leq i$. Further, we have

$$\Phi_i \Phi_j \subseteq \Phi_{i+j},$$

and since $1 \in \Phi_0$, any unit $\alpha \in \Phi_{i>0}$ would have an inverse $\alpha^{-1} \in \Phi_{-i<0}$, so that

$$\left(\pi_0 E_n^{hU}\right)^\times \cong (\Phi_0/\Phi_1)^\times \cong \mathbb{W}(\mathbb{F}_{p^n})^\times,$$

and in the limit, we have

$$(\pi_0 F_n)^\times \cong \mathbb{W}(\mathbb{F}_{p^n}).$$

□

10.4 On the torsion-free rank of $\pi_*\mathcal{P}\mathrm{ic}_n$

By Theorem 10.2.15, we can compute the rationalization of $\pi_*\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp})$ via a spectral sequence

$$\begin{aligned} E_2^{s,t} \otimes \mathbb{Q} = & H^s_{\mathrm{cont}}\left(\mathbb{G}_n; \left(\pi_t \mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^\delta\right) \otimes \mathbb{Q} \\ & \Downarrow \\ & \pi_{t-s}\mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}) \otimes \mathbb{Q}. \end{aligned}$$

When $t \geq 2$, by Proposition 10.3.1,

$$\pi_t \mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right) = \pi_{t-1}\mathrm{GL}_1(\varinjlim_U E_n^{hU}) = \pi_{t-1}\mathrm{GL}_1(F_n) = \pi_{t-1}F_n$$

is torsion.

Furthermore, for $s \geq 1$, $E_2^{s,t}$ is torsion by [Ser79][Prop. 8, Cor. 3] since $\mathbb{G}_n$ is profinite. As a result, the differentials

$$d_r \otimes \mathbb{Q} : E_r^{s,t} \otimes \mathbb{Q} \to E_r^{s+r,t+r-1} \otimes \mathbb{Q}$$

are zero for all $r \geq 2$.

Putting this all together, we have

$$\bigoplus_t E_2^{t,t} \otimes \mathbb{Q} = E_2^{0,0} \otimes \mathbb{Q},$$

so that since $E_2 \otimes \mathbb{Q} = E_\infty \otimes \mathbb{Q}$, the torsion-free rank of Pic_n matches that of

$$\begin{aligned} E_2^{0,0} &= H^0_{\mathrm{cont}}\left(\mathbb{G}_n; \mathrm{Pic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)^{\delta}\right) \\ &= \left(\varinjlim_U \mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^{\mathbb{G}_n}. \end{aligned}$$

That is to say:

Theorem 10.4.1.

$$\mathrm{Pic}_n \otimes \mathbb{Q} = \pi_0 \mathcal{P}\mathrm{ic}_n \otimes \mathbb{Q} \cong \left(\varinjlim_U \mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^{\mathbb{G}_n} \otimes \mathbb{Q},$$

or equivalently,

$$\mathrm{Pic}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \pi_0 \mathcal{P}\mathrm{ic}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \left(\varinjlim_U \mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^{\mathbb{G}_n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For $n \geq 2$, we know that the rank of Pic_n is at least 2, with families generated by $L_{K(n)}S^0$ and the determinantal sphere $S\langle \det \rangle$, and it is conjectured that this is also an upper bound:

Conjecture 10.4.2 (Hopkins)**.** For $n \geq 1$, the rank of Pic_n over $\mathbb{Z}_p$ is 2.

Remark 10.4.3. There is a well-defined map

$$\varphi : \mathrm{Pic}_n \to \left(\varinjlim_U \mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^{\mathbb{G}_n}$$

given by

$$X \mapsto \varinjlim_U \left(X \otimes_{L_{K(n)}S^0} E_n^{hU} \right)$$

which, by Theorem 10.4.1, is a rational isomorphism. The action of $\mathbb{G}_n/U$ on $\mathrm{Pic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}))$ is by sending an E_n^{hU}-module M with E_n^{hU}-action μ to a module M^g whose underlying spectrum is M, with action μ^g defined by

$$\begin{array}{ccc} E_n^{hU} & \xrightarrow{\mu} & \mathrm{End}(M) \\ g\big\uparrow & \nearrow_{\mu^g} & \\ E_n^{hU}. & & \end{array}$$

To see that φ is indeed a map to the $\mathbb{G}_n$-fixed points, we need that

$$(X \otimes_{L_{K(n)}S^0} E_n^{hU})^g \cong X \otimes_{L_{K(n)}S^0} E_n^{hU}.$$

We have

$$(X \otimes_{L_{K(n)}S^0} E_n^{hU})^g \cong X \otimes_{L_{K(n)}S^0} \left(E_n^{hU}\right)^g,$$

so it suffices to show that $E_n^{hU} \cong \left(E_n^{hU}\right)^g$ as E_n^{hU}-modules. This isomorphism is evidently given by $g \in \mathbb{G}_n/U \subseteq \mathrm{End}(E_n^{hU})$ with inverse g^{-1}.

In particular, each module category $\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})$ includes the invertible E_n^{hU}-modules

$$L_{K(n)}S^{\alpha} \otimes_{L_{K(n)}S^0} E_n^{hU}$$

for $\alpha \in \mathbb{Z}_p$ (with inverse $L_{K(n)}S^{-\alpha} \otimes_{L_{K(n)}S^0} E_n^{hU}$). These are fixed at every stage by $\mathbb{G}_n$ and evidently correspond to the $\mathbb{Z}_p$-family in Pic_n generated by $L_{K(n)}S^0$. Similarly,

$$S\langle \det \rangle^{\otimes \alpha} \otimes_{L_{K(n)}S^0} E_n^{hU}$$

is fixed as an element of $\mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\left(E_n^{hU}\right)\right)$ by $\mathbb{G}_n/U$, so that it represents an element in

$$\left(\varinjlim_U \mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^{\mathbb{G}_n}.$$

Further investigation into the right hand side of the equalities in Theorem 10.4.1 may be useful in resolving Conjecture 10.4.2.

Theorem 10.4.4. *$\pi_*\mathcal{P}\mathrm{ic}_n$ is torsion for $* \geq 2$.*

Proof. As before, since $\mathbb{G}_n$ is profinite,

$$E_2^{s,t} \otimes \mathbb{Q} = H^s_{\mathrm{cont}}\left(\mathbb{G}_n; \left(\pi_t \mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right)\right)^\delta\right) \otimes \mathbb{Q}$$

is torsion for $s \geq 1$ by [Ser79, Proposition 8, Corollary 3]. By Proposition 10.3.1, for $t \geq 2$,

$$\pi_t \mathcal{P}\mathrm{ic}\left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})\right) = \pi_{t-1}\mathrm{GL}_1(\varinjlim_U E_n^{hU}) = \pi_{t-1}\mathrm{GL}_1(F_n) = \pi_{t-1}F_n$$

is torsion. So, $E_2^{s,t} \otimes \mathbb{Q} = 0$ except possibly $s = 0, t = 0$, and $s = 0, t = 1$, and since $d_r \otimes \mathbb{Q} = 0$ for all r,

$$E_\infty^{s+t} \otimes \mathbb{Q} = 0$$

for all $s + t \geq 2$, giving the result. □

By the same line of reasoning, we have the following result:

Theorem 10.4.5. $\pi_1\mathcal{P}\mathrm{ic}_n \otimes \mathbb{Q} \cong \mathbb{Q}_p$.

Proof. We have

$$\begin{aligned}
\pi_1 \mathcal{P}\mathrm{ic}_n \otimes \mathbb{Q} &\cong H^0_{\mathrm{cont}}\left(\mathbb{G}_n; \left((\pi_0 F_n)^\times\right)^\delta\right) \otimes \mathbb{Q} \\
&\cong \left((\pi_0 F_n)^\times\right)^{\mathbb{G}_n} \otimes \mathbb{Q} \\
&\cong \left(\mathbb{W}(\mathbb{F}_{p^n})^\times\right)^{\mathbb{G}_n} \otimes \mathbb{Q} \\
&\cong \left(\mathbb{Z}_p^\times\right)^{\mathbb{S}_n} \otimes \mathbb{Q} \\
&\cong \mathbb{Z}_p^\times \otimes \mathbb{Q} \\
&\cong \mathbb{Q}_p
\end{aligned}$$

□

10.5 Example: Height $n = 1$

We now run this entire program at height $n = 1$, showing that $\pi_1 \mathcal{P}\mathrm{ic}_1 \otimes \mathbb{Q} \cong \mathbb{Q}_p$ at all primes, in agreement with the result of Theorem 10.4.5.

When $n = 1$, since $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$, we can take our family of open subgroups $U \subseteq \mathbb{G}_1$ to be the linearly ordered set $\{U_i\}_{i \geq 2}$ with $U_i = 1 + p^i \mathbb{Z}_p$. Then, following the conventions of §10.3, $Z_i = U_i$ so that $U_i/Z_i = 1$ and the Lyndon-Hochschild-Serre Spectral Sequence associated with the inclusion $Z_i \to U_i$ is of no computational value, and we are reduced to computing $\pi_* F_1$ via the Homotopy Fixed Point Spectral Sequence

$$H^s_{\mathrm{cont}}(U_i; \pi_r E_1) \implies \pi_{r-s} E_1^{hU_i},$$

and then taking the limit over i. Since the cohomological dimension of U_i is 1, we need only consider the cases of $s = 0$ and $s = 1$, so that we have

$$(\pi_0 E_1)^{U_i} \oplus \underbrace{(\pi_1 E_1)_{U_i}}_{=0} \implies \pi_0 E_1^{hU_i}$$

and

$$\underbrace{(\pi_1 E_1)^{U_i}}_{=0} \oplus (\pi_2 E_1)_{U_i} = \mathbb{Z}_p/p^{\varepsilon_{2,i}} \cong \mathbb{Z}/p^{i+\epsilon} \implies \pi_1 E_1^{hU_i}.$$

Taking limits over the U_i, we find that

$$\pi_0 F_1 = \varinjlim (\pi_0 E_1)^{U_i} = \varinjlim \pi_0 E_1 = \pi_0 E_1 = \mathbb{Z}_p$$

and

$$\pi_1 F_1 = \varinjlim \mathbb{Z}/p^{\varepsilon_{2,i}} = \mathbb{Q}_p/\mathbb{Z}_p$$

since $\varepsilon_{2,i} = \mathrm{val}_p((1+p^i)^2 - 1) = \mathrm{val}_p(2p^i + p^{2i}) \geq i$. (For p odd, this is equality.) So, $\pi_1 F_1$ is torsion as expected.

We then have

$$H^0_{\text{cont}}(\mathbb{G}_1; \left(\pi_0 F_1^{\times}\right)^{\delta}) = \left((\pi_0 E_1)^{\times}\right)^{\mathbb{G}_1} = (\pi_0 E_1)^{\times} = \mathbb{Z}_p^{\times}$$

and

$$\begin{aligned} H^1_{\text{cont}}(\mathbb{G}_1; (\pi_1 F_1)^{\delta}) &\cong \mathrm{Hom}_{\text{cont}}(\mathbb{G}_1, \pi_1 F_1^{\delta}) \\ &\cong \mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^{\times}, (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta}). \end{aligned}$$

For odd p,

$$\mathbb{Z}_p^{\times} \otimes \mathbb{Q} = (\underbrace{\mu_{p-1}}_{\text{torsion}} \oplus \underbrace{(1 + p\mathbb{Z}_p)^{\times}}_{\cong \mathbb{Z}_p}) \otimes \mathbb{Q} \cong \mathbb{Q}_p$$

and for $p = 2$,

$$\mathbb{Z}_2^{\times} \otimes \mathbb{Q} = (\underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{torsion}} \oplus \underbrace{(1 + 4\mathbb{Z}_2)^{\times}}_{\cong \mathbb{Z}_2}) \otimes \mathbb{Q} \cong \mathbb{Q}_2.$$

Finally, since $\mathbb{Z}_p^{\times}$ is compact, its image under a continuous map into $(\mathbb{Q}_p/\mathbb{Z}_p)^{\delta}$ must be finite and therefore $\mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^{\times}, (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta})$ is torsion and we conclude that $\pi_1 \mathcal{P}\mathrm{ic}_1 \otimes \mathbb{Q} \cong \mathbb{Q}_p$ is of rank 1.

Chapter 11

Further results and directions for future research

11.1 Understanding of $\mathrm{Pic}_n^{\mathrm{alg}}$ and the exotic Picard group

11.1.1 The algebraic Picard group $\mathrm{Pic}_n^{\mathrm{alg}}$

The algebraic Picard group $\mathrm{Pic}_n^{\mathrm{alg}}$ contains an index 2 subgroup $\mathrm{Pic}_n^{\mathrm{alg},0}$ which can be computed as

$$\mathrm{Pic}_n^{\mathrm{alg},0} \cong H^1_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 E_n)^\times).$$

The $E_2^{1,1}$ term in our main spectral (Theorem 10.2.15) sequence computing

$$P_* = \pi_* \mathcal{P}\mathrm{ic} \left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}) \right)^{h\mathbb{G}_n}$$

is

$$H^1_{\mathrm{cont}}(\mathbb{G}_n; ((\pi_0 F_n)^\times)^\delta),$$

contributing to P_0 and being entirely torsion. We then have the following commutative diagram, where the horizontal arrows are given by localization

$$F_n \to L_{K(n)}F_n \simeq E_n$$

and the vertical arrows are given by the natural continuous maps on the coefficients:

$$\begin{array}{ccc} H^1_{\mathrm{cont}}(\mathbb{G}_n; ((\pi_0 F_n)^\times)^\delta) & \longrightarrow & H^1_{\mathrm{cont}}(\mathbb{G}_n; ((\pi_0 E_n)^\times)^\delta) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 F_n)^\times) & \longrightarrow & H^1_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 E_n)^\times) = \mathrm{Pic}_n^{\mathrm{alg},0}. \end{array} \tag{11.1}$$

The vertical arrows are inclusions. To see this, write

$$H^1_{\mathrm{cont}}(G; M) \cong \frac{\text{continuous crossed homomorphisms } G \to M}{\text{continuous principal crossed homomorphisms } G \to M}$$

and let φ be a continuous principal crossed homomorphism $G \to M$ so that

$$[\varphi] = 0 \in H^1_{\mathrm{cont}}(G; M).$$

That is, $\varphi = f_m$ for some $m \in M$, where

$$f_m(g) = gm - m.$$

Then, the preimage of $[\varphi]$ in $H^1_{\mathrm{cont}}(G; M^\delta)$ is either empty (when $\varphi : G \to M^\delta$ is not continuous) or $[\varphi] = [f_m]$, now regarded as a continuous principal crossed homomorphism $G \to M^\delta$, and therefore is zero in $H^1_{\mathrm{cont}}(G; M^\delta)$.

At height $n = 1$, diagram (11.1) is

$$\begin{array}{ccc} H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times; (\mathbb{Z}_p^\times)^\delta) & \longrightarrow & H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times; (\mathbb{Z}_p^\times)^\delta) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times; \mathbb{Z}_p^\times) & \longrightarrow & H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times; \mathbb{Z}_p^\times) = \mathrm{Pic}_1^{\mathrm{alg},0}. \end{array} \tag{11.2}$$

For p odd, $H^1_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \cong \mathbb{Z}_p^\times \cong \mathbb{Z}_p \oplus \mathbb{Z}/(p-1)\mathbb{Z}$. Further,

$$\begin{aligned} H^1_{\text{cont}}(\mathbb{Z}_p^\times, (\mathbb{Z}_p^\times)^\delta) \cong & \operatorname{Hom}_{\text{cont}}(\mathbb{Z}_p, \mathbb{Z}_p^\delta) \\ & \oplus \underbrace{\operatorname{Hom}_{\text{cont}}(\mu_{p-1}, \mu_{p-1})}_{\cong \mu_{p-1}} \\ & \oplus \underbrace{\operatorname{Hom}_{\text{cont}}(\mu_{p-1}, \mathbb{Z}_p^\delta)}_{=0} \\ & \oplus \underbrace{\operatorname{Hom}_{\text{cont}}(\mathbb{Z}_p, \mu_{p-1})}_{0}. \end{aligned}$$

Since $\mathbb{Z}_p$ is compact, its image under any continuous homomorphism to $\mathbb{Z}_p^\delta$ must be finite. Since there are no non-trivial finite subgroups of $\mathbb{Z}_p$, we conclude that

$$\operatorname{Hom}_{\text{cont}}(\mathbb{Z}_p, \mathbb{Z}_p^\delta) = 0,$$

and diagram (11.1) becomes

$$\begin{array}{ccc} \mathbb{Z}/(p-1)\mathbb{Z} & \longrightarrow & \mathbb{Z}/(p-1)\mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}_p \oplus \mathbb{Z}/(p-1)\mathbb{Z} & \longrightarrow & \mathbb{Z}_p \oplus \mathbb{Z}/(p-1)\mathbb{Z} = \operatorname{Pic}_1^{\text{alg},0}. \end{array} \tag{11.3}$$

The vertical maps are inclusion of the $\mathbb{Z}/(p-1)\mathbb{Z}$-summand. For a fixed U, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_0 E_1^{hU} & \xrightarrow{\sim} & \pi_0 F_1 \\ & \searrow^{\sim} & \downarrow L_{K(n)} \\ & & \pi_0 E_1, \end{array}$$

where the top map is given by the universal property of $\pi_0 F_1$ as the colimit

$$\pi_0 F_1 = \varinjlim_U \pi_0 E_1^{hU},$$

and the natural map $\pi_0 E_1^{hU} = (\pi_0 E_1)^U \to \pi_0 E_1$ is an equivalence by Proposition 10.3.4. As a result, the map $(\pi_0 F_n)^\times \to (\pi_0 E_n)^\times$ is an equivalence. So, we have a commutative square

$$\begin{array}{ccc} H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times;(\mathbb{Z}_p^\times)^\delta) & \xrightarrow{\sim} & H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times;(\mathbb{Z}_p^\times)^\delta) \\ \big\downarrow & & \big\downarrow \\ H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times;\mathbb{Z}_p^\times) & \xrightarrow{\sim} & H^1_{\mathrm{cont}}(\mathbb{Z}_p^\times;\mathbb{Z}_p^\times) = \mathrm{Pic}_1^{\mathrm{alg},0}, \end{array} \tag{11.4}$$

where both compositions

$$E_2^{1,1} = H^1_{\mathrm{cont}}(\mathbb{G}_1, ((\pi_0 F_n)^\times)^\delta) \to \mathrm{Pic}_1^{\mathrm{alg},0}$$

represent the inclusion of the torsion subgroup

$$\mathrm{Tor}(\mathrm{Pic}_1^{\mathrm{alg},0}) \hookrightarrow \mathrm{Pic}_1^{\mathrm{alg},0}.$$

Similarly, at $n = 1$, $p = 2$, we have

$$H^1_{\mathrm{cont}}(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times) \cong \mathbb{Z}_2^\times \oplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

and

$$\begin{aligned} H^1_{\mathrm{cont}}(\mathbb{Z}_2^\times; \left(\mathbb{Z}_2^\times\right)^\delta) \cong & \underbrace{\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_2, \mathbb{Z}_2^\delta)}_{=0} \\ & \oplus \underbrace{\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})}_{\mathbb{Z}/2\mathbb{Z}} \\ & \oplus \underbrace{\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}_2^\delta)}_{=0} \\ & \oplus \underbrace{\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_2, \mathbb{Z}/2\mathbb{Z})}_{\cong \mathbb{Z}/2\mathbb{Z}}. \end{aligned}$$

So, we have

$$\begin{array}{ccc} H^1_{\mathrm{cont}}(\mathbb{G}_1; \left((\pi_0 F_n)^\times\right)^\delta\right) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \mathrm{Pic}_1^{\mathrm{alg},0}, \end{array} \tag{11.5}$$

and again, both composites represent the inclusion of the torsion subgroup

$$\mathrm{Tor}(\mathrm{Pic}_1^{\mathrm{alg},0}) \hookrightarrow \mathrm{Pic}_1^{\mathrm{alg},0}.$$

However, when $n > 1$, we have by Theorem 10.3.6

$$(\pi_0 F_n)^\times \cong \mathbb{W}(\mathbb{F}_{p^n})^\times,$$

so that the map

$$(\pi_0 F_n)^\times \hookrightarrow (\pi_0 E_n)^\times = (\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]])^\times$$

will not be an isomorphism. Nevertheless, it does induce injections $H^1_{\text{cont}}(\mathbb{G}_n; -)$, so that every map in

$$\begin{array}{ccc} H^1_{\text{cont}}(\mathbb{G}_n; ((\pi_0 F_n)^\times)^\delta) & \hookrightarrow & H^1_{\text{cont}}(\mathbb{G}_n; ((\pi_0 E_n)^\times)^\delta) \\ \downarrow & & \downarrow \\ H^1_{\text{cont}}(\mathbb{G}_n; (\pi_0 F_n)^\times) & \hookrightarrow & H^1_{\text{cont}}(\mathbb{G}_n; (\pi_0 E_n)^\times) = \text{Pic}_n^{\text{alg},0} \end{array} \tag{11.6}$$

is an injection and we can still ask whether

$$E_2^{1,1} = H^1_{\text{cont}}\left(\mathbb{G}_n; \left((\pi_0 F_n)^\times\right)^\delta\right) \cong \text{Tor}\left(\text{Pic}_n^{\text{alg, }0}\right),$$

or if this relationship breaks down for $n > 1$. In any case, we have an injection

$$E_2^{1,1} \hookrightarrow \text{Tor}(\text{Pic}_n^{\text{alg},0}).$$

Explicitly, we have

$$\begin{aligned} H^1_{\text{cont}}(\mathbb{G}_n; (\pi_0 F_n)^\times) &\cong H^1_{\text{cont}}(\mathbb{G}_n; \mathbb{W}(\mathbb{F}_{p^n})^\times) \\ &\cong H^1_{\text{cont}}(\mathbb{S}_n; \mathbb{Z}_p^\times) \\ &\cong \text{Hom}_{\text{cont}}(\mathbb{S}_n, \mathbb{Z}_p^\times) \\ &\cong \text{Hom}_{\text{cont}}(\mathbb{S}_n^{\text{ab}}, \mathbb{Z}_p^\times). \end{aligned}$$

The first equality is by Proposition 10.3.6, the second is from the equality

$$(\mathbb{W}(\mathbb{F}_{p^n})^\times)^{\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} = \mathbb{Z}_p^\times,$$

and the third is by recognizing that the action of $\mathbb{S}_n$ on $\mathbb{Z}_p^\times$ is trivial. The last equality holds because because homomorphisms $\mathbb{S}_n \to \mathbb{Z}_p^\times$ factor through the abelianization $\mathbb{S}_n^{\text{ab}}$ because $\mathbb{Z}_p^\times$ is abelian.

Similarly,

$$E_2^{1,1} = H^1_{\mathrm{cont}}(\mathbb{G}_n; ((\pi_0 F_n)^\times)^\delta) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{S}_n^{\mathrm{ab}}, (\mathbb{Z}_p^\times)^\delta).$$

At height $n = 1$, $\mathbb{S}_n = \mathbb{Z}_p^\times$ is abelian. By [Hen17, Props. 5.2, 5.3], we have

$$\mathbb{S}_n^{\mathrm{ab}} \cong H_1(\mathbb{S}_n; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p \oplus (\mathbb{Z}/p\mathbb{Z})^n; & p \geq 3 \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z})^{n+1}; & p = 2. \end{cases}$$

As a result, we can compute

$$E_2^{1,1} \cong \begin{cases} 0; & p \geq 3 \\ (\mathbb{Z}/2\mathbb{Z})^{n+2}; & p = 2. \end{cases} \tag{11.7}$$

Similarly,

$$H^1_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 F_n)^\times) = \begin{cases} \mathbb{Z}_p; & p \geq 3 \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z})^{n+2}; & p = 2, \end{cases} \tag{11.8}$$

so that diagram (11.1) is

$$\begin{array}{ccc} 0 & \hookrightarrow & H^1_{\mathrm{cont}}(\mathbb{G}_n; ((\pi_0 E_n)^\times)^\delta) \\ \downarrow & & \downarrow \\ \mathbb{Z}_p & \hookrightarrow & H^1_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 E_n)^\times) = \mathrm{Pic}_n^{\mathrm{alg},0} \end{array} \tag{11.9}$$

for $p \geq 3$, $n > 1$, and

$$\begin{array}{ccc} (\mathbb{Z}/2\mathbb{Z})^{n+2} & \hookrightarrow & H^1_{\mathrm{cont}}(\mathbb{G}_n; ((\pi_0 E_n)^\times)^\delta) \\ \downarrow & & \downarrow \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z})^{n+2} & \hookrightarrow & H^1_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 E_n)^\times) = \mathrm{Pic}_n^{\mathrm{alg},0} \end{array} \tag{11.10}$$

for $p = 2$, $n > 1$.

11.1.2 The exotic Picard group κ_n

Define Pic_n^0 via the pullback diagram

$$\begin{array}{ccccc} \kappa_n & \longrightarrow & \mathrm{Pic}_n^0 & \longrightarrow & \mathrm{Pic}_n^{\mathrm{alg},0} \\ \downarrow = & & \downarrow & \lrcorner & \downarrow \\ \kappa_n & \longrightarrow & \mathrm{Pic}_n & \longrightarrow & \mathrm{Pic}_n^{\mathrm{alg}}, \end{array}$$

where $\kappa_n = \ker(\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}})$ denotes the subgroup of exotic elements. The above means that $E_\infty^{1,1} \subseteq E_2^{1,1}$ is contributing (via φ^{-1} of Remark 10.4.3) to the *non-exotic* torsion in Pic_n^0. As a result, contributions to κ_n must be from $E_\infty^{s,s}$ for $s \neq 1$.

By [Hea15, Theorem 4.4.1], for $p > 2$, κ_n is torsion. (When $p = 2$, κ_n is shown via explicit computations to be torsion at heights 1 and 2. See §9.4.) This in turn shows that

$$\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Pic}_n \leq \mathrm{rank}_{\mathbb{Z}_p}\mathrm{Pic}_n^{\mathrm{alg}},$$

We can show that this is indeed an equality: In [Hea21], Heard constructs a spectral sequence (for all primes p) with

$$\widetilde{E}_2^{s,t} = \begin{cases} \mathbb{Z}/2\mathbb{Z}; & s = t = 0 \\ \mathrm{Pic}_n^{\mathrm{alg},0}; & s = t = 1, \\ H^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_{t-1}E_n); & t \geq 2 \end{cases}$$

with differential d_r of bidegree $(r, r-1)$ computing Pic_n in the case $s = t$. By the proof of Proposition 10.3.1, $H^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_{t-1}E_n)$ is torsion for $t \geq 2$. So, rationally, this is

$$\widetilde{E}_2^{s,t} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \begin{cases} 0; & s = t = 0 \\ \mathrm{Pic}_n^{\mathrm{alg},0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p; & s = t = 1, \\ 0; & t \geq 2, \end{cases}$$

so that

$$\mathrm{Pic}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathrm{Pic}_n^{\mathrm{alg},0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{11.11}$$

and

$$\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Pic}_n = \mathrm{rank}_{\mathbb{Z}_p}\mathrm{Pic}_n^{\mathrm{alg}}. \tag{11.12}$$

This also shows that κ_n is torsion for all primes p at all heights n. This does not, however, imply that κ_n is finite.

11.1.3 An alternate notion of exoticness

Consider the map φ of Remark 10.4.3. The kernel (which is torsion) is given by

$$\ker\varphi = \{X \in \mathrm{Pic}_n \mid X \otimes_{L_{K(n)}S^0} E_n^{hU} \cong_{E_n^{hU}} E_n^{hU} \text{ for sufficiently small U}\},$$

where $\cong_{E_n^{hU}}$ denotes equivalence as E_n^{hU}-modules, and U is taken to be an open normal subgroup of $\mathbb{G}_n$. We can therefore filter $\ker\varphi$ over such $U \trianglelefteq \mathbb{G}_n$ by "*U-exotic*" elements:

$$\kappa_U := \ker(\mathrm{Pic}_n \to \mathrm{Pic}(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU})))$$

When $U = \mathbb{G}_n$,

$$\kappa_{\mathbb{G}_n} = \{X \mid X \otimes_{L_{K(n)}S^0} L_{K(n)S^0} \cong L_{K(n)}S^0\} = \{L_{K(n)}S^0\}$$

is trivial, and for $U \supseteq V$, $\kappa_U \subseteq \kappa_V$, with

$$\ker\varphi = \varinjlim_{U} \kappa_U.$$

One might ask what relation (if any) there is between the κ_U and the traditional group $\kappa_n = \ker(\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}})$ of exotic elements of Pic_n.

11.2 Comparison with results of [BSSW23]

In [BSSW23], the authors compute

$$\pi_* L_{K(n)}S^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \ldots, \zeta_n)$$

to be an exterior $\mathbb{Q}_p$ algebra on generators ζ_i in degree $1-2i$. For a fixed open subgroup $U \trianglelefteq \mathbb{G}_n$, we have a homotopy fixed point spectral sequence

$$H^s(\mathbb{G}_n/U; \pi_* E_n^{hU}) \implies \left(\pi_{t-s}E_n^{hU}\right)^{h\mathbb{G}_n/U} \cong \pi_{t-s}E_n^{h\mathbb{G}_n} \cong \pi_{t-s}L_{K(n)}S^0.$$

The limit of these spectral sequences is a spectral sequence with

$$E_2^{s,t} = H^s_{\mathrm{cont}}(\mathbb{G}_n; (\pi_t F_n)^\delta),$$

and by an identical arguement to that made in Subsection 10.2.2 about the limit of these spectral sequences, $E_2^{s,t} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ converges to $\pi_{t-s} L_{K(n)} S^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

$$E_2^{s,t} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^s_{\mathrm{cont}}(\mathbb{G}_n; (\pi_t F_n)^\delta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \implies \pi_{t-s} L_{K(n)} S^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

(This result actually holds pre-rationalization by [DL14, Theorem 1.1].) By Theorem 10.3.1,

$$\pi_{*>0} F_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong 0,$$

and by Theorem 10.3.5,

$$\pi_0 F_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{W}(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The methods of this dissertation thus allow us to identify

$$\begin{aligned} \pi_{*\geq 0} L_{K(n)} S^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &\cong H^0_{\mathrm{cont}}(\mathbb{G}_n; (\pi_0 F_n)^\delta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\cong (\mathbb{W}(\mathbb{F}_{p^n}))^{\mathbb{G}_n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\cong \mathbb{Q}_p, \end{aligned}$$

which is in agreement with the results of [BSSW23]. It might be possible to further adapt the methods used here to compute the negative homotopy groups of the rationalization of $L_{K(n)} S^0$.

11.3 Better identification of the abutment of the colimit spectral sequence

In Theorem 10.2.15, we identified the abutment of the colimit spectral sequence

$$E_r^{*,*} = \varinjlim_U E_{r,U}^{*,*}$$

as

$$P_* = \pi_* \mathcal{P}\mathrm{ic} \left(\varinjlim_U \mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}(E_n^{hU}) \right)^{h\mathbb{G}_n},$$

with

$$P_* \otimes \mathbb{Q} \cong \pi_* \mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}) \otimes \mathbb{Q}.$$

We now state a conjecture which if true would help simplify the statement of Theorem 10.2.15:

Conjecture 11.3.1. The abutment P_* of $E_r^{*,*} = \varinjlim_U E_{r,U}^{*,*}$ of Theorem 10.2.15 is the direct limit of the abutments of the $E_{r,U}^{*,*}$. That is,

$$P_* \cong \pi_* \mathcal{P}\mathrm{ic}(L_{K(n)}\mathbf{Sp}).$$

Remark 11.3.2. We have good reason to believe that Conjecture 11.3.1 holds. In [DL14, Theorem 1.1], Davis and Lawson prove the existence of a very similar spectral sequence with

$$\widehat{E}_2^{s,t} = H^s_{\mathrm{cont}}(\mathbb{G}_n; (\pi_t F_n)^\delta) = \varinjlim_U H^s(\mathbb{G}_n/U; \pi_t F_n) \implies \pi_{t-s} L_{K(n)} S^0,$$

and for $t \geq 2$, we have

$$E_2^{s,t} = H^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_{t-1} F_n) = \widehat{E}_2^{s,t-1}.$$

Further, for fixed U, there is a spectral sequence

$$\widehat{E}_{2,U}^{s,t-1} = H^s(\mathbb{G}_n/U; \pi_t E_n^{hU}) \implies \left(\pi_{t-s} E_n^{hU}\right)^{h\mathbb{G}_n/U} \cong \pi_{t-s} E_n^{h\mathbb{G}_n} \cong \pi_{t-s} L_{K(n)} S^0,$$

so that in that very similar situation, the colimit of the abutments is the abutment of the colimit of the spectral sequences.

11.4 Extending results of Westerland to $p = 2$ case

In the construction of an invertible spectrum $Z \in \mathrm{Pic}_n$ of [Wes17], a restriction to odd primes ($p \geq 3$) was made. Can this construction, as well as other results from [Wes17] be extended to the case of $p = 2$? Many of the necessary inputs involving the Morava

K-theory of Eilenberg-Mac Lane spaces from [RW80] extend to the case of $p = 2$ by [JW85], with the exception of the global Hopf ring structure on $K(n)_*K(\mathbb{Z}/p^j, *)$, and the results of Hopkins-Lurie in [HL13] on the Morava K-theory of Eilenberg-Mac Lane spaces via Dieudonné theory seems likely to help answer in the affirmative the question of whether $\mathbb{G}_n$ acts on the Morava Modules of [Wes17, 3.21] via the determinant.

11.5 Questions related to generalizations of [HMS94]

Suppose we have an A_∞ spectrum E and a space X with $E_*X = E_*(\Sigma^\infty_+ X) = E_*$ along with a map

$$\xi : X \to BGL_1(S^0).$$

We can form the associated Thom spectrum $M\xi$. If this is E-orientable (in the sense that the Thom spectrum associated to the composite

$$X \xrightarrow{\xi} BGL_1(S^0) \to BGL_1(E)$$

is orientable), then we find via the Thom isomorphism that $E_*(M\xi)$ is an invertible E_*-module. By [BR05], when E is E_∞, there is an injective map

$$\Phi : \mathrm{Pic}(E_*) \to \mathrm{Pic}(E)$$

from invertible E_*-modules to invertible E-module spectra. $\mathrm{Pic}(E)$ is called *algebraic* if Φ is an isomorphism. [MS16, 2.4.7] provides conditions for an E_∞ ring E to have an algebraic Picard group. Following [ABG$^+$13, 2.9], this construction for arbitrary X gives elements of the Picard ∞-groupoid $\mathcal{P}ic(E)$ of invertible E-modules and equivalences between them, with

$$\pi_0 \mathcal{P}ic(E) = \mathrm{Pic}(E)$$

being the Picard group. A few questions arise:

- When $\mathrm{Pic}(E)$ is algebraic, can we construct all elements of $\mathrm{Pic}(E)$ in this way?

- For $E = K(n)$, this construction, combined with an orientation $M\xi \to K(n)$ realizes $K(n)_*(M\xi)$ as an invertible $K(n)_*$-module, which by Theorem 9.2.1 means that $M\xi \in \mathrm{Pic}_n$. Can we find explicit families of such (X, ξ) to give a lower bound on the rank of Pic_n over $\mathbb{Z}_p$ which is greater than 2? (We already have the families topologically generated by $L_{K(n)}S^1$ and $L_{K(n)}S\langle\det\rangle$.) By [RW80, 11.1], for $p > 2$,

$$K(n)_*K(\mathbb{Z}/p^j\mathbb{Z}, q) \cong K(n)_*$$

for $q > n$, $j > 0$. In particular, $K(\mathbb{Z}/p\mathbb{Z}, n+1)$ is a good candidate as a base space. By [JW85], the same holds at $p = 2$.

Remark 11.5.1. Modifying this approach, for $R_n = E_n^{S\mathbb{G}_n^{\pm}}$, being R_n-orientable guarantees that

$$R_n \wedge M\xi \simeq R_n \wedge \Sigma^\infty_+ X$$

so that

$$K(n)_*R_n \otimes_{K(n)_*} K(n)_*(M\xi) \cong K(n)_*R_n \otimes_{K(n)_*} K(n)_*(\Sigma^\infty_+ X).$$

By [Wes17, 3.12], $K(n)_*R_n$ is non-zero. It is therefore faithfully flat over $K(n)_*$ (as all $K(n)_*$-modules are free), and we find

$$K(n)_*(M\xi) \cong K(n)_*(\Sigma^\infty_+ X) \cong K(n)_*,$$

so that $M\xi \in \mathrm{Pic}_n$.

- If this method fails to increase this lower bound on the rank of Pic_n, can we describe conveniently the spectra $M\xi$ in terms of S^1 and $S\langle\det\rangle$? Can we place an upper bound on the rank of Pic_n?

- For other fixed-point spectra $E = E_n^{hG}$, can we produce Thom spectra whose Morava K-theory is readily computable? When do these give elements of Pic_n?

11.6 Questions about descent

For $E = K(n)$ and $E = E_n$, we know that the collection of isomorphism classes of invertible E-local spectra forms a set. Can we compare the groups $\mathrm{Pic}(L_E\mathbf{Sp})$ and $\mathrm{Pic}(L_E S^0)$? For $E = K(n)$, this amounts to asking whether there is a relationship between Pic_n and $\mathrm{Pic}(L_{K(n)}S^0) = \mathrm{Pic}(E_n^{h\mathbb{G}_n})$. By [MS16, §3.3], we do have Galois descent: For G finite and $A \to B$ a faithful G-Galois extension of E_∞ rings (in the sense of [Rog08]),

$$\mathcal{P}\mathrm{ic}(A) \simeq \mathcal{P}\mathrm{ic}(B^{hG}) \simeq \mathcal{P}\mathrm{ic}(B)^{hG},$$

and by Proposition 10.2.12, we have a localized version:

$$\mathrm{Pic}\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\, L_{K(n)}S^0\right) \simeq \mathrm{Pic}\left(\left(\mathrm{Mod}_{L_{K(n)}\mathbf{Sp}}\, E_n^{hU}\right)^{h\mathbb{G}_n/U}\right)$$

for $U \trianglelefteq \mathbb{G}_n$ a finite index normal subgroup.

References

[ABG$^+$08] Matthew Ando, Andrew J Blumberg, David J Gepner, Michael J Hopkins, and Charles Rezk. Units of ring spectra and Thom spectra. *arXiv preprint arXiv:0810.4535*, 2008.

[ABG$^+$13] Matthew Ando, Andrew J Blumberg, David Gepner, Michael J Hopkins, and Charles Rezk. An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology. *Journal of Topology*, 7(3):869–893, 2013.

[Ati61] Michael F Atiyah. Characters and cohomology of finite groups. *Publications Mathématiques de l'IHÉS*, 9:23–64, 1961.

[Bak00] Andrew Baker. I_n-local Johnson-Wilson spectra and their Hopf algebroids. *Doc. Math*, 5:351–364, 2000.

[BB19a] Tobias Barthel and Agnès Beaudry. Chromatic structures in stable homotopy theory. *arXiv preprint arXiv:1901.09004*, 2019.

[BB19b] Tobias Barthel and A Bousfield. On the comparison of stable and unstable p-completion. *Proceedings of the American Mathematical Society*, 147(2):897–908, 2019.

[BBG^{+}22] Agnes Beaudry, Irina Bobkova, Paul G Goerss, Hans-Werner Henn, Viet-Cuong Pham, and Vesna Stojanoska. The exotic $K(2)$-Local Picard Group at the Prime 2. *arXiv preprint arXiv:2212.07858*, 2022.

[BD10] Mark Behrens and Daniel Davis. The homotopy fixed point spectra of profinite Galois extensions. *Transactions of the American Mathematical Society*, 362(9):4983–5042, 2010.

[BF15] Tobias Barthel and Martin Frankland. Completed power operations for Morava E-theory. *Algebraic & Geometric Topology*, 15(4):2065–2131, 2015.

[BK72] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.

[BK73] AK Bousfield and DM Kan. A second quadrant homotopy spectral sequence. *Transactions of the American Mathematical Society*, 177:305–318, 1973.

[Bou79] Aldridge K Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[Bou03] AK Bousfield. Cosimplicial resolutions and homotopy spectral sequences in model categories. *Geometry & Topology*, 7(2):1001–1053, 2003.

[BR05] Andrew Baker and Birgit Richter. Invertible modules for commutative $\mathbb{S}$-algebras with residue fields. *manuscripta mathematica*, 118(1):99–119, 2005.

[BSSW23] Tobias Barthel, Tomer M Schlank, Nathaniel Stapleton, and Jared Weinstein. On the rationalization of the $K(n)$-local sphere. 2023.

[Dav06] Daniel G Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *Journal of Pure and Applied Algebra*, 206(3):322–354, 2006.

[Dev17] Sanath K Devalapurkar. The Lubin-Tate stack and Gross-Hopkins duality. *arXiv preprint arXiv:1711.04806*, 2017.

[DH95] Ethan S Devinatz and Michael J Hopkins. The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts. *American Journal of Mathematics*, 117(3):669–710, 1995.

[DH04] Ethan S Devinatz and Michael J Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[DHS88] Ethan S Devinatz, Michael J Hopkins, and Jeffrey H Smith. Nilpotence and stable homotopy theory I. *Annals of Mathematics*, 128(2):207–241, 1988.

[DL14] Daniel G Davis and Tyler Lawson. A descent spectral sequence for arbitrary $K(n)$-local spectra with explicit E_2-term. *Glasgow Mathematical Journal*, 56(2):369–380, 2014.

[GH05] Paul Goerss and Michael Hopkins. Moduli spaces of commutative ring spectra. *London Mathematical Society Lecture Note Series*, 315:151, 2005.

[GHMR05] Paul Goerss, H-W Henn, Mark Mahowald, and Charles Rezk. A resolution of the $K(2)$-local sphere at the prime 3. *Annals of Mathematics*, pages 777–822, 2005.

[GHMR14] Paul Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk. On Hopkins' Picard groups for the prime 3 and chromatic level 2. *Journal of Topology*, 8(1):267–294, 2014.

[Goe08] Paul G Goerss. Quasi-coherent sheaves on the moduli stack of formal groups. *arXiv preprint arXiv:0802.0996*, 2008.

[Hea15] Drew Heard. Morava modules and the $K(n)$-local Picard group. *Bulletin of the Australian Mathematical Society*, 92(1):171–172, 2015.

[Hea21] Drew Heard. The $\mathrm{Sp}_{k,n}$-local stable homotopy category. *arXiv preprint arXiv:2108.02486*, 2021.

[Hen17] Hans-Werner Henn. A mini-course on Morava stabilizer groups and their cohomology. In *Algebraic Topology*, pages 149–178. Springer, 2017.

[HL13] Michael Hopkins and Jacob Lurie. Ambidexterity in $K(n)$-local stable homotopy theory. *preprint*, 2013.

[HMS94] Michael J Hopkins, Mark Mahowald, and Hal Sadofsky. Constructions of elements in Picard groups. *Contemporary Mathematics*, 158:89–89, 1994.

[Hov93] Mark Hovey. Bousfield localization functors and Hopkins' zeta conjecture. 1993.

[HS98] Michael J Hopkins and Jeffrey H Smith. Nilpotence and stable homotopy theory II. *Annals of mathematics*, 148:1–49, 1998.

[HS99a] Mark Hovey and Hal Sadofsky. Invertible spectra in the $E(n)$-local stable homotopy category. *Journal of the London Mathematical Society*, 60(1):284–302, 1999.

[HS99b] Mark Hovey and Neil P Strickland. *Morava K-theories and localisation*, volume 666. American Mathematical Soc., 1999.

[JW75] David Copeland Johnson and W Stephen Wilson. BP operations and Morava's extraordinary K-theories. *Mathematische Zeitschrift*, 144(1):55–75, 1975.

[JW85] David Copeland Johnson and W Stephen Wilson. The Brown-Peterson homology of elementary p-groups. *American Journal of Mathematics*, 107(2):427–453, 1985.

[Kar10] Nasko Karamanov. On Hopkins' Picard group Pic_2 at the prime 3. *Algebraic & Geometric Topology*, 10(1):275–292, 2010.

[Kra08] Henning Krause. Localization theory for triangulated categories. *arXiv preprint arXiv:0806.1324*, 2008.

[Kuh87a] Nicholas J Kuhn. The mod p K-theory of classifying spaces of finite groups. *Journal of Pure and Applied Algebra*, 44(1-3):269–271, 1987.

[Kuh87b] Nicholas J Kuhn. The Morava K-theories of some classifying spaces. *Transactions of the American Mathematical Society*, 304(1):193–205, 1987.

[Lad13] Olivier Lader. *Une résolution projective pour le second groupe de Morava pour $p \geq 5$ et applications*. PhD thesis, Université' de Strasbourg, 2013.

[Law19] Tyler Lawson. An introduction to Bousfield localization. 2019.

[Laz55] Michel Lazard. Sur les groupes de lie formels à un paramètre. *Bulletin de la Société Mathématique de France*, 83:251–274, 1955.

[Les95] Kathryn Lesh. Hybrid spaces with interesting cohomology. *Transactions of the American Mathematical Society*, 347(9):3247–3262, 1995.

[Lur10] Jacob Lurie. Chromatic homotopy theory. *Lecture notes online*, 2010.

[Lur17] Jacob Lurie. Higher Algebra, 2017.

[Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Advances in Mathematics*, 291:403–541, 2016.

[May77] J Peter May. E_∞ ring spaces and E_∞ ring spectra. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. *Lecture Notes in Mathematics*, 577, 1977.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*. Number 58. Cambridge University Press, 2001.

[Mit97] Stephen A Mitchell. Hypercohomology spectra and Thomason's descent theorem. *Algebraic K-Theory (Toronto, ON, 1996)*, 16:221–277, 1997.

[MLC+96] J Peter May, LG Lewis, M Cole, G Comezana, S Costenoble, AD Elmendorf, and JPC Greenlees. *Equivariant homotopy and cohomology theory: Dedicated to the memory of Robert J. Piacenza*. Number 91. American Mathematical Soc., 1996.

[MS16] Akhil Mathew and Vesna Stojanoska. The Picard group of topological modular forms via descent theory. *Geometry & Topology*, 20(6):3133–3217, 2016.

[Pst18] Piotr Pstrągowski. Chromatic Picard groups at large primes. *arXiv preprint arXiv:1811.05415*, 2018.

[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75(6):1293–1298, 11 1969.

[Qui71] Daniel Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Mathematics*, 7(1):29–56, 1971.

[Rav76] Douglas C Ravenel. The structure of Morava stabilizer algebras. *Inventiones mathematicae*, 37(2):109–120, 1976.

[Rav84] Douglas C Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106(2):351–414, 1984.

[Rav92] Douglas C Ravenel. *Nilpotence and periodicity in stable homotopy theory*. Princeton University Press, 1992.

[Rav03] Douglas C Ravenel. *Complex cobordism and stable homotopy groups of spheres*. American Mathematical Soc., 2003.

[Rog08] John Rognes. *Galois Extensions of Structured Ring Spectra/Stably Dualizable Groups: Stably Dualizable Groups*, volume 192. American Mathematical Soc., 2008.

[RW80] Douglas C Ravenel and W Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *American Journal of Mathematics*, 102(4):691–748, 1980.

[Ser51] Jean-Pierre Serre. Homologie singulière des espaces fibrés. *Ann. of Math*, 54(2):425–505, 1951.

[Ser79] Jean-Pierre Serre. *Galois Cohomology*. Springer, 1979.

[SR05] Dennis Parnell Sullivan and Andrew Ranicki. *Geometric Topology: Localization, Periodicity and Galois Symmetry: the 1970 MIT Notes*, volume 8. Springer, 2005.

[Sta63] James Stasheff. A classification theorem for fibre spaces. *Topology*, 2(3):239–246, 1963.

[Str98] Neil P Strickland. Morava E-theory of symmetric groups. *arXiv preprint math/9801125*, 1998.

[Wes17] Craig Westerland. A higher chromatic analogue of the image of J. *Geometry & Topology*, 21(2):1033–1093, 2017.

[Wil98] John S Wilson. *Profinite Groups*, volume 19. Clarendon Press, 1998.

[Wür77] Urs Würgler. On products in a family of cohomology theories associated to the invariant prime ideals of BP. *Commentarii mathematici Helvetici*, 52:457–482, 1977.

[Wür91] Urs Würgler. Morava K-theories: A survey. In *Algebraic Topology Poznań 1989*, pages 111–138. Springer, 1991.