ON THE SENSITIVITY OF SOLUTIONS OF HYPERBOLIC EQUATIONS TO THE COEFFICIENTS

Gang Bao
Department of Mathematics
University of Florida
Gainesville, FL 32611

William W. Symes
Department of Computational & Applied Mathematics
Rice University
Houston, Texas 77251-1892

Abstract

The goal of this work is to determine appropriate domain and range of the map from the coefficients to the solutions of the wave equation for which its linearization or formal derivative is bounded and the properties of the coefficients on which the bound depends. Such information is indispensable in the study of the inverse (coefficient identification) problem *via* smooth optimization methods. The main result of this paper is an explicit microlocal Sobolev estimate for the linearized forward map. In view of results of Rakesh [19] for the smooth coefficient case, the order of our regularity result is optimal.

Our proof is based on the method of nonsmooth microlocal analysis, in particular various results on propagation of singularities, the method of progressing wave expansions, microlocal study of solutions of the transport equations, study of conormal properties of the fundamental solution, and a duality technique.

1 Introduction

Linear acoustic wave equation governs many physical processes such as seismic and acoustic wave propagation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - \nabla\sigma \cdot \nabla\right) u = f \ . \tag{1.1}$$

Here $\sigma = \sigma(x)$ is the logarithm of the density, $c = c(x)$ is the sound speed of the medium, and $f = f(x,t)$ is the source term which introduces the energy to the problem. If σ, c and f are given along with appropriate side conditions, the forward (or direct) problem is to determine $u = u(x,t)$, the excess pressure. For appropriate choices of σ, c, and f, u is determined uniquely by standard linear hyperbolic theory of partial differential equations (*p.d.e.*). Thus the problem stated above defines a map from the coefficients to the solution of the wave equation. In this paper, we study an aspect of the *regularity* of this map, and especially of its composition with the trace on a time-like hypersurface.

Throughout this work we shall restrict ourselves to the special case of constant velocity c. We believe that the ideas in this work may be extended to cover some more general cases.

To fix ideas, write $x \in \mathbf{R}^n$ as (x', x_n), where $x' \in \mathbf{R}^{n-1}$, $x_n \in \mathbf{R}$. We assume that the problem is set in the whole space $\mathbf{R}^n$ and $u = 0$ in the past ($t < 0$). Take $f(x,t) = \delta(x,t)$ as an ideal point source. Thus u is the retarded fundamental solution:

$$\Box u - \nabla\sigma \cdot \nabla u = \delta(x,t)\,, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}\,, \tag{1.2}$$

$$u = 0\,, \quad t < 0\,, \tag{1.3}$$

where $\Box$ is defined to be $\partial_t^2 - \Delta$, and Δ is the Laplacian.

It is some what easier to understand the sensitivity of the solution to distant perturbations of the coefficients. Thus we will assume that σ and its perturbations are supported in the half space $\{x_n > 0\}$ and study the solution near the boundary $\{x_n = 0\}$.

Define the *forward map* F as:

$$F : \sigma \to (\phi u)\,|_{x_n=0}\,, \tag{1.4}$$

where $\phi \in C_0^\infty(\mathbf{R}^{n+1})$ is supported inside the conoid $\{t > |x|\}$ and near $\{x_n = 0\}$.

F is nonlinear. As a first step toward understanding the regularity of F, we study the formal linearization (or formal derivative) DF, with respect to the reference state (σ_0, u_0). The first order perturbation theory gives, for a small change $\delta\sigma$, the following problem for the resulting change δu in u:

$$\Box\delta u - \nabla\sigma_0 \cdot \nabla\delta u = \nabla\delta\sigma \cdot \nabla u_0\,, \tag{1.5}$$

$$\delta u = 0\,, \quad t < 0\,. \tag{1.6}$$

The formal derivative $DF(\sigma_0)$ is given by

$$DF(\sigma_0)\delta\sigma = (\phi\delta u)\,|_{x_n=0}\,. \tag{1.7}$$

It is our main goal in this work to determine appropriate spaces of the domain and range of F for which

the formal derivative DF is bounded.

We obtain:

Theorem 1.1 *Assume $l > 3/2$, and that $s > l + 9/2$ for $n = 2$, $s > l + 3n/2 + 1$ for $n \geq 3$. Then for σ_0, $\delta\sigma \in C_0^\infty(\{x_n > 0\})$,*

$$||DF(\sigma_0)\delta\sigma||_l \leq C||\delta\sigma||_{l+\frac{n-1}{2}}\,, \tag{1.8}$$

where the constant C depends on the $||\sigma_0||_s$ and the support of ϕ, but is independent of $\delta\sigma$.

The study of the forward map is motivated by the *inverse problem* which arises in reflection seismology, oil exploration, ground-penetrating radar, etc. A highly over simplified version of the inverse problem is to determine the coefficient σ by knowing additional boundary value conditions of u. Since the inverse problem is just to invert the functional relation F, we are naturally interested in all the properties of this forward map.

To understand the problem, let us look at a simple exploration seismology experiment: Near the surface of the earth, a seismic source is fired at some point (point source). The seismic waves propagate into the earth. Since the earth's structure varies (as do its physical properties) part of the energy of the wave will be reflected back to the surface and can be measured. The inverse problem is to deduce the interior properties of the earth from the recorded data.

A simple model of this reflection seismic inverse problem in this context is: given data $F_{data}(x', t)$, find a coefficient $\sigma(x)$ so that

$$F(\sigma) = F_{data}$$

or perhaps minimizing the error $(F_{data} - F(\sigma))$ in some norm.

Numerical solution of this problem by means of Newton's method and its relatives, such as the quasi-Newton, conjugate gradient, and variable metric methods, requires a choice of Banach space structure in the space of models σ and in the space of data $F(\sigma)$ (see e.g. Kantorovich and Akilov [16]), in such a way that F is regular. This fact accounts for our reliance on the L^2-based Sobolev spaces in this work. The simplest regularity property of F is boundedness of DF, which is discussed in this paper. We believe that similar arguments will establish smoothness of F and allow investigation of coercive properties of DF, as is required by the theory of optimization.

When the spatial dimension is one or c and σ depend only on x_n (layered problem) there is a large literature available. For a similar problem in which the medium was assumed to be excited by an impulsive load on the surface $\{x_n = 0\}$ instead of point sources, the properties of the forward map have been studied fairly satisfactorily by Symes and others (see Symes [26] for references). It was shown by Symes that, in the constant wave speed case, the forward map defines a C^1-*diffeomorphism* between open sets in certain Hilbert spaces by applying the method of geometrical optics together with energy estimates.

When the spatial dimension $n > 1$ and c, σ depend on all space variables (nonlayered problem), very little is known in mathematics. Symes [24, 25], Sacks and Symes [22], Rakesh [19], and Sun [23] have some partial results. The difficulties are essentially due to the ill-posed nature of the timelike hyperbolic Cauchy problem and the presence of nonsmooth coefficients. For the one dimensional wave equation, both coordinate directions are spacelike, which indicates that the problem is hyperbolic with respect to both directions. Apparently, this is not the case when the spatial dimension is larger than one.

Rakesh in [19] studied a related linearized velocity inversion problem with constant density and point sources. Assuming smooth background velocity, he obtained both upper and lower bounds for the linearized forward map. The essential observation in Rakesh's work is that DF is a Fourier integral operator (see also Beylkin [7]). The calculus of Fourier integral operators employed in Rakesh's work is not applicable to the nonsmooth reference velocity case since the linearized forward map is a Fourier integral operator only when the reference velocity is smooth. Nonetheless, the regularity estimate for DF in Theorem 1.1 (loss of $(n-1)/2$ derivatives) is exactly the same as that proved in [19], and is optimal.

In [24], Symes gave a pair of examples, based on the geometric optics construction, which show that both $DF(1)$ and $DF(1)^{-1}$ are unbounded for a slightly different problem. As the examples show, within the Sobolev scales no strengthening or weakening of topologies of the domain and range can make both DF and DF^{-1} bounded. This fact also implies a strategy of regularization: Change the topology in the domain so that DF becomes bounded, then ask for optimal regularization of DF^{-1} in the sense of best possible lower bound estimate for DF. In both examples of Symes, the unboundedness was caused by rapid oscillation of σ in the x'-direction or the tangential directions, hence the problem is actually "partially well-posed", i.e., only more smoothness of the coefficients in tangential directions (essentially grazing ray directions) will be required to cure the difficulty. For this reason, the results of [22] and [23] were formulated using the anisotropic Sobolev spaces $H^{m,s}(\mathbf{R}^n)$ or Hörmander spaces.

In Theorem 4.1 of [22], Sacks and Symes showed by using the method of sideways energy estimates that for a linearized density determination problem with constant velocity and plane wave sources, DF is bounded from $H^{1,1}$ to H^1, provided that the reference coefficient is in $H^{1,s}$ for some $s > n+2$. They also proved the injectivity of DF. An extension of their reasoning shows that DF is bounded from $H^{l,1}$ to H^l provided that σ is in $H^{l,s}$ for $s > n+2$. Since $H^{l,s} \subset H^{l+s}$ and $H^{l,s} \not\subset H^q$ for $q < l+s$, the regularity condition on σ_0 in Theorem 1.1 is compatible with that of [22]. The bounds on DF are compatible as well, allowing for the difference between plane wave and point sources. Our method is completely different from theirs. In particular, we believe that our method could be extended to study the velocity inversion problem, *i.e.*, to determine $c(x)$ when the density is a known constant. In fact, we have recently obtained several results in [3] that would be necessary to solve this more difficult problem.

Even though the coefficients are always assumed to be smooth in this paper, our method works equally well in the case of nonsmooth coefficients. In particular, the method does not require the reference density σ_0 to be smooth. Regularity results and some simple interpolation arguments should yield a proof to the nonsmooth case. A detailed study of this and results on continuity and differentiability of the forward map will be reported elsewhere.

Throughout, C serves as a generalized positive constant the precise value of which is not needed.

2 Preliminaries

In this section, we state some basic results that will be frequently used in this work. Only those relatively new results will be proved.

The first was originally established by Bony [8] and was extended by Meyer [17]. See also Beals [4] for a different proof.

Proposition 2.1 *Suppose that for some $(x_0, \xi_0) \in T^*(\mathbf{R}^n)\backslash 0$, $u \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$, $n/2 < s \le r \le 2s - n/2$, and $g \in C^\infty$, then*

$$g(x, u) \in H^s \cap H^r_{m\ell}(x_0, \xi_0) \, .$$

The next is an algebraic property of the classic Sobolev spaces, whose proof may be found in [2] or [4].

Proposition 2.2 *(Generalized Schauder's Lemma) If* $u \in H^{s_1}(\mathbf{R}^n)$ *and* $v \in H^{s_2}(\mathbf{R}^n)$, *with* $s_1 + s_2 \geq 0$. *Then*

$$uv \in H^{min(s_1,s_2,s_1+s_2-n/2+\delta)} \quad \textit{for any } \delta > 0 \ .$$

We need the following standard result for hyperbolic *p.d.e*, as well as the estimates involving in its proof. See, for example, Chazarain-Piriou [9] for the idea of the proof. The following is the version stated in Beals [5].

Lemma 2.1 *(Linear Energy Inequality) Let* $p(x, D)$ *be a partial differential operator of order* m *on* $\mathbf{R}^{n+1}$, *strictly hyperbolic with respect to the plane* $\{x_{n+1} = 0\}$, *and let* u *satisfy* $p(x, D)u = f(x)$. *If* $f \in H^{s-m+1}_{loc}(\mathbf{R}^{n+1})$ *and* $u \in H^s_{loc}(x : |x_{n+1}| \leq \epsilon)$ *for some* $\epsilon > 0$, *then* $u \in H^s_{loc}(\mathbf{R}^{n+1})$.

Concerning the microlocal ellipticity, the following Gårding type inequality is very useful.

Lemma 2.2 *Assume that* $Q_1 \in OPS^{m_1}$, $Q_2 \in OPS^{m_2}$, *with* $m_1, m_2 \in \mathbf{R}$. *Furthermore assume that* Q_2 *is elliptic on* $ES(Q_1)$. *Then for any* $r \in \mathbf{R}$, Ω *and* Ω' *two open bounded sets of* $\mathbf{R}^n$ *with* $\Omega \subset\subset \Omega'$, *and* $u \in C_0^\infty(\mathbf{R}^n)$,

$$||Q_1 u||_{s,\Omega} \leq C||Q_2 u||_{s+m_1-m_2,\Omega'} + C||u||_r \ .$$

Proof. Let Ω_1 and Ω_2 be open sets with $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega'$. Construct a cut-off function $\phi \in C_0^\infty(\Omega_2)$ with $\phi = 1$ on Ω_1.

The assumption Q_2 is elliptic on $ES(Q_1)$ implies that a $\psi.d.o$ R, a parametrix of Q_2 on $ES(Q_1)$, may be found such that

$$Q_1 R Q_2 = Q_1 + K \tag{2.1}$$

with K a smoothing operator.

Having defined ϕ, we can now rewrite

$$Q_1 R Q_2 u = Q_1 R \phi Q_2 u + Q_1 R(1-\phi) Q_2 u \ .$$

It follows that, for any r,

$$\begin{aligned} ||Q_1 R Q_2 u||_{s,\Omega} &\leq ||Q_1 R \phi Q_2 u||_{s,\Omega} + ||Q_1 R(1-\phi)Q_2 u||_{s,\Omega} \\ &\leq C||\phi Q_2 u||_{s+m_1-m_2,\mathbf{R}^n} + C||Q_2 u||_r \\ &\leq C||Q_2 u||_{s+m_1-m_2,\Omega'} + C||u||_r \ . \end{aligned}$$

where to obtain the second inequality, we have used that $\phi = 1$ on $\Omega_1 \supset\supset \Omega$.

It is obvious, from (2.1),

$$||Q_1 R Q_2 u||_{s,\Omega} \geq ||Q_1 u||_{s,\Omega} - ||Ku||_{s,\Omega} \ .$$

The proof is then complete. □

The next proposition is also concerned with the microlocal ellipticity, whose proof may be given as a simple exercise of the calculus of $\psi.d.o.s$. We shall leave it to the reader.

Proposition 2.3 *Let A and B be ψ.d.o.s of the same oredr. Suppose that B is elliptic on $ES(A)$, then there exists $C \in OPS^0$, such that*

$$A = CB + smoothing\ operators$$

or

$$A = BC + smoothing\ operators\,.$$

With the presence of nonsmooth coefficients, one always has to deal with the product of two distributions of limited regularity and the commutator of an operator with a function with limited smoothness. Here we state an extended Rauch's lemma and two commutator lemmas that established in [2].

Lemma 2.3 *Suppose that $\Omega = \Omega_0 \times \Omega_1 \subset\subset \mathbf{R}^{n_0} \times \mathbf{R}^{n-n_0}$ $(1 \le n_0 \le n)$, $\Omega' \subset\subset \Omega$, and $n_0/2 < s$, $0 \le l \le s$, q and $q < l + s - n_0/2$. Suppose that $Q \in S^0(\Omega')$, $\tilde{Q} \in S^0(\Omega)$ elliptic on $ES(Q)$, and $Q_0 \in S^0(\Omega_0)$ satisfies that: $(x, y, \xi, \eta) \in ES(\tilde{Q}), \xi \ne 0 \Longrightarrow (x, \xi) \in ES(Q_0)$. Then there exists a constant $C > 0$ so that for $u \in C_0^\infty(\Omega_0)$ and $v(x, y) \in C_0^\infty(\Omega)$,*

$$||Quv||_{q,\Omega'} \le C(||u||_{s,\Omega_0} + ||Q_0 u||_{q,\Omega_0})(||v||_{l,\Omega} + ||\tilde{Q}v||_{q,\Omega}).$$

Remark. If $n_0 = n$, the lemma implies the original Rauch's lemma in [20].

Lemma 2.4 *Suppose that $\Omega' \subset\subset \Omega \subset\subset \mathbf{R}^n$, and assume $p_1(x, D_x) \in OPS^1$, $b_0(x, D_x) \in OPS^0$ are properly supported, $s > n/2 + 1$, $0 \le l \le s$, and $a \in H^s_{loc}$. There exists $C > 0$ so that for $v \in H^l(\Omega)$,*

$$||[b_0, ap_1]v||_{l,\Omega'} \le C||v||_{l,\Omega)}\,.$$

Lemma 2.5 *Suppose that $\Omega = \Omega_0 \times \Omega_1 \subset\subset \mathbf{R}^{n_0} \times \mathbf{R}^{n_1}$, $\Omega' \subset\subset \Omega$, $p_1 \in S^1(\Omega)$, $b_0 \in S^0(\Omega)$, $a \in H^s(\Omega_0)$ with $s > n_0/2 + 1$, $0 \le l \le s$, $r < l + s - n_0/2 - 1$. Suppose that $q_0 \in S^0(\Omega_0)$ and $q_0(x, D_x)a \in H^r(\Omega_0)$. Suppose that $q \in S^0(\Omega)$ satisfies: $(x, y, \xi, \eta) \in ES(q)$, $\xi \ne 0 \Longrightarrow (x, \xi) \in ES(q_0)$, and suppose $b_0' \in S^0(\Omega)$ so that q is elliptic on $ES(b_0')$. Then there exists $C > 0$ so that for $v \in C_0^\infty(\Omega)$,*

$$\begin{aligned} ||[b_0, ap_1]v||_{l,\Omega'} &\le C||v||_{l,\Omega}\,, \\ ||b_0'[b_0, ap_1]v||_{r,\Omega'} &\le C(||v||_{l,\Omega} + ||qv||_{r,\Omega})\,. \end{aligned}$$

Remark. If $n_0 = n$ then the condition relating q, b_0' is simpler: $q = q_0$ and b should be elliptic on $ES(q_0)$.

3 Propagation of Regularity

In this section, we shall derive an estimate out of Hörmander's theorem on propagation of singularities. The main results concerns regularity in t direction of the fundamental solution for the wave equation. Recall that u_0 solves the model problem (1.2) and (1.3) for $\sigma = \sigma_0$.

Theorem 3.1 *Suppose that* $1+n/2<s$ *and* $\sigma_0 \in H^s(\mathbf{R}^n)$. *Then for* $l<s-n+1/2$

$$\partial_t^l u_0 \in L^2_{loc}(U) ,$$

where $U=\{\mathbf{R}^n \times (0,T_1)\} \cap \{t>|x|\}$ $(T_1>0)$. *And for* $\phi \in C_0^\infty(U)$, *the following estimate holds*

$$||\phi \partial_t^l u_0|| \le C , \tag{3.1}$$

where the constant C *depends on* ϕ *and* $||\sigma_0||_s$.

3.1 Propagation of regularity with estimates

In order to establish Theorem 3.1, we need the following results. Lemma 3.1 gives an estimate based on Nirenberg's proof [18] of Hörmander's theorem which describes the propagation of regularity along bicharacteristics. With nonsmooth coefficients, only a limited amount of regularity propagates. It indicates that an estimate may be derived near any bicharacteristic, hence near the characteristic variety of operator $\Box = \partial_t^2 - \Delta$. We then proceed in Lemma 3.2 to argue that in the elliptic region of the operator $\Box$ an estimate may also be formed.

Let $\Pi : T^*(\Omega_0) \to \Omega_0$ denote the projection of $T^*(\Omega_0)$ onto its base space.

Lemma 3.1 *Suppose that* ψ, $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $s>n/2$, $k<s+2-n/2$, $\sigma_0 \in H^{s+1}_{comp}(\mathbf{R}^n)$, β *is a null bicharacteristic strip for* $\Box$, *and* $\Omega \subset\subset \mathbf{R}^{n+1}$ *satisfies* $\Pi\beta \cap \bar{\Omega} = \emptyset$. *Then there exist* $B \in OPS^0$ *with* $ES(B)$ *supported in an arbitrarily small conic neighborhood of* β *and* $C>0$ *so that any* $w \in H^1_{loc}(\mathbf{R}^{n+1})$ *vanishing for large* t *and satisfying*

$$\Box w - \nabla\sigma_0 \cdot \nabla w = f \in L^2(\Omega),$$

satisfies in addition

$$||\psi B \phi w||_k \le C||f||_0.$$

Proof. The standard energy estimate for the (variable-coefficient) wave equation implies that for any $T \in \mathbf{R}$

$$||w||_{H^1([T,\infty)\times\mathbf{R}^n)} \le C||f||_{L^2(\Omega)},$$

where C depends on σ_0 and on T.

Select T such that

$$t<T \Rightarrow (x,t) \notin \Omega \cup supp(\phi) \cup \text{supp } (\psi) .$$

Select $\phi_1 \in C^\infty(\mathbf{R})$ so that

$$\phi_1(t) = \begin{cases} 1, & t>T-k, \\ 0, & t<T-k-1 . \end{cases}$$

Then $w_1 = \phi_1 w \in H^1_{comp}(\mathbf{R}^{n+1})$ satisfies

$$\Box w_1 - \nabla\sigma_0 \cdot \nabla w_1 = f_1 \in L^2_{comp}(\mathbf{R}^{n+1})$$

with

$$||f_1||_0 \leq C||f||_0$$

so

$$||w_1||_1 \leq C||f||_0$$

and moreover $f_1 \equiv f$ for $t \geq T-k$.

The construction described in Nirenberg [18], pp. 44-45, produces a sequence $\{B_i\} \subset OPS^0$, $j = 1, \cdots, 2k-2$, having the following properties:

(1) $ES(B_j)$ is contained in a small conic neighborhood of β, and in particular $\Pi ES(B_j) \cap \bar{\Omega} = \emptyset$, $j = 1, \cdots, 2k-1$;

(2) $[\Box, B_j] \in OPS^0$;

(3) B_j is elliptic on $ES(B_{j+1})$, $j = 1, \cdots, 2k-2$.

Let $T_1, R > 0$ be so large that

$$[T, T_1] \times B_R(\mathbf{R}^n) \supset \Omega \cup \text{supp } (\phi) \cup \text{supp } (\psi),$$

where $B_R(\mathbf{R}^n)$ is the ball in $\mathbf{R}^n$ centered at the origin of radius R. Also make $R > 0$ large enough that the support of w does not intersect the cylinder $[T-k+1, \infty) \times \partial B_R$.

Set

$$\Omega_j = [T-k+j+1, T_1+k-j] \times B_{R+k-j}(\mathbf{R}^n)$$

for $j = [0, k]$.

Let $A_j = B_{2j-1}B_{2j}$, $j = 1, \cdots, k-1$, and $A_0 = I$. We claim that

$$||\frac{\partial^j}{\partial t^j} A_j w_1||_{1,\Omega_j} \leq C||f||_0 \ , \quad j = 0, \cdots, k-1 \ .$$

For $j = 0$, this just the consequence of the standard energy estimate stated above. Thus assume the conclusion for $j \leq k-2$.

We may assume that the essential support of the B_j (hence of the A_j) is so small that ∂/∂_t is elliptic there. Thus Lemma 2.2 implies

$$\begin{aligned} ||B_{2j+1}w_1||_{j+1,\Omega_{j+1/2}} &\leq C||\partial^j/\partial t^j A_j w_1||_{1,\Omega_j} + C||w_1||_1 \\ &\leq C||f||_0, \end{aligned}$$

by the induction hypothesis.

On the other hand,

$$(\Box - \nabla\sigma_0 \cdot \nabla)B_{2j+1}w_1 = [\Box, B_{2j+1}]w_1 + [B_{2j+1}, \nabla\sigma_0 \cdot \nabla]w_1 + B_{2j+1}f_1 \ .$$

By construction, $B_{2j+2}[\Box, B_{2j+1}] \in OPS^0$; moreover B_{2j+1} is elliptic on $ES(B_{2j+2}[\Box, B_{2j+1}])$, so another application of Lemma 2.2 gives

$$\begin{aligned} ||\frac{\partial^{j+1}}{\partial t^{j+1}} B_{2j+2}[\Box, B_{2j+1}]w_1||_{0,\Omega_{j+1}} &\leq C||B_{2j+1}w_1||_{j+1,\Omega_{j+1/2}} + C||w_1||_1 \\ &\leq C||f||_0. \end{aligned}$$

The microlocal version of the Commutator Lemma (Lemma 2.5) gives

$$\begin{aligned} ||B_{2j+2}[\nabla\sigma_0\cdot\nabla, B_{2j+1}]w_1||_{j+1,\Omega_{j+1}} &\leq C(||w_1||_{1,\Omega_{j+1/2}} + ||B_{2j+1}w_1||_{j+1,\Omega_{j+1/2}}) \\ &\leq C||f||_0 . \end{aligned}$$

The correspondence with the items in the statement of the Commutator Lemma is:

Lemma 2.5	**Here**
Ω	$\Omega_{j+1/2}$
Ω'	Ω_{j+1}
v	w_1
p_1	∇
a	$\nabla\sigma_0$
s	s
l	1
r	$j+1 < s-n/2$
n_0	n
q	B_{2j+1}
q_0	I
b_0	B_{2j+1}
b_0'	B_{2j+2}.

Thus, from Lemma 2.5

$$||B_{2j+2}[\nabla\sigma_0\cdot\nabla, B_{2j+1}]w_1||_{j+1,\Omega_{j+1}} \leq C(||w_1||_1 + ||B_{2j+1}w_1||_{j+1,\Omega_{j+1/2}}),$$

provided that $j+1 < s-n/2$.

Since we are only considering $j+1 \leq k-1$ and have assumed $k < s-n/2+1$, we obtain the conclusion above.

As noted above $f_1 = f$ for $t > T-k$, in particular $f_1 = f$ on Ω_j, $j \in [0, k-1]$. Since

$$B_{2j+1}f_1 = B_{2j+1}f + B_{2j+1}(f_1 - f) ,$$

$\Pi ES(B_{2j+1})$ is disjoint from $\Omega \supset supp(f)$, and Ω_j is disjoint from $supp(f_1-f)$, it follows that

$$||B_{2j+1}f_1||_{j+1,\Omega_{j+1}} \leq C||f||_0.$$

On the other hand,

$$\begin{aligned} ||[B_{2j+2}, \Box]B_{2j+1}w_1||_{j+1,\Omega_{j+1}} &\leq C||B_{2j+1}w_1||_{j+1,\Omega_{j+1/2}} + C||w_1||_{1,\mathbf{R}^n} \\ &\leq C||f||_0, \end{aligned}$$

$$||[\nabla\sigma_0\cdot\nabla, B_{2j+2}]B_{2j+1}w_1||_{j+1,\Omega_{j+1}} \leq C||B_{2j+1}w_1||_{j+1,\Omega_{j+1/2}} \leq C||f||_0,$$

so we obtain ($A_{j+1} = B_{2j+2}B_{2j+1}$)

$$||(\Box - \nabla\sigma_0\cdot\nabla)A_{j+1}w_1||_{j+1,\Omega_{j+1}} \leq C||f||_0.$$

In particular,

$$||(\Box - \nabla\sigma_0 \cdot \nabla)\frac{\partial^{j+1}}{\partial t^{j+1}} A_{j+1} w_1||_{0,\Omega_{j+1}} \le C||f||_0.$$

Since both the hyperplane $\{t = T_1 + k - j - 1\}$ and the cylinder boundary $\partial\Omega_{j+1}$ of the set $[T - k + j + 2, T_1 + k - j - 1] \times \partial B_{R+k-j-1}$ are disjoint from the support of w_1, for every $r \in \mathbf{R}$ the H^r norms of the traces of $\partial_t^{j+1} A_{j+1} w_1$ on these sets, and the traces of all derivatives, are bounded by multiples of $||w_1||_1 \le C||f||_0$. So we can regard $\frac{\partial^{j+1}}{\partial_t^{j+1}} A_{j+1} w_1$ as the solution of a mixed problem for $\Box - \nabla\sigma_0 \cdot \nabla$ in Ω_{j+1}, with Cauchy data at $t = T_1 + k - j - 1$ and Dirichlet data, say, on the cylinder boundary, and all of this data is smooth, *i.e.*, is bounded in H^r in terms of $||f||_0$. According to the standard energy estimate,

$$||\frac{\partial^{j+1}}{\partial t^{j+1}} A_{j+1} w_1||_{1,\Omega_{j+1}} \le C||f||_0,$$

which establishes the induction step.

Now set $B = B_{2k-1}$. Note that $\phi_1 = 1$ on $supp(\phi)$

$$B\phi w = \phi B w_1 + [B, \phi] w_1 \ .$$

By construction, A_{k-1} is elliptic on $ES(B)$ and $ES([B, \phi])$, hence by Lemma 2.2

$$||\psi B\phi w||_k \le C||\frac{\partial^{k-1}}{\partial t^{k-1}} A_{k-1} w_1||_{1,\Omega_{k-1}} + C||w_1||_1 \le C||f||_0 \ .$$

□

Corollary 3.1 *Suppose that* ψ, $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $s > n/2$, $k < s - n/2 + 2$, $\sigma_0 \in H^{s+1}_{comp}(\mathbf{R}^n)$. *Suppose that* γ *is a set of null bicharacteristic strips for* $\Box$, *and* $\Omega \subset\subset \mathbf{R}^{n+1}$ *satisfies* $\Pi\gamma \cap \bar{\Omega} = \emptyset$. *Then there exists* $Q \in OPS^0$ *with* $ES(Q)$ *supported in an arbitrarily small conic neighborhood of* γ *and* $C > 0$ *so that for* $w \in H^1_{loc}(\mathbf{R}^{n+1})$ *vanishing for large* t *and satisfying*

$$\Box w - \nabla\sigma_0 \cdot \nabla w = f \in L^2(\Omega)$$

satisfies in addition

$$||\psi Q\phi w||_k \le C||f||_0 \ .$$

Proof. For every null bicharacteristic strip of the set γ, Lemma 3.1 indicates that a ψ.*d.o.* B of order zero may be found so that

$$||\psi B\phi w||_k \le C||f||_0 \ .$$

Now Q may be constructed as $Q = \sum B$. Moreover, the local compactness of the unit sphere ensures that the summation is finite. □

Lemma 3.2 *Suppose that ψ, $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $s > n/2$, $k < s - n/2 + 2$, $\sigma_0 \in H^{s+1}_{comp}(\mathbf{R}^n)$, P is a ψ.d.o. of order zero such that a conic neighborhood of its essential support is contained in the microlocal elliptic region of $\Box$, and $\Omega \subset\subset \mathbf{R}^{n+1}$ satisfies $\Pi P \cap \bar{\Omega} = \emptyset$. Then there exists a constant $C > 0$ so that any $w \in H^1_{loc}(\mathbf{R}^{n+1})$ vanishing for large t and satisfying*

$$\Box w - \nabla\sigma_0 \cdot \nabla w = f \in L^2(\Omega)$$

satisfies

$$||\psi P \phi w||_k \leq C||f||_0$$

where the constant C depends on σ_0, k, P, ϕ, and ψ, but not on w.

Proof. The proof is based on the same type of arguments as in the proof of last lemma.

Once again, select T such that

$$t < T \Rightarrow (x,t) \notin \Omega \cup supp(\phi) \cup \text{supp } (\psi) \ .$$

Select $\phi_1 \in C^\infty(\mathbf{R})$ so that

$$\phi_1(t) = \begin{cases} 1, & t > T - k, \\ 0, & t < T - k - 1 \ . \end{cases}$$

Then $w_1 = \phi_1 w \in H^1_{comp}(\mathbf{R}^{n+1})$ satisfies

$$\Box w_1 - \nabla\sigma_0 \cdot \nabla w_1 = f_1 \in L^2_{comp}(\mathbf{R}^{n+1}) \tag{3.2}$$

with

$$||f_1||_0 \leq C||f||_0$$

so

$$||w_1||_1 \leq C||f||_0,$$

and moreover $f_1 \equiv f$ for $t \geq T - k$.

Let $T_1, R > 0$ be so large that

$$[T, T_1] \times B_R(\mathbf{R}^n) \supset \Omega \cup \text{supp } (\phi) \cup \text{supp } (\psi),$$

where $B_R(\mathbf{R}^n)$ is the ball in $\mathbf{R}^n$ centered at the origin of radius R.

Set

$$\Omega_j = [T - j + 1, T_1 + j] \times B_{R+j}(\mathbf{R}^n)$$

for $j = [0, k]$.

Now since $\Box$ is elliptic in a small conic neighborhood of $ES(P)$, we can construct a sequence of $\psi.d.o.$ $\{P_i\} \in OPS^0$, $i = 0, 1, \cdots, 2k-2$, such that:

(1) $\Box$ is elliptic in a small conic neighborhood of $ES(P_i)$, and $\Pi ES(P_i) \cap \bar{\Omega} = \emptyset$, $i = 0, 1, \cdots, 2k-2$;

(2) P_{i+1} is elliptic on $ES(P_i)$, $i = 0, \cdots, 2k-3$, in particular P_0 is elliptic on $ES(P)$.

A simple application of Lemma 2.2 gives

$$||\psi P\phi w||_k \leq C||P_0 w_1||_{k,\Omega_0} + ||w_1||_1 \ .$$

Therefore, it suffices to show that

$$||P_0 w_1||_{k,\Omega_0} \leq C||f||_0,$$

which may be established by the following "bootstrap" argument.

Applying P_0 to both sides of (3.2), we find

$$\Box P_0 w_1 = [\Box, P_0]w_1 + [P_0, \nabla\sigma_0 \cdot \nabla]w_1 + \nabla\sigma_0 \cdot \nabla P_0 w_1 + P_0 f_1 \ . \tag{3.3}$$

From the ellipticity of $P_1\Box$ on $ES(P_0)$, a Gårding type inequality yields

$$||P_0 w_1||_{k,\Omega_0} \leq C||P_1\Box P_0 w_1||_{k-2,\Omega_1} + C||w_1||_1 \ ,$$

or from (3.3)

$$\begin{aligned} ||P_0 w_1||_{k,\Omega_0} \quad \leq \quad & C(||P_1[\Box, P_0]w_1||_{k-2,\Omega_1} + ||P_1[P_0, \nabla\sigma_0 \cdot \nabla]w_1||_{k-2,\Omega_1} + \\ & ||P_1\nabla\sigma_0 \cdot \nabla P_0 w_1||_{k-2,\Omega_1} + ||f||_0) \ . \end{aligned}$$

Therefore an application of Lemma 2.5 and the extended Rauch's lemma Lemma 2.3 yields

$$\begin{aligned} ||P_0 w_1||_{k,\Omega_0} \quad \leq \quad & C_1||P_2 w||_{k-1,\Omega_2} + C_2(||w_1||_1 + ||P_2 w_1||_{k-2,\Omega_2}) \\ & +C_3(||w_1||_1 + ||P_2 w_1||_{k-1,\Omega_2}) \\ \leq \quad & C||f||_0 + C||P_2 w_1||_{k-1,\Omega_2} \ . \end{aligned}$$

Here constants C_2 and C_3 depend on $||\nabla\sigma_0||_s$ for $k - 2 + n/2 < s$.

Now we may continue this process to obtain the following estimate

$$||P_{2j} w_1||_{k-j,\Omega_j} \leq C||f||_0 + C||P_{2j+2} w_1||_{k-j-1,\Omega_{2j+2}},$$

for $j = 0, 1, \cdots, k-2$.

Then the proof is complete by knowing that

$$||P_{2j+2} w_1||_{1,\Omega_{2k-2}} \leq ||f||_0 \ .$$

□

3.2 Regularity of the fundamental solution: Proof of Theorem 3.1

We study the regularity of u_0 through its dual problem. To simplify the arguments on the dual problem, we make use of the symmetric form of (1.2)($u = u_0$ and $\sigma = \sigma_0$) by introducing $\rho(x) = e^{-\sigma_0}$. Then (1.2) becomes

$$\begin{aligned} \Box_1 u_0 = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)]u_0 = \frac{1}{\rho}\delta(t)\delta(x), \\ u_0 = 0 \quad t < 0 \ . \end{aligned} \tag{3.4}$$

Now let us look at a dual problem to (3.4),

$$\begin{array}{c}\Box_1 w = [\frac{1}{\rho}\partial_t^2 - \nabla\cdot(\frac{1}{\rho}\nabla)]w = \Psi,\\ w = 0 \quad t >> T_1\ ,\end{array} \tag{3.5}$$

where $\Psi \in C_0^\infty(\Omega)$ with $\Omega = \{\mathbf{R}^n \times (0, T_1)\} \cap \{t > |x| + \epsilon_0\}$, for $\epsilon_0 > 0$ small. Note that, this equation may be reformulated as

$$\begin{array}{c}\Box_1' w = \Box w - \nabla\sigma_0 \cdot \nabla w = e^{-\sigma_0}\Psi,\\ w = 0 \quad t >> T_1\ .\end{array} \tag{3.6}$$

Thus if we can show that for any $\Psi \in C_0^\infty(\Omega)$

$$|(\partial_t^l u_0, \Psi)| \leq C||\Psi||_0\ , \tag{3.7}$$

then it can be concluded that

$$||\partial_t^l u_0||_{0,\Omega} \leq C\ . \tag{3.8}$$

From (3.4), integration by parts leads

$$\begin{aligned} |(\partial_t^l u_0, \Psi)| &= |(\Box_1 \partial_t^l u_0, w)| \\ &= |(\frac{1}{\rho}\delta(t)\delta(x), \partial_t^l w)| \\ &\leq C|(\partial_t^l w)(0,0)|\ . \end{aligned}$$

The trace theorem (see for example [27]) yields that

$$|(\partial_t^l u_0, \Psi)| \leq C||\phi_1 w||_{l+(n+1)/2} \tag{3.9}$$

with $\phi_1 \in C_0^\infty(\Omega_1)$, Ω_1 a small neighborhood of the origin and $\Omega_1 \cap supp(\Psi) = \emptyset$.

Construct two $\psi.d.o.$ Q_1, $Q_2 \in OPS^0(\mathbf{R}^{n+1})$, such that

- $Q_1 + Q_2 = R$; where R is an elliptic $\psi.d.o.$ of order zero in Ω_1;
- $\Pi supp(q_i) \cap supp(\Psi) = \emptyset$, for $i = 1, 2$;
- $ES(Q_2)$ is a small conic neighborhood of set of null bicharacteristics of the wave operator $\Box$ passing over Ω_1;
- Q_1 is microlocally smoothing on the null bicharacteristics passing over Ω_1.

Therefore, with (3.9), we have

$$|(\partial_t^l u_0, \Psi)| \leq C||Q_1\phi_1 w||_{l+(n+1)/2,\Omega_1} + C||Q_2\phi_2 w||_{l+(n+1)/2,\Omega_1}, \tag{3.10}$$

here the expression makes sense because the domain of dependence for w and the pseudo-local properties of Q_1 and Q_2.

Now, we can apply Corollary 3.1 to obtain that

$$||Q_2 w||_{l+(n+1)/2,\Omega_1} \leq C||\Psi||_0 \ . \tag{3.11}$$

Lemma 3.2 yields

$$||Q_1 w||_{l+(n+1)/2,\Omega_1} \leq C||\Psi||_0, \tag{3.12}$$

where the constants here depend on $||\sigma_0||_s$ with $s > max\{1+n/2, l+n-1/2\}$.

Therefore, we have shown

$$|(\partial_t^l u_0, \Psi)| \leq C||\phi_1 w||_{l+(n+1)/2} \leq C||\Psi|| \ . \tag{3.13}$$

which completes the proof. □

4 Regularity of the Transport Equations

Consider a problem related to the model problem,

$$\begin{array}{l} (\Box - \nabla\sigma_0 \cdot \nabla)v_0 = \delta^{-\frac{n-1}{2}}(t)\delta(x) \ , \\ v_0 = 0 \ , \quad t < 0 \ . \end{array} \tag{4.1}$$

Hadamard's construction leads to the progressing wave expansion for v_0,

$$v_0 = \sum_{k=0}^{l_1-1} b_k(x) S_k(t - r(x)) + R(x,t), \tag{4.2}$$

where $r(x) = |x|$, $S_0(t-r(x)) = H(t-r(x))$ is the Heaviside function, $S_k' = S_{k-1}$ $(k \geq 1)$, and $\{b_k\}$ solve the so-called transport equations, for $k = 1, \cdots, s$,

$$2\nabla r \cdot \nabla b_0 + (\Delta r + \nabla r \cdot \nabla\sigma_0) b_0 = 0, \tag{4.3}$$

$$2\nabla r \cdot \nabla b_k + (\Delta r + \nabla r \cdot \nabla\sigma_0) b_k = \Delta b_{k-1} + \nabla\sigma_0 \cdot \nabla b_{k-1}, \tag{4.4}$$

for $k = 1, \cdots, s$. Moreover, the remainder term R satisfies

$$\begin{array}{l} (\Box - \nabla\sigma_0 \cdot \nabla)R = (\Delta + \nabla\sigma_0 \cdot \nabla) b_{l_1-1} S_{l_1-1}(t - r(x)) \ , \\ R = 0 \ , \quad t < 0 \ . \end{array} \tag{4.5}$$

Clearly away from the origin, R is the smoother part in the expansion.

This section is devoted to the regularity study of the solutions of the above transport equations.

Let us first simplify the transport equations (4.3) and (4.4) by introducing two functions α and α_0, such that

$$\nabla r \cdot \nabla\alpha_0 = \Delta r/2 \ , \quad \text{and } \alpha = \sigma_0/2 + \alpha_0 \ .$$

It is obvious that α is nothing more than a smooth perturbation of $\sigma_0/2$, away from the origin.

We can then rewrite the transport equations as

$$2\nabla r\cdot\nabla(b_0e^{\alpha}) = 0, \tag{4.6}$$
$$2\nabla r\cdot\nabla(b_ke^{\alpha}) = e^{\alpha}(\Delta b_{k-1}+\nabla\sigma_0\cdot\nabla b_{k-1})\,. \tag{4.7}$$

Observe that all the transport equations have the same principal part $2\nabla r\cdot\nabla$ which is a smooth vector field away from the origin. Therefore in order to understand the regularity of the solutions, it is essential to study the properties of this vector field.

Let V denote the vector field $\nabla r\cdot\nabla$ and $Char(\nabla r\cdot\nabla)=\{(x,\xi)\in T^*(\mathbf{R}^n),\ \nabla r\cdot\xi=0\}$.

Lemma 4.1 *Assume that u is a smooth function with $supp(u)\subset\{|x|>\delta\}$ for some $\delta>0$. Assume also that $\phi\in C_0^\infty(\mathbf{R}^n)$ with $\phi=1$ on Ω and $supp(\phi)\subset\subset\Omega'$, where Ω and Ω' are bounded open sets in $\mathbf{R}^n$. Then there exist a $Q\in OPS^0$ which is elliptic on $Char(V)$, $[Q,V]\in OPS^{-\infty}$, and $\phi'\in C_0^\infty(\mathbf{R}^n)$, $\phi'>0$ on $supp(\phi)$, such that for $s\in\mathbf{R}$ the following estimates*

$$||u||_{s,\Omega} \leq C||Q\phi'Vu||_{s,\Omega'}+C||Vu||_{s-1,\Omega'}+C||u||_{\tau,\Omega'}, \tag{4.8}$$
$$||Q\phi u||_{s,\Omega} \leq C||Q\phi Vu||_{s,\Omega}+C||u||_{\tau,\Omega'}, \tag{4.9}$$

hold for any $\tau\in\mathbf{R}$, where the constants are independent of u.

Remark. Since the vector field V is singular at $x=0$, the assumption $supp(u)\subset\{|x|>\delta\}$ is essential to make sense of the whole discussion here.

Proof. Following Nirenberg's construction (in the proof of Theorem 6 in [18]), one can construct operators Q and $P\in OPS^0$ with properties:

- $[Q,V]\in OPS^{-\infty}$;
- Q is elliptic on a small conic neighborhood γ of $Char(V)$;
- $P+Q$ is elliptic and;
- $ES(P)\cap\gamma=\emptyset$.

Let $\{\Omega_i\}_{i=0}^{j}$ be a sequence of bounded open sets, such that

$$\Omega=\Omega_0\subset\subset supp(\phi)\subset\subset\Omega_1\subset\subset\cdots\subset\subset\Omega_{j-1}\subset\subset\Omega_j=\Omega',$$

here j is the smallest integer with $s-j\leq k$.

Correspondingly, one can construct a sequence of functions $\{\phi_i\}_{i=o}^{j-1}$ that satisfy: $\phi_0=\phi$, $\phi_i\in C_0^\infty(\mathbf{R}^n)$; $\phi_i=1$ on $supp(\phi_{i-1})\cup\Omega_i$ for $i=1,\cdots,j-1$; $\phi_j>0$ on $supp(\phi_{j-1})$; and $supp(\phi_i)\subset\subset\Omega_{i+1}$, for $i=1,\cdots,j-1$.

Because of the ellipticity of $R+Q$, Gårding's inequality (see *f.g.* Taylor [27]) yields

$$\begin{aligned}||\phi u||_{s,\Omega} &\leq C||(R+Q)\phi u||_{s,\Omega}+C||\phi u||_{\tau,\Omega}\\ &\leq C||R\phi u||_{s,\Omega}+C||Q\phi u||_{s,\Omega}+c||\phi u||_{\tau,\Omega}\,.\end{aligned} \tag{4.10}$$

Now since V is elliptic on $ES(R)$, our Gårding type result Lemma 2.2 gives

$$\begin{aligned} ||R\phi u||_{s,\Omega} &\leq C||V\phi u||_{s-1,\Omega_1} + C||\phi u||_{\tau,\Omega_1} \\ &\leq C||[V,\phi]u||_{s-1,\Omega_1} + C||\phi V u||_{s-1,\Omega_1} + C||\phi u||_{\tau,\Omega_1} \\ &\leq C||\phi_1 u||_{s-1,\Omega_1} + C||\phi V u||_{s-1,\Omega_1} + C||\phi u||_{\tau,\Omega_1} \ . \end{aligned} \tag{4.11}$$

Next note that $V = \frac{\partial}{\partial r}$. Thus along the radial direction $r = |x|$, the standard method of energy estimates may be applied to get

$$\begin{aligned} ||Q\phi u||_{s,\Omega} &\leq ||Q\phi V u||_{s,\Omega} + ||QV(\phi)u||_{s,\Omega} + ||[V,Q]\phi u||_{s,\Omega} \\ &\leq ||Q\phi V u||_{s,\Omega} + C||u||_{\tau,\Omega_1}, \end{aligned} \tag{4.12}$$

where to obtain the second inequality, we have used the facts $[V,Q] \in OPS^{-\infty}$. The estimate $||QV(\phi)u||_{s,\Omega} \leq C||u||_{\tau,\Omega_1}$ follows from the assumption $\phi = 1$ (hence $V\phi = 0$) on Ω.

Substitutes estimates (4.11) and (4.12) into (4.10), we have

$$||\phi u||_{s,\Omega} \leq C||\phi_1 u||_{s-1,\Omega_1} + C||Q\phi V u||_{s,\Omega} + C||\phi V u||_{s-1,\Omega_1} + C||u||_{\tau,\Omega_1} \ . \tag{4.13}$$

Now, we can repeat the above procedure to estimate $||\phi_1 u||_{s-1,\Omega_1}$. Actually, the same idea allows us to obtain estimates

$$||\phi u||_{s-i,\Omega_i} \leq C||\phi_{i+1} u||_{s-i-1,\Omega_{i+1}} + C||Q\phi_i V u||_{s-i,\Omega_i} + C||\phi_i V u||_{s-i-1,\Omega_{i+1}} + C||u||_{\tau,\Omega_{i+1}} \tag{4.14}$$

for $i = 1, \cdots, j-1$.

Combining all of the estimates in (4.14), we have

$$\begin{aligned} ||\phi u||_{s,\Omega} &\leq C||\phi_j u||_{s-j,\Omega_j} + C\sum_{i=0}^{j-1} ||Q\phi_i V u||_{s-i,\Omega_i} + C\sum_{i=0}^{j-1} ||\phi_i V u||_{s-i-1,\Omega_{i+1}} + C\sum_{i=0}^{j-1} ||u||_{\tau,\Omega_{i+1}} \\ &\leq C\sum_{i=0}^{j-1} ||Q\phi_i V u||_{s-i,\Omega_i} + C\sum_{i=0}^{j-1} ||\phi_i V u||_{s-i-1,\Omega_{i+1}} + C||u||_{\tau,\Omega'} \end{aligned} \tag{4.15}$$

because $s - j \leq r$ and $\Omega_j = \Omega'$.

It follows from our construction of $\{\Omega_i\}$ and $\phi_i > 0$ on $\cup_{i=0}^{j-1} supp(\phi_i)$ that

$$\begin{aligned} ||\phi u||_{s,\Omega} &\leq C\sum_{i=0}^{j-1} ||Q\phi_i V u||_{s-i,\Omega_i} + C||Vu||_{s-1,\Omega'} + C||u||_{\tau,\Omega'} \\ &= C\sum_{i=0}^{j-1} ||Q\phi_i\phi_j^{-1}\phi_j V u||_{s-i,\Omega_i} + C||Vu||_{s-1,\Omega'} + C||u||_{\tau,\Omega'} \\ &\leq C\sum_{i=0}^{j-1}(||[Q,\phi_i\phi_j^{-1}]\phi_j V u||_{s-i,\Omega_i} + ||\phi_i\phi_j^{-1} Q\phi_j V u||_{s-i,\Omega_i}) + C||Vu||_{s-1,\Omega'} + C||u||_{\tau,\Omega'} \\ &\leq C||Q\phi_j V u||_{s,\Omega'} + C||Vu||_{s-1,\Omega'} + C||u||_{\tau,\Omega'} \ . \end{aligned} \tag{4.16}$$

Thus by choosing $\phi' = \phi_j$, we have completed the proof of (4.9). □

Having established Lemma 4.1, we are ready to study the solutions of the transport equations, $\{b_i\}(i = 0, \cdots, l_1 - 1)$. Unfortunately, Lemma 4.1 is not directly applicable essentially due to the fact that $\{b_i\}$ are not necessarily supported away from the origin. However, the assumption made in Section 1, $\delta\sigma = 0$ near $\{x_n = 0\}$ indicates that no perturbation of the coefficient σ_0 will take place near the origin. Since our primary concern is the linearized problem in this work, σ_0 may therefore be assumed to be constant σ_0^c near $\{x_n = 0\}$ for convenience. Suppose the solutions of the corresponding transport equations of the problem obtained by replacing σ_0 by σ_o^c are $e_0, e_1, \cdots, e_{l_1-1}$. Obviously, they are smooth functions, and moreover

$$d_i = e_i - b_i, \quad \text{for } i = 0, \cdots, l_1 - 1$$

are all supported away from the origin. Hence Lemma 4.1 becomes applicable.

It is easy to write down the equations for d_k

$$\begin{aligned} 2\nabla r \cdot \nabla(d_0 e^{q_0}) &= (\nabla r \cdot \nabla \sigma_0) b_0 e^{q_0}, &(4.17)\\ 2\nabla r \cdot \nabla(d_k e^{q_0}) &= (\nabla r \cdot \nabla \sigma_0) b_k e^{q_0} + \Delta d_{k-1} e^{q_0} - (\nabla \sigma_0 \cdot \nabla b_{k-1}) e^{q_0}, &(4.18)\end{aligned}$$

for $k = 1, \cdots, l_1 - 1$.

Let us first examine d_0. Equation (4.17) may be rewritten as

$$2\nabla r \cdot \nabla(d_0 e^{q_0 + \sigma_0/2}) = (\nabla r \cdot \nabla \sigma_0) e_0 e^{q_0 + \sigma_0/2}$$

then energy estimates together with Schauder's Lemma yield, for any bounded open set $\Omega \subset \mathbf{R}^n$

$$||d_0 e^{q_0 + \sigma_0}||_{i,\Omega} \leq C,$$

where C depends on $||\sigma_0||_s$ for $s > n/2 + 1$ and $s \geq i$. Thus

$$||b_0||_{i,\Omega} \leq C, \tag{4.19}$$

where again the constant C depends on $||\sigma_0||_s$.

Next, equation (4.18) may be rewritten as

$$\begin{aligned} \nabla r \cdot \nabla(d_k e^{q_0}) &= \frac{1}{2}(\Delta - 2\nabla q_0 \cdot \nabla + |\nabla q_0|^2 - \Delta q_0)(d_{k-1} e^{q_0}), \quad \text{for } k = 1, \cdots, l_1 - 1,\\ &= P_2(d_{k-1} e^{q_0}), &(4.20)\end{aligned}$$

where $P_2 = \frac{1}{2}(\Delta - 2\nabla q_0 \cdot \nabla + |\nabla q_0|^2 - \Delta q_0)$ a second order differential operator. To establish our regularity theorem of the transport equations, we need the following result.

Proposition 4.1 *Let Q and u be defined as in Lemma 4.1. Assume that Ω is a bounded open set of $\mathbf{R}^n$, $\phi \in C_0^\infty(\mathbf{R}^n)$. Then there exist $Q' \in OPS^0$ which is elliptic on $Char(V)$ and $[Q', V] \in OPS^{-\infty}$, $\Omega' \supset\supset \Omega \cup supp\ (\phi)$, and $\phi' \in C_0^\infty(\mathbf{R}^n)$, such that*

$$||Q\phi P_2 u||_{s,\Omega} \leq C||Q'\phi' u||_{s+2,\Omega'} + ||u||_{\tau,\Omega'}$$

holds for any $\tau \in \mathbf{R}$.

Proof. Let $\phi' \in C_0^\infty(\Omega')$ and $\phi' = 1$ on $supp(\phi)$. Then

$$Q\phi P_2 u = Q\phi P_2 \phi' u \ .$$

Construct a $\psi.d.o.$ Q' of order zero with properties: $Q' = 1$ on $ES(Q)$ and $[Q', V] \in OPS^{-\infty}$.

It follows that

$$Q\phi P_2 u = Q\phi P_2 Q'\phi' u + Q\phi P_2(I - Q')\phi' u$$

but the operator $Q\phi P_2(I - Q')$ is an smoothing operator. Therefore,

$$\begin{aligned} ||Q\phi P_2 u||_{s,\Omega} &\leq ||Q\phi P_2 Q'\phi' u||_{s,\Omega} + ||Q\phi P_2(I - Q')\phi' u||_{s,\Omega} \\ &\leq ||Q'\phi' u||_{s+2,\Omega'} + ||u||_{\tau,\Omega'} \end{aligned}$$

which concludes the proof. □

Applications of Lemma 4.1, Proposition 4.1, and the transport equations (4.20) together with the estimate of b_0 (4.19) will result in the following theorem. Since the proof is straightforward, we shall omit it.

Theorem 4.1 *Suppose Ω is a bounded open subset of $\mathbf{R}^n$. Then*

$$||b_i||_{I_i,\Omega} \leq C_{i,I_i} \ \ for \ \ i = 0, \cdots, l_1 - 1,$$

where the constants C_{i,I_i} depend on $H^{s_i} \cap H^{\tau_i}_{m\ell}(Char(V))$-norm of σ_0 with $I_i, s_i, \tau_i \in \mathbf{R}$ satisfying $s_i > n/2 + 1$, $s_i \geq I_i + i$, and $\tau_i \geq max\{I_i + 2i, s_i\}$.

5 Conormal Properties of the Wave Equation

This section is concerned with the conormal properties of the wave operator, such properties are of great importance in the understanding of progressing wave expansions. To demonstrate the ideas, we use the following wave equation:

$$\begin{aligned} &(\partial_t^2 - \Delta - \nabla\sigma_0 \cdot \nabla) u(x,t) = a(x) S(t - r(x)) \ , \\ &u = 0 \ , \quad t < 0 \ , \end{aligned} \tag{5.1}$$

where again $r(x) = |x|$.

To study the conormal properties for the wave equation, it is crucial to have various commutator estimates. We begin with some useful identities. It seems that the polar coordinates are particularly suitable for this study. Throughout this section, we shall assume $n > 2$. Similar analysis will go through in the case of $n = 2$. Also, x is always assumed to be away from the origin. In our application, the origin would not cause any trouble because $\delta\sigma = 0$ near the origin.

Introduce the standard polar coordinates for $n > 2$ with variables $r = |x|$, θ_1, θ_2, $\cdots$, θ_{n-1}:

$$\begin{aligned} x_1 &= r sin\theta_{n-1} sin\theta_{n-2} \cdots sin\theta_2 cos\theta_1 \\ x_2 &= r sin\theta_{n-1} sin\theta_{n-2} \cdots sin\theta_2 sin\theta_1 \end{aligned}$$

$$
\begin{aligned}
&\cdot\\
&\cdot\\
&\cdot\\
x_{n-1} &= r sin\theta_{n-1} cos\theta_{n-2}\\
x_n &= r cos\theta_{n-1}\ .
\end{aligned}
$$

Then $T_1 = \partial_t + \partial_r$. Denote $T_{i+1} = \partial_{\theta_i}$ for $i = 1, 2, \cdots, n-1$, and $T_{n+1} = \partial_t - \partial_r$. Note that T_i $(i = 1, 2, \cdots, n)$ form a basis of the tangent space to the characteristic surface $\{t = |x|\}$.

In polar coordinates, the Laplacian has the following form

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_\theta\ ,$$

where Δ_θ is the $(n-1)$ dimensional angular Laplacian.

Proposition 5.1 *The following identity holds*

$$[\Box, T_1] = \frac{2}{r}\Box - \frac{2}{r}(\partial_t - \partial_r)T_1 + \frac{n-1}{r^2}\partial_r\ . \tag{5.2}$$

Proof. From the above expression of the Laplacian, we have

$$
\begin{aligned}
[\Box, T_1] &= [\partial_t^2 - (\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_\theta), \partial_t + \partial_r]\\
&= -\frac{n-1}{r^2}\partial_r - \frac{2}{r^3}\Delta_\theta\\
&= \frac{2}{r}\Box - \frac{2}{r}(\partial_t^2 - \partial_r^2) + \frac{n-1}{r^2}\partial_r\ .
\end{aligned}
$$

□

Combining Proposition 5.1 with Leibnitz's rule, and knowing that T_1 and $\partial_t - \nabla r \cdot \nabla$ commute, we have the following result.

Lemma 5.1 *There exist smooth functions $\{\alpha_i(r)\}$ and $\{\beta_i(r)\}$ such that the following identity holds*

$$\Box T_1^k = \tilde{T}_1^{\,k}\Box + \sum_{i=0}^{k-1}\alpha_i(\partial_t - \nabla r\cdot\nabla)T_1^{k-i} + \sum_{i=0}^{k-1}\beta_i\nabla T_1^{k-i}\ ,$$

where $\tilde{T}_1 = T_1 + 2/r$.

Next, we want to study the commutator of the Laplacian and ∂_{θ_i} or the commutator of the angular Laplacian Δ_θ and ∂_{θ_i}. It is easy to see that Δ_θ has the following expression:

$$
\begin{aligned}
\Delta_\theta = \ & r^2\sum_{i=1}^{n}[\sum_{j=1}^{n-1}(\frac{\partial\theta_j}{\partial x_i})(\frac{\partial r}{\partial x_i})\partial^2_{\theta_j r} + \sum_{j=1}^{n-1}(\frac{\partial\theta_j}{\partial x_i})_r(\frac{\partial r}{\partial x_i})\partial_{\theta_j}\\
&+ \sum_{k=1}^{n-1}\{(\frac{\partial r}{\partial x_i})(\frac{\partial\theta_k}{\partial x_i})\partial^2_{\theta_k r} + \sum_{j=1}^{n-1}(\frac{\partial\theta_j}{\partial x_i})(\frac{\partial\theta_k}{\partial x_i})\partial^2_{\theta_j\theta_k} + \sum_{j=1}^{n-1}(\frac{\partial\theta_j}{\partial x_i})_{\theta_k}(\frac{\partial\theta_k}{\partial x_i})\partial_{\theta_j}\}]\ .
\end{aligned} \tag{5.3}
$$

Because of Lemma 5.1, the differential operators that involve ∂_r can be handled similarly as other lower order operators in estimates. The only trouble some term is

$$\sum_{i=1}^{n}\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}(\frac{\partial\theta_j}{\partial x_i})(\frac{\partial\theta_k}{\partial x_i})\partial^2_{\theta_j\theta_k} = \sum_{k=1}^{n-1}\sum_{j=1}^{n-1}[\sum_{i=1}^{n}(\frac{\partial\theta_j}{\partial x_i})(\frac{\partial\theta_k}{\partial x_i})]\partial^2_{\theta_j\theta_k} \ .$$

The following proposition follows immediately from orthogonal properties in the expression of the derivatives of θ with respect to x_i.

Proposition 5.2 *If $j \neq k$, then*

$$\sum_{i=1}^{n}(\frac{\partial\theta_j}{\partial x_i})(\frac{\partial\theta_k}{\partial x_i}) = 0.$$

Also

$$\sum_{i=1}^{n}(\frac{\partial\theta_j}{\partial x_i})(\frac{\partial r}{\partial x_i}) = 0.$$

Introduce a vector $\lambda \in \mathbf{R}^n$ that satisfies

$$\lambda = (r^2 sin^2\theta_{n-1}\cdots sin^2\theta_2, r^2 sin^2\theta_{n-1}\cdots sin^2\theta_3, \cdots, r^2 sin^2\theta_{n-1}, r^2, 1) \ .$$

Then similar to Proposition 5.2, one can prove the following identity.

Proposition 5.3 *For $j = 1, 2, \cdots, n-1$,*

$$\sum_{i=1}^{n}(\frac{\partial\theta_j}{\partial x_i})^2 = \frac{1}{\lambda_j} \ .$$

Therefore, the last two propositions imply that

$$\Delta_\theta = r^2\sum_{j=1}^{n-1}\{\sum_{i=1}^{n}[(\frac{\partial\theta_j}{\partial x_i})_r(\frac{\partial r}{\partial x_i})\partial_{\theta_j} + \sum_{k=1}^{n-1}(\frac{\partial\theta_k}{\partial x_i})_{\theta_j}(\frac{\partial\theta_j}{\partial x_i})\partial_{\theta_k}] + \frac{1}{\lambda_j}\partial^2_{\theta_j^2}\} \ .$$

In particular, Δ_θ is independent of r as used in the proof of Proposition 5.1.

Next, we introduce the so called Anisotropic Sobolev spaces or Hömander's spaces, $\mathcal{H}^{m,s}(\mathbf{R}^{n+1})$, which are defined originally in Hömander [13] as

$$\mathcal{H}^{m,s}(\mathbf{R}^k) = \{f \in \mathcal{D}',\ D^\alpha_{x',x_k}f \in L^2(\mathbf{R}^k),\ \forall\ \alpha = (\alpha_1, \alpha_1, \cdots, \alpha_k), |\alpha| \le m+s, \alpha_k \le m\}$$

where $D^\alpha_{x',x_k} = D^{\alpha_1,\cdots,\alpha_{k-1}}_{x'}D^{\alpha_k}_{x_k}$.

For convenience, we state the following results, see [13] for a proof of Proposition 5.4.

Proposition 5.4 *Suppose $m > 1/2$ and $m+s > k/2$. Then*

$$\mathcal{H}^{m,s} \subset L^\infty(\mathbf{R}^k) \cap C^0(\mathbf{R}^k) \ \ \textit{continuous inclusion} \ .$$

For the equation (5.1), we have the following conormal regularity result. Recall that T_i, $i = 1, ..., n$ are the vector fields tangential to the hypersurface $\{t = r(x)\}$.

Theorem 5.1 *Suppose that, in (5.1), $S \in H^{m-1}_{loc}$ and $a(x)$ is a smooth function. Suppose also that $k \geq 0$, $p \geq m + k$, $p > k + n/2 + 1$, $q \geq m + k - 1$, and $q > k + n/2$. Then for $\{i_1, \cdots, i_k\} \subset \{1, 2, \cdots, n\}$ and $\phi(x,t) \in C^\infty_0(\mathbf{R}^{n+1})$*

$$T_{i_1} \cdots T_{i_k} u \in H^m_{loc}$$

or

$$u \in H^{m,k}_{loc}.$$

In addition,

$$||\phi T_{i_1} \cdots T_{i_k} u||_m \leq C_m$$

with the constant C_m depending on $||a||_q$ and $||\sigma_0||_p$.

On the proof of Theorem 5.1. Clearly, in order to prove Theorem 5.1, it is sufficient to show that

$$T_{i_1} \cdots T_{i_k} u \in H^m_{loc} \ , \tag{5.4}$$

where $\{i_j\}$ are not necessarily distinct. The relation (5.4) may be proved by induction by applying the method of energy estimates and commutator results above. The key fact is that T_i for $i = 1, \cdots, n$ are tangential vector fields to $\{t = |x|\}$. In other words,

$$T_j[a(x)S(t-r)] = [T_j a(x)]S(t-r)$$

where $i = 1, \cdots, n$. We shall skip the formal proof. However, for the sake of completeness, let us point out how the regularity assumptions on the coefficients a and σ_0 are determined. From the discussions above, it is evident that the highest derivative of a involving in the estimate of $T^k u$ is the k-th. Thus $||\phi T_{i_1} \cdots T_{i_k} u||_s$ should be bounded by $||\phi T_{i_1} \cdots T_{i_k} aS||_{m-1}$. But by using Schauder's lemma and the assumption that $S \in H^{m-1}_{loc}$, we have for $\tau \geq m-1$ and $\tau > n/2$ that

$$\begin{aligned} ||\phi T_{i_1} \cdots T_{i_k} aS||_{m-1} &\leq C||\psi T_{i_1} \cdots T_{i_k} a||_\tau ||\phi S||_m \\ &\leq C||\psi a||_q \end{aligned}$$

where $\psi \in C^\infty_0(\mathbf{R}^n)$, $q = \tau + k$, *i.e.*, $q \geq m + k - 1$ and $q > k + n/2$.

Concerning σ_0, the dependence comes from verifying that

$$T_{i_1} \cdots T_{i_k} \nabla \sigma_0 \cdot \nabla u \in H^{s-1}_{loc}$$

or

$$(T_{i_1} \cdots T_{i_k} \nabla)\sigma_0 \cdot \nabla u \in H^{s-1}_{loc} \ .$$

Note that the hypotheses on $S(t-r)$, a, and the method of energy estimates imply that $u \in H^m_{loc}$. Therefore, Schauder's Lemma yields, for $\tau \geq m-1$ and $\tau > n/2$,

$$\begin{aligned} ||\phi_0(T_{i_1} \cdots T_{i_k} \nabla)\sigma_0 \cdot \nabla u||_{m-1} &\leq C||\psi_0(T_{i_1} \cdots T_{i_k} \nabla)\sigma_0||_\tau ||\phi_1 u||_m \\ &\leq C||\sigma_0||_{\tau+k+1} ||\phi_1 u||_s \end{aligned}$$

where $\psi_0 \in C^\infty_0(\mathbf{R}^n)$, ϕ_0 and $\phi_1 \in C^\infty_0(\mathbf{R}^{n+1})$. Hence the constant in the final estimate depends on $||\sigma_0||_p$ with $p \geq m + k$ and $p > k + n/2 + 1$. □

6 Proof of the Main Theorem

Recall the linearized problem corresponding to the reference state (u_0, σ_0), for $(t,x) \in \mathbf{R}^{n+1}$, $x = (x', x_n)$,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)\delta u &= \nabla\delta\sigma \cdot \nabla u_0 \ , \\ \delta u = 0 \ , \quad t &< 0 \ , \end{aligned} \tag{6.1}$$

where u_0 is the solution of the model problem corresponding to the reference density σ_0. The linearized forward map can be defined as

$$DF(\sigma_0)\delta\sigma = (\phi\delta u)\,|_{x_n=0} \ , \tag{6.2}$$

where $\phi(x,t) \in C_0^\infty(\mathbf{R}^{n+1})$ is supported inside the conoid $\{t > |x|\}$, and near $\{x_n = 0\}$.

Consider a problem related to the linearized problem,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)v &= \nabla\delta\sigma \cdot \nabla v_0 \ , \\ v = 0 \ , \quad t &< 0 \ , \end{aligned} \tag{6.3}$$

where $\delta u = \partial_t^{\frac{n-1}{2}} v$ and v_0 solves

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)v_0 &= \delta^{-\frac{n-1}{2}}(t)\delta(x) \ , \\ v_0 = 0 \ , \quad t &< 0 \ . \end{aligned} \tag{6.4}$$

Observe that for $l \in \mathbf{R}$, there exists $\hat{\phi} \in C_0^\infty(\mathbf{R}^{n+1})$ supported inside the characteristic surface and near $\{x_n = 0\}$ so that

$$\begin{aligned} ||DF(\sigma_0)\delta\sigma||_l &= \|\,(\phi\delta u)\,|_{x_n=0}\|_l \\ &\leq C||(\hat{\phi}v)\,|_{x_n=0}\,||_{l_1}, \end{aligned} \tag{6.5}$$

where l_1 denotes $l + (n-1)/2$. Thus the real challenge here is to get an appropriate trace regularity estimate for v on a time-like hypersurface $\{x_n = 0\}$.

Before getting into the details of the proof, let us first make the following general remarks on this theorem:

The estimate (1.8) has a similar form to a Rakesh's theorem (Theorem 2.5 in [19]). Actually, we conjecture that a formal extension of our proof here could lead to an elementary proof of his theorem. On the contrary, the principal tool of Rakesh's proof, calculus of Fourier integral operators, is not available when the reference density is nonsmooth.

Our approach enjoys the beauty of the method of energy estimates, that is, it possesses useful information on various parameters involved in the estimates.

Next, we present a trace regularity result. Recall the assumption made in Section 1, $supp(\delta\sigma) \subset \{|x_n| > \epsilon\}$, for $\epsilon > 0$ small.

Lemma 6.1 *Assume that $s > 3 + n/2$, $1 \leq l_1 \leq s$, and v solves problem (6.3) then there is a $\phi_0 \in C_0^\infty$ supported near $supp(\hat{\phi})$ such that the following estimate holds,*

$$||(\hat{\phi}v)|_{x_n=0}||_{l_1} \leq C||\phi_0 v||_{l_1} \ , \tag{6.6}$$

where C is a constant depending on the $H^s \cap H^{l_1+1}_{m\ell}(K)$-norm of σ_0, but is independent of $\delta\sigma$.

Proof. This lemma is a direct application of Theorem 3.1 in [2] by taking into account of the fact that ϕ and $\delta\sigma$ have disjoint supports. □

For simplicity, we shall also assume that l_1 is an integer. Without further difficulty, the proof may be extended formally to cover the general case.

Now let us restrict $\hat{\phi} \in C_0^\infty(\mathbf{R}^{n+1})$ to being supported inside the characteristic surface and the set $\{x_n < \epsilon/2\}$. Multiplying $\hat{\phi}$ to both sides of equation (6.3), we have

$$\begin{aligned} &\Box \hat{\phi} v = \hat{\phi} \nabla \sigma_0 \cdot \nabla v + [\Box, \hat{\phi}] v \ , \\ &v = 0 \ , \quad t < 0 \ . \end{aligned} \tag{6.7}$$

Here we have used the fact that according to the assumption (A), $\hat{\phi}$ and $\delta\sigma$ have disjoint supports, so that $\hat{\phi} \nabla \delta\sigma \cdot \nabla v_0 = 0$.

Having Lemma 6.1, the estimate of $||(\hat{\phi} v)|_{x_n=0}||_{l_1}$ may be reduced to estimating $||\phi_0 v||_{l_1}$. We shall first discuss the estimates of the t-derivatives of v. By the end of this section, it will be shown that the rest may be dominated by the t-derivatives of v.

The rest of this section is devoted to the estimate of $||\phi_0 v||_{l_1}$.

We study the regularity of v through a dual problem. It is then convenient to look at the symmetric form of (6.3), for $\rho(x) = e^{-\sigma_0}$

$$\begin{aligned} &\Box_1 v = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)] v = \frac{1}{\rho} \nabla \delta\sigma \cdot \nabla v_0 \ , \\ &v = 0 \ , \quad t < 0 \ . \end{aligned} \tag{6.8}$$

Introduce a dual problem to (6.8)

$$\begin{aligned} &\Box_1' w = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)] w = \Psi \ , \\ &w = 0 \ , \quad t >> T_0 \ , \end{aligned} \tag{6.9}$$

where $\Psi \in C_0^\infty(\Omega_0)$ with $\Omega_0 \subseteq \{|x_n| < \epsilon\} \cap \{t \in (0, T_0), t > |x| + \epsilon\}$ for some $\epsilon > 0$.

Thus if we can show that for any $\Psi \in C_0^\infty(\Omega_0)$

$$|(\partial_t^{l_1} v, \Psi)| \leq C ||\delta\sigma||_{l_1} \, ||\Psi||_0 \ , \tag{6.10}$$

then it can be concluded that

$$||\partial_t^{l_1} v_2||_{0,\Omega_0} \leq C ||\delta\sigma||_{l_1} \ . \tag{6.11}$$

Integration by parts gives

$$\begin{aligned} (\partial_t^{l_1} v, \Psi) &= (\nabla \delta\sigma \cdot \nabla \partial_t^{l_1} v_0, w) \\ &= (\nabla \delta\sigma \cdot \nabla v_0, \partial_t^{l_1}(\phi_1 w)) \ , \end{aligned} \tag{6.12}$$

where $\phi_1 \in C_0^\infty(\mathbf{R}^{n+1})$, ϕ_1 is supported in a small neighborhood of $supp(w) \cap supp(\nabla\delta\sigma \cdot \nabla \partial_t^{l_1} v_0)$, and $\phi_1 = 1$ on $supp(w) \cap supp(\nabla\delta\sigma \cdot \nabla\partial_t^{l_1} v_0)$.

Construct Q_0, Q_1 and $Q_2 \in OPS^0$, such that

- $supp(q_0)$ is strictly contained in the light cone $\{t \geq r(x)\}$; q_1 and q_2 are supported near the light cone;
- $ES(Q_1) \subseteq$ a conic neighborhood of $Char(\partial_t + \nabla r \cdot \nabla)$;
- $ES(Q_2) \subseteq \{\omega + \nabla r \cdot \xi \neq 0\}$;
- $Q_0 + Q_1 + Q_2 = I$.

Hence

$$\partial_t^{l_1}(\phi_1 w) = \partial_t^{l_1} Q_0(\phi_1 w) + \partial_t^{l_1} Q_1(\phi_1 w) + \partial_t^{l_1} Q_2(\phi_1 w) \ .$$

Therefore, (6.12) becomes

$$(\partial_t^{l_1} v, \Psi) = E_0 + E_1 + E_2 \tag{6.13}$$

and E_i are defined by

$$\begin{aligned} E_0 &= (\nabla\delta\sigma \cdot \nabla v_0, \partial_t^{l_1} Q_0(\phi_1 w)) \ , & (6.14)\\ E_1 &= (\nabla\delta\sigma \cdot \nabla v_0, \partial_t^{l_1} Q_1(\phi_1 w)) \ , & (6.15)\\ E_2 &= (\nabla\delta\sigma \cdot \nabla v_0, \partial_t^{l_1} Q_2(\phi_1 w)) \ . & (6.16) \end{aligned}$$

We shall estimate these three terms separately.

6.1 The estimate of E_0

We apply Theorem 3.1 to the study of E_0. By observing the notation that P^* represents the formal adjoint of an operator P, it is obvious that

$$\begin{aligned} E_0 &= (\phi_2 \partial_t^{l_1} v_0, (\nabla\delta\sigma \cdot \nabla)^* Q_0(\phi_1 w)) \\ &\leq ||\phi_2 \partial_t^{l_1} v_0|| \ ||(\nabla\delta\sigma \cdot \nabla)^* Q_0(\phi_1 w)|| \ , \end{aligned}$$

where $\phi_2 \in C_0^\infty(\mathbf{R}^{n+1})$ is supported strictly inside the light cone and $\phi_2 = 1$ on $supp(q_0)$.

Theorem 3.1 implies that

$$||\phi_2 \partial_t^{l_1} v_0|| \leq C$$

with the constant C depending on $||\sigma_0||_s$ for $s > l + n - 1/2$.

An application of Schauder's Lemma yields, since $l_1 - 1 > n/2$, that

$$\begin{aligned} |E_0| &\leq C||\nabla\delta\sigma Q_0(\phi_1 w)||_1 \\ &\leq C||\nabla\delta\sigma||_{l_1-1}||Q_0(\phi_1 w)||_1 \\ &\leq C||\delta\sigma||_{l_1}||\phi_1 w||_1 \\ &\leq C||\delta\sigma||_{l_1}||\Psi|| \ , \end{aligned} \tag{6.17}$$

where to obtain the second inequality, we have used the assumption that $l > 3/2$ or $l_1 - 1 > n/2$.

6.2 The estimate of E_1

According to the progressing wave expansion (4.2) in last section,

$$\nabla\delta\sigma \cdot \nabla v_0 = -b_0 \nabla\delta\sigma \cdot \nabla r \; \delta(t-r) + \nabla\delta\sigma \cdot \nabla b_0 \; H(t-r) + \nabla\delta\sigma \cdot \nabla R_0 \; , \tag{6.18}$$

where $b_0(x)$ solves the first transport equation of v_0, (4.3), and $R_0(x,t)$ solves

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla) R_0 &= (\Delta + \nabla\sigma_0 \cdot \nabla) b_0 \; H(t - r(x)) \; , \\ R_0 = 0 \; , &\quad t < 0 \; . \end{aligned} \tag{6.19}$$

It follows that

$$E_1 = (-\psi b_0 \nabla\delta\sigma \cdot \nabla r \; \delta(t-r) + \psi \nabla\delta\sigma \cdot \nabla b_0 \; H(t-r), \partial_t^{l_1} Q_1(\phi_1 w)) + (\phi_1 \nabla\delta\sigma \cdot \nabla R_0, \partial_t^{l_1} Q_1(\phi_1 w))$$

with $\psi \in C_0^\infty(\mathbf{R}^n)$, $\psi(x) = 1$ on $\Pi_x(supp(\nabla\delta\sigma \cdot \nabla v_0) \cap supp(\partial_t^{l_1} Q_1(\phi_1 w)))$.

Therefore, properties of distributions give

$$\begin{aligned} E_1 \;\; &\leq \;\; (-\psi b_0 \nabla\delta\sigma \cdot \nabla r, (\partial_t^{l_1} Q_1(\phi_1 w))_{t=r}) + (\psi \nabla\delta\sigma \cdot \nabla b_0, (\partial_t^{l_1 - 1} Q_1(\phi_1 w))_{t=r}) \\ &\quad + ((\phi_1 \nabla\delta\sigma \cdot \nabla R_0, \partial_t^{l_1} Q_1(\phi_1 w)) \; . \end{aligned}$$

The Cauchy-Schwartz inequality and the trace theorem yield

$$\begin{aligned} |E_1| \;\; &\leq \;\; ||\psi b_0 \nabla\delta\sigma \cdot \nabla r|| \; ||Q_1(\phi_1 w)||_{l_1 + 1/2} + ||\psi \nabla\delta\sigma \cdot \nabla b_0|| \; ||Q_1(\phi_1 w)||_{l_1 - 1/2} \\ &\quad + ||\phi_1 \nabla\delta\sigma \cdot \nabla R_0|| \; ||Q_1(\phi_1 w)||_{l_1} \; . \end{aligned}$$

Because of the construction of Q_1, the result on propagation of singularities and Lemma 3.1 imply that for $i = 0, 1/2$, and 1

$$||Q_1 \phi_1 w||_{l_1 - i + 1/2} \leq C_i ||\Psi||_0 \; ,$$

where C_i depending on $||\sigma_0||_{s_i}$, for $s_i > l_1 + (n-1)/2 - i$. Thus

$$\begin{aligned} |E_1| \;\; &\leq \;\; C(||\psi b_0 \nabla\delta\sigma \cdot \nabla r|| + ||\psi \nabla\delta\sigma \cdot \nabla b_0|| + ||\phi_1 \nabla\delta\sigma \cdot \nabla R_0||) \; ||\Psi| \\ &\leq \;\; C ||\delta\sigma||_{l_1} ||\Psi|| + C ||\phi_1 \nabla\delta\sigma \cdot \nabla R_0|| \; ||\Psi|| \end{aligned}$$

with C depending on $||\sigma_0||_s$, $s > l_1 + (n-1)/2 = l + n - 1$.

Thus the problem has been reduced to estimate

$$||\phi_1 \nabla\delta\sigma \cdot \nabla R_0|| \leq C ||\delta\sigma||_{l_1} ||\bar{\phi}_1 R_0||_1$$

for some $\bar{\phi}_1 \in C_0^\infty(\mathbf{R}^{n+1})$, where to get the above estimate we have used the assumption that $l_1 - 1 > n/2$ and Schauder's lemma.

Knowing that R_0 satisfies (6.19), the method of energy estimates gives

$$||\phi_1 R_0||_1 \leq C \; ,$$

where the constant C depends on $||\tilde{\phi}_1 (\Delta + \nabla\sigma_0 \cdot \nabla) b_0||$ hence on $||\sigma_0||_s$ with $s > max\{1 + n/2, 2\}$, for some $\tilde{\phi}_1 \in C_0^\infty(\mathbf{R}^{n+1})$.

Therefore,

$$|E_1| \leq C ||\delta\sigma||_{l_1} ||\Psi|| \; . \tag{6.20}$$

6.3 The estimate of E_2

Since $\partial_t + \nabla r \cdot \nabla$ is elliptic on $ES(\partial_t^{l_1} Q_2)$, Proposition 2.3 implies that there exists a $\psi.d.o.$ $\tilde{Q}_2$ of order zero, such that

$$\partial_t^{l_1-i} Q_2 = (\partial_t + \nabla r \cdot \nabla)^{l_1-i} \tilde{Q}_2 + K_i \ , \tag{6.21}$$

where K_i are smoothing operators and for $i = 0, 1, 2$.

Once again, the progressing wave expansion leads to

$$\nabla \delta\sigma \cdot \nabla v_0 = \sum_{i=0}^{2} a_i(x) S_{i-1}(t - r(x)) + \nabla \delta\sigma \cdot \nabla R_1(x,t) \ , \tag{6.22}$$

where $S_{-1}(t - r(x)) = \delta(t - r(x))$, $S_0(t - r(x)) = H(t - r(x))$, $S_1' = H$, and

$$a_0 = -b_0 \nabla \delta\sigma \cdot \nabla r \ , \tag{6.23}$$
$$a_1 = \nabla \delta\sigma \cdot \nabla b_0 - b_1 \nabla \delta\sigma \cdot \nabla r \ , \tag{6.24}$$
$$a_2 = \nabla \delta\sigma \cdot \nabla b_1 \ . \tag{6.25}$$

Here $\{b_i\}$ satisfy the transport equations for v_0, (4.3) and (4.4), and R_1 solves

$$\begin{array}{l} (\Box - \nabla\sigma_0 \cdot \nabla) R_1 = (\Delta + \nabla\sigma_0 \cdot \nabla) b_1 S_1(t - r(x)) \ , \\ \quad R_1 = 0 \ , \quad t < 0 \ . \end{array} \tag{6.26}$$

Denote $T_1 = \partial_t + \nabla r \cdot \nabla$.

We can then rewrite E_2 as, for some $\psi \in C_0^\infty(\mathbf{R}^n)$,

$$\begin{aligned} E_2 &= \sum_{i=0}^{2} (\psi a_i S_{i-1}, \partial_t^{l_1} Q_2(\phi_1 w)) + (\phi_1 \nabla \delta\sigma \cdot \nabla R_1, \partial_t^{l_1} Q_2(\phi_1 w)) \\ &= \sum_{i=0}^{2} (\psi a_i, (\partial_t^{l_1-i} Q_2(\phi_1 w))_{t=r}) + (\phi_1 \nabla \delta\sigma \cdot \nabla R_1, \partial_t^{l_1} Q_2(\phi_1 w)) \ . \end{aligned}$$

Notice that

$$(\partial_t^{l_1-i} Q_2(\phi_1 w))_{t=r}) = (\nabla r \cdot \nabla)^{l_1-i} (Q_2(\phi_1 w))_{t=r}) \ .$$

We may use (6.21) to get

$$\begin{aligned} |E_2| &\le \sum_{i=0}^{2} ||((\nabla r \cdot \nabla)^{l_1-i-1})^* \psi a_i|| \ ||(T_1 Q_2(\phi_1 w))_{t=r})|| + ||({T_1}^{l_1-1})^* \phi_1 \nabla \delta\sigma \cdot \nabla R_1|| \ ||T_1 \tilde{Q}_2(\phi_1 w)|| \\ &\le C \sum_{i=0}^{2} ||\psi||_{l_1-i-1} ||(Q_2(\phi_1 w))_{t=r})||_1 + ||({T_1}^{l_1-1})^* \phi_1 \nabla \delta\sigma \cdot \nabla R_1|| \ ||\phi_1 w||_1 \ . \end{aligned}$$

Applying the generalized Schauder's Lemma and the assumption that $l > 3/2$ or $l_1 - 1 > n/2$, we have

$$\begin{aligned} ||\psi a_0||_{l_1-1} &\le C||\delta\sigma||_{l_1} ||\psi_0 b_0||_{l_1-1} \le C_0 ||\delta\sigma||_{l_1} \ , \\ ||\psi a_1||_{l_1-2} &\le C||\delta\sigma||_{l_1} (||\psi_1 b_0||_{l_1-1} + ||\psi_1 b_1||_{l_1-2}) \le C_1 ||\delta\sigma||_{l_1} \ , \\ ||\psi a_2||_{l_1-3} &\le C||\delta\sigma||_{l_1} ||\psi_2 b_1||_{l_1-2} \le C_1 ||\delta\sigma||_{l_1} \ , \end{aligned}$$

where $\psi_i \in C_0^\infty(\mathbf{R}^n)$. By Theorem 4.1, the constants C_0 and C_1 depend on $||\sigma_0||_{s_0}$ and $||\sigma_0||_{s_1}$ respectively, where s_0 $s_1 > 1 + n/2$ and $s_0 \geq l_1 - 1$, $s_1 \geq l_1$.

Since $ES(\tilde{Q}_2) \subset \{\omega + \nabla r \cdot \xi \neq 0\}$, a trace regularity result (Corollary 2 in [2]) implies the existence of a $\phi_1' \in C_0^\infty(\mathbf{R}^{n+1})$ such that

$$||(\tilde{Q}_2\phi_1 w)_{t=r(x)}||_1 \leq C||\phi_1' w||_1 \leq C||\Psi||_0 \,.$$

Therefore

$$|E_2| \leq C||\delta\sigma||_{l_1}||\Psi|| + C||({T_1}^{l_1-1})^*\phi_1\nabla\delta\sigma\cdot\nabla R_1||\,||\Psi|| \,. \tag{6.27}$$

Thus is suffices to estimate $||({T_1}^{l_1-1})^*\phi_1\nabla\delta\sigma\cdot\nabla R_1||$.

It is easy to see that

$$({T_1}^{l_1-1})^* = (-1)^{l_1-1}(T_1+\Delta r)^{l_1-1}$$

and

$$(T_1+\Delta r)^{l_1-1} = T_1^{l_1-1} + \sum_{i=1}^{l_1-1}\alpha_i T_1^{l_1-1-i}$$

with smooth coefficients $\alpha_i(x)$.

Hence

$$||({T_1}^{l_1-1})^*\phi_1\nabla\delta\sigma\cdot\nabla R_1|| \leq ||T_1^{l_1-1}\phi_1\nabla\delta\sigma\cdot\nabla R_1|| + \sum_{i=1}^{l_1-1} C||T_1^{l_1-1-i}\phi_1\nabla\delta\sigma\cdot\nabla R_1|| \,. \tag{6.28}$$

Obviously it suffices to study the first term on the right hand side of (6.28). Moreover, the fact that T_1 is a differential operator assures that it is sufficient to estimate, for $\phi \in C_0^\infty(\mathbf{R}^{n+1})$,

$$||\phi T_1^{l_1-1}\nabla\delta\sigma\cdot\nabla R_1|| \,. \tag{6.29}$$

Therefore, it remains to estimate

$$\sum_{i=0}^{l_1-1}||\phi T_1^{l_1-1-i}\nabla\delta\sigma\cdot T_1^i\nabla R_1|| = ||\phi\nabla\delta\sigma\cdot T_1^{l_1-1}\nabla R_1|| + \sum_{i=0}^{l_1-2}||\phi T_1^{l_1-1-i}\nabla\delta\sigma\cdot T_1^i\nabla R_1|| \,. \tag{6.30}$$

Note that although $R_1 \in H^2_{loc}$ from the progressing wave expansion, as we shall show, the terms on the right hand side of the above inequalities are still well defined due to the conormal properties of R_1. However, because of the lack of control on higher order normal derivatives of the characteristic surface, R_1 is only in H^2_{loc}. It is relatively easier to estimate the first term on the right hand side. In fact, by using the assumption $l_1 - 1 > n/2$ and Schauder's lemma, we have

$$||\phi\nabla\delta\sigma\cdot T_1^{l_1-1}\nabla R_1|| \leq C||\delta\sigma||_{l_1}||\phi T_1^{l_1-1}\nabla R_1||.$$

An application of Theorem 5.1 indicates that $R_1 \in H^{1,l_1-1}_{loc}$. Thus

$$||\phi\nabla\delta\sigma\cdot T_1^{l_1-1}\nabla R_1|| \leq C||\delta\sigma||_{l_1}$$

with the constant C depending on $||\sigma_0||_{l+n+5/2}$ from Theorem 5.1.

In order to estimate the second term on the right hand side of (6.30), we need a different approach. The idea is to show that $\phi T_1^i \nabla R_1$ $(0 \le i \le l_1 - 2)$ is bounded. In fact, if this is the case then

$$\begin{aligned} \sum_{i=0}^{l_1-2} \|\phi T_1^{l_1-1-i} \nabla \delta\sigma \cdot T_1^i \nabla R_1\| &\le C \sum_{i=0}^{l_1-2} \|T_1^{l_1-1-i} \nabla \delta\sigma\| \\ &\le C\|\delta\sigma\|_{l_1} . \end{aligned}$$

According to Proposition 5.4, it suffices to show that

$$R_1 \in H_{loc}^{2,s+l_1-2}, \quad s > (n-1)/2 .$$

This can be done by an application of Theorem 5.1. In fact, Theorem 5.1 and Proposition 5.4 imply that the term $\phi T_1^i \nabla R_1$ is bounded by a constant depending on $\|\sigma_0\|_p$ with $p \ge s + l_1$ and $p > s + l_1 + n/2 - 1$, and on $\|\psi(\Delta + \nabla\sigma_0 \cdot \nabla) b_1\|_q$ with $q \ge s + l_1 - 1$ and $q > s + l_1 + n/2 - 2$, for $\psi \in C_0^\infty$.

Further, since $l_1 - 1 > n/2$, Schauder's lemma gives

$$\|\psi(\Delta + \nabla\sigma_0 \cdot \nabla) b_1\|_q \le C\|b_1\|_{q+2} + C\|\sigma_0\|_{q+1}\|\psi b_1\|_{q+1},$$

which is bounded by a constant depending on $\|\sigma_0\|_{q+4}$ with $q + 4 > l + 3n/2 + 1$ by Theorem 4.1.

Combining the discussions above, we finally obtain

$$|E_2| \le C\|\delta\sigma\|_{l_1}\|\Psi\| \tag{6.31}$$

with the constant C depending on $\|\sigma_0\|_\tau$ for $\tau > l + n + 5/2$ and $\tau > l + 3n/2 + 1$.

6.4 A useful Proposition

Until now, according to (6.17,6.20,6.31), we have shown that

$$\|\partial_t^{l_1} v\|_{0,\Omega} \le C\|\delta\sigma\|_{l_1} \tag{6.32}$$

under the hypotheses of Theorem 1.1.

To complete our proof, it suffices to show that

$$\|v\|_{l_1,\Omega_0} \le C\|\partial_t^{l_1} v\|_{0,\Omega_1} + C\|\delta\sigma\|_{l_1} , \tag{6.33}$$

where $\Omega \subset\subset \Omega_1 \subset \mathbf{R}^{n+1}$, and both Ω_0 and Ω_1 are near $\{x_n = 0\}$.

Construct a $\psi.d.o.$ A of order zero:

- $a = 1$ on $\{|w| \ge \epsilon|\xi|\}$;
- $ES(A) \subset \{|w| \ge \epsilon_0|\xi|, \ \epsilon > \epsilon_0\}$.

Let $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $\phi = 1$ on Ω_0, and $supp(\phi) \subset\subset \Omega_1$. Then

$$\phi v = A\phi v + (I - A)\phi v \ .$$

Since $\partial_t^{l_1}$ is elliptic on $ES(A)$, for $\Omega_0 \subset\subset \Omega$

$$||A\phi v||_{l_1,\Omega_0} \leq C||\partial_t^{l_1}\phi v||_{0,\Omega} + C||\phi v||_{0,\Omega} \ .$$

On the other hand, $\Box = \partial_t^2 - \Delta$ is elliptic on $ES(I - A)$ as well as a small conic neighborhood of $ES(I - A)$. Furthermore, one can also ask that ϕ and $\delta\sigma$ have disjoint supports. Then the same idea as in the proof of Lemma 3.2 establishes the following result.

Proposition 6.1 *Assume that $s > max\{l_1 + n/2 - 2, n/2\}$, then the estimate*

$$||(I - A)\phi v||_{l_1,\Omega_0} \leq C||v||_{0,\Omega}$$

holds for some constant C depending on $||\sigma_0||_{s+1}$.

The above discussion and Proposition 6.1 lead to the estimate (6.33), which completes the proof of Theorem 1.1.

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