FUNCTIONAL EQUATIONS IN THE PROBLEM OF BOUNDEDNESS OF STOCHASTIC BRANCHING DYNAMICS

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Abstract. A general model of a branching random walk in $\mathbf{Z}$ (called in the paper stochastic branching dynamics) is considered, where the branching and displacements occur with probabilities determined by the position of a parent particle. A necessary and sufficient condition is given for the random variable

$$M = \sup_{n\geq 0} \max_{1\leq k\leq N_n} X_{n,k}$$

to be finite. Here $X_{n,k}$ is the position of the k-th particle in the n-th generation. The condition is stated in terms of a naturally arising linear functional equation.

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0. Introduction and Results

0.1. Suppose that at times $n = 0, 1, \ldots$ a population of individuals is observed, placed on the one-dimensional lattice $\mathbf{Z}$. After the unit time each individual disappears, giving birth to a random number of offspring that are randomly distributed

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along the lattice. The number of offspring of a given individual and their positions may depend on the place where they were born, but not on the pre-history of the process. The offspring of different individuals are created and positioned on $\mathbf{Z}$ independently. At time zero there is a single individual at site $x \in \mathbf{Z}$. Following [1], we call the random process arising here stochastic branching dynamics. The state of the population in stochastic branching dynamics is described by the positions of individuals. Thus, stochastic branching dynamics is a random process $\mathbf{X} = \{X_n, n = 0, 1, \dots\}$ in a space of counting measures on $\mathbf{Z}$. We denote by $X_n(z)$ the number of individuals at time n positioned at $z \in \mathbf{Z}$. We assume that process $\mathbf{X}$ is time-homogeneous Markov.

Set

$$M = \sup_{n \geq 0} \ \sup \left[z \in \mathbf{Z} : \ X_n(z) \geq 1 \right] \tag{0.0}$$

and

$$F(x, y) = \mathbf{Pr}\left(M < y \,\middle|\, X_0 = \delta_x \right), \ \ x, y \in \mathbf{Z}, \tag{0.1}$$

where δ_x is the Dirac measure at site x. Denote by $\mathcal{U}$ the set of functions u: $\mathbf{Z} \times \mathbf{Z} \to [0, 1]$, such that $u(x, y) = 0$ when $x \geq y$. Function F is a maximal solution of an equation

$$u(x, y) = \mathbf{1}_{\{x<y\}} \mathbf{E} \left(\prod_z u(z, y)^{X_1(z)} \,\middle|\, X_0 = \delta_x \right), \ \ u \in \mathcal{U}; \tag{0.2}$$

see Theorems 2.1 and 2.2 below. Here and below, $0^0 = 1$.

Thus the (natural) question when branching dynamics is bounded in probability (that is, random variable M is proper), i.e.

$$\mathbf{Pr}\left(M < \infty \,\middle|\, X_0 = \delta_x \right) \ = \ 1, \ \ x \in \mathbf{Z}, \tag{0.3}$$

is reduced to the question when the maximal solution F to equation (0.2) obeys

$$\lim_{y \to \infty} F(x, y) = 1. \tag{0.4}$$

The question of boundedness of a stochastic branching dynamics was discussed, in various terms, by a number of authors, see, e.g., [2, 3, 4] (and the references therein), where the space-homogeneous case was considered. The general case is more difficult; an example of a non-homogeneous dynamics was considered in [5]. Some aspects of non-homogeneous diffusion branching dynamics are discussed in [6] (see also the references therein, in particular, the earlier papers by the authors of [6]). In this paper we give a necessary and sufficient condition for (0.4) to hold, in terms of a *linear* equation (see equation (0.6) below) that is naturally associated with (0.2). The paper is a logical continuation of previous papers by the authors (see the references in [5]). The advantage of our approach is that linear equations are much easier to deal with, and our result admits a generalisation to the case of stochastic branching dynamics on a general graph.

0.2. As was noted, for stochastic branching dynamics X_n are counting measures: $X_n(A)$ is a non-negative integer number. However, for the above problem of the maximal solution F to equation (0.2), such a restriction seems unnatural. Let $\mathcal{M}$ denote the space of non-negative measures μ on $\mathbf{Z}$ satisfying the following conditions:

1^0. $\quad 1 \leq \mu(\mathbf{Z}) < \infty.$

2^0. For each $z \in \mathbf{Z}$, either $\mu(z) = 0$ or $\mu(z) \geq 1$. Here and below $\mu(y)$ denotes the measure of a one-point set $\{y\}$.

Clearly, every counting measure on $\mathbf{Z}$ belongs to $\mathcal{M}$. We endow $\mathcal{M}$ with the topology of vague convergence.

Let $\{P_x,\ x \in \mathbf{Z}\}$ be a family of probability measures on $\mathcal{M}$. Consider an equation in space $\mathcal{U}$:

$$u(x,y) = \mathbf{1}_{\{x<y\}} \mathbf{E}_x \left(\prod_z u(z,y)^{\mu(z)} \right), \quad u \in \mathcal{U}; \tag{0.5}$$

$\mathbf{E}_x$ denotes the expectation in P_x. Let F be the maximal solution of equation (0.5). [It always exists: see Theorem 2.1.] We are interested in the question: what are the conditions on $\{P_x\}$ under which function F satisfies (0.4) for any $x \in \mathbf{Z}$?

0.3. We need to introduce several concepts. We say that a site $b \in \mathbf{Z}$ is accessible from site $a \in \mathbf{Z}$ in one step if

$$P_a\big(\mu(b) > 0\big) > 0.$$

We say that b is accessible from a in n steps if there exists a finite sequence of sites $x_0 = a,\ x_1,\ .,\ x_s = b$, where x_i is accessible from x_{i-1} in one step. Consider the following conditions on $\{P_x\}$:

(I) For any $x \in \mathbf{Z}$, $\mathbf{E}_x\big(\mu(\mathbf{Z})^2\big) < \infty$.

(II) For any $a, b \in \mathbf{Z}$, b is accessible from a in a finite number of steps (possibly different for different a and b).

(III) There exists a partition of $\mathbf{Z}$ into finite (lattice) intervals $\Delta_i = \{z_i, z_i + 1, \ldots, z_{i+1} - 1\}$, $i \in \mathbf{Z}$, such that

(III.1) the set of points accessible from $z \in \Delta_i$ in one step is contained in $\bigcup\limits_{j=i-1}^{\infty} \Delta_j$,

(III.2) the product $\prod\limits_{r=0}^{\infty} \alpha(r)$ is divergent, where $\alpha(r) = \min \Big[\mathbf{E}_x\big(\mu(\mathbf{Z})\big), x \in \Delta_r \Big]$.

(IV) For any $a \in \mathbf{Z}$, the set of points accessible from a in one step is finite.

The main result of the paper is

Theorem 1. *Assume that for a family $\{P_x\}$ conditions (I) to (IV) are fulfilled. Let F be the maximal solution of (0.5). Then (0.4) holds iff there exists a positive solution f_0 to a linear equation*

$$f_0(x) = \sum_z f_0(z)\mathbf{E}_x\big(\mu(z)\big), \tag{0.6}$$

with

$$\lim_{x\to\infty} f_0(x) = \infty. \tag{0.7}$$

Without assumptions (I)–(IV), the above condition is still suffiecient for (0.4).

0.4. If, for any $x \in \mathbf{Z}$, measure P_x is concentrated on counting measures $\mu \in \mathcal{M}$, we have stochastic branching dynamics on $\mathbf{Z}$, and Theorem 1 gives an answer to the above question of when random variable M is proper. If stochastic branching dynamics are space-homogeneous, Theorem 1 follows from results in [2, 3, 4]. As was noted, in paper [5] a particular case of a non-homogeneous branching dynamics was considered, where the distribution of the number of offspring of an individual at site x does not depend on x and their positions are independent of each other and take values x and $x \pm 1$ (with probabilities that may depend on x).

0.5. In Section 1 we introduce the operation h^μ (here, $\mu \in \mathcal{M}$ and $h : \mathbf{Z} \to [0,1]$) and discuss its properties. In Section 2 we construct a particular solution to (0.5) and prove that it is maximal; we also show that function F, introduced in (0.1), coincides with the maximal solution. In Section 3 we check the sufficiency of the existence of a positive solution to (0.6)–(0.7) for (0.4). In Section 4 we construct, for a given family $\{P_x\}$, a Markov process $\mathbf{X}^0 = \{X_n^0, n = 0, 1, \dots\}$ and establish its properties, in terms of measures P_x. Finally, in Section 5 we check the necessity of the existence of a positive solution to (0.6)–(0.7).

Among conditions (I)–(IV), (III.1) seems the least natural. However, the appendix gives an example of a family of counting measures P_x for which all conditions hold, except for (III.1), and the assertion of Theorem 1 fails; in this example, (0.4) holds, but there is no solution to (0.6) with property (0.7).

0.6. Throughout the paper $\mathcal{H}$ denotes the set of functions $h : \mathbf{Z} \to [0,1]$ and notation $\mathcal{U}$ is used from **0.1.** If φ is a function on $\mathbf{Z} \times \mathbf{Z}$ and $y \in \mathbf{Z}$, φ_y denotes a function on $\mathbf{Z}$ defined by $\varphi_y(x) = \varphi(x,y)$. We also set $\mathbf{1}_< = \mathbf{1}_{x<y}$. The ordering $u \le v$ and the convergence $u_n \to u$ (or $\lim\limits_{n\to\infty} u_n = u$) mean that the corresponding relations hold pointwise everywhere in the domain of the corresponding functions. If μ and φ are a measure and a function on $\mathbf{Z}$, we denote by $\langle \mu\varphi \rangle$ the integral of φ in μ:

$$\langle \mu\varphi \rangle = \sum_z \varphi(z)\mu(z).$$

Finally we set $\langle \mu \rangle = \langle \mu 1 \rangle = \mu(\mathbf{Z})$.

1. Operation h^μ

1.1. Lemma 1.1. *Assume two sequences of numbers, m_1, ..., $m_k \ge 1$ and x_1, ..., $x_k \in [0,1]$, are given, and a variable $\lambda \in [0,1]$. Then*

$$\prod_1^k \left(1 - \lambda x_i\right)^{m_i} = 1 - \lambda \sum_1^k m_i x_i + \frac{\lambda^2}{2(1-s)^2} \left(\sum_1^k m_i x_i \right)^2 \beta, \tag{1.1}$$

where $0 < s < \lambda$ and $0 \le \beta \le 1$.

Proof of Lemma 1.1. Denote the LHS of (1.1) by $\pi(\lambda)$. Differentiating in λ and using conditions $m_i \ge 1$ and $x_i \in [0,1]$ yields $\pi''(\lambda) > 0$. In addition,

$$\pi'(\lambda) = -\pi(\lambda) \sum_1^k \frac{m_i x_i}{1 - \lambda x_i}$$

and

$$\pi''(\lambda) = \pi(\lambda)\left(\left(\sum_1^k \frac{m_i x_i}{1-\lambda x_i}\right)^2 - \sum_1^k \frac{m_i x_i^2}{(1-\lambda x_i)^2}\right).$$

Together with

$$\pi(\lambda) = \pi(0) + \lambda\pi'(0) + \frac{\lambda^2}{2}\pi''(s), \quad 0 < s < \lambda,$$

this implies (1.1).

1.2. For a function $h \in \mathcal{H}$ and a measure $\mu \in \mathcal{M}$ we define a number h^μ by

$$h^\mu = \prod_z h(z)^{\mu(z)}. \tag{1.2}$$

Owing to 1^0 and 2^0, all but a finite number of terms in the product are equal to 1. Note that the map $(h,\mu) \mapsto h^\mu$ is continuous.

Lemma 1.2. *If* $f \in \mathcal{H}$, $\mu \in \mathcal{M}$ *and* $\lambda \in [0,1]$, *then*

$$(1-\lambda f)^\mu = 1 - \lambda\langle\mu f\rangle + \frac{\lambda^2}{2(1-s)^2}(\langle\mu f\rangle)^2\beta, \tag{1.3}$$

where $0 < s < \lambda$ and $0 \le \beta \le 1$.

Proof of Lemma 1.2. By 1^0 and 2^0, the support of measure μ is finite. Suppose supp $\mu = \{z_1,\ldots,z_k\}$. Set $\mu(z_i) = m_i$. From 2^0 it follows that $m_i \ge 1$, $i = 1,\ldots,k$. Denoting $x_i = f(z_i)$, we obtain (1.3) from Lemma 1.1.

Lemma 1.3. *Suppose that* $f \in \mathcal{H}$, $\mu \in \mathcal{M}$ *and a set* $S \supset$ *supp* μ. *Denote* $l =$ *sup* $[\, f(z) : z \in S\,]$. *Then the representation*

$$(1-f)^\mu = 1 - \langle\mu f\rangle + \frac{(\langle\mu f\rangle)^2}{2(1-s)^2}\beta \tag{1.4}$$

holds, where $0 < s < l$ *and* $0 \le \beta \le 1$.

Proof of Lemma 1.3. Consider a function

$$f_1(z) = \min\left(1, \frac{f(z)}{l}\right).$$

It is clear that $f_1 \in \mathcal{H}$. For f_1, write expansion (1.3) with $\lambda = l$ (clearly, $l \leq 1$). We obtain

$$(1 - lf_1)^\mu = 1 - l\langle \mu f_1 \rangle + \frac{l^2(\langle \mu f_1 \rangle)^2}{2(1-s)^2}\beta,$$

where $0 < s < l$ and $0 \leq \beta \leq 1$. Since f_1 coincides, on S, with f/l, and $\langle \mu f_1 \rangle$ and h^μ depend on the values of f_1 and h on supp μ only, we obtain (1.4).

Lemma 1.4. *For any* $h \in \mathcal{H}$ *and* $\mu \in \mathcal{M}$, *the following bound holds:*

$$h^\mu \leq \frac{\langle \mu h \rangle}{\langle \mu \rangle}. \tag{1.6}$$

Proof of Lemma 1.4. As was noted, supp μ is finite. As before, set $S = \{z_1, \ldots, z_k\}$, $\mu(z_i) = m_i$, $h(z_i) = x_i$, $i = 1, ..., k$, and $m = \sum_1^k m_i$. Then

$$h^\mu = \prod_1^k x_i^{m_i}. \tag{1.6}$$

A well-known inequality

$$\prod_1^k x_i^{r_i} \leq \sum_1^k r_i x_i \tag{1.7}$$

holds for any $x_i \in [0,1]$ and $r_i \in [0,1]$, with $\sum_1^k r_i = 1$. Taking $r_i = m_i/m$ and noting that $m \geq 1$, we derive (1.5) from (1.6) and (1.7).

2. Existence of the maximal solution to (0.5)

2.1. Consider the following non-linear operator L on space $\mathcal{U}$:

$$(Lu)(x,y) = \mathbf{1}_< \mathbf{E}_x(u_y^\mu). \tag{2.1}$$

Operator L preserves the order between functions: if $u \leq v$ then

$$Lu \leq Lv. \tag{2.2}$$

Equation (0.5) now takes the form

$$Lu = u, \quad u \in \mathcal{U}. \tag{2.3}$$

Set

$$F_0(x,y) = \mathbf{1}_< , \tag{2.4}$$

and

$$F_{n+1}(x,y) = \big(LF_n\big)(x,y), \quad n = 0,1,\dots . \tag{2.5}$$

Theorem 2.1. *Functions* F_n *form a non-increasing sequence of functions from* $\mathcal{U}$. *The limit*

$$F = \lim_{n\to\infty} F_n \tag{2.6}$$

gives the maximal solution to (2.3) in the sense that, if $u \in \mathcal{U}$ *is an arbitrary solution of (2.3), then*

$$u \le F. \tag{2.7}$$

Proof of Theorem 2.2. Obviously, $F_n \in \mathcal{U}$, $n = 0,1, \dots$. Hence, $F_1 \le \mathbf{1}_< = F_0$. Applying operator L and using (2.2), we obtain that sequence $\{F_n\}$ is non-increasing. Letting $n \to \infty$, in (2.5), and using (2.6), we conclude that F satisfies (2.3). It remains to check (2.7); fix an arbitrary solution, $u \in \mathcal{U}$, to (2.3). Then $u \le \mathbf{1}_< = F_0$. Again applying operator L and using (2.2) and (2.5), we obtain $u \le F_n$. Letting $n \to \infty$ yields the result.

2.2. Suppose that for any $x \in \mathbf{Z}$ P_x is concentrated on counting measures $\mu \in \mathcal{M}$. As was noted, family $\big\{P_x\big\}$ determines stochastic branching dynamics. In fact, for $\mathcal{A} \subseteq \mathcal{M}$, setting

$$\mathbf{Pr}\Big(X_1 \in \mathcal{A} \,\Big|\, X_0 = \delta_x\Big) = P_x(\mathcal{A}) \tag{2.8}$$

leads, via the independence, to a Markov process $\mathbf{X} = \{X_n,\ n = 0,1,\dots\}$, on the space of counting measures, with the initial state δ_x. Denoting

$$M_n = \max_{1\le k\le n} \ \max\,[z \in \mathbf{Z} :\ X_k(z) \ge 1], \tag{2.9}$$

we observe that, for the random variable M (see (0.0)),

$$M = \sup_n \ M_n. \tag{2.10}$$

It is plain that

$$M_n \nearrow M, \quad \text{for} \quad n \to \infty. \tag{2.11}$$

By construction,

$$\mathbf{Pr}\Big(M_0 < y \bigm| X_0 = \delta_x\Big) = \mathbf{1}_< = F_0(x,y).$$

Using the Markov property of the dynamics, it is easy to check that probabilities $\mathbf{Pr}\Big(M_n < y \bigm| X_0 = \delta_x\Big)$ satisfy (2.5). Therefore

$$F_n(x,y) = \mathbf{Pr}\Big(M_n < y \bigm| X_0 = \delta_x\Big), \ \ n = 0, 1, \dots . \tag{2.12}$$

Letting $n \to \infty$ and using (2.11) leads to the following theorem:

Theorem 2.2. *If measure P_x is concentrated, for any $x \in \mathbf{Z}$, on counting measures, and F is the maximal solution to (2.3), then*

$$F(x,y) = \mathbf{Pr}\ \Big(M < y \bigm| X_0 = \delta_x\Big).$$

3. Proof of Theorem 1: sufficiency

3.1. Consider an operator K taking a function $u \in \mathcal{U}$ to a function on $\mathbf{Z} \times \mathbf{Z}$:

$$(Ku)(x,y) = \mathbf{E}_x\big(u_y^\mu\big), \ \ x, y \in \mathbf{Z}. \tag{3.1}$$

Observe that K preserves the inequality: if $u \le v$ then

$$Ku \le Kv. \tag{3.2}$$

Set $u = 1 - v$. Taking, in Lemma 1.3, $f = v_y$ and $S = \mathbf{Z}$, we deduce from (1.4) that

$$u_y^\mu \ge 1 - \langle \mu v_y \rangle.$$

Hence, for any $u = 1 - v \in \mathcal{U}$,

$$(Ku)(x,y) \ge 1 - \mathbf{E}_x\big(\langle \mu v_y \rangle\big). \tag{3.3}$$

Theorem 3.1. *Suppose that there exists a positive solution to (0.6), with property (0.7). Then, for the maximal solution, F, to (0.5), for any x relation (0.4) holds.*

3.2. *Proof of Theorem 3.1.* Let f_0 be a positive solution of (0.6)–(0.7). Given $y \in \mathbf{Z}$, denote

$$a(y) = \inf \left[f_0(x) : \ x \geq y \right]. \tag{3.4}$$

Following from (0.7), the infimum in (3.4) is attained. Therefore, as $f_0 > 0$, then $a > 0$. Furthermore, function a is non-decreasing, and

$$\lim_{y \to \infty} a(y) = \infty. \tag{3.5}$$

Consider a function u_0 on $\mathbf{Z} \times \mathbf{Z}$ given by

$$u_0(x, y) = \max \left(0, \ 1 - \frac{f_0(x)}{a(y)} \right). \tag{3.6}$$

It is easy to check that $u_0 \in \mathcal{U}$. Set $u_0 = 1 - v$, then from (3.3) we obtain

$$(Ku_0)(x, y) \geq 1 - \mathbf{E}_x \big(\langle \mu v_y \rangle \big). \tag{3.7}$$

Since $u_0(x, y) \geq 1 - \dfrac{f_0(x)}{a(y)}$, we have

$$v_y \leq \frac{1}{a(y)} f_0.$$

From (0.6) we deduce that

$$\mathbf{E}_x \big(\langle \mu v_y \rangle \big) \leq \frac{1}{a(y)} \mathbf{E}_x \big(\langle \mu f_0 \rangle \big) = \frac{f_0(x)}{a(y)}.$$

Therefore, according to (3.7),

$$(Ku_0)(x, y) \geq 1 - \frac{f_0(x)}{a(y)}.$$

But $(Ku_0)(x, y) \geq 0$; thus

$$(Ku_0)(x, y) \geq \max \left(0, \ 1 - \frac{f_0(x)}{a(y)} \right) = u_0(x, y).$$

Multiplying the last inequality by $\mathbf{1}_<$ and using the fact that $u_0(x, y) = 0$ for $x \geq y$, we obtain

$$Lu_0 \geq u_0. \tag{3.8}$$

Now set $u_{n+1} = Lu_n$, $n = 0, 1, \ldots$. From (3.8) and (2.2) it follows that $\{u_n\}$ is a non-decreasing sequence from $\mathcal{U}$, bounded by $\mathbf{1}_<$. Therefore, there exists the limit

$$\lim_{n \to \infty} u_n = u.$$

It is plain that $u \in \mathcal{U}$, $u \geq u_0$, and that u satisfies (2.3) (which is equivalent to (0.5)). By Theorem 2.1, $F \geq u$, and hence

$$F \geq u_0. \tag{3.9}$$

According to (3.5), for any $x \in \mathbf{Z}$,

$$\lim_{y \to \infty} u_0(x, y) = 1.$$

The last relation, together with (3.9), implies (0.4). This completes the proof of sufficiency.

4. Process $\mathbf{X}^0$

4.1. Consider a pair of linear operators, Q and Q^0, acting on functions $f: \mathbf{Z} \to \mathbf{R}$ and defined by

$$Qf(x) = \mathbf{E}_x \langle \mu f \rangle \tag{4.1}$$

and

$$Q^0 f(x) = \mathbf{E}_x \frac{\langle \mu f \rangle}{\langle \mu \rangle}. \tag{4.2}$$

These operators are determined by non-negative kernels q and q^0:

$$Qf(x) = \sum_y q(x, y) f(y) \tag{4.3}$$

and

$$Q^0 f(x) = \sum_y q^0(x, y) f(y). \tag{4.4}$$

Properties 1^0 and 2^0 guarantee that, for any $\mu \in \mathcal{M}$, $\langle \mu \rangle \geq 1$. Therefore, for any $x, y \in \mathbf{Z}$,

$$0 \leq q^0(x, y) \leq q(x, y). \tag{4.5}$$

Operator Q^0 takes the unit function $\mathbf{1}$ to itself; thus q^0 is a stochastic kernel:

$$\sum_y q^0(x, y) = 1. \tag{4.6}$$

Hence q^0 determines a time-homogeneous Markov chain $\mathbf{X}^0 = \{X_n^0, n = 0, 1, \dots\}$ on $\mathbf{Z}$. So, for any function $f: \mathbf{Z} \to \mathbf{R}$,

$$\mathbf{E}_x \left(f(X_1^0) \mid X_0^0 = x \right) = \mathbf{E}_x \frac{\langle \mu f \rangle}{\langle \mu \rangle}. \tag{4.8}$$

4.2. Process $\mathbf{X}^0$ is a particular case of stochastic branching dynamics on $\mathbf{Z}$, with the number of offspring equal to one. [The corresponding random measure at time n coincides with the Dirac measure $\delta_{X^0_n}$.] One can introduce, for process $\mathbf{X}^0$, all objects and concepts used earlier for stochastic branching dynamics. We use the same notation, with upper index 0. So, according to (2.9) and (2.10),

$$M^0_n = \max\ \big(X^0_k :\ 0 \le k \le n\big), \qquad M^0 = \sup\ \big(X^0_n :\ n \ge 0\big).$$

Operators K and L become

$$(K^0 u)(x,y) = \mathbf{E}_x u_y\big(X^0_1\big) = Q^0 u_y(x), \ \ u \in \mathcal{U}, \tag{4.9}$$

and

$$(L^0 u)(x,y) = \mathbf{1}_< \mathbf{E}_x u_y\big(X^0_1\big) = \mathbf{1}_< Q^0 u_y(x), \ \ u \in \mathcal{U}. \tag{4.10}$$

Using Theorems 2.1 and 2.2, we arrive at

Theorem 4.1. *Function*

$$F^0(x,y) = \mathbf{Pr}\ \big(M^0 < y \bigm| X^0_0 = x\big) \tag{4.11}$$

is the maximal solution of the equation

$$u = L^0 u, \quad u \in \mathcal{U}. \tag{4.12}$$

We need the following lemma:

Lemma 4.2. *For any function* $u \in \mathcal{U}$,

$$Ku \le K^0 u, \ \ u \in \mathcal{U}, \tag{4.13}$$

and

$$Lu \le L^0 u, \ \ u \in \mathcal{U}. \tag{4.14}$$

Proof of Lemma 4.2. Operators K and L (and K^0 and L^0) differ by factor $\mathbf{1}_<$. Therefore, it is sufficient to prove (4.13) only. By Lemma 1.4,

$$u^\mu_y \le \frac{\langle \mu u_y \rangle}{\langle \mu \rangle}. \tag{4.15}$$

For the expectations in measure P_x we obtain, by using (3.1), (4.2), (4.9) and (4.15),

$$(Ku)(x,y) \le \mathbf{E}_x \frac{\langle \mu u_y \rangle}{\langle \mu \rangle} = P^0 u_y(x) = (K^0 u)(x,y).$$

Corollary 4.3. *The following inequality holds:*

$$F \le F^0. \tag{4.16}$$

Proof. Consider a sequence of functions u_0, u_1, ., defining

$$u_0 = F, \;\; u_{n+1} = Lu_n, \;\; n = 0, 1, \ldots .$$

It follows from (4.15) that u_n form a non-decreasing sequence from $\mathcal{U}$. Any function from $\mathcal{U}$ is bounded by $\mathbf{1}_<$. Therefore, u_n converge, as $n \to \infty$, to a limit $u \in \mathcal{U}$. Function u satisfies (4.12) and $u \ge u_0 = F$. Hence $F^0 \ge F$.

From (4.16) we immediately get

Corollary 4.4. *If the maximal solution* F *of equation (0.5) satisfies (0.4) then, for any* $x \in \mathbf{Z}$,

$$\mathbf{Pr}\left(M^0 < \infty \;\middle|\; X_0^0 = x \right) = 1. \tag{4.17}$$

4.3. It is easy to check that the set of the lattice sites accessible in one step from a site $a \in \mathbf{Z}$ coincides with the corresponding set for process $\mathbf{X}^0$. Thus conditions (II), (III.1) and (IV) are equivalent to similar conditions stated in terms of process $\mathbf{X}^0$:

(II′) Any site $b \in \mathbf{Z}$ is accessible, for process $\mathbf{X}^0$, from any other site $a \in \mathbf{Z}$ in finitely many steps.

(III.1′) For the intervals Δ_i figuring in (III), for any i the set of sites accessible for $\mathbf{X}^0$ in one step from $z \in \Delta_i$ is contained in $\bigcup_{j=i-1}^{\infty} \Delta_j$.

(IV) For any $a \in \mathbf{Z}$, the set of points accessible for $\mathbf{X}^0$ in one step from a is finite.

Recall that a function f on $\mathbf{Z}$ is called excessive for process $\mathbf{X}^0$ if $f(z) \geq 0$, $z \in \mathbf{Z}$, and f obeys

$$Q^0 f \leq f. \tag{4.18}$$

Lemma 4.5. *Under condition (II′), any excessive function not identical to zero is strictly positive.*

Proof of Lemma 4.5. Let O denote the set of zeros of an excessive function f. It follows from (4.18) that if $x_0 \in O$ then all sites accessible for $\mathbf{X}^0$ in one step from x_0 belong to O. Thus, either O is empty or it coincides with $\mathbf{Z}$.

Lemma 4.6. *Suppose that for process* $\mathbf{X}^0$ *conditions (II′) and (III.1′) hold, and for some* $x_0 \in \mathbf{Z}$,

$$\mathbf{Pr}\left(M^0 < \infty \mid X_0^0 = x_0\right) = 1. \tag{4.19}$$

Also suppose that f *is an excessive function for* $\mathbf{X}^0$. *Then, for any* $i \in \mathbf{Z}$,

$$\min\ \big(f(z):\ z \in \Delta_i\big) \leq \min\ \big(f(z):\ z \in \Delta_{i+1}\big). \tag{4.20}$$

Proof of Lemma 4.6. Denote

$$A_i = \cup_{j=i}^{\infty} \Delta_j.$$

Owing to (III.1′),

$$\mathbf{Pr}\left(X_1^0 \in A_i \mid X_0^0 \in A_{i+1}\right) = 1. \tag{4.21}$$

Given $l > i$, denote by $g_l(x)$, $x \in A_i$, the probability that process $\mathbf{X}^0$, starting at site x, hits A_l earlier than Δ_i. Function g_l takes value zero on Δ_i, value one on A_l and satisfies, on $A_i^l = A_{i+1} \setminus A_l = \bigcup_{j=i+1}^{l-1} \Delta_j$, an equation

$$Q^0 g_l(x) = g_l(x). \tag{4.22}$$

Now suppose that (4.20) fails: for some $z_0 \in \Delta_{i+1}$,

$$f(z_0) < m, \tag{4.23}$$

where $m = \min\ (f(z) : z \in \Delta_i)$. Consider a function φ, on set A_i, given by

$$\varphi(x) = \frac{f(x)}{m} + g_l(x).$$

Function φ is ≥ 1 on $A_l \cup \Delta_i$; on A_i^l it obeys

$$Q^0 \varphi(x) \leq \varphi(x).$$

In view of (4.21) and the minimum principle for the excessive functions (see, e.g., [7]), $\varphi(x) \geq 1$ for $x \in A_i$. Hence

$$g_l(x) \geq 1 - \frac{f(x)}{m}. \tag{4.24}$$

Condition (II$'$) then implies that if (4.19) holds for some starting point x_0 it holds for any other starting point, in particular for the starting point z_0 from (4.23). On the other hand, using (4.23) and (4.24), we obtain

$$\mathbf{Pr}\left(M^0 = \infty \mid X_0^0 = z_0\right) \geq \lim_{l \to \infty} g_l(z_0) \geq 1 - \frac{f(z_0)}{m} > 0,$$

which contradicts (4.19).

5. Proof of Theorem 1: necessity

Throughout this section we assume that properties (I)–(IV) (and hence (II$'$)–(IV$'$)) are valid.

5.1. Lemma 5.1. *If, for the maximal solution of (0.5), relation (0.4) holds then any non-negative solution of (0.6), which is not identical to zero, is strictly positive and obeys (0.7).*

Proof of Lemma 5.1. Let f be a solution to (0.6), not identical to zero. Then

$$f = Qf, \tag{5.1}$$

where Q is the operator defined in (4.1). It follows from (4.3)–(4.5) and (5.1) that f is an excessive function for $\mathbf{X}^0$. By virtue of Lemma 4.5, $f > 0$ on $\mathbf{Z}$.

It remains to check that f obeys (0.7). To this end, set

$$\widehat{f}(r) = \min\ \left(f(z) : z \in \Delta_r\right),$$

where Δ_r, $r \in \mathbf{Z}$, are the intervals from (III). By using Corollary 4.4 and Lemma 4.6, we find that $\widehat{f}(r)$ is non-decreasing in r. Let x_r be a site in Δ_r, where f takes value $\widehat{f}(r)$. Then, using (5.1) and (III.1′), we have

$$\widehat{f}(r) = f(x_r) = Qf(x_r) \geq \widehat{f}(r-1)\mathbf{E}_{x_r}\big(\langle\mu\rangle\big) = \widehat{f}(r-1)\mathbf{E}_{x_r}\big(\mu(\mathbf{Z})\big).$$

Therefore,

$$\widehat{f}(r) \geq \widehat{f}(r-1)\alpha(r),$$

cf. (III.2). For $k > 0$, we have

$$\widehat{f}(k) \geq \widehat{f}(0)\prod_1^k \alpha(r),$$

which, in view of (III.2), yields (0.7).

5.2. Set

$$F = 1 - G$$

and

$$H(x,y) = \frac{G(x,y)}{G(0,y)}.$$

Lemma 5.2. *For any* $a \in \mathbf{Z}$, $H(a,y)$ *is bounded in* y.

Proof of Lemma 5.2. Using (4.14), we find that

$$F(x,y) = (LF)(x,y) \leq (L^0F)(x,y).$$

Thus,

$$G(x,y) \geq (K^0G)(x,y), \quad x < y. \tag{5.2}$$

[Operators L^0 and K^0 are defined in (4.10) and (4.9), respectively.] According to (II′), there exists a sequence of sites $x_0 = 0$, x_1, ..., $x_n = a$ for which

$$q^0(x_i, x_{i+1}) > 0, \quad i = 0, \ldots, n-1, \tag{5.3}$$

where q^0 is defined in (4.4).

Assume that $y > \max\ \big(x_0, \ldots, x_{n-1}\big)$. Using (4.9) and (5.2), we obtain that

$$G(x_i, y) \geq \sum_z q^0(x_i, z)G(z,y) \geq q^0(x_{i+1}, y)G(x_{i+1}, y), \quad i = 0, \ldots, n-1.$$

This yields

$$G(0,y) = G(x_0,y) \geq G(x_n,y) \prod_{i=0}^{n-1} q^0(x_i,x_{i+1}).$$

The last inequality, together with (5.3) and the fact that $G(x_n,y) = G(a,y)$, gives the assertion of the lemma.

5.3. From Lemma 5.2 it follows that there exists a sequence y_1, ..., y_n, ... such that for any x there exists a finite limit

$$\lim_{n\to\infty} H(x,y_n) = f_0(x). \tag{5.4}$$

Theorem 5.3. *If the maximal solution F of equation (0.5) satisfies (0.4) then there exists a positive solution to (0.6)–(0.7).*

Proof of Theorem 5.3. We will check that function f_0 defined by (5.4) is a positive solution to (0.6)–(0.7). Fix a point $x_0 \in \mathbf{Z}$ and assume that x_0 belongs to an interval Δ_i . Let A denote the set of points accessible from x_0 in one step. According to condition (IV), set A is finite. Setting

$$l_n = \max\ \big(G(x,y_n):\ x \in A\big), \tag{5.5}$$

write (0.4) in the equivalent form

$$\lim_{y\to\infty} G(x,y) = 0, \quad x \in \mathbf{Z}, \tag{5.6}$$

and conclude that

$$\lim_{n\to\infty} l_n = 0. \tag{5.7}$$

Suppose that $y_n > x_0$. From (0.5) we obtain that

$$F(x_0,y_n) = \mathbf{E}_{x_0}\left(F^{\mu}_{y_n}\right). \tag{5.8}$$

In equality (5.8), one can assume that supp $\mu \subset A$. Thus we can use Lemma 1.3, setting $f = G_{y_n}$ and $S = A$. Then, by virtue of (1.4) and (5.5),

$$F^{\mu}_{y_n} = 1 - \langle \mu G_{y_n} \rangle + \frac{\langle \mu G_{y_n} \rangle^2 \beta_n}{2(1-l_n)^2},$$

where $0 \le \beta_n \le 1$.

Therefore,

$$\mathbf{E}_{x_0}\left(F^{\mu}_{y_n}\right) = 1 - \mathbf{E}_{x_0}\langle \mu G_{y_n}\rangle + \frac{1}{2(1-l_n)^2}\mathbf{E}_{x_0}\left(\langle \mu G_{y_n}\rangle^2 \beta_n\right).$$

Together with (5.8), (4.1) and (4.3) this yields

$$G(x_0, y_n) = \sum_{z\in A} q(x_0, z)G(z, y_n) - \frac{1}{2(1-l_n)^2}\mathbf{E}_{x_0}\left(\langle \mu G_{y_n}\rangle^2 \beta_n\right)$$

and, after dividing by $G(0, y_n)$,

$$H(x_0, y_n) = \sum_{z\in A} q(x_0, z)H(z, y_n) - \frac{G(0,y_n)}{2(1-l_n)^2}\mathbf{E}_{x_0}\left(\langle \mu H_{y_n}\rangle^2 \beta_n\right). \tag{5.9}$$

As before, one can assume in equality (5.9) that $\operatorname{supp}\,\mu \subset A$. By Lemma 5.2, $\dfrac{\langle \mu H_{y_n}\rangle}{\langle \mu\rangle}$ is bounded by a constant c that may depend on x_0, but not on n. Thus,

$$\mathbf{E}_{x_0}\left(\langle \mu H_{y_n}\rangle^2 \beta_n\right) \le c^2 \mathbf{E}_{x_0}\left(\langle \mu\rangle^2\right),$$

and, by virtue of (5.9),

$$H(x_0, y_n) = \sum_{z\in A} q(x_0, z)H(z, y_n) - \frac{G(0,y_n)}{2(1-l_n)^2}c_n, \tag{5.10}$$

where c_n is bounded by $c^2\mathbf{E}_{x_0}\big(\langle\mu\rangle^2\big)$. By virtue of condition (I), $\mathbf{E}_{x_0}\big(\langle\mu\rangle^2\big) = \mathbf{E}_{x_0}\big(\mu(\mathbf{Z})^2\big) < \infty$.

Letting in (5.10) $n \to \infty$ and using (5.4), (5.6) and (5.7), we obtain, by virtue of (4.1) and (4.3), that

$$f_0(x_0) = \sum_{z\in A} q(x_0, z) f_0(z) = \mathbf{E}_{x_0}\langle \mu f_0\rangle.$$

Therefore f_0 satisfies (0.6). It is plain that $f_0 \ge 0$. Using Lemma 5.1 completes the proof of Theorem 5.3.

Theorem 1 follows immediately from Theorems 3.1 and 5.3.

6. Appendix

6.1. We construct here an example of stochastic brancing dynamics where the family $\{P_x\}$ satisfies all conditions (I)–(IV) but (III.1), and the assertion of Theorem 1 fails. That is, the maximal solution F of equation (0.5) satisfies (0.4) whereas problem (0.6)–(0.7) does not have a solution. We start by analyzing simple stochastic branching dynamics on $\mathbf{Z}$. Assume that, in branching dynamics, each individual produces, after the unit time, precisely two offspring. If the individual is positioned at time n at site $x \in \mathbf{Z}$, then at time $n+1$ its offspring are positioned independently of each other at sites $x+1$ with probability p and $x-1$ with probability q. It is easy to check that in this case all conditions (I)–(IV) are fulfilled. Equation (0.6) takes the form

$$f_0(x) = 2\big(pf_0(x+1) + qf_0(x-1)\big),$$

and its general solution is

$$f_0(x) = C_1\lambda_1^x + C_2\lambda_2^x,$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$1 = 2\left(p\lambda + \frac{q}{\lambda}\right). \tag{6.1}$$

According to Theorem 1, relation (0.4) (and, in view of Theorem 2.2, also (0.3)) holds iff (6.1) possesses a root >1. This condition is equivalent to

$$\inf_{\lambda\geq 1}\left(\lambda p + \frac{q}{\lambda}\right) \leq \frac{1}{2}, \tag{6.2}$$

or to

$$p \leq \frac{1}{2} - \frac{\sqrt{3}}{4}. \tag{6.3}$$

6.2. To modify the above example, assume that if the individual is positioned at site $x < 0$ then, as before, each of its two offspring is positioned, independently of each other, at site $x+1$ with probability p and at site $x-1$ with probability $q = 1-p$. On the other hand, if $x \geq 2$ and x is even then each of the offspring is positioned, again independently of each other, with probability q at $x-2$, with probability $p-\epsilon$ at $x+2$ and with probability ϵ at $x-1$. Here, $\epsilon > 0$ is small enough. If $x \geq 1$ and x is odd then with probability one both offspring are positioned at site 0. Finally, if $x = 0$ then each of two offspring, still independently of each other, is positioned with probability q at -1 and with probability p at site 2.

It is not hard to check that, for the family of probability measures P_x which corresponds to modified stochastic branching dynamics, all properties (I)–(IV) are valid, except for (III.1). Comparing these dynamics to those from **6.1**, we see that, if (6.3) holds, then (0.3) holds, which implies (0.4). But for odd $x \geq 1$, relation (0.6) takes the form $f_0(x) = 2f_0(0)$. Therefore, for any solution of (0.6), relation (0.7) fails to hold. Thus, for modified dynamics (0.4) is fulfilled, but there is no solution to (0.6) – (0.7).

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