A NOTE ON THE UNIQUENESS FOR AN INVERSE DIFFRACTION PROBLEM

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IMA Preprint Series # 1220

March 1994

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Abstract. In this note, a local uniqueness result is proved for an inverse diffraction problem. Moreover, an explicit characterization for the "locality" is given.

1. Introduction. Consider a time-harmonic electromagnetic plane wave incident on a periodic structure in IR^2. The periodic structure is the surface of some perfectly reflecting material or conductor. The inverse problem is to determine the periodic structure or the shape of the interface from the scattered field. In this paper, a local uniqueness theorem is proved by using one measurement. Our theorem indicates that any two surface profiles are identical if they generate the same scattering fields (or patterns) and the area in between the two profiles are sufficiently small. Moreover, the smallness of the area is characterized explicitly in terms of a condition which relates k, the period, and the maximum of the difference in height allowed for the two profiles. This work is motivated by the study of optimal design problems of gratings where one wishes to design a grating (or periodic) structure that generates some specified scattered field.

The scattering of electromagnetic waves in a periodic structure has recently received much attention. We refer to [2], [6], and [3] for existence and uniqueness results, and to [7] for applications. The inverse scattering problem in periodic structure was studied in [3] and [4], [5] when the index of refraction is real above the structure, and in [1] in the case of lossy medium (the index of refraction is complex) above the structure. In [5], Kirsch proved a uniqueness theorem by using many various incident waves. For one incident plane wave, Dobson [4] proved a local uniqueness theorem by an analytic argument. The proof of the uniqueness theorem in this work is given by an extension of the continuation idea in [1]. Moreover, by estimating the eigenvalue, one can get precise idea on how close the profiles have to be for uniqueness to hold.

2. The main result. Let the scattering profile (object) be described by the curve $\Gamma = \{(x_1, x_2) : x_2 = f(x_1)\}$ with a periodic function f of period $\Lambda > 0$. The function f is supposed to be sufficiently smooth, for example of $C^{1,1}$. The space below Γ is filled with some perfectly reflecting material. Let $\Omega = \{x \in IR^2 : x_2 > f(x_1), x_1 \in IR\}$ be filled with a material whose index of refraction is a real constant $k > 0$. Suppose that a plane wave given by $u_I = e^{i\alpha x_1 - i\beta x_2}$ is incident on Γ from the top. Here $\alpha = k sin\theta$, $\beta = k cos\theta$, and $-\pi/2 < \theta < \pi/2$ is the incident angle. The scattering of time harmonic electromagnetic waves in the TE (transverse electric) mode can then be modeled by the

Helmholtz equation with a homogeneous Dirichlet boundary condition

$$(\triangle + k^2)u = 0, \quad \text{in } \Omega, \tag{1}$$

$$u|_\Gamma = 0 \, . \tag{2}$$

We seek for quasiperiodic solutions to this problem, *i.e.* the solution u such that $ue^{-i\alpha x_1}$ is Λ-periodic for every x_2. To completely specify the boundary value problem, we need to impose a radiation condition.

Since $ue^{-i\alpha x_1}$ is Λ-periodic, one can expand u in a Fourier series

$$u(x_1, x_2) = \sum_{n \in Z} u_n(x_2) e^{i\alpha_n x_1}, \quad \alpha_n = n + \alpha \, , \tag{3}$$

where $u_n = \frac{1}{\Lambda}\int_0^\Lambda u(x_1, x_2) e^{-i\alpha_n x_1} dx_1$. The radiation condition that we impose is the boundedness of u as x_2 tends to infinity. More precisely, we insist that u is composed of bounded outgoing plane waves plus the incident wave u_I. Let

$$\beta_n = e^{\theta_n/2} |k^2 - \alpha_n^2|^{1/2},$$

with $\theta_n = arg(k^2 - \alpha_n^2)$, $0 \le \theta_n < 2\pi$, and $\beta_0 = \beta$. The radiation condition and the knowledge of the fundamental solution as in [2] yield

$$u(x_1, x_2) = \sum_{n \in Z} a_n e^{i\beta_n x_2 + i\alpha_n x_1} + u_I \, , \quad x_2 > max\{f(x_1)\} \tag{4}$$

where the coefficients $\{a_n\}$ are complex scalars. We shall assume $k^2 \ne \alpha_n^2$ to avoid waves that propagate along the x_1 axis.

For the direct scattering problem, questions on existence and uniqueness are well understood, see for example [2] and [6]. Basically the scattering problem (1), (2), and (4) specified above has a unique quasiperiodic solution, for all but a discrete set of the frequencies or k. Since k is fixed constant, in this note, we shall always assume that the direct scattering problem has a unique solution. The inverse problem can be stated as follows: Let T be a fixed constant such that $T > max\{f(x_1)\}$. Suppose that u (quasiperiodic) solves the scattering problem (1), (2), and (4). Determine $f(x_1)$ by the knowledge of $u(x_1, T)$, *i.e.* the trace of u.

More precisely, suppose that $u_j(x_1, x_2)$ $(j = 1, 2)$ are Λ-quasiperiodic and solve the scattering problem (1), (2), and (5) with respect to the profiles $f_j(x_1)$, where the functions f_j are Λ-periodic. Let $T > max\{f_1(x_1), f_2(x_1)\}$ be a fixed constant. Denote $h = max\{f_1(x_1), f_2(x_1)\} - min\{f_1(x_1), f_2(x_1)\}$.

THEOREM 2.1. *Assume that* $u_1(x_1, T) = u_2(x_1, T)$ *and* h *satisfies* $k^2 < 2[h^{-2} + \Lambda^{-2}]$. *Then* $f_1(x_1) = f_2(x_1)$.

A general global uniqueness may not be possible, as long as k is real, even in the simplest case with a plane wave incident on a flat surface. In that case, the solution of the scattering problem can be written down explicitly. The nonuniqueness is obvious since the scattering fields will remain the same when one moves the flat surface up or down in certain multiples of the wavelength.

3. Some auxiliary results. The following results are crucial in the proof of Theorem 2.1.

LEMMA 3.1. *Let $T > max\{f(x_1)\}$, and u be the quasiperiodic solution that solves the scattering problem (1), (2), and (4). Then there is a pseudodifferential operator B of order one such that*

$$\frac{\partial u}{\partial n}|_{x_2=T} = B(u|_{x_2=T}) - 2i\beta e^{-i\beta T + i\alpha x_1} \ . \tag{5}$$

Proof. From the expressions (3), (4), and the radiation condition, we can compute the Fourier components u_n explicitly. For $\Omega_T = \{x_2 \geq T > max f(x_1)\}$,

$$u_n(x_2) = \begin{cases} u_n(T)e^{i\beta_n(x_2-T)}, & n \neq 0, \text{ in } \Omega_T, \\ u_0(T)e^{i\beta_0(x_2-T)} + e^{-i\beta_0 x_2} - e^{i\beta_0(x_2-2T)}, & n = 0, \text{ in } \Omega_T. \end{cases} \tag{6}$$

Further

$$\frac{\partial u_n}{\partial n}|_{x_2=T} = \begin{cases} i\beta_n u_n(T), & n \neq 0, \\ i\beta_0 u_0(T) - 2i\beta_0 e^{-i\beta_0 T}, & n = 0. \end{cases} \tag{7}$$

Therefore

$$\frac{\partial u}{\partial n}|_{x_2=T} = \sum_{n\in Z} i\beta_n u_n(T) e^{i\alpha_n x_1} - 2i\beta_0 e^{-i\beta_0 T + i\alpha x_1} \ . \tag{8}$$

Define for $g \in H^{1/2}$

$$B(g) = \sum_{n\in Z} i\beta_n g_n e^{i\alpha_n x_1}, \tag{9}$$

where $g_n = \frac{1}{\Lambda}\int_0^\Lambda g(x_1,x_2)e^{-i\alpha_n x_1}dx_1$. Clearly, the "Dirichlet-Neumann" operator B is a pseudodifferential operator of order one. □

LEMMA 3.2. *Suppose that $v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\triangle v + k^2 v = 0 \ , \quad in\ \Omega \ . \tag{10}$$

Let $I \in \Omega$ be a segment. Suppose also that $v|_I = 0$, $\frac{\partial v}{\partial \gamma}|_I = 0$ where γ is the normal to I. Then $v = 0$ in $\bar{\Omega}$.

The proof of Lemma 3.2 may be given by an application of Holmgren's uniqueness theorem and some continuation argument.

LEMMA 3.3. *(Poincaré's inequality) Let $D \subset IR^n$ be a bounded domain in $\{x \in IR^n, a_i \leq x_i \leq b_i, i = 1,\cdots,n\}$, where $b_i > a_i$ for $i = 1,2,\cdots,n$. Then for $v \in H_0^1(D)$,*

$$||\nabla u||^2 \geq 2\sum(b_i - a_i)^{-2}||u||^2. \tag{11}$$

Proof. It suffices to prove (11) for $u \in C_0^\infty(D)$. Further, one can extend u to the whole space IR^n so that $u \in C_0^\infty(IR^n)$. Write

$$u(x) = \int_{a_1}^{x_1} u_{x_1} dx_1$$

then

$$|u|^2 \leq (x_1 - a_1) \int_{a_1}^{b_1} |u_{x_1}|^2 dx_1$$

or

$$\int_{a_1}^{b_1} |u(x)|^2 dx_1 \leq \frac{(b_1 - a_1)^2}{2} \int_{a_1}^{b_1} |u_{x_1}|^2 dx_1 \ .$$

Therefore, let D_1 be the projection of D on $\{x_1 = 0\}$,

$$\begin{aligned} \int_D |u(x)|^2 dx &= \int_{D_1} dx_2 \cdots dx_n \int_{a_1}^{b_1} |u(x)|^2 dx_1 \\ &\leq \frac{(b_1 - a_1)^2}{2} \int_D |u_{x_1}|^2 dx \ . \end{aligned}$$

Similarly, for $i = 2, \cdots, n$,

$$\int_D |u(x)|^2 dx \leq \frac{(b_i - a_i)^2}{2} \int_D |u_{x_i}|^2 dx \ .$$

The conclusion follows by combining the above inequalities. □

4. Proof of Theorem 2.1. Theorem 2.1 may be proved by contradiction. Let us assume that $f_1(x_1)$ and $f_2(x_1)$ are two different functions. Denote $v = u_1 - u_2$, $f(x_1) = max\{f_1(x_1), f_2(x_1)\}$, $\Omega_j = \{(x_1, x_2) : f_j(x_1) < x_2\}$, and $\Omega = \{(x_1, x_2) : f(x_1) < x_2\}$. Then

$$v|_{\partial\Omega} = \begin{cases} 0, & \text{for } f_1(x_1) = f_2(x_1), \\ -u_2(x_1, f_1(x_1)), & f_1(x_1) > f_2(x_1), \\ u_1(x_1, f_2(x_1)), & f_1(x_1) < f_2(x_1) \ . \end{cases} \tag{12}$$

It follows from Lemma 2.1 and $u_1|_{x_2=T} = u_2|_{x_2=T}$ that

$$\frac{\partial u_1}{\partial n}|_{x_2=T} = \frac{\partial u_2}{\partial n}|_{x_2=T}$$

or

$$\frac{\partial v}{\partial n}|_{x_2=T} = 0.$$

Therefore,

$$\triangle v + k^2 v = 0, \text{ in } \Omega, \ v \in C^2(\Omega) \cap C(\bar{\Omega}) \ , \tag{13}$$

$$v|_{x_2=T} = 0 \ , \ \frac{\partial v}{\partial n}|_{x_2=T} = 0 \ . \tag{14}$$

Applying Lemma 3.2 to the equation for v deduces

$$v = 0 \ , \text{ in } \Omega \ .$$

In particular,

$$u_2(x_1, f_1(x_1)) = 0 \ , \quad u_1(x_1, f_2(x_1)) = 0 \ . \tag{15}$$

W.l.o.g., we may assume that $f_1(x_1) \geq f_2(x_1)$ for some x_1. Denote the region between Ω and Ω_1 by $\Omega_0 = \Omega_1 - \Omega$. According to the boundary condition of u_1 and (15)

$$\triangle u_1 + k^2 u_1 = 0, \ \text{ in } \Omega_0, \ u_1 \in C^2(\Omega_0) \cap C(\bar{\Omega}_0) \ , \tag{16}$$

$$u_1|_{\partial\Omega_0} = 0 \ . \tag{17}$$

Next we want to show that k^2 is not an eigenvalue to the problem

$$\triangle w + \lambda w = 0, \ \text{ in } \Omega_0,$$
$$w|_{\partial\Omega_0} = 0 \ .$$

It is well known that the spectrum of this problem is $\{\lambda_j\}_{j=1}^{\infty}$ where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots <$$

with possible finite multiplicities. In particular, the Rayleigh-Ritz characterization gives

$$\lambda_1 = min_{\phi \in H_0^1(\Omega_0)} \frac{\int_{\Omega_0} |\nabla \phi|^2 dx}{\int_{\Omega_0} \phi^2 dx} \ .$$

Since $\Omega_0 \subset [0, \Lambda] \times [min\{f_1, f_2\}, max\{f_1, f_2\}]$, from Lemma 3.3,

$$\lambda_1 \geq 2(h^{-2} + \Lambda^{-2}) > k^2$$

i.e., k^2 can not be an eigenvalue of the problem, it follows that

$$u_1 = 0 \ , \ \text{ in } \Omega_0 \ . \tag{18}$$

Thus $u_1 = 0$ in Ω_1. Similarly, one can show that

$$u_2 = 0 \ , \ \text{ in } \Omega_2 \ . \tag{19}$$

But this contradicts the identity (6) since $\beta \neq 0$ and the assumption that $-\pi/2 < \theta < \pi/2$. Hence

$$f_1(x_1) = f_2(x_1)$$

which concludes the proof of Theorem 2.1.

Acknowledgements. I thank Dr. J. Allen Cox at the Honeywell, Inc. Systems and Research Center for motivating my interest in diffractive optics. This research was partially supported by NSF and a grant from Honeywell, Inc.

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