SINGLE PHASE ENERGY MINIMIZERS FOR MATERIALS WITH NONLOCAL SPATIAL DEPENDENCE

By

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For Materials With
Nonlocal Spatial Dependence

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Abstract

We consider a one-dimensional body and assume that the total stored energy functional depends not only on the local strain field but also on the spatial average of the strain field over the body, weighted with an influence kernel. We investigate the problem of minimizing the total stored energy subject to given end displacements. The general existence theory for this problem is reviewed. Then, we narrow our study and concentrate on certain fundamental aspects of nonlocal spatial dependence by restricting our consideration to the case of a convex local energy and an exponential-type influence function for the nonlocal part. We find explicit solutions and show their characteristic properties as a function of the parameter that measures the extent of influence in the nonlocal kernel. We then study in detail the behavior that results when the total stored energy functional looses its coercivity. In this case, issues concerning the local and global stability of extremal fields are considered.

Contents

1   Introduction.

Consider a one dimensional body $\mathcal{B}$ represented in its reference configuration by the point set

$$\mathcal{B} := (0, l) \subset I\!R,$$

where $I\!R$ represents the set of all real numbers. Let $x \in \mathcal{B}$ denote a material point of $\mathcal{B}$ and let a deformation of $\mathcal{B}$ be given by a continuous mapping $y(\cdot) \in \mathcal{C}(\mathcal{B}, I\!R)$. Then, $u(x) := y(x) - x$ is the associated displacement field for all $x \in \mathcal{B}$ and we let $e(\cdot) := u'(\cdot)$ denote the strain field. A common hypothesis in the theory of materials is to assume that $\mathcal{B}$ supports a certain *total stored energy* functional $E(\cdot) : \mathcal{D} \longrightarrow I\!R \cup \{\infty\}$ where $\mathcal{D}$ is a given class of continuous displacement fields in $\mathcal{C}(\mathcal{B}, I\!R)$. In such theories, the stable equilibrium states of the body $\mathcal{B}$ are then supposed to be those displacement fields that minimize $E(\cdot)$ over $\mathcal{D}$ subject to given boundary conditions. In this work we are interested in the fixed displacement problem, for which $\Delta \in I\!R^+ := \{x \in I\!R \;:\; x \geq 0\}$ is given, and we seek a displacement field $u_0(\cdot) \in \mathcal{A}_>(\mathcal{B})$ such that

$$E(u_0) \;=\; \min_{u \in \mathcal{A}_>(\mathcal{B})} E(u), \tag{1}$$

where

$$\mathcal{A}_>(\mathcal{B}) := \{u(\cdot) \in \mathcal{D} \;:\; \int_{\mathcal{B}} e(x)\, dx \;=\; \Delta \text{ and } e(x) > -1 \text{ for a.e. } x \in \mathcal{B}\}. \tag{2}$$

Here, we require $e(x) > -1$ to ensure the invertablility of the deformation $y(\cdot)$. The particular functional form of the total stored energy functional $E(\cdot)$ will invariably affect the class of displacements $\mathcal{D}$ over which the minimum is sought. Any useful set $\mathcal{D}$ must be rich enough to guarantee existence of an energy minimizer, while at the same time be sufficiently focused to allow application of direct methods in the calculus of variations.

A variety of forms for the total stored energy functional $E(\cdot)$ have been considered in recent years. In [6, 8][1], the body $\mathcal{B}$ is assumed to be elastic and the total stored energy functional consists of the integral over $\mathcal{B}$ of a sole nonconvex integrand composed with the strain field $e(\cdot)$. In that case, the set $\mathcal{D}$ may be taken to be the Sobolev space $W^{1,p}(\mathcal{B})$, $1 < p < \infty$, and the problem (1) generally leads to highly nonunique minimizers. In fact, for certain $\Delta > 0$ there are minimizers that oscillate wildly between two different values of strain on subsets of the body $\mathcal{B}$ that are fixed only in measure. A method of removing such indeterminateness is to modify the total stored energy functional so as to include a penalty regularization that participates in the minimization and, in a sense, forms the basis of a *selection* criterion. This approach has been employed in two different, but related, attempts by Gurtin, Slemrod and Carr [3] and by Walter [13]. In the first, higher order gradients are introduced into the total stored energy functional, thereby creating a penalty (through increased energy) for the formation of regions where rapid changes in the strain field occur. In the second approach, the dependence of the total stored energy on a localized gradient is generalized and replaced by dependence on the spatial average of the strain field weighted with an influence kernel. This approach requires less hypotheses on the local smoothness of admissable strain fields and, in a first order approximation for smooth fields, is arguably reducible to the gradient model. We shall adopt the nonlocal approach here.

Specifically, we suppose that the total stored energy functional for $\mathcal{B}$, is of the form

$$E_\delta(u) := \int_\mathcal{B} \{W(e(x)) - e(x) \int_\mathcal{B} f_\delta(x - z)e(z)dz\}dx, \tag{3}$$

where $e(x) := u'(x)$ for every $x \in \mathcal{B}$. Here, $W(\cdot)$ represents the local pointwise contribution of the strain field to the total stored energy of $\mathcal{B}$. This portion of $E_\delta(\cdot)$ usually is referred to as the *bulk* or *local energy* (Figures 1a and 1b). In general, $W(\cdot) : I\!\!R \to I\!\!R^+ \cup \{\infty\}$ is assumed to satisfy

Hypothesis 1 (Bulk Energy). *$W(\cdot) : I\!\!R \to I\!\!R^+ \cup \{\infty\}$ is a continuously differentiable function such that*

(i)
$$\lim_{e \to \infty} W(e) = \lim_{e \to \infty} W'(e) = \infty, \tag{4}$$

(ii) $\exists\, a, b \in (0, \infty)$ such that $0 < a < b < \infty$ and $W(\cdot)$ is convex on $(-\infty, a] \cup [b, \infty)$.

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[1] The work of [6] concerns the question of coexistent phases in Gibbsian thermostatics and thermodynamics and is more general than implied here. In particular, it addresses the elementary *thermo*elastic bar problem for nonconvex energies.
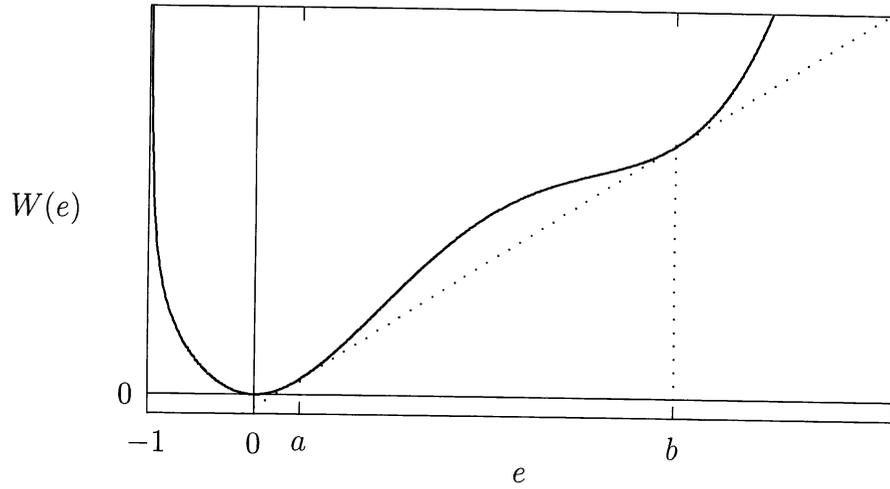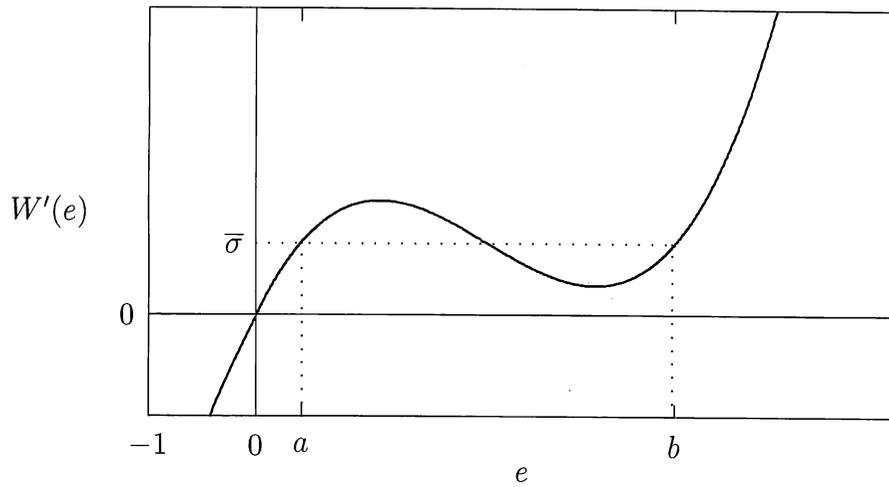
Figure 1a. The bulk stored energy $W(e)$.

Figure 1b. The derivative $W'(e)$ of the bulk stored energy $W(e)$.

The second term in (3) represents the nonlocal part of the constitutive assumption and expresses the spatial strain dependence via an *influence function* $f_\delta(\cdot)$. Given $x \in \mathcal{B}$, the

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function $f_\delta(x - z)$ in (3) serves as a weight on the relative influence of the material strain $e(z)$ at a point z in a δ-neighborhood of x. Specifically, $f_\delta(\cdot)$ behaves as a weighted *identity approximation* and is assumed to satisfy

Hypothesis 2 (Spatial Energy). *For* $2 \leq p < \infty$, $f_\delta(\cdot) \in L^q(I\!R)$ *is a nonnegative, monotonically decreasing even function with the following properties:*

(i) $\exists\, \overline{a} > 0$ *such that* $\int_{I\!R} f_\delta(x)\,dx = \overline{a}$, $\forall \delta > 0$ *(fixed total spatial influence),*

(ii) $\|f_\delta(x - \cdot)\|_{q,\mathcal{B}} \to 0$, *uniformly* $\forall x \in \mathcal{B}$ *as* $\delta \to \infty$,

(iii) $\forall \phi(\cdot) \in L^p(I\!R)$, $\int_{I\!R} f_\delta(x - y)\phi(y)\,dy \to \overline{a}\phi(x)$ *in* $L^q(I\!R)$ *as* $\delta \to 0$ *(identity approx.),*

where $1 < q \leq 2$ *satisfies* $1/q + 1/p = 1$.

The requirement that $f_\delta(\cdot)$ is monotonically decreasing in Hypothesis 2 reflects the idea that the ability of a point $z \in \mathcal{B}$ to affect the local stored energy at a point $x \in \mathcal{B}$ decreases as the distance between the two points increases. The parameter $\delta \in (0, \infty)$ is a measure of the localization of the influence; a particular choice for the influence function should reflect the view that for small δ, a large percentage of the mass of the function $f_\delta(\cdot)$ is concentrated at the origin. Figure 2 illustrates a possible influence function that is compatible with this hypothesis. This is the essence of statement (iii) in Hypothesis 2.
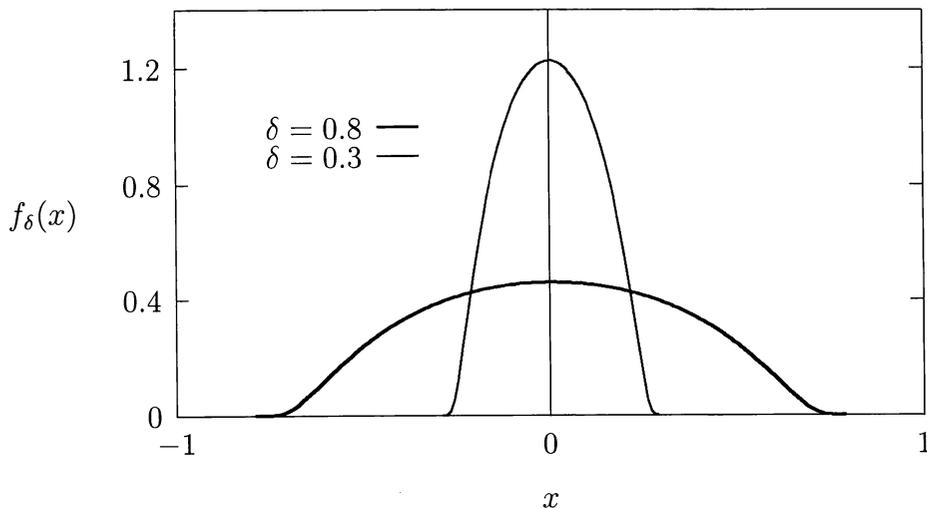

Figure 2. A possible influence function: $f_\delta(x) := \frac{1}{\delta} e^{\left\{\left(\left(\frac{x}{\delta}\right)^2 - 1\right)^{-1}\right\}}$ for $|x| < \delta = 0.3, 0.8$.

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To more clearly distinguish the difference between the local and nonlocal parts of the total stored energy functional, it is instructive to introduce an alternative form of (3). For this, we let $f_\delta(\cdot)$ be as in Hypothesis 2, define

$$a_\delta(x) := \int_\mathcal{B} f_\delta(x - z)\, dz, \qquad (5)$$

and suppose that $e(\cdot) \in L^p(\mathcal{B})$. Then a simple application of the Fubini-Tonelli theorem and the evenness of $f_\delta(\cdot)$ allows us to rewrite $E_\delta(\cdot)$ as

$$E_\delta(u) = \int_\mathcal{B} \{ W(e(x)) - a_\delta(x)\, e^2(x) + \frac{1}{2} \int_\mathcal{B} f_\delta(x - z)(e(x) - e(z))^2\, dz \}\, dx. \qquad (6)$$

The spatial kernel in (6) now is clearly seen to be zero on any strain field that is constant. In fact, the nonlocal convolution in (6) is exactly that portion of the energy functional that penalizes the total stored energy for strain fields that exhibit nonconstant behavior. Moreover, if $u(\cdot) \in W^{1,p}(\mathcal{B})$ is given, independent of δ, then it follows from the total stored energy (3) and Hypothesis 2, (ii) and (iii), that

$$\lim_{\delta \to 0} E_\delta(u) = \int_\mathcal{B} W(u'(x)) - \bar{a} u'(x)^2\, dx \qquad (7)$$

and

$$\lim_{\delta \to \infty} E_\delta(u) = \int_\mathcal{B} W(u'(x))\, dx. \qquad (8)$$

We wish to consider the following fixed displacement minimization problem: Given $\Delta \in \mathbb{R}^+$ and $p \in [2, \infty)$, determine $u_0(\cdot) \in \mathcal{A}^{1,p}_>(\mathcal{B})$ such that

$$E_\delta(u_0) = \min_{u \in \mathcal{A}^{1,p}_>(\mathcal{B})} E_\delta(u), \qquad (9)$$

where

$$\mathcal{A}^{1,p}_>(\mathcal{B}) := \{ u(\cdot) \in W^{1,p}(\mathcal{B}) : \int_\mathcal{B} u'(x)\, dx = \Delta \text{ and } e(x) > -1 \text{ for a.e. } x \in \mathcal{B} \}, \qquad (10)$$

and $W^{1,p}(\mathcal{B})$ is the usual Sobolev space of functions such that for some $p \in [2, \infty)$, both the function $u(\cdot)$ and its generalized derivative $u'(\cdot) = e(\cdot)$ are elements of $L^p(\mathcal{B})$. We say that a displacement field $u(\cdot) : \mathcal{B} \to \mathbb{R}$ is admissable if $u(\cdot) \in \mathcal{A}^{1,p}_>(\mathcal{B})$. In order to assure that there exists a minimizing sequence for (9) which posesses a weak limit in $\mathcal{A}^{1,p}_>(\mathcal{B})$, we

must make two assumptions. The first is a *coercivity hypothesis* on the total stored energy functional $E_\delta(\cdot)$. This hypothesis provides for a uniform bound on any minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathcal{B})$, and thereby ensures that such a sequence is precompact in $W^{1,p}(\mathcal{B})$. In that case, there is a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $u_0(\cdot) \in W^{1,p}(\mathcal{B})$ [5, 9]. Specifically, we require $E_\delta(\cdot)$ to satisfy

Hypothesis 3 (Coercivity). *$\exists\, p \in [2,\infty)$, $\sigma_1 > 0$ and $d_1 \in \mathbb{R}$ such that $\forall\, e \in \mathbb{R}$*

$$W(e) - \beta|e|^2 \geq \sigma_1|e|^p + d_1\,, \tag{11}$$

where

$$\beta := \sup_{x\in\mathcal{B}} a_\delta(x).$$

In order to guarantee that a solution of (9) is also locally invertible (i.e., $e(x) > -1$ for a.e. $x \in \mathcal{B}$), and consequently residing in $\mathcal{A}_{>}^{1,p}(\mathcal{B})$, we also assume the following

Hypothesis 4 (Stiffness). *The bulk energy function $W(\cdot) : \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$ has the form*

$$W(e) = \left\{ \begin{array}{lll} W(e) \in \mathbb{R}^+ & : & e \in (-1,\infty) \\[2mm] \infty & : & e \in (-\infty,-1] \end{array} \right\}, \tag{12}$$

where, in particular,

$$\lim_{e\downarrow-1} W(e) = -\lim_{e\downarrow-1} W'(e) = \infty. \tag{13}$$

The *stiffness hypothesis* penalizes a deformation $y(\cdot)$ by assigning an infinite amount of bulk energy when distinct points in the reference configuration $\mathcal{B}$ occupy identical positions in the deformed configuration $y(\mathcal{B})$. Therefore, if $W(\cdot)$ satisfies Hypothesis 4, any displacement field $u(\cdot) \in W^{1,p}(\mathcal{B})$ which satisfies $|E_\delta(u)| < \infty$, necessarily has the property that $u'(x) > -1$ for a.e. $x \in \mathcal{B}$. In the remainder of this section, we require $W(\cdot)$ to have the form as given in (12).

The question of existence of a solution to (9) is difficult to answer if the bulk energy $W(\cdot)$ is nonconvex. This is because, in this case, the first portion of the energy functional (3) is not generally weakly lower-semicontinuous (w.l.s.c.) on $W^{1,p}(\mathcal{B})$. However, since $f_\delta(\cdot) \in L^q(\mathcal{B})$, it follows from slight modifications of known theorems [5, 9, 14], that the nonlocal portion of $E_\delta(\cdot)$ is weakly continuous on $W^{1,p}(\mathcal{B})$. This observation, and an awareness of classical approaches in existence theory [5, 7], directs us to consider the auxillary *relaxed problem*. Let

$$W^*(e) := \sup \{g(e) : g(e) \le W(e) \quad \forall e \in I\!\!R \text{ and } g(\cdot) \text{ is convex }\},$$

and define

$$E_\delta^*(u) := \int_\mathcal{B} \{W^*(e(x)) - e(x) \int_\mathcal{B} f_\delta(x - z)e(z)dz\}dx. \tag{14}$$

The relaxed problem associated with (9) is then to determine $u_0 \in \mathcal{A}_>^{1,p}(\mathcal{B})$ such that

$$E_\delta^*(u_0) = \min_{u \in \mathcal{A}_>^{1,p}(\mathcal{B})} E_\delta^*(u). \tag{15}$$

Regarding (15), since $W^*(\cdot)$ is convex, the local portion of (14) is now w.l.s.c. on $W^{1,p}(\mathcal{B})$ [5, 9], and therefore, the relaxed energy $E_\delta^*(u)$ is as well. Consequently, we may prove [9] the following

Theorem 1 (Relaxed Existence).[2] *Let (4), (11), and (12) hold for some $p \in [2, \infty)$ and let $f_\delta(\cdot) \in L^q(I\!\!R)$ satisfy Hypothesis 2. Then,*

(i) $\displaystyle\inf_{u \in \mathcal{A}_>^{1,p}(\mathcal{B})} E_\delta^*(u) =: l_p^* = l_p := \inf_{u \in \mathcal{A}_>^{1,p}(\mathcal{B})} E_\delta(u) \in I\!\!R,$

(ii) $\exists \, u_0(\cdot) \in \mathcal{A}_>^{1,p}(\mathcal{B})$, *and a sequence* $u_k(\cdot) \in \mathcal{A}_>^{1,p}(\mathcal{B})$ *such that*

(a) $u_k \rightharpoonup u_0$ *in* $W^{1,p}(\mathcal{B})$,

(b) $E_\delta^*(u_0) = \displaystyle\lim_{k \to \infty} E_\delta(u_k) = l_p.$

The derivation of the classical Euler-Lagrange equations as necessary conditions for problems (9) and (15) becomes a nontrivial matter when the stiffness Hypothesis 4 is assumed. Two difficulties are encountered in the classical derivation. The first is that given a function $u(\cdot) \in \mathcal{A}_>^{1,p}(\mathcal{B})$ and a possible variation $u(\cdot, \epsilon) := u(\cdot) + \epsilon\eta(\cdot)$ for $\eta(\cdot) \in W^{1,\infty}(\mathcal{B})$, it is not immediate that $u(\cdot, \epsilon) \in \mathcal{A}_>^{1,p}(\mathcal{B})$ when ϵ is sufficiently small. In particular, it is not clear

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[2]We note that the hypothesis on $f_\delta(\cdot)$ is only a sufficient condition; it may be weakened. In particular, aside from evenness, all that is needed from Hypothesis 2 is that $f_\delta(\cdot) \in L^q(I\!\!R)$. Moreover, if we relax definition (10) to include noninvertible deformations, replacing $\mathcal{A}_>^{1,p}(\mathcal{B})$ by

$$\mathcal{A}^{1,p}(\mathcal{B}) := \{u(\cdot) \in W^{1,p}(\mathcal{B}) : \int_\mathcal{B} u'(x)\,dx = \Delta\},$$

then Theorem 1 also holds as stated, provided we drop the stiffness Hypothesis 4.

that $e(\cdot, \epsilon) > -1$ for a.e. $x \in \mathcal{B}$ holds for $\epsilon \to 0$. Fortunately, this problem can be circumvented with a clever choice of variations [1]. The second problem is more serious. Since we do not know that $e(\cdot)$ is bounded below *away* from -1, there is no reason to believe that $W'(e(\cdot, \epsilon)) \in L^1(\mathcal{B})$ as $\epsilon \to 0$. This presents a serious obstacle to the necessary step of interchanging the differentiation of (3) with respect to ϵ and the first integral operation. For the purely local problem with no spatial dependence, this difficulty was circumvented in [1, 2], and in [9] we determine necessary and sufficient conditions, similar to those in [1], to ensure that this process may be undertaken. These conditions are given in the following

Theorem 2 (Euler-Lagrange).[2]   *Let (4) and (12) hold for some $p \in [2, \infty)$. Let $f_\delta(\cdot)$ satisfy Hypothesis 2, and assume that $\exists M, C > 0$ such that $|W'(e)| \leq C|e|^p$, $\forall e \in [M, \infty)$. Suppose $u_0(\cdot) \in \mathcal{A}_>^{1,p}(\mathcal{B})$ is such that $E_\delta(u_0) = l_p$. Then, the following three conditions are equivalent:*

(i) $\exists \epsilon_0 > 0$ such that

$$\int_{\{-1 < u_0' < -1 + \epsilon_0\}} |W'(u_0'(x))| \, dx < \infty,$$

(ii) $\exists \sigma \in \mathbb{R}$ such that for a.e. $x \in \mathcal{B}$

$$W'(e_0(x)) - 2 \int_{\mathcal{B}} f_\delta(x - z) \, e_0(z) \, dz = \sigma, \qquad (16)$$

(iii) $\operatorname*{ess\,inf}_{x \in \mathcal{B}} e_0(x) > -1$ and $\operatorname*{ess\,sup}_{x \in \mathcal{B}} e_0(x) < \infty$.

Observe that if the hypotheses of Theorem 2 hold and $u_0(\cdot)$ is a solution of either (9) or (15), then, because of (16), $e_0(\cdot)$ must necessarily be nonconstant. This property is an immediate and obvious departure from solutions of problem (1) when the energy functional is purely local, even including higher order gradients [3, 6, 8]. In these works, it is shown that when $\Delta/l \notin [a, b]$, the global minimizer for (1) is a constant.

In solving (9), we have found [9] that a further necessary condition for minimizers of $E_\delta(\cdot)$ results as a direct consequence of (i) in Theorem 1:

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[2]As in Theorem 1, the stated hypotheses may be relaxed. Specifically, regarding (4), (12), and Hypothesis 2, we essentially need $f_\delta(\cdot) \in L^q(\mathbb{R})$, and $W(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ to be continuously differentiable to guarantee the truth of the theorem. Also, if we relax definition (10) to include noninvertible deformations, replacing $\mathcal{A}_>^{1,p}(\mathcal{B})$ by $\mathcal{A}^{1,p}(\mathcal{B})$ (see footnote in Theorem 1), then conclusion (iii)$_1$ does not apply and conclusions (ii) and (iii)$_2$ in Theorem 2 follow as direct implications of the stated hypothesis, except that in the formal statement of Theorem 2 we should require $|W'(e)| \leq C|e|^p$ for every $|e| \in [M, \infty)$, and exclude (12).

Corollary 1 (Weierstrass Condition). *Let $u_0(\cdot)$ be as in Theorem 1 and suppose that the hypotheses of that theorem hold. Then, for a.e. $x \in \mathcal{B}$, $\forall \lambda \in (0,1)$ and $\forall \epsilon_1, \epsilon_2 \in \mathbb{R}$ such that $e_0(x) = \lambda \epsilon_1 + (1 - \lambda) \epsilon_2$,*

$$W(e_0(x)) \leq \lambda W(\epsilon_1) + (1 - \lambda) W(\epsilon_2). \tag{17}$$

In particular, the Weierstrass condition requires that any solution of (9) must posess a strain field $e_0(\cdot)$ whose range nontrivially intersects $(-1, a] \cup [b, \infty)$ almost everywhere in the body $\mathcal{B}$ (see Figures 1a and 1b). This places a restriction on the type of strain discontinuities a solution $u_0(\cdot)$ may support. These restrictions are called *jump conditions* and, for simple discontinuities, are summarized in

Corollary 2 (Weierstrass Jump Conditions). *Let the hypotheses of Theorem 2 and Corollary 1 be satisfied. Suppose that $x_0 \in \mathcal{B}$ is such that*

$$e_0(x_0 - 0) = e_1 \neq e_2 = e_0(x_0 + 0)$$

holds. Then,

(i) $W'(e_1) = W'(e_2) =: \overline{\sigma}$,

(ii) $\overline{\sigma}(e_1 - e_2) = W(e_1) - W(e_2)$.

This corollary implies that a simple discontinuity in a minimizing strain field must occur at the Maxwell stress, denoted by $\overline{\sigma}$ here and in Figures 1a and 1b.

In the remainder of this paper, we shall restrict ourselves to a specific *convex* bulk energy $W(\cdot)$ and an exponential influence function $f_\delta(\cdot)$. This approach isolates the nonlocal properties of the body from any effects that could be attributed to a nonconvexity in $W(\cdot)$, and allows us to solve problem (9) explicitly. We believe that this opens a clear pathway to understand the fundamental effects of nonlocal spatial dependence.

2   The particular model: A necessary differential equation.

We now focus on the minimization problem (9) in the case that the bulk energy $W(\cdot)$ is a positive definite quadratic form with minimum at zero. We also assume an exponential form

for the nonlocal influence function $f_\delta(\cdot)$ which satisfies Hypothesis 2. Thus, in the remainder of this work we take $W(e) = ke^2$, where $k > 0$, and define

$$f_\delta(\eta) = \frac{a}{\delta} e^{\frac{-|\eta|}{\delta}} \tag{18}$$

for all $\eta \in I\!\!R$, and nonzero $a, \delta \in I\!\!R^+$. Since $W(\cdot)$ does not exhibit any singular behavior at $x = -1$, we will take the class of admissable displacement fields for the minimization problem (9) to be as in (10) *except* that the condition of $e(x) > -1$ for a.e. $x \in \mathcal{B}$ is omitted. In this case, the stiffness hypothesis (12) is not relevant, and in the remainder of this work we replace (9) by the following minimization problem: For $p \in [2, \infty)$, determine $u_0(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ such that

$$E_\delta(u_0) = \min_{u \in \mathcal{A}^{1,p}(\mathcal{B})} E_\delta(u), \tag{19}$$

where

$$\mathcal{A}^{1,p}(\mathcal{B}) := \{u(\cdot) \in W^{1,p}(\mathcal{B}) \ : \ \int_\mathcal{B} u'(x)\, dx = \Delta\}. \tag{20}$$

Since $|W'(e)| \le 2ke^2$ for every $|e| \ge 1$, and $f_\delta(\cdot) \in L^q(I\!\!R)$ for every $q \in (1, \infty)$, Theorem 2 then implies that if $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is a solution of (19) for $p = 2$, then $u(\cdot) \in \mathcal{A}^{1,\infty}(\mathcal{B})$ and there exists $\sigma \in I\!\!R$ such that the following Euler-Lagrange integral equation holds:

$$T(x) := 2ke(x) - 2\int_\mathcal{B} f_\delta(x - z)e(z)\, dz = \sigma \quad \text{for a.e. } x \in \mathcal{B}. \tag{21}$$

It is clear that any solution $u(\cdot)$ of (19) may be adjusted on a set of Lebesgue measure zero so that it still solves (19) and satisfies (21) for every point $x \in \mathcal{B}$. Thus, for all $x \in \mathcal{B}$, it follows that

$$e(x) = \frac{1}{k} \int_0^x f_\delta(x - z)e(z)\, dz + \frac{1}{k} \int_x^l f_\delta(x - z)e(z)\, dz + \frac{\sigma}{2k}, \tag{22}$$

and because of the form of $f_\delta(\cdot)$ in (18), we see that $u(\cdot)$ must be at least of class $C^3(\mathcal{B})$, thereby justifying the following argument: Observe that by differentiating (21) and use of (18) we readily obtain

$$T'(x) = 2ke'(x) + \frac{2}{\delta} \int_0^x f_\delta(x - z)e(z)\, dz - \frac{2}{\delta} \int_x^l f_\delta(x - z)e(z)\, dz.$$

Again, by use of (21) we see that

$$T'(x) + \frac{1}{\delta} T(x) = 2ke'(x) + \frac{1}{\delta} 2ke(x) - \frac{4}{\delta} \int_x^l f_\delta(x - z)e(z)\, dz. \tag{23}$$

and

$$T'(x) - \frac{1}{\delta} T(x) = 2ke'(x) - \frac{1}{\delta} 2ke(x) + \frac{4}{\delta} \int_0^x f_\delta(x - z)e(z)\, dz. \tag{24}$$

Finally, by adding (23) and (24) together and differentiating once more, we find

$$\frac{d}{dx}(T'(x)) = 2ke''(x) + \frac{4}{\delta} f_\delta(0)\, e(x) - \frac{2}{\delta^2} \int_0^l f_\delta(x - z)e(z)\, dz, \tag{25}$$

which, because of (18), (21), and the fact that $T \equiv \sigma$, produces the following ordinary differential equation for $e(\cdot)$:

$$e''(x) + \frac{1}{k}\left(\frac{2\,a - k}{\delta^2}\right) e(x) = -\frac{\sigma}{2\,k\,\delta^2}. \tag{26}$$

This ordinary differential equation represents a *necessary* condition on any solution of (21), and it is easily solved by standard methods, yielding a two parameter family of solutions. These parameters combine with σ to give three free constants at this stage in the solution to the minimization problem (19). In the next section, this nonuniqueness will be removed by returning to the original Euler-Lagrange equation (21) to gain *sufficient* conditions for its solvability. Of course, it is also essential to note that any solution $u(\cdot) \in \mathcal{A}^{1,\infty}(\mathcal{B})$ of (21) is required to satisfy the fixed displacement condition

$$\int_\mathcal{B} e(x)\, dx = \Delta. \tag{27}$$

If we apply the standard Taylor expansion argument of "approximation", as was proposed and advanced by van der Waals [12], to the nonlocal part of the functional (6) we are guided to replace the functional in (6) by the gradient model

$$E_\delta^\#(u) := \int_\mathcal{B} \{W(e(x)) - a_\delta(x)e^2(x) + b_\delta(x)[e'(x)]^2\}\, dx \tag{28}$$

at the lowest non-trivial order of approximation, where

$$b_\delta(x) := \frac{1}{2} \int_\mathcal{B} (z - x)^2 f_\delta(x - z)\, dz. \tag{29}$$

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If δ is sufficiently small it can be shown from (5), (18) and (29) that $a_\delta(x) \simeq 2a$ and $b_\delta(x) \simeq 2a\delta^2$ except near the boundary of $\mathcal{B}$. In any case, for $W(e) = ke^2$, the Euler-Lagrange equation associated with the minimization of (28) subject to the constraint (27) is a regular second order ordinary differential equation for $e(x)$ containing an unknown Lagrange multiplier constant. The natural boundary conditions will require $e'(x) = 0$ on $\partial\mathcal{B}$, and these, along with (27), will determine the three constants of this theory. Since the more complete nonlocal theory does not require $e'(x)$ to vanish at the boundary $\partial\mathcal{B}$, we expect that at least in a neighborhood of $\partial\mathcal{B}$ the "approximation" will break down. Moreover, if $W(\cdot)$ is nononvex, there is strong evidence [10] to suggest that similar breakdowns of the boundary layer type occur even in the interior of $\mathcal{B}$. Later, we give examples of the nonlocal theory for convex $W(\cdot)$ which show that large strain gradients do occur at the boundary $\partial\mathcal{B}$. If, in (28), $a_\delta(x)$ and $b_\delta(x)$ are simply replaced by constants then for convex $W(\cdot)$ the strain field that minimizes (28) subject to (27) is constant.

3 The structure of admissible extremals.

The differential equation introduced in (26) has solutions whose form depends on the sign of the expression $2a - k$. Here, we show that in each of the three cases $k > 2a$, $k = 2a$, and $k < 2a$, the solutions of (26) are compatible with the Euler-Lagrange equation (21), if and only if a certain 2×3 system of linear equations for three arbitrary constants is satisfied. These equations, coupled with the fixed displacement condition(27), yield a 3×3 linear system whose solutions are guaranteed to solve the Euler-Lagrange equation (21).

For the remainder of this work, it is convenient to introduce the material parameter $\omega > 0$ defined by

$$\omega^2 \equiv \left| \frac{k - 2a}{k\delta^2} \right|, \tag{30}$$

and to let $c_1, c_2 \in I\!\!R$ denote the arbitrary constants that are contained in the general solution of (26).

Case (1) $k > 2a$: Here, (26) implies that

$$e(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} + \frac{\sigma}{2(k - 2a)}, \tag{31}$$

and this, with (18) and (21), yields

$$T(x) = \sigma + \left(\frac{2a c_1}{1 + \omega\delta} + \frac{2a c_2}{1 - \omega\delta} + \frac{a\sigma}{k - 2a} \right) e^{\frac{-x}{\delta}}$$

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$$+ \left(\frac{2 a c_1}{1 - \omega\delta} e^{\omega l} + \frac{2 a c_2}{1 + \omega\delta} e^{-\omega l} + \frac{a\sigma}{k - 2a} \right) e^{\frac{(x-1)}{\delta}} = \sigma. \tag{32}$$

Here, we note that $\omega\delta - 1 \neq 0$ since $a > 0$. Now, since (32) must hold for every $x \in B$, it follows that the coefficients of $e^{\frac{-x}{\delta}}$ and $e^{\frac{(x-1)}{\delta}}$ must vanish. This provides two of the three equations we are seeking, and the fixed displacement condition (27), when applied to (31), yields the third. Thus, we have the following linear system for determining c_1, c_2, and σ:

$$\frac{1 + \omega\delta}{2a} c_1 + \frac{1 - \omega\delta}{2a} c_2 + \frac{a}{k - 2a} \sigma = 0,$$

$$\frac{2 a e^{\omega l}}{1 - \omega\delta} c_1 + \frac{2 a e^{-\omega l}}{1 + \omega\delta} c_2 + \frac{a}{k - 2a} \sigma = 0, \tag{33}$$

$$\frac{\omega}{e^{\omega l} - 1} c_1 + \frac{\omega}{1 - e^{-\omega l}} c_2 + \frac{1}{2(k - 2a)} \sigma = \nabla.$$

Any solution (c_1, c_2, σ) of this system, when coupled with (31), necessarily solves (21) and the associated fixed displacement condition (27).

Case (2) $k = 2a$: When $k = 2a$, (26) reduces to a trivial form which has the general solution

$$e(x) = -\frac{\sigma}{4 k \delta^2} x^2 + c_1 x + c_2. \tag{34}$$

Upon substituting (34) and (18) into (21) and simplifying as in Case (1) above, we get

$$T(x) = \sigma + \left(-2a\delta c_1 + 2 a c_2 - \frac{\sigma a}{k} \right) e^{\frac{-x}{\delta}}$$

$$+ \left((2 a l + 2 a \delta) c_1 + 2 a c_2 - \frac{\sigma a}{2 k \delta^2} (l^2 + 2\delta l + 2\delta^2) \right) e^{\frac{(x-1)}{\delta}} = \sigma. \tag{35}$$

Again, by recognizing that the coefficients of $e^{\frac{-x}{\delta}}$ and $e^{\frac{(x-1)}{\delta}}$ in (35) must vanish, and by applying the fixed displacement condition (27), we find the following linear system for determining c_1, c_2, and σ:

$$-2\,a\,\delta\,c_1 \;+\; 2\,a\,c_2 \;-\; \frac{a}{k}\,\sigma \;=\; 0,$$

$$(2\,a\,l \;+\; 2\,a\,\delta)\,c_1 \;+\; 2\,a\,c_2 \;-\; \frac{\sigma\,a}{2\,k\,\delta^2}\left(l^2 \;+\; 2\,\delta\,l \;+\; 2\,\delta^2\right) \;=\; 0, \tag{36}$$

$$\frac{l^2}{2}\,c_1 \;+\; l\,c_2 \;-\; \frac{l^3}{12\,k\,\delta^2}\,\sigma \;=\; \Delta.$$

Any solution (c_1, c_2, σ) of this system, when coupled with (34), necessarily solves (21) and the associated fixed displacement condition (27).

Case (3) $k < 2a$: Proceeding as before, we see that when $k < 2a$, the general solution of (26) is

$$e(x) \;=\; c_1\,\sin\omega x \;+\; c_2\cos\omega x \;+\; \frac{\sigma}{2\,(k\,-\,2\,a)}. \tag{37}$$

Upon substituting (37) and (18) into (21) and preforming a significant amount of tedious integration and simplification, we get

$$
\begin{aligned}
T(x) \;=\;\; & \sigma \;+\; \left(k\,c_1 \;-\; k\,\omega\,\delta\,c_2 \;+\; \frac{a}{(k\,-\,2\,a)}\,\sigma\right) e^{\frac{-x}{\delta}}\\[2mm]
& +\; (k\cos\omega l \;-\; k\,\delta\,\omega\,\sin\omega l)\,c_1\,e^{\frac{(x-l)}{\delta}}\\[2mm]
& +\; \left((k\,\sin\omega l \;+\; k\,\omega\,\delta\,\cos\omega l)\,c_2 \;+\; \frac{a}{(k\,-\,2\,a)}\,\sigma\right) e^{\frac{(x-l)}{\delta}}\\[2mm]
\;=\;\; & \sigma. \tag{38}
\end{aligned}
$$

As in the previous two cases, it follows that the coefficients of $e^{\frac{-x}{\delta}}$ and $e^{\frac{(x-l)}{\delta}}$ in (38) must vanish. This, together with the fixed displacement condition (27) applied to (37), then gives the following system for determining c_1, c_2, and σ:

$$k\,c_1 \;-\; k\,\omega\,\delta\,c_2 \;+\; \frac{a}{(k\,-\,2\,a)}\,\sigma \;=\; 0,$$

$$
\begin{aligned}
(k\cos\omega l \;-\; k\,\delta\,\omega\,\sin\omega l)\,c_1 & \\
+\; (k\,\sin\omega l \;+\; k\,\omega\,\delta\,\cos\omega l)\,c_2 \;+\; \frac{a}{(k\,-\,2\,a)}\,\sigma & \;=\; 0, \tag{39}
\end{aligned}
$$

$$\frac{\sin\omega l}{\omega}\,c_1 \;+\; \frac{1\,-\,\cos\omega l}{\omega}\,c_2 \;+\; \frac{l}{2\,(k\,-\,2\,a)}\,\sigma \;=\; \Delta.$$

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As in the Cases (1) and (2), any solution of this system, when coupled with (37), necessarily solves the Euler-Lagrange equation (21) and the fixed displacement condition (27).

For easy reference, it will be convenient to denote the coefficient matrices of the linear systems (33), (36) and (39) for the Cases (1), (2), and (3), respectively, as A_i where $i = 1, 2, 3$. Then, by defining

$$\mathbf{d} := \begin{pmatrix} 0 \\ 0 \\ \Delta \end{pmatrix} \quad , \quad \mathbf{c} := \begin{pmatrix} c_1 \\ c_2 \\ \sigma \end{pmatrix}, \tag{40}$$

the relevant linear systems for the respective Cases (1), (2), and (3) may be written in the compact form

$$A_i\,\mathbf{c} = \mathbf{d} \quad \text{for} \quad i = 1, 2, 3. \tag{41}$$

In view of (41), if we wish to exhibit a unique solution of (21) and (27), then we must be assured of the existence of the inverse matrices A_i^{-1} for $i = 1, 2, 3$. In the next section, we see, unequivocally, that A_i^{-1} exists for the Cases (1) and (2), i.e., $i = 1, 2$. However, some interesting existence and stability questions arise for the Case (3), i.e., $i = 3$. These issues will be considered in Section 5.

4   Analysis of minimizers in the Cases (1) and (2): $k \geq 2a$.

In this section we study the existence and uniqueness of solutions to the minimization problem (19) for the Cases (1) and (2). To begin, we note that in these two cases the coercivity hypothesis (11) is satisfied, for $p = 2$, by the stored energy functional (3) with $W(e) = ke^2$ and $f_\delta(\cdot)$ given by (18). To see this, we first observe that (5) and (18) yield

$$a_\delta(x) = 2a - a(e^{\frac{-x}{\delta}} + e^{\frac{(x-l)}{\delta}}). \tag{42}$$

Thus,

$$k - a_\delta(x) = (k - 2a) + a(e^{\frac{-x}{\delta}} + e^{\frac{(x-l)}{\delta}}), \tag{43}$$

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and since $a(e^{\frac{-x}{\delta}} + e^{\frac{(x-l)}{\delta}})$ is bounded below away from zero on $\mathcal{B}$, it follows that for $W(e) = ke^2$ the coercivity hypothesis (11) holds for $p = 2$. Further, since $W(\cdot)$ is convex and $f_\delta(\cdot) \in L^q(\mathcal{B})$ for every $q \in (1, \infty)$, Theorem 1 implies that there exists a solution $u(\cdot) \in \mathcal{A}^{1,2}(\mathcal{B})$ of the minimization problem (19) if $k \geq 2a$. Theorem 2 then implies that $u(\cdot) \in \mathcal{A}^{1,\infty}(\mathcal{B})$, and also that $u(\cdot)$ satisfies the Euler-Lagrange equation (21); thus it must have one of the smooth forms generated in Section 3. To determine whether any of these forms actually solves the minimization problem (19) for Cases (1) and (2), we must then solve (41) for i=1, 2. From Theorem 2, we note that since any solution $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ of (19) must necessarily reside in $\mathcal{A}^{1,\infty}(\mathcal{B})$, and $\mathcal{A}^{1,\infty}(\mathcal{B}) \subset \mathcal{A}^{1,p}(\mathcal{B})$ when $p \in (0, \infty)$, then $l_p = l_\infty$ for every $p \in [2, \infty)$. Therefore, any solution $u(\cdot)$ of (19) for $p = 2$ actually solves (19) for every $p \in [2, \infty]$. Thus, for the remainder of this work we will assume without loss of generality that $p \in [2, \infty]$, keeping in mind that our initial analysis required $p = 2$.

Case (1) $k > 2a$ ($i = 1$): In this case, a straightforward calculation using A_1 as defined in (41), and inferred from (33), gives

$$
\begin{aligned}
det A_1 \;=\; & 2\,a^2\left(-1 + e^{l\,\omega} + \delta\,\omega + e^{l\,\omega}\,\delta\,\omega\right) \\[2ex]
& \times \left\{ \frac{\left(2 - 2\,e^{l\,\omega} + l\,\omega + e^{l\,\omega}\,l\,\omega - l\,\delta\,\omega^2\right)}{e^{l\,\omega}\,(2\,a - k)\,\omega\,(-1 + \delta\,\omega)^2\,(1 + \delta\,\omega)^2} \right. \\[2ex]
& \left. + \frac{\left(e^{l\,\omega}\,l\,\delta\,\omega^2 - 2\,\delta^2\,\omega^2 + 2\,e^{l\,\omega}\,\delta^2\,\omega^2\right)}{e^{l\,\omega}\,(2\,a - k)\,\omega\,(-1 + \delta\,\omega)^2\,(1 + \delta\,\omega)^2} \right\}.
\end{aligned}
\tag{44}
$$

Then, because of (30) and the inequalities $a > 0$, $k > 0$, we see that $\delta\omega \neq 1$, and it follows that the denominator of (44) is strictly negative. Also, since $\omega > 0$, the first term of the product in (44) is nonzero as well. Now, let $G(\omega)$ be defined as the numerator in the second term of the product in (44), and note that, equivilantly,

$$
G(\omega) \;=\; (e^{l\omega} - 1)(l\omega^2\delta + 2\omega^2\delta^2) + \psi(\omega),
\tag{45}
$$

where $\psi(\omega) := (l\omega - 2)e^{l\omega} + l\omega + 2$. Then, it follows that $\psi(0) = \psi'(0) = 0$, and that $\psi''(\omega) > 0$ for every $\omega > 0$; thus, $\psi(\omega) > 0$ on $(0, \infty)$. Combining this with the positivity of $G(\omega) - \psi(\omega)$, we see that $G(\omega) > 0$ for every $\omega > 0$. Therefore, the determinant of A_1 is nonzero for every $\omega > 0$ and A_1^{-1} exits and is unique for every $\omega > 0$ when $k > 2a$. Thus, $\mathbf{c} = A_1^{-1}\mathbf{d}$ uniquely solves (41), and with (40) we conclude that (31) generates the unique solution of (19) for every $p \in [2, \infty]$. Specifically, while the computation of A_1^{-1} is long and tedious, because of the simple structure of $\mathbf{d}$, it is straightforward to determine $\mathbf{c}$, i.e., c_1, c_2 and σ. We find

$$c_1 = -\frac{2\,a\,\omega}{k\,G(\omega)}\,\Delta\,,$$

$$c_2 = -\frac{2\,a\,\omega\,e^{l\omega}}{k\,G(\omega)}\,\Delta\,, \tag{46}$$

$$\sigma = \frac{2\,(k-2\,a)\,\omega\,\left(1+e^{l\,\omega}-\delta\,\omega+e^{l\,\omega}\,\delta\,\omega\right)}{G(\omega)}\,\Delta\,.$$

When (46) is substituted into (31) and simplified, we obtain the following minimizing field for (19) in the Case (1):

$$e_{1,\delta}(x) := -\frac{4\,\omega\,a\,\Delta}{k\,G(\omega)}\,e^{\frac{l\omega}{2}}\cosh\{\omega(x-\frac{l}{2})\} + \frac{\sigma}{2\,(k-2\,a)}\,. \tag{47}$$

Clearly, $e_{1,\delta}(\cdot)$ is smooth and symmetric about $x = l/2$ for all $\delta > 0$, and it follows from (30), (46)$_3$, and (47) that $e_{1,\delta}(\cdot) > 0$ on $\mathcal{B}$ if and only if $\Delta > 0$.

It is of interest to consider the asymptotic behavior of (47) as $\delta \to 0$; i.e., as the spatial dependence of the material becomes more locally concentrated. To do this, we first recall from (30) that $\omega^{-1} = O(\delta)$ as $\delta \to 0$. Then, (45) yields

$$G(\omega)\omega^{-1}e^{-\omega l} = l(1 + \delta\omega) + O(\omega^{-1}), \tag{48}$$

and with (30) and (46) we obtain

$$c_1 = \frac{-2\,a\,e^{-l\omega}}{k\,(l(1+\delta\omega)+O(\omega^{-1}))}\Delta\,,$$

$$c_2 = \frac{-2\,a}{k\,(l(1+\delta\omega)+O(\omega^{-1}))}\Delta\,, \tag{49}$$

$$\sigma = \frac{2\,(k-2\,a)\,\left(\left(1+e^{-l\omega}\right)+\delta\,\omega\left(1-e^{-l\omega}\right)\right)}{l(1+\delta\omega)+O(\omega^{-1})}\Delta\,.$$

Then, using (31), we find

$$e_{1,0}(x) := \lim_{\delta \to 0} e_{1,\delta}(x) = \left\{ \begin{array}{lcl} \frac{\Delta}{l} & : & x \in (0,l) \\[2mm] \frac{\Delta}{l}\delta\omega & : & x = 0, l \end{array} \right\}, \tag{50}$$

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which, of course, also follows from (47). We note, from (30), that the discontinuity exhibited in (50) at the boundary points is nontrivial, approaching a maximum of Δ/l as $k \downarrow 2a$ and a minimum of 0 as $k \to \infty$. In order to manifest this behavior as $\delta \to 0$, the strain field $e_{1,\delta}(\cdot)$ of (47) supports high gradients in localized regions near $\partial \mathcal{B}$ while exhibiting nearly a constant value of Δ/l in the interior of $\mathcal{B}$ for small positive δ. These high gradient boundary layers become sharply focused in the limit $\delta \to 0$ as seen in Figure 3.
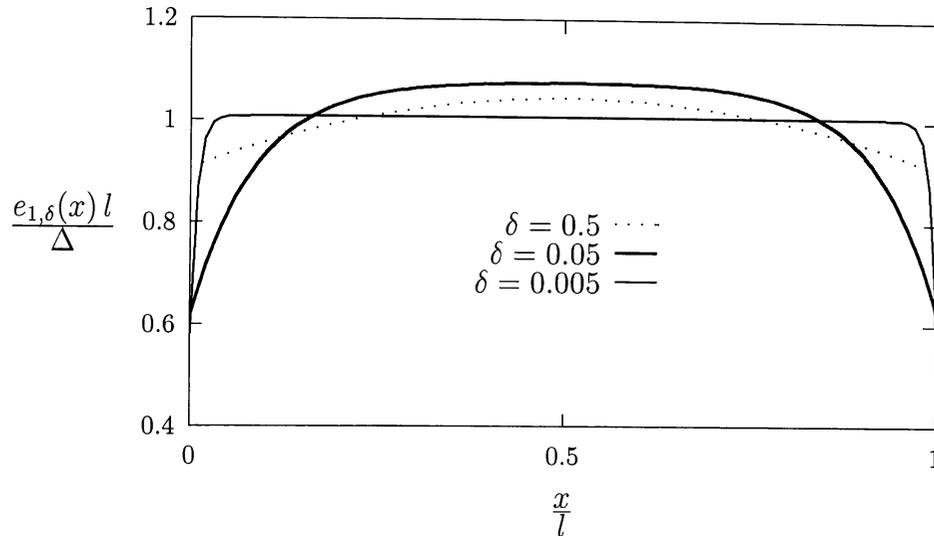
Figure 3. The minimizing strain field of (47) for $k = 3a$ and for various values of the influence parameter δ.

It is straightforward to show [9, 14] that $f_\delta(\cdot)$ given in (18) satisfies Hypothesis 2 with $\bar{a} = 2a$. Therefore, using (7), for $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$, we observe that

$$E_0(u) := \lim_{\delta \to 0} E_\delta(u) = \int_\mathcal{B} (k - 2a)u'(x)^2 \, dx. \tag{51}$$

The unique solution of the minimization problem (19) for the functional $E_0(\cdot)$ is well known for the case $k > 2a$; it is the constant strain field Δ/l [6, 8]. Thus, we conclude that the limit of $e_{1,\delta}(\cdot)$ as $\delta \to 0$ coincides with the minimizer of $E_0(\cdot)$ over $\mathcal{A}^{1,p}(\mathcal{B})$ everywhere except for $x \in \partial \mathcal{B}$. This assertion also holds if we replace $E_0(\cdot)$ with the analagous gradient model of Gurtin, Slemrod and Carr [9].

Now, we wish to show that the asymptotic behaviour of the minimizing displacement field $u_{1,\delta}(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ associated with (47) respects that of the functional E_δ in the limit as $\delta \to 0$, in the sense that

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$$\lim_{\delta \to 0} E_\delta(u_{1,\delta}) = E_0(\lim_{\delta \to 0} u_{1,\delta}).$$ (52)

To do this, we first observe, from Theorem 2, that if $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ solves the Euler-Lagrange equation (21), we have

$$
\begin{aligned}
E_\delta(u) &= \int_\mathcal{B} \{ku'(x)^2 - u'(x) \int_\mathcal{B} f_\delta(x - z)u'(z)\, dz\}\, dx \\
&= \frac{\sigma}{2} \int_\mathcal{B} u'(x)\, dx \\
&= \frac{1}{2}\Delta\sigma.
\end{aligned}
$$ (53)

Then, recalling $(49)_3$, we see that

$$\lim_{\delta \to 0} E_\delta(u_{1,\delta}) = (k - 2a)\frac{\Delta^2}{l},$$ (54)

which allows us to conclude (52).

Note, from $(46)_3$, that for a given displacement Δ the stress σ is linear in the gross displacement Δ. If we temporarily rewrite $(46)_3$ as

$$\sigma = \phi(\delta)\, \Delta,$$ (55)

for a fixed l, k, and a, then it readily follows that $\partial\phi/\partial\delta > 0$. Thus, for a given gross displacement $\Delta > 0$, if the material is replaced by one of lesser spatial influence (i.e., if δ is decreased) then the level of stress necessary for equilibrium is decreased. We conclude that a focusing of the spatial dependence through the influence function $f_\delta(\cdot)$ by reducing the value of δ acts as a softening agent in the effective stiffness modulus of the material[3] (see Figure 4). The limiting value of σ as $\delta \to 0$ is $2(k - 2a)\Delta/l$; exactly the value predicted by solving (19) for $E_0(\cdot)$.

In the limiting case $\delta \to \infty$, the identity approximating property of the spatial influence function $f_\delta(\cdot)$ is essentially nullified by (ii) of Hypothesis 2 on the spatial energy. This asymptotics identifies the limiting form of the minimizing displacement field as the effect of spatial dependence is "turned off". Analogous to the previous discussion, we find

---

[3]Similarly, for a fixed l, a, and δ, we have that $\partial\phi/\partial k > 0$, and therefore when the material is replaced by one of greater elastic stiffness k the level of stress necessary for equilibrium is increased. This hardening effect is illustrated in Figure 4 by the uniform raising of the stress-influence graphs as k is increased relative to a.

$$\lim_{\delta \to \infty} e_{1,\delta}(x) = \frac{\Delta}{l}, \tag{56}$$

and

$$\lim_{\delta \to \infty} E_\delta(u_{1,\delta}) = E_\infty(\frac{\Delta}{l}), \tag{57}$$

where, according to (8), we have defined

$$E_\infty(u) := \int_B ku'(x)^2 \, dx \tag{58}$$

for $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$. In this case, no discontinuity appears at the boundary points of $\mathcal{B}$ in the limit. This is due to the diminishing magnitude of the nonlocal spatial term in $E_\delta(\cdot)$ as $\delta \to \infty$.
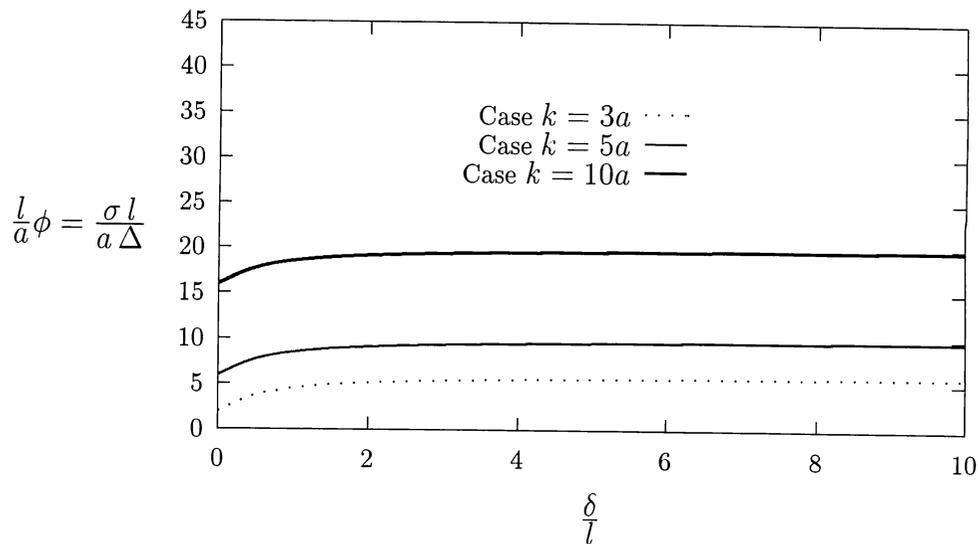
Figure 4. The normalized effective stiffness modulus $\phi(\delta)$ of (55) for the cases $k = 3a$, $k = 5a$, and $k = 10a$.

Case (2) $k = 2a$ ($i = 2$): Here, we recall A_2 as defined in (41), and inferred from (33), to obtain

$$det A_2 = -\frac{a^2 l (l + 2\delta)(l^2 + 6l\delta + 12\delta^2)}{3}. \tag{59}$$

Clearly, $det A_2 \neq 0$ and A_2^{-1} exists and is unique for every $\delta \geq 0$. Hence, as in Case (1), $\mathbf{c} = A_2^{-1}\mathbf{d}$ uniquely solves (41), and with (40) we find that (34) generates the unique solution of the minimization problem (19) for every $p \in [2, \infty]$. We find that $\mathbf{c}$, i.e., c_1, c_2, and σ, is given by

$$c_1 = \frac{6}{l^2 + 6l\delta + 12\delta^2}\Delta,$$

$$c_2 = \frac{6\delta(l + 2\delta)}{l^3 + 6l^2\delta + 12l\delta^2}\Delta, \tag{60}$$

$$\sigma = \frac{24k\delta^2}{l^3 + 6l^2\delta + 12l\delta^2}\Delta.$$

When this is substituted into (34) and simplified we find that

$$e_{2,\delta}(x) := \frac{12\delta^2 + 6l\delta + \frac{3}{2}l^2 - 6(x - \frac{l}{2})^2}{l^2 + 6l\delta + 12\delta^2}\frac{\Delta}{l} \tag{61}$$

is the minimizing strain field for (19). Because $\Delta > 0$, it is clear from (61) that $e_{2,\delta}(\cdot) > 0$ on the entire body $\mathcal{B}$. As in Case (1), we note that $e_{2,\delta}(\cdot)$ is smooth, symmetric and even about $x = l/2$.

The asymptotics for this case behave somewhat differently than in the previous case. First, the limiting strain field in $\mathcal{B}$ as $\delta \to 0$ is not a constant field as it is in Case (1), but rather it is a positive, inverted parabola, symmetric about $x = l/2$. Specifically, we have from (61) that

$$e_{2,0}(x) := \lim_{\delta \to 0} e_{2,\delta}(x) = \frac{3l^2 - 12(x - \frac{l}{2})^2}{2l^2}\frac{\Delta}{l}. \tag{62}$$

From (61), we see that the maximum strain for any given $\delta > 0$ is achieved at $x = l/2$ and, according to (62), this value increases to $3\Delta/2l$ at $\delta = 0$. Similarly, the minimum positive strain is achieved for $x \in \partial\mathcal{B}$ and this value decreases to 0 at $\delta = 0$ (see Figure 5). Recalling (7), we note that in this case $E_0(u) \equiv 0$ for all $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$. Thus, every displacement field $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is a solution to the minimization problem (19) for $E_0(\cdot)$, including the constant solution Δ/l as in Case (1). In this case ($k = 2a$), the effect of localizing the spatial influence by letting $\delta \to 0$ produces a negating effect on the coercivity of the functional $E_\delta(\cdot)$ that is induced by the bulk energy $W(\cdot)$. This competition appears

to act as a selection mechanism in choosing the particular limiting strain field $e_{2,0}(\cdot)$. The fact that $E_0(\cdot) \equiv 0$ is respected in the energy asymptotics is seen via (53) and (60), i.e.,

$$\lim_{\delta \to 0} E_\delta(u_{2,\delta}) = E_0\big(\lim_{\delta \to 0} u_{2,\delta}\big) = 0. \tag{63}$$

To continue, we observe from $(60)_3$ that the stress σ is linear in Δ. Moreover, similar to the behavior as found in Case (1), here we find the same softening behavior in $\mathcal{B}$ due to the localization as $\delta \to 0$ of the nonlocal influence. In this case, however, the limiting effective modulus is zero.
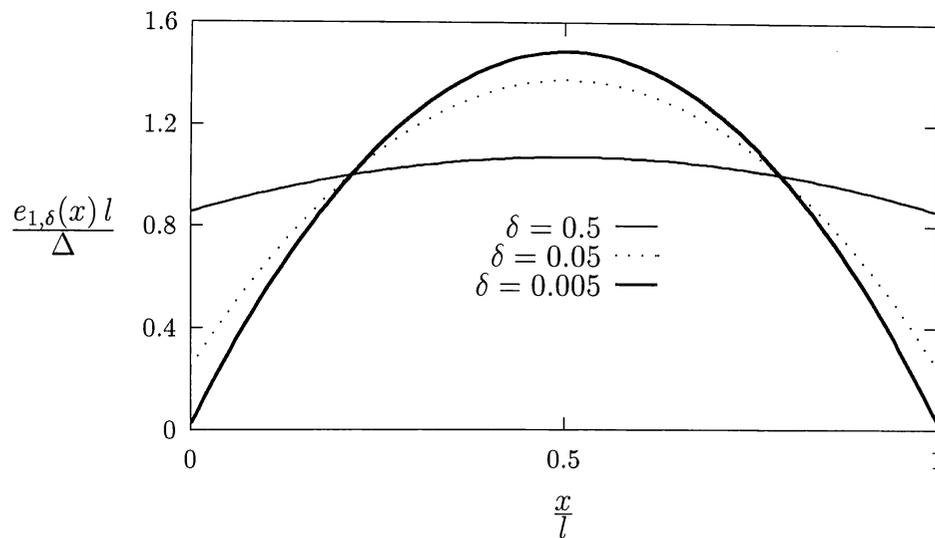
Figure 5. The minimizing strain field of (61) for $k = 2a$ and for various values of the influence parameter δ.

In the alternative limit $\delta \to \infty$ of zero spatial dependence, the limiting strain field and total stored energy are identical to those recorded in (56) and (57) for the Case (1).

5   The minimization problem in the Case (3): $k < 2a$.

Contrary to the straightforward analysis in Cases (1) and (2) given above, this case presents a number of difficulties to determining a minimizer of problem (19), if, in fact, one exists. The most pressing problem is that for k sufficiently less than $2a$, an unavoidable violation of the coercivity hypothesis (11) occurs, thereby disallowing the application of Theorem 1 and its consequential conclusion of existence. Moreover, due to the highly transcendental

nature of the elements of the coefficient matrix A_3, as defined in (41) and inferred from (39), even the determination of an extremal field of the form (37), that solves the Euler-Lagrange equation (21) becomes problematic. In fact, when $k < 2a$, we shall obtain a positive, strictly decreasing sequence $\{\hat{\delta}_n\}_{n \in I\!N}$, of values of the influence parameter δ, corresponding to certain zeros of $det A_3$, such that for each n, $\lim_{\delta \to \hat{\delta}_n} |E_\delta(u_{3,\delta})| = \infty$, where $u_{3,\delta}(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is any extremal field that solves (21). Moreover, when $\delta = \hat{\delta}_n$, we show that there does not exist a field $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ that satisfies (21). Also, by studying (21) as a Fredholm integral equation of the second kind, we shall identify a second sequence of values of the influence parameter δ, corresponding to other zeros of $det A_3$, which completely characterize the extremal fields $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ that do solve (21).

5.1 General extremal fields: Existence and asymptotics.

First some remarks concerning coercivity which, according to the coercivity hypothesis (11), is related to the aggregate sign of $k - a_\delta(\cdot)$ on the body $\mathcal{B}$. Recalling (43), we see that $k - a_\delta(\cdot)$ is bounded below away from zero on $(0, l)$ whenever $2a - k < 2ae^{\frac{-l}{2\delta}}$. This is equivalent to the condition that

$$\delta^{-1} \in \left(0\,,\, \frac{-2}{l}\ln(1 - \frac{k}{2a})\,\right) \;=:\; (0, \frac{1}{\delta^*}\,). \tag{64}$$

Thus, δ^* is the greatest lower bound to the set of all $\delta > 0$ such that the functional $E_\delta(\cdot)$ is coercive. It follows that when $\delta \leq \delta^*$, the coercivity Hypothesis 3 fails and, as a consequence, the existence theorem, Theorem 1, cannot be applied to such a situation. This, of course, does not mean that a solution of (19) does not exist for $\delta \leq \delta^*$, but rather, the loss of coercivity motivates a more delicate and detailed analysis of the existence question. We begin with some preliminary asymptotic limits.

From (39) and (41), we easily find that $det A_3$ has the form

$$det A_3 \;=\; 8a^2 \;-\; (8a^2 + \frac{4al}{\delta}(2a - k))\cos\omega l \;-\; (4al\omega\,(k - a) - 8a^2\delta\omega)\sin\omega l. \tag{65}$$

Of course, the system (41) has a unique solution $\mathbf{c} = A_3^{-1}\mathbf{d}$ for c_1, c_2, and σ when the above expression is nonzero. To explore this possibility, we note from the definition of ω in (30) that $det A_3$ has an infinite number of zeros in any neighborhood of $\delta = 0$, and that these zeros possess a limit point at $\delta = 0$. Further, we note that for $r := l/\delta$,

$$\left.\frac{\partial(det A_3)}{\partial r}\right|_{r=0} = 4a^2(4\frac{k}{a} + 16\frac{a}{k} - 16), \tag{66}$$

which vanishes only when $k = 2a$. Since $(det A_3)_{r=0} = 0$, we then may conclude that whenever $0 < k < 2a$ there exists a $\bar{\delta} > 0$ such that for any $\delta \in (\bar{\delta}, \infty)$, $det A_3$ does not vanish. Hence, for δ sufficiently large and finite, (41) is uniquely solvable for $\mathbf{c}$. In any case, when $det A_3 \neq 0$, we find

$$
\begin{aligned}
c_1 &= \frac{2a\Delta\omega(2a\delta\omega(1 + \cos\omega l) + 2a\sin\omega l)}{det A_3}, \\[2ex]
c_2 &= \frac{4a\Delta\omega\sin(\frac{l\omega}{2})(2a\delta\omega\cos(\frac{\omega l}{2}) + 2a\sin(\frac{\omega l}{2}))}{det A_3}, \\[2ex]
\sigma &= \frac{2k(2a - k)\omega\Delta(4a\delta\omega\cos\omega l - 2a(\delta^2\omega^2 - 1)\sin\omega l)}{det A_3}.
\end{aligned}
\tag{67}
$$

We denote the resulting extremal strain field of (37) by $e_{3,\delta}(\cdot)$ and its associated displacement field in $\mathcal{A}^{1,p}(\mathcal{B})$ by $u_{3,\delta}(\cdot)$.

In the limiting case $\delta \to \infty$, we know from (30) that $\omega = O(1/\delta)$, and from (65) we obtain

$$
det A_3 = \left(\frac{k - 2a}{\omega\delta}4al + 8a^2\delta\omega l\right)\omega + O(\omega^2).
\tag{68}
$$

This, with (67) and the form for $e_{3,\delta}(\cdot)$ given in (37), then justifies the conclusion that for every $x \in \mathcal{B}$,

$$
e_{3,\infty}(x) := \lim_{\delta \to \infty} e_{3,\delta}(x) = \frac{\Delta}{l}.
\tag{69}
$$

While this result follows the same pattern as that observed earlier in the Cases (1) and (2) corresponding to $k \geq 2a$, when an attempt is made to evaluate the limit of $e_{3,\delta}(\cdot)$ as $\delta \to 0$, we encounter problems. One complication is due to the large number of zeros for $det A_3$ present in any neighborhood of zero, and another is that $\lim_{\delta \to 0} \omega = \infty$. These two facts, coupled with (67) and (37) indicate that the extremals of (21) will possess a highly oscillatory spatial behavior and unbounded character with respect to δ as δ approaches 0. This is what we see in Figure 6.

25
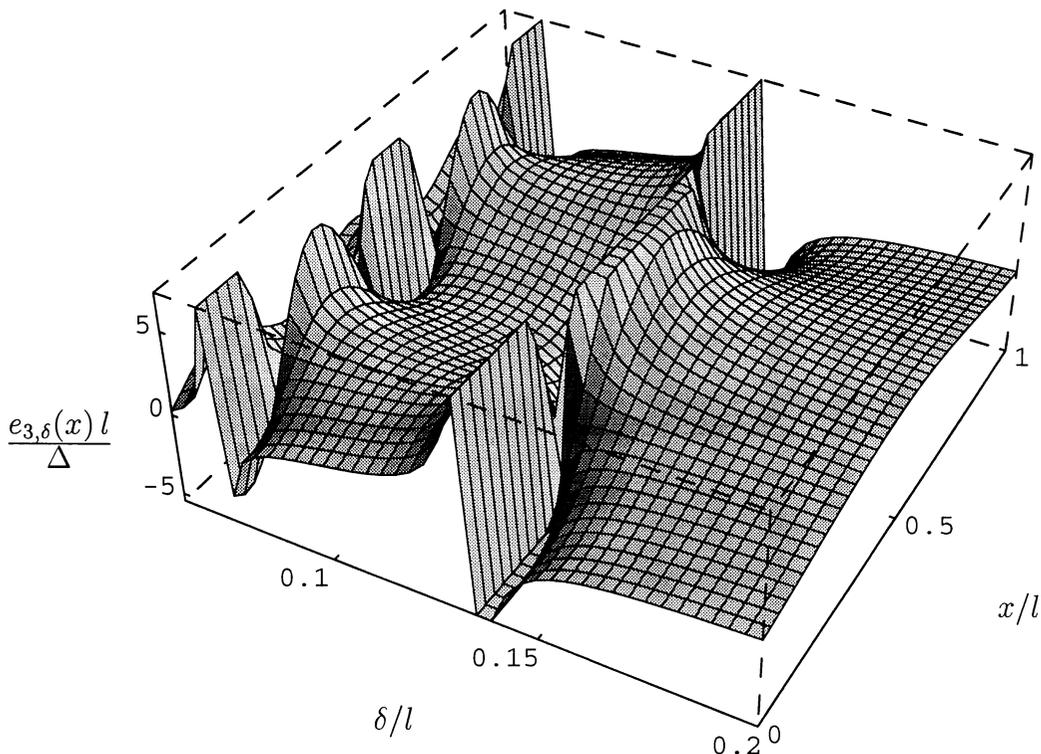
Figure 6. Solution of the Euler-Lagrange equation (21) and the fixed displacement
condition (27) for $k = a$.

With a view toward obtaining a complete characterization of the fields $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$
that solve (21) in the present case when $k < 2a$, we observe that (21) is a Fredholm integral
equation of the second kind, which naturally suggests that we first analyze the associated
homogeneous problem. For this purpose, let $\phi(\cdot) \in L^1(\mathcal{B})$, and define

$$\pounds_\delta \phi(x) := \int_{\mathcal{B}} f_\delta(x - z)\phi(z)\, dz\,. \tag{70}$$

Then, for $\sigma = 0$, (21) becomes

$$e(x) = \frac{1}{k}\pounds_\delta e(x)\,, \tag{71}$$

and from (37) and (39) we see that the general solution of (71) is of the form

$$e(x) = c_1 \cos \omega x + c_2 \sin \omega x, \tag{72}$$

where the pair (c_1, c_2) must nontrivially solve the system

$$kc_1 - k\omega\delta c_2 = 0,$$
$$(k\cos\omega l - k\omega\delta\sin\omega l)c_1 + (k\omega\delta\cos\omega l + k\sin\omega l)c_2 = 0. \tag{73}$$

Clearly, a necessary and sufficient condition that this be the case is that $det A = 0$, where A is the coefficient matrix for (73). Since

$$det A = (2\delta\omega\cos\omega l - (\delta^2\omega^2 - 1)\sin\omega l)k^2, \tag{74}$$

it follows that $det A = 0$ if and only if

$$\left\{ \begin{array}{lll} \omega l = (n - \frac{1}{2})\pi & : & k = a\,; \, n \in I\!N \\[2mm] \tan\omega l = \dfrac{2\delta\omega}{\delta^2\omega^2 - 1} & : & k \neq a \end{array} \right\}. \tag{75}$$

Recalling (30), we see that (75) places a restriction on the parameters k, a, δ, and l in order for (71) to be solvable; we say that the ordered set $\Lambda := \{k, a, \delta, l\} \in I\!R^4$ is an *eigenset* if its elements combine so as to satisfy $det A = 0$. It will be convenient to separate the possible eigensets into the following two types: Λ *is a Type I eigenset if*

$$\Lambda \in \{ \{k, a, \delta, l\} \, : \, \sin\omega l = -\frac{k}{a}\delta\omega \text{ and } \cos\omega l = -\frac{k}{2a}(\delta^2\omega^2 - 1) \}; \tag{76}$$

Λ *is a Type II eigenset if*

$$\Lambda \in \{ \{k, a, \delta, l\} \, : \, \sin\omega l = \frac{k}{a}\delta\omega \text{ and } \cos\omega l = \frac{k}{2a}(\delta^2\omega^2 - 1) \}. \tag{77}$$

Note that in the case $k = a$, where $\delta\omega = 1$ according to (30), this separation corresponds, respectively, to the choice $n = $ even or $n = $ odd in (75).

In general, then, (71) is solvable if and only if Λ is an eigenset. In that case, (73) requires $\ker A = \mathrm{span}\{(\delta\omega, 1)\}$, and so the solution (72) must have the form

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$$e(x) = c_2(\delta\omega\cos\omega x + \sin\omega x). \tag{78}$$

Moreover, we see from (76)-(78) that

$$\int_{\mathcal{B}} e(x)\,dx = \frac{c_2}{\omega}\{\delta\omega\sin\omega l - \cos\omega l + 1\} = \left\{ \begin{array}{lll} 0 & : & \Lambda \text{ is a Type I eigenset} \\[2ex] 2\frac{c_2}{\omega} & : & \Lambda \text{ is a Type II eigenset} \end{array} \right\}. \tag{79}$$

Let us now require $e(\cdot)$ to satisfy the fixed displacement condition (27). It is clear from (79) that when Λ is of Type II, (71) has a unique solution which satisfies (27) for every $\Delta \in \mathbb{R}$ and that solution is identically zero if and only if $\Delta = 0$. Explicitly, we have

$$e_{3,\delta}(x) = \frac{\omega\Delta}{2}(\delta\omega\cos\omega x + \sin\omega x). \tag{80}$$

When Λ is a Type I eigenset, we conclude from (79) that there is a one prameter family of solutions of (71), all satisfying the zero fixed displacement condition (i.e., $\Delta = 0$), of the form given in (78). We note, from (53) and the fact that here we are setting $\sigma = 0$, that the total stored energy of the family (78) is zero, independent of c_2.

Having completely solved the homogeneous problem, (21) with $\sigma = 0$, and characterized those fields that satisfy the fixed displacement condition (27), we now use these results to completely analyze the case $\sigma \neq 0$. Thus, we now consider

$$e(x) = \frac{1}{k}\mathcal{L}_\delta e(x) + \frac{\sigma}{2k}, \tag{81}$$

coupled with the associated fixed displacement condition (27). Recalling the expression for $det A_3$ in (65), we see that

$$det A_3 = \left\{ \begin{array}{lll} 0 & : & \Lambda \text{ is a Type I eigenset} \\[2ex] 16a^2 & : & \Lambda \text{ is a Type II eigenset} \end{array} \right\}. \tag{82}$$

Thus, if Λ is a Type II eigenset, there is a unique solution of (81) of the form (37), (67), which satisfies the fixed displacement condition (27) for every $\Delta \in \mathbb{R}$; that solution is nonconstant, and is identically zero if and only if $\Delta = 0$. In any case, however, by $(67)_3$ it follows that $\sigma = 0$, and this reduces (81) to the homogeneous case (71). Unfortunately, when $\{k, a, \delta, l\}$ is not an eigenset, the general problem of determining whether or not $det A_3$ is zero is algebraically complicated and not readily solvable. Recalling the general form for A_3 as implied by (39) and (41), we instead have the following

Lemma 1. *The linear equation*

$$A_3 \mathbf{c} = \mathbf{d}, \tag{83}$$

as defined in (39) and (41), has a solution $\mathbf{c} = (c_1, c_2, \sigma)$ *for* $\mathbf{d} = (0, 0, \Delta) \neq \mathbf{0}$ *if and only if either* $\Lambda = \{k, a, \delta, l\}$ *is a Type I eigenset or* $\det A_3 \neq 0$.

Proof. Let $\Delta \neq 0$ and suppose that $\mathbf{x} = (x_1, x_2, x_3) \in \ker A_3^T$. Then, $\langle \mathbf{d}, \mathbf{x} \rangle = \Delta x_3$ and (83) will have a solution if and only if $x_3 = 0$. Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $I\!R^3$. Thus, with $A_3^T \mathbf{x} = \mathbf{0}$ it readily follows that

$$x_3 = 0 \Leftrightarrow x_1 + x_2 = 0, \tag{84}$$

and, if $\mathbf{x} \neq \mathbf{0}$, that

$$\delta\omega + \sin\omega l + \delta\omega\cos\omega l = 0, \tag{85}$$

and

$$1 - \cos\omega l + \delta\omega\sin\omega l = 0. \tag{86}$$

But, according to (76), this implies that Λ is a Type I eigenset. On the other hand, if $\ker A_3^T = \{\mathbf{0}\}$, then $\mathbf{x} = \mathbf{0}$, and we must have $\det A_3 \neq 0$. Thus, we conclude that

$$\mathbf{d} \perp \ker A_3^T \Leftrightarrow \{\Lambda \text{ is a Type I eigenset or } \det A_3 \neq 0\} \tag{87}$$

to complete the proof. $\square$

Since we are interested here in the situation $\sigma \neq 0$, and since σ must vanish if Λ is a Type II eigenset, then according to this lemma, we need only consider the two possibilities: Either $\det A_3 \neq 0$ and Λ is not an eigenset, in which case the unique solution of (81) and the fixed displacement condition (27) is given by (37) and (67), or Λ is a Type I eigenset. For the latter case, we know that $\det A_3 = 0$ and we see from the definition of A_3 in (39) and (41), and ω in (30) that

$$A_3 = \begin{pmatrix} k & -k\delta\omega & \dfrac{a}{k-2a} \\ k & -k\delta\omega & \dfrac{a}{k-2a} \\ -\dfrac{k\delta}{a} & \dfrac{k\delta^2\omega}{a} & \dfrac{l}{2(k-2a)} \end{pmatrix}. \tag{88}$$

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Performing a standard Gaussian reduction and simplifying we arrive at two results for the case when Λ is of Type I. First, by (37), (39), (41) and (88), the general solution of (81) and the fixed displacement condition (27) is given by the one parameter family

$$e(x) = c_2(\delta\omega\cos\omega x + \sin\omega x) - \frac{2a\Delta}{k(l+2\delta)}\cos\omega x + \frac{\Delta}{l+2\delta}. \tag{89}$$

Secondly, we find

$$\sigma = \frac{2(k-2a)\Delta}{l+2\delta}. \tag{90}$$

Recalling that $k < 2a$, we see that the stress is of *opposite* sign to the displacement, which is contrary to physical reasoning and suggests that such an equilibrium state would be unstable. It is of interest to observe from (53) and (90) that when Λ is a Type I eigenset, the total stored energy $E_\delta(u)$ for an extremal $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$, as implied by $u'(x) = e(x)$ and (89), is non positive, finite, independent of c_2, and bounded below.

5.2 Stability analysis.

In order to better understand the instabilities in the material model for $k < 2a$, we next observe the following elementary properties of Type I eigensets. Suppose that the three parameters $0 < k < 2a$ and $l > 0$ appearing in the ordered set $\Lambda = \{k, a, \delta, l\}$ are given and fixed. Then, a brief inspection of the definition in (75) and (76) implies that there is a strictly decreasing sequence $\wp_I := \{\delta_{(n)}\}_{n\in I\!N}$ of positive, real numbers such that each collection $\{k, a, \delta_{(n)}, l\}$ is a Type I eigenset. Moreover, $\wp_I$ is bounded above, and $\lim_{n\to\infty}\delta_{(n)} = 0$. Thus, $\delta_{(1)} \in \wp_I$ is the largest member of the sequence. Similarly, if instead we hold k, a, and δ fixed in the ordered set Λ, then there is a strictly increasing sequence $\aleph_I := \{l_{(n)}\}_{n\in I\!N}$ of positive, real numbers such that each collection $\{k, a, \delta, l_{(n)}\}$ is also a Type I eigenset. In this case, $\aleph_I$ is bounded below by $l_{(1)}$ with $\lim_{n\to\infty}l_{(n)} = \infty$. Identical observations based upon (75) and (77) also hold for Type II eigensets, whose relevant sequences we denote by $\wp_{II} := \{\delta^{(n)}\}_{n\in I\!N}$ and $\aleph_{II} := \{l^{(n)}\}_{n\in I\!N}$, respectively. It so happens that for every $n \in I\!N$, we have $\delta_{(n)} \in (\delta^{(n+1)}, \delta^{(n)})$ and $l_{(n)} \in (l^{(n)}, l^{(n+1)})$, which roughly stated says that "between" any two Type I eigensets there is a Type II eigenset.

It is clear from (82) that $detA_3$ vanishes at Type I eigensets but not at the eigensets of Type II. On the other hand, it follows from (30), (65) and (76) that

$$\frac{1}{a^2}\frac{d(detA_3)}{d\delta}\bigg|_{\delta=\delta_{(n)}} = \frac{4(2a-k)}{\delta_{(n)}^2}\left(2 + \frac{1}{\delta_{(n)}}\right) > 0. \tag{91}$$

Thus we see that for each $n \in I\!N$ there must be a $\hat{\delta}_{(n)} \in (\delta_{(n+1)}, \delta_{(n)})$ such that $detA_3(\hat{\delta}_{(n)}) =$

0. By (53), (67)$_3$, and (74) we know that if $u_{3,\delta}(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is an extremal such that $det A_3 \neq 0$, then

$$E_\delta(u_{3,\delta}) = \frac{2ak\omega\Delta^2(2a - k)det A}{det A_3}. \tag{92}$$

Since $det A$ is continuous in δ and $det A(\hat{\delta}_{(n)}) \neq 0$, we may conclude from (91) and (92) that for every $n \in I\!N$,

$$\lim_{\delta \to \hat{\delta}_{(n)}} E_\delta(u_{3,\delta}) = \pm\infty. \tag{93}$$

This highly irregular behavior of an extremal field, and our earlier conclusion concerning the boundedness of $E_\delta(\cdot)$ on extremals when Λ is a Type I eigenset, suggests that there is a $\delta_0 \in (0, \delta_{(1)})$ such that for every $\delta < \delta_0$, the functional $E_\delta(\cdot)$ will cease to be bounded below in $\mathcal{A}^{1,p}(\mathcal{B})$. If this is the case, then the material will be globally unstable when $\delta < \delta_0$. This is the essence of

Theorem 3. *Let $0 < k < 2a$ and $l > 0$ be given. Then, $\exists\, \delta_0 \in (0, \delta_{(1)})$ such that $\forall\, \delta \in (0, \delta_0)$,*

$$\inf_{u \in \mathcal{A}^{1,p}(\mathcal{B})} E_\delta(u) = -\infty. \tag{94}$$

Proof. Let $0 \leq x_1 < x_2 < x_3 \leq l$ be given. Define

$$e_\lambda(x) := \left\{ \begin{array}{ccc} \lambda & : & x \in [x_1, x_2] \\ \phi(\lambda) & : & x \in (x_2, x_3] \\ 0 & : & x \in \mathcal{B} - [x_1, x_3] \end{array} \right\}, \tag{95}$$

where $\lambda \in I\!R$, and $\phi(\lambda)$ is determined so that (95) satisfies the fixed displacement condition (27), i.e.,

$$\lambda(x_2 - x_1) + \phi(\lambda)(x_3 - x_2) = \Delta. \tag{96}$$

Hence, if we let $u_\lambda(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ be such that $u'_\lambda(\cdot) = e_\lambda(\cdot)$, then with the aid of (95) and (6), for the particular model of Section 2, we find that

$$\begin{aligned} E_\delta(u_\lambda) &= (k - 2a)\alpha\lambda^2 + (k - 2a)\beta[\phi(\lambda)]^2 \\ &\quad - 2a\delta\left(\lambda^2(e^{-\frac{\alpha}{\delta}} - 1) + [\phi(\lambda)]^2(e^{-\frac{\beta}{\delta}} - 1)\right) - 2a\delta\lambda\phi(\lambda)P_\delta, \end{aligned} \tag{97}$$

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where $\alpha := x_2 - x_1$, $\beta := x_3 - x_2$ and $P_\delta := (1 - e^{-\frac{\alpha}{\delta}} - e^{-\frac{\beta}{\delta}} + e^{-\frac{(\alpha+\beta)}{\delta}})$. Upon using (96) we see that

$$
\begin{aligned}
E_\delta(u_\lambda) \ = \ & c(k, a, \alpha, \beta, \delta, \Delta) + \{\frac{4a\delta\alpha\Delta}{\beta^2}(e^{-\frac{\beta}{\delta}} - 1) - \frac{2\alpha(k - 2a)\Delta}{\beta} - \frac{2a\delta\Delta P_\delta}{\beta}\} \lambda \\
& + \{(k - 2a)\alpha + \frac{(k - 2a)}{\beta}\alpha^2 - 2a\delta(e^{-\frac{\alpha}{\delta}} - 1) \\
& - \frac{2a\delta\alpha^2}{\beta^2}(e^{-\frac{\beta}{\delta}} - 1) + \frac{2a\alpha\delta P_\delta}{\beta}\} \lambda^2,
\end{aligned}
\tag{98}
$$

where $c(k, a, \alpha, \beta, \delta, \Delta)$ is a real constant. Now, for simplicity, we assume that $\alpha = \beta = l/2$. Then, the coefficient of λ^2 in (98) reduces to the form

$$
h(\delta) := al\{(\frac{k}{a} - 2) - \frac{4\delta}{l}(e^{-\frac{l}{2\delta}} - 1) + \frac{2\delta}{l}(e^{-\frac{l}{2\delta}} - 1)^2\},
\tag{99}
$$

and we see that

$$
\lim_{\delta \to \infty} h(\delta) = kl > 0 \quad , \quad \lim_{\delta \to 0} h(\delta) = (\frac{k}{a} - 2)al < 0,
\tag{100}
$$

for every $0 < k < 2a$, and that $h'(\delta) < 0$ for every $\delta \in (0, \infty)$.

In the remainder of this proof, we shall show that there is a single positive root δ_0 of $h(\delta) = 0$ which is less than $\delta_{(1)}$ and such that for any $\delta \in (0, \delta_0)$, $h(\delta)$ is negative. Then, for any such δ it will follow from (98) that $E_\delta(u_\lambda)$ is unbounded below in the limit as $\lambda \to \infty$, as claimed in (94).[3] Specifically, let us consider $h(\delta)/(al)$ from (99) as a *continuous* function $g(\cdot, \cdot)$ of the two variables $(k/a, l/\delta) \in \Omega := (0, 2) \times (0, \infty) \subset I\!\!R^2$. Then, it follows from (100) that the point set $\Omega_0 := \{(k/a, l/\delta) : h(\delta) = 0\}$ consists of a single, plane curve that separates Ω into two disjoint, connected, open sets; namely $\Omega_+ := \{(k/a, l/\delta) : g(k/a, l/\delta) > 0\}$ and $\Omega_- := \{(k/a, l/\delta) : g(k/a, l/\delta) < 0\}$, where $\Omega = \Omega_0 \cup \Omega_+ \cup \Omega_-$. The pointset Ω_0 is determined by a simple numerical computation, and this is shown in Figure 7. For fixed parameters $0 < k < 2a$ and $l > 0$ and $\delta \in (0, \delta_0)$, the function $h(\delta)$ is indeed negative. Similarly, we may use (30) to write (76) in terms of the variables $(k/a, l/\delta) \in \Omega$ and obtain the plane curve $\Omega_{(1)}$ that corresponds to the eigenset $\{k, a, \delta_{(1)}, l\}$ of Type I which contains the *largest* member $\delta_{(1)}$ in the decreasing sequence $\wp_I$. We also have shown this in Figure 7, where it is clear that for fixed parameters $0 < k < 2a$ and $l > 0$ we have $\delta_0 \in (0, \delta_{(1)})$.$\square$

---

[3]Certainly, for k and a such that $0 < k < 2a$ and $\alpha, \beta > 0$, one may use (98) to numerically determine the maximum value of δ_0 such that (94) holds for a field of the form (95). This computation is easily completed when k and a are given. We performed a number of numerical computations for various choices of the parameters α and β, all of which indicated that the value of δ_0 is maximized when $\alpha = \beta = l/2$. We have no formal proof of this, however.
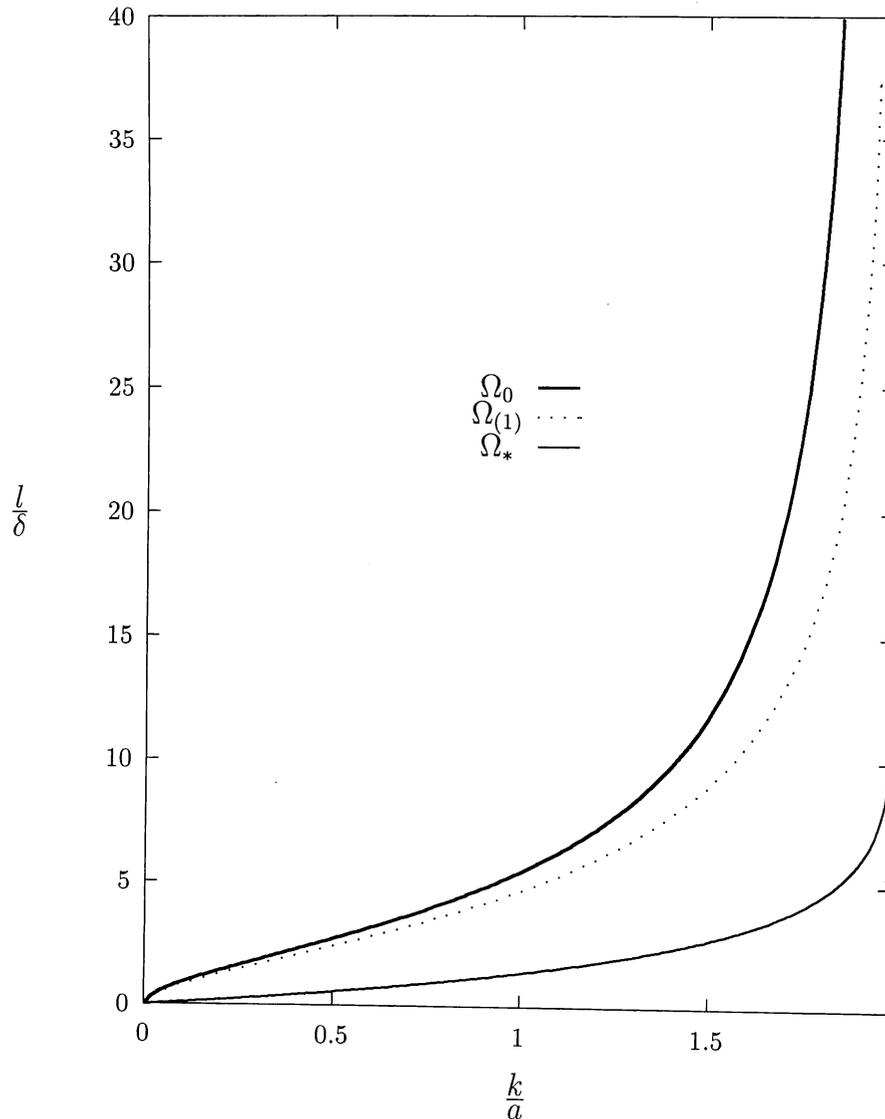
Figure 7. Admissible values of l/δ and k/a for (i) the curve Ω_0 defined by $h(\delta) = 0$ from (99); (ii) the curve $\Omega_{(1)}$ which defines the *maximal* Type I eigenset $\{k, a, \delta_{(1)}, l\}$ from (30) and (76); (iii) the curve Ω_* determined by the coercivity condition (64) for δ^*.

Given k, a, and l such that $0 < k < 2a$ and $l > 0$, Theorem 3 provides sufficient conditions on the parameter $\delta > 0$ to ensure that the total stored energy functional $E_\delta(\cdot)$ is unbounded below on $\mathcal{A}^{1,p}(\mathcal{B})$. Superimposed on Figure 7 is a graph, which we denote by Ω_*, of l/δ^* vrs. k/a from the coercivity condition (64). Recall, δ^* is the greatest lower bound to the set of all positive δ where the functional $E_\delta(\cdot)$ is coercive. From this figure we see that for fixed k, a, and l, as δ runs through a decreasing sequence, the functional $E_\delta(\cdot)$ first looses its coercivity at $\delta = \delta^*$ and then, after passing $\delta = \delta_0 < \delta_{(1)} < \delta^*$, it is unbounded below on $\mathcal{A}^{1,p}(\mathcal{B})$. In

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the remainder of this work we shall investigate the sign of the second variation of $E_\delta(\cdot)$ in an attempt to determine the stability character of $E_\delta(\cdot)$ in the "stability gap" $[\delta_0, \delta^*]$, i.e., between Ω_0 and Ω_* in Figure 7. We begin by considering this question for the case when $\Lambda = \{k, a, \delta, l\}$ is a Type I eigenset corresponding to either the maximal $\delta = \delta_{(1)}$ in $\wp_I$ or, equivalantly, the minimal $l = l_{(1)}$ in $\aleph_I$. These values lie on the curve $\Omega_{(1)}$ in Figure 7.

First, suppose $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is a solution of the minimization problem (19), and introduce the set of *null fields*

$$\mathcal{A}_0^{1,p}(\mathcal{B}) := \{\eta(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B}) \ : \ \Delta = 0\}. \tag{101}$$

Then, for every $\epsilon > 0$ and null field $\eta(\cdot)$, we see that $u_\epsilon(\cdot) := u(\cdot) + \epsilon\eta(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$. Necessary conditions for $u(\cdot)$ to solve (19) are

$$\left.\frac{\partial E_\delta(u_\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = 0 \quad ; \quad \left.\frac{\partial^2 E_\delta(u_\epsilon)}{\partial \epsilon^2}\right|_{\epsilon=0} \geq 0. \tag{102}$$

The first condition in (102) reduces essentially to the statement of Theorem 1 and the necessity of the Euler-Lagrange equation (21). Given $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$, we observe that the second condition in (102) reduces to

$$\delta^2 E_\delta(u; \eta) := \left.\frac{\partial^2 E_\delta(u_\epsilon)}{\partial \epsilon^2}\right|_{\epsilon=0} = 2 \int_{\mathcal{B}} \{k\eta'^2(x) - \eta'(x)\pounds_\delta\eta'(x)\}\, dx \geq 0, \tag{103}$$

where the expression $\delta^2 E_\delta(u; \eta)$ denotes the second variation of the functional $E_\delta(\cdot)$ at $u(\cdot)$ in the direction $\eta(\cdot)$. We say that a solution $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ of the Euler-Lagrange equation (21) is *locally stable* if for every $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ we have $\delta^2 E_\delta(u; \eta) > 0$. If instead $\delta^2 E_\delta(u; \eta) < 0$, then we say that the given displacement field is *unstable*. We emphasize here that the positivity of the second variation of $E_\delta(\cdot)$ at a field $u(\cdot)$ does not necessarily guarantee that the field $u(\cdot)$ is *globally stable* in the sense that it is a solution of the complete minimization problem (19).

To determine those $\delta > 0$ for which there is a null field $\eta(\cdot)$ with the property that $\delta^2 E_\delta(u; \eta) < 0$, we minimize the second variation (103) over the class of null functions $\mathcal{A}_0^{1,p}(\mathcal{B})$. Since $\delta^2 E_\delta(u; \eta)$ is a positive scalar multiple of the original total stored energy $E_\delta(\cdot)$, we have, by Theorem 2, that if $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ is an extremal of the second variation, then there exists a $\mu \in I\!R$ such that

$$\eta'(x) = \frac{1}{k} \int_{\mathcal{B}} f_\delta(x - z)\eta'(z)\, dz + \frac{\mu}{2k}. \tag{104}$$

Now suppose $\Lambda = \{k, a, \delta, l\}$ is a Type I eigenset. Then, by our previous discussion concerning (71)-(80) we know that there is a nontrivial null function $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ such that $\eta(\cdot)$

solves (104) when $\mu = 0$. It is then immediate from (103) that in this case $\delta^2 E_\delta(u;\eta) = 0$. This extremal function $\eta(\cdot)$ can now be used as the building block for a classical "Jacobi-type" stability argument. Recalling the sequences $\wp_I = \{\delta_{(n)}\}_{n\in I\!N}$ and $\aleph_I = \{l_{(n)}\}_{n\in I\!N}$ corresponding to Type I eigensets, we have the following

Theorem 4 (Jacobi). *Let $0 < k < 2a$ and $\delta > 0$ be given. Let $l_{(1)} > 0$ be the minimal element of $\aleph_I$, so that $\{k, a, \delta, l_{(1)}\}$ is a Type I eigenset, and let $l > l_{(1)}$. Suppose that $E_\delta(\cdot)$ denotes the total stored energy functional for the body $\mathcal{B} := (0, l)$, and that $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ satisfies the Euler-Lagrange equation (21). Then, $\exists\, \eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ such that*

$$\delta^2 E_\delta(u;\eta) < 0. \tag{105}$$

Proof. From the previous discussion concerning (71), there is a $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B}_{(1)})$ such that

$$\eta'(x) = \frac{1}{k} \int_{\mathcal{B}_{(1)}} f_\delta(x-z)\eta'(z)\, dz \tag{106}$$

for every $x \in \mathcal{B}_{(1)}$, where $\mathcal{B}_{(1)} := (0, l_{(1)}) \subset \mathcal{B}$. Let $\hat{\eta}(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ be defined such that

$$\hat{\eta}'(x) := \left\{ \begin{array}{lll} \eta'(x) & : & x \in (0, l_{(1)}) \\[2mm] 0 & : & x \in [l_{(1)}, l) \end{array} \right\}. \tag{107}$$

Then, using (103), (106) and (107) we see that

$$\delta^2 E_\delta(u;\hat{\eta}) = 2\int_{\mathcal{B}_{(1)}} \left\{ k\eta'^2(x) - \eta'(x) \int_{\mathcal{B}_{(1)}} f_\delta(x-z)\eta'(z)\, dz \right\} dx = 0. \tag{108}$$

While (108) shows that $\delta^2 E_\delta(u;\cdot)$ evaluated at $\hat{\eta}(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ is zero, we will now show that $\delta^2 E_\delta(u;\cdot)$ is not minimized at $\hat{\eta}(\cdot)$ over the set $\mathcal{A}_0^{1,p}(\mathcal{B})$. This will establish (105).

To carry this out, we suppose, for contradiction, that $\hat{\eta}(\cdot)$ is a minimizer of $\delta^2 E_\delta(u;\cdot)$ over $\mathcal{A}_0^{1,p}(\mathcal{B})$. Then, the first variation condition (104) requires $\hat{\eta}(\cdot)$ to satisfy

$$\hat{\eta}'(x) = \frac{1}{k} \int_{\mathcal{B}} f_\delta(x-z)\hat{\eta}'(z)\, dz + \rho \tag{109}$$

for all $x \in \mathcal{B}$ and for some real constant ρ. Because of (107), this yields the following necessary conditions:

$$\eta'(x) = \frac{1}{k} \int_{\mathcal{B}_{(1)}} f_\delta(x - z)\eta'(z)\, dz + \rho \quad \forall\, x \in \mathcal{B}_{(1)}, \tag{110}$$

and

$$\int_{\mathcal{B}_{(1)}} f_\delta(x - z)\eta'(z)\, dz + \rho = 0 \quad \forall\, x \in [l_{(1)}, l). \tag{111}$$

Using (106), we then see that $\rho = 0$, so that if $\hat{\eta}(\cdot)$ is to be a minimizing field, then (111) must hold with $\rho = 0$. But, by analogy between (71) and (106), and the general solution (78) of (71), we know that $\eta'(\cdot)$ must be of the form $\eta'(x) = c(\delta\omega \cos \omega x + \sin \omega x)$, where c is an arbitrary constant. Thus, carrying out the relevant integration using (18), we find that

$$\int_{\mathcal{B}_{(1)}} f_\delta(x - z)\eta'(z)\, dz = -k\delta\omega c e^{-\frac{x - l_1}{\delta}} \neq 0. \tag{112}$$

Since this contradicts (111) for $\rho = 0$, we conclude that $\hat{\eta}(\cdot)$ *does not* satisfy the first variation condition (109), and therefore it cannot minimize $\delta^2 E_\delta(u; \cdot)$ over $\mathcal{A}_0^{1,p}(\mathcal{B})$. In light of (108), the proof is completed. $\square$

In physical terms, Theorem 4 says that when all other parameters are fixed with $k < 2a$, if the length l of $\mathcal{B}$ is large enough $(> l_{(1)})$ then an extremal of $E_\delta(\cdot)$ in $\mathcal{A}^{1,p}(\mathcal{B})$ is unstable. Alternatively, if all other parameters except the influence parameter δ are fixed, as above, then such an extremal is unstable if δ is small enough $(< \delta_{(1)})$. This is the content of

Corollary 3. *Let $0 < k < 2a$ and $l > 0$ be given. Let $\delta_{(1)} > 0$ be the maximal element of the sequence $\wp_I$, so that $\{k, a, \delta_{(1)}, l\}$ is a Type I eigenset, and let $0 < \delta < \delta_{(1)}$. Suppose that $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ satisfies the Euler-Lagrange equation (21) for $E_\delta(\cdot)$. Then, $\exists\, \eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ such that*

$$\delta^2 E_\delta(u; \eta) < 0. \tag{113}$$

Proof. Let $\gamma \in (0, 1)$ be such that $\delta = \gamma\delta_{(1)}$ and observe from (21) that $u'(\cdot) = e(\cdot)$ is supposed to satisfy

$$e(x) = \frac{1}{k} \int_{\mathcal{B}} f_\delta(x - z)e(z)\, dz + \alpha \quad \forall x \in \mathcal{B}, \tag{114}$$

for some $\alpha \in \mathbb{R}$. Let $\bar{x} := x/\gamma$ and $\bar{z} := z/\gamma$ and define $\hat{e}(\bar{x}) := e(\gamma\bar{x})$. Then, recalling (18), we may rewrite (114) in the form

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$$\hat{e}(\bar{x}) \;=\; \frac{1}{k}\int_{\mathcal{B}_{(\bar{l})}} f_{\delta_{(1)}}(\bar{x}-\bar{z})\hat{e}(\bar{z})\,d\bar{z} \;+\; \alpha \tag{115}$$

for every $\bar{x} \in (0,\bar{l})$, where $\bar{l} := l/\gamma$ and $\mathcal{B}_{(\bar{l})} := (0,\bar{l}) \supset \mathcal{B}$. Thus, by applying Theorem 2 we see that $\hat{e}(\cdot)$ satisfies the Euler-Lagrange equation for the functional

$$\bar{E}_{\delta_{(1)}}(u) \;:=\; \int_{\mathcal{B}_{(\bar{l})}} \{ke^2(x) \;-\; e(x)\int_{\mathcal{B}_{(\bar{l})}} f_{\delta_{(1)}}(x-z)e(z)\,dz\}\,dx\,, \tag{116}$$

defined for all $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B}_{(\bar{l})})$. Since $\{k,\,a,\,\delta_{(1)},\,l\}$ is a Type I eigenset and, in the present case, $\mathcal{B} \subset \mathcal{B}_{(\bar{l})}$ for $\gamma \in (0,1)$, we may conclude using Theorem 4 that there is a $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B}_{(\bar{l})})$ such that

$$\delta^2 \bar{E}_{\delta_{(1)}}(\hat{u};\eta) < 0, \tag{117}$$

where $\hat{u}(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B}_{(\bar{l})})$, with $\hat{u}'(\cdot) = \hat{e}(\cdot)$, satisfies the Euler-Lagrange equation (115). Finally, by recalling (18) and making a standard change of variables of intergration in the expression

$$\delta^2 \bar{E}_{\delta_{(1)}}(\hat{u};\eta) \;=\; 2\int_{\mathcal{B}_{(\bar{l})}} \{k\eta'^2(x) \;-\; \eta'(x)\int_{\mathcal{B}_{(\bar{l})}} f_{\delta_{(1)}}(x-z)\eta'(z)\,dz\}\,dx, \tag{118}$$

we observe that

$$\begin{aligned}
\delta^2 \bar{E}_{\delta_{(1)}}(\hat{u};\eta) \;&=\; \frac{2}{\gamma}\int_{\mathcal{B}} \{k\check{\eta}'^2(x) \;-\; \check{\eta}'(x)\int_{\mathcal{B}} f_\delta(x-z)\check{\eta}'(z)\,dz\}\,dx, \\
&=\; \frac{1}{\gamma}\delta^2 E_\delta(u;\check{\eta}),
\end{aligned} \tag{119}$$

where $\check{\eta}'(x) := \eta'(x/\gamma)$ for every $x \in \mathcal{B}$. Then, since it is clear that $\check{\eta}(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$, we see from (117), (119) and the fact that $\gamma > 0$, that (113) holds. $\square$

The previous theorem and corollary prove that if k, a, and l are given such that $0 < k < 2a$ and $l > 0$, and if $0 < \delta < \delta_{(1)}$, where $\delta_{(1)}$ is uniquely determined from (75) and (76) as the largest element of $\wp_I$, then the extremal strain fields derived in Section 3 and given by (37) cannot correspond to local minima of $E_\delta(\cdot)$ over $\mathcal{A}^{1,p}(\mathcal{B})$. Similarly, if the material paramters k, a and $\delta > 0$ are given such that $0 < k < 2a$ and $\delta > 0$, then the model becomes unstable if the reference length l of the body $\mathcal{B}$ is chosen sufficiently large ($l > l_{(1)}$). Given such suitable arbitrary values of k, a, and l, we observe from Figure 7 that $\delta_{(1)}$ is strictly less than δ^*. Thus, as δ runs through a decreasing sequence of values, the functional $E_\delta(\cdot)$ first looses coercivity at δ^* and then for all $\delta < \delta_{(1)}$ we now know, according to Corollary 3, that the

model is unstable. The value of $\delta = \delta_0$, corresponding to the curve Ω_0 in Figure 7 where the model shows global instability, is less that $\delta_{(1)}$ and the ordering $\delta_0 < \delta_{(1)} < \delta^*$ is natural. The "stability gap" referred to above in which the stability character of $E_\delta(\cdot)$ is unknown has now been narrowed to those δ in the interval $[\delta_{(1)}, \delta^*]$, i.e., between $\Omega_{(1)}$ and Ω_* in Figure 7. In fact, we will now narrow this gap further and show that the *maximal* member $\delta^{(1)}$ of the sequence $\wp_{II}$ corresponding to Type II eigensets $\{k, a, \delta^{(n)}, l\}$ provides a lower bound such that for all $\delta > \delta^{(1)}$ the second variation $\delta^2 E_\delta(u; \cdot)$ at the solution $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ of the Euler-Lagrange equation (21) is positive. Moreover, we will see that $\delta^{(1)}$ is *below* δ^* and above $\delta_{(1)}$ so that the remaining "stability gap" is again narrowed to those δ that belong to the interval $[\delta_{(1)}, \delta^{(1)})$.

We observed earlier, following (104), that the second variation $\delta^2 E_\delta(u; \cdot)$ vanishes in a particular direction at $\delta = \delta_{(1)}$, and that there is a direction in which it is negative for every $\delta < \delta_{(1)}$. Based upon an approach discussed in a remark at the end of this paper, it is possible that the "stability gap" can be further narrowed and, in fact, possibly eliminated. For arbitrary k, a, and l such that $0 < k < 2a$ and $l > 0$, we show that there is a $\overline{\delta} \in (\delta_{(1)}, \delta^{(1)})$ such that the second variation is strictly positive on $\mathcal{A}_0^{1,p}(\mathcal{B})$ for every $\delta \in (\overline{\delta}, \infty)$, thereby narrowing the "stability gap" to $[\delta_{(1)}, \overline{\delta}) \subset [\delta_{(1)}, \delta^{(1)}]$. To proceed, we examine the eigenvalue problem for the homogeneous problem (21) (i.e., $\sigma = 0$), with a goal toward using the equivilancy between the eigenvalue problem and its associated minimization problem to conclude the positivity of the second variation $\delta^2 E_\delta(u; \cdot)$ on an extremal field $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$.

Consider, for $\lambda \in I\!R$ and $\phi(\cdot) \in L^2(\mathcal{B})$,

$$\phi(x) = \frac{\lambda}{k} \mathcal{L}_\delta \phi(x) \tag{120}$$

for all $x \in \mathcal{B}$. By analogy with the Euler-Lagrange equation (21) in Section 2, we conclude that a necessary condition for $\phi(\cdot)$ to solve (120) is that $\phi(\cdot)$ satisfy

$$\phi''(x) + \frac{1}{k\delta^2}(2a\lambda - k)\phi(x) = 0 \quad \forall x \in \mathcal{B}. \tag{121}$$

The general solutions of (121) contain two arbitrary, real constants c_1 and c_2, and separate into three classes depending upon the sign of $2a\lambda - k$. Let C_i denote the upper left 2×2 submatrix of A_i, as inferred from equations (33), (36), and (39), and defined in (41), except that everywhere in A_i we replace a by $a\lambda$. Then, by the discussion in Section 3, it is clear that solutions of (121) solve (120) if and only if the coefficient matrix C_i has a nontrivial kernel, where here the subscript $i = 1, 2, 3$ refers to the cases $2a\lambda < k$, $2a\lambda = k$, and $2a\lambda > k$, respectively. We easily see from the linear systems (33) and (36) that if $2a\lambda \leq k$, then $\ker C_1 = \ker C_2 = \{\mathbf{0}\}$. Therefore, a solution of (120) is possible only if $2a\lambda > k$. Thus, by introducing $\kappa > 0$ through

$$\kappa^2 := \frac{1}{\delta^2}\left(\frac{2a\lambda}{k} - 1\right), \tag{122}$$

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we recall from (73) and (74) that $\ker C_3 \neq \{0\}$ if and only if

$$2\delta\kappa\cos\kappa l = (\delta^2\kappa^2 - 1)\sin\kappa l. \tag{123}$$

In this case, similar to our analysis in the first half of this section, we infer that there is a strictly increasing sequence of eigenvalues $\Im := \{\lambda_n\}_{n \in I\!N}$, having a minimum element, denoted by λ_1, and satisfying $\lim_{n\to\infty} \lambda_n = \infty$. Moreover, we may separate the members of $\Im$ into two distinct classes depending upon whether n is odd;

$$\left\{ \sin\kappa l = \frac{k}{a\lambda}\kappa\delta \quad ; \quad \cos\kappa l = \frac{k}{2a\lambda}(\kappa^2\delta^2 - 1) \right\}, \tag{124}$$

or whether n is even;

$$\left\{ \sin\kappa l = -\frac{k}{a\lambda}\kappa\delta \quad ; \quad \cos\kappa l = -\frac{k}{2a\lambda}(\kappa^2\delta^2 - 1) \right\}. \tag{125}$$

The eigenfunctions for (120), which we denote by $\phi_n(\cdot)$, are therefore of the form

$$\phi_n(x) := c_n(\kappa_n\delta\cos\kappa_n x + \sin\kappa_n x) \tag{126}$$

where κ_n has the obvious definition from (122). Also, we conclude from (122), (124), (125) and (126) that

$$\int_{\mathcal{B}} \phi_n(x)\,dx = \left\{ \begin{array}{lll} 0 & : & n \text{ is even} \\ 2\frac{c_n}{\kappa_n} & : & n \text{ is odd} \end{array} \right\}. \tag{127}$$

Since $2a\lambda_n > k > 0$ for every $n \in I\!N$, it follows that $\Im \subset (0,\infty)$, and therefore, the linear Hilbert-Schmidt operator $\mathcal{L}_\delta(\cdot)$ is symmetric and positive definite [4]. We may assume without loss of generality that for every $n \in I\!N$, the eigenfunctions $\phi_n(\cdot)$ have been normalized so that $\|\phi_n\|_2 = 1$, and in this case we have $c_n^2 = (la\lambda_n/k + \delta)^{-1}$. Consequently, since the eigenvalues λ_n are distinct, we have that $\langle\phi_i, \phi_j\rangle = \delta_{ij}$, where δ_{ij} is the ususal Kronecker delta, and $\langle\cdot, \cdot\rangle$ denotes the usual inner product on $L^2(\mathcal{B})$ [4]. Then, if we define

$$\mathcal{D}_n := \{p(\cdot) \in L^2(\mathcal{B}) \ : \ \langle p, \phi_k\rangle = 0 \text{ for } k = 1, \cdots, n-1\}, \tag{128}$$

it follows from the extremal properties of eigenvalues for operators of this type [4, 11] that for every $n \in I\!N$ and nontrivial $p(\cdot) \in \mathcal{D}_n$,

$$0 < \langle \pounds_\delta p, p \rangle \leq \frac{k}{\lambda_n} \|p\|_2^2, \tag{129}$$

where equality holds if and only if $p(\cdot) \propto \phi_n(\cdot)$. Now, we recall the sequences $\wp_I$ ane $\wp_{II}$ which characterized, respectively, the Type I and Type II eigensets $\{k, a, \delta_{(n)}, l\}$ as defined in (30), (75), (76) and (77) for $n \in I\!N$. Then, according to (122) and (123) the sequence of eigenvalues $\mathfrak{S}$ possesses the following elementary properties relative to $\wp_I$ and $\wp_{II}$, which we state without proof.

Lemma 2. *Let $0 < k < 2a$ and $l > 0$ be given, and choose $\delta_{(m)} \in \wp_I$ and $\delta^{(n)} \in \wp_{II}$. Then,*

(i) if $n \in I\!N$ is fixed, λ_n strictly increases as a function of δ,

(i) $\delta = \delta_{(m)} \Leftrightarrow \lambda_{2m} = 1$,

(ii) $\delta = \delta^{(n)} \Leftrightarrow \lambda_{2n-1} = 1$.

Then, using (129), we have the following

Theorem 5. *Let $0 < k < 2a$ and $l > 0$ be given, and suppose that $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is a solution of the Euler-Lagrange equation (21) for $\delta \geq \delta^{(1)}$. Then, for every $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$,*

$$\delta^2 E_\delta(u; \eta) > 0. \tag{130}$$

Proof. Suppose $\delta \geq \delta^{(1)}$, choose $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$, and assume without loss of generality that $\|\eta'\|_2 = 1$. Then, by (103), (128) and (129), we observe that

$$\begin{aligned}
\delta^2 E_\delta(u; \eta) &= 2 \int_\mathcal{B} \{ k\eta'^2(x) - \eta'(x) \pounds_\delta \eta'(x) \} \, dx \\
&\geq 2 \int_\mathcal{B} \{ k\eta'^2(x) - \eta'(x) \frac{k}{\lambda_1} \eta'(x) \} \, dx \\
&= 2k\left(1 - \frac{1}{\lambda_1}\right). \tag{131}
\end{aligned}$$

By Lemma 2, we may then conclude that $\delta^2 E_\delta(u; \eta) \geq 0$. However, by (129), we see that the inequality in (131) is strict unless $\eta'(\cdot) = \phi_1(\cdot)$, and this cannot happen since $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ and $\phi_1(\cdot)$ satisfies $(127)_2$. $\square$

In Figure 8 we show all of the information contained in Figure 7 and in addition we illustrate a graph, denoted by $\Omega^{(1)}$, of $l/\delta^{(1)}$ vrs. k/a for the Type II eigenset which corresponds

to the maximal value of $\delta = \delta^{(1)}$. Earlier, we observed that $\delta_{(1)} < \delta^{(1)}$ and this is certainly confirmed in Figure 8. More insightful, however, is the observation that $\delta^{(1)} < \delta^*$, where, recall, δ^* defines the coercivity limit in the sense that $E_\delta(\cdot)$ is coercive only if $\delta > \delta^*$. Because of Theorem 5, this means that for k, a and l given such that $0 < k < 2a$ and $l > 0$, and for any $\delta \geq \delta_{(1)}$ the solution $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ of the Euler-Lagrange equation (21) is locally stable, even though for $\delta \in (\delta^{(1)}, \delta^*]$ the existence theorem, Theorem 1, for the minimum problem (19) does not apply. With this result, we have narrowed the "stability gap" to those δ in the interval $[\delta_{(1)}, \delta^{(1)})$, as is shown in Figure 8.
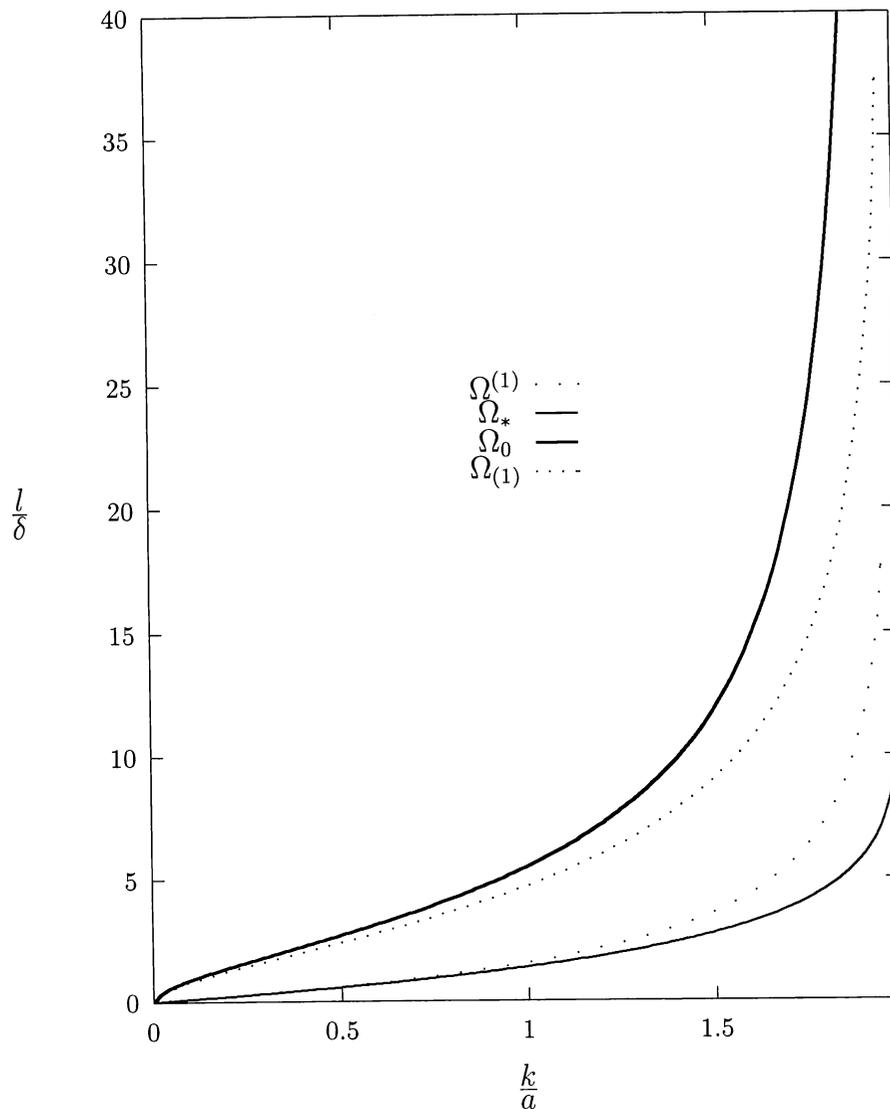
Figure 8. Stability diagram showing loss of coercivity Ω_*, the stability gap bounded by the dotted curves $\Omega_{(1)}$ and $\Omega^{(1)}$, the loss of local stability $\Omega_{(1)}$, and the loss of global stability Ω_0.

In this gap we have not determined the stability character of $E_\delta(\cdot)$.

Remark in closing: A possible approach to the remaining problem of characterizing the "stability gap" is to use the eigenfunction theory developed above in an attempt to either increase from $\delta_{(1)}$ the threshold for local instability, or to decrease from $\delta^{(1)}$ the threshold for local stability. In particular, suppose $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ is a solution of the Euler-Lagrange equation (21), and define null functions $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ by

$$\eta(x) := \int_0^x \{b_1 \phi_1(\xi) + w(\xi)\} \, d\xi, \tag{132}$$

where $b_1 \in \mathbb{R}$, $\phi_1(\cdot)$ is the eigenfunction associated with λ_1, and $w(\cdot) \in \mathcal{D}_1$. Then, the definition of $\mathcal{A}_0^{1,p}(\mathcal{B})$ coupled with (122) and (127) implies that $w(\cdot)$ has nonzero mass, and that

$$b_1 = -\frac{1}{2}\kappa_1 (\frac{la\lambda_1}{k} + \delta) \int_\mathcal{B} w(x) \, dx \tag{133}$$

holds. By substituting (132) and (133) into the second variation (103), and recalling (120) and (129), we observe that

$$\delta^2 E_\delta(u; \eta) = \sigma_{\lambda_1} \left(\int_\mathcal{B} w(x) \, dx \right)^2 + k\|w\|_2^2 - \langle \mathcal{L}_\delta w, w \rangle, \tag{134}$$

$$\geq \sigma_{\lambda_1} \left(\int_\mathcal{B} w(x) \, dx \right)^2 + k(1 - \frac{1}{\lambda_2})\|w\|_2^2, \tag{135}$$

where

$$\sigma_{\lambda_1} := \frac{k}{4}(1 - \frac{1}{\lambda_1})(\frac{la\lambda_1}{k} + \delta)(\frac{2a\lambda_1}{k} - 1). \tag{136}$$

It then follows from Lemma 2 that, given k, a, and l such that $0 < k < 2a$ and $l > 0$, the expression σ_{λ_1} given by (136) is nonpositive for all $\delta \in [\delta_{(1)}, \delta^{(1)}]$. In this case, we conclude from (135) and the Cauchy-Schwartz inequality that

$$\delta^2 E_\delta(u; \eta) \geq (\sigma_{\lambda_1} l + k(1 - \frac{1}{\lambda_2}))\|w\|_2^2 \tag{137}$$

must hold for every $\delta \in [\delta_{(1)}, \delta^{(1)}]$ and $w(\cdot) \in \mathcal{D}_1$. Moreover, with the aid of Lemma 2, we see that there is a $\overline{\delta} \in (\delta_{(1)}, \delta^{(1)})$ with the property that for every $\delta \in (\overline{\delta}, \delta^{(1)}]$ the expression $\sigma_{\lambda_1} l + k(1 - \lambda_2^{-1})$ is positive. Therefore, since $L^2(\mathcal{B}) = \mathcal{D}_1 \oplus \{\alpha\phi_1(\cdot) : \alpha \in \mathbb{R}\}$, we may conclude from Theorem 5 that

$$\delta^2 E_\delta(u; \eta) > 0 \tag{138}$$

holds for every $\eta(\cdot) \in \mathcal{A}_0^{1,p}(\mathcal{B})$ and $\delta \in (\overline{\delta}, \infty)$, and therefore, the extremal field $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ of (21) is locally stable. For the particular choice of material parameters $k = a$ and $l = 1$, this analysis yields a marked improvement on $\delta^{(1)}$ as a lower bound for local stablility. For this case, we have

$$
\begin{aligned}
\delta^* &= (2ln2)^{-1} &\simeq\ & 0.7213, \\
\\
\delta^{(1)} &= \frac{2}{\pi} &\simeq\ & 0.6366, \\
\\
\delta_{(1)} &= \frac{2}{3\pi} &\simeq\ & 0.2122,
\end{aligned}
\tag{139}
$$

and, by numerical analysis of the expression $\sigma_{\lambda_1} l + k(1 - \lambda_2^{-1})$ in (135), we find that $\overline{\delta} \simeq 0.35$. This means that for any $\delta > 0.35$ the extremal field $u(\cdot) \in \mathcal{A}^{1,p}(\mathcal{B})$ that solves the Euler-Lagrange equation (21) is locally stable. Earlier, we had found this to be true for any $\delta \geq \delta^{(1)}$ and so an improvement has been accomplished. To completely answer the question of stablility for the remaining "stability gap" $[\delta_{(1)}, \overline{\delta}]$, a more thorough study of (134) should be completed, but we do not attempt this here.

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