HOMOGENIZATION ON LATTICES: SMALL PARAMETER LIMITS, H-MEASURES, AND DISCRETE WIGNER MEASURES

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Abstract. We fully characterize the small-parameter limit for a class of lattice models with two-particle long or short range interactions with no "exchange energy." One of the problems we consider is that of characterizing the continuum limit of the classical magnetostatic energy of a sequence of magnetic dipoles on a Bravais lattice, (letting the lattice parameter tend to zero). In order to describe the small-parameter limit, we use *discrete Wigner transforms* to transform the stored-energy which is given by the double convolution of a sequence of (dipole) functions on a Bravais lattice with a kernel, homogeneous of degree $-\gamma$ with $\gamma \geq N$ with the cancellation property, as the lattice parameter tends to zero. By rescaling and using Fourier methods, discrete Wigner transforms in particular, to transform the problem to one on the torus, we are able to characterize the small-parameter limit of the energy depending on whether the dipoles oscillate on the scale of the lattice, oscillate on a much longer lengthscale, or converge strongly. In the case where $\gamma > N$, the result is simple and can be characterized by an integral with respect to the Wigner measure limit on the torus. In the case where $\gamma = N$, oscillations essentially on the scale of the lattice must be separated from oscillations essentially on a much longer lengthscale in order to characterize the energy in terms of the Wigner measure limit on the torus, an H-measure limit, and the limiting magnetization. We show that the classical magnetostatic energy with added lattice-induced anisotropies corresponds to oscillations essentially on a much larger lengthscale than that of the lattice and note that this energy is nonlocal in character. We also show that if the square of a suitable extension of the dipoles to $\Re^N$ is precompact in L^1 then the part of the limiting energy which corresponds to oscillations essentially on the scale of the lattice is local in the sense that the energy of short lengthscale oscillations generates a finitely additive finite signed measure on Borel sets. Examples are also given where the dipoles concentrate on Lebesgue null-sets and the corresponding energy of short lengthscale oscillations is not local. Several extensions are discussed as well as applications to magnetostatics.

Key words. Harmonic Analysis, Pseudo-Differential Operators, Homogenization, Magnetism, Separation of Scales, H-measures, Wigner measures, Fourier Methods, Lattice Models.

1. Introduction.

Lattice models have been used extensively in the study of mechanics in fields ranging from classical magnetostatics to the quantum mechanical structures of materials, (e.g., Bloch [2]), to the modeling of coherent structures (Toda [20], among others). Lattice models are studied especially in their relation to continuum models through the use of small-parameter limits, (e.g., Vogelius [21], Fujiki et al [7], De'Bell and Whitehead [4], James-Müller [10], Khachaturyan [13], etc). Small-parameter limits have been studied in the context of Γ-convergence, where some stored-energy is minimized for every value of the lattice parameter, or in the more general context, where compactness is assumed a priori and the limiting behavior is studied without the more restrictive assumption of energy-minimization. In both the contexts of Γ-convergence and more general small-parameter limits, continuum limits are seen

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to be a means of extending basic principles to the modeling of macroscopic bodies, and conversely to relate experimental measurements on large bodies to the microscopic properties of underlying material lattices.

The goal of this paper is to use Fourier methods and rescalings, (*discrete Wigner transforms*), to transform small-parameter limits of a class of lattice models with both long-range and short-range two-particle interactions with no "exchange energy," whose energies are given by convolutions, into equivalent convergence problems on tori, then to separate differing scales of oscillation for dipole sequences so as to completely characterize the limiting energies. Our model problem involves determining the limiting magnetostatic energy of a lattice of magnetic dipoles with no exchange energy and no anisotropy energy other than that intrinsic to the lattice. Some limited Γ-convergence results for specific models of lattices of magnetic dipoles were found by Khachaturyan [13] where he let particles interact by both long-range interactions and short-range "exchange energies," (which penalize dipole moment oscillations on the same lengthscale as that of the lattice). Vogelius [21] described the Γ-convergence of more general electrical networks, (rather than lattices), but only under the influence of specific short-range "exchange energies." Without considering small-parameter limits, Khachaturyan [14] and Chouliourous-Pouget [3] also used lattice models to describe the evolution of magnetic moments with more complicated Hamiltonians (higher-order interactions). Using finite Fourier transforms and theta-functions as a means of approximating small-parameter limits without actually refining the lattice, Fujiki et al [7] and De'Bell-Whitehead [4] described certain lattice-induced anisotropies for dipole-dipole interactions. More recently, James and Müller [10] have used singular integrals to characterize small-parameter limits of a particular lattice energy for lattices of magnetic dipoles oscillating only on the specific lengthscales in three dimensions with long-range interactions and no exchange energy.

Our study improves upon the results of James-Müller in that we are able to completely characterize the full small-parameter limit of the "no exchange" energy (including lattice-induced anisotropies) for lattices of magnetic dipoles which are allowed to oscillate on any lengthscale in any number of dimensions. We are also able to consider the limiting energy induced by kernels in the class of all smooth functions with the cancellation property, which are homogeneous of degree $-\gamma$ where $\gamma \geq N$, N being the dimension. We reposit the problem as one on a torus, relying upon the work of S. Wainger [22] on the Fourier transforms of kernels on lattices. In the process, we introduce *discrete Wigner measures,* the continuous analogs of which have been used by Gérard [9] and Lions-Paul [15] to describe semiclassical limits of Schrödinger equations. After transforming the problem to one on the torus, we cannot pass to the limit directly; it is first necessary to separate different lengthscales of oscillation. Using *essential filters* to separate various lengthscales, we are able to characterize the limiting energy completely by three complementary tools: the limiting magnetization; the H-measure limit, (see Tartar [19], Gérard [8]), to represent oscillations on a much larger lengthscale than that of the lattice; and the discrete Wigner measure limit, used to represent oscillations only on the scale of the lattice.

The "no-exchange" energy of a sequence of dipoles d_λ on the Bravais lattice $\mathcal{L}_\lambda = \lambda\mathcal{L}$ for $\lambda > 0$ is given by the classical formulation:

$$e_\lambda = \sum_{\mathcal{L}_\lambda, x \neq y} K(x-y) d_\lambda(x) d_\lambda(y),$$

where $K(z)$ is a matrix function which has the cancellation property in z and which is homogeneous of degree $-\gamma$ for $\gamma \geq N$. We show that the limiting energy can be completely characterized by the discrete homogeneous Wigner measure μ on the torus, the H-measure μ^H of a filtered extension of the sequence d_λ to $\Re^N$, and the weak limit of this extended sequence d. The energy e_λ converges to

$$e_\lambda \to e = \int_{T^N} \check{K}(\xi) d\mu(\xi) + \int_{S^{N-1}} G(n) d\mu^H(n) + \int_{\Re^N} G(\xi) \hat{d}(\xi) \otimes \overline{\hat{d}(\xi)} d\xi,$$

where

$$\check{K}(\xi) = \sum_{\mathcal{L}_1 \setminus \{0\}} e^{-2\pi i \xi \cdot m} K(m),$$

is the Fourier transform of K on the torus, and in the case where $\gamma = -N$, the correction energy G (the lattice-induced anisotropy) is given by asymptotical behavior of the Fourier transform of K near the origin, which can be calculated explicitly and is homogeneous of degree zero.

The model kernel K, the *Helmholtz* kernel,

$$K_{ij}(y) = \frac{c(N)}{|y|^N}\left(N \frac{y_i y_j}{|y|^2} - \delta_{ij}\right),$$

which is given by $K_{ij} = -\nabla_i \nabla_j \Delta^{-1}(\delta_0)$, is used in magnetostatics to determine the stored energy in the field induced by a magnetic dipole. Convolution by K on $\Re^N$ corresponds to projections onto gradient vector fields, (i.e., one part of the Helmholtz decomposition). James and Müller studied the homogenization of the energy given by this kernel in three dimensions. We will consider this kernel among others in any dimension.

We begin our study with a discussion of preliminaries in section 2. We also give an overview of some of the complications of our approach. Additionally, we describe the lattice rescaling and the use of discrete homogeneous Wigner measures in our problem, meanwhile transforming our problem to one on the torus. In section 3, we consider sequences of dipoles which are allowed to oscillate only on a much larger lengthscale than that of the lattice (*weak-long oscillations*), and thus are able to characterize their limiting behavior completely in terms of double convolutions over $\Re^N$. We consider, in section 4, sequences of dipoles which are allowed to oscillate only on the scale of the lattice (*weak-short oscillations*), and show that the limiting energy of these sequences is characterized by an integral on the torus with respect to the Wigner measure limit. Since most sequences of dipoles may oscillate on various lengthscales, in section 5 we use filters and diagonalization arguments, (*essential filters*), in order to separate sequences of dipoles into the parts which oscillate essentially on the scale of the lattice and those parts which oscillate esentially on a much larger lengthscale. We similarly decompose the energy and are thus able to fully describe limiting energies.

In sections 6 and 7, we describe the unusual behavior of limiting energies of weak-short oscillations. In section 6, we use the theory of measure filters developed by Firoozye and Šverák [5] to give an example of a sequence which oscillates on the scale of the lattice and to describe the limiting energy of a localization of this sequence to a given domain. We show that the limiting energy acts as a measure on this given domain. In section 7, we show that this example is not unique and that if the square of an extension of the sequence of dipoles to $\Re^N$ does not concentrate on a Lebesgue null-set and no mass escapes to infinity, the limiting energy induces a finitely additive finite signed measure on $\Re^N$ (i.e., there is no long-range interaction energy). By inducing a measure, we mean that if d_λ is a sequence of dipoles oscillating only on the scale of the lattice, then localizing the sequence by multiplying by a characteristic function, $\chi_\Omega\, d_\lambda$ gives a limiting energy $e(\Omega)$, parameterized by the set Ω. This energy is then a finitely additive measure on Ω. Also, in section 7 we give examples of sequences which do concentrate and whose corresponding energies are not additive.

In section 8, we apply our results to the case considered by James-Müller [10], Khachaturyan [13], De'Bell and Whitehead [4], and Fujiki et al [7], of magnetostatics in $\Re^3$. We analyze lattice-induced anisotropies for several types of lattices and explain the significance of the weak-long part of the energy. We then describe the importance of the locality of the weak-short part of the energy and its implications.

Our analysis of separation of scales may be extended easily to other small-parameter limits in $\Re^N$ where there is a preferred lengthscale, such the characterization of limits of the sort:

$$\int\int \lambda^{-N} K(\frac{x-y}{\lambda}) f_\lambda(x) f_\lambda(y) dx dy,$$

where $||f_\lambda||_{L^2} \leq C$, and K is not of any particular homogeneity, but $\hat{K} \in C_0(\Re^N \setminus \{0\})$, (or $\hat{K} \in C_0(\Re^N)$), and the asymptotics of $\hat{K}$ at the origin are known. Most of our results for convolutions on lattices are immediately applicable to problems of this type, and in fact the analogous statements are much simpler in the context of convolutions over $\Re^N$.

Our characterization of the limiting energy is complete because of our ability to separate scales of oscillation. Our techniques of rescaling, using discrete Wigner transforms, essential filters, and blowup techniques, are very general and can be applied to a variety of problems of convolutions on lattices. We easily see that our analysis also applies to a large variety of short-range interactions as well. The considerations of "exchange energies" is equally simple within this framework.

2. Preliminaries and Rescalings. As a starting point, we will introduce the discrete Wigner transform and show its use in the context of convolutions on lattices. Let $\mathcal{L}$ be a Bravais Lattice in $\Re^N$ ($\mathcal{L} = \{\sum_1^N r_i e^i : r_i \in \mathcal{Z}\}$, where $\{e^i\}_{i=1}^N$ is an orthogonal basis for $\Re^N$ normalized so that the unit cell, $U = \{\sum s_i e^i : s^i \in [0,1]\}$ has $vol(U) = 1$).

Let d_λ be a sequence of functions in $l^2(\mathcal{L}_\lambda;\Re^M)$, where $\mathcal{L}_\lambda = \lambda \cdot \mathcal{L}$ is the rescaled lattice with parameter λ. The energy of a sequence of dipoles d_λ on the Bravais lattice $\mathcal{L}_\lambda$ for $\lambda > 0$ is given by the classical formulation:

$$e_\lambda = \sum_{\mathcal{L}_\lambda, x\neq y} K(\frac{x+y}{2}, x-y) d_\lambda(x) d_\lambda(y),$$

where $K(x,z)$ is a matrix function which is smooth in the inhomogeneity x and has the cancellation property in z and which is homogeneous of degree $-\gamma$ for $\gamma \geq N$ in z. Introducing new variables,

$$\begin{aligned} e_\lambda &= \sum_{r\in\mathcal{L}_{\lambda/2}, z\in\mathcal{L}\setminus\{0\}} K(r,\lambda z) d_\lambda(r+\lambda z/2) d_\lambda(r-\lambda z/2) \\ &= \sum_{r\in\mathcal{L}_{\lambda/2}, z\in\mathcal{L}\setminus\{0\}} \lambda^{-\gamma} K(r,z) d_\lambda(r+\lambda z/2) d_\lambda(r-\lambda z/2) \\ &= \sum_{r\in\mathcal{L}_{\lambda/2}} \int_{T^N} \breve{K}(r,\xi) \lambda^{-\gamma} \breve{U}_\lambda(r,\xi) d\xi, \end{aligned}$$

where $\breve{K}(r,\xi) = \sum_{z\in\mathcal{L}\setminus\{0\}} K(r,z) e^{-2\pi i \xi\cdot z}$ and

$$\breve{U}_\lambda(r,\xi) = \sum_{z\in\mathcal{L}} d_\lambda(r+\lambda z/2) d_\lambda(r-\lambda z/2) e^{-2\pi i z\cdot\xi},$$

for $(r,\xi) \in \mathcal{L}_{\lambda/2} \times T^N$ is the *discrete Wigner transform* of the sequence d_λ and has the property that $\breve{U}_\lambda \in l^1(\mathcal{L}_{\lambda/2}) \times L^1(T^N)$. It also satisfies the property

$$\sum_{r\in\mathcal{L}_{\lambda/2}} \breve{U}_\lambda(r,\xi) = |\breve{d}_\lambda|^2(\xi),$$

if $\tilde{d}_\lambda \in l^1(\mathcal{L}_\lambda)$, where $\breve{d}_\lambda(\xi) = \sum_{\mathcal{L}} d_\lambda(\lambda n) e^{-in\cdot\xi}$, (see Folland [6] for other properties of the continuous Wigner transform). Markowich recently used the Wigner transform for converting Schrödinger equations into Vlasov-Liouville equations [16]. Gérard [9] and Lions-Paul [15] have expanded upon this use, introducing the corresponding continuous Wigner measure solutions to Vlasov-Liouville equations as a tool for obtaining semiclassical limits for Schrödinger equations.

We will be concerned instead with spatially homogeneous kernels. Let K be a kernel homogeneous of degree $-\gamma$ with $\gamma \geq N$, $K(z) = \frac{1}{|z|^\gamma} K'(\frac{z}{|z|})$, with $K' \in C^\infty(S^{N-1};\Re^{M\times M})$ having the cancellation property:

$$\int_{S^{N-1}} K'(s) ds = 0.$$

We are interested in the energy,

$$e_\lambda = \sum_{\substack{n,m\in\mathcal{L}_\lambda \\ n\neq m}} K(n-m) d_\lambda(n) d_\lambda(m), \tag{1}$$

and its small parameter limit, where we will assume a priori that a suitable extension of d_λ to $\Re^N$:

$$\tilde{d}_\lambda(x) \stackrel{def}{=} \sum_{m\in\mathcal{L}_\lambda} \chi_{m+\lambda U}(x) d_\lambda(m)\lambda^{-(N+\gamma)/2}, \tag{2}$$

is uniformly bounded (weakly compact), i.e., $||\tilde{d}_\lambda||_{L^2(\Re^N)} \le C$, where U is the unit cell of the lattice $\mathcal{L}_1 = \mathcal{L}$, normalized so that $|U| = 1$, and $\chi_{m+\lambda U}(x)$ is the characteristic function of a translated and scaled cell. The choice of scaling in $\tilde{d}_\lambda$ is exactly that which insures that e_λ will remain bounded in the limit. As in all homogenization problems, we are interested primarily in the relationship between the limiting energy, $e_\lambda \to e$, and the limiting magnetization, $\tilde{d}_\lambda \rightharpoonup d$, where the convergence is weak-$L^2(\Re^N)$.

Let us begin by rescaling:

$$\begin{aligned} e_\lambda &= \sum_{\substack{n,m\in\mathcal{L}_\lambda\\ n\neq m}} K(n-m) d_\lambda(n) d_\lambda(m) \\ &= \sum_{n,m\in\mathcal{L}_1} K\Big(\lambda(n-m)\Big) d_\lambda(\lambda n) d_\lambda(\lambda m) \\ &= \sum_{n,m\in\mathcal{L}_1} K(n-m)\Big(\lambda^{-\gamma/2} d_\lambda(\lambda n)\Big)\ \Big(\lambda^{-\gamma/2} d_\lambda(\lambda m)\Big) \\ &= \sum_{n,m\in\mathcal{L}} K(n-m) h_\lambda(n) h_\lambda(m), \end{aligned} \tag{3}$$

where $h_\lambda(n) = \lambda^{-\gamma/2} d_\lambda(\lambda n)$ is defined on $\mathcal{L}_1$ and we note that

$$||h_\lambda||_{l^2(\mathcal{L})} = ||\tilde{d}_\lambda||_{L^2(\Re^N)}.$$

Using Plancherel's theorem, (3) can be written as

$$e_\lambda = \int_Q \breve{K}(\xi)\, \breve{h}_\lambda(\xi) \otimes \overline{\breve{h}_\lambda(\xi)}\, d\xi, \tag{4}$$

where

$$\breve{K}(\xi) = \sum_{n\neq 0, n\in\mathcal{L}} K(n) e^{-2\pi i n\cdot\xi},$$

and $\breve{h}_\lambda(\xi) = \sum_{n\in\mathcal{L}_1} h_\lambda(n) e^{-2\pi i n\cdot\xi}$ are functions periodic on the unit cell Q of the reciprocal lattice. We will not make any distinction between the unit cell Q and the N-torus T^N below, and will use Q to represent both. We have chosen the notation $\breve{f}$ to denote the Fourier transform of functions defined on the lattice $\mathcal{L}$, and we will reserve the notation $\hat{f}$ for the Fourier transform of functions defined on $\Re^N$. The relationship between $\breve{h}_\lambda(\xi)$ and the Fourier transform over $\Re^N$ of $\tilde{d}_\lambda(x)$ will be useful to us at a later point:

$$\breve{h}_\lambda(\xi)\hat{\chi}_U(\xi) = S_\lambda^{-1}[\widehat{\tilde{d}_\lambda}](\xi), \tag{5}$$

where χ_U is the characteristic function of the unit cell U of the lattice $\mathcal{L}$, and

$$S_\lambda[f](x) = \lambda^{N/2} f(\lambda x), \tag{6}$$

is an isometry on $L^2(\Re^N)$. We note that $\widehat{S_\lambda[f]} = S_\lambda^{-1}[\hat{f}]$.

To show existence of the limit, we note that $|| \, |\check{h}_\lambda|^2||_{L^1(Q)} = ||\tilde{d}_\lambda||^2_{L^2(\Re^N)} \le C^2$ and that, due to the homogeneity and cancellation property, $\check{K} \in L^\infty(Q)$. In fact we can say more about $\check{K}$ following Wainger, ([22], see also Stein-Weiss [18], ch. 7, §6.1):

Theorem 1 (Wainger,1965). *Let $K(x) = |x|^{-\gamma} K'(x/|x|)$ for $\gamma \ge N$, with $K' \in C^\infty(S^{N-1})$ and*

$$\int_{S^{N-1}} K'(s) ds = 0,$$

and, without loss of generality, assume that

$$K'(s) = \sum_{l=1}^{\infty} \sum_m a_{l,m} Y_{l,m}(s),$$

where $Y_{l,m}$ is a spherical harmonic of degree l and $N \le \gamma < N + \Lambda$, where Λ is the smallest integer for which $l < \Lambda$ implies that $a_{l,m} = 0$ for all m. Let

$$\psi(t) = \begin{cases} 0 & \text{for } t \le \frac{1}{2} \\ 1 & \text{for } t \ge 1 \end{cases},$$

with $\psi \in C^\infty(-\infty, \infty)$ and $0 \le \psi(t) \le 1$ for all t. Let

$$F_\epsilon(\xi) = \Big(\psi(|x|) e^{-\epsilon|x|} |x|^{-\gamma} K'\Big(\frac{x}{|x|}\Big) \Big)^\vee.$$

Then $\lim_{\epsilon \to 0} F_\epsilon(\xi)$ exists for all $\xi \ne 0$. Letting $F(\xi) = \lim_{\epsilon\to 0} F_\epsilon(\xi)$, we find that F is infinitely differentiable for all $\xi \ne 0$. Also, for any integer r, $|F_\epsilon(\xi)| = O(|\xi|^{-r})$ uniformly in $\epsilon \ge 0$ as $\xi \to \infty$. Moreover, we have that F is bounded at the origin and at $\xi = 0$,

$$F(\xi) = |\xi|^{\gamma - N} \tilde{K}_{\gamma,N}\Big(\frac{\xi}{|\xi|}\Big) + E(\xi),$$

where

$$\tilde{K}_{\gamma,k}(s) = \pi^{\gamma-N} \sum_{l=\Lambda}^{\infty} \sum_m (-i)^l a_{l,m} \frac{\Gamma(\frac{1}{2}(N+l-\gamma))}{\Gamma(\frac{1}{2}(l+\gamma))} Y_{l,m}(s),$$

is smooth on S^{N-1}, with $E(\xi) = O(|\xi|^\Lambda) + o(1)$ as $|\xi| \to 0$. Also, F_ϵ is dominated by an L^1 function and

$$\hat{F}(x) = \psi(|x|)|x|^{-\gamma} K'\Big(\frac{x}{|x|}\Big).$$

An easy application of Poisson's summation formula to Theorem 1 gives the following theorem, which we will make extensive use of below.

THEOREM 2 (WAINGER,1965). *Let $K(x)$ be as in Theorem 1. Let*

$$f_\epsilon(\xi) = \sum_{m\neq 0} e^{-2\pi i m\cdot\xi} e^{-\epsilon|m|}|m|^{-\gamma} K'\Big(\frac{m}{|m|}\Big).$$

Then $\lim_{\epsilon\to 0} f_\epsilon(\xi)$ exists for all ξ which are not lattice points. Letting $\breve{K}(\xi) = \lim_{\epsilon\to 0} f_\epsilon(\xi)$, we find that $\breve{K} \in C_{per}(Q\setminus\{0\}) \cap L^\infty(Q)$ and $\breve{K}$ is bounded at the origin and

$$\breve{K}(\xi) = F(\xi) + L(\xi),$$

where F is as in Theorem 1 and $L \in C^\infty(Q)$, with

$$L(\xi) = \sum_{n\neq 0} F(\xi+n).$$

Moreover, $\breve{K}$ has the following asymptotics at the origin:

$$\breve{K}(\xi) = |\xi|^{\gamma-N}\tilde{K}_{\gamma,N}\Big(\frac{\xi}{|\xi|}\Big) + S + L'(\xi),$$

where $\tilde{K}_{\gamma,N}$ is as defined in Theorem 1 and

$$S = \lim_{\epsilon\to 0}\sum_{\mathcal{L}_1\setminus\{0\}} e^{-\epsilon|m|}|m|^{-\gamma}K'\Big(\frac{m}{|m|}\Big),$$

is a lattice dependent constant, and $|L'(\xi)| = |L(\xi) - S| = o(1)$ as $|\xi| \to 0$. Furthermore $\widehat{\breve{K}}(n) = K(n)$.

From this point on, we will refer to the asymptotics at the origin for $\breve{K}(\xi)$ as $G(\xi)$, i.e.,

$$G(\xi) \stackrel{def}{=} |\xi|^{\gamma-N}\tilde{K}_{\gamma,N}\Big(\frac{\xi}{|\xi|}\Big) + S, \tag{7}$$

where $\tilde{K}_{\gamma,N}$ and S are as defined in Theorem 1 and 2, respectively.

A simple corollary to Theorem 2 is that $\breve{K} \in U(Q)$, the subset of the bidual of $C_{per}(Q)$, $C^{**}(Q)$, of universally integrable functions, since it is easy to see that it can be represented both as $\breve{K} = \inf\sup G_{\epsilon,\delta}$ and as $\breve{K} = \sup\inf L_{\epsilon,\delta}$, where $G_{\epsilon,\delta}, L_{\epsilon,\delta} \in C_{per}(Q)$, (see Kaplan [12]). Thus $\breve{K}$ can be integrated against any finite Radon measure $\mu \in \mathcal{M}(Q) = C^*_{per}(Q)$. If $\gamma > N$ then $\breve{K} \in C_{per}(Q)$.

In order to represent the limiting energy, we first rewrite (4) as

$$e_\lambda = \int \breve{K}(\xi)d\mu_\lambda(\xi),$$

where the measure $d\mu_\lambda(\xi) = \breve{h}_\lambda(\xi)\otimes\overline{\breve{h}_\lambda(\xi)}d\xi$ is the *discrete homogeneous Wigner measure* induced on the torus. The mass of the Wigner measure μ_λ is uniformly bounded since

$$||\mu_\lambda||_{\mathcal{M}(Q)} = ||\breve{h}_\lambda||^2_{L^2(Q)} = ||\tilde{d}_\lambda||^2_{L^2(\Re^N)} \le C^2.$$

Thus, we can use the compactness of the unit ball in the vague topology on measures to extract a subsequence, which we call μ_λ such that $\mu_\lambda \stackrel{*}{\rightharpoonup} \mu$. We call μ the *discrete Wigner measure limit* of h_λ. Note that μ is an $M \times M$ Hermitian matrix of measures, i.e., $\mu_{ij} = \mu_{ji}*$ which is positive definite in the sense that $\int_Q \sum_{i,j}^M \mu_{ij}\phi_i\phi_j^* \geq 0$ for $\phi_i \in C_{per}(Q)$, $i = 1, \ldots, M$.

In the case where $\gamma > N$, the Wigner measure limit tells us everything we need to know about the limiting energy; since $\breve{K} \in C_{per}(Q)$ in this case, we will be able to pass to the vague limit directly. Some properties of the limiting energy will be described in sections 3 and 4. If we also know that no mass is "lost at infinity," condition (32), and that the sequence $|\tilde{d}_\lambda|^2$ is strongly precompact in L^1, we will also show in section 7 that the limiting energy generates a finitely additive finite signed measure on $\Re^N$ in the sense that the sequence $\tilde{d}_\lambda\ \chi_\Omega$ generates the limiting energy $e(\Omega)$ which is a measure on the Borel set Ω.

The situation is much more complicated in the case when $\gamma = N$. It is clear that we cannot just represent the limiting energy e by the limiting Wigner measure μ alone, since $\breve{K} \notin C_{per}(Q)$, so we cannot pass to the vague limit under the integral. But since $\breve{K}$ is well behaved everywhere except at the origin, all is not lost, and a more refined analysis of the sequence $\tilde{d}_\lambda$ is necessary. We will see in section 4 that the Wigner measure limit is sufficient to describe the limiting energy only in the case when the oscillations of d_λ are essentially on the scale of the lattice. If the sequence $\tilde{d}_\lambda$ oscillates only on a much larger scale or converges strongly, the Wigner measure limit will be a point mass at the origin—exactly where $\breve{K}$ fails to be continuous. Therefore, in this case the Wigner measure limit will tell us nothing of the limiting energy. Conversely, we will find that the part of the Wigner measure limit singular with respect to a point mass at the origin depends only on those oscillations essentially on the scale of the lattice. The complications involved with characterizing the limiting energy in the weak-long case and the blowup techniques used to determine this energy will be described in section 3.

We remark that in the case where $\gamma < N$ (when $\breve{K} \notin L^\infty(Q)$), boundedness of the energy requires much more complicated bounds on $\tilde{d}_\lambda$ and on the growth (or decay) of $\hat{\tilde{d}}_\lambda(\xi)$ as $|\xi| \to 0$. We will not be discussing these cases of homogenization on lattices in the paper because the nature of these bounds is entirely different from those bounds we have stated above.

3. Long Waves and Strong Convergence. Following James and Müller ([10]), we say that $\tilde{d} \rightharpoonup d$ converges *weak-long* if the L^2-modulus of continuity, $\Omega_2[\tilde{d}_\lambda](\lambda)$, converges to zero, i.e.,

$$\Omega_2[\tilde{d}_\lambda](\lambda) = \sup_{\zeta \in B_{\rho\lambda}} ||\tilde{d}_\lambda(\cdot + \zeta) - \tilde{d}_\lambda(\cdot)||_{L^2(\Re^N)} \to 0 \tag{8}$$

as $\lambda \to 0$, where ρ is large enough so that $B_\rho \cap \mathcal{L}_1$ contains N independent vectors (i.e, $U \subset B_\rho$). If $\tilde{d}_\lambda \to d$ strongly in $L^2(\Re^N)$, then $d_\lambda \rightharpoonup d$ weak-long as well.

We note the following properties of the modulus of continuity (see, for example [18], §1.1):

$$\Omega_2[f](c) \leq 2||f||_{L^2(\Re^N)}, \tag{9a}$$

for any $c > 0$,

(9b) $$||\rho_\lambda * f - f||_{L^2} \leq \Omega_2[f](\lambda),$$

by Minkowski's inequality, where $\rho_\lambda(x) = (1/\lambda^N)\rho(x/\lambda)$ is an approximation of the identity with ρ supported in the unit ball (i.e., $\rho \in C_0(B_1)$, $\int \rho(x)dx = 1$),

(9c) $$\Omega_2[f](c \cdot \lambda) \leq c \cdot \Omega_2[f](\lambda),$$

where c is a positive integer, and

(9d) $$\Omega_2[f](a) \leq \Omega_2[f](b),$$

when $a \leq b$. Note too that $\Omega_2[S_\lambda[f]](1) = \Omega_2[f](\lambda)$, where S_λ is the isometry on $L^2(\Re^N)$ defined in (6). We also define the modulus of continuity for functions defined on the lattice $\mathcal{L}$, i.e.,

$$\Omega_2[h](1) = \sup_{\zeta \in \mathcal{L} \cap B_\rho} ||h(\cdot + \zeta) - h(\cdot)||_{l^2(\mathcal{L})}.$$

Note that properties (9a–9d) are also true of the modulus of continuity on the lattice, so we can prove the following lemma:

LEMMA 1. *The moduli of continuity* $\Omega_2[h_\lambda](1)$ *and* $\Omega_2[\tilde{d}_\lambda](\lambda)$ *are equivalent, i.e.,*

$$\Omega_2[h_\lambda](1) \leq \Omega_2[\tilde{d}_\lambda](\lambda) \leq c\,\Omega_2[h_\lambda](\lambda),$$

where $c \geq 1$.

Proof. Recall that $S_\lambda[f](x) = \lambda^{N/2} f(x/\lambda)$. Since $\Omega_2[\tilde{d}_\lambda](\lambda) = \Omega_2[S_\lambda[\tilde{d}_\lambda]](1)$, and since $S_\lambda[\tilde{d}_\lambda](x) = \sum_{\mathcal{L}} h_\lambda(m)\chi_{m+U}(x)$, we find that

$$\begin{aligned}
\Big(\Omega_2[h_\lambda](1)\Big)^2 &= \sup_{\zeta \in B_\rho \cap \mathcal{L}} \sum |h_\lambda(m+\zeta) - h_\lambda(m)|^2 \\
&\leq \sup_{\zeta \in B_\rho} \int |\sum h_\lambda(m)\chi_{m+U}(x+\zeta) - \sum h_\lambda(m)\chi_{m+U}(x)|^2 dx \\
&= \Big(\Omega_2[S_\lambda[\tilde{d}_\lambda]](1)\Big)^2 \\
&= \Big(\Omega_2[\tilde{d}_\lambda](\lambda)\Big)^2.
\end{aligned}$$

To prove the opposite inequality, we first fix x and ζ. Then

$$|\sum h_\lambda(m)\chi_{m+U}(x+\zeta) - \sum h_\lambda(m)\chi_{m+U}(x)|^2 = \begin{cases} 0 & \text{if } x \text{ and } x+\zeta \text{ are in the same cell,} \\ |h_\lambda(m') - h_\lambda(m)|^2 & \text{otherwise,} \end{cases}$$

with m' a neighboring lattice point to m. Thus,

$$|\sum h_\lambda(m)\chi_{m+U}(x+\zeta) - \sum h_\lambda(m)\chi_{m+U}(x)|^2 \leq \sum_{\substack{m'=m+\zeta \\ \zeta \in B_\rho \cup \mathcal{L}}} |h_\lambda(m') - h_\lambda(m)|^2,$$

giving us,

$$\Big(\Omega_2[S_\lambda[\tilde{d}_\lambda](1)\Big)^2 \le \sum_m \sum_{\zeta\in B_\rho\cap\mathcal{L}_1} |h_\lambda(m+\zeta)-h_\lambda(m)|^2\,|U|$$
$$\le \sup_{\zeta\in B_\rho\cap\mathcal{L}_1} \sum_m |h_\lambda(m+\zeta)-h_\lambda(m)|^2\,|U|\cdot\#\{B_\rho\cup\mathcal{L}_1\},$$

where # is the cardinality function. □

We remark that, although h_λ is uniformly bounded in $l^2(\mathcal{L}_1)$, and thus has a weakly convergent subsequence, ($h_\lambda \rightharpoonup h$ weak-$l^2(\mathcal{L}_1)$), in the case of weak-long oscillations this weak limit contains no information; rescaling gives us that

$$\Omega_2[h_\lambda](1)\to 0,$$

and by lower semi-continuity of the norm, we see that $\Omega_2[h](1)=0$, thus h must be a constant and since it is also in $l^2(\mathcal{L})$, the constant must be zero. The measures μ_λ, being "concentration" measures, may contain more information. Heuristically, that h_λ converges to a constant implies that $h_\lambda * h_\lambda$ converges to a constant and thus that μ_λ converges to a point-mass at the origin.

Let us make this rigorous:

Lemma 2. *If $\tilde{d}_\lambda \rightharpoonup d$ weak-long or strong, then the Wigner measure limit μ has the property that $0\notin supp\,|\mu|$.*

Proof. By properties (9b) and (9c), $\Omega_2[h_\lambda](1)\to 0$ as $\lambda\to 0$ implies that

$$\sum |h_\lambda-\rho*h_\lambda|^2 = \int_Q |\breve{h}_\lambda(\xi)|^2(1-\breve{\rho}(\xi))^2 d\xi$$
$$= \int_Q (1-\breve{\rho}(\xi))^2 tr\{d\mu_\lambda\}(\xi)$$
$$\longrightarrow 0$$

for any $\rho\in l^1(\mathcal{L}_1)$ with finite support and $\sum_{\mathcal{L}}\rho=1$ (i.e., for any trigonometric poynomial $\breve{\rho}$ with $\breve{\rho}(0)=1$), where $tr\{d\mu_\lambda\}(\xi)$ is the trace of the matrix of measures $\mu_\lambda(\xi)$. Thus, by passing to the limit,

$$\int_Q (1-\breve{\rho}(\xi))^2 d\mu(\xi)=0,$$

for all such trigonometric polynomials. But, since the trigonometric polynomials are dense in the uniform topology in $C_{per}(Q)$,

$$\int_Q (1-\psi_\epsilon(\xi)) tr\{d\mu\}(\xi)=0,$$

for any $\psi_\epsilon\in C_0(B_{2\epsilon})$ with $0\le\psi_\epsilon\le 1$, $B_{2\epsilon}\subset Q$, and $\psi\equiv 1$ in B_ϵ, i.e., $\operatorname{supp} tr\{\mu\}\cap(Q\setminus B_\epsilon)=\emptyset$ for all $\epsilon>0$, which by the positive definiteness of μ implies that $supp|\mu_{ij}|\cap(Q\setminus B_\epsilon)=\emptyset$ for all $i,j=1,\dots,M$, where $|\mu_{ij}|$ is the total variation of μ_{ij}. □

Remark 1. *Using similar methods to the above, we can also show that*

$$\lim_{\lambda\to 0}\int_{\Re^N}(1-\psi_{\epsilon/\lambda})(\xi)\widehat{\tilde{d}_\lambda}(\xi)\otimes\overline{\widehat{\tilde{d}_\lambda}(\xi)}d\xi=0,$$

for any $\epsilon > 0$ where $\psi_{\epsilon/\lambda}(\xi) = \psi((\lambda/\epsilon)\xi)$ with $\psi \in C_0(B_2)$, $0 \leq \psi \leq 1$ and $\psi \equiv 1$ inside B_1.

We can use the last lemma to prove the following:

LEMMA 3. *If $\tilde{d}_\lambda \rightharpoonup d$ weak-long or strong, then the Wigner measure limit μ is a point mass at the origin, i.e.,*

$$\mu_\lambda \overset{*}{\rightharpoonup} \mu = \delta_0 \cdot \lim_{\lambda \to 0} \int_{\Re^N} \tilde{d}_\lambda(x) \otimes \tilde{d}_\lambda(x) dx.$$

Conversely, if $\mu_\lambda \overset{*}{\rightharpoonup} c \cdot \delta_0$, then condition (8) is satisfied.

Proof. Let f be a continuous periodic function on Q. Then

$$\int_Q f(\xi) d\mu_\lambda(\xi) = \int f(\xi)\, \check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)}\, d\xi. \tag{10}$$

Let $\psi_\epsilon \in C_0(B_{2\epsilon})$ with $0 \leq \psi_\epsilon \leq 1$ and $\psi_\epsilon(\xi) \equiv 1$ for $\xi \in B_\epsilon$, where $\epsilon > 0$ is fixed. Since f is continuous, $\lim_{\epsilon \to 0} \sup_{\xi \in B_\epsilon} |f(\xi) - f(0)| = 0$, and then

$$\begin{aligned}
\Big| \int_Q \Big(f(\xi) - f(0) \Big) \{d\mu_\lambda\}(\xi) \Big| &\leq \int |f(\xi) - f(0)| \psi_\epsilon(\xi) tr\{d\mu_\lambda\}(\xi) \\
&\qquad + \int |f(\xi) - f(0)| (1 - \psi_\epsilon)(\xi) tr\{d\mu_\lambda\}(\xi) \\
&\leq \sup_{B_\epsilon} |f(\xi) - f(0)| \int |\check{h}_\lambda|^2(\xi) d\xi \\
&\qquad + 2||f||_{C(Q)} \int (1 - \psi_\epsilon)(\xi) |\check{h}_\lambda|^2 d\xi \\
&\leq \sup_{B_\epsilon} |f(\xi) - f(0)|\; C^2 \\
&\qquad + 2||f||_{C(Q)} \int (1 - \psi_\epsilon)(\xi) |\check{h}_\lambda|^2 d\xi,
\end{aligned} \tag{11}$$

since $||\check{h}_\lambda||_{L^2} \leq C$. Applying Lemma 2, we see that the second term on the right of (11) tends to zero for arbitrary ϵ. Letting $\epsilon \to 0$ we see that the first term on the right-hand side of (11) also tends to zero. Thus,

$$\begin{aligned}
\int f(\xi) d\mu_\lambda(\xi) &\to f(0) \lim_\lambda \int_Q d\mu_\lambda(\xi) \\
&= f(0) \cdot \lim_\lambda \int_Q \check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)} d\xi \\
&= f(0) \cdot \lim_\lambda \int_{\Re^N} \tilde{d}_\lambda(x) \otimes \tilde{d}_\lambda(x) dx,
\end{aligned}$$

where we have used the fact that $1 \in C_{per}(Q)$ to show that the limit in fact exists and Plancherel's theorem and the definition of h_λ to show the connection between $\check{h}_\lambda$ and $\tilde{d}_\lambda$.

To prove the converse, first let us transform the weak-long condition on the lattice to a condition on the torus, i.e., we wish to show that if $\mu_\lambda \overset{*}{\rightharpoonup} c \cdot \delta_0$, weak-$\star$ in the sense of measures, then

$$\lim_{\lambda \to 0} \Omega_2[h_\lambda](1) = \lim_{\lambda \to 0} \max_{\zeta \in B_\rho \cap \mathcal{L}_1} \int_Q (e^{-i\zeta \cdot \xi} - 1)^2 d\mu_\lambda(\xi) = 0.$$

First note that the maximum is over finitely many points $\zeta \in B_\rho \cap \mathcal{L}_1$, and we can change the order of the maximum and the limit. Since $\mu_\lambda \overset{*}{\rightharpoonup} c \cdot \delta_0$ weak-$\star$ in the sense of measures, $g_\zeta(\xi) = (e^{-i\zeta\cdot\xi} - 1)^2$ is continuous and $g_\zeta(0) = 0$, $\int_Q (e^{-i\zeta\cdot\xi} - 1)^2 d\mu_\lambda \to 0$. Thus, trivially, $\Omega_2[h_\lambda](1) \to 0$. □

So, when $\gamma = N$ in the cases of weak-long or strong convergence, the Wigner measure limit is concentrated exactly where $\breve{K}$ is discontinuous, and thus offers no useful information for characterizing the limiting energy. Fortunately, by a blowup argument, we are able to say more:

Theorem 3. *Suppose that $\tilde{d}_\lambda \rightharpoonup d$ weak-long or strong $L^2(\Re^N)$. Then, if $\gamma > N$,*

$$e_\lambda = \int_Q \breve{K}(\xi) d\mu_\lambda(\xi) \to \lim_\lambda \int \langle S\tilde{d}_\lambda(x), \tilde{d}_\lambda(x)\rangle dx.$$

If $\gamma = N$ and $G(\xi) = \breve{K}_{\gamma,N}(\xi/|\xi|) + S$ (as defined in equation 7),

$$e_\lambda = \int_Q \breve{K}(\xi) d\mu_\lambda(\xi) \to \lim_{\lambda\to 0} \int_{\Re^N} G(\xi)\ \widehat{\tilde{d}_\lambda}(\xi) \otimes \overline{\widehat{\tilde{d}_\lambda}(\xi)}\ d\xi, \tag{12}$$

where the right-hand side of (12) can be represented by an H-measure μ_H and the weak limit, i.e.,

$$e = \int_{S^{N-1}} G(s) d\mu_H(s) + \int_{\Re^N} G(\xi)\hat{d}(\xi) \otimes \overline{\hat{d}(\xi)} d\xi.$$

Proof. Before proceeding, note that $||G||_{C^0(\Re^N)} \le ||\breve{K}||_{C^0(Q\setminus\{0\})}$. First, let us show that

$$\Big|\int_Q \breve{K}(\xi)\ \check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)} d\xi - \int_Q G(\xi)\check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)} d\xi\Big| \tag{13}$$

converges to zero. We can bound (13) above by

$$\begin{aligned} &\le \int_Q |\breve{K}(\xi) - G(\xi)||\check{h}_\lambda(\xi)|^2 d\xi \\ &= \int_Q \rho_\epsilon(\xi)|\breve{K}(\xi) - G(\xi)|\ |\check{h}_\lambda(\xi)|^2 d\xi + \int_Q (1-\rho_\epsilon)(\xi)|\breve{K}(\xi) - G(\xi)|\ |\check{h}_\lambda(\xi)|^2 d\xi \\ &\le \sup_{\xi\in B_\epsilon} |\breve{K}(\xi) - G(\xi)|C^2 + 2||\breve{K}||_{L^\infty} \int (1-\rho_\epsilon)(\xi)|\check{h}_\lambda|^2 d\xi, \end{aligned} \tag{14}$$

where $\rho_\epsilon \in C_0(B_{2\epsilon})$, $0 \le \rho_\epsilon \le 1$ and $\rho_\epsilon \equiv 1$ inside B_ϵ. By Lemma 3, for any $\epsilon > 0$, the second term on the right of (14) tends to zero as $\lambda \to 0$. Since ϵ was arbitrary, we let it tend toward zero, making the first term on the right of (14) go to zero too by Theorem 2.

In the case where $\gamma > N$, we find that $G \equiv S + O(|\xi|^{\gamma-N})$, thus

$$\begin{aligned} \Big|\int_Q (G(\xi) - S)\,\check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)} d\xi\Big| &\le \int_Q \rho_\epsilon(\xi)|G(\xi)|\ |\check{h}_\lambda(\xi)|^2 d\xi \\ &\quad + \int_Q (1-\rho_\epsilon)(\xi)|G(\xi)|\ |\check{h}_\lambda(\xi)|^2 d\xi \\ &\le \epsilon^{\gamma-N}C^2 + ||G||_{L^\infty} \int (1-\rho_\epsilon)(\xi)|\check{h}_\lambda|^2 d\xi, \end{aligned} \tag{15}$$

where ρ_ϵ is as before. Again we note that the second term on the right of (15) tends to zero as $\lambda \to 0$ for any $\epsilon > 0$ by Lemma 3. Letting $\epsilon \to 0$, we find that the first term on the right of (15) also tends to zero and thus, $e_\lambda \to \lim \int \langle \check{h}_\lambda, \check{h}_\lambda \rangle d\xi$, which by the definition of h_λ and the Plancherel's theorem, gives the stated result.

In the case where $\gamma = N$, the result is slightly more difficult. We must use equation (5) and the fact that $\hat{\chi}_U$ is twice differentiable at the origin, (in fact it is analytic by Paley-Wiener), $\hat{\chi}_U(\eta) = 1 + o(|\eta|)$ as $\eta \to 0$ since $|U| = 1$ and U is symmetric about the origin. Then, by using the homogeneity of G and changing variables,

$$\int_Q G(\xi)[\hat{\chi}_U(\lambda\eta)]^2 \check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)} d\xi = \int \chi_{Q\cdot(1/\lambda)}(\eta) G(\eta) \widehat{\tilde{d}}_\lambda(\eta) \otimes \overline{\widehat{\tilde{d}}_\lambda(\eta)}\, d\eta.$$

Incorporating this,

$$\begin{aligned}
&\left| \int_{\Re^N} \Big(\chi_{Q\cdot(1/\lambda)}(\eta) - 1 \Big)\, G(\eta)\, \widehat{\tilde{d}}_\lambda(\eta) \otimes \overline{\widehat{\tilde{d}}_\lambda(\eta)} d\eta \right| \\
&\qquad \le \int \rho_{\epsilon/\lambda}(\eta) \Big| \chi_{Q\cdot(1/\lambda)}(\eta) - 1 \Big|\, |\widehat{\tilde{d}}_\lambda|^2\, |G(\eta)| d\eta \\
&\qquad\quad + \int (1 - \rho_{\epsilon/\lambda})(\eta) \Big| \chi_{Q\cdot(1/\lambda)}(\eta) - 1 \Big|\, |\widehat{\tilde{d}}_\lambda|^2\, |G(\eta)| d\eta \\
&\qquad \le ||G||_{L^\infty} \Big(C^2 \cdot \sup_{\xi \in B_{2\epsilon}} \Big| \chi_Q(\xi) - 1 \Big| \\
&\qquad\quad + \Omega_2[\tilde{d}_\lambda](\epsilon^{-1}) \Big). \qquad (16)
\end{aligned}$$

By choosing ϵ small enough and then letting $\lambda \to 0$, we see that the right of (16) tends to zero. By the continuity of $\hat{\chi}_U$ at the origin we can see that

$$\int_Q |G(\xi)([\hat{\chi}_U(\lambda\eta)]^2 - 1) \check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(\xi)}| d\xi$$

also tends to zero. Thus we have proved the result.

Note that, because of the homogeneity of G, the limit

$$\lim_{\lambda \to 0} \int_{\Re^N} G(\eta) \widehat{\tilde{d}}_\lambda(\eta) \otimes \overline{\widehat{\tilde{d}}_\lambda(\eta)}\, d\eta \qquad (17)$$

can be represented by an integral with respect to a homogeneous H-measure induced by $(\tilde{d}_\lambda - d) \rightharpoonup 0$ in L^2, (see Tartar [19], and Gérard [8]) and by an integral of the weak-limit, i.e.,

$$e = \int_{S^{N-1}} G(n) d\mu^H(n) + \int_{\Re^N} G(\eta) \hat{d}(\eta) \otimes \overline{\hat{d}(\eta)} d\eta.$$

In the case where $\tilde{d}_\lambda \to d$ strongly in $L^2(\Re^N)$, (17) becomes

$$\int_{\Re^N} G(\eta)\, \hat{d}(\eta) \otimes \overline{\hat{d}(\eta)}\, d\eta.$$

□

Thus, in the case of weak-long or strong convergence, we retrieve the classical continuum energy with some extra lattice-induced anisotropy terms given by S.

4. Weak-Short Oscillations. We say that a sequence $\tilde{d}_\lambda \rightharpoonup d$ weak-$L^2(\Re^N)$ converges *weak-short* if

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0}\sum_{m\in\mathcal{L}}|\Phi_{1/\epsilon} * h_\lambda|^2 = 0, \tag{18}$$

where $\Phi_{1/\epsilon}(x) = (\epsilon)^N\Phi(\epsilon x)$, with $\Phi(x) = \pi^{-\alpha}\Gamma(\alpha+1)|x|^{-N/2-\alpha}J_{N/2+\alpha}(2\pi|x|)$ with $J_{N/2+\alpha}$ a Bessel function of degree $N/2+\alpha$ for $\alpha > (N-1)/2$ fixed. Note that Φ is the kernel used for taking Bochner-Riesz means, (see Stein and Weiss [18], Chapter 4, Theorem 4.15), and that

$$\phi(\xi) = \hat{\Phi}(\xi) = \begin{cases} (1-|\xi|^2)^\alpha & \text{if } |\xi| \le 1 \\ 0 & \text{if } |\xi| \ge 1. \end{cases} \tag{19}$$

We will see below that our choice of Φ is one of convenience; any other $\Phi(x)$ with nice decay at infinity which has $\phi(\xi) = \hat{\Phi}(\xi)$ monotone decreasing in $|\xi|$, and continuous with bounded support and with $\phi(0) = 1$ would suffice in our definition, and the conditions of bounded support and monotonicity can also be relaxed. We will see in Remark 3 that in order for (18) to be satisfied, oscillations in $\tilde{d}_\lambda$ must be essentially only on the scale of the lattice and that this condition is in direct contrast to condition (8). Although our terminology is borrowed from James and Müller [10], our definition of *weak-short* is different from theirs; they define it to be *not weak-long*, which would thus allow oscillations on the scale of the lattice as well as on much longer lengthscales. The utility of our definition will become apparent from the results presented in this section, and those in section 5 where we demonstrate that any sequence can be separated into its weak-short and weak-long parts.

We will see in Remark 2 that condition (18) is equivalent to the following:

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int_{\Re^N}|\Phi_{\lambda/\epsilon} * \tilde{d}_\lambda|^2 dx = 0,$$

where $\Phi_{\lambda/\epsilon}(x) = (\epsilon/\lambda)^n\Phi(\epsilon x/\lambda)$ is as defined above.

We claim that under the above conditions, the Wigner measure limit μ of the sequence $\tilde{d}_\lambda$ is singular with respect to a point mass at the origin. Let us make this rigorous:

Lemma 4. *Let $\tilde{d}_\lambda \rightharpoonup d$ converge weak-short, i.e., let h_λ satisfy (18), then $\mu_\lambda = \check{h}_\lambda \otimes \overline{\check{h}_\lambda} d\xi \overset{*}{\rightharpoonup} \mu$ where $\mu \perp \delta_0$, i.e., $\int \psi(\xi/\epsilon)d\mu(\xi) \to 0$ as $\epsilon \to 0$ for $\psi \in C_0(B_1)$.*

Proof. Because of nice decay at infinity, Φ satisfies the hypotheses for Poisson's summation formula, thus, taking the Fourier transform of condition (18),

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int_Q|\phi_\epsilon(\xi)\check{h}_\lambda(\xi)|^2 = \lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int_Q|\phi_\epsilon(\xi)|^2 tr\{d\mu_\lambda\}(\xi) = 0, \tag{20}$$

since

$$\sum_{\mathcal{L}}\epsilon^N\Phi(\epsilon m)e^{-2\pi m\cdot\xi} = \sum_{\mathcal{L}^*}\phi\Big(\frac{\xi+m}{\epsilon}\Big) \overset{def}{=} \phi_\epsilon(\xi),$$

where $\mathcal{L}^*$ is the reciprocal lattice to $\mathcal{L}$ and where ϕ has support in the unit ball. For ϵ small enough we see that $\phi_\epsilon(\xi) = \phi(\xi/\epsilon)$ for $\xi \in Q$.

Because $\mu_\lambda \stackrel{*}{\rightharpoonup} \mu$ and $\phi_\epsilon \in C_{per}(Q)$, the limit in λ in (20) exists and we have that

$$\lim_\epsilon \int_Q |\phi(\xi/\epsilon)|^2 tr\{d\mu\}(\xi) = 0.$$

Due to monotonicity in ϵ, the limit in ϵ exists. We also see that for any $\rho \in C_0(B_1)$, $\lim_\epsilon \int_Q \rho(\xi/\epsilon) tr\{d\mu\}(\xi) = 0$, and due to positive definiteness, this is also true for μ_{ij} and $|\mu_{ij}|$ for $i, j = 1, M$, where $|\mu_{ij}|$ is the total variation of μ_{ij}. Thus $|\mu_{ij}| \perp \delta_0$. □

REMARK 2. *We can now easily see that condition (18) is equivalent to the following condition:*

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0} \int |\Phi_{\lambda/\epsilon} * \tilde{d}_\lambda|^2 dx = 0, \tag{21}$$

where $\Phi_{\lambda/\epsilon}(x) = (\epsilon/\lambda)^N \Phi(\epsilon x/\lambda)$. *By rescaling (21) and using Plancherel's theorem, we see that*

$$\begin{aligned}
\frac{1}{2}\int |\Phi_{1/\epsilon} * S_\lambda[\tilde{d}_\lambda]|^2 dx &= \frac{1}{2}\int_{|\xi|\le\epsilon} (1-|\xi/\epsilon|^2)^{2\alpha} |S_\lambda^{-1}[\widehat{\tilde{d}_\lambda}]|^2 d\xi \\
&\le \int_{|\xi|\le\epsilon} (1-|\xi/\epsilon|^2)^{2\alpha} |\check{h}_\lambda|^2 d\xi = \sum_{\mathcal{L}} |\Phi_{1/\epsilon} * h_\lambda|^2 \\
&\le 2\int_{|\xi|\le\epsilon} (1-|\xi/\epsilon|^2)^{2\alpha} |S_\lambda^{-1}[\widehat{\tilde{d}_\lambda}]|^2 d\xi = 2\int |\Phi_{1/\epsilon} * S_\lambda[\tilde{d}_\lambda]|^2 dx,
\end{aligned}$$

for ϵ *small, where we have used (5) and the fact that* $\hat{\chi}_U(\xi)$ *is continuously differentiable at the origin, and* $\hat{\chi}_U(\xi) = 1 + o(|\xi|)$ *for* $|\xi| \to 0$. *Passing to the limits we see the equivalence immediately.*

REMARK 3. *We note that the much more easily verifiable condition,*

$$\limsup_{\rho\to\infty} \lim_{\lambda\to 0} \sum_{m\in\mathcal{L}} \Big| \frac{1}{\rho^N} \sum_{k\in B_\rho(m)\cap\mathcal{L}} h_\lambda \Big|^2 = 0, \tag{22}$$

implies (18). Note too that in order for condition (22) to hold, oscillations must be only on the scale of the lattice and that this is in direct contrast to condition (8).

To show that condition (22) implies (18), we take the Fourier transform of (22):

$$\sum_{m\in\mathcal{L}} \Big| \frac{1}{\rho^N} \sum_{k\in B_\rho(m)\cup\mathcal{L}} h_\lambda(k) \Big|^2 = \int |f_\rho(\xi)\check{h}_\lambda(\xi)\,|^2 d\xi,$$

where $f_\rho(\xi) = \sum_{n\in B_\rho\cap\mathcal{L}} (1/\rho^N) e^{-2\pi i n\cdot\xi}$ *is a sequence of trigonometric polynomials which have* $f_\rho(0) = 1$ *and* $f'_\rho(0) = 0$, *with* $||f_\rho||_{L^2(Q)} \to 0$ *as* $\rho \to \infty$. *We claim that there exists a constant* $M > 0$ *such that for all* $\rho > 0$ *there exists an* ϵ_0 *and for any* $\epsilon \le \epsilon_0$, *we have the following inequality:*

$$\Big(1 - \Big|\frac{\xi}{\epsilon}\Big|^2\Big)^\alpha \le M|f_\rho(\xi)|, \tag{23}$$

for all $|\xi| \leq \epsilon$*. If we choose* $\epsilon^{1/2}\rho = 1$ *and rescale, we obtain the inequality*

$$\Big(1 - \Big|\frac{\xi}{\sqrt{\epsilon}}\Big|^2\Big)^\alpha \leq M\Big|\sum \epsilon^{N/2} \sum_{B_\delta \cup \mathcal{L}_{\sqrt{\epsilon}}} e^{-2\pi i m\cdot\xi}\Big|, \tag{24}$$

for all $|\xi| \leq \epsilon^{1/2}$*. The right hand side of (24) converges pointwise to* $|\hat{\chi}_{B_1}|$*. Since* $\hat{\chi}_{B_1}$ *is bounded away from zero in a neighborhood of the origin, and because of the monotonicity of the left-hand side of (23) in* ϵ *we see that inequality (23) must be true as according to the claim. Thus, condition (22), a much easier condition is seen to imply (18).*

We use methods similar to those used to prove Lemma 4 to show that in the case of weak-short convergence the Wigner measure μ gives all the information necessary to find the limiting energy.

Theorem 4. *If* $\tilde{d}_\lambda \rightharpoonup d$ *weak-short, then*

$$e_\lambda \to \int \check{K}(\xi)d\mu(\xi) = \lim_{\epsilon\to 0}\int (1-\psi(\xi/\epsilon))\check{K}(\xi)d\mu(\xi) \tag{25a}$$

$$= \lim_{\epsilon\to 0}\int K_\epsilon(\xi)d\mu(\xi) \tag{25b}$$

$$= \lim_{\epsilon\to 0}\int f_\epsilon(\xi)d\mu(\xi) \tag{25c}$$

where $\psi(\xi) \in C_0(B_1)$ *with* $\psi(0) = 1$, $0 \leq \psi \leq 1$,

$$K_\epsilon(\xi) = \frac{1}{|B_\epsilon|^N}\int_{B_\epsilon(\xi)} \check{K}(y)dy,$$

and where f_ϵ *is the regularization defined in Theorem 2.*

Proof. First, let us start with the easiest regularization of $\check{K}$ in (25a), which effectively carves out the origin, i.e.,

$$\begin{aligned}\Big|\int \check{K}(\xi)d\mu_\lambda(\xi) - \int \check{K}(\xi)(1-\psi_\epsilon(\xi))d\mu(\xi)\Big| &\leq \Big|\int (1-\psi_\epsilon)\check{K}(\xi)(d\mu_\lambda(\xi) - d\mu(\xi))\Big| \\ &\quad + \int |\psi_\epsilon|\;|\check{K}| tr\{d\mu_\lambda\}(\xi) \\ &\leq \Big|\int (1-\psi_\epsilon)\check{K}(d\mu_\lambda(\xi) - d\mu(\xi))\Big| \\ &\quad + ||\check{K}||_{L^\infty}\int |\psi_\epsilon| tr\{d\mu_\lambda\}(\xi).\end{aligned} \tag{26}$$

Note that $\check{K}(1-\psi_\epsilon) \in C_{per}(Q)$ for all $\epsilon > 0$ and, thus by the vague convergence of μ_λ to μ, for λ small, the first term on the right-hand side of (26) is arbitrarily small. Condition (18) implies that the second term on the right-hand side of (26) is arbitrarily small for λ and ϵ small. Thus, the right hand side of (26) tends to zero.

Continuing, we now turn to the justification of (25b). We see that $K_\epsilon(\xi) \to \check{K}(\xi)$ pointwise on Q, and thus $K_\epsilon \overset{*}{\rightharpoonup} \check{K}$ weak-$\star$ in $C^{**}_{per}(Q)$, (see Kaplan [12], §54.2), and also uniformly on $Q \setminus B_\delta$ for $\delta > 0$. Using this, we estimate the following:

$$\begin{aligned}\Big|\int \check{K}(\xi)(1-\phi(\xi/\delta))d\mu(\xi) - \int K_\epsilon d\mu\Big| &\leq \Big|\int_Q (\check{K} - K_\epsilon)d\mu\Big| + \Big|\int \check{K}\phi(\xi/\delta)d\mu\Big| \\ &\leq \Big|\int_Q (\check{K} - K_\epsilon)d\mu\Big| + ||\check{K}||_{L^\infty(d\mu)}\int \phi(\xi/\delta)d\mu.\end{aligned}$$

We first claim that $||\breve{K}||_{L^\infty(d\mu)} < \infty$, since $||\breve{K}||_{C^{**}_{per}} < \infty$ (i.e., it is bounded in the sup norm). We know that $\int(K_\epsilon - \breve{K})d\mu \to 0$ due to weak-$\star$ convergence in $C^{**}_{per}(Q)$. We also know that $\int \phi(\xi/\delta)d\mu$ tends to zero from the fact that ϕ is bounded with compact support and from condition (18). Thus, we have justified (25b).

Finally, the justification of (25c) is quite similar to that of (25b). Theorem 2 states that $f_\epsilon \to \breve{K}$ pointwise on $Q \setminus \{0\}$, and thus that $f_\epsilon(1-\phi_\delta) \overset{*}{\rightharpoonup} \breve{K}(1-\phi_\delta)$ weak-$\star$ in $C^{**}_{per}(Q)$ for all $\delta > 0$. Thus,

$$\begin{aligned} \Big|\int \breve{K}(1-\phi(\xi/\delta))d\mu - \int f_\epsilon d\mu\Big| &\leq \Big|\int(\breve{K} - f_\epsilon)(1-\phi(\xi/\delta))d\mu\Big| + \int |\phi_\delta|\,|f_\epsilon| d\mu \\ &\leq \Big|\int(\breve{K} - f_\epsilon)(1-\phi(\xi/\delta))d\mu\Big| + ||f_\epsilon||_{L^\infty(d\mu)}\int |\phi_\delta| d\mu. \end{aligned} \tag{27}$$

Again $\left|\int(\breve{K} - f_\epsilon)(1-\phi(\xi/\delta))d\mu\right|$ tends to zero as $\epsilon \to 0$ for all $\delta > 0$ by weak-$\star$ convergence. We also claim that $||f_\epsilon||_{L^\infty(d\mu)} < \infty$ independent of ϵ because it is uniformly bounded in the sup norm and thus, by condition (18), that the right-hand side of (27) tends to zero, completing the proof. □

Concluding, we see that the Wigner measure limit μ on the torus fully describes the limiting energy. Moreover, the Wigner measure limit gives us the limiting energy explicitly if we have complete knowledge of the pointwise behavior of $\breve{K}$. In section 7 we will describe some further properties of the energy of weak-short oscillations.

5. Separation of Scales. We are able to show that general sequences can be decomposed or "filtered" into their weak-long and weak-short parts and the energy can thus be decomposed into the constituent parts. This is important since most sequences $\tilde{d}_\lambda$ neither oscillate on the scale of the lattice alone nor on a much longer lengthscale alone and thus, neither the analysis for weak-short nor for weak-long oscillations applies in the form given in sections 3 or 4.

Our method for decomposing the sequence of dipoles into its constituent parts is to find an "essential" filter, one which filters out the oscillations which are essentially on the scale of the lattice and calls them the weak-short part, and also filters out the oscillations essentially on a much larger scale and calls these the weak-long part.

We will prove the following:

Theorem 5. *Let $\tilde{d}_\lambda \rightharpoonup d$ weakly in L^2 and let $\{\mu_\lambda\}_{\lambda>0}$, (with $\mu_\lambda \overset{*}{\rightharpoonup} \mu$ weak-$\star$ in the sense of measures), be the induced sequence of Wigner measures on the torus Q. Then there is a decomposition*

$$\mu_\lambda = \mu^L_\lambda + \mu^S_\lambda,$$

where $\mu^L_\lambda \overset{*}{\rightharpoonup} a_0 \cdot \delta_0$ weak-$\star$ in the sense of measures and $\mu^S_\lambda \overset{*}{\rightharpoonup} \mu^S$ weak-$\star$ where $\mu^S \perp \delta_0$ and $\mu = a_0 \cdot \delta_0 + \mu^S$. Moreover, μ^L_λ (or the corresponding filtered sequence of dipoles, $\tilde{d}^L_\lambda \rightharpoonup d$) induces an H-measure ν_H with total mass equal to a_0 so that the total energy converges to

$$e_\lambda \to \int_{S^{N-1}} G(s)d\nu_H(s) + \int_Q \breve{K}(\xi)\mu^S(\xi) + \int_{\Re^N} G(\xi)\hat{d}(\xi)\otimes\overline{\hat{d}(\xi)}d\xi.$$

Proof. Let $\phi_\delta(\xi) = \phi(\xi/\delta)$ be as defined in equation (19). This will be our filter. Since we are unable to decide a priori which frequencies to place into the weak-short part and which into the weak-long part, we use the intermediary step:

$$\begin{aligned}\mu_\lambda &= \mu_\lambda^{S,\delta} + \mu_\lambda^{L,\delta} \\ &= (1-|\phi_\delta|^2)\mu_\lambda + |\phi_\delta|^2\mu_\lambda.\end{aligned}$$

We will show that $\mu_\lambda^{S,\delta}$ satisfies the weak-short condition (18) and $\mu_\lambda^{L,\delta}$ satisfies the weak-long condition (8) if we choose $\delta = \delta(\lambda)$ appropriately. The rest of the proof will be devoted to this choice of $\delta(\lambda)$.

First let us check that $\mu_\lambda^{L,\delta}$ satisfies (8): fixing $\epsilon > 0$,

$$\lim_{\lambda\to 0}\int (1-|\phi_\epsilon|^2)d\mu_\lambda^{L,\delta} = \lim_{\lambda\to 0}\int (1-|\phi_\epsilon|^2)|\phi_\delta|^2 d\mu_\lambda = O(\alpha|\delta/\epsilon|^2||\mu_\lambda||).$$

So we must choose $\delta = \delta(\lambda) \to 0$ as $\lambda \to 0$, which gives us that

$$\lim_{\lambda\to 0}\int (1-|\phi_\epsilon|^2)d\mu^{L,\delta(\lambda)} = 0,$$

for all $\epsilon > 0$, verifying (8). We remark that (8) is even more easily verifiable if we test against trigonometric polynomials $\rho_\epsilon \in C_0(B_{2\epsilon})$ with $\rho_\epsilon \equiv 1$ inside B_ϵ.

Second, let

$$a_0 = \lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int |\phi_\epsilon(\xi)|^2 d\mu_\lambda(\xi),$$

(we note that the limit in λ exists because $\phi_\epsilon(\xi) = \sum_{\mathcal{L}^*}\phi_\epsilon(\xi+m)$ since $\mathrm{supp}\phi_\epsilon \subset Q$ implying that $\phi_\epsilon \in C_{per}(Q)$ and because the limit in ϵ exists from the monotonicity of ϕ_ϵ). From a diagonalization argument of Attouch ([1], Lemma 1.15 and Corollary 1.16), we can choose $\epsilon = \epsilon(\lambda) \to 0$ as $\lambda \to 0$ such that $\lambda \ll \epsilon(\lambda)$ and

$$a_0 = \lim_{\lambda\to 0}\int_Q |\phi_{\epsilon(\lambda)}|^2 d\mu_\lambda.$$

Letting $\delta(\lambda) = \epsilon(\lambda)$ as defined above, then the weak-short condition (18), can be tested:

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int |\phi_\epsilon|^2 d\mu_\lambda^{S,\delta(\lambda)} &= \lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int |\phi_\epsilon|^2(1-|\phi_{\delta(\lambda)}|^2)d\mu_\lambda \\ &= \lim_{\epsilon\to 0}\lim_{\lambda\to 0}\Big(\int |\phi_\epsilon|^2 d\mu_\lambda - \int |\phi_{\delta(\lambda)}|^2|\phi_\epsilon|^2 d\mu_\lambda\Big) \\ &= a_0 - a_0 = 0,\end{aligned}$$

since for $|\xi| \leq \delta(\lambda) < \epsilon$,

$$\begin{aligned}\lim_\epsilon\lim_\lambda\Big(1-\Big|\frac{\delta(\lambda)}{\epsilon}\Big|^2\Big)^{2\alpha}\int |\phi_{\delta(\lambda)}|^2 d\mu_\lambda &= \lim_\epsilon\lim_\lambda\Big(1-\Big|\frac{\delta(\lambda)}{\epsilon}\Big|^2\Big)^{2\alpha}\int\Big(1-\Big|\frac{\xi}{\delta(\lambda)}\Big|^2\Big)^{2\alpha}d\mu_\lambda \\ &\leq \lim_\epsilon\lim_\lambda\int\Big(1-\Big|\frac{\xi}{\delta(\lambda)}\Big|^2\Big)^{2\alpha}\Big(1-\Big|\frac{\xi}{\epsilon}\Big|^2\Big)^{2\alpha}d\mu_\lambda\end{aligned}$$

$$= \lim_{\epsilon} \lim_{\delta} \int |\phi_{\delta(\lambda)}|^2 |\phi_\epsilon|^2 d\mu_\lambda$$

$$\leq \lim_{\epsilon} \lim_{\lambda} \int \Big(1 - \Big|\frac{\xi}{\delta(\lambda)}\Big|^2\Big)^{2\alpha} d\mu_\lambda$$

$$= \lim_{\epsilon} \lim_{\delta} \int |\phi_{\delta(\lambda)}|^2 d\mu_\lambda \tag{28}$$

and since $\delta(\lambda) \to 0$ as $\lambda \to 0$, $(\delta(\lambda) \ll \epsilon)$, we see that both upper and lower bounds in (28) converge to a_0. Thus, for $\delta(\lambda)$ as chosen above, $\mu_\lambda^{S,\delta(\lambda)}$ satisfies condition (18) and $\mu_\lambda^{L,\delta(\lambda)}$ satisfies condition (8). Letting $\mu_\lambda^S = \mu_\lambda^{S,\delta(\lambda)}$ and $\mu_\lambda^L = \mu_\lambda^{L,\delta(\lambda)}$, we obtain the desired decomposition.

We can trivially decompose the energy:

$$e_\lambda = \int \check{K} d\mu_\lambda^S + \int \check{K} d\mu_\lambda^L,$$

The weak-short part is easily dealt with by the methods of section 4. The weak-long part is also easily dealt with once we realize that

$$|\hat{\chi}_U|^2 \mu_\lambda^L = |\hat{\chi}_U|^2 |\phi_{\delta(\lambda)}|^2 \check{h}_\lambda \otimes \overline{\check{h}_\lambda}$$
$$= |\phi_{\delta(\lambda)}|^2 S_\lambda^{-1}[\hat{\tilde{d}}_\lambda] \otimes \overline{S_\lambda^{-1}[\hat{\tilde{d}}_\lambda]}$$
$$= S_\lambda^{-1}[\hat{\tilde{d}}^L{}_\lambda] \otimes \overline{S_\lambda^{-1}[\hat{\tilde{d}}^L{}_\lambda]}$$

where $\hat{\tilde{d}}^L{}_\lambda(\xi) = \phi(\lambda\xi/\delta(\lambda))\hat{\tilde{d}}_\lambda(\xi)$. We also know that $\check{\phi}_{\delta(\lambda)/\lambda}(x) = (\delta(\lambda)/\lambda)^N \Phi(\delta(\lambda)x/\lambda) = \Phi_{\delta(\lambda)/\lambda}(x)$ Thus, μ_λ^L is a sequence of Wigner measures generated by the sequence $\tilde{d}_\lambda^L = \Phi_{\delta(\lambda)/\lambda} * \tilde{d}_\lambda$. Since $||\tilde{d}_\lambda^L||_{L^2} < \infty$ by Minkowski's inequality, a subsequence is weakly convergent. In fact, $\tilde{d}_\lambda^L \rightharpoonup d$; it is easy to see that because $\delta(\lambda) \gg \lambda$, $\hat{\tilde{d}}^L{}_\lambda - \hat{\tilde{d}}_\lambda \overset{*}{\rightharpoonup} 0$ weak-$\star$ in the sense of measures, and thus weakly in L^2. Thus, the sequence $\tilde{d}_\lambda^L - d \rightharpoonup 0$ generates an H-measure, μ_H^L, (generally not equal to the H-measure generated by $\tilde{d}_\lambda - d \rightharpoonup 0$), with $\int_{S^{N-1}} d\mu_H^L = a_0$. We can see that the theory of section 3 then applies and we get the desired energy decomposition,

$$e_\lambda \to \int_Q \check{K}(\xi) d\mu^S(\xi) + \int_{S^{N-1}} G(n) d\mu_H^L(n) + \int_{\Re^N} G(\xi)\hat{d}(\xi) \otimes \overline{\hat{d}(\xi)} d\xi.$$

We note that the weak-short part of the energy can also be represented by the singular integral: $\lim_{\epsilon\to 0} \int \check{K} \phi_\epsilon d\mu$, where $\phi_\epsilon \in C_0(B_\epsilon)$ and $\phi(\xi) \equiv 1$ for $\xi \in B_{\epsilon/2}$. $\square$

6. An Example of Weak-Short Oscillations. We already know that the energy associated with weak-long oscillations converges to a double convolution over $\Re^N$ and that this is nonlocal by nature, i.e., if $\tilde{d}_\lambda$ converges weak-long then the limiting energy associated with $\chi_{\Omega_1}\tilde{d}_\lambda$ and that associated with $\chi_{\Omega_2}\tilde{d}_\lambda$ do not sum to give the energy associated with $(\chi_{\Omega_1} + \chi_{\Omega_2})\tilde{d}_\lambda$, where Ω_1 and Ω_2 are disjoint sets. There is an interaction energy for assembling multi-bodied systems of weak-long dipoles. We can also characterize this weak-long energy explicitly with the use of H-measures. The weak-short part of the energy is not so easily characterized, since we would need to know $\check{K}$ completely on Q, (whereas Wainger's results [22] only give us complete knowledge of its asymptotics around the origin). However, we are able to characterize the behavior of the weak-short part of the energy as being essentially local in nature when the sequence $|\tilde{d}_\lambda|^2$ is strongly precompact in L^1 and no mass escapes to infinity, (which we will do in section 7). We provide this example, a generalization of example 8.4 from James and Müller ([10]), as a motivation for this approach.

Let $\Psi(x)$ satisfy the hypotheses of Poisson's summation formula, i.e., $|\Psi(x)| \leq A(1+|x|)^{-n-\delta}$ and $\hat{\Psi} = \psi$ with $|\psi(\xi)| \leq A'(1+|\xi|)^{-n-\delta}$, and let μ be a measure on the torus, Q, with Fourier series:

$$\mu(\xi) = \sum a(m) e^{-2\pi i m\cdot\xi}.$$

We will consider a kernel K which is homogeneous of degree $-\gamma = -N$ and the sequence of dipoles,

$$d_\lambda(m) = \lambda^N \Psi(m) a(m/\lambda).$$

The corresponding rescaled function h_λ is given by $h_\lambda(m) = \lambda^{N/2}\Psi(\lambda m)a(m)$. Note that $\tilde{d}_\lambda$ is a weak-short sequence.

By Poisson's summation formula,

$$\sum_{\mathcal{L}} \lambda^{N/2}\Psi(\lambda m) e^{-2\pi i m\cdot\xi} = \sum_{\mathcal{L}^*} \frac{1}{\lambda^{N/2}} \psi\Big(\frac{\xi+m}{\lambda}\Big) \stackrel{def}{=} L_\lambda(\xi),$$

where $\mathcal{L}^*$ is the reciprocal lattice to $\mathcal{L}$. Then $\check{h}_\lambda = L_\lambda * \mu$. In a similar fashion to Firoozye and Šverák, ([5]), we prove the following:

Theorem 6. *Under the above hypotheses, where also μ can be decomposed into its atomic and diffuse parts, i.e., $\mu = \mu_a + \mu_d$, where $\mu_a = \sum a_i \delta_{x^i}$, the Wigner measure sequence $\mu_\lambda = \check{h}_\lambda \otimes \overline{\check{h}_\lambda} d\xi$ converges weak-$\star$ in the sense of measures to $\int |\Psi|^2 dx \sum a_i^2 \delta_{x^i}$.*

Proof. We note first that $h_\lambda = L_\lambda * \mu = \sum a_m \lambda^{N/2}\Psi(\lambda m) e^{-2\pi i m\cdot x}$ and that since Ψ decreases rapidly at infinity and $a_m \in l^\infty(\mathcal{L})$, the sum converges absolutely. We can also bound L_λ uniformly in L^2, i.e.,

$$\begin{aligned} \int_Q |L_\lambda|^2 dx &= \sum_{\mathcal{L}} |\lambda^{N/2}\Psi(\lambda m)|^2 \\ &= \lambda^N \sum_{\mathcal{L}_\lambda} |\Psi(m)|^2, \end{aligned}$$

and, since $\Psi \in L^2 \cap C^0$, this Riemann sum converges to $||\Psi||^2_{L^2}$, thus the Riemann sums are uniformly bounded (and also will be bounded if $\phi = \chi_\Omega$ with Ω a bounded domain) Using Jensen's inequality, we find that

$$||\,|\check{h}_\lambda|^2||_{L^1(Q)} = ||L_\lambda * \mu||^2_{L^2(Q)} \leq ||L_\lambda||^2_{L^2(Q)} \cdot ||\mu||^2_{\mathcal{M}(Q)}.$$

So we know that, after passing to a subsequence, $|\check{h}_\lambda|^2 \overset{*}{\rightharpoonup} \nu$ weakly in the sense of measures. Also it is simple to show that $h_\lambda \rightharpoonup 0$ weakly in l^2.

First, we show that $|L_\lambda|^2 \overset{*}{\rightharpoonup} \delta_0 \cdot \int |\Psi|^2 dx$ weak-$\star$ in the sense of measures. Let $f \in C_{per}(Q)$. Then

$$\begin{aligned} \int_Q |L_\lambda|^2 f(x)dx &= \int_Q |\sum_{\mathcal{L}} \lambda^{-N/2} \psi\Big(\frac{x+m}{\lambda}\Big)|^2 f(x)dx \\ &= \int_{Q\cdot(1/\lambda)} |\sum \psi(y+m/\lambda)|^2 f(\lambda y)dy \\ &= \int_{Q\cdot(1/\lambda)} |\psi(y) + O(\lambda^{N+\delta})|^2 f(\lambda y)dy, \end{aligned} \tag{29}$$

since $\sum \psi(y+m/\lambda) = \psi(y) + \sum_{m\neq 0} \psi(y+m/\lambda)$ and

$$\sum_{m\neq 0} \psi(y+m/\lambda) \leq \sum_{m\neq 0} \frac{A}{(1+|y+m/\lambda|)^{N+\delta}} \leq C\lambda^{N+\delta},$$

for $y \in Q_{1/\lambda}$. Finally, the right hand side of (29) converges to $\int |\psi|^2 dx \cdot f(0)$ by the dominated convergence theorem.

We are now prepared to prove the theorem. Let μ have the decomposition into diffuse and atomic parts, $\mu = \mu_a + \mu_d$ where $\mu_a = \sum a_i \delta_{x_i}$. For the time being, suppose that $\mu_d = 0$. Then

$$\begin{aligned} \int_Q |\check{h}_\lambda|^2 f(x)dx &= \int_Q |L_\lambda * \mu|^2 f(x)dx \\ &= \int_Q \Big|\int (\lambda^{-N/2}\psi((x-y)/\lambda) + O(\lambda^{N/2+\delta})) \cdot \sum a_i \delta_{x_i}(y)dy\Big|^2 f(x)dx \\ &= \int_Q \Big|\sum \lambda^{-N/2}\psi((x-x_i)/\lambda)a_i + O(\lambda^{N/2+\delta}\sum |a_i|)\Big|^2 f(x)dx \\ &= \int_Q \sum a_i^2 \lambda^{-N} |\psi((x-x_i)/\lambda)|^2 f(x) \\ &\qquad + \lambda^{-N} \int_Q \sum_{i\neq j} a_i a_j \psi((x-x_i)/\lambda)\psi((x-x_j)/\lambda) f(x)dx + O(\lambda^{2\delta}), \end{aligned} \tag{30}$$

and we see that the first term on the right of equation (30) converges to $\sum a_i^2 f(x_i)$. The second term on the right of (30) converges to zero because of the growth conditions at infinity, (see Firoozye-Šverák, [5], Theorem 1).

If $\mu_d \neq 0$ then we have two other terms:

$$\int |L_\lambda * (\mu_a + \mu_d)|^2 f(x)dx = \int (|L_\lambda * \mu_a|^2 + |L_\lambda * \mu_d|^2 + 2\langle L_\lambda * \mu_a, L_\lambda * \mu_d\rangle) f(x)dx. \tag{31}$$

Following Firoozye-Šverák, we use Fubini's theorem to expand the second term thus:

$$\int\int\int L_\lambda(x-y)L_\lambda(x-z)f(x)dxd\mu_d(y)d\mu_d(z) = \int F_\lambda(y,z)d(\mu_d \times \mu_d)(y,z),$$

where $F_\lambda(y,z) = \int L_\lambda(x-y)L_\lambda(x-z)f(x)dx$. It is quite simple to see that $F_\lambda(y,z)$ concentrates on the diagonal $\Delta = \{(y,z) : y = z\}$. We also know that $(\mu_d \times \mu_d)[\Delta] \neq 0$

is equivalent to $\mu_d(\{x\}) \neq 0$ for some point $x \in Q$, which is clearly impossible. Thus the second term in equation (31) converges to zero. We deal with the third term in equation (31) similarly, noting that $(\mu_a \times \mu_d)[\Delta] = 0$ as well. Thus, we have proven the theorem.

□

REMARK 4. *We note that the above result holds for Ψ a characteristic function as well, since $|L_\lambda * \mu|^2$ is still uniformly bounded in $L^1(Q)$ for $\Psi \in L^2(\Re^N)$, and all other results are easily seen to be extendable to the general case. This gives us the "locality" of the energy. In other words, we have proved the following:*

COROLLARY 7. *Let $\mu(\xi)$ be a finite measure on the torus, with $\mu = \mu_a + \mu_d$, where $\mu_a = \sum a_i \delta_{x^i}$, with Fourier coefficients $a(m)$. Let the dipole moments $d_\lambda(m) = \lambda^N \chi_\Omega(m) a(m/\lambda)$, where χ_Ω is the characteristic function of a finite volume set, Ω. Then the stored-energy associated with the sequence of dipoles $\{d_\lambda\}$ is*

$$e(\Omega) = \int_Q \check{K}(\xi) d\mu_a(\xi) \cdot vol(\Omega),$$

a σ-finite measure.

Of course, this example of "locality" of the weak-short part of the energy leads one to suspect that locality is a general quality of weak-short oscillations. We will see below that this is indeed true but only under special circumstances.

7. The Locality of the Weak-Short Energy. The weak-long part of the energy is represented via limits of convolutions over $\Re^N$ and is non-local in that if we were to assemble a system of two magnetic dipoles each with weak-long internal oscillations, the energy of the system is different from the energy of the separate components. The example of the preceeding section leads one naturally to the question of whether the energy given by weak-short oscillations is local, i.e., additive. We will see in this section that under the condition of "no mass-loss at infinity," (a condition that the example of the preceding section does not satisfy),

$$\lim_{\rho \to \infty} \sup_{\lambda} \int_{|x| \geq \rho} |\tilde{d}_\lambda|^2 dx = 0, \tag{32}$$

and that if the sequence $|\tilde{d}_\lambda|^2$ is strongly precompact in L^1, (a condition which the example of the preceding section satisfies), the energy of weak-short oscillations is indeed local. In fact, it will generate a measure.

Before we proceed, we must phrase the concept of "locality" in a mathematically rigorous language. Suppose $\tilde{d}_\lambda$ is a sequence of dipoles converging weak-short in L^2. Then, as we have seen above, the energy $e_\lambda \to \int_Q \check{K}(\xi) d\mu(\xi)$, where μ is the Wigner measure limit of the sequence d_λ. Let χ_Ω be the characteristic function of a set $\Omega \subset \Re^N$. Then $\chi_\Omega \tilde{d}_\lambda$ converges weak-short again and induces a Wigner measure μ_Ω and the corresponding energies converge $e_\lambda(\Omega) \to \int_Q \check{K}(\xi) d\mu_\Omega(\xi) = e(\Omega)$. We will show that the energy is "local" in the sense that the energy parameterized by Ω, $e(\Omega)$ is in fact a measure on Ω.

We will prove the following:

THEOREM 8. *Let $\tilde{d}_\lambda$ converge weak-short in $L^2(\Re^N)$, assume that no mass escapes to infinity, condition (32), and $\gamma = N$. Let $\Omega \subset \Re^N$. Then $\tilde{d}_\lambda \chi_\Omega$ also converges weak-short and induces a Wigner measure limit on the torus, μ_Ω (where Ω is a parameter). The limiting energy $e(\Omega)$ is given by $e(\Omega) = \int_Q \check{K}(\xi) d\mu_\Omega(\xi)$. This energy $e(\Omega)$ is a finitely additive finite signed measure on Borel sets in $\Re^N$. The same result holds in the case where $\gamma > N$ when $\tilde{d}_\lambda$ is a sequence of dipoles converging weak-short, weak-long, or strong satisfying condition (32).*

Before proceeding to the proof of this theorem, we need several approximation lemmas. First we will show that our kernel can be approximated by functions of bounded support, i.e., kernels whose Fourier transform on the torus are trigonometric polynomials.

LEMMA 5. *Let $\tilde{d}_\lambda$ be a sequence of dipoles. Then the energy associated with a kernel of degree $-\gamma$ where $\gamma > N$ can be approximated arbitrarily closely by kernels of finite support. If $\tilde{d}_\lambda$ converges weak-short and $\gamma = N$ then the same approximation result holds.*

Proof. Consider the energy associated with the kernel f with finite support, i.e., where $\check{f}$ is a trigonometric polynomial:

$$\begin{aligned} e^f_\lambda &= \sum_{\mathcal{L}_\lambda} f(\frac{x-y}{\lambda}) h_\lambda(x/\lambda) h_\lambda(y/\lambda) \\ &= \sum_{\mathcal{L}} f(x-y) h_\lambda(x) h_\lambda(y) \\ &= \int_Q \check{f}(\xi) \check{h}_\lambda(\xi) \otimes \overline{\check{h}_\lambda(x)} d\xi \\ &\to \int_Q \check{f}(\xi) d\mu(\xi) = e^f, \end{aligned}$$

where μ is the Wigner measure limit of the sequence h_λ. Since trigonometric polynomials are dense in the space $C_{per}(Q)$ under uniform convergence, e^f can be made arbitrarily close to e^K for any $\check{K} \in C_{per}(Q)$, (e.g., for any K homogeneous of degree $-\gamma$ where $\gamma > N$). In the case where $\gamma = N$, we recall from Theorem 4 that

$$e = \lim_{\epsilon \to 0} \int (1 - \psi(\xi/\epsilon)) \check{K}(\xi) d\mu(\xi),$$

where $\psi(\xi) \in C_0(B_1)$ with $\psi(0) = 1$ and $0 \le \psi \le 1$. The function $(1 - \psi(\xi/\epsilon))\check{K}(\xi) \in C_{per}(Q)$ and thus can be approximated arbitrarily closely in the uniform norm by trigonometric polynomials. Thus, e^K can be approximated arbitrarily closely by e^f where f is a kernel of finite support when the sequence of dipoles converges weak-short. □

We will also show that the domains can be approximated:

LEMMA 6. *Suppose $\tilde{d}_\lambda$ is a sequence bounded in L^2 with $|\tilde{d}_\lambda|^2$ strongly precompact in L^1. Then the limiting energy $e^f(B)$ associated with $\chi_B(x)\tilde{d}_\lambda(x)$, where B is a set of finite volume, and $f \in l^1(\mathcal{L})$, converges to zero when $|B|$ converges to zero.*

Proof. The energy $e_\lambda(\chi_B)$ is given by

$$|e_\lambda(\chi_B)| = |\sum_{\mathcal{L}} f(x-y) \chi_B(\lambda x) \chi_B(\lambda y) h_\lambda(x) h_\lambda(y)|$$

$$\leq ||f||_{l^1(\mathcal{L})} \sum_{\mathcal{L}} |\chi_B(\lambda x) h_\lambda(x)|^2$$
$$\leq ||\check{f}||_{l^1(\mathcal{L})} \int_B |\tilde{d}_\lambda|^2 dx,$$

where we have used Minkowski's theorem in the second line. We know that for any $\epsilon > 0$ there is some $\delta > 0$ such that if $|B| < \delta$,

$$\int_B |\tilde{d}_\lambda|^2 dx \leq \epsilon,$$

and we have proved the result. □

Finally, let us show that for finite kernels and domains which are bounded away from one another, the energy is additive:

Lemma 7. *Let f be a kernel of finite support. Let $\tilde{d}_\lambda$ satisfy condition (32) and let $\{A_i\}$ be a family of sets such that $dist(A_i, A_j) \geq \epsilon_0 > 0$. Then $e^f(\sum \chi_{A_i}) = \sum e^f(\chi_{A_i})$.*

Proof. Since f has finite support, let us suppose that $supp\ f \subseteq B_R$ for R finite. The energy $e^f_\lambda(\sum \chi_{A_i})$ then can be expanded as

$$e^f_\lambda(\sum \chi_{A_i}) = \sum e^f_\lambda(\chi_{A_i}) + \sum_{i \neq j} \sum_{\mathcal{L}_\lambda} f(\frac{x-y}{\lambda}) \chi_{A_i}(x) \chi_{A_j}(y) h_\lambda(x/\lambda) h_\lambda(y/\lambda).$$

We only need to show that the cross-terms converge to zero. Because of the finite support of f, the summand contributes to the cross-terms only if $|x - y| < R\lambda$. But $\chi_{A_i}(x)\chi_{A_j}(y)$ is non-zero only if $|x - y| \geq \epsilon_0$. Thus, if we choose λ small enough so that $\lambda < R/\epsilon_0$, the cross-terms vanish and we have shown the result. □

Finally, we are able to prove theorem 8.

Proof. We must show that $e(\Omega)$ is finitely additive on disjoint Borel sets and that $e(\emptyset) = 0$. The latter statement is trivial. Also, we note that $|e(\Omega)| \leq |e|$ where e is the limiting energy associated with the whole sequence $\tilde{d}_\lambda$. Thus $e(\cdot)$ will be a finite measure.

We will begin by combining Lemmas 5 and 7 to show additivity of the energy on sets which are a positive distance apart. Let $\{A_i\}$ be a family of Borel sets which satisfies $dist(A_i, A_j) \geq \epsilon_0 > 0$ for $i \neq j$. Let μ_i^A be the Wigner measure generated by the sequence $\chi_{A_i}\tilde{d}_\lambda$. Then, from Lemma 7, we know that

$$\int \check{f}(\xi) d\mu^{\bigcup A_i}(\xi) = \sum \int \check{f}(\xi) d\mu^{A_i},$$

for all trigonometric polynomials $\check{f}$. Since Lemma 5 shows us that we can approximate the weak-short part of the energy given by the kernel $\check{K}$ arbitrarily closely by that given by trigonometric polynomials, we obtain by completion,

$$\int \check{K}(\xi) d\mu^{\bigcup A_i}(\xi) = \sum \int \check{K}(\xi) d\mu^{A_i}.$$

For sets which may touch the approximation is much harder. Since, as we have seen above, all the properties of e^f can be generalized to e^K for kernels K homogeneous of degree $-\gamma$, we will drop the notation e^f and use e instead.

First let us reduce the case of touching domains to the same problem where the supports are finite. The sequence $\tilde{d}_\lambda$ satisfies condition (32), or, restated in terms of h_λ,

$$\lim_{\rho\to\infty} \sup_{\lambda>0} \sum_{x\in\mathcal{L}_\lambda\cup B_\rho} |h_\lambda(x/\lambda)|^2 = 0.$$

Thus, the energy associated with kernel f can be expanded thus:

$$\begin{aligned} e_\lambda &= \sum_{\mathcal{L}_\lambda\cap B_\rho} f(\frac{x-y}{\lambda})h_\lambda(x/\lambda)h_\lambda(y/\lambda) + \sum_{\mathcal{L}_\lambda\cap B_\rho^c} f(\frac{x-y}{\lambda})h_\lambda(x/\lambda)h_\lambda(y/\lambda) \\ &= \sum_{\mathcal{L}_\lambda\cap B_\rho} f(\frac{x-y}{\lambda})h_\lambda(x/\lambda)h_\lambda(y/\lambda) + ||f||_{l^1(\mathcal{L}_1)}\epsilon, \end{aligned}$$

where ϵ can be made arbitrarily small as $\rho \to \infty$.

Fix $\rho > 0$. Let A and B be two disjoint Borel sets. Let $A_i \subset A$ and $B_j \subset B$ be a sequence of approximating sets such that $dist(A_i, B_j) > 0$ for all i, j and $vol((A \setminus A_i) \cap B_\rho) \to 0$ and $vol((B \setminus B_j) \cap B_\rho) \to 0$ as $i, j \to \infty$. From Lemma 6, we know that $|e(A\cap B_\rho) - e(A_i\cap B_\rho)| \to 0$ and similarly for the difference of $e(B\cap B_\rho)$ and $e(B_i\cap B_\rho)$. From Lemma 7, we know that

$$e((A_i \cup B_j) \cap B_\rho) = e(A_i \cap B_\rho) + e(B_j \cap B_\rho),$$

for all i, j. Letting $i, j \to \infty$, we see that

$$e((A \cup B) \cap B_\rho) = e(A \cap B_\rho) + e(B \cap B_\rho).$$

Similarly, we can show for any set C that $e(C) = e(C \cap B_\rho) + e(C \cap B_\rho^c)$. Finally,

$$\begin{aligned} e(A \cup B) &= e((A \cup B) \cap B_\rho) + e((A \cup B) \cap B_\rho^c) \\ &= e(A \cap B_\rho) + e(B \cap B_\rho) + e((A \cup B) \cap B_\rho^c) \\ &= e(A) + e(B) + e((A \cup B) \cap B_\rho^c) - e(A \cap B_\rho^c) - e(B \cap B_\rho^c) \\ &= e(A) + e(B) + O(3\epsilon||f||_{l^\infty(\mathcal{L})}), \end{aligned}$$

and letting $\rho \to \infty$ (and $\epsilon \to 0$) we see that we have additivity of the energy. Since $e^f(A \cup B) = e^f(A) + e^f(B)$ for all f of finite support, or rephrased on the torus,

$$\int \check{f} d\mu^{A\cup B} = \int \check{f} d\mu^A + \int \check{f} d\mu^B,$$

for all trigonometric polynomials $\check{f}$. Similarly we can extend this to any finite family of disjoint sets. We can then use Lemma 5, letting the trigonometric polynomials approach our kernel $\check{K}$ uniformly, (or uniformly on $Q \setminus B_\delta$ for all $\delta > 0$), extending the result to K. Thus, we have shown that $e(\cdot)$ is a finitely additive finite signed measure on $\Re^N$. □

Remark 5. *The methods of the proof of Theorem 8 are not extendible to the case of infinite unions or intersections of Borel sets, which calls into question whether $e(\cdot)$ is in fact a σ-finitely additive measure on $\Re^N$.*

Finally, we shall give several examples of sequences h_λ for which the corresponding sequences $|\tilde{d}_\lambda|^2$ are not precompact in L^1 and which $e(\cdot)$ is not a measure.

EXAMPLE 1. *First of all, let* $h_\lambda(n) = h(n) = \prod sign(\cos(\pi n_i/2))\chi_Q$ *for* Q *a cube centered at the origin and* $n \in \mathcal{Z}^N$ *and note that* $\sum h_\lambda = 0$. *The corresponding sequence,*

$$\tilde{d}_\lambda(x) = \sum \lambda^{-N/2} \prod sign(\cos(\pi n_i/2\lambda))\chi_{\lambda Q}\chi_{n+\lambda U}(x)$$

satisfies

$$\sup_\lambda \int_{B_\epsilon} |\tilde{d}_\lambda|^2 dx = c > 0$$

for all $\epsilon > 0$, *so* $|\tilde{d}_\lambda|^2 \overset{*}{\rightharpoonup} c\delta_0$, *weak-*$\star$ *in the sense of measures. Note that the sequence* $\tilde{d}_\lambda$ *is weak-short because* $\check{h}_\lambda = \check{h}$ *is a trigonometric polynomial and thus, the Wigner measure* μ *does not charge the origin. We will consider the energy induced by a kernel of finite support,*

$$f(n) = \begin{cases} \prod \sigma(n_i) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0, \end{cases}$$

where $\sigma(1) = \sigma(-1) = 1 = -\sigma(0)$ *and* $\sigma = 0$ *otherwise. Let* $A = \{x : x_n \leq 0\}$ *and* $B = \{x : x_n > 0\}$. *Then*

$$\begin{aligned} e(A \cup B) - e(A) - e(B) &= 2 \sum_{x \in A, y \in B} f(x-y)h(x)h(y) \\ &= c' 3^N \#\{y \in \partial A \cap \partial B \cap Q \cap \mathcal{Z}^N\}. \end{aligned}$$

In other words, $e(\cdot)$ *is not additive, and there is a nonzero interaction energy between the sets* A *and* B.

EXAMPLE 2. *As a second counterexample, in two dimensions let* $h_\lambda = h$ *with*

$$1 = h_1(0,0) = h_1(1,0) = -h_1(0,1) = -h_1(0,-1),$$

with $h_1 = 0$ *elsewhere and* $h_2 = 0$. *Again, we have a concentration at the origin, i.e.,* $|\tilde{d}_\lambda|^2 \overset{*}{\rightharpoonup} 4 \cdot \delta_0$ *weak-*$\star$ *in the sense of measures. Let* K_{ij} *be the Helmholtz energy,*

$$K_{ij}(x) = \frac{c}{|x|^2}\Big(2\frac{x_i x_j}{|x|^2} - \delta_{ij}\Big).$$

Letting A *and* B *be defined as in Example 1, we see that*

$$e(A \cup B) - e(A) - e(B) = 2 \sum K(x-y)h(x)h(y) = 2c'.$$

EXAMPLE 3. *Let* f *be as in Example 1 and*

$$h_\lambda(n) = \lambda \prod_1^N sign(\cos(\pi n_i/2\lambda))\chi_{Q_\lambda},$$

where $Q_\lambda = \{x : |x_i| \leq c/\lambda^{2/N}, i = 1, \ldots, N-1, |x_N| \leq c\}$. *This time* $|\tilde{d}_\lambda|^2$ *concentrates on the hyperplane* $x_N = 0$. *The sequence* h_λ *is weak-short because it satisfies condition (22). Again,*

$$e_\lambda(A \cup B) - e_\lambda(A) - e_\lambda(B) = 2 \sum f(x-y)h_\lambda(x)h_\lambda(y) \geq c > 0.$$

8. Applications to Magnetostatics. The original motivation of this study was a problem in magnetostatics. In this problem we will let $d_\lambda(\cdot)$ represent a sequence of dipoles on the lattices $\mathcal{L}_\lambda$, which are bounded in the sense that

$$\tilde{d}_\lambda = \sum_{m\in\mathcal{L}_\lambda} \chi_{m+\lambda U}(x) d_\lambda(m)\lambda^{-3}$$

is uniformly in $L^2(\Re^3)$ (moreover, magnetic dipoles are generally taken to be of unit length). Since averages are preserved by the association of $\tilde{d}_\lambda$ to d_λ, i.e.,

$$d_\lambda(n) = \int_{n+\lambda U} \tilde{d}_\lambda(x)dx,$$

for $n \in \mathcal{L}_\lambda$, we say that d_λ represents the experimental measurements of the underlying dipole field $\tilde{d}_\lambda$ with refinement given by λ. The energy associated with the lattice of magnetic dipoles is given by

$$e_\lambda = \sum_{\substack{n,m\in\mathcal{L}_\lambda \\ n\neq m}} K(n-m)d_\lambda(n)d_\lambda(m),$$

where

$$K_{ij}(n) = \frac{1}{4\pi|n|^3}\Big(3\frac{n_i n_j}{|n|^2} - \delta_{ij}\Big),$$

is the field induced by one dipole. Actually, $K_{ij} = -\nabla_i\nabla_j\Delta^{-1}(\delta_0)$ is the Helmholtz kernel (convolution of a vector function in $L^2(\Re^3)$ with K acts as a projection into gradient vector fields in L^2, and $\hat{K} = -\frac{\xi\otimes\xi}{|\xi|^2}$). And, K has the "cancellation property" required by Theorem 2. In fact, K is of the form $K(n) = \frac{c}{|n|^3}P_2(\frac{n}{|n|})$, where P_2 are spherical harmonics of degree two. Thus, we can apply Wainger's results directly to find that

$$\breve{K}_{ij}(\xi) = -\frac{\xi_i\xi_j}{|\xi|^2} + \frac{1}{3}\delta_{ij} + S + L'(\xi),$$

with $L' \in C^\infty(Q)$ and $|L'(\xi)| = o(1)$ as $|\xi| \to 0$, and

$$S = \lim_{\epsilon\to 0}\sum_{\mathcal{L}\setminus\{0\}} e^{-\epsilon|m|}K(m).$$

We note, as did James and Müller, that in the case of a cubic lattice, $S = 0$, (i.e., the anisotropy terms are given by $(1/3)\delta_{ij}$ alone). Fujiki et al [7] and De'Bell and Whitehead [4] also studied this lattice homogenization problem using discrete Fourier transforms and theta functions. Although they never actually pass to the limit, but instead approximate the limit by taking a fine-scale lattice, they show some agreement in their analysis of the lattice-induced anisotropy, (which is diagonal in the case of cubic lattices).

In the case of weak-long or strong convergence, we merely apply Theorem 3 to obtain that the limiting energy

$$e = \lim_{\lambda} (K * \tilde{d}_\lambda, \tilde{d}_\lambda) + \frac{1}{3}||\tilde{d}_\lambda||^2 + (S\tilde{d}_\lambda, \tilde{d}_\lambda),$$

as did James and Müller. In the case of weak-short convergence or a mixture of the two types of sequences, we get the stronger results of being able to represent the limiting energy explicitly by the use of the weak limit, a homogeneous H-measure and a homogeneous Wigner measure on the torus. Moreover, if the sequence of dipoles does not concentrate on a Lebesgue null-set, (e.g., if $||\tilde{d}_\lambda||_{L^\infty} \leq C$, which is a frequently made assumption in magnetostatics), then the weak-short part of the energy, as represented by the Wigner measure on the torus, is known to be local. The only nonlocality enters into the weak-long interaction. Thus, the energy given by fine scale oscillations is some sort of "internal variable" for each magnetic body and does not affect interactions between several nonintersecting magnetic bodies. The interaction energy is completely classical in nature, (since it involves convolution with the Helmholtz kernel). Measurements of magnetic energy taken from outside the body which are based entirely upon change of energy in the system will effectively see nothing of the local weak-short energy.

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