# A STEFAN PROBLEM FOR A REACTION-DIFFUSION SYSTEM

AVNER FRIEDMAN*, DAVID S. ROSS† AND JIANHUA ZHANG‡

**Abstract.** The paper deals with a Stefan problem for a system of three weakly coupled semilinear parabolic equations. The system describes dissolution of a spherical particle in solution. The dissolved species $A$ reacts chemically with species $B$ already in the solution, thereby forming species $C$. Species $C$ diffuses in the solution and some of it adsorbs to the particle's boundary and gradually shuts down the dissolution. It is shown that the mathematical model has a unique solution with finite shut-down time. When the reaction rate $K$ increases to infinity, the limit model should exhibit phase separation between $A$ and $B$ and it thus has two free boundaries: the particle's boundary, and the $A-B$ interface. It is proved, in the case in which $A$ and $B$ diffuse at the same rate, that the solution with finite $K$ converges to the solution of the limit problem, and that the $A$ phase in the limit problem disappears in finite time.

§**1. The model.** Consider a solid spherical particle composed of chemical $A$ with uniform concentration $A^*$. The particle is in a solution of chemical $B$. As the particle dissolves, the $A$ that enters the solution reacts with $B$ to form chemical $C$. Then $C$ diffuses in the solution and some of it reaches the solid particle and adsorbs to its surface. The presence of the adsorbed $C$ inhibits the dissolution, and ultimately shuts it down entirely.

Assuming radially symmetric data and radially symmetric functions $A, B, C$, we denote by $r = R(t)$ the radius of the solid sphere at time $t$. Then the equations

$$\frac{\partial A}{\partial t} = D_A \Delta A - KAB \ , \tag{1.1}$$

$$\frac{\partial B}{\partial t} = D_B \Delta B - KAB \ , \tag{1.2}$$

$$\frac{\partial C}{\partial t} = D_C \Delta C + KAB \ , \tag{1.3}$$

hold in $\{r > R(t)\}$, where $K$ is the reaction rate and $D_A, D_B, D_C$ are the diffusion coefficients. These equations indicate that $A$ and $B$ are lost in a second-order reaction in which $C$ is formed, and all three species diffuse. In the standard mass-action model of chemical kinetics, the concentrations are all expressed in moles/liter, and the coefficient $K$, the second-order reaction rate, is expressed in liters/(mole-sec). Then $KAB$ is the number of moles per liter per second that undergo reaction; in our case, $A$ and $B$ are consumed, $C$ is created, the same number of moles of $A$ and $B$ are lost, and this number of moles of $C$ is created. A nice reference for this material is the book by Erdi and Toth [4].

Next,

$$\frac{dR}{dt} = \alpha \frac{\partial A}{\partial r} \quad \text{on} \quad r = R(t) \tag{1.4}$$

where $\alpha$ is a positive constant, i.e., the rate at which the radius of the particle decreases is proportional to the flux of species $A$ away from the particle. We also have

$$\frac{\partial B}{\partial r} = 0 \quad \text{on} \quad r = R(t) \ , \tag{1.5}$$

* Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota 55455,

† Applied Mathematics and Statistics Group, Computational Science Laboratory, Eastman Kodak Company, Rochester, New York 14650–2205

‡ University of Minnesota, School of Mathematics, Minneapolis, Minnesota 55455

i.e., there is no flux of $B$ through the particle's surface and $B$ does not undergo any surface reaction.

The adsorption of $C$ to the surface is proportional to the local saturation; it is given by an empirical law $D_C \partial C/\partial r = \gamma C^n$ for some positive constants $\gamma$, $n$ (see [13, pp. 104–105]); for definiteness we take $n = 4$, that is,

$$D_C \frac{\partial C}{\partial r} = \gamma C^4 \ . \tag{1.6}$$

However, all the results of this paper remain valid with minor changes if we replace $\gamma C^4$ by any other monotone increasing function $f(C)$ with $f(0) = 0$, $f(C) > 0$ for $C > 0$.

The boundary conditions at $r = \infty$ are

$$A(\infty, t) = 0, \ B(\infty, t) = B^* \ , \qquad C(\infty, t) = 0 \ , \tag{1.7}$$

where $B^*$ is a positive constant.

We now impose initial conditions. First,

$$R(0) = R_0 > 0 \ . \tag{1.8}$$

Next we assume that

$$A(r, 0) = A_0(r) \ , \ B(r, 0) = B_0(r) \ , \ C(r, 0) = C_0(r) \tag{1.9}$$

for $r > R_0$ where $A_0, B_0, C_0$ are approximately $0, B^*, 0$ (i.e., initially mostly only $B$ is present in the solution and its concentration is nearly uniform). We also assume that the initial conditions are smooth and fit smoothly with the boundary conditions:

$$\begin{cases} A_0 \in C^{2+\nu}[R_0, \infty), \ A_0'(r) \le 0, \ A_0(R_0) = A^* \ , \\ A_0(r) = 0 \quad \text{if } r > R_0 + \delta_1 \text{ for some } \delta_1 > 0 \ ; \end{cases} \tag{1.10}$$

$$\begin{cases} B_0 \in C^{2+\nu}[R_0, \infty), \ B_0(r) \ge 0, \ B_0'(r) \ge 0 \ , \\ B_0(r) = B^* \text{ if } r \ge R_0 + \delta_2 \quad \text{for some} \ \ \delta_2 > 0 \ ; \end{cases} \tag{1.11}$$

$$\begin{cases} C_0(r) \in C^2[R_0, \infty), \ C_0(r) \ge 0 \ , \\ C_0(r) = 0 \quad \text{if } r > R_0 + \delta_3 \text{ for some } \delta_3 > 0, \\ D_C \dfrac{\partial C_0}{\partial r} = \gamma C_0^4 \quad \text{at} \quad r = R_0 \ , \end{cases} \tag{1.12}$$

for some $0 < \nu < 1$.

We finally need to determine the boundary condition for $A$ at the particle's surface. The flux of $A$ from the particle depends on the amount of $C$ that is adsorbed to the surface. On a portion of $\{r = R\}$ where there is no adsorbed $C$, $A = A^*$, the saturation concentration of $A$; local thermodynamic equilibrium is established instantaneously. On portion which is fully covered by $C, \partial A/\partial r = 0$, i.e., the dissolution shuts down. This is actually a microscopic statement, which we shall now "average." We shall use the "weighted average"

$$\zeta(t) = \frac{\beta \int_0^t R^2(s) D_C \frac{\partial C}{\partial r}|_{r=R(s)} ds + \delta}{R^2(t)} \tag{1.13}$$

where $\beta$ is a positive empirical parameter, and $\delta$ is a small positive parameter such that $\delta / R_0^2 < 1$. We then impose the boundary condition

$$-\zeta_0(t) D_A \frac{\partial A}{\partial r} + (1 - \zeta_0(t))(A - A^*) = 0 \quad \text{on} \quad r = R(t) \tag{1.14}$$

where

$$\zeta_0(t) = \min\{\zeta(t), 1\} \ . \tag{1.15}$$

Thus the dissolution shuts down as soon as $\zeta(t)$ becomes equal to 1. This boundary condition has the basic properties demanded by the physical problem; it reduces to the Dirichlet condition in the absence of adsorbed $C$, it reduces to the Neumann condition when the surface is covered, and it makes a continuous monotone transition between these two conditions as a function of the fraction of surface area that is covered.

Remark 1.1. The parameter $\delta$ in (1.13) ensures that $\zeta(0) > 0$ and therefore the boundary condition in (1.14) does not degenerate at $t = 0$. All the results of this paper, however, except uniqueness, extend to the case $\delta = 0$ by simply going to the limit with $\delta \to 0$. If $\delta = 0$, the solution is not smooth at $(R_0, 0)$ and our proof of uniqueness (for the case $\delta > 0$) does not carry through.

For additional information on the model see [9; Chap. 18].

## §2. The main results.

Definition 2.1. We refer to the system (1.1)–(1.15) as Problem $(P)$. By a solution to Problem $(P)$ we mean $(A, B, C, R)$ satisfying (1.1)–(1.15) in the classical case; in particular, $R(t)$ is continuously differentiable for all $t \geq 0$.

Definition 2.2. Suppose $T^*$ is such that

$$\zeta(t) < 1 \quad \text{if} \quad t < T^* \ , \qquad \zeta(t) \geq 1 \quad \text{if} \quad t \geq T^* \ .$$

Then we call $T^*$ the *shut-down time*.

Note that (1.4) and (1.14) reduce to

$$R(t) = R(T^*), \ A_r(R(T^*), t) = 0 \quad \text{for} \quad t > T^* \ . \tag{2.1}$$

Theorem 2.1. *There exists a unique solution of $(P)$, and it has the following properties: (i) $R(t) > 0$, $R'(t) \leq 0$ and $\zeta'(t) > 0$ for all $t > 0$; (ii) it has a finite shut-down time $T^*$ and $R'(t) < 0$ if $0 < t < T^*$, and (iii) $R$ and $\zeta$ belong to $C^{1+\mu}[0, \infty) \cap C^\infty(0, T^*)$ for any $0 < \mu < 1$.*

We are interested in the case of fast reaction, that is, $K \gg 1$. This motivates the study of the solution

$$(A_K, \ B_K, \ C_K, \ R_K) \quad \text{of} \quad (P)$$

as $K \to \infty$. It can be shown that

$$\int_0^T \int_{R_K(t)}^{\infty} A_K B_K \le \frac{\text{const.}}{K} \ ,$$

and that

$$\frac{\partial}{\partial r} \ A_K \le 0, \ \ \frac{\partial}{\partial r} \ B_K \ge 0 \ .$$

It follows, formally, that the limits

$$\begin{aligned} A &= \lim A_K, \ \ B = \lim B_K, \ \ C = \lim C_K \ , \\ R &= \lim R_K \quad \text{and} \quad \zeta = \lim \zeta_K \end{aligned} \tag{2.2}$$

are such that $AB = 0$, i.e.,

$$\begin{aligned} A(r,t) &> 0 \quad \text{if} \quad R(t) < r < S(t) \ , \\ &= 0 \quad \text{if} \quad r > S(t) \ , \end{aligned} \tag{2.3}$$

$$\begin{aligned} B(r,t) &> 0 \quad \text{if} \quad r > S(t) \\ &= 0 \quad \text{if} \quad R(t) < r < S(t) \end{aligned} \tag{2.4}$$

for some function $S(t)$; furthermore,

$$\frac{\partial A}{\partial t} = D_A \Delta A \quad \text{if} \quad R(t) < r < S(t), \quad t > 0 \ , \tag{2.5}$$

$$\frac{\partial B}{\partial t} = D_B \Delta B \quad \text{if} \quad S(t) < r < \infty, \quad t > 0 \ , \tag{2.6}$$

$$\frac{\partial C}{\partial t} = D_C \Delta C - \left( D_A \frac{\partial A}{\partial r}\Big|_{r=S(t)} \right) \delta(r - S(t)) \quad \text{if} \quad R(t) < r < \infty, \ t > 0 \ , \tag{2.7}$$

$$\frac{dR}{dt} = \alpha \frac{\partial A}{\partial r}\Big|_{r=R(t)} \ , \tag{2.8}$$

$$- \zeta_0 D_A \frac{\partial A}{\partial r} + (1 - \zeta_0)(A - A^*) = 0 \quad \text{at} \quad r = R(t) \tag{2.9}$$

where $\zeta_0$ is defined by (1.13), (1.15),

$$D_C \frac{\partial C}{\partial r} = \gamma C^4 \quad \text{on} \quad r = R(t) \ , \tag{2.10}$$

$$A(S(t), t) = 0 \ , \tag{2.11}$$

$$B(S(t), t) = 0 \ , \tag{2.12}$$

$$D_A \frac{\partial A}{\partial r} = -D_B \frac{\partial B}{\partial r} \quad \text{on} \quad r = S(t) \; ; \tag{2.13}$$

since $A+B \to C$, the generation of $C$ which occurs at $r = S(t)$ is at the same rate as $D_A \partial A/\partial r$, or $-D_B \partial B/\partial r$, which explains both (2.13) and the source term in (2.7).

We finally have the initial conditions:

$$\begin{aligned} A(r,0) &= A_0(r) \quad \text{if} \quad R(0) < r < S(0) \; , \\ B(r,0) &= B_0(r) \quad \text{if} \quad S(0) < r < \infty \; , \\ C(r,0) &= C_0(r) \quad \text{if} \quad R(0) < r < \infty \; , \end{aligned} \tag{2.14}$$

$$S(0) = r_0 \quad \text{where } r_0 \text{ is such that } A_0(r_0) = B_0(r_0) \tag{2.15}$$

($r_0$ is uniquely determined), and the conditions at $r = \infty$:

$$B(\infty, t) = B^*, \; C(\infty, t) = 0 \; . \tag{2.16}$$

Definition 2.3. We shall denote the problem $(P)$ by $(P_K)$ and refer to the system (2.5)–(2.16) as Problem $(P_\infty)$. By a solution to Problem $(P_\infty)$ we mean $(A, B, C, R, S, T_f)$ such that all the equations are satisfied in the classical case for $0 < t < T_f$, and

$$R(t) < S(t) < \infty \quad \text{if} \quad t < T_f \; ,$$

$$S(t) - R(t) \to 0 \quad \text{if} \quad t \to T_f \; ;$$

in particular, $R(t)$ and $S(t)$ are continuously differentiable for $0 < t < T_f$, and continuous for $0 \le t \le T_f$, and $A_r(r,t)$ is continuous for $R(t) \le r \le S(t)$.

The curve $r = S(t)$ is the interface between the separated phases $A$ and $B$, and $T_f$ is the *final time*, i.e., the time at which phase $A$ has totally disappeared.

Theorem 2.2. *Assume that $D_A = D_B$ and $C_0(r) \not\equiv 0$. Then there exists a unique solution to Problem $(P_\infty)$, and it has the following properties: (i) $R(t) > 0$, $R'(t) \le 0$ and $\zeta'(t) > 0$ for all $0 < t < T_f$, and $R'(t) < 0$ as long as $\zeta(t) < 1$; (ii) $T_f < \infty$, and (iii) $R, S$ and $\zeta$ belong to $C^0[0, T_f] \cap C^{1+\mu}[0, T_f) \cap C^\infty(0, T_f)$.*

Theorem 2.3. *Assume that $D_A = D_B$. Then, as $K \to \infty$, the limits in (2.2) exist where $(A, B, C, R, S, T_f)$ is the solution to Problem $(P_\infty)$; the convergence of $A_K, B_K, C_K$ is uniform in any compact subset of*

$$\{R(t) < r < S(t), \; 0 \le t < T_f\} \cup \{S(t) < r < \infty, \; 0 \le t < T_f\} \; , \tag{2.17}$$

*and the convergence of $R_K$ and $\zeta_K$ is uniform for $0 \le t < T_f$.*

Theorem 2.1 is proved in Sections 3–6, and Theorems 2.2, 2.3 are proved in Sections 7–10.

Remark 1. Reaction-diffusion systems of the form

$$A_t = D_A \Delta A - KAB, \quad B_t = D_B \Delta B - KAB \tag{2.18}$$

with $K \to \infty$ have been studied in [1] [12]. Evans [5] considered (2.18) in a fixed cylinder $\Omega \times (0,T)$ under the assumptions

$$\frac{\partial A}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \quad \frac{\partial B}{\partial n} = 0 \text{ on } \partial\Omega, \quad A_0 B_0 = 0 \tag{2.19}$$

where $A_0, B_0$ are the nonnegative initial values for $A$ and $B$ respectively. He proved that, as $K \to \infty, A \to u^+$ and $B \to u^-$ where $u$ is the solution of

$$u_t = \text{div}(a(u)\nabla u) = 0 \quad \text{in} \quad \Omega \times (0,T)$$

with $\partial u/\partial n = 0$ on $\partial\Omega$ and $u|_{t=0} = A_0 - B_0$; here

$$a(u) = D_A \quad \text{if} \quad u > 0\ , \ a(u) = D_B \quad \text{if} \quad u < 0\ .$$

(Uniqueness for the limit problem was established in [2].) His proof relies heavily on the assumptions in (2.19) and does not seem to extend to the present case where we have a moving boundary $r = R(t)$ and different boundary conditions than in (2.19).

REMARK 2. For the general study of the Stefan problem in $n$-dimensions we refer to [8] [10] and the references therein. In the case of radially symmetric solutions, existence, uniqueness and regularity have been established by several methods (see [6] [7] and the references therein). In the standard Stefan problem one assumes that $A$ vanishes on the free boundary. The condition (1.14) is called a "kinetic" condition. A Stefan problem for one heat equation in one-dimension with kinetic condition, was studied by Visintin [14] and Xie [15].

## §3. Local existence for (P).

In this section we prove:

THEOREM 3.1. *There exists a solution $(A,B,C,R)$ of Problem $(P)$ for $0 < t < T$, where $T$ is some small positive number.*

*Proof*. The proof is based on a fixed-point argument. Set, for any $N_0 > 0$,

$$K_R = \{R(t) \in C^{0,1}[0,T], R(0) = R_0,\ -M \le \dot{R}(t) \le 0 \text{ a.e.}\}\ ,$$

$$K_\zeta = \{\zeta(t) \in C^{0,1}[0,T], \frac{\delta}{R_0^2} \le \zeta(t) \le N_0,\ 0 \le \dot{\zeta}(t) \le N \text{ a.e.}\}$$

where

$$M = \frac{\alpha}{D_A}\,\frac{A^* R_0^2}{\delta}$$

and $N$ is a positive constant to be determined. We endow $K_R$ and $K_\zeta$ with the $C^0[0,T]$-norm; then $K_R \times K_\zeta$ is a compact set in $C^0[0,T] \times C^0[0,T]$.

For each $(R(t), \zeta(t)) \in K_R \times K_\zeta$ there exists a unique solution $(A(r,t),\ B(r,t))$ of (1.1), (1.2), (1.5), (1.14), (1.7) with the initial conditions as in (1.9); since the parabolic system is weakly coupled, such a solution exists for any given time $T$. By the maximum principle,

$$0 \le A(r,t) \le A^*,\ 0 \le B(r,t) \le B^*\ . \tag{3.1}$$

We next prove that

$$A_r(r,t) \le 0,\ B_r(r,t) \le 0\ . \tag{3.2}$$

If we differentiate (1.1), (1.2) with respect to $r$, we get a coupled system of parabolic equations

$$\frac{\partial}{\partial t}A_r - \mathcal{L}A_r = -KAB_r \ ,$$
$$\frac{\partial}{\partial t}B_r - \mathcal{L}B_r = -KBA_r$$

where $\mathcal{L}$ is an elliptic operator. On $t = 0$ and on the boundary $r = R(t)$ we have $A_r \le 0,\ B_r \ge 0$. We approximate $A_r, B_r$ by solutions $A_r^\varepsilon, B_r^\varepsilon$ satisfying the same parabolic system, with initial and boundary conditions given by

$$A_r^\varepsilon = A_r - \varepsilon,\ B_r^\varepsilon = B_r + \varepsilon \ .$$

Then $A_r^\varepsilon < 0,\ B_r^\varepsilon > 0$ for $R(t) \le r < \infty, 0 \le t \le T$. Indeed, otherwise there is a smallest $t_0$ such that $A_r^\varepsilon \le 0,\ B_r^\varepsilon \ge 0$ for $R(t) \le r < \infty,\ 0 \le t \le t_0$, and $A_r^\varepsilon = 0$ or $B_r^\varepsilon = 0$ at some point $(r_0, t_0)$. This is a contradiction to the strong maximum principle applied to $A_r^\varepsilon$ or to $B_r^\varepsilon$.

If we now let $\varepsilon \to 0$, we obtain the assertion (3.2).

Motivated by (1.4), (1.14), we now define

$$\overline{R}(t) = R_0 + \frac{\alpha}{D_A}\int_0^t \frac{1-\zeta_0(s)}{\zeta_0(s)}[A(R(s),s) - A^*]ds \ . \tag{3.3}$$

Next we consider the parabolic equation (1.3) in $\overline{R}(t) < r < \infty,\ 0 < t < T$ with boundary conditions (1.6) and $C(\infty, t) = 0$, and with initial conditions $C(r, 0) = C_0(r)$. Since the functions $KA^*B^*t + \widehat{C}\quad (\widehat{C} = \sup C_0)$ and 0 are supersolution and sub-solution, respectively, the existence of a solution can be established by a fixed-point argument (cf. [6; Chap. 7, Sec. 5]). Uniqueness follows by a comparison principle [6].

We now define

$$\overline{\zeta}(t) = \frac{\beta\gamma \int_0^t \overline{R}^2(s)C^4(\overline{R}(s), s) + \delta}{\overline{R}^2(t)} \tag{3.4}$$

and consider the mapping $W$:

$$W(R(t),\ \zeta(t)) = (\overline{R}(t),\ \overline{\zeta}(t)) \ .$$

If we show that $W$ has a fixed point in $K_R \times K_\zeta$, then this yields a solution to Problem $(P)$.

LEMMA 3.2. *$W$ maps $K_R \times K_\zeta$ into itself.*

*Proof*. From (3.3), (3.4) we get

$$\frac{d}{dt}\ \overline{R}(t) = \frac{\alpha}{D_A}\ \frac{1-\zeta_0(t)}{\zeta_0(t)}\ (A - A^*), \tag{3.5}$$

$$\frac{d}{dt}\ \overline{\zeta}(t) = \beta\gamma C^4(\overline{R}(t), t) - \frac{2}{\overline{R}(t)}\ \overline{\zeta}(t)\frac{d\overline{R}(t)}{dt} \ . \tag{3.6}$$

From (3.5) we see that

$$-M \le \frac{d\overline{R}(t)}{dt} \le 0 \tag{3.7}$$

Since $C \le KA^*B^*t + \widehat{C}$ we find from (3.4) that $\overline{\zeta}(t) \le N_0$ if $T$ is small enough. From (3.6) and (3.7) it then follow that

$$0 < \frac{d}{dt}\overline{\zeta}(t) \le \widehat{N}$$

provided $\widehat{N}$ is a sufficiently large number independent of $N$ (see the definition of $K_\zeta$). Hence, choosing $N = \widehat{N}$ we see that $(\overline{R}, \overline{\zeta})$ belongs to $K_R \times K_\zeta$.

LEMMA 3.3. *$W$ is continuous (when $K_R \times K_\zeta$ is endowed with the $C^0[0,T] \times C^0[0,T]$ norm).*

*Proof*. If we use the transformation $\widehat{r} = r - R(t)$ in order to flatten the boundary $r = R(t)$, we get a new parabolic equation for $A$ where a new term $\dot{R}A_r$ is added to the heat operator. On the lateral boundary $\widehat{r} = 0$

$$-A_r + a(t)A = b(t)$$

where $a, b$ are Lipschitz continuous and their Lipschitz constants are bounded independently of $(R, \zeta)$ in $K_R \times K_\zeta$. We can therefore apply $L^p$-estimates [10] to deduce that

$$\|A_r\|_{L^p(\Omega_T)} + \|A_{rr}\|_{L^p(\Omega_T)} + \|A_t\|_{L^p(\Omega_T)} \le C_1 \tag{3.8}$$

for any $1 < p < \infty$, where

$$\Omega_T = \{R(t) \le r < \infty,\ 0 \le t \le T\}$$

and $C_1$ is a constant depending on $p$ but not on $(R, \zeta)$ and $T$. By Sobolev's imbedding [11] we then have the Hölder estimate

$$\|A_r\|_{C^\mu(\Omega_T)} \le C_2 \tag{3.9}$$

for some $0 < \mu < 1$ and $C_2$ independent of $T$ and $(R, \zeta)$.

We now proceed to prove that $W$ is continuous.

Suppose $(R_n, \zeta_n)$ and $(R, \zeta)$ belong to $K_R \times K_\zeta$ and $R_n \to R,\ \zeta_n \to \zeta$ in the $C^0[0,T]$-norm. We need to prove that

$$W(R_n, \zeta_n) \to W(R, \zeta)\ .$$

Define $A_n, B_n, \overline{\zeta}_n, \overline{R}_n$ and $C_n$ corresponding to $R_n, \zeta_n$, so that $W(R_n, \zeta_n) = (\overline{R}_n, \overline{\zeta}_n)$. Applying the estimates (3.8), (3.9) to $A_n$ and similar estimates to $B_n$ and $C_n$, we can easily show that any subsequence of $n$'s has a subsequence for which

$$A_n \to \widetilde{A}\ ,\ \ B_n \to \widetilde{B}$$

where $\widetilde{A}, \widetilde{B}$ satisfy the same parabolic system in $R(t) < r < \infty,\ 0 < t < T$ which $A, B$ satisfy. By uniqueness it follows that $\widetilde{A} = A$ and $\widetilde{B} = B$. Therefore

$$A_n \to A,\ B_n \to B \quad \text{uniformly in} \quad \rho_n(t) \le r < \infty,\ 0 \le t \le T \tag{3.10}$$

where $\rho_n(t) = \max\{R_n(t),\ R(t)\}$.

We proceed to prove that

$$\begin{aligned}\overline{R}_n(t) - \overline{R}(t) = \frac{\alpha}{D_A}\Big\{ &\int_0^t \frac{1-\zeta_{n,0}(s)}{\zeta_{n,0}(s)}[A_n(R_n(s),s) - A^*]ds \\ &- \int_0^t \frac{1-\zeta_0(s)}{\zeta_0(s)}[A(R(s),s) - A^*]ds\Big\} \to 0\end{aligned} \tag{3.11}$$

where $\zeta_{n,0} = \min\{\zeta_n, 1\}$. By (3.9)

$$|A_n(R_n(s),s) - A_n(\rho_n(s),s)| \le C_1|R_n(s) - R(s)|^\mu \ ,$$

$$|A(R(s),s) - A(\rho_n(s),s)| \le C_1|R_n(s) - R(s)|^\mu \ .$$

Using also (3.10) we conclude that

$$\Big|\frac{\alpha}{D_A}\int_0^t \frac{1-\zeta_{n,0}(s)}{\zeta_{n,0}(s)}[A_n(R_n(s),s) - A(R(s),s)]ds\Big| \le C_2\int_0^t |R_n(s) - R(s)|^\mu ds + \varepsilon_n$$

where $\varepsilon_n \to 0$ uniformly in $t$, as $n \to \infty$. Since $\zeta_n(t) \to \zeta(t)$ uniformly, we deduce from the expression for $\overline{R}_n - \overline{R}$ in (3.11) that

$$|\overline{R}_n(t) - \overline{R}(t)| \to 0 \quad \text{uniformly in} \quad t \in [0,T] \ .$$

Similarly we can prove that $C_n \to C$ for $\overline{R}(t) < r < \infty$ , $0 \le t \le T$ and $\overline{\zeta}_n(t) \to \overline{\zeta}(t)$ uniformly in $0 \le t \le T$, and this completes the proof of Lemma 3.3.

From Lemmas 3.2, 3.3 we have that $W$ maps the compact set $K_R \times K_\zeta$ into itself and is continuous. By the Schauder fixed-point theorem, $W$ has a fixed point $(R, \zeta)$ and this completes the proof of Theorem 3.1.

## §4. Uniqueness.

In this section we prove:

**Theorem 4.1.** *For any $T > 0$, Problem $(P)$ has at most one solution.*

*Proof*. Suppose there are two solutions $(A_i, B_i, C_i, R_i, \zeta_i)$ $\ (i = 1, 2)$. Set

$$\widehat{A}_i(x,t) = A_i(r,t),\ \widehat{B}_i(x,t) = B_i(r,t),\ \widehat{C}_i(x,t) = C_i(r,t)$$

where $x = r - R_i(t)$. Then

$$\begin{aligned}&\widehat{A}_{i,t} = D_A\left(\widehat{A}_{i,xx} + \frac{2}{x+R_i}\ \widehat{A}_{i,x}\right) + \dot{R}_i\widehat{A}_{i,x} - K\widehat{A}_i\widehat{B}_i \\ &\qquad\qquad\qquad\qquad \text{in} \quad Q_T = \{x > 0,\ t > 0\}\ , \\ &-D_A\zeta_{i,0}\widehat{A}_{i,x} + (1-\zeta_{i,0})(\widehat{A}_i - A^*) = 0 \quad \text{for} \quad x = 0\ ,\ t > 0\ , \\ &\widehat{A}_i(x,0) = A_0(x+R_0) \quad \text{for} \quad x > 0\ , \\ &\lim_{x\to\infty}\widehat{A}_i(x,t) = 0 \quad \text{for} \quad t > 0\ .\end{aligned} \tag{4.1}$$

Similar systems can be written for $\widehat{B}_i$ and $\widehat{C}_i$. Set

$$u = \widehat{A}_1 - \widehat{A}_2,\ v = \widehat{B}_1 - \widehat{B}_2,\ w = \widehat{C}_1 - \widehat{C}_2$$

and

$$\rho = R_1 - R_2\ ,\ \eta = \zeta_1 - \zeta_2\ .$$

Then

$$\dot{\rho}(t) = \frac{\alpha}{D_A}\left[\frac{1-\zeta_{1,0}(t)}{\zeta_{1,0}(t)}(\widehat{A}_1(x,0) - A^*) - \frac{1-\zeta_{2,0}(t)}{\zeta_{2,0}(t)}(\widehat{A}_2(x,0) - A^*)\right]$$

so that

$$|\dot{\rho}(t)| \le C[|u(0,t)| + |\eta(t)|]\ . \tag{4.2}$$

Next we easily estimate from (1.13),

$$|\eta(t)| \le C\left[\|\rho\|_{L^\infty(0,T)} + \|w(0,t)\|_{L^\infty(0,T)}\right]\ . \tag{4.3}$$

Since $\rho(0) = 0$, (4.2) yields

$$\|\rho\|_{L^\infty(0,T)} \le CT\left[\|u(0,t)\|_{L^\infty(0,T)} + \|\eta\|_{L^\infty(0,T)}\right]\ . \tag{4.4}$$

Substituting this into (4.3), we get

$$\|\eta\|_{L^\infty(0,T)} \le \frac{CT}{1-CT}\left[\|u(0,t)\|_{L^\infty(0,T)} + \|w(0,t)\|_{L^\infty(0,T)}\right] \tag{4.5}$$

if $T$ is such that $CT < 1$. Finally, using (4.5) in (4.4) we get

$$\|\rho\|_{L^\infty(0,T)} \le CT\left[\|u(0,t)\|_{L^\infty(0,T)} + \|w(0,t)\|_{L^\infty(0,T)}\right]\ . \tag{4.6}$$

From (4.1) we see that $u$ satisfies a parabolic system

$$\begin{aligned}
&u_t = D_A u_{xx} + \left(\frac{2D_A}{x+R_1} + \dot{R}_1\right)u_x - K\widehat{B}_1 u + F \quad \text{in} \quad Q_T\ ,\\
&-D_A\zeta_{1,0}u_x + (1-\zeta_{1,0})u = G \quad \text{if} \quad x=0,\ t>0\ ,\\
&u(x,0) = 0,\ x>0\\
&\lim_{x\to\infty} u(x,t) = 0\ ,\ t>0\ .
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
|F| &\le C\left[\|\dot{\rho}\|_{L^\infty(0,T)} + \|v\|_{L^\infty(Q_T)}\right]\\
&\le C\left[\|u\|_{L^\infty(Q_T)} + \|v\|_{L^\infty(Q_T)} + \|w\|_{L^\infty(Q_T)}\right]
\end{aligned}$$

by (4.2), (4.5), and

$$|G| \le C|\eta(t)| \le CT\left[\|u\|_{L^\infty(Q_T)} + \|w\|_{L^\infty(Q_T)}\right]\ .$$

It is easy to see that the function

$$C(t+T)\cdot\left[\|u\|_{L^\infty(Q_T)}+\|v\|_{L^\infty(Q_T)}+\|w\|_{L^\infty(Q_T)}\right]$$

is a supersolution to (4.7), so that

$$\|u\|_{L^\infty(Q_T)}\le CT\left[\|u\|_{L^\infty(Q_T)}+\|v\|_{L^\infty(Q_T)}+\|w\|_{L^\infty(Q_T)}\right]\ .$$

The same estimate can similarly be established for $v$ and for $w$. Hence if $T$ is small enough then $u=v=w=0$ in $Q_T$. We can now proceed step-by-step to prove that $u=v=w=0$ for all $t$, as long as the two solutions exist.

§**5. Global existence.** In this section we prove that there exists a solution to Problem $(P)$ for all time. We first recall that as long as $\zeta(t)\le 1$

$$\dot R(t)=\frac{\alpha}{D_A}\,\frac{1-\zeta(t)}{\zeta(t)}(A-A^*)\ , \tag{5.1}$$

$$\dot\zeta(t)=\beta\gamma C^4(R(t),t)-\frac{2\zeta(t)}{R(t)}\ \dot R(t)\ . \tag{5.2}$$

Hence

$$\dot R(t)<0\ ,\ \dot\zeta(t)>0\ . \tag{5.3}$$

Lemma 5.1. *There exists a positive constant $R_*$ such that*

$$R(t)\ge R_*\quad\text{as long as}\quad \zeta(t)\le 1\ . \tag{5.4}$$

*Proof*. Take $\lambda$ and $\varepsilon_1$ positive and small such that

$$C^4(R(t),t)\le\varepsilon_1\quad\text{and}\quad \frac{1}{2}\ R_0<R(t)$$

for all $\lambda\le t\le 2\lambda$. For $t>2\lambda$

$$\zeta(t)\ge\frac{\beta\gamma\varepsilon_1\lambda+\delta}{R^2(t)}$$

Hence $R(t)>R_*$ as long as $\zeta(t)\le 1$, where $R_*=(\beta\gamma\varepsilon_1\lambda)^{1/2}$.

Remark 5.1. Note that the lower bound $R_*$ is independent of the regularizing parameter $\delta$; cf. Remark 1.1.

Theorem 5.2. *There exists a global solution to Problem $(P)$, and $R\in C^{1+\mu}[0,\infty)\cap C^\infty(0,T^*)$, for any $0<\mu<1$.*

*Proof*. Suppose we already have a solution for $0\le t\le T$ where $T$ is a positive number, not necessarily small. By Lemma 5.1, $R(t)\ge R_*$ for $0\le t\le T$.

Since $R(t)\ge R_*>0$ ($R_*$ independent of $t$), a review of the proof of local existence shows that the solution can be extended to $0\le t\le T+T_0$ provided $T_0$ is a small positive constant depending only on an a priori bound on $\sup|\dot R(t)|$. By (5.1) $|\dot R(t)|$ is indeed uniformly bounded (independently on $t$), and therefore the solution to Problem $(P)$ can be extended step-by-step to all $t > 0$.

To prove the a priori regularity of $R$ and $\zeta$, we perform a change of variables

$$\widehat{A}(x,t) = A(r,t),\ \widehat{B}(x,t) = B(r,t),\ \widehat{C}(x,t) = C(r,t)$$

where $x = r - R(t)$. Then $\widehat{A}$ satisfies, for any $T > 0$,

$$\begin{aligned}
&\widehat{A}_t = D_A\left(\widehat{A}_{xx} + \frac{2}{x+R}\,\widehat{A}_x\right) + \dot{R}A_x - K\widehat{A}\widehat{B} \quad \text{in} \quad Q_T\ ,\\
&-D_A\zeta_0\widehat{A}_x + (1-\zeta_0)(\widehat{A} - A^*) = 0 \quad \text{for} \quad x = 0,\ \ 0 < t < T\ ,\\
&\widehat{A}(x,0) = A_0(x+R_0)\ ,\ x > 0\ ,\\
&\lim_{x\to\infty}\widehat{A}(x,t) = 0\ ,\ 0 < t < T\ .
\end{aligned} \tag{5.5}$$

Since $\dot{R}(t)$ is uniformly bounded, the same is true of $\dot{\zeta}(t)$, by (5.2) (Recall that $C(R(t),t)$ is bounded by $KA^*B^*T + \widehat{C}$). We can therefore apply the $L^p$ parabolic estimates [10] to $\widehat{A}$ and conclude that

$$\int\limits_{Q_T}(|\widehat{A}_r| + |\widehat{A}_{rr}| + |\widehat{A}_t|)^p \le C_{p,T}$$

for any $p > 1$. This implies that

$$\widehat{A} \in C^{\mu,\mu/2}_{r,t}(Q_T) \quad \text{for} \quad 0 < \mu < 1\ ,$$

and yields the $C^{1+\mu/2}[0,T]$ regularity of $R(t)$.

Similar $L^p$ estimates can be established for $\widehat{B}$ and $\widehat{C}$ and then, from (5.2), one deduces the $C^{1+\mu/2}[0,T]$ regularity of $\zeta(t)$.

The above arguments can be used to prove step-by-step the $C^{1+\mu}[0,T]$ and $C^\infty(0,T)$ regularity of $R(t)$ and $\zeta(t)$ for any $0 < T \le T^*$

## §6. Finite shut-down.

LEMMA 6.1. *If $\zeta(t) < 1$ for all $t > 0$ then*

$$\lim_{t\to\infty}\zeta(t) = 1 \tag{6.1}$$

and

$$\lim_{t\to\infty}\dot{R}(t) = 0\ . \tag{6.2}$$

*Proof*. The idea of the proof is to show by comparison (with linear problems) that both $A$ and $B$ do not go to zero as $t\to\infty$, and the same is then true of $C$; this implies that $\zeta(t)\to\infty$ as $t\to\infty$, which is a contradiction. Set

$$\rho = \lim_{t\to\infty}R(t) > 0\ ,\quad \sigma = \lim_{t\to\infty}\zeta(t)\ .$$

We shall assume that $\sigma < 1$ and derive a contradiction. We begin by introducing the solution $\widetilde{A}, \widetilde{B}$ to

$$
\begin{aligned}
&\widetilde{A}_t = D_A \Delta \widetilde{A} - K B^* \widetilde{A} \quad \text{for} \quad r > R(t),\ t > 0\ , \\
&-D_A \zeta(t) \widetilde{A}_r + (1 - \zeta(t))(\widetilde{A} - A^*) = 0 \quad \text{if} \quad r = R(t),\ t > 0\ , \\
&\widetilde{A}(r, 0) = A_0(r) \quad \text{if} \quad r > R_0\ , \\
&\lim_{r \to \infty} \widetilde{A}(r, t) = 0 \quad \text{for any} \quad t > 0\ ,
\end{aligned} \tag{6.3}
$$

and

$$
\begin{aligned}
&\widetilde{B}_t = D_B \Delta \widetilde{B} - K A^* \widetilde{B} \quad \text{for} \quad r > R(t),\ t > 0\ , \\
&\widetilde{B}_r = 0 \quad \text{if} \quad r = R(t),\ t > 0\ , \\
&\widetilde{B}(r, 0) = B_0(r) \quad \text{if} \quad r > R_0\ , \\
&\lim_{r \to \infty} \widetilde{B}(r, t) = B^* \quad \text{for any} \quad t > 0\ .
\end{aligned} \tag{6.4}
$$

By comparison, $\widetilde{A} \leq A,\ \widetilde{B} \leq B$.

Let $\widetilde{A}_\infty,\ \widetilde{B}_\infty$ denote the solutions to the limit problems ($t \to \infty$) (6.3) and (6.4):

$$
\begin{aligned}
&D_A \Delta \widetilde{A}_\infty - K B^* \widetilde{A}_\infty = 0\ ,\ \rho < r < \infty\ , \\
&-D_A \sigma \widetilde{A}_{\infty, r} + (1 - \sigma)(\widetilde{A}_\infty - A^*) = 0\ ,\ r = \rho\ , \\
&\lim_{r \to \infty} \widetilde{A}_\infty(r) = 0
\end{aligned} \tag{6.5}
$$

and

$$
\begin{aligned}
&D_B \Delta \widetilde{B}_\infty - K A^* \widetilde{B}_\infty = 0\ ,\ \rho < r < \infty\ , \\
&\widetilde{B}_{\infty, r}(\rho) = 0\ , \\
&\lim_{r \to \infty} \widetilde{B}_\infty(r) = B^*\ .
\end{aligned} \tag{6.6}
$$

Note that, since $\sigma < 1$, $\widetilde{A}_\infty(r) \geq 0$ for all $r > \rho$; $\widetilde{B}_\infty(r)$ is positive for all $r \geq \rho$.

We shall prove that the functions

$$
\widetilde{u}(r, t) = \widetilde{A}(r, t) - \widetilde{A}_\infty(r),\ \ \widetilde{v}(r, t) = \widetilde{B}(r, t) - \widetilde{B}_\infty(r)
$$

converge to zero as $t \to \infty$.

The function $\widetilde{u}$ satisfies

$$
\begin{aligned}
&\widetilde{u}_t = D_A \Delta \widetilde{u} - K B^* \widetilde{u} \quad \text{if} \quad r > R(t), t > 0 \\
&-D_A \sigma \widetilde{u}_r + (1 - \sigma) \widetilde{u} = \widetilde{\varepsilon}(t) \quad \text{if} \quad r = R(t),\ t > 0
\end{aligned} \tag{6.7}
$$

where $\widetilde{\varepsilon}(t) \to 0$ as $t \to \infty$. For any $\varepsilon > 0$,

$$
\frac{1}{\sigma}\ |\widetilde{\varepsilon}(t)| < \varepsilon \quad \text{if} \quad t > t_\varepsilon\ ,
$$

and the function

$$\widetilde{U}(r,t) = A^* e^{-KB^*(t-t_\varepsilon)} + \frac{\varepsilon}{r}(R(t_\varepsilon))^2 \tag{6.8}$$

is then a supersolution of (6.7) in $r > R(t),\ t > t_\varepsilon$, which majorizes $|\tilde{u}|$ both on $t = t_\varepsilon$ and as $r \to \infty$. Hence, by comparison,

$$|\tilde{u}(r,t)| \le \widetilde{U}(r,t)\ .$$

Taking $t \to \infty$ and recalling that $\varepsilon$ is arbitrary, we conclude that

$$\sup_{r \ge R(t)} |\widetilde{A}(r,t) - \widetilde{A}_\infty(r)| \to 0 \quad \text{if} \quad t \to \infty\ . \tag{6.9}$$

Similarly, the function $v$ satisfies

$$\tilde{v}_t = D_B \Delta \tilde{v} - KA^* \tilde{v} \quad \text{for} \quad r > R(t),\ t > 0\ ,$$

$$v_r(R(t),t) \to 0 \quad \text{if} \quad t \to \infty$$

and, with a supersolution similar to (6.8), we conclude that

$$\sup_{r \ge R(t)} |\widetilde{B}(r,t) - \widetilde{B}_\infty(r)| \to 0 \quad \text{if} \quad t \to \infty\ . \tag{6.10}$$

Since $\widetilde{A}_\infty(r) > 0,\ \widetilde{B}_\infty(r) > 0$ for $r > \rho$, it follows from (6.9), (6.10) that

$$\widetilde{A}(r,t)\widetilde{B}(r,t) \ge c_0 \chi_{[a,b]} \quad \text{for} \quad t \ge t_* \tag{6.11}$$

where $c_0$ is a positive constant, $\rho < a < b < \infty$, and $t_*$ is sufficiently large.

Consider the parabolic problem

$$\begin{aligned} &\widetilde{C}_t = D_C \Delta \widetilde{C} + K c_0 \chi_{[a,b]} \quad \text{if} \quad r > R(t),\ t > t_*\ , \\ &D_C \widetilde{C}_r = \gamma \widetilde{C}^4 \quad \text{if} \quad r = R(t),\ t > t_*\ , \\ &\widetilde{C}(r,t_*) = 0 \quad \text{if} \quad r > R(t_*)\ , \\ &\lim_{r\to\infty} \widetilde{C}(r,t) = 0 \quad \text{if} \quad t > t_*\ . \end{aligned} \tag{6.12}$$

Since $AB \ge \widetilde{A}\widetilde{B}$, it follows from (6.11), by comparison, that

$$C(r,t) \ge \widetilde{C}(r,t)\ . \tag{6.13}$$

Denote by $\widetilde{C}_\infty$ the solution to

$$\begin{aligned} &D_C \Delta \widetilde{C}_\infty + K c_0 \chi_{[a,b]} = 0\ , \quad r > \rho\ , \\ &D_C \widetilde{C}_{\infty,r} = \gamma \widetilde{C}_\infty^4\ ,\ r = \rho\ , \\ &\widetilde{C}_\infty(\infty) = 0\ ; \end{aligned} \tag{6.14}$$

clearly $\widetilde{C}_\infty(r) > 0$ for $r \geq \rho$. We shall evaluate the difference

$$\tilde{w}(r,t) = \widetilde{C}(r,t) - \widetilde{C}_\infty(r) \ . \tag{6.15}$$

It satisfies

$$\begin{aligned} &\tilde{w}_t = D_C \Delta w \quad \text{if} \quad r > R(t),\ t > t_* \ , \\ &-D_C w_r + a(t) w = \varepsilon(t) \quad \text{if} \quad r = R(t),\ t > 0 \ , \\ &\lim_{r\to\infty} w(r,t) = 0 \quad \text{if} \quad t > t_* \ , \end{aligned}$$

where

$$\varepsilon(t) = D_C \widetilde{C}_{\infty,r}(R(t)) - \gamma \widetilde{C}^4_\infty(R(t)) \to 0 \quad \text{if} \quad t \to \infty$$

and

$$a(t) = (\gamma \widetilde{C}^4 - \gamma \widetilde{C}^4_\infty)/(\widetilde{C} - \widetilde{C}_\infty) > 0 \ .$$

As in the proof of (6.9) we now construct, for any $\varepsilon > 0$, a supersolution (cf. (6.8))

$$\widetilde{W}(r,t) = \frac{M_\varepsilon}{t^{3/2}}\, e^{-\frac{r^2}{4t\sqrt{D_C}}} + \frac{\varepsilon}{r}(R(t_\varepsilon))^2 \quad \text{for} \quad t > t_\varepsilon \tag{6.16}$$

where $D_A|\varepsilon(t)| < \varepsilon$ if $t > t_\varepsilon$. By comparison,

$$|w(r,t)| \leq \widetilde{W}(r,t) \quad \text{if} \quad r > R(t),\ t > t_\varepsilon \ .$$

Recalling (6.15) we conclude that

$$\widetilde{C}(R(t),t) \to \widetilde{C}_\infty(\rho) \quad \text{if} \quad t \to \infty$$

and, by (6.13),

$$C(R(t),t) \geq \nu \quad \text{if} \quad t \geq \tilde{t} \qquad (\nu = \frac{1}{2}\, \widetilde{C}_\infty(\rho) > 0) \tag{6.17}$$

for some $\tilde{t} > 0$.

Since $D_C C_r = \gamma C^4$ at $(R(t),t)$, and since $0 < R_* \leq R(t) < R_0$, we can use (6.17) in (1.13) to deduce that $\zeta(t) \to \infty$ if $t \to \infty$; this is a contradiction to the assumption that $\zeta(t) < 1$ for all $t$. We have thus proved that $\sigma = 1$, i.e., (6.1) holds. The assertion (6.2) then follows from (5.1) and (6.1).

To prove that there is a finite shut-down time we may proceed by contradiction as before, namely, we assume that $\zeta(t) < 1$ for all $t > 0$, and then wish to derive a contradiction by using (1.13). This however requires a good lower bound on $C(R(t),t)$ in case $\sigma = \lim_{t\to\infty} \zeta(t)$ is equal to 1 (Note that if $\sigma = 1$ then $\widetilde{A} = 0$ and the previous proof does not provide a useful lower bound on $C(R(t),t)$.) The desired bound is provided in the following lemma.

Lemma 6.2. *If $\zeta(t) < 1$ for all $t > 0$ then there exists a positive constant $M$ such that*

$$C_r(R(t),t) \geq \frac{M}{t^6} \tag{6.18}$$

*for all $t$ large.*

*Proof*. For any $t_0 > 0$, $C(r, t_0) > 0$ for all $r \ge R(t_0)$. Choose $R(t_0) < a < b < \infty$ such that

$$C(r, t_0) > \varepsilon_0 \chi_{[a,b]}(r) \equiv c_0(r) \tag{6.19}$$

for some $\varepsilon_0 > 0$, and define

$$\widetilde{C}(r,t) = \int_{\mathbb{R}^3} \frac{e^{-\frac{|x-y|^2}{4\sqrt{D_C}\ (t-t_0)}}}{(t-t_0)^{3/2}}\ \varepsilon c_0(|y|) dy\ , \quad 0 < \varepsilon < 1 \tag{6.20}$$

for $r = |x| > R(t),\ t > t_0$. Since $\widetilde{C}(R(t_0), t_0) = 0$,

$$C(R(t), t) > \widetilde{C}(R(t), t) \quad \text{if} \quad t_0 \le t \le t_1$$

for some $t_1 > t_0$. On the other hand, for $r = R(t),\ t \ge t_1$ we have

$$-D_C \widetilde{C}_r + \gamma \widetilde{C}^4 \le -D_C \varepsilon \int_{\{a<|y|<b\}} \frac{|y| - R(t)}{(4\pi)^{3/2} 2\sqrt{D_C}\ (t-t_0)^{5/2}}\ e^{-\frac{|R(t)-y|^2}{4\sqrt{D_C}\ (t-t_0)}}$$
$$+ C_1 \varepsilon^4 < 0$$

if $\varepsilon$ is sufficiently small. Recalling also (6.19), (6.20) and noting that $\widetilde{C}$ is a subsolution of (1.3), we conclude, by comparison, that if $\varepsilon$ is small enough then

$$C(r,t) > \widetilde{C}(r,t) \quad \text{if} \quad t > t_0\ ,$$

from which (6.18) follows upon using (1.6).

We finally prove:

**Theorem 6.3**. *There is a finite shut-down time $T^*$.*

*Proof*. Suppose $\zeta(t) < 1$ for all $t > 0$. We can write, for $r = R(t)$,

$$\dot{\zeta}(t) = \beta D_C C_r - \zeta(t) \frac{2\dot{R}(t)}{R(t)} = \beta D_C C_r - \frac{2\alpha}{R(t)}\ \zeta(t) A_r$$
$$= \beta D_C C_r - \frac{2\alpha}{R(t)} (1 - \zeta(t))(A - A^*)\ .$$

Setting

$$\eta(t) = \zeta(t) - 1, \quad q(t) = \frac{2\alpha}{R(t)} (A^* - A(R(t), t)$$

we can write

$$\dot{\eta} + q(t)\eta = \beta D_C C_r(R(t), t) \equiv p(t)\ , \tag{6.21}$$

so that,

$$\eta(t) = e^{-\int\limits_0^t q(\tau)d\tau} \left[ \eta(0) + \int\limits_0^t p(\tau) e^{-\int\limits_\tau^t q(s)ds} d\tau \right] . \tag{6.22}$$

Consider first the case that there exists a positive constant $\alpha_0$ and a sequence $t_n \to \infty$ such that

$$\frac{1}{t_n} \int\limits_0^{t_n} (A^* - A(R(\tau),\tau))d\tau \geq \alpha_0 \quad \text{as} \quad t_n \to \infty \ ; \tag{6.23}$$

then,

$$\int\limits_0^{t_n} q(\tau)d\tau \geq \frac{2\alpha}{R_0}\ \alpha_0 t_n \quad \text{as} \quad t_n \to \infty \ .$$

By using (6.18) in (6.22), we get

$$\begin{aligned} \eta(t_n) &\geq \eta(0) e^{-\frac{2\alpha\alpha_0}{R_0}\ t_n} + \int\limits_{t_0}^{t_n} \frac{M}{\tau^6}\ e^{-\frac{2\alpha A^*}{R_*}(t_n - \tau)} d\tau \\ &\geq \eta(0) e^{-\frac{2\alpha\alpha_0}{R_0}\ t_n} + \frac{1}{2}\ \frac{R_*}{2\alpha A^*}\ \frac{M}{t_n^6} \end{aligned}$$

for some $t_0 > 0$ and all $t_n$ sufficiently large. This is a contradiction since $\eta(t) = \zeta(t) - 1 < 0$ for all $t > 0$.

It remains to consider the case where (6.23) is not satisfied for any $\alpha_0 > 0$, that is,

$$\frac{1}{t} \int\limits_0^t A(R(s),s)ds \to A^* \quad \text{as} \quad t \to \infty \ . \tag{6.24}$$

Since

$$A_r(R(t),t) = \frac{1}{\alpha}\ \dot{R}(t) \to 0$$

by Lemma 6.1, we can derive, by comparison,

$$A(r,t) \leq \widetilde{W}(r,t)$$

for any $\varepsilon > 0$ and $t > t_\varepsilon$, where $\widetilde{W}$ is the function introduced in (6.16). It follows that

$$A(R(t),t) \to 0 \quad \text{if} \quad t \to \infty \ ,$$

a contradiction to (6.24).

§7. **Asymptotic estimates as $K \to \infty$.** We now study the behavior of the solution $(A_K, B_K, C_K, R_K, \zeta_K)$ as $K \to \infty$, assuming that

$$D_B = D_A \ . \tag{7.1}$$

Recall that

$$0 \le A_K \le A^* \ , \ 0 \le B_K \le B^* \tag{7.2}$$

and

$$\frac{\partial}{\partial r} A_K \le 0 \ , \ \frac{\partial}{\partial r} B_K \ge 0 \ , \ \dot{R}_K(t) \le 0 \ . \tag{7.3}$$

LEMMA 7.1. *There exists a positive constant $M$ such that*

$$- M \le \frac{\partial}{\partial r} A_K, \ \frac{\partial}{\partial r} B_K \le M \ , \ -\alpha M \le \dot{R}_K(t) \tag{7.4}$$

*for all $K$.*

*Proof*. Consider the function $u = A_K - B_K$. It satisfies

$$\begin{aligned} & u_t = D_A u \quad \text{if} \quad r > R_K(t), \ t > 0 \\ & -\zeta_{K,0}(t) D_A u_r = (1 - \zeta_{K,0}(t))(A^* - A) \quad \text{if} \quad r = R_K(t), \ t > 0 \ , \\ & u(r,0) = A_0(r), \ r > R_0 \ , \\ & u(\infty, t) = 0 \ , \ t > 0 \end{aligned} \tag{7.5}$$

where $\zeta_{K,0} = \min\{\zeta_K, 1\}$. Applying the maximum principle to $u_r$ we deduce that

$$- M \le u_r \le 0 \tag{7.6}$$

where $M$ is a constant independent of $K$. But then, upon recalling also (7.3),

$$(A_{K,r})^2 = A_{K,r} u_r + A_{K,r} B_{K,r} \le A_{K,r} u_r \le M |A_{K,r}|$$

so that $|A_{K,r}| \le M$. The proof that $|B_{K,r}| \le M$ is similar. Finally, $\dot{R}_K = \alpha A_{K,r} \le -\alpha M$.

LEMMA 7.2. *There exist positive constants $N_0, N$ independent of $K$ such that, for any $T > 0$,*

$$\int_0^T \int_{R_K(t)}^\infty K A_K B_K r^2 dr dt \le N_0 + NT \tag{7.7}$$

*for all $K$.*

*Proof*. Integrating the equation

$$K A_K B_K = D_A \Delta A_K - \frac{\partial}{\partial t} A_K \tag{7.8}$$

over $R_K(t) < r < \infty$, $0 < t < T$ and using Lemma 7.1, (7.7) readily follows.

In order to obtain uniform Hölder estimates on $A_K, B_K$, we consider the function $v = A_K B_K$. It satisfies:

$$\begin{aligned} v_t &= D_A \Delta v - 2A_{K,r}B_{K,r} - K(A_K + B_K)v \;, \quad r > R_K(t) \;, \\ -\zeta_{K,0}(t) D_A v_r &= (1 - \zeta_{K,0}(t))(A^* - A_K)B_K \;, \quad r = R_K(t) \;, \\ v(r,0) &= A_0(r)B_0(r), \quad r > R_0 \;, \\ v(\infty, t) &= 0 \;, \; t > 0 \;. \end{aligned} \tag{7.9}$$

Lemma 7.3. *For any $T > 0$ and any compact set $\Omega_T$ in $\{R_K(t) < r < \infty \;, \; 0 \le t \le T\}$ whose distance to $r = R_K(t)$ is $\ge c_* > 0$, there exists a constant $M$ depending on $T$ and $c_*$ (but not on $K$) such that*

$$\int_{\Omega_T} [K A_K B_K (A_K + B_K)]^2 r^2 dr dt \le M \;. \tag{7.10}$$

*Proof*. Let $\xi(r,t)$ be a cutoff function such that $\xi = 1$ in $\Omega_T$ and $\xi = 0$ outside $\left(\frac{1}{2}\, c_*\right)$-neighborhood of $\Omega_T$. Multiplying the differential equation for $v$ by $\xi^2 K v r^2$ and integrating, we obtain

$$\int_{R_K(t)}^{\infty} \xi^2 K \frac{v^2}{2}\, r^2 dr + \int_0^T \int_{R_K(t)}^{\infty} \xi^2 K (v_r)^2 r^2 dr dt$$
$$+ \int_0^T \int_{R_K(t)}^{\infty} \xi^2 K^2 v^2 (A_K + B_K) r^2 dr dt \le I$$

where

$$I = \int_0^T \int_{R_K(t)}^{\infty} \Big[ |\xi \xi_t| K v^2 + 2\xi |\xi_r| \; |v_r| K v + 2|A_{K,r} B_{K,r}| \xi^2 K v \Big] r^2 dr dt$$

is bounded independently of $K$, by Lemmas 7.1 and 7.2; this implies the assertion (7.10).

Lemma 7.4. *Let $\Omega_T$ be as in Lemma 7.3. Then there exists a constant $M$ depending on $T$ and $c_*$ (but not on $K$) such that*

$$\|A_K\|_{C^{1/4,1/8}(\Omega_T)} \le M \;, \tag{7.11}$$

$$\|B_K\|_{C^{1/2,1/8}(\Omega_T)} \le M \;. \tag{7.12}$$

That means that the $A_K$ and $B_K$ are uniformly Hölder continuous (in $\Omega_T$) with exponent 1/4 in $r$ and exponent 1/8 in $t$.

*Proof*. By Lemmas 7.2, 7.3,

$$v_t - D_A \Delta v \quad \text{is in} \quad L^2(\Omega_T) \;,$$

uniformly in $K$. By $L^2$ estimates it then follows that

$$\|v\|_{W_2^{2,1}(\Omega_T)} \le M$$

with a slightly smaller set $\Omega_T$ and a larger constant $M$, both independent however of $K$. By Sobolev's imbedding [11] we then have

$$\|v\|_{C^{1/2,1/4}(\Omega_T)} \le M \tag{7.13}$$

with yet another constant $M$.

The function $u = A_K - B_K$ satisfies (7.5), and we can apply $L^p$ estimates to deduce $u$ also satisfies the estimate (7.13). Thus both $A_K B_K$ and $A_K - B_K$ belong to $C^{1/2,1/4}(\Omega_T)$, uniformly in $K$. Since

$$A_K + B_K = (4v + u^2)^{1/2}$$

it follows that $A_K + B_K$ is in $C^{1/4,1/8}(\Omega_T)$, and the same then holds for $A_K$ and $B_K$.

**Theorem 7.5.** *Let $\tilde{\Omega}_T$ be any compact which is contained in $R_K(t) < r < \infty$, $0 \le t < T$ for all $K$ sufficiently large. Then*

$$\lim_{K\to\infty} A_K(r,t)B_K(r,t) = 0 \tag{7.14}$$

uniformly in $(r,t) \in \tilde{\Omega}_T$.

*Proof*. By Lemma 7.2

$$\iint\limits_{\tilde{\Omega}_T} A_K B_K r^2 dr dt \le \frac{M}{K} \to 0$$

if $K \to \infty$. Since $A_K B_K$ is uniformly Hölder continuous in $\tilde{\Omega}_T$ with exponent and coefficient independent of $K$, (7.14) follows.

§**8. Asymptotic limits as $K \to \infty$.** The estimates of §7 show that for any sequence $K_n' \to \infty$ there is a subsequence $K_n$ such that, as $K_n \to \infty$,

$$R_{K_n}(t) \to R(t) \in Lip[0,T] \quad \text{in} \quad C^0[0,T] \ , \tag{8.1}$$

$$A_{K_n}(r,t) \to A(r,t) \in C^{1/4,1/8}(Q_T) \quad \text{in} \quad C^0(Q_T) \ , \tag{8.2}$$

$$B_{K_n}(r,t) \to B(r,t) \in C^{1/4,1/8}(Q_T) \quad \text{in} \quad C^0(Q_T) \ , \tag{8.3}$$

$$K_n A_{K_n} B_{K_n} \to f \quad \text{in the sense of weak convergence of measures,} \tag{8.4}$$

for any $0 < T < \infty$, where

$$Q_T = \{(r,t); r > R(t),\ 0 \le t < T\} \ ,$$

and $f$ is a measure.

From Theorem 7.5 we have

$$A(r,t)B(r,t) = 0 \ . \tag{8.5}$$

The functions $A, B$ both satisfy the equation

$$w_t - D_A \Delta w + f = 0 \quad \text{in} \quad \mathcal{D}'(Q_T) \ , \tag{8.6}$$

whereas the function $u = A - B$ satisfies

$$u_t - D_A \Delta u = 0 \quad \text{in} \quad Q_T \tag{8.7}$$

since each of the functions $A_{K_n} - B_{K_n}$ satisfies this equation.

By (7.3) it follows that $u_r \le 0$ in $Q_T$ and then, by the strong maximum principle,

$$u_r < 0 \quad \text{in} \quad Q_T \ . \tag{8.8}$$

It follows that there exists a curve $r = S(t)$ with $S(t) \in C^{1+\nu}[0,T] \cap C^\infty(0,T)$ such that

$$\begin{aligned} &u(r,t) > 0 \quad \text{if} \quad r < S(t) \ , \\ &u(r,t) < 0 \quad \text{if} \quad r > S(t) \ ; \end{aligned} \tag{8.9}$$

here $\nu$ is as in (1.10), (1.11). Since $A_0 - B_0$ is positive at $r = R_0$ and negative at $r = \infty$,

$$R_0 < S(0) < \infty \ . \tag{8.10}$$

Take $T$ such that

$$S(t) > R(t) \quad \text{for all} \quad 0 \le t \le T \ . \tag{8.11}$$

For $r < S(t)$ we have $u > 0$, or $A > B$. Since $AB = 0$, it follows that $B = 0$. Similarly $A = 0$ if $r > S(t)$; thus

$$\begin{aligned} &A(r,t) = 0 \quad \text{if} \quad r > S(t) \ , \\ &B(r,t) = 0 \quad \text{if} \quad r < S(t) \ . \end{aligned} \tag{8.12}$$

In any closed domain in $\{r < S(t),\ t < T\}$ we have $B = 0$ and then, by (8.6) with $w = B, \quad f = 0$. Similarly $f = 0$ if $r > S(t)$. It follows that $f$ is a measure supported on $r = S(t),\ 0 < t < T$. In particular,

$$A_t - D_A \Delta A = 0 \quad \text{if} \quad r < S(t), \quad 0 < t < T \ .$$

Since also $A(S(t),t) = 0$ and $S(t)$ is smooth, regularity results for the heat equation imply that $A$ is in $C^{1+\nu}$ in $R(t) \le r \le S(t),\ 0 \le t \le T$ and in $C^\infty$ in $R(t) \le r \le S(t),\ 0 < t \le T$.

Next, from (8.6) for $w = A$,

$$-\int\int A(-\varphi_t - D_A \Delta \varphi) r^2 dr dt = \int\int f \varphi r^2 dr dt \tag{8.13}$$

for any test function $\varphi$ in $Q_T$. Using the fact that $A = 0$ if $r > S(t)$, and integrating by parts in (8.13) we find that

$$\int\int f \varphi r^2 dr dt = -\int_0^T S^2(t) D_A \ A_r(S(t) - 0, t) \varphi(S(t), t) dt \ .$$

It follows that

$$f(r,t) = D_A A_r(S(t)-0,0)\delta(r-S(t)) \ . \tag{8.14}$$

THEOREM 8.1. *For any sequence $K_n' \to 0$ there is a subsequence $K_n \to \infty$ such that the solutions of $(P_{K_n})$ converge to a solution of $(P_\infty)$ uniformly in compact subsets of $\{r > R(t),\ \ 0 \le t \le T\}$*

*Proof*. We have already proved most of the theorem. Since $S(t) > R(t)$ for $0 \le t \le T$, it follows that $A$ satisfies the two boundary conditions (2.8), (2.9), where $\zeta = \lim\limits_{K\to\infty} \zeta_K$. Using the $C^{1+\mu}$-regularity of $A$ (which one obtains by the same argument as for Problem $(P_K)$) as well as the $C^{1+\mu}$-regularity of $R_K(t)$ and of the $A_K$ near $r = R_K(t)$, we can deduce that $C_{K_n} \to C$ uniformly near $r = R(t)$ and that $C$ satisfies the boundary condition (2.10). The remaining assertions of Theorem 8.1 have already been proved.

Denote by $T_f$ be the supremum of all $T$'s for which (8.11) holds. We claim:

$$\text{if } T_f < \infty \text{ then } S(t) \to R(T_f) \text{ if } t \to T_f \ . \tag{8.15}$$

Indeed, if $\overline{\lim}\ S(t) > R(T_f)$, then (8.11) holds for $T > T_f$ (since $u$ is smooth and $u_r < 0$ in $\{r > R(t),\ t > 0\}$. On the other hand, if the limit $S(T_f - 0)$ does not exist then $u_r(r, T_f)$ will vanish on a nonempty interval, contradicting the inequality $u_r < 0$.

In §9 we shall prove uniqueness for Problem $(P_\infty)$; this implies that the convergence asserted in Theorem 8.1 is not just for a subsequence $K_n \to \infty$ but for all $K \to \infty$.

In §10 we shall prove that $T_f < \infty$, and this will conclude the proof of all the assertions made in Theorems 2.2 and 2.3.

## §9. Uniqueness for $(P_\infty)$. In this section we prove:

THEOREM 9.1. *Assume that $D_A = D_B$. Then there exists at most one solution to Problem $(P_\infty)$.*

*Proof*. We begin with some remarks on the regularity of any solution $(A, B, C, R, S)$. Consider the function

$$u = \begin{cases} A & \text{if} \quad R(t) \le r \le S(t) \\ -B & \text{if} \qquad r > S(t) \end{cases}$$

for all $t < T$, where $T < T_f$, the final time of the solution. Then

$$u_t = D_A\ \Delta u \ ,$$

and $u = A$ in a neighborhood of $\{(R(t),t),\ 0 < t < T\}$. From this and the boundary conditions for $A$ and $C$ at $r = R(t)$ we easily deduce, as in earlier sections, that $R(t)$ and $\zeta(t)$ belong to $C^{1+\mu}[0,T]$ and $C(r,t)$ belongs to $C^{1+\mu}$ in $R(t) \le r < S(t),\ \ 0 \le t \le T$, for some $0 < \mu < 1$, and

$$-M < u_r < 0 \quad \text{for} \quad r > R(t),\ 0 \le t \le T \ ; \tag{9.1}$$

furthermore, the function $S(t)$ defined by

$$r = S(t) \quad \text{if} \quad u(r,t) = 0$$

is in $C^{1+\mu}[0,T]$ with

$$\dot{S}(t) = -\frac{u_t(S(t),t)}{u_r(S(t),t)} \ .$$

(Note that $u_t \in C^\mu$ with $\mu = \nu$ up to $t = 0$ since $A_0$ and $B_0$ belong to $C^{2+\nu}$). The function $C(r,t)$ is $C^\infty$ off $r = S(t)$. Near $r = S(t)$ the regularity of $C(r,t)$ is the same as the regularity of the special solution of (2.7):

$$\widetilde{C}(|x|,t) = \int_0^t \int_{|y|=R(\tau)} \frac{e^{-\frac{|x-y|^2}{\sqrt{D_C}\,4(t-\tau)}}}{(4\pi^2 D_C(t-\tau))^{3/2}} u_r(S(\tau),\tau)d\sigma_y d\tau \ . \tag{9.2}$$

This is a single layer potential and, since $u_r(S(\tau),\tau)$ is in $C^\mu$, this potential is in $C^\beta$ (across $r = S(t)$) for any $0 < \beta < 1$, and its derivative from each side of $r = S(t)$ is uniformly continuous (the proof is similar to the proof of Lemma 1 in [6; p. 217]). Thus in particular,

$$|C_r|_{L^\infty} \le M' \ , \quad M' \quad \text{constant.} \tag{9.3}$$

Suppose now that $(A_i, B_i, C_i, R_i, S_i)$ are solutions of Problem $(P_\infty)$ for $i = 1,2$, and set

$$u_i(r,t) = \begin{cases} A_i(r,t) & \text{if} \quad R_i(t) \le r \le S_i(t) \\ -B_i(r,t) & \text{if} \quad S_i(t) < r < \infty \end{cases}$$

for $0 \le t \le T$ where $T$ is such that

$$S_i(t) - R_i(t) \ge \lambda > 0 \quad \text{for } 0 \le t \le T \text{ and } i = 1,2 \tag{9.4}$$

where $x = r - R_i(t)$, and

$$\tilde{u}(x,t) = \hat{u}_1(x,t) - \hat{u}_2(x,t) \ ,$$
$$\widetilde{C}(x,t) = \widehat{C}_1(x,t) - \widehat{C}_2(x,t) \ .$$

Set

$$\Omega_\tau = \{0 < x < \infty, \ 0 < t < \tau\} \quad \text{for any} \quad \tau \le T \ .$$

The functions $\tilde{u}, \widetilde{C}$ satisfy (for simplicity we take $D_A = D_C = 1$):

$$\begin{aligned}
&\tilde{u}_t = \tilde{u}_{xx} + \left(\frac{2}{x+R_1} + \dot{R}_1\right)\tilde{u}_x + \left[\frac{2}{x+R_1} + \dot{R}_1 - \frac{2}{x+R_2} - \dot{R}_2\right]\tilde{u}_{2,x} \quad \text{in} \quad \Omega_T \ , \\
&-\zeta_{1,0}\tilde{u}_x + (1-\zeta_{1,0})\tilde{u} = (\zeta_{1,0} - \zeta_{2,0})[\tilde{u}_{2,x} + \tilde{u}_2 - A^*] \quad \text{if} \quad x = 0, \ 0 < t < T \ , \\
&\tilde{u}(x,0) = 0 \ , \quad x > 0 \ , \\
&\tilde{u}(\infty,t) = 0 \ , \quad 0 < t < T \ ,
\end{aligned} \tag{9.5}$$

and

$$\widetilde{C}_t = \widetilde{C}_{xx} + \left(\frac{2}{x+R_1} + \dot{R}_1\right)\widetilde{C}_x + \left[\frac{2}{x+R_1} + \dot{R}_1 - \frac{2}{x+R_2} - \dot{R}_2\right]\widehat{C}_{2,x}$$

$$+\widehat{u}_{2,x}(S_2 - R_2, t)\delta(x + R_2 - S_2) - \widehat{u}_{1,x}(S_1 - R_1, t)\delta(x + R_1 - S_1) \quad \text{in} \quad \Omega_T \ ,$$

$$-\widetilde{C}_x + \gamma(\widehat{C}_1 + \widehat{C}_2)(\widehat{C}_1^2 + \widehat{C}_2^2)\widetilde{C} = 0 \quad \text{if} \quad x = 0, \quad 0 < t < T \ , \tag{9.6}$$

$$\widetilde{C}(x, 0) = 0 \ , \quad x > 0 \ ,$$

$$\widetilde{C}(\infty, t) = 0 \ , \quad 0 < t < T \ .$$

By the maximum principle,

$$\|\widetilde{u}\|_{L^\infty(\Omega_\tau)} \le \|\zeta_1 - \zeta_2\|_{L^\infty(0,\tau)} + N\tau\left[\|R_1 - R_2\|_{L^\infty(0,\tau)} + \|\dot{R}_1 - \dot{R}_2\|_{L^\infty(0,\tau)}\right] \tag{9.7}$$

for any $0 < \tau < T$, where $N$ is a constant independent of $\tau$; in the sequel we shall denote by $N$ any such constant.

Differentiating in $x$ the differential equation in (9.5), we obtain a parabolic equation for $\widetilde{u}_x$. Using the maximum principle, we find that

$$\|\widetilde{u}_x\|_{L^\infty(\Omega_\tau)} \le N\|\zeta_1 - \zeta_2\|_{L^\infty(\Omega_\tau)} + N\tau\left[\|R_1 - R_2\|_{L^\infty(0,\tau)} + \|\dot{R}_1 - \dot{R}_2\|_{L^\infty(0,\tau)}\right] . \tag{9.8}$$

Since $R_1(0) = R_2(0)$ we also have

$$\|R_1 - R_2\|_{L^\infty(0,\tau)} \le \tau\|\dot{R}_1 - \dot{R}_2\|_{L^\infty(0,\tau)} \ . \tag{9.9}$$

From the boundary conditions for $\widetilde{u}_i$ at $x = 0$,

$$|\dot{R}_1 - \dot{R}_2| = \alpha|\widehat{u}_{1,x}(0,t) - \widehat{u}_{2,x}(0,t)|$$

$$= \alpha\left|\frac{1-\zeta_{1,0}}{\zeta_{1,0}}(\widehat{u}_1 - A^*) - \frac{1-\zeta_{2,0}}{\zeta_{2,0}}(\widehat{u}_2 - A^*)\right| ,$$

so that

$$\|\dot{R}_1 - \dot{R}_2\| \le N\left[\|\widetilde{u}\|_{L^\infty(\Omega_\tau)} + \|\zeta_1 - \zeta_2\|_{L^\infty(0,\tau)}\right] \ . \tag{9.10}$$

Substituting (9.9), (9.10) into (9.7), (9.8) and choosing $\tau < 1/(2N)$, we get

$$\|\widetilde{u}\|_{L^\infty(\Omega_\tau)} + \|\widetilde{u}_x\|_{L^\infty(\Omega_\tau)} \le N\|\zeta_1 - \zeta_2\|_{L^\infty(0,\tau)} \ . \tag{9.11}$$

Next, from the definition (1.14), (1.15) of $\zeta_{i,0}, \zeta_i$ we deduce that

$$\|\zeta_1 - \zeta_2\|_{L^\infty(0,\tau)} \le N\tau\|\widetilde{C}(0,t)\|_{L^\infty(0,\tau)} + N\|R_1 - R_2\|_{L^\infty(0,\tau)} \ , \tag{9.12}$$

and we need to estimate $\widetilde{C}(0,t)$.

Denote by $G(x,y,t,s)$ the Green function for the problem (see [3])

$$w_t = w_{xx} + \left(\frac{2}{x+R_1} + \dot{R}_1\right)w_x \quad \text{in} \quad \Omega_T \ ,$$

$$-w_x + \gamma(\widehat{C}_1 + \widehat{C}_2)(\widehat{C}_1^2 + \widehat{C}_2^2)w = 0 \quad \text{if} \quad x = 0, \quad 0 < t < T \ ,$$

$$w(\infty, t) = 0, \quad 0 < t < T \ .$$

Then (using (9.6)) we can represent $\widetilde{C}$ in the form

$$\widetilde{C}(|x|,t) = \int_0^t \int_0^\infty G(x,y,t,s)\left[\frac{2}{y+R_1(s)} + \dot{R}_1(s) - \frac{2}{y+R_2(s)} - \dot{R}_2(s)\right]\widehat{C}_{2,y}(y,s)dyds$$

$$+\int_0^t G(x,S_2(s)-R_2(s),t,s)\widehat{u}_{2,x}(S_2(s)-R_2(s),s)ds$$

$$-\int_0^t G(x,S_1(s)-R_1(s),t,s)\widehat{u}_{1,x}(S_1(s)-R_1(s),t-s)ds\ .$$

$$\equiv J_1 + J_2 + J_3\ .$$

We can write

$$|J_2(0,t)+J_3(0,t)| \le \int_0^t |G(0,S_2-R_2,t,s)||\widehat{u}_{2,x}(S_2-R_2,s) - \widehat{u}_{1,x}(S_1-R_1,s)|$$

$$+\int_0^t |\widehat{u}_{1,x}(S_1-R_1,s)||G(0,S_2-R_2,t,s) - G(0,S_1-R_1,t,s)|$$

where $S_i = S_i(s),\ R_i = R_i(s)$. The difference in the first integral is equal to

$$[\widehat{u}_{2,x}(S_2-R_2,s) - \widehat{u}_{2,x}(S_1-R_1,s)] + \widetilde{u}_x(S_1-R_1,s)\ .$$

Therefore, if $0 < t < \tau$,

$$|J_2(0,t) - J_3(0,t)| \le N\left[\|S_1-S_2\|_{L^\infty(0,\tau)} + \|R_1-R_2\|_{L^\infty(0,\tau)} + \|\widetilde{u}_x\|_{L^\infty(\Omega_\tau)}\right]\ .$$

Using the regularity result (9.3), we can immediately estimate also $|J_1(0,t)|$ by the $L^\infty$ norm of $|R_1-R_2| + |\dot{R}_1 - \dot{R}_2|$. Hence

$$(9.13)\qquad |\widetilde{C}(0,t)| \le N\left[\|\dot{R}_1-\dot{R}_2\|_{L^\infty(0,\tau)} + \|S_1-S_2\|_{L^\infty(0,\tau)} + \|\widetilde{u}_x\|_{L^\infty(\Omega_\tau)}\right]\ .$$

Next we need to estimate $S_1 - S_2$. By the mean value theorem,

$$\widehat{u}_1(S_1-R_1,t) - \widehat{u}_1(S_2-R_2,t) = \widehat{u}_{1,x}(y,t)\cdot(S_1-R_1-S_2+R_2)$$

where $y$ is a point between $S_1 - R_1$ and $S_2 - R_2$. Since $|u_{1,x}| \ge \nu_0 > 0$ ($\nu_0$ constant which depends on the $\lambda$ in (9.4)), we get

$$|S_1(t) - S_2(t)| \le |R_1-R_2| + \frac{1}{\nu_0}|\widehat{u}_1(S_1-R_1,t) - \widehat{u}_1(S_2-R_2,t)|$$

$$= |R_1-R_2| + \frac{1}{\nu_0}|\widehat{u}_2(S_2-R_2,t) - \widehat{u}_1(S_2-R_2,t)|$$

$$\le \|R_1-R_2\|_{L^\infty(0,\tau)} + \frac{1}{\nu_0}\ \|\widetilde{u}\|_{L^\infty(\Omega_\tau)}\ .$$

Substituting this into (9.13) and using the result in (9.12), we find after using also (9.10) and (9.11), that (9.12) becomes

$$\|\zeta_1 - \zeta_2\|_{L^\infty(0,\tau)} \le N\tau \|\zeta_1 - \zeta_2\|_{L^\infty(0,\tau)} \ .$$

Hence $\zeta_1(t) = \zeta_2(t)$ if $0 \le t \le \tau$, $\tau$ small. This implies (by (9.11) and (9.10)) that $u_1 \equiv u_2$, $R_1 \equiv R_2$ and the two solutions coincide if $0 \le t \le \tau$. We similarly proceed step-by-step to complete the proof of uniqueness.

## §10. $T_f < \infty$.

In this section we prove:

**Theorem 10.1.** *If $D_A = D_B$ and $C_0(r) \not\equiv 0$ then $T_f < \infty$.*

We shall denote by $T_\infty^*$ the smallest $t$ (if existing) such that $\zeta(t) \ge 1$, and first prove:

**Lemma 10.2.** *If $T_f = \infty$ then $T_\infty^* < \infty$.*

*Proof.* Suppose the assertion is not true; then $\zeta(t) < 1$ for all $t \ge 0$. Since also $T_f = \infty$, $A(r,t) = u(r,t)$ for all $(r,t)$ in a neighborhood of $\{(R(t),t),\ 0 \le t < \infty\}$ and, consequently,

$$D_A \zeta u_r + (1-\zeta)(u - A^*) = 0 \quad \text{if} \quad r = R(t), \quad t > 0 \ , \tag{10.1}$$

$$u_r = \alpha \dot{R} \quad \text{if} \quad r = R(t),\ t > 0 \ . \tag{10.2}$$

where $u = \lim\limits_{K\to\infty} (A_K - B_K)$. Set

$$\sigma = \lim_{t\to\infty} \zeta(t); \quad \text{then} \quad \sigma \le 1 \ .$$

If $\sigma < 1$ then we can use the comparison argument as in the proof of (6.7) to deduce that

$$|u(r,t) - u_\infty(r)| \le \widetilde{W}(r,t) \quad \text{for all} \quad t > 0 \ , \tag{10.3}$$

where $\widetilde{W}$ is defined in (6.16) and $u_\infty(r)$ is the harmonic function in $\rho < r < \infty$ where ($\rho = \lim\limits_{t\to\infty} R(t)$) satisfying

$$-\, D_A \sigma u_{\infty,r}(\rho) + (1-\sigma)(u_\infty(\rho) - A^*) = 0 \ , \quad u_\infty(\infty) = -B^* \ . \tag{10.4}$$

Next we argue as in Lemma 6.2 (with $t_0 = 0$ in (6.19)) and deduce that

$$C_{K,r}(R_K(t),t) \ge \frac{M}{t^6} \qquad (M > 0, \quad t \ge 1) \ . \tag{10.5}$$

where $M$ is independent of $K$. Hence

$$C_r(R(t),t) \ge \frac{M}{t^6} \qquad (M > 0, \quad t \ge 1) \ . \tag{10.6}$$

Using this estimate and (10.1), (10.2), we can proceed as in the proof of Theorem 6.3, deriving (6.22) for $\eta(t) = \zeta(t) - 1$, and concluding that if

$$\frac{1}{t_n} \int_0^{t_n} (A^* - u(R(\tau),\tau))d\tau \ge \alpha_0 > 0 \tag{10.7}$$

for a sequence $t_n \to \infty$, then $\eta(t_n) > 0$ for $t_n$ large enough, which is a contradiction to the assumption that $\zeta(t) < 1$ for all $t > 0$. Hence

$$\frac{1}{t} \int_0^t u(R(\tau), \tau) d\tau \to A^* \quad \text{as} \quad t \to \infty \ . \tag{10.8}$$

This together with (10.3) implies that $u_\infty(\rho) = A^*$. Hence $u_\infty$ takes it maximum at $r = \rho$ and, by the maximum principle, $u_{\infty,r}(\rho) < 0$. However, since $u_\infty(\rho) = A^*$, we must also have $u_{\infty,r}(\rho) = 0$ by (10.4), which is a contradiction. We conclude that $\sigma = 1$, i.e., $\zeta(\infty) = 1$ and then, by (10.1),

$$u_r(R(t), t) \to 0 \quad \text{if} \quad t \to \infty \ . \tag{10.9}$$

By (10.9) and a comparison argument (as in the proof of Lemma 6.1) we then get

$$u(r,t) \le \widetilde{W}(r,t)$$

where $\widetilde{W}$ is defined in (6.16); therefore

$$u(R(t), t) \to 0 \quad \text{if} \quad t \to \infty \ . \tag{10.10}$$

Observe next that the argument that led from (10.5) to (10.8) is independent of whether $\sigma < 1$ or $\sigma = 1$. Thus (10.8) still holds and this is a contradiction to (10.10). The proof that $T^*_\infty$ is finite is thereby complete.

*Proof of Theorem 10.1.* Suppose $T_f = \infty$. Then, by Lemma 10.2, $T^*_\infty < \infty$. Therefore

$$u_r(\overline{R}, t) = 0 \quad \text{if} \quad t > T^*_\infty \quad (\overline{R} = R(T^*_\infty)) \ .$$

Since also $u_t = D_A \Delta u$ and $u(\infty, t) = -B^*$, it follows by the comparison argument that

$$|u(r,t) + B^*| \le M \frac{e^{-\frac{r^2}{\sqrt{D_A}\, 4t}}}{t^{3/2}} \quad \text{if} \quad t > T^*_\infty \ .$$

Consequently, $u(r,t) < 0$ for all $r \ge R(t)$, $t \ge T$ provided $T$ is sufficiently large, and this is a contradiction to the assumption that $T_f = \infty$ (which implies that $u$ remains positive for all $t > 0$).

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