On the upper semicontinuity of the set of solutions of differential inclusions with a small parameter in the derivative*

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Abstract

We consider an initial value problem for a differential inclusion with a scalar parameter ε in a part of the derivatives. We give conditions under which the problem with $\varepsilon = 0$ has Lipschitz continuous solutions; moreover, the map "$\varepsilon \to$ set of solutions that are Lipschitz continuous with a fixed sufficiently large Lipschitz constant" is upper semicontinuous at $\varepsilon = 0^+$ in $C[0,1] \times C[\delta, 1]$ for any $\delta \in (0,1]$.

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Consider the following initial value problem for a singularly perturbed differential inclusion

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in F(x(t), y(t), t), \quad x(0) = x^0, y(0) = y^0, \tag{1}$$

where $x(t) \in \mathbf{R}^n, y(t) \in \mathbf{R}^m, t \in [0,1], x^0$ and y^0 are given vectors, ε is a nonnegative real parameter, and F is a set-valued map from $\mathbf{R}^n \times \mathbf{R}^m$ into itself. For $\varepsilon > 0$ we employ the standard definition for a solution of (1): a function (x, y) is a solution of (1) if it is absolutely continuous on the interval [0, 1] and satisfies (1) for almost every $t \in [0,1]$. For $\varepsilon = 0$ we have an inclusion of a mixed type

$$\begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix} \in F(x(t), y(t), t), \quad x(0) = x^0. \tag{2}$$

We say that (x, y) is a solution of (2) if x is absolutely continuous on [0, 1], y is measurable on [0, 1], and (x, y) satisfies (2) for a.e. $t \in [0,1]$. With this definition the initial condition for y becomes meaningless, therefore it is dropped.

Let $Z(\varepsilon)$ be the set of solutions of (1) for an $\varepsilon > 0$ and let $Z(0)$ be the set of solutions of (2). In this paper we study continuity properties of the set-valued map $\varepsilon \to Z(\varepsilon)$ at $\varepsilon = 0^+$. Throughout the paper we denote by $< \cdot, \cdot >$ the usual scalar product in $\mathbf{R}^n$, $|\cdot|$ is the Euclidean norm, $\sigma(\cdot, A)$ is the support function to a set $A \subset \mathbf{R}^n$: $\sigma(x, A) = \sup_{y \in A} < x, y >$, $|A| = \sup_{x \in A} |x|$, $\bar{F}(z, t) = \{y \in \mathbf{R}^m : (x, y) \in F(z, t)\}$, i.e. $\bar{F}(z, t)$ is the y-part of $F(z, t)$, $C[a, b]$ is the space of continuous vector functions equipped with the supremum norm on the interval $[a, b]$, $L^p[a, b], 1 \le p \le +\infty$, is the space of p-integrable functions on $[a, b]$, a.e. means "almost every".

There is a fundamental theorem referred as Tikhonov's theorem [13] dealing with continuity properties with respect to ε at $\varepsilon = 0^+$ of a unique solution $z(\varepsilon) = (x(\varepsilon), y(\varepsilon))$ to a nonlinear singularly perturbed differential equation (i.e. (1) with F single-valued). This theorem states that, under certain conditions, the function $\varepsilon \to x(\varepsilon)$ is continuous at $\varepsilon = 0^+$ in the norm of the space $C[0,1]$ and $\varepsilon \to y(\varepsilon)$ is continuous at $\varepsilon = 0^+$ in $C[\delta, 1]$ for any fixed $\delta \in (0,1]$. Predecessors of Tikhonov's theorem are contained in [5], [9] and a corrected proof is given in [6], [7]. For a thorough discussion of Tikhonov's theorem see the monographs [10] and [14].

Singular perturbation techniques have been extensively developed for the purposes of control theory, see e.g. [8]. For a linear control systems, i.e. when the map F in (1) has the form $F(z,t) = A(t)z + B(t)U$, where $A(t)$ and $B(t)$ are matrices and U is a compact set, it was shown in [3] that the reachable set at fixed time $t \in (0,1]$ (that is, the set of values at t of solutions $z \in Z(\varepsilon)$) has a Hausdorff limit as $\varepsilon \to 0^+$ which is, typically, larger than the set of values $z(t)$ of $z \in Z(0)$. This result indicates that if the map $\varepsilon \to Z(\varepsilon)$ is not single-valued, then it may be not upper semicontinuous at $\varepsilon = 0^+$ with respect to the pointwise convergence. For a discussion of continuity properties of reachable sets and sets of trajectories of linear control systems see [4], Chapter 7. In [2] we showed that the solution map Z is upper semicontinuous at $\varepsilon = 0^+$ in $C[0,1]$ for the x variable and in $L^2[0,1]$-weak for the y provided that the graph of $F(x,\cdot,t)$ is convex. A similar results was announced recently in [11]. We also proved in [2] that if we require $\bar{F}$ be single-valued, then the map Z is upper semicontinuous in $C[0,1] \times L^2[0,1]$ at $\varepsilon = 0^+$. Veliov [15] proved lower semicontinuity of the solution map Z at $\varepsilon = 0^+$ in $C[0,1] \times L^2[0,1]$ under conditions that are related to (but different from) the conditions used in the present paper.

Given $L > 0$ and $\varepsilon \geq 0$ let us define the set

$$Z_L(\varepsilon) = \{(x,y) \in Z(\varepsilon) : (x,y) \text{ is Lipschitz continuous in } [0,1] \times [\sqrt{\varepsilon},1] \text{ with a Lipschitz constant } L\}.$$

In this paper we show that for a sufficiently large L the map $\varepsilon \to Z_L(\varepsilon)$ is upper semicontinuous at $\varepsilon = 0^+$ in the Tikhonov-type metric $C[0,1] \times C[\delta,1], \delta \in (0,1]$. The difficult part of our proof is to establish the existence of Lipschitz continuous solutions of the reduced inclusion (2) corresponding to $\varepsilon = 0$.

We prove the following

Theorem. *Suppose that:*

(i) For every $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$ and for a.e. $t \in [0,1]$ the set $F(x,y,t)$ is nonempty, compact, and convex. The map $F(\cdot,t)$ is upper semicontinuous in $\mathbf{R}^n \times \mathbf{R}^m$ for a.e. $t \in [0,1]$ and $F(x,y,\cdot)$ is measurable in $[0,1]$ for every $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$;

(ii) There exists a positive constant ρ such that

$$|F(x,y,t)| \leq \rho(1 + |x| + |y|)$$

for every $(x, y) \in \mathbf{R}^{n+m}$ *and for a.e.* $t \in [0, 1]$*;*

(iii) There exist positive constants μ *and* ν *such that*

$$\sigma(y_1 - y_2, \bar{F}(x_1, y_1, t_1)) - \sigma(y_1 - y_2, \bar{F}(x_2, y_2, t_2))$$
$$\leq -\mu \mid y_1 - y_2 \mid^2 + \nu(\mid x_1 - x_2 \mid + \mid t_1 - t_2 \mid) \mid y_1 - y_2 \mid$$

for every $(x_i, y_i, t_i) \in \mathbf{R}^n \times \mathbf{R}^m \times [0, 1], i = 1, 2.$

Then for every $\varepsilon_0 > 0$ *there exists a constant* L *such that for every* $\varepsilon \in (0, \varepsilon_0]$ *the set* $Z_L(\varepsilon)$ *is nonempty and compact in* $C[0, 1] \times C[\sqrt{\varepsilon}, 1]$*; moreover, for every* $\delta \in (0, 1]$ *the map* $\varepsilon \to Z_L(\varepsilon)$ *is upper semicontinuous in* $C[0, 1] \times C[\delta, 1]$.

Assumptions (i) and (ii) are standard conditions for the existence of solutions of differential inclusions, see e.g. [1], Chapter 2. Assumption (iii) is a one-side Lipschitz condition combined with a stability-type condition. For example, suppose that the y-part of (1) has the form

$$\varepsilon \dot{y}(t) \in f(y(t)) + V(x(t), t), \tag{3}$$

where f is a function and V is a set-valued map. In this case condition (iii) holds if the map V is Lipschitz continuous with respect to the Hausdorff metric and f is dissipative; that is,

$$< y_1 - y_2, f(y_1) - f(y_2) > \leq -\mu |y_1 - y_2|^2$$

for some $\mu > 0$ and every $y_1, y_2 \in \mathbf{R}^m$. In particular, if f is linear, $f(y) = Ay$, then this condition is fulfilled if the eigenvalues of the matrix A have negative real parts. We note that various prototypes of condition (iii) are common in the singular perturbation literature.

We summarize the main step of the proof of the theorem in the following

Lemma. *On the assumptions of the theorem for every* $\varepsilon_0 > 0$ *there exists a constant* c *such that for every* $\varepsilon \in (0, \varepsilon_0]$ *there is a solution* $(x_\varepsilon, y_\varepsilon)$ *of (1) with the following property: for every* $\tau \in [0, 1]$ *and for every* $t \in [0, 1 - \tau]$

$$|x_\varepsilon(t + \tau) - x_\varepsilon(t)| \leq c\tau, \tag{4}$$
$$|y_\varepsilon(t + \tau) - y_\varepsilon(t)| \leq c\tau \left(1 + \frac{1}{\varepsilon} \exp\left(-\mu \frac{t}{\varepsilon}\right)\right). \tag{5}$$

Proof of lemma. We present a proof consisting of three steps.

Step 1. Condition (i) implies that for every $\varepsilon > 0$ the set $Z(\varepsilon)$ of solutions of (1) is nonempty, see [1] p. 58. Let us choose $\varepsilon_0 > 0$. We prove first that $\cup_{\varepsilon\in(0,\varepsilon_0]} Z(\varepsilon)$ is contained in a bounded set in $C[0,1]$. Let $\varepsilon \in (0,\varepsilon_0]$, let $(x_\varepsilon, y_\varepsilon) \in Z(\varepsilon)$, and let $t \in [0,1]$ be such that (1) holds at t. Multiplying (1) by $(x_\varepsilon(t), 0)$ and using condition (i), we obtain

$$\begin{aligned} |x_\varepsilon(t)|\frac{d}{dt}|x_\varepsilon(t)| &\leq \sigma((x_\varepsilon(t),0), F(x_\varepsilon(t), y_\varepsilon(t), t)) \\ &\leq \rho(1 + |x_\varepsilon(t)| + |y_\varepsilon(t)|)|x_\varepsilon(t)|. \end{aligned} \tag{6}$$

Multiplying (1) by $(0, y_\varepsilon(t))$ and using conditions (ii) and (iii) we have

$$\begin{aligned} \varepsilon|y_\varepsilon(t)|\frac{d}{dt}|y_\varepsilon(t)| &\leq \sigma(y_\varepsilon(t), \bar{F}(x_\varepsilon(t), y_\varepsilon(t), t)) \\ &\leq -\mu|y_\varepsilon(t)|^2 + \sigma(y_\varepsilon(t), \bar{F}(x_\varepsilon(t), 0, t)) \\ &\leq -\mu|y_\varepsilon(t)|^2 + \rho(1 + |x_\varepsilon(t)|)|y_\varepsilon(t)|. \end{aligned} \tag{7}$$

Define the set $T = \{t \in [0,1] : |x_\varepsilon(t)| > 0\}$. If $t \in T$, then from (6)

$$\frac{d}{dt}|x_\varepsilon(t)| \leq \rho(1 + |x_\varepsilon(t)| + |y_\varepsilon(t)|). \tag{8}$$

If $t \notin T$ and $|x_\varepsilon|$ is differentiable at t, then $d(|x_\varepsilon(t)|)/dt = 0$. Hence, (8) holds for a.e. $t \in [0,1]$. From (7) we obtain in the same way that

$$\varepsilon\frac{d}{dt}|y_\varepsilon(t)| \leq -\mu|y_\varepsilon(t)| + \rho(1 + |x_\varepsilon(t)|). \tag{9}$$

for a.e. $t \in [0,1]$. Let us denote

$$\alpha(t) = \varepsilon\frac{d}{dt}|y_\varepsilon(t)| + \mu|y_\varepsilon(t)|.$$

Then

$$|y_\varepsilon(t)| = \exp(-\mu\frac{t}{\varepsilon})|y^0| + \frac{1}{\varepsilon}\int_0^t \exp(-\mu\frac{t-s}{\varepsilon})\alpha(s)ds,$$

and from (9)

$$\alpha(t) \leq \rho(1 + |x_\varepsilon(t)|).$$

Hence,

$$|y_\varepsilon(t)| \leq \exp(-\mu\frac{t}{\varepsilon})|y^0| + \frac{\rho}{\varepsilon}\int_0^t \exp(-\mu\frac{t-s}{\varepsilon})(1 + |x_\varepsilon(s)|)\,ds. \tag{10}$$

Integrating (8) we have

$$|x_\varepsilon(t)| \le |x^0| + \rho \int_0^t (1 + |x_\varepsilon(s)| + |y_\varepsilon(s)|)\, ds. \tag{11}$$

Substituting (10) in (11) and exchanging the order of integration, we obtain that for every $t \in [0,1]$

$$\begin{aligned} |x_\varepsilon(t)| &\le |x^0| + \rho \int_0^t (1 + |x_\varepsilon(s)|)\, ds + \rho \int_0^t \exp(-\mu \frac{s}{\varepsilon}) |y^0|\, ds \\ &+ \frac{\rho^2}{\varepsilon} \int_0^t \int_0^s \exp(-\mu \frac{s-\tau}{\varepsilon})(1 + |x_\varepsilon(\tau)|)\, d\tau\, ds \\ &\le |x^0| + \rho + \rho \int_0^t |x_\varepsilon(s)|\, ds + \frac{\rho}{\mu}|y^0|\varepsilon + \frac{\rho^2}{\mu} \int_0^t (1 + |x_\varepsilon(\tau)|)\, d\tau \\ &\le |x^0| + \rho\theta + \frac{\rho}{\mu}|y^0|\varepsilon + \rho\theta \int_0^t |x_\varepsilon(s)|\, ds, \end{aligned}$$

where $\theta = 1 + \rho/\mu$. Then the Gronwall lemma implies that for every $t \in [0,1]$

$$|x_\varepsilon(t)| \le (|x^0| + \rho\theta + \frac{\rho}{\mu}|y^0|\varepsilon) \exp(\rho\theta).$$

Denoting $M(\varepsilon) = (|x^0| + \rho\theta + \varepsilon\rho|y^0|/\mu)\exp(\rho\theta)$, by (10) we get

$$|y_\varepsilon(t)| \le \exp(-\mu\frac{t}{\varepsilon})|y^0| + \frac{\rho}{\mu}(1 - \exp(-\mu\frac{t}{\varepsilon})) \max_{s\in[0,1]}(1 + |x_\varepsilon(s)|) \le |y^0| + \frac{\rho}{\mu}(1 + M(\varepsilon))$$

for every $t \in [0,1]$. Hence

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \sup_{z\in Z(\varepsilon)} \sup_{t\in[0,1]} |z(t)| \le c_0,$$

where $c_0 = M(\varepsilon_0)(1+\rho/\mu) + \rho/\mu + |y^0|$. From this estimate and condition (ii) we conclude that if $(x_\varepsilon, y_\varepsilon)$ is a solution of (1), then x_ε is Lipschitz continuous with respect to t in $[0,1]$ with a Lipschitz constant $l_x = \rho(1 + 2c_0)$, i.e. (4) holds, and y_ε is Lipschitz continuous with respect to t in $[0,1]$ with a Lipschitz constant l_x/ε.

Step 2. Let $\varepsilon \in (0, \varepsilon_0]$ be fixed and let N be a natural number. Let $h = 1/N$. We construct by induction an absolutely continuous function

$(x_\varepsilon^h, y_\varepsilon^h)$ on $[0,1]$ such that $(x_\varepsilon^h, y_\varepsilon^h)$ is a solution of (1) and has the following property: for every $k = 1, \cdots, N-1$ and for a.e. $t \in [kh, (k+1)h]$

$$\varepsilon \frac{d}{dt}|y_\varepsilon^h(t) - y_\varepsilon^h(t-h)| \leq -\mu |y_\varepsilon^h(t) - y_\varepsilon^h(t-h)| + \nu(|x_\varepsilon^h(t) - x_\varepsilon^h(t-h)| + h). \quad (12)$$

Let $(x_\varepsilon, y_\varepsilon) \in Z(\varepsilon)$ and let $(x_\varepsilon^h(t), y_\varepsilon^h(t)) = (x_\varepsilon(t), y_\varepsilon(t))$ for $t \in [0,h]$. Suppose that $(x_\varepsilon^h, y_\varepsilon^h)$ is already determined on the interval $[0, kh]$ for some $k \geq 1$. Given $(x,y) \in \mathbf{R}^{n+m}$ and $t \in [kh, (k+1)h]$ for which $\dot{y}_\varepsilon^h(t-h)$ exists, let us define the set

$$\begin{aligned} \Gamma_\varepsilon^{kh}(x,y,t) \;\; = \;\; & \{(p,q) \in \mathbf{R}^{n+m} : \varepsilon < y_\varepsilon^h(t-h) - y, \dot{y}_\epsilon^h(t-h) - q/\varepsilon > \\ & \leq -\mu |y_\varepsilon^h(t-h) - y|^2 + \nu(|x_\varepsilon^h(t-h) - x| + h)|y_\varepsilon^h(t-h) - y|\}. \end{aligned}$$

The set $\Gamma_\varepsilon^{kh}(x,y,t)$ is closed and convex for every $(x,y) \in \mathbf{R}^{n+m}$ and a.e. $t \in [0,1]$, the map $\Gamma_\varepsilon^{kh}(\cdot, t)$ has closed graph for a.e. $t \in [0,1]$ and $\Gamma_\varepsilon^{kh}(x,y,\cdot)$ is measurable for every $(x,y) \in \mathbf{R}^{n+m}$; for the last property see [12]. Since F is compact-valued, for every $(x,y) \in \mathbf{R}^{n+m}$ and for a.e. $t \in [kh, (k+1)h]$ there exists $q \in \bar{F}(x,y,t)$ such that

$$< y_\varepsilon^h(t-h) - y, q >= \sigma(y_\varepsilon^h(t-h) - y, \bar{F}(x,y,t)).$$

Taking into account that $\varepsilon \dot{y}_\varepsilon^h(t-h) \in \bar{F}(x_\varepsilon^h(t-h), y_\varepsilon^h(t-h), t-h)$ for a.e. $t \in [kh, (k+1)h]$, we obtain that for every $(x,y) \in \mathbf{R}^{n+m}$ and for a.e. $t \in [kh, (k+1)h]$ there exists $(p,q) \in F(x,y,t)$ such that

$$\begin{aligned} & \varepsilon < y_\varepsilon^h(t-h) - y, \dot{y}_\varepsilon^h(t-h) - q/\varepsilon > \\ & \leq \sigma(y_\varepsilon^h(t-h) - y, \bar{F}(x_\varepsilon^h(t-h), y_\varepsilon^h(t-h), t-h)) - \sigma(y_\varepsilon^h(t-h) - y, \bar{F}(x,y,t)). \end{aligned}$$

Then, using (iii), we have

$$\begin{aligned} & \varepsilon < y_\varepsilon^h(t-h) - y, \dot{y}_\varepsilon^h(t-h) - q/\varepsilon > \\ & \leq -\mu |y_\varepsilon^h(t-h) - y|^2 + \nu |y_\varepsilon^h(t-h) - y|(|x_\varepsilon^h(t-h) - x| + h). \end{aligned}$$

Hence, the set

$$\Sigma_\varepsilon^{kh}(x,y,t) = \Gamma_\varepsilon^{kh}(x,y,t) \cap F(x,y,t)$$

is nonempty for every $(x,y) \in \mathbf{R}^{n+m}$ and for a.e. $t \in [kh, (k+1)h]$. The map Σ_ε^{kh} is nonempty-, compact- and convex-valued and the graph of $\Sigma_\varepsilon^{kh}(\cdot, t)$ is closed for a.e. $t \in [kh, (k+1)h]$. By (i) $F(\cdot, t)$ maps bounded sets in $\mathbf{R}^{n+m}$ into bounded sets in $\mathbf{R}^{n+m}$, hence so does $\Sigma_\varepsilon^{kh}(\cdot, t)$. Thus $\Sigma_\varepsilon^{kh}(\cdot, t)$ is upper semicontinuous in $\mathbf{R}^{n+m}$ for a.e. $t \in [kh, (k+1)h]$. The map $\Sigma_\varepsilon^{kh}(x, y, \cdot)$ is the intersection of two measurable maps, hence it is measurable for every $(x, y) \in \mathbf{R}^{n+m}$. Applying Theorem 5.2 from [1], we obtain that the initial-value problem

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{y}(t) \end{pmatrix} \in \Sigma_\varepsilon^{kh}(x(t), y(t), t) \text{ for a.e. } t \in [kh, (k+1)h],$$
$$x(kh) = x_\varepsilon^h(kh), \quad y(kh) = y_\varepsilon^h(kh),$$

has a solution $(\bar{x}_\varepsilon^h, \bar{y}_\varepsilon^h)$. Taking $(x_\varepsilon^h, y_\varepsilon^h) = (\bar{x}_\varepsilon^h, \bar{y}_\varepsilon^h)$ for $t \in [kh, (k+1)h]$ we complete the induction step. Thus there exists a solution $(x_\varepsilon^h, y_\varepsilon^h)$ of (1) satisfying the condition (12).

Step 3. By Step 1 x_ε^h is Lipschitz continuous uniformly in ε; that is, for every $t \in [0, 1-h]$

$$|x_\varepsilon^h(t+h) - x_\varepsilon^h(t)| \le l_x h, \tag{13}$$

while $y_\varepsilon^h(\cdot)$ is Lipschitz continuous with a Lipschitz constant l_x/ε. From (12) and (13)

$$\varepsilon \frac{d}{dt} |y_\varepsilon^h(t) - y_\varepsilon^h(t-h)| \le -\mu |y_\varepsilon^h(t) - y_\varepsilon^h(t-h)| + \nu(1 + l_x)h.$$

This inequality and the Lipschitz continuity of $y_\varepsilon^h(\cdot)$ yield that for any $t \in [h, 1]$

$$\begin{aligned} |y_\varepsilon^h(t) - y_\varepsilon^h(t-h)| &\le \exp(-\mu \frac{t-h}{\varepsilon}) |y_\varepsilon^h(h) - y_\varepsilon^h(0)| \\ &+ \frac{\nu(l_x+1)h}{\varepsilon} \int_h^t \exp(-\mu \frac{t-s}{\varepsilon})\, ds \\ &\le \frac{l_x h}{\varepsilon} \exp(-\mu \frac{t-h}{\varepsilon}) + \frac{\nu}{\mu}(l_x+1)h. \end{aligned}$$

That is,

$$|y_\varepsilon^h(t) - y_\varepsilon^h(t-h)| \le c_1 h \left(1 + \frac{1}{\varepsilon} \exp(-\mu \frac{t-h}{\varepsilon})\right), \tag{14}$$

where $c_1 = \max\{l_x, \nu(1+l_x)/\mu\}$. Let $h = 1/N, N = 1, 2, \cdots$. From Step 1, (13), and (14) we conclude that for every fixed $\varepsilon \in (0, \varepsilon_0]$ the sequence

$\{(x_\varepsilon^h, y_\varepsilon^h), h = 1, 1/2, 1/3, \cdots\}$ of solutions of (1) is bounded in $C[0,1]$ and equicontinuous; hence, by Arzela theorem, it has a subsequence which is convergent in $C[0,1]$ to some $(x_\varepsilon, y_\varepsilon)$. Using the upper semicontinuity of F it is a standard observation that the limit $(x_\varepsilon, y_\varepsilon)$ is a solution to (1).

Let us choose $\tau \in [0,1]$ and a sequence of natural numbers l_N such that $l_N/N \to \tau$ as $N \to \infty$. We extend y_ε^h on $[-1,2]$ assuming that it is constant outside $(0,1)$. Let $t \in [0,1]$. From (14) we have

$$\begin{aligned} |y_\varepsilon^h(t + l_N h) - y_\varepsilon^h(t)| &\le \sum_{j=1}^{l_N} |y_\varepsilon^h(t + jh) - y_\varepsilon^h(t + (j-1)h)| \\ &\le c_1 l_N h + \frac{c_1 h}{\varepsilon} \exp{(-\mu \frac{t}{\varepsilon})} \sum_{j=1}^{l_N} \exp{(-\mu \frac{(j-1)h}{\varepsilon})} \\ &\le c_1 l_N h \left(1 + \frac{1}{\varepsilon} \exp{(-\mu \frac{t}{\varepsilon})}\right). \end{aligned}$$

Then

$$\begin{aligned} |y_\varepsilon(t+\tau) - y_\varepsilon(t)| &\le |y_\varepsilon(t+\tau) - y_\varepsilon^h(t+\tau)| + |y_\varepsilon^h(t+\tau) - y_\varepsilon^h(t)| + |y_\varepsilon^h(t) - y_\varepsilon(t)| \\ &\le |y_\varepsilon^h(t+\tau) - y_\varepsilon^h(t)| + \theta^N \\ &\le c_1 l_N h \left(1 + \frac{1}{\varepsilon} \exp{(-\mu \frac{t}{\varepsilon})}\right) + \theta^N, \end{aligned}$$

where $\theta^N \to 0$ as $N \to \infty$. Since $l_N h \to \tau$ as $h \to 0$, from the last inequality we obtain (5).

Proof of theorem. Let $\varepsilon_k \in (0, \varepsilon_0], k = 1, 2, \cdots, \varepsilon_k \to 0$, and let (x_k, y_k) be a corresponding sequence of solutions of (1) satisfying (4) and (5). We extend y_k on the interval $[-1,2]$ assuming that it is constant outside $(0,1)$. From (5) we have

$$|y_k(\sqrt{\varepsilon_k} + \tau) - y_k(\sqrt{\varepsilon_k})| \le c\tau$$

for every $\tau \in [0,1]$. For $t \in [-1,2]$ define

$$\tilde{y}_k(t) = \begin{cases} y_k(t) & \text{for } \sqrt{\varepsilon_k} \le t \le 2, \\ y_k(\sqrt{\varepsilon_k}) & \text{otherwise.} \end{cases}$$

Then for every t and τ in $[0,1]$

$$|\tilde{y}_k(t+\tau) - \tilde{y}_k(t)| \le c\tau. \tag{15}$$

The sequence $(x_k, \tilde{y}_k)$ is uniformly bounded and equicontinuous, hence from Arzela theorem it has a subsequence which is convergent in $C[0,1]$ to some (x_0, y_0). (4) and (15) imply that the limit (x_0, y_0) is Lipschitz continuous on $[0,1]$. Since $\{\dot{x}_k\}_{k=1}^{\infty}$ is bounded in L^∞ then it is sequentially compact in L^1-weak. Without loss of generality, let $(x_k, \tilde{y}_k) \to (x_0, y_0)$ in $C[0,1]$ and $\dot{x}_k \to \dot{x}_0$ in L^1-weak as $k \to \infty$. Choose arbitrary $(p,q) \in \mathbf{R}^{n+m}$ and a measurable set $\Delta \subset [0,1]$. Then we have

$$\int_{\Delta \setminus [0,\sqrt{\varepsilon_k}]} (< p, \dot{x}_k(t) > + < q, \varepsilon_k \dot{y}_k(t) >)\, dt$$
$$\leq \int_{\Delta \setminus [0,\sqrt{\varepsilon_k}]} \sigma((p,q), F(x_k(t), y_k(t), t))\, dt$$
$$= \int_{\Delta} \sigma((p,q), F(x_k(t), \tilde{y}_k(t), t))\, dt - \int_{\Delta \cap [0,\sqrt{\varepsilon_k}]} \sigma((p,q), F(x_k(t), \tilde{y}_k(t), t))\, dt.$$

From (15) it follows that $\dot{\tilde{y}}_k$ is bounded in $L^\infty[0,1]$, hence $\varepsilon_k \dot{\tilde{y}}_k \to 0$ in $L^\infty[0,1]$ as $k \to \infty$. Using the conditions (i) and (ii) and the boundedness of $(x_k, \tilde{y}_k)$ in $C[0,1]$ we obtain

$$\begin{aligned}
\int_{\Delta} < p, \dot{x}_0(t) > dt &= \lim_{k\to\infty} \int_{\Delta \setminus [0,\sqrt{\varepsilon_k}]} < p, \dot{x}_k(t) > dt \\
&\leq \mathrm{limsup}_{k\to\infty} \int_{\Delta} \sigma((p,q), F(x_k(t), \tilde{y}_k(t), t))\, dt \\
&- \lim_{k\to\infty} \int_{\Delta \cap [0,\sqrt{\varepsilon_k}]} \sigma((p,q), F(x_k(t), \tilde{y}_k(t), t))\, dt \\
&\leq \int_{\Delta} \sigma((p,q), F(x_0(t), y_0(t), t))\, dt.
\end{aligned}$$

Hence (x_0, y_0) is a solution of (2). Clearly, (x_0, y_0) is Lipschitz continuous, i.e. $Z_L(0) \neq \emptyset$ for some sufficiently large L. Suppose that Z_L is not upper semicontinuous at $\varepsilon = 0^+$ in $C[0,1] \times C[\delta,1]$ for some $\delta \in (0,1]$. Then there exists an $\alpha > 0$ and sequences $\varepsilon_k \to 0^+, z_k \in Z_L(\varepsilon_k)$ such that the $C[0,1] \times C[\delta,1]$-distance from z_k to $Z_L(0)$ is greater than α for $k = 1, 2, \cdots$. By repeating the argument of the first part of the proof of theorem we find a subsequence of z_k which is convergent in $C[0,1] \times C[\delta,1]$ to an element of $Z_L(0)$. The obtained contradiction completes the proof.

From the above proof one can extract a lower bound for the constant L.

From (14) we obtain that for every $t \in [\sqrt{\varepsilon}, 1-h]$

$$|y_\varepsilon^h(t+h) - y_\varepsilon^h(t)| \leq c_1 h \left(1 + \frac{1}{\varepsilon} \exp{(-\frac{\mu}{\sqrt{\varepsilon}})} \right),$$

where the constant c_1 is defined in (14). Then y_ε^h is Lipschitz continuous in $[\sqrt{\varepsilon}, 1]$ with a Lipschitz constant

$$l_y = c_1 \left(1 + \frac{1}{\varepsilon_0} \exp{(-\frac{\mu}{\sqrt{\varepsilon_0}})} \right).$$

Thus one can take $L \geq l_x + l_y$.

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