Homogenization of two phase emulsions with surface tension effects

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Abstract

We consider an emulsion of two Stokes fluids, one of which is periodically distributed in the form of small spherical bubbles. The effects of surface tension on the bubble boundaries are modelled mathematically, as in the work of G. I. Taylor, by a jump only in the normal component of the traction. For a given volume fraction of bubbles we consider the two-scale convergence, and in the fine phase limit we find that the bulk flow is described by an anisotropic Stokes fluid. The effective viscosity tensor is consistent with the bulk stress formula obtained by Batchelor [2]. We find a new dual variational principle describing the effective viscosity and obtain bounds on the energy dissipation rate for periodic anisotropic emulsions.

1 Introduction

There are two mechanical aspects to the motion of a suspension: the dependance of the bulk properties of the suspension upon the microstructure and the evolution of the microstructure determined by the bulk flow (see

Batchelor [2], Hinch and Leal [8]). The focus of this paper lies in the first aspect of suspension mechanics as applied to concentrated periodic suspensions of fluid drops.

We consider flows of suspensions of n fluid drops in a second fluid. In such flows, the velocities of the drops must be determined simultaneously with the flow. For the case of fixed drops, the problem reduces to the one studied in Lipton and Avellaneda [12]. The bubbles are assumed to be small with respect to macroscopic length scales. We model the effects of surface tension on the bubble boundary using the zeroth order approximation introduced by Taylor [17] . In this approximation the bubbles are assumed spherical and only the normal component of the traction is allowed to jump at the bubble interface.

We suppose that at the initial time the bubbles are periodically distributed in the emulsion. The scale of the period is assumed to be of the same order as the bubble diameter. Since the characteristic length scales of the body force and flow region are much greater than the local period it follows that periodicity is preserved in the flow.

It is assumed that the flow is quasistatic and satisfies the time stationary Stokes equations at each instant. Under these hypothesis the flow equations of the emulsion are given by those derived by Keller, Rubenfeld and Molyneux [9].

We show that as the period of the suspension approaches zero the associated family of flows approach a *homogenized* flow with velocity field satisfying the stationary Stokes equations where the constitutive relation is given by an anisotropic viscosity that depends upon the microscopic geometry.

The emulsion considered here is given by a simple cubic lattice of spherical drops. The associated effective viscosity is therefore cubically symmetric and of the form:

$$2\mu^H_{ijkl} = 2\mu^S \mathbf{P}^S_{ijkl} + 2\mu^D \mathbf{P}^D_{ijkl} \tag{1.1}$$

where $\mathbf{P}^S$ is the projection onto off digonal strain rates and $\mathbf{P}^D$ is the projection onto digonal trace free strain rates (see Thurston [19]). We remark that our homogenization result applies immediately to any periodic arrangement of spherical drops.

The paper is outlined as follows. In Section 2 the flow problem for the emulsion is formulated.

The homogenization result that describes the asymptotic behaviour of the emulsion is stated in Section 3.

In Section 4, by means of two-scale expansions we show that as the period goes to zero, the pressure develops a singular and a regular part. The singular part is shown to be piecewise constant and to satisfy Laplace's formula for surface the tension in the case of spherical bubbles. The regular part participates in the bulk Stokes equations. Using formal two-scale expansions, we recover the homogenized flow equations and the equation for the effective viscosity tensor.

In Section 5 we give the rigorous proof of the two-scale convergence. We first normalize the pressure in order to get a uniform $L^2(\Omega)$ estimate for the sequence of pressures, in Section 5.1. Global and local conservation of mass equations are obtained for the homogenized velocity and its first corrector in Section 5.2. In Section 4.3 we obtain the local kinematic condition on the bubble interface by identifying the two scale limit of a suitably normalized velocity field with respect to the local bubble velocity. The two-scale convergence method is applied to the momentum balance equations to obtain the local balance laws for the homogenized stress (see Lemma 5.9.). These results, together with those in Sections 5.3 and 5.4 give the local problem. In Section 5.6 we obtain the homogenized momentum equation and the formula for the effective viscosity. Unlike problems with continuity of the traction across phases, the effective property for this problem contains a term encoding the effects of the work done against the bubble boundary due to viscous forces. The effective viscosity obtained here is shown to be consistent with the bulk stress formula obtained by Batchelor [2] in Section 6. We develop a new dual variational principle describing the effective viscosity and use it to obtain a harmonic mean bound on the energy dissipation rate for periodic anisotropic emulsions. We observe that the bounds on the effective viscosity for isotropic suspensions given in Keller, Rubenfeld and Molyneux [9], can be applied to the anisotropic case considered here.

Homogenization and bounds for anisotropic emulsions, neglecting the effects of surface tension can be found in Kohn and Lipton [10]. Bounds for isotropic emulsions neglecting surface tension were first derived in Hashin [7].

Homogenization results for periodic suspensions of solid particles can be found in Lewy and Palencia [11]. Flow past elastic or viscoelastic skeletons were considered in Cioranescu and Saint Jean Paulin [4] and respectively in Ene and Vernescu [6].

2 Formulation

We consider a bounded domain Ω in $\Re^3$, containing an emulsion of two fluids. The viscosities of the bubbles and of the surrounding fluid are μ_1 and μ_2 respectively with $0 < \mu_1 < \mu_2$.

The local fluid velocity is denoted by $v(x)$. We consider the local strain rate tensor $e(v) = (\nabla v + \nabla v^T)/2$ and the local stress tensor $\sigma = 2\mu e(v) - pI$ where p is the local pressure and:

$$\mu = \begin{cases} \mu_1 & \text{in the bubbles,} \\ \mu_2 & \text{in the continuous fluid phase.} \end{cases} \tag{2.1}$$

For a prescribed body force f the equations of motion in each phase are:

$$div\sigma + f = 0 \tag{2.2}$$

and the incompressibility condition is:

$$div\ v = 0. \tag{2.3}$$

On the boundary of Ω a no-slip condition is imposed.

Following Taylor [17] and others (Cox [5], Schowalter, Chaffey and Brenner [15]) , we assume that the fluid velocity is continuous across the bubble surfaces. For a suspension of n bubbles, we denote the velocity of the center of mass of the i-th bubble, $1 \leq i \leq n$, by V^i. The kinematic condition on the bubble surface Γ^i is given by:

$$v \cdot n = V^i \cdot n. \tag{2.4}$$

The associated dynamic condition is given by:

$$[\sigma n] = [\sigma n] \cdot n\ n \tag{2.5}$$

where n is the exterior unit normal to the bubble surface Γ_i. Here the notation [ ] denotes the jump of the bracketed quantity across the bubble surface.

The balance of forces on each bubble B^i is given by:

$$\int_{B^i} f dx + \int_{\Gamma^i} \sigma n ds = 0 \tag{2.6}$$

Here the surface integral of the normal stress is evaluated on the exterior of the bubble. The above condition is easily seen from equations (2.2) and (2.5) to be equivalent to:

$$\int_{\Gamma^i} [\sigma n] ds = 0 \tag{2.7}$$

The balance of torque on each bubble B^i is given by:

$$\int_{B^i} x \times f dx + \int_{\Gamma^i} x \times \sigma n ds = 0 \tag{2.8}$$

In view of (2.2), (2.5) and (2.7) and the fact that the bubbles are spherical equation (2.8) is automatically satisfied.

The flow problem is to simultaneously find the flow v, pressure p and the bubble velocities $V^i, i = 1, \ldots, n$ satisfying (2.1) through (2.6). We remark that the problem given by (2.1)-(2.6) is a specialization of the suspension flow problem formulated in Keller, Rubenfeld and Molyneux [9], to emulsions.

We observe that the bubble velocities are related to the flow by the following:

$$V^i = \frac{1}{\mid B^i \mid} \int_{B^i} v dx \tag{2.9}$$

This follows immediately from (2.4) and the identity:

$$\int_{B^i} v_j dx = \int_{\Gamma^i} v \cdot n x_j ds \tag{2.10}$$

which holds for divergence free flows.

3 Homogenization Result

We suppose that at some instant in time the emulsion is periodic with the ratio between the period and the characteristic length of the domain given by ϵ. We consider a unit periodic reference emulsion of bubbles B^i with centers specified by the vectors r^i, such that r^0 coincides with the origin. The bubbles of the ϵ-periodic emulsion are denoted by $B^{i\epsilon}$ and their centers are given by ϵr^i. Thus the coordinates of a point in the emulsion will be given by:

$$x = \epsilon r^i + \epsilon y \tag{3.1}$$

with $y \in (-\frac{1}{2}, \frac{1}{2})^3$.

The emulsion is equivalently characterized by an ϵ-periodic viscosity μ^ϵ given by:

$$\mu^\epsilon = \mu(\frac{x}{\epsilon}), \text{ where } \mu(y) = \mu_1\chi_1(y) + \mu_2\chi_2(y) \tag{3.2}$$

where χ_1 and χ_2 are the characteristic functions of the bubble and of the surrounding fluid in the unit period cell $Q = (-1/2, 1/2)^3$.

We consider the associated family of emulsion flow problems with solutions $v^\epsilon, p^\epsilon, V^{i\epsilon}$ satisfying:

$$\begin{aligned}
div\sigma^\epsilon + f &= 0 \qquad \text{in } \Omega - \cup_i \partial B^{i\epsilon} \\
\sigma^\epsilon &= 2\mu^\epsilon e(v^\epsilon) - p^\epsilon I \qquad \text{in } \Omega \\
div v^\epsilon &= 0 \qquad \text{in } \Omega \\
[v^\epsilon] &= 0 \qquad \text{on } \Gamma^{i\epsilon} = \partial B^{i\epsilon} \\
v^\epsilon \cdot n &= V^{i\epsilon} \cdot n \qquad \text{on } \Gamma^{i\epsilon} \\
[\sigma^\epsilon n] &= [\sigma^\epsilon n] \cdot n\, n \qquad \text{on } \Gamma^{i\epsilon} \\
\int_{\Gamma^{i\epsilon}} [\sigma^\epsilon n] ds &= 0 \\
v^\epsilon &= 0 \qquad \text{on } \partial\Omega
\end{aligned} \tag{3.3}$$

We observe that the pressure can be adjusted by a constant in each bubble and still satisfy (3.3).

To obtain the asymptotic behavior of the flow, we define the following local problem:

$$\begin{aligned}
div_y \tau^{ij} &= 0 \qquad \text{in } Q - \Gamma \\
div_y v^{ij} &= 0 \qquad \text{in } Q \\
[v^{ij}] &= 0 \qquad \text{on } \Gamma \\
(\frac{1}{2}(\delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell} - \frac{2}{3}\delta_{ij}\delta_{\ell m})y_\ell + v^{ij}_m)n_m &= \frac{1}{\mid B \mid}(\int_B v^{ij}_m dy) \cdot n_m \\
[\tau^{ij} n] &= [\tau^{ij} n] \cdot nn \qquad \text{on } \Gamma \\
\int_{\partial B} [\tau^{ij} n] ds_y &= 0 \qquad \text{on } \Gamma
\end{aligned} \tag{3.4}$$

where:

$$\tau^{ij}_{\ell m} = 2\mu(\frac{1}{2}(\delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}) - \frac{1}{3}\delta_{ij}\delta_{\ell m} + e_{\ell m}(v^{ij})) - p^{ij}\delta_{\ell m}$$

and

$$\int_Q p^{ij}dy = 0.$$

Here v^{ij} is a Q periodic vector field, and p^{ij} is normalized such that its average is zero.

We introduce a security region S inside the unit period containing the bubble B, and denote by B_+ the set:

$$B_+ = S - \overline{B} \tag{3.5}$$

The region homothetic to B_+ about the bubble $B^{i\epsilon}$ is denoted by $B_+^{i\epsilon}$.

We introduce a normalized pressure field $\overline{p}^\epsilon$ by subtracting off a constant pressure $C^{i\epsilon}$ inside each bubble given by:

$$C^{i\epsilon} = \frac{1}{\mid B^{i\epsilon} \mid}\int_{B^{i\epsilon}} p^\epsilon dx - \frac{1}{\mid B_+^{i\epsilon} \mid}\int_{B_+^{i\epsilon}} p^\epsilon dx \tag{3.6}$$

From earlier remarks, the normalized pressure also satisfies the emulsion flow problem (3.3). For each ϵ the constant $C^{i\epsilon}$ is a measure of the difference between the average pressures of the fluid inside and outside the bubble. We show in Lemma 4.5 that the normalized pressure is uniformly bounded in $L^2(\Omega)$.

The asymptotic behavior of the flow is described by the following homogenization result:

Theorem 3.1 *As ϵ tends to zero, we have: for any body force f in $H^{-1}(\Omega)$, the sequence of flow fields and pressures $(v^\epsilon, \overline{p}^\epsilon)$ converges weakly in $H_0^1(\Omega))^3 \times L^2(\Omega)$ to (v^0, q) satisfying the* homogenized *flow equation given by:*

$$\begin{aligned} div\sigma^H + f &= 0 \quad in\ \Omega \\ \sigma^H_{ij} &= 2\mu^H_{ijkl}e_{kl}(v^0) - q\delta_{ij}\ in\Omega \\ divv^0 &= 0\ in\Omega \\ v^0 &= 0 \quad on\ \partial\Gamma \end{aligned} \tag{3.7}$$

where the effective viscosity μ^H_{ijkl} is defined by:

$$2\mu^H_{lkij} = \int_Q \left(\tau^{ij}_{lk} - \frac{1}{3}\delta_{lk}\tau^{ij}_{pp}\right) dy - \int_\Gamma \left([\tau^{ij}_{lm}n_m]y_k - \frac{1}{3}[\tau^{ij}_{pm}n_m]y_p\delta_{lk}\right) ds \tag{3.8}$$

with $\mu^H_{ijkl} = \mu^H_{jikl} = \mu^H_{klij}$.

4 Two-scale asymptotic expansions

In this section we formally establish the homogenized flow equations for the emulsion using the method of two-scale asymptotic expansions (see Sanchez-Palencia [14] and Bensoussan, Lions and Papanicolaou [3]).

To fix ideas we consider a unit periodic reference emulsion with drop centers specified by the vectors r^l such that r^0 is coincident with the origin. The drop centers of the ϵ-periodic emulsion are given by ϵr^l. Then the coordinates of a point in the emulsion will be given by:

$$x = \epsilon r^l + \epsilon y \tag{4.1}$$

with $y \epsilon (-\frac{1}{2}, \frac{1}{2})^3$.

The expansions for velocity and pressure are given by:

$$v^\epsilon(x) = v^0(x) + \epsilon v^1(x,y) + ... \tag{4.2}$$

$$p^\epsilon(x) = \epsilon^{-1} p^{-1}(x,y) + p^0(x,y) + ... \tag{4.3}$$

Therefore the expansion for the stress is:

$$\sigma^\epsilon(x) = \epsilon^{-1}\sigma^{-1}(x) + \sigma^0(x,y) + \epsilon\sigma^1(x,y) + ... \tag{4.4}$$

where:

$$\sigma^{-1} = -p^{-1}(x,y)I, \tag{4.5}$$

$$\sigma^0 = -p^0(x,y)I + 2\mu(e_x(v^0) + e_y(v^1)), \tag{4.6}$$

$$\sigma^1 = -p^1(x,y)I + 2\mu(e_x(v^1) + e_y(v^2)) \tag{4.7}$$

and so on.

We start by identifying σ^{-1}. From (4.4) and (2.2) it follows that:

$$\nabla_y p^{-1} = 0 \tag{4.8}$$

and therefore p^{-1} is piecewise constant in y and upon rescaling

$$p^{-1}(x,y) = T(x)\chi_1(y) \tag{4.9}$$

where χ_1 is the indicator function of the bubble in the unit cell Q.

In what follows we deduce the asymptotic forms for the kinematic and dynamic conditions on the bubble surface. These are used in the determination of the effective viscosity and homogenized flow equations.

We will assume in what follows that the velocity field is sufficiently smooth and expand the velocity terms $v^0, v^1, \ldots$ in Taylor series around the center of the bubbles:

$$v^k(x) = v^{(k)}(\epsilon r^l) + \frac{\partial v^k}{\partial x_j}(\epsilon r^l)(x_j - \epsilon r^l_j) + \ldots \tag{4.10}$$

for $k = 0, 1, \ldots$. Substitution of these expansions in the kinematic conditions (2.4) and (2.9) gives at order ϵ^0 the obvious identity:

$$v^0(\epsilon r^l) \cdot n = v^0(\epsilon r^l) \cdot n \tag{4.11}$$

at order ϵ^1 we have:

$$(\frac{\partial v^0}{\partial x_j}(\epsilon r^l) y_j + v^1(\epsilon r^l, y)) \cdot n = \frac{1}{\mid B \mid} \int_B (\frac{\partial v^0}{\partial x_j}(\epsilon r^l) y_j + v^1(\epsilon r^l, y)) dy. \tag{4.12}$$

Due to symmetry the first term in the right member is zero and we obtain:

$$(\frac{\partial v^0}{\partial x_j}(\epsilon r^l) y_j + v^1(\epsilon r^l, y)) \cdot n = \frac{1}{\mid B \mid} \int_B v^1(\epsilon r^l, y) \cdot n dy, \tag{4.13}$$

noting that as ϵ tends to zero, the emulsion becomes arbitrarily fine we replace ϵr^l by x and write:

$$(\frac{\partial v^0}{\partial x_j}(x) y_j + v^1(x, y)) \cdot n = \frac{1}{\mid B \mid} \int_B v^1(x, y) \cdot n dy, \tag{4.14}$$

We can apply a similar method to expand the balance of forces (2.7). Let us consider Taylor expansions for the stress, inside and outside the bubble of the form

$$\sigma^\epsilon_{in} = \epsilon^{-1} T(x) \chi_1(y) + \sigma^0_{in}(\epsilon r^i, y) + \frac{\partial \sigma^0_{in}}{\partial x_l}(\epsilon r^i, y)(x_l - \epsilon r^i_l) +$$
$$\epsilon \sigma^1_{in}(\epsilon r^i, y) + + \epsilon \frac{\partial \sigma^1_{in}}{\partial x_l}(\epsilon r^i, y)(x_l - \epsilon r^i_l) + \ldots \tag{4.15}$$

$$\sigma^{\epsilon}_{out} = \sigma^0_{out}(\epsilon r^i, y) + \frac{\partial \sigma^0_{out}}{\partial x_l}(\epsilon r^i, y)(x_l - \epsilon r^i_l) + \epsilon \sigma^1_{out}(\epsilon r^i, y) + $$
$$+ \epsilon \frac{\partial \sigma^1_{out}}{\partial x_l}(\epsilon r^i, y)(x_l - \epsilon r^i_l) + ... \qquad (4.16)$$

where σ^{ϵ}_{out} is smoothly extended inside the bubble. Then from (2.7) and (4.15) and (4.16) it follows that:

$$\int_{\partial B} Tn = 0 \qquad (4.17)$$

$$\int_{\partial B} [\sigma^0 n] = 0 \qquad (4.18)$$

$$\int_{\partial B} [\frac{\partial \sigma^0}{\partial x_i} y_i + \sigma^1] n = 0 \qquad (4.19)$$

We now substitute (4.2), (4.4) in (2.2), (2.3), (2.5) and together with (4.15), (4.19) and (4.20) to obtain an explicit form for p^{-1}, the local problem and the homogenized problem.

To identify the p^{-1} term and the local problem we note that from the ϵ^{-1} and ϵ^0 perturbations of the balance of momentum and mass we have:

$$div_x \sigma^{-1} + div_y \sigma^0 = 0 \qquad (4.20)$$

$$div_x v^0 + div_y v^1 = 0 \qquad (4.21)$$

and from (2.5) and the continuity of velocity it follows that:

$$[\sigma^0 n] = [\sigma^0 n] \cdot nn \qquad (4.22)$$

$$[v^0] = 0, \;\; [v^1] = 0. \qquad (4.23)$$

Integrating (4.20) in the y variable, over the unit cell and integrating by parts applying (4.9) and (4.18) we find:

$$\nabla_x T(x) = 0 \qquad (4.24)$$

Hence

$$p^{-1}(x,y) = T\chi_1(y) \tag{4.25}$$

where T is a constant in both variables, and (4.21) becomes:

$$div_y \sigma^0 = 0. \tag{4.26}$$

Therefore:

$$p^\epsilon = T\chi_1(y)\epsilon^{-1} + p^0(x,y) + ...$$

The first term in the series is phisically interpreted as Laplace's formula for the jump in pressure on the surface of a spherical bubble of radius $a\epsilon$. Due to the piecewise constant nature of the singular term, it does not affect the bulk flow, as the system (3.3) is invariant under a constant pressure change in each phase.

A similar integration by parts of (4.22) over the unit cell, yields

$$div_x v^0 = 0 \tag{4.27}$$

from which it follows that

$$div_y v^1 = 0 \tag{4.28}$$

Collecting our observations we find that for each point x, v^1 solves the following flow problem in the unit cell:

$$\begin{aligned}
div_y(2\mu(y)e_x(v^0) + 2\mu(y)e_y v^1) &= \nabla_y p^0 && (4.29)\\
div_y v^1 &= 0 && (4.30)\\
[v^1] &= 0 && (4.31)\\
(\frac{\partial v^0}{\partial x_i} y_i + v^1)\cdot n &= \frac{1}{\mid B \mid}\int_B v^1(x,y)dy \cdot n && (4.32)\\
[\sigma^0 n] &= [\sigma^0 n]\cdot nn && (4.33)\\
\int_B [\sigma^0] n &= 0 && (4.34)
\end{aligned}$$

We observe that on the surface of the sperhical bubble, the normal vector $n = y$ and only the summetric part of $\frac{\partial v_j^0}{\partial x_i}$ appears in the contraction $\frac{\partial v_j^0}{\partial x_i} y_i n_j$; i.e:

$$\frac{\partial v_j^0}{\partial x_i} y_i n_j = e_{ij}(v^0) y_i n_j. \tag{4.35}$$

Therefore we see that the solution v^1, p^0 of (4.29)-(4.34) depends linearly on $e(v^0)$ and we write:

$$v^1 = v^{ij}(y) e_{ij}(v^0) + w(x) \tag{4.36}$$
$$p^0 = p^{ij}(y) e_{ij}(v^0) + q(x) \tag{4.37}$$

where v^{ij}, p^{ij} are solutions of the local problem:

$$
\begin{aligned}
div_y \tau^{ij} &= 0 \qquad \text{in } Q - \Gamma && (4.38)\\
div_y v^{ij} &= 0 \qquad \text{in } Q && (4.39)\\
[v^{ij}] &= 0 \qquad \text{on} \Gamma && (4.40)\\
(\frac{1}{2}(\delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell} - \frac{2}{3}\delta_{ij}\delta_{\ell m}) y_\ell + v_m^{ij}) n_m &= \frac{1}{\mid B \mid}(\int_B v_m^{ij} dy) \cdot n_m && (4.41)\\
[\tau^{ij} n] &= [\tau^{ij} n] \cdot nn \qquad \text{on} \Gamma && (4.42)\\
\int_{\partial B} [\tau^{ij} n] ds_y &= 0 && (4.43)
\end{aligned}
$$

where:

$$\tau_{\ell m}^{ij} = 2\mu(\frac{1}{2}(\delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}) - \frac{1}{3}\delta_{ij}\delta_{\ell m} + e_{\ell m}(v^{ij})) - p^{ij}\delta_{\ell m} \tag{4.44}$$

In view of (4.44) we may write σ^0 as

$$\sigma_{\ell m}^0 = e_{xij}(v^0) \tau_{\ell m}^{ij} - q\delta_{\ell m} \tag{4.45}$$

To obtain the homogenized problem we consider the ϵ^0 perturbation for the balance of momentum:

$$div_x \sigma^0 + div_y \sigma^1 + f = 0. \tag{4.46}$$

Integrating (4.46) in the y variable over the unit cell yields:

$$\int_{\partial Q} \sigma^1 n ds_y + \int_{\partial B} [\sigma^1 n] ds_y + div_x \int_Q \sigma^0 dy + f = 0 \tag{4.47}$$

The first term vanishes by periodicity. From (4.19) we see that

$$\int_{\partial B} [\sigma^1_{ij} n_j] ds_y = - \int_{\partial B} [\frac{\partial \sigma^0_{ij}}{\partial x_\ell} y_\ell n_j] ds_y. \tag{4.48}$$

From (4.47) together with (4.48) and (4.45) we obtain the homogenized momentum equation:

$$\frac{\partial}{\partial x_k}(2\mu^H_{\ell kij} e_{xij}(v^0) - q\delta_{\ell k}) = f_\ell \tag{4.49}$$

where:

$$2\mu^H_{\ell kij} = \int_Q (\tau^{ij}_{\ell k} - \frac{1}{3}\delta_{\ell k} tr\tau^{ij}) dy - \int_{\partial B} [(\tau^{ij}_{\ell m} - \frac{1}{3}\delta_{\ell k} tr\tau^{ij}) y_k n_m] ds_y \tag{4.50}$$

5 Two-scale Convergence Proof of the Homogenization Result

5.1 Pressure Extension and Estimates

The emulsion flow problem (3.3) has the equivalent variational formulation: for $f \in L^2(\Omega)$, find $v^\epsilon \in \mathbf{V}^\epsilon$ such that:

$$\int_\Omega 2\mu^\epsilon e(v^\epsilon) : e(w) dx = \int_\Omega f w dx, \text{ for any } w \in \mathbf{V}^\epsilon \tag{5.1}$$

where $\mathbf{V}^\epsilon$ is the closed subspace of $(H^1_0(\Omega))^3$ given by:

$$\mathbf{V}^\epsilon = \{w \in (H^1_0(\Omega))^3 \mid divw = 0 \text{ in } \Omega, w \cdot n = W^{i\epsilon} \cdot n \text{ on } \Gamma^{i\epsilon}, W^{i\epsilon} = \frac{1}{\mid B^{i\epsilon} \mid} \int_{B^{i\epsilon}} w dx\} \tag{5.2}$$

We remark that the last equation in (5.2) is a compatibility condition for a divergence free vector field with normal component constant on $\Gamma^{i\epsilon}$.

The existence and uniqueness of the solution of the emulsion flow problem follows from the Lax-Milgram lemma.

It also follows immediately from (5.1) that the velocity field is bounded uniformly in ϵ:

$$\| v^\epsilon \|_{H_0^1(\Omega)} < C \tag{5.3}$$

The pressure gradients delivered by (5.1) are linear functionals on subspaces of $H_0^1(\Omega)$. Following the ideas of Tartar [16] we construct an extension for the pressure gradient as a linear functional on $H_0^1(\Omega)$. We show as in [12] that the extension is equivalent to a suitable normalization of pressures.

Lemma 5.1 *There exists a restriction operator:*

$$R^\epsilon \in \mathcal{L}[(H_0^1(\Omega))^3; H^\epsilon] \tag{5.4}$$

satisfying:

1. $R^\epsilon u = u,$ *for any* $u \in H^\epsilon$ (5.5)
2. *if* $div u = 0$ *then* $div R^\epsilon u = 0$
3. $\epsilon \| \nabla R^\epsilon u \|_{L^2(\Omega)} + \| R^\epsilon u \|_{L^2(\Omega)} \leq C(\epsilon \| \nabla u \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)})$

here:

$$H^\epsilon = \{ w \in (H_0^1(\Omega))^3 \mid w \cdot n = W^{i\epsilon} \cdot n \ on \ \Gamma^{i\epsilon} \}.$$

Proof. Let us introduce for any $u \in H^1(Q), v^+ \in H^1(B_+), q^+ \in L^2(B_+)$ and $v \in H^1(B), q \in L^2(B)$ solutions of the following nonhomogeneous Stokes problems:

$$\begin{aligned} \Delta v^+ &= \Delta u - \nabla q^+, \text{ in } B_+ \\ div v^+ &= div u + C^+, \text{ in } B_+ \\ v^+ &= u \text{ on } \Gamma^+ = \partial S \\ v^+ \cdot n &= V \cdot n \text{ on } \Gamma \\ v^+ \cdot \tau &= u \cdot \tau \text{ on } \Gamma \end{aligned} \tag{5.6}$$

$$
\begin{aligned}
\Delta v &= \Delta u - \nabla q \text{ in } B \\
divv &= divu + C \text{ in } B \\
v \cdot n &= V \cdot n \text{ on } \Gamma \\
v \cdot \tau &= u \cdot \tau \text{ on } \Gamma
\end{aligned}
\tag{5.7}
$$

where τ is any tangent vector to Γ and

$$V = \frac{1}{\mid B \mid} \int_B (u + (divu)x)dx \tag{5.8}$$

$$C^+ = \frac{1}{\mid B_+ \mid} \int_\Gamma u \cdot nds \tag{5.9}$$

$$C = -\frac{1}{\mid B \mid} \int_\Gamma u \cdot nds \tag{5.10}$$

The existence and uniqueness of the solutions for (5.5) and (5.6) follow from general results for non-homogeneous Stokes problems (see Teman [18])

We define the restriction operator on $(H^1(Q))^3$:

$$R(u) = \begin{cases} u & \text{in} \quad Q - S \\ v^+ & \text{in} \quad B_+ \\ v & \text{in} \quad B \end{cases}$$

and applying R to every εQ period we define R^ϵ.

The first property of the restriction follows from the uniqueness of the solutions of (5.5) and (5.6) and from the fact that for any $u \in H^1(Q)$ with $u \cdot n = U \cdot n$ on Γ we have that $V = U$.

Since the normal component of Ru is continuous across Γ_+ and Γ, and $divu = 0$ the second property follows immediately from (5.8) and (5.9).

From standard trace and lift estimates we obtain:

$$\parallel R(u) \parallel_{H^1(Q)} \leq C \parallel u \parallel_{H^1(Q)}$$

and the third property is obtained by rescaling.$\square$

Substitution of smooth test fields in $\mathbf{V}^\epsilon$ with support in each phase in (5.1) delivers pressure fields p^ϵ in each phase such that:

$$div(2\mu_2 e(v^\epsilon) - p^\epsilon I) = f \text{ in } H^{-1}(\Omega - \cup_i \overline{B}^{i\epsilon}) \tag{5.11}$$

$$div(2\mu_1 e(v^\epsilon) - p^\epsilon I) = f \text{ in } H^{-1}(B^{i\epsilon}) \tag{5.12}$$

Definition 5.2 *The normalized pressure $\overline{p}^\epsilon$ is defined by:*

$$\overline{p}^\epsilon = p^\epsilon - \sum_i C^{i\epsilon}\chi_{B^{i\epsilon}} \tag{5.13}$$

where the constants $C^{i\epsilon}$ are given by:

$$C^{i\epsilon} = -\frac{1}{\mid B_+^{i\epsilon} \mid}\int_{B_+^{i\epsilon}} p^\epsilon dx + \frac{1}{\mid B^{i\epsilon} \mid}\int_{B^{i\epsilon}} p^\epsilon dx \tag{5.14}$$

Lemma 5.3 *The normalized pressure gradient $\nabla\overline{p}^\epsilon$ is in $H^{-1}(\Omega)$ and satisfies*

$$\int_\Omega \overline{p}^\epsilon divu dx = \int_\Omega \overline{p}^\epsilon div R^\epsilon(u) dx = \int_\Omega p^\epsilon div R^\epsilon(u) dx \tag{5.15}$$

for any $u \in H_0^1(\Omega)$.

Proof. Indeed:

$$\begin{aligned}\int_\Omega (\overline{p}^\epsilon divu &- \overline{p}^\epsilon div R^\epsilon(u))dx = \\ &= \sum_i \Bigg(\int_{B_+^{i\epsilon}} p^\epsilon(divu - divv^+)dx \\ &+ \int_{B^{i\epsilon}} (p^\epsilon + C^{i\epsilon})(divu - divv)dx \Bigg) = \\ &= \sum_i (-\int_{B_+^{i\epsilon}} p^\epsilon C^+ dx - \int_{B^{i\epsilon}} (p^\epsilon + C^{i\epsilon})C dx) = 0\end{aligned}$$

Here the last equality follows from (5.8), (5.9) and (5.14).

The second equality in (5.15) is a direct consequence of the definition of the normalized pressure and the restriction operator. □

We denote by $\overline{\sigma}^\epsilon$ the stress associated with the normalized pressure (see Definition 5.2) i.e.:

$$\overline{\sigma}^\epsilon = -\overline{p}^\epsilon I + 2\mu^\epsilon e(v^\epsilon)$$

Remark 5.4 *Since the solution of the emulsion flow problem is unique up to a constant pressure in each phase, we see that we may replace σ^ϵ by $\overline{\sigma}^\epsilon$ in (3.3).*

We are now in a position to estimate the normalized pressure.

Lemma 5.5 *The normalized pressure is uniformly bounded in $L^2(\Omega)$:*

$$\| \overline{p}^\epsilon \|_{L^2(\Omega)|\Re} \leq C. \tag{5.16}$$

PROOF. We first estimate $\epsilon\overline{p}^\epsilon$ and then use this to prove the uniform bound on the sequence $\overline{p}^\epsilon$.

Indeed, it follows from the emulsion flow problem (3.3) and its variational formulation (5.1) that

$$- \, div\overline{\sigma}^\epsilon = f \tag{5.17}$$

as regular distributions in each phase, thus for u in $(H_0^1(\Omega))^3$ we have

$$(div\overline{\sigma}^\epsilon, R^\epsilon(u)) = -(f, R^\epsilon(u)). \tag{5.18}$$

Integrating by parts using an adequate Stokes formula (c.f. Temam [18]) yields:

$$\int_\Omega p^\epsilon div R^\epsilon(u) dx = \int_\Omega 2\mu^\epsilon e(v^\epsilon) : e(R(u)) - \int_\Omega f \cdot R(u) \tag{5.19}$$

Combining Lemma 5.3 with (5.19) and the estimate $(5.5)_3$ we find that the normalized pressure gradient satisfies

$$\| \nabla\overline{p}^\epsilon \|_{H^{-1}(\Omega)} \leq \epsilon^{-1} C. \tag{5.20}$$

Thus from [18] it follows that

$$\| \epsilon\overline{p}^\epsilon \|_{L^2(\Omega)|\Re} \leq C. \tag{5.21}$$

Multiplying (3.3) by $w^\epsilon\phi$ where $\phi \in \mathcal{D}(\Omega)$, and $w^\epsilon(x) = w(\frac{x}{\epsilon})$ with $w \in (H^1_{per}(Q))^3$ and integrating by parts we obtain the identity:

$$\int_\Omega \overline{\sigma}^\epsilon : e(w^\epsilon\phi) dx - \sum_i \int_{\Gamma^{i\epsilon}} [\overline{\sigma}^\epsilon n] \cdot w^\epsilon \phi ds = \int_\Omega f w^\epsilon \phi dx \tag{5.22}$$

Here the integral over $\Gamma^{i\epsilon}$ is understood in the sense of traces: $\overline{\sigma}^\epsilon n \in (H^{-1/2}(\Gamma^{i\epsilon}))^3$ and $w^\epsilon \phi \in (H^{1/2}(\Gamma^{i\epsilon}))^3$. Choosing $w \cdot n = 0$ on Γ, it follows from $(3.3)_6$ that the second term in (5.22) vanishes and multiplication of the result by ϵ gives:

$$\int_\Omega \overline{p}^\epsilon (div_y w)^\epsilon \phi dx = -\epsilon \int_\Omega f w^\epsilon \phi dx + \int_\Omega 2\mu^\epsilon e(v^\epsilon) : ((e_y w)^\epsilon \phi \quad - \quad \epsilon w^\epsilon \cdot \nabla \phi) dx - \\ - \int_\Omega \epsilon \overline{p}^\epsilon w^\epsilon \cdot \nabla \phi dx \tag{5.23}$$

where $(e_y w)^\epsilon = (e_y w)(x/\epsilon)$ and $(div_y w)^\epsilon = (div_y w)(x/\epsilon)$. It now follows from the uniform $L^2(\Omega)$ bound on $\epsilon \overline{p}^\epsilon$ and (5.3) that the right hand side of (5.23) is uniformly bounded. Therefore:

$$\left| \int_\Omega \overline{p}^\epsilon (div_y w)^\epsilon \phi dx \right| < C \tag{5.24}$$

for all $\phi \in \mathcal{D}(\Omega), w \in (H^1_{per}(Q))^3, w \cdot n = 0$ on Γ. This is equivalent to:

$$\left| \int_\Omega \overline{p}^\epsilon s^\epsilon \phi dx \right| < C \tag{5.25}$$

for all $\phi \in \mathcal{D}(\Omega), s \in L^2_{per}(Q), \int_B s dy = \int_{Q-B} s dy = 0$. By a density argument, for ϵ fixed, we may choose $s^\epsilon \phi$ to approximate $\overline{p}^\epsilon$ and obtain the estimate:

$$\| \overline{p}^\epsilon \|_{L^2(\Omega)|\Re} \leq C. \tag{5.26}$$

□

5.2 Convergence of the Conservation of Mass

The sequence v^ϵ of flow fields is uniformly bounded in $H^1_0(\Omega)$, therefore it follows from the two-scale convergence introduced by Nguetseng [13] (see also Allaire [1]) that there exists $v^0 \in H^1(\Omega)$ and $v^1 \in L^2(\Omega, H^1_{per}(Q))$ such that

$$v^\epsilon \rightharpoonup v^0, \text{ weakly in } H^1(\Omega) \text{ and} \tag{5.27}$$

$$\frac{\partial v^\epsilon}{\partial x_i} \rightsquigarrow \frac{\partial v^0}{\partial x_i} + \frac{\partial v^1}{\partial y_i} \tag{5.28}$$

where the convergence indicated in (5.28) is in the two-scale sense, i.e. for $u^\epsilon \in L^2(\Omega), u^0 \in L^2(\Omega, L^2_{per}(Q)), u^\epsilon \rightsquigarrow u^0$ if and only if:

$$\int_\Omega u^\epsilon w^\epsilon \phi dx \to \int_{\Omega\times Q} u^0(x,y)w(y)\phi(x)dydx \qquad (5.29)$$

for any $w \in L^2_{per}(Q), w^\epsilon(x) = w(\frac{x}{\epsilon})$ and $\phi \in \mathcal{D}(\Omega)$. To expedite the presentation the symbol "$\rightsquigarrow$" will be used to indicate two-scale convergence.

Applying the two scale convergence to the conservation of mass law $(3.2)_3$ gives

$$div_x v^0 + div_y v^1 = 0 \qquad (5.30)$$

Integration in the y variable of (5.30) over the unit cell Q yields from periodicity:

$$div_x v^0 = 0, \;\; div_y v^1 = 0 \qquad (5.31)$$

5.3 Convergence of the Kinematic Condition on the Bubble Interface

In this section we rigorously prove the asymptotic behavior of the kinematic condition $(3.3)_5$ as ϵ tends to zero. In view of (5.27) and (5.28) the asymptotic behavior of the kinematic condition is given by the:

Theorem 5.6 *The limits v^0 and v^1 delivered by the two-scale convergence of the emulsion flow fields v^ϵ satisfy:*

$$\left(\frac{\partial v^0_j}{\partial x_i} y_i + v^1_j(x,y)\right) n_j = \frac{1}{\mid B \mid}\left(\int_B v^1_j dy\right) n_j \qquad (5.32)$$

for y on Γ and x in Ω.

The proof of Theorem 5.6 follows immediately from the following two lemmas.

Lemma 5.7 *Let $\{v^\epsilon\}$ be a bounded sequence in $H^1(\Omega)$ and let $\{v^\epsilon\}$ be an $L^2(\Omega)$ approximation to $\{v^\epsilon\}$, i.e.:*

$$V^\epsilon = \sum_i \left(\frac{1}{\mid B^{i\epsilon} \mid}\int_{B^{i\epsilon}} v^\epsilon dx\right)\chi(Q^{i\epsilon}) \qquad (5.33)$$

then there exists $c_i \in L^2(\Omega)$ such that:

$$\frac{1}{\epsilon}(v_i^\epsilon - V_i^\epsilon) \rightsquigarrow \frac{\partial v_j^0}{\partial x_i} y_j + v_i^1(x,y) + c_i(x) \tag{5.34}$$

PROOF. From (5.34) and Poincare's inequality we have:

$$\| v^\epsilon - V^\epsilon \|_{L^2(\Omega)} \leq C\epsilon \tag{5.35}$$

and thus from the two-scale convergence theorem, there exists $s \in L^2(\Omega, L^2_{per}(Q))$ such that:

$$\frac{1}{\epsilon}(v^\epsilon - V^\epsilon) \rightsquigarrow s \tag{5.36}$$

In order to identify s, we introduce $w \in L^2(Q)$, such that $divw \in L^2(Q), w \cdot n = 0$ on ∂Q and write the following identity:

$$\frac{1}{\epsilon}\int_\Omega (v^\epsilon - V^\epsilon)(div_y w)^\epsilon \phi dx = \int_\Omega (v^\epsilon - V^\epsilon) div\ w^\epsilon \phi dx \tag{5.37}$$

for any $\phi \in \mathcal{D}(\Omega)$. The left hand side converges from (5.36) to:

$$\int_{\Omega\times Q} s(x,y) div_y w \phi(x) dy dx \tag{5.38}$$

and integration by parts on the right hand side yields:

$$\int_\Omega (v^\epsilon - V^\epsilon) div\ w^\epsilon \phi dx = -\int_\Omega \frac{\partial v^\epsilon}{\partial x_i} w_i^\epsilon \phi dx - \int_\Omega (v^\epsilon - V^\epsilon) w^\epsilon \cdot \nabla\phi \tag{5.39}$$

which by (5.35) and two-scale convergence has the limit:

$$-\int_{\Omega\times Q} \left(\frac{\partial v^0}{\partial x_i} + \frac{\partial v^1}{\partial y_i} \right) w_i(y) \phi(x) dy dx \tag{5.40}$$

Equating (5.38) and (5.40) we see that:

$$\frac{\partial s}{\partial y_i} = \frac{\partial v^0}{\partial x_i} + \frac{\partial v^1}{\partial y_i} \tag{5.41}$$

and the lemma is proved. □

Lemma 5.8 *Let $\{v^\epsilon\}$ and $\{V^\epsilon\}$ satisfy the conditions of the previous lemma and moreover*

$$div\ v^\epsilon = 0,\ v^\epsilon \cdot n = V^\epsilon \cdot n\ on\ \partial B^{i\epsilon} \tag{5.42}$$

then on ∂B:

$$\left(\frac{\partial v_j^0}{\partial x_i} y_i + v_j^1(x,y)\right) n_j = \frac{1}{\mid B \mid}\left(\int_B v_j^1 dy\right) n_j \tag{5.43}$$

Proof. We let χ_B denote the characteristic function of the bubble B and consider the identity given by:

$$\begin{aligned}\frac{1}{\epsilon}\int_\Omega (v_j^\epsilon &- V_j^\epsilon)\left(\frac{\partial w}{\partial y_j}\chi_B\right)^\epsilon \phi \\ &= \sum_i \left(\int_{\partial B^{i\epsilon}} (v_j^\epsilon - V_j^\epsilon) n_j w^\epsilon \phi dx - \int_{B^{i\epsilon}} div(v^\epsilon - V^\epsilon) w^\epsilon \phi dx\right. \\ &- \left.\int_{B^{i\epsilon}} (v_j^\epsilon - V_j^\epsilon) w^\epsilon \frac{\partial \phi}{\partial x_j} dx\right) \end{aligned} \tag{5.44}$$

Applying the hypotheses (5.42), and using the estimate (5.35) we see that the limit of the right hand side is zero.

Passing to the limit on the left hand side using Lemma 5.7 we find:

$$\int_{\Omega \times B} \left(\frac{\partial v_i^0}{\partial x_j} y_j + v_i^1 + c_i\right) \frac{\partial w}{\partial y_i} \phi(x) dy dx = 0 \tag{5.45}$$

Thus

$$\left(\frac{\partial v_i^0}{\partial x_j} y_j + v_i^1\right) n_i = C \cdot n \text{ on } \partial B \tag{5.46}$$

Since C is constant in y and the vector:

$$\frac{\partial v^0}{\partial x_j} y_j + v^1 \tag{5.47}$$

is from (5.30) divergence free in y, it follows from (2.10) that:

$$C = \frac{1}{\mid B \mid}\left(\int_B v^1 dy\right) \tag{5.48}$$

and the lemma follows.□

5.4 Derivation of the Cell Problem

In this section we obtain the local balance laws of the flow.

From the estimates (5.3) and (5.26) on the velocity field and extended pressure we have $2\mu^\epsilon e(v^\epsilon)$ and $\overline{p}^\epsilon$ bounded in $L^2(\Omega)$. Thus from two-scale convergence:

$$\overline{p}^\epsilon \rightsquigarrow p^0, \;\; \overline{\sigma}^\epsilon \rightsquigarrow \sigma^0 = -p^0 I + 2\mu(y)(e_x(v^0) + e_y(v^1)). \tag{5.49}$$

Lemma 5.9 *The two-scale limit of the stress satisfies:*

$$div_y \sigma^0 = 0, \;\; \int_\Gamma [\sigma^0 n] ds = 0, \;\; [\sigma^0 n] = ([\sigma^0 n] \cdot n) n \tag{5.50}$$

where n is the unit outer normal to Γ

Proof. We first observe that for any function $\phi \in \mathcal{D}(\Omega)$, the step function approximation

$$\phi^\epsilon = \sum_i \phi(\epsilon r^i) \chi(Q^{i\epsilon}) \tag{5.51}$$

converges to ϕ in $L^\infty(\Omega)$. Thus from the Hölder inequality we have that:

$$\int_\Omega w^\epsilon \phi^\epsilon div \overline{\sigma}^\epsilon dx - \int_\Omega w^\epsilon \phi div \overline{\sigma}^\epsilon dx \to 0 \tag{5.52}$$

as $\epsilon \to 0$, since w^ϵ is bounded in $L^2(\Omega)$ and $div \overline{\sigma}^\epsilon = f$ in $(L^2(\Omega - \Gamma^{i\epsilon}))^3$.

We now multiply (3.3) by $w^\epsilon \phi^\epsilon$. Here we shall take $w \in (H^1_{per}(Q))^3$, such that $w = 0$ on $\partial Q, w \cdot n = W \cdot n$ on Γ with W constant and ϕ^ϵ as defined above. Integrating by parts we obtain the identity:

$$\int_\Omega \overline{\sigma}^\epsilon : e(w^\epsilon) \phi^\epsilon dx = \int_\Omega f w^\epsilon \phi^\epsilon \tag{5.53}$$

Multiplying (5.53) by ϵ and taking the limit as ϵ goes to zero gives:

$$\int_{\Omega \times Q} \sigma^0(x,y) : e_y(w) \phi(x) dy dx = 0. \tag{5.54}$$

Thus we see that

$$div_y \sigma^0 = 0 \tag{5.55}$$

for y in B and $Q - \overline{B}$. Equation $(5.50)_2$ follows by integrating (5.54) over the period cell. Finally integrating by parts and choosing w with support on Γ and taking arbitrary tangential variations gives:

$$[\sigma^0 n] = ([\sigma^0 n] \cdot n)n \tag{5.56}$$

for y on Γ. □

Collecting the results of Lemma 5.9, equations (5.31) and (5.43), we find that for each point x, v^1 solves the following flow problem in the unit cell:

$$\begin{aligned}
div_y(2\mu(y)e_x(v^0) + 2\mu(y)e_y v^1) &= \nabla_y p^0 \quad \text{in } Q - \Gamma && (5.57)\\
div_y v^1 &= 0 \quad \text{in } Q && (5.58)\\
[v^1] &= 0 \quad \text{on } \Gamma && (5.59)\\
(\frac{\partial v^0}{\partial x_i} y_i + v^1) \cdot n &= \frac{1}{\mid B \mid} \int_B v^1(x,y)dy \cdot n \quad \text{on } \Gamma && (5.60)\\
[\sigma^0 n] &= [\sigma^0 n] \cdot n n \text{on } \Gamma && (5.61)\\
\int_\Gamma [\sigma^0 n] &= 0 && (5.62)
\end{aligned}$$

We observe that on the surface of the sperhical bubble, the normal vector $n = y$ and only the symmetric part of $\frac{\partial v_j^0}{\partial x_i}$ appears in the contraction $\frac{\partial v_j^0}{\partial x_i} y_i n_j$; i.e:

$$\frac{\partial v_j^0}{\partial x_i} y_i n_j = e_{ij}(v^0) y_i n_j. \tag{5.63}$$

Therefore we see that, for each x, the solution v^1, p^0 of (5.57)-(5.62) depends linearly on $e(v^0)$ and we write:

$$v^1 = v^{ij}(y)e_{ij}(v^0) + w(x) \tag{5.64}$$

$$p^0 = p^{ij}(y)e_{ij}(v^0) + q(x) \tag{5.65}$$

where v^{ij}, p^{ij} are solutions of the local problem:

$$\begin{aligned}
div_y \tau^{ij} &= 0 \quad \text{in } Q - \Gamma && (5.66)\\
div_y v^{ij} &= 0 \quad \text{in } Q && (5.67)\\
[v^{ij}] &= 0 \quad \text{on } \Gamma && (5.68)
\end{aligned}$$

$$(\frac{1}{2}(\delta_{i\ell}\delta_{jm}+\delta_{im}\delta_{j\ell}-\frac{2}{3}\delta_{ij}\delta_{\ell m})y_\ell+v_m^{ij})n_m = \frac{1}{|B|}(\int_B v_m^{ij}dy)\cdot n_m \quad (5.69)$$

$$[\tau^{ij}n] = [\tau^{ij}n]\cdot n n \text{on } \Gamma \quad (5.70)$$

$$\int_\Gamma [\tau^{ij}n]ds_y = 0 \quad (5.71)$$

where:

$$\tau_{\ell m}^{ij} = 2\mu(\frac{1}{2}(\delta_{i\ell}\delta_{jm}+\delta_{im}\delta_{j\ell})-\frac{1}{3}\delta_{ij}\delta_{\ell m}+e_{\ell m}(v^{ij}))-p^{ij}\delta_{\ell m} \quad (5.72)$$

and

$$\int_Q p^{ij}dy = 0. \quad (5.73)$$

In view of (5.72) we may write σ^0 as

$$\sigma_{\ell m}^0 = e_{xij}(v^0)\tau_{\ell m}^{ij} - q\delta_{\ell m} \quad (5.74)$$

5.5 Homogenized Momentum Equation and Formula for the Effective Viscosity

Theorem 5.10 *The two scale limit of the stress satisfies the macroscopic momentum balance law given by:*

$$\int_\Omega \left(\int_Q \sigma_{il}^0 dy - \int_\Gamma [\sigma_{ij}^0 n_j]y_l ds_y\right) e_{il}v dx = \int_\Omega f\cdot v dx \quad (5.75)$$

for all $v\in (H_0^1(\Omega))^3$.

PROOF. Multiplying (3.3) by $v\in\mathcal{D}(\Omega)$, we obtain:

$$-\sum_i \int_{\Gamma^{i\epsilon}} [\overline{\sigma}^\epsilon n]\cdot v ds + \int_\Omega \overline{\sigma}^\epsilon : e(v)dx = \int_\Omega f\cdot v \quad (5.76)$$

It is evident from (5.49) that the second term has the limit:

$$\int_{\Omega\times Q} \sigma^0 : e(v)dxdy \quad (5.77)$$

To compute the limit of the first term we introduce $w \in L^2(Q)$ such that for a given constant W, we have $w = W$ on Γ and $w = 0$ on ∂Q. For a given $v \in \mathcal{D}(\Omega)$ we introduce the piecewise constant approximation:

$$V^\epsilon = \sum_i v(\epsilon r^i)\chi(Q^{i\epsilon}) \tag{5.78}$$

In the sequel we use the fact that $v - V^\epsilon$ converges to zero $L^\infty(\Omega)$ as ϵ tends to zero. Multiplying the first term in (5.76) by W, writing $v = V^\epsilon + (v - V^\epsilon)$ and application of $(3.3)_7$ gives:

$$-\sum_i W \int_{\Gamma^{i\epsilon}} [\overline{\sigma}^\epsilon n] \cdot v ds = -\sum_i \int_{\Gamma^{i\epsilon}} W[\overline{\sigma}^\epsilon_{im} n_m] \cdot (v_i - V_i^\epsilon) ds \tag{5.79}$$

Integrating by parts on each pave $Q^{i\epsilon}$ we obtain:

$$\sum_i \left(\int_{Q^{i\epsilon}} \frac{\partial w^\epsilon}{\partial x_m} \overline{\sigma}^\epsilon_{im} (v_i - V_i^\epsilon) dx - \int_{Q^{i\epsilon}} w^\epsilon div \overline{\sigma}^\epsilon \cdot (v - V^\epsilon) dx \quad - \right.$$
$$\left. - \int_{Q^{i\epsilon}} w^\epsilon \overline{\sigma}^\epsilon_{im} \frac{\partial (v_i - V_i^\epsilon)}{\partial x_m} dx \right) \tag{5.80}$$

Here the contribution on the cell boundary $\partial Q^{i\epsilon}$ vanishes as $w^\epsilon = 0$ there. Passing to the limit in (5.80) we see that the second term converges to zero and the third reduces to:

$$-\sum_i \int_{Q^{i\epsilon}} w^\epsilon \overline{\sigma}^\epsilon_{im} \frac{\partial v_i}{\partial x_m} dx \to -\int_{\Omega \times Q} \sigma^0_{im}(x,y) w(y) \frac{\partial v_i}{\partial x_m} dx dy \tag{5.81}$$

For a given $\epsilon > 0$ we have for each pave $Q^{i\epsilon}$ the estimate:

$$\max_{x \in Q^{i\epsilon}} \left| (v_i - V_i^\epsilon) - \partial_l v_i(x)(x_l - r_l^{i\epsilon}) \right| < \epsilon^2 C \tag{5.82}$$

Here the constant C is independent of ϵ and the estimate follows from Taylor's formula and from the uniform Lipshitz continuity of $\partial_{x_l} v_i$. Therefore we can approximate the first term by:

$$\begin{aligned} -\sum_i \int_{Q^{i\epsilon}} \frac{\partial w^\epsilon}{\partial x_m} \overline{\sigma}^\epsilon_{im} (x_l - r_l^{i\epsilon}) \frac{\partial v_i}{\partial x_l} dx &= \\ = -\sum_i \int_{Q^{i\epsilon}} \left(\frac{\partial w}{\partial x_m} \right)^\epsilon \overline{\sigma}^\epsilon_{im} y_l \frac{\partial v_i}{\partial x_l} dx &= \\ = -\int_\Omega \overline{\sigma}^\epsilon_{im} \left(y_l \frac{\partial w}{\partial y_m} \right)^\epsilon \frac{\partial v_i}{\partial x_l} dx & \end{aligned} \tag{5.83}$$

and passage to the limit gives:

$$-\int_{\Omega\times Q}\sigma^0_{im}(x,y)\frac{\partial w}{\partial y_m}y_l\frac{\partial v_i}{\partial x_l}dxdy \tag{5.84}$$

In this way we see that (5.79) converges to

$$-\int_{\Omega\times Q}\sigma^0_{im}(x,y)\frac{\partial w}{\partial y_m}y_l\frac{\partial v_i}{\partial x_l}dxdy-\int_{\Omega\times Q}\sigma^0_{im}(x,y)w(y)\frac{\partial v_l}{\partial x_m}dxdy \tag{5.85}$$

Integration by parts in the first term of (5.85) and applying (5.50) gives:

$$\lim_{\epsilon\to 0}\left(-\sum_i W\int_{\Gamma^{i\epsilon}}[\overline{\sigma}^\epsilon n]\cdot v ds\right)=-W\int_\Omega\frac{\partial v_i}{\partial x_l}\left(\int_\Gamma[\sigma^0_{ij}n_j]y_l ds_y\right)dx \tag{5.86}$$

Setting $W=1$ in (8.12) it follows from (5.80) and (5.81) that the limit of (5.79) is:

$$\int_\Omega\left(\int_Q\sigma^0_{il}dy-\int_\Gamma[\sigma^0_{ij}n_j y_l]ds_y\right)e_{il}v dx=\int_\Omega f\cdot v dx \tag{5.87}$$

and the theorem follows from density of $(\mathcal{D}(\Omega))^3$ in $(H^1_0(\Omega))^3$. $\square$

It follows immediately from Theorem 5.10 that we can identify the deviatoric part of the homogenized stress as:

$$\sigma^H_{lk}-\frac{1}{3}\sigma^H_{ii}\delta_{lk}=\int_Q\left(\sigma^0_{lk}-\frac{1}{3}\sigma^0_{ii}\delta_{lk}\right)dy-\int_\Gamma\left([\sigma^0_{lj}n_j]y_k-\frac{1}{3}[\sigma^0_{ij}n_j]y_i\delta_{lk}\right)ds \tag{5.88}$$

and from (5.74) we see that this is related to the homogenized strain rate $e(v^0)$ through the effective viscosity tensor μ^H given by:

$$\sigma^H_{lk}-\frac{1}{3}\sigma^H_{ii}\delta_{lk}=2\mu^H_{lkij}e_{ij}v^0 \tag{5.89}$$

where

$$2\mu^H_{lkij}=\int_Q\left(\tau^{ij}_{lk}-\frac{1}{3}\delta_{lk}\tau^{ij}_{pp}\right)dy-\int_\Gamma\left([\tau^{ij}_{lm}n_m]y_k-\frac{1}{3}[\tau^{ij}_{pm}n_m]y_p\delta_{lk}\right)ds \tag{5.90}$$

From the Theorem 5.10 and equation (5.74) the hydrostatic part of the homogenized stress σ^H is given by the pressure $q(x)$ i.e.:

$$\frac{1}{3}\sigma^H_{ii}\delta_{lk}=q(x)\delta_{lk} \tag{5.91}$$

Collecting our results we observe that Theorem 3.1 follows from Theorem 5.10, equations (5.31), (5.90) and (5.91).

6 Remarks on the effective viscosity

We extend, by periodicity, the solution v^{ij}, p^{ij} of the cell problem (3.4) to $\Re^3$ and for any constant strain rate ξ we form the linear combination $\varphi = \xi_{ij} v^{ij}$. From linearity we see that ϕ is the periodic solution to the emulsion flow problem on $\Re^3$ given by:

$$\begin{aligned} div\tau &= 0 \qquad \text{in } Q - \Gamma \\ \tau(y) &= 2\mu(e(\varphi) + \xi) - qI \qquad \text{in } Q \\ div\varphi &= 0 \qquad \text{in } Q \\ [\varphi] &= 0 \qquad \text{on } \Gamma = \partial B \\ (\varphi_i + \xi_{ij} y_j) n_i &= \frac{1}{\mid B \mid}(\int_B (\varphi_i + \xi_{ij} y_j)) \cdot n_i \qquad \text{on } \Gamma \\ [\tau n] &= [\tau n] \cdot n\, n \qquad \text{on } \Gamma \\ \int_\Gamma [\tau n] &= 0 \end{aligned} \tag{6.1}$$

For this problem the formal averaging methods given in Batchelor [2] we see that the bulk deviatoric stress Σ is related to the bulk deviatoric strain ξ through the formula:

$$\Sigma_{ij} = 2\mu_2 \xi_{ij} - 2\mu_2 \int_\Gamma (\varphi_i + \xi_{ik} y_k) n_j ds$$
$$-2\mu_2 \int_\Gamma (\varphi_j + \xi_{jk} y_k) n_i ds + \int_\Gamma (\tau_{ik} y_i - \frac{1}{3}\delta_{ij} \tau_{mk} y_m) n_k ds \tag{6.2}$$

We remark that the effective dissipation rate $\Sigma : \xi$ agrees with the dissipation rate associated with the homogenized effective viscosity described in Theorem 3.1.

To see this we contract equation (3.8) with ξ_{ij}:

$$2\mu^H_{ijkl} \xi_{kl} = \int_Q 2\mu(e_{ij}(\varphi) + \xi_{ij}) dy -$$
$$- \int_\Gamma (y_i [\tau n] \cdot n n_j - \frac{1}{3}\delta_{ij} [\tau n] \cdot y) ds_y. \tag{6.3}$$

and using the local problem (6.1) we find that:

$$\mu^H \xi : \xi = A^1 + A^2 \tag{6.4}$$

where:

$$A^1 = \int_Q 2\mu(e(\varphi)+\xi) : \xi dy \tag{6.5}$$

$$A^2 = \int_Q 2\mu(e(\varphi)+\xi) : e(\varphi) dy \tag{6.6}$$

Integration by parts and (6.5) shows that:

$$A^1 = \int_\Gamma [\tau_{ij}\ n_j]\xi_{ik} y_k dy + \int_{\partial Q} \tau_{ij} n_j \xi_{ik} y_k ds \tag{6.7}$$

and

$$A^2 = \int_\Gamma [\xi_{ij} n_j]\varphi_i ds \tag{6.8}$$

From (6.7) and (6.8) it follows that:

$$A^1 + A^2 = \int_Q \chi_2(y) 2\mu_2 \xi : \xi dy - \int_\Gamma \mu_2(\varphi_i n_j + \varphi_j n_i)\xi_{ij} ds + \int_\Gamma \xi_{ij} n_j \xi_{ik} y_k ds \tag{6.9}$$

where the boundary integrals are computed from the exerior. Adding and subtracting the integral

$$\int_Q \chi_1(y) 2\mu_2 \xi : \xi dy = \int_\Gamma 2\mu_2(\xi_{ik} y_k n_j + \xi_{jk} y_k n_i)\xi_{ij} ds \tag{6.10}$$

to (6.9) we obtain

$$A^1 + A^2 = \Sigma\ :\ \xi \tag{6.11}$$

and the claim follows.

The homogenized effective viscosity admits a variational formulation. Indeed, it follows immediately from the extremum principles of Keller, Rubenfeld and Molyneux [9] and the local problem (6.1) that

$$2\mu^H \xi\ :\ \xi = \min_{\psi \epsilon C} \int_Q 2\mu(y)(e(\psi)+\xi)\ :\ (e(\psi)+\xi) dy \tag{6.12}$$

where the class of periodic test fields C is given by:

$$C = \{\psi_i \in (H^1(Q))^3 \mid \psi \text{ is periodic in } Q,\ div\psi = 0 \text{ in } Q,$$
$$(\psi + \xi_{ij} y_j) n_i = \frac{1}{\mid B \mid}(\int_B \psi_i dy) n_i \text{ on } \Gamma\}. \tag{6.13}$$

One also obtains from the analysis of Keller et. al.[9] the complementary variational principle:

$$2\mu^H \xi \ : \ \xi = \max_{\sigma \epsilon K} \{ - \int_Q (2\mu(y))^{-1} (\sigma - \frac{1}{3} tr\sigma I) \ : \ (\sigma - \frac{1}{3} tr\sigma I) dy + \\ + 2 \int_{\partial Q} \xi_{ik} y_k \sigma_{il} n_l ds \} \quad (6.14)$$

where K is given by:

$$K = \{ \sigma \in (L^2(Q))^{3\times 3} \mid \sigma_{ij} = \sigma_{ji}, \ \sigma \text{ periodic in } Q, div\sigma = 0 \text{ in } Q, \\ [\sigma n] = [\sigma n] \cdot nn \text{ on } \Gamma \} \quad (6.15)$$

In addition one has a new dual variational principle given by:

$$(2\mu^H)^{-1} \overline{\sigma} : \overline{\sigma} = \min_{\sigma \in H} \int_Q (2\mu)^{-1} (\sigma - \frac{1}{3} tr\sigma I) : (\sigma - \frac{1}{3} tr\sigma I) dy \quad (6.16)$$

where:

$$H = \{ \sigma \in K \mid \int_Q (\sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma) dy - \int_\Gamma ([\sigma n] \cdot n n_i y_j - \frac{1}{3} \delta_{ij} [\sigma n] \cdot n) ds_y = \overline{\sigma} \}. \quad (6.17)$$

This principle involves the constant tensor $\overline{\sigma}$. Here $\overline{\sigma}$ is the difference between the volume average stress, and the surface energy tensor (cf. Batchelor [2]). This principle follows from the standard variational principle (6.14). To see this we observe that (6.14) can be written as:

$$2\mu^H \xi : \xi = \max_{\sigma \in K} \{ - \int_Q (2\mu)^{-1} (\sigma - \frac{1}{3} tr\sigma I) : (\sigma - \frac{1}{3} tr\sigma I) dy + \\ + 2 \int_Q \xi : \sigma dy - 2 \int_\Gamma [\sigma n] \cdot n \xi_{ij} n_i y_j ds \} \quad (6.18)$$

Noting the linearity of the solution φ in ξ, we expedite the presentation by introducing the tensor (M_{ijkl}) defined by:

$$M_{ijkl} \xi_{kl} = \int_\Gamma [(2\mu(e(\varphi) + \xi) - qI)n] \cdot n n_i y_j ds. \quad (6.19)$$

The variational principle (6.14) is stationary for:

$$\sigma^* = 2\mu(e(\varphi) + \xi) - qI \tag{6.20}$$

and from (6.3) we have:

$$\int_Q (\sigma^* - \frac{1}{3} tr\sigma^* I) dy = 2\mu^H \xi + (M\xi - \frac{1}{3} tr(M\xi) I). \tag{6.21}$$

The maximum in (6.18) is unchanged under the added constraints:

$$\int_\Gamma ([\sigma n] \cdot n n_i y_j - \frac{1}{3}\delta_{ij}[\sigma n] \cdot n) ds_y = (M\xi - \frac{1}{3} tr(M\xi) I)_{ij}, \tag{6.22}$$

$$\int_Q \sigma dy = 2\mu^H \xi + M\xi - \frac{1}{3} tr(M\xi) I \tag{6.23}$$

It now follows that:

$$2\mu^H \xi : \xi = \min\{ \int_Q (2\mu)^{(-1)} (\sigma - \frac{1}{3} tr\sigma I) : (\sigma - \frac{1}{3} tr\sigma I) dy. \tag{6.24}$$

where the minimum is taken over all σ in K satisfying (6.24) and (6.25). For this choice we have:

$$\overline{\sigma} = 2\mu^H \xi. \tag{6.25}$$

Thus elimination of ξ in (6.24) and noting that ξ is arbitrary yields the desired result (6.16).

For the periodic emulsion treated here, the effective viscosity is characterized by two shear moduli μ^S and μ^D (see equation (1.1)). In what follows we obtain upper and lower bounds on each shear modulus.

Bounds on the effective energy dissipation rate for an anisotropic periodic emulsion can be obtained upon substitution of suitable trial fields into the variational principles (6.12), (6.14) and (6.16). Noting that the harmonic mean of the two component viscosities is given by:

$$h = (\int_Q (2\mu)^{-1} dy)^{-1} \tag{6.26}$$

we have, for all ξ, the elementary bound:

$$h\xi : \xi \leq 2\mu^H \xi : \xi. \tag{6.27}$$

which implies:

$$h \leq 2\mu^S, \;\; h \leq 2\mu^D \tag{6.28}$$

These bounds follow from (6.16) with the choice σ=constant.

Denoting the functionals appearing in (6.12) and (6.14) by $E_\xi(\psi)$ and $H_\xi(\sigma)$ respectively, one immediately has:

$$H_\xi(\sigma) \leq 2\mu^H \xi \cdot \xi \leq E_\xi(\psi) \tag{6.29}$$

for any choice of trial fields $\sigma \in H$ and $\psi \in C$. We observe here that the upper and lower bounds on the effective viscosity originally developed in Keller, Rubenfeld and Molyneux [9] for isotropic suspensions apply to the anisotropic case treated here.

Using trial fields introduced by Keller, Rubenfeld and Molyneux [9], one may obtain bounds on the energy dissipation rate for the periodic emulsion, given by a simple cubic lattice of spheres of radius $a < 1/2$. These bounds are seen to agree with the results of G. I. Taylor [17] in the low volume fraction limit.

Indeed we introduce a security sphere of radius 1/2 inscribed in the unit cell Q. We denote the region inside the sphere by S.The trial field $\bar{\psi}$ is chosen such that

$$\bar{\psi} = 0, \quad \text{for } y \text{ in } Q - S \tag{6.30}$$

and $\bar{\psi}$ solves the flow problem inside S given by:

$$\begin{aligned} div\tau &= 0 \qquad \text{in } S \\ \tau(y) &= 2\mu(e(\bar{\psi}) + \xi) - qI \qquad \text{in } S \\ div\bar{\psi} &= 0 \qquad \text{in } Q \\ [\bar{\psi}] &= 0 \qquad \text{on } \Gamma \\ \bar{\psi} \cdot n &= \xi_{ij} y_j n_i \qquad \text{on } \Gamma \\ [\tau n] &= [\tau n] \cdot n \; n \qquad \text{on } \Gamma \\ \int_\Gamma [\tau n] &= 0 \\ \bar{\psi} &= 0 \qquad \text{on } S \end{aligned} \tag{6.31}$$

The associated upper bound is given by:

$$E_\xi(\bar{\psi}) = \int_S 2\mu(y)(e(\bar{\psi}) + \xi) : (e(\bar{\psi}) + \xi) dy + \left(1 - \frac{\pi}{6}\right) 2\mu_1 \mid \xi \mid^2 \tag{6.32}$$

Similarly a trial field $\bar{\sigma}$ is chosen such that:

$$\bar{\sigma} = 2\mu_1\xi, \quad \text{for } y \text{ in } Q - S \tag{6.33}$$

and

$$\bar{\sigma} = 2\mu(y)(e(\Phi) + \xi) - qI \qquad \text{in } S \tag{6.34}$$

where Φ solves the problem given by (6.31) with condition $(6.31)_8$ replaced by

$$(2\mu(y)(e(\Phi) + \xi) - qI) \cdot n = 2\mu_1\xi \cdot n \text{ on } S. \tag{6.35}$$

The associated lower bound is given by:

$$H_\xi(\bar{\sigma}) = \int_S (2\mu(y))^{-1}(\bar{\sigma} - \frac{1}{3}tr\bar{\sigma}) : (\bar{\sigma} - \frac{1}{3}tr\bar{\sigma})dy + \left(1 + \frac{\pi}{6}\right) 2\mu_1 \mid \xi \mid^2 . \tag{6.36}$$

The solutions of the flow problem for $\bar{\psi}$ and $\bar{\sigma}$ are substituted into (6.32), (6.36) which together with (6.29) yields:

$$\left(L(a) + (1 + \frac{\pi}{6})\right) \mid \xi \mid^2 \leq 2\mu^H\xi \cdot \xi \leq \left(U(a) + (1 - \frac{\pi}{6})\right) \mid \xi \mid^2 \tag{6.37}$$

Here $L(a)$ and $U(a)$ are given by:

$$L(a) = \frac{\pi\mu_1[6(\eta - 1)\lambda^{10} + 5(5\eta - 2)\lambda^7 - 42\eta\lambda^5 + 3(5\eta + 2)\lambda^3 - 4(1 + \eta)]}{3[4(1 - \eta)\lambda^10 + 5(5\eta - 2)\lambda^7 - 42\eta\lambda^5 + 5(5\eta + 2)\lambda^3 - 4(1 + \eta)]} \tag{6.38}$$

$$U(a) = \frac{-\pi\mu_1[160(1 - \eta)\lambda^{10} - 200(2 - 5\eta)\lambda^7 - 1680\eta\lambda^5 + 130(2 + 5\eta)\lambda^3 + 190(1 + \eta)]}{15[-48(1 - \eta)\lambda^{10} - 40(2 - 5\eta)\lambda^7 - 336\eta\lambda^5 + 45(2 + 5\eta)\lambda^3 + 38(1 + \eta)]} \tag{6.39}$$

where $\lambda = 2a$ and $\eta = \mu_2/\mu_1$. Equations (6.38) and (6.39) correspond to those derived in Keller, Rubenfeld and Molyneux [9]. It follows from (6.37) that:

$$L(a) + 1 + \frac{\pi}{6} \leq 2\mu^D \leq U(a) + 1 - \frac{\pi}{6} \tag{6.40}$$

$$L(a) + 1 + \frac{\pi}{6} \leq 2\mu^S \leq U(a) + 1 - \frac{\pi}{6} \tag{6.41}$$

For a simple cubic lattice, the bubble volume fraction is given by $\theta_2 = 4/3\pi a^3$, and in the dilute limit one sees that to first order in the concentration θ_2 that

$$L(a) + 1 + \frac{\pi}{6} \simeq U(a) + 1 - \frac{\pi}{6} \simeq \mu_1 + \theta_2 \left(\frac{\mu_1 + 5/2\mu_2}{\mu_1 + \mu_2}\right) \tag{6.42}$$

which agrees with the result of G. I. Taylor [17].

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