Geometry of positive scalar curvature on complete Riemannian manifold

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Abstract

The study of the interplay of geometry, topology, and curvature lower bound is an important topic in differential geometry. Many progresses have been made on the manifolds with sectional curvature or Ricci curvature bounded below over the past fifth years ([29, 42, 63]). However, many problems related to the scalar curvature remain conjectural [25, 26, 33, 40, 55, 62, 68] and see the website https://www.spp2026.de/.

In this thesis, first, we study the interplay of the geometry and positive scalar curvature on a complete, non-compact manifold with non-negative Ricci curvature. In three-dimensional manifold, we prove a minimal volume growth, an estimate of integral of scalar curvature, and a width estimate. In general dimensional manifold, we obtain a volume growth of a geodesic ball.

Next, we study the geometry of the mean convex domain in $\mathbb{R}^n$. Then, we prove that for every three-dimensional Riemannian manifold with non-negative Ricci curvature and strictly mean convex boundary, there exists a Morse function so that each connected component of its level sets has a uniform diameter bound, which depends only on the lower bound of mean curvature. This gives an upper bound of Uryson 1-width for those three manifolds with boundary, which answered a question raised by Gromov for three-dimensional case in [31].

Finally, we extend a comparison theorem of minimal Green functions in [52] to harmonic functions on complete non-compact three-dimensional manifolds with compact connected boundary. This yields an upper bound on the integral related to the scalar curvature on complete, non-parabolic three-dimensional manifolds.

Contents

Chapter 1

Introduction

My research mainly focus on the studies of the geometry and topology of the scalar curvature on complete Riemannian manifolds and their surrounding topics. I would arrange the thesis as follows: I will introduce part of my research contributions over the past four years in Chapter 1. Then, I will introduce the backgrounds and developments of the scalar curvature on complete Riemannian manifolds, and many aspects of the scalar curvature in geometric analysis and topology and their interplay in Chapter 2. Finally, I will give the details of the proof of the theorems in the Chapter 1 at the end of this thesis.

1.1 Geometry of positive scalar curvature on complete manifolds

This section is a combination of the two works [86, 88]. An important topic in geometric analysis is to understand the interplay between curvature and geometry. One of the classical and widely popularized results in this aspect is the Myers and Cheng's maximal diameter theorem.

Theorem 1.1.1 (Maximal Diameter Theorem, [12, 54]). *Any complete Riemannian manifold* (M^n, g) *with the Ricci curvature* $Ric(g) \geqslant n-1$ *has* $\mathrm{Diam}(M) \leqslant \pi$ *with equality if and only if* M *is a round sphere.*

From the perspective of size geometry [35], i.e., diameter, volume, Uryson width, Filling Radius, injectivity radius, etc. are called the size quantities of a Riemannian manifold. Myers and Cheng's maximal diameter theorem indicates that the positivity of Ricci curvature completely controls its distance spread in all directions, i.e., diameter. Here, we primarily focus on the size of Riemannian manifold and metric structure of Riemannian manifold. A natural problem is that how we could generalize theorem 1.1.1 to the scalar curvature in some sense. It is clear that positive scalar curvature on a Riemannian manifold can not determine its own distance spread fully. For instance, $(\mathbb{S}^2 \times \mathbb{R}^{n-2}, g)$ has scalar curvature 2 but no control of the diameter if g is the standard direct product of Riemannian metric. Based on this basic example, we could never expect that the positivity of scalar curvature on a Riemannian manifold can fully control its size. In fact, the most promising expectation is that the positivity of scalar curvature on a Riemannian manifold should have control on the size, which is only related to 1 or 2 dimensional quantities. In this direction, Gromov conjectures that

Conjecture 1.1.2 (Gromov, [24])**.** *Let (M^n, g) be a complete non-compact manifold with the scalar curvature $Sc(g) \geqslant n(n-1)$. Then the macroscopic dimension of M satisfies*

$$macrodim(M) \leqslant n - 2.$$

It remains open even for $marcodim(M) \leqslant n-1$. In the view of geometric dimension theory, Conjecture 1.1.2 is equivalent to the statement that M can be approximated by $n-2$ dimensional polyhedrons within finite distance. For the definition of macroscopic dimension, the reader can refer to the references [24, 35], and we will not use the definition of macroscopic dimension directly. However, in the spirit of it, Conjecture 1.1.2 provides the insight for the following conjectures and the results of our paper.

Together with Theorem 1.1.1, it is proved that for the conjugate radius $\mathrm{conj}(M)$ of a Riemannian manifold M,

Theorem 1.1.3. *[Maximal Conjugate Radius Theorem, [22]] Let (M^n, g) be a closed Riemannian manifold with the scalar curvature $Sc(g) \geqslant n(n-1)$. Then $\mathrm{conj}(M) \leqslant \pi$, and equality holds if and only M is isometric to the round sphere $\mathbb{S}^n$.*

Since the injectivity radius $\mathrm{inj}(M) \leqslant \mathrm{conj}(M)$, we obtain that Theorem 1.1.3 implies that for any closed Riemannian manifold with $Sc(g) \geqslant n(n-1)$, $\mathrm{Inj}(M) \leqslant \pi$, and equality holds if and only M is isometric to the round sphere $\mathbb{S}^n$, which we would like to name as maximal injectivity radius theorem. For the details of the proof of Theorem 1.1.3, see [22].

To my best knowledge, it remains open whether maximal conjugate or injectivity radius still hold on any complete non-compact Riemannian manifold. In fact, this question is deeply related to Conjecture 1.1.2. Here, we apply the techniques used in the case of closed Riemannian manifold to the complete, non-compact Riemannian manifold (M^n, g) and then obtain a local estimate on the integral of scalar curvature.

Proposition 1.1.4. *Let (M^n, g) be a complete, non-compact Riemannian manifold. Then,*

- *Suppose that $B(p, R) \subset M$ is a geodesic ball with center $p \in M$ and radius $r > 0$. If $Ric(g) \geqslant 0$ on $B(p, R)$, then we obtain,*

$$\int_{B(p,R-c)} Sc \leqslant n(n-1)(\frac{\pi}{c})^2 \mathrm{vol}B(p, R), \forall c \leqslant \mathrm{conj}(M); \tag{1.1.1}$$

- *If $Sc(g) \geqslant n(n-1)$ on M and $Ric(g) \geqslant 0$ on M, then* $\mathrm{inj}(M) \leqslant \mathrm{conj}(M) \leqslant \pi$.

The conjugate radius estimate indicates that positive scalar curvature does imply that a Riemannian manifold is curved or becomes thinner under the assumption of non-negative Ricci curvature and strictly positive scalar curvature. However, we still do not know whether Proposition 1.1.4 holds without any assumptions on non-negative Ricci curvature. On the one hand, if one can construct a complete, non-compact Riemannian manifold with positive scalar curvature and its injectivity radius is infinity, then it will deduce a negative answer to Conjecture 1.1.2; on the other hand, the local estimate (1.1.1) indicates that the average of the integral of scalar curvature is bounded above in terms of the lower bound of the conjugate radius. Also, it implies that the volume growth and positive scalar curvature are intertwined locally. However, the interplay is still a mystery on a global scale. Many years ago, Yau proposed the following problem, which is involved with the volume growth and positivity of scalar curvature on a complete, non-compact Riemannian manifold with non-negative Ricci curvature.

Problem 1.1.5 (Yau [83])**.** *Let (M^n, g) be a complete, non-compact manifold with non-negative Ricci curvature and $B(p, r) \subset M$ a geodesic ball with center $p \in M$ and radius* *r. Do we have*

$$\limsup_{r\to\infty} r^{2-n}\int_{B(p,r)} Sc < \infty? \tag{1.1.2}$$

In fact, Yau proposed a more general version of this problem that is involved with the $\sigma_k, k = 1, 2, \cdots, n$ of Ricci tensor in [83]. Unfortunately, Yang [80] constructs a counterexample on Kähler manifold to prove that the general version of Yau's Problem 1.1.5 does not hold for $k = 1, 2, \cdots, n-1$, Xu [79] obtains an estimate involved with the integral of scalar curvature towards the Problem 1.1.5 in the case of three-dimensional Riemannian manifold by using the monotonicity formulas of Colding and Minicozzi [16]. However, Problem 1.1.5 remains open. In fact, it has been shown [61] that the inequality (1.1.2) holds if we impose a strong curvature condition non-negative sectional curvature instead of non-negative Ricci curvature. Also, Naber [55] asks the non-collapsing version of Yau's Problem 1.1.5 that is a baby version, and propose a local version of Yau's Problem 1.1.5. Here, we propose a baby version of Yau's Problem 1.1.5 that is worthwhile of investigating as well.

Problem 1.1.6. *Suppose that (M^n, g) is a complete, non-compact Riemannian manifold with $Ric(g) \geqslant 0$ and $Sc(g) \geqslant 1$. Do we have*

$$\limsup_{r\to\infty} \frac{\mathrm{vol}(B(p,r))}{r^{n-2}} < \infty?$$

On the one hand, this problem can be regarded as a baby version of Yau's Problem 1.1.5: if Problem 1.1.5 holds, then Problem 1.1.6 holds; on the other hand, from the perspective of size geometry, this problem can provide more valid evidence for Gromov's Conjecture 1.1.2, and it can be considered as a quantitative version of 1.1.2 in the category of Riemannian manifolds with non-negative Ricci curvature.

Here, this study primarily focuses on the case of three-dimensional, complete, non-compact Riemannian manifolds. In fact, there are abundant of studies on three dimensional Riemannian manifolds, including the case of compact or non-compact. For the closed three dimensional Riemannian manifolds, the topological classification of three dimensional Riemannian manifolds is clear according to Poincáre Conjecture. Moreover, the proof of Thurston's Geometrization conjecture [56–58] shows that a closed three dimensional Riemannian manifold admits a metric with positive scalar curvature if and only if it is a connected sum of spherical 3-manifolds and some copies of $S^1 \times S^2$.

For complete non-compact three-dimensional Riemannian manifolds with non-negative Ricci curvature, Liu [46] proves that it is either diffeomorphic to $\mathbb{R}^3$ or its universal cover splits. Therefore, there are not many three-dimensional Riemannian manifolds with the properties that they admit a complete Riemannian metric with non-negative Ricci curvature and positive scalar curvature. Topologically, they are either $\mathbb{R}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. In a recent progress, Wang [74] shows that any complete, non-compact contractible three-dimensional Riemannian manifold with non-negative scalar curvature is homeomorphic to $\mathbb{R}^3$.

However, our goal in the paper is to study the geometry of a complete, non-compact, three-dimensional Riemannian manifold that positive scalar curvature has influence on, rather than the topology of a complete, non-compact, three-dimensional Riemannian manifold. According to the splitting theorem of complete, non-compact Riemannian manifold with non-negative Ricci curvature [9], we primarily focus on the geometry of positive scalar curvature on $\mathbb{R}^3$.

First, we have the following observation on Yau's Problem 1.1.5

Theorem 1.1.7. *Let (M^3, g) be a complete, non-compact three-dimensional Riemannian manifold with a pole p and $Ric(g) \geqslant 0$. Then*

$$\limsup_{r\to\infty} \frac{1}{r} \int_{B(p,r)} Sc \leqslant 20\pi. \tag{1.1.3}$$

This estimate confirms that Yau's Problem 1.1.5 holds in the special case of complete, non-compact three-dimensional Riemannian manifolds with a pole. But, our assumption on the manifolds with pole is very artificial. For Yau's Problem 1.1.5, we will do more studies in our future works.

Moreover, we consider the baby version of Yau's Problem 1.1.5, then we obtain

Theorem 1.1.8. *Let (M^3, g) be a complete, non-compact, three-dimensional Riemannian manifold with $Ric(g) \geqslant 0$ and $Sc(g) \geqslant 6$. Then, for any $p \in M$, we obtain*

$$\limsup_{R\to\infty} \frac{vol(B(p,R))}{R} < \infty, \tag{1.1.4}$$

if $vol(B(q,1)) \geqslant \epsilon > 0$ for any $q \in M$.

Yau [81] proved that any complete, noncompact manifolds with non-negative Ricci curvature have at least linear volume growth. Combining Yau's result with our Theorem 1.1.8, we obtain that any complete, non-compact three-dimensional manifold with non-negative Ricci curvature and strictly positive scalar curvature has linear volume growth or, namely, minimal volume growth. At this moment, it's known that Theorem 1.1.8 holds as $n = 3$([53]) without assumption on the volume non-collpased.

In higher dimension, we obtain the following volume estimate in higher dimensional case,

Theorem 1.1.9. *Let* (M^n, g) *be a complete, non-compact Riemannian manifold with* $Ric(g) \geqslant 0$ *and* $Sc(g) \geqslant n(n-1)$*. Then for any* $p \in M$*, we obtain*

- *For any* $q \in M$*, then there exists a constant* c_n *such that*

$$vol(B(p, R)) \leqslant c_n R^{n-1};$$

- *if* $\mathrm{inj}(M) \geqslant \epsilon > 0$*, then*

$$\limsup_{R\to\infty} \frac{vol(B(p, R))}{R^{n-2}} < \infty.$$

In fact, Anderson proved that [3] any complete, non-compact Riemannian manifold (M^n, g) with positive Ricci curvature has $b_1(M) \leqslant n-3$ and the rank of any free Abelian subgroup of $\pi_1(M)$ is at most $n-3$ for which the estimate is optimal. From Corollary 1.1.9, we also expect that non-negative Ricci curvature and strictly positive scalar curvature on a complete, non-compact Riemannian manifold would imply $b_1(M) \leqslant n-3$ and it is optimal as well. For relevant result, you may refer to [14]. In fact, it will be interesting to study the dimension of harmonic functions with linear growth on a complete, non-compact manifold with non-negative Ricci curvature and strictly positive scalar curvature on higher dimensional Riemannian manifolds.

Remark 1.1.10. *In fact, Gromov* [23] *stated that, under the assumption that* $K(g) \geqslant 0, Sc(g) \geqslant n(n-1)$ *without any details, then*

$$\sup_{p\in M} vol(B(p, r)) \leqslant c_n r^{n-2}. \tag{1.1.5}$$

For the proof of (1.1.5)*, you may refer to* [61]*. Furthermore, Gromov* [23] *conjectures that the volume estimate* 1.1.5 *holds if we only* $Ric(g) \geqslant 0$*. Here, our results can be regarded as a step to prove Gromov's conjecture under an extra condition volume non-collapse as* $n = 3$ *or injectivity radius non-collapse as* $n \geqslant 4$*.*

Moreover, a complete, non-compact Riemannian manifold is said to be non-parabolic if it admits a positive Green function, otherwise it is said to be parabolic. By a result of Varopoulos [73], a complete, non-compact Riemannian manifold with $Ric(g) \geqslant 0$ is non-parabolic if and only if

$$\int_1^\infty \frac{r}{vol(B(p,r))} dr < \infty.$$

Hence, the following conclusion is deduced

Corollary 1.1.11. *Let* (M^3, g) *be a complete, non-compact three-dimensional Riemannian manifold with* $Ric(g) \geqslant 0$*,* $Sc(g) \geqslant 6$ *and* $vol(B(p,1)) \geqslant \epsilon > 0$ *for all* $p \in M$*. Then* (M^3, g) *is parabolic.*

According to the result in [69], (M^3, g) admits no any nontrivial harmonic functions with polynomial growth unless it splits. In fact, we would like to believe that: on a complete, non-compact, n-dimensional Riemannian manifold with non-negative Ricci curvature and strictly positive scalar curvature, then the dimension of harmonic function with linear growth should be less or equal to $n-2$ for any $n \geqslant 4$.

Finally, let's study the width of the manifold. Sormani [70] proves for any complete non-compact manifold with non-negative Ricci curvature and minimal volume growth, Busemann function is proper. However, we can not expect that the diameter of the level set of Busemann function has a uniformly bound there and many examples are illustrated in [70]. Here, we prove an upper bound on the width of the manifold.

Theorem 1.1.12. *Let* (M^3, g) *be a complete, non-compact three-dimensional Riemannian manifold with* $Ric(g) \geqslant 0$*,* $Sc(g) \geqslant 2$ *and* $vol(B(p,1)) \geqslant \epsilon > 0$*. Then, there exists a constant* c *and continuous function* $f : M \to \mathbb{R}$ *such that for any* $r \in \mathbb{R}$*,*

$$diam(f^{-1}(r)) \leqslant c.$$

Remark 1.1.13. *Theorem 1.1.12 indicates that any 3 dimensional manifold* (M^3, g) *with non-negative Ricci curvature and positive scalar curvature will wander around a line. Hence, in the large scale,* M *is a one dimensional line* $\mathbb{R}$*. In fact, by the proof of Theorem 1.1.12, we know* f *can be a Lipschitz function such that* $Lip(f) \leqslant 1$*. Besides, it is possible that* $f^{-1}(r)$ *for some* r *may not be connected, and there is no reason to expect that* $f^{-1}(r)$ *for all* $t \in \mathbb{R}$ *are connected. Hence, for the case of discounted level set, we also count the distance among different connected components of the level set in the theorem.*

$$diam(f^{-1}(r)) = \sup_{x,y}\{d_g(x,y) : f(x) = f(y) = r\}.$$

Here, d_g *is the distance induced by the Riemannian metric* g *on* M*. Finally, by the language of Uryson width, Theorem 1.1.12 implies that* $width_1(M) \leqslant c$ *and* $macrodim(M^3) = 1$.

The article is organized as follows. In Section 3.1, we introduce some preliminary materials for the proofs of main theorems and then prove Proposition 1.1.4. In Section 3.2, we prove Theorem 1.1.7, 1.1.8, Corollary 1.1.9 and Theorem 1.1.12. Now, let's briefly outline the proof of main results in this work. For the proof of Theorem 1.1.7: first, by using the stability of the geodesic ray, we deduce an upper bound on the integral of Ricci curvature in the direction of the normal vector field over the geodesic ball centered at the pole. Then, we make use of the geometrically relative Bochner formula and Gauss Bonnet formula on a geodesic sphere to obtain an upper bound on the integral of scalar curvature. In fact, the geometrically relative Bochner formula plays a vital role in the proof to get rid of the second fundamental form of the geodesic sphere and Theorem 1.1.7 will be much stronger than Proposition 1.1.4. For the proof of Theorem 1.1.8: we first prove Lemma 3.1.12 which relates volume growth to the Euclidean split of Ricci limiting space. Then, we argue our theorem by contradiction. By combining Lemma 3.1.12 with Cheeger-Colding theory and the result of M-T on the characterization of three-dimensional Ricci limiting space, we obtain that, topologically, the Ricci limiting space can only be: $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$, then, we rule out both cases by using the torus band estimate. Hence, we prove a minimal volume growth under strictly positive scalar curvature. Here, we avoid defining a generalized scalar curvature on the Ricci limiting space, which is a hard question for the author. Instead, we use the Lipschitz structure of positive scalar curvature proved by G-WXY to rule out 2 cases of Ricci limiting space. Essentially, our argument works whenever the Ricci limiting space is a topological manifold and hence we have a corresponding result in the higher dimensional Riemannian manifolds with stronger condition assumption. In fact, the method of using Lipschitz structure of strictly positive scalar curvature is quite new to study the positivity of scalar curvature on complete Riemannian manifolds with non-negative Ricci curvature.

1.2 Uryson width of three-dimensional mean convex domain with non-negative Ricci curvature

The section is from the original paper [78] collaborated with Zhichao Wang in 2021. In [31], Gromov proposed the following conjecture:

Conjecture 1.2.1 (Gromov [31])**.** *Suppose that* $X \subset \mathbb{R}^n$ *is a smooth domain such that* $H_{\partial X} \geqslant n-1$*. Then there exists a continuous self-map* $R : X \to X$ *such that*

- *the image* $R(X) \subset X$ *has topological dimension* $n-2$*;*
- $\mathrm{dist}(x, R(x)) \leqslant c_n$ *for all* $x \in X$*, with the best expected* $c_n = 1$*.*

Recall that the *Uryson k-width* $\mathrm{width}_k(M)$ of a Riemannian manifold M is the infimum of the real numbers $d \geqslant 0$, such that there exist a k-dimensional polyhedral space P^k and a continuous map $f : M \to P^k$ with

$$\mathrm{diam}_M f^{-1}(p) \leqslant d, \quad \text{for all} \ \ p \in P^k,$$

where $\mathrm{diam}_M(\cdot)$ denotes the diameter of the subset of M. Clearly, Conjecture 1.2.1 implies that the Uryson $(n-2)$-widths of mean convex domains in Euclidean spaces are bounded from above by a constant relying on their mean curvature lower bounds. In this paper, we give a direct proof of such an upper bound. More generally, our result holds for all three-dimensional mean convex domains with non-negative Ricci curvature.

Theorem 1.2.2. *Suppose that* $(M, \partial M, g)$ *is a complete (possibly non-compact) three dimensional Riemannian manifold with* $\mathrm{Ric}(g) \geqslant 0$ *and* $H_{\partial M} \geqslant 1$*. Then there exists a* *smooth Morse function* $f : M \to \mathbb{R}$ *such that for any* t *and* x, y *in the same connected component of* $f^{-1}(t)$,

$$\mathrm{dist}_M(x, y) < 117.$$

In particular, if M *is a smooth domain in* $\mathbb{R}^3$ *with* $H_{\partial M} \geqslant 1$, *then the upper bound can be improved to be* 49.

We remark that the condition of non-negative Ricci curvature can not be relaxed to non-negative scalar curvature due to the following example.

Example 1.2.3. *Let* $B_R(x)$ *be the Euclidean ball in* $\mathbb{R}^3$ *centered at* x *with radius* R. *Then* $(\mathbb{R}^3 \backslash B_{1/3}(0), g)$ *is a Riemannian manifold with positive scalar curvature, where*

$$g_{ij} = (4 + \frac{4}{r})^4 \delta_{ij}, \ r > 0.$$

Here r *is the distance function to the origin with respect to the Euclidean metric* δ. *Moreover, its boundary* $\partial B_{1/3}(0)$ *has mean curvature* $H = 3 > 1$ *with respect to the outward normal vector field. However, outside a sufficiently large ball, the manifold is close to the Euclidean spaces, which has infinite Uryson 1-width.*

Constructing a singular foliation by surfaces of controlled size has been successfully used to understand the structure of three dimensional manifolds with positive scalar curvature. Gromov-Lawson [28] obtained an upper bound of Uryson 1-width for simply connected Riemannian manifolds by considering the level sets of distance function to a fixed point. For closed manifolds with nonnegative Ricci and positive scalar curvature, Marques-Neves [48] proved a sharp bound on the area of the maximal leaves. In a recent work [44], Liokumovich-Maximo proved that every closed three manifold with positive scalar curvature admits singular foliations so that each leave has controlled diameter, area and genus. In [33] and the Lemma §3.10, Property A there, Gromov proved the Uryson 1-width upper bound for three-dimensional complete (possibly non-compact) Riemannian manifolds X with positive scalar curvature and $H_1(X; \mathbb{Q}) = 0$. Our method in this paper does not require any topological conditions and can probably be applied to all 3-manifolds with positive scalar curvature.

Challenges and ideas

The main challenge in our paper is to decompose the manifold into *geometrically prime regions*, i.e. those regions where each closed curve bounds a surface relative to a connected boundary component. The idea is to cut the manifold along two-sided stable free boundary minimal surfaces. Unlike the argument in [44], we don't have a bumpy metric theorem for non-compact manifold with the non-negative Ricci curvature preserved. Nevertheless, we can find countably many stable free boundary minimal surfaces so that after cutting along them, each connected component contains only "trivial" two-sided stable ones which are isotopic to one of the boundary components. Then for those connected components that are not geometrically prime, we are going to apply min-max theory in the "core region" (see [67] by A. Song) to find an index one free boundary minimal surfaces that subdivides the components into two geometrically prime regions.

However, this "core region" could be non-compact and there is no general min-max theory for such manifolds. In this paper, to deal with non-compact manifolds, we take a sequence of compact domains that exhaust the manifold; c.f. [67, §3.2]. We perturb the metric in the neighborhood of the new boundary so that the new boundary component becomes a stable free boundary minimal surface. However, the perturbation will also produce more stable free boundary minimal surfaces. More importantly, the diameter bounds for stable/index one surfaces can not be preserved anymore since the Ricci curvature will not be non-negative with respect to the new metric. Fortunately, as these compact domains exhausting the non-compact manifold, the new stable free boundary minimal surfaces are far away from a fixed compact domain. Then by cutting along those surfaces with small area in a suitable order, one can obtain a sequence of compact "core regions" converging to the non-compact domain in the Gromov-Hausdorff topology. Moreover, these "core regions" satisfy a weak Frankel property, which is directly from cutting process. By applying the min-max theory to these compact "core regions", one can construct a sequence of two-sided free boundary minimal surfaces with index one. By the weak Frankel property, these min-max surfaces should intersect a given domain, which implies the limit of this sequence of surfaces is non-empty. Hence such a sequence of surfaces are actually free boundary minimal surfaces with respect to the original metric. This gives the desired surfaces.

The remaining issue is to adapt Gromov-Lawson' trick to each geometrically prime region. Then we require a uniform diameter bound for the two-sided stable/index one free boundary surfaces that we have cut. Recall that the length of boundaries of these surfaces have been bounded by Ambrozio-Buzano-Carlotto-Sharp [2, Lemma 48], which is still far from the diameter bound. In this paper, we obtain a general *radius bound* (i.e. distance bound from interiors to boundaries) for any smooth surfaces with bounded mean curvature. Suppose on the contrary that there exists a surface-with-boundary that has large radius. Then regarding such a surface as a barrier, by a minimizing process in a relative homology class, there is a stable constant mean curvature surface with mean curvature 1, whose possible boundaries are far away from an interior point. By applying Schoen-Yau's trick [65] here, such a cmc surface has a uniform radius bound, which implies that it is closed. Clearly, in Riemannian manifolds with non-negative Ricci curvature, there is no closed stable cmc surface. Such a contradiction gives the radius bound for any surface-with-boundary. Combining with the length bound of boundaries of free boundary minimal surfaces with index one, we then obtain the diameter upper bounds for these surfaces.

Outline

The proof of this section will be organized as follows. In Section 4.1, we prove a "radius" bound for each embedded surface with bounded mean curvature. Then combining the length estimates in [2] and [7], we obtain a diameter upper bound for two-sided free boundary minimal surface with index less than or equal to 1. In the second part of this section, we state the diameter estimates for the level sets of distance functions in geometrically prime regions. In Section 4.2, we decompose three-dimensional manifolds with non-negative Ricci and strictly positive mean curvature into countably many geometrically prime regions. The most technical part is Proposition 4.2.1, where we will use the min-max theory to produce free boundary minimal surfaces of index one. Finally, in Section 4.3, we construct the desired function in each geometrically prime region and then glue them together to get the desired function. For the sake of completeness, we adapt Gromov-Lawson's trick in Section 4.4, which is parallel to Lemma 4.1 in [44].

1.3 Comparison theorem and integral of scalar curvature on three manifolds

The results in this section from my original paper [87]. Munteanu-Wang [52] prove a comparison theorem of minimal Green function on complete, non-parabolic, three-dimensional Riemannian manifolds with a minor topological condition.

Theorem 1.3.1. *([52]) Let (M, g) be a complete non-compact three-dimensional manifold with non-negative scalar curvature. Assume that M has one end and its first Betti number $b_1(M) = 0$. If M is non-parabolic and the minimal positive Green's function $G(x) = G(p, x)$ satisfies $\lim_{x\to\infty} G(x) = 0$, then*

$$\frac{d}{dt}\left(\frac{1}{t}\int_{l(t)} |\nabla G|^2 - 4\pi t\right) \leqslant 0,$$

for all $t > 0$. Moreover, equality holds for some $T > 0$ if and only if the super level set $\{x \in M, G(x) > T\}$ is isometric to a ball in the Euclidean space $\mathbb{R}^3$.

Here, we generalize Theorem 1.3.1 from the minimal Green function to the harmonic functions on complete, non-compact three-dimensional Riemannian manifolds with compact and connected boundary.

In this paper, (M^n, g) is always a complete, non-compact, oriented Riemannian manifold with connected and compact boundary ∂M, let Δ be the Beltrami-Laplacian operator defined on M, i.e., $\Delta = tr(\nabla^2)$. Let f be the solution of following Dirichlet boundary problem

$$\Delta f = 0 \text{ on } M, \ f|_{\partial M} = 1.$$

Note that by [42], M is either parabolic or non-parabolic. In this paper, we always the following:

1. If M is a non-parabolic Riemannian manifold, then $\lim_{x\to\infty} f(x) = 0$.
2. If M is a parabolic Riemannian manifold, then $\lim_{x\to\infty} f(x) = \infty$.

Notice that f is always assumed to satisfy (1) and(2) throughout the paper.

Then, for any $a, b \in \mathbb{R}, b > a$, define

$$L^f(a, b) = \{p \in M : a \leqslant f(p) \leqslant b\} \text{ and } l^f(t) = \{p \in M : f(p) = t\}.$$

Denote $l(t) = l^f(t)$ if there is no confusion in the context. Moreover, we know that $l(t)$ is compact. i.e., the harmonic functions with conditions assumed above are proper. Furthermore, by Morse theory, the collections of regular values of f is open and dense in $\mathbb{R}$. For any irregular value t, f must have $\nabla f(x) = 0$ for some point $x \in l(t)$ and the set $\{x \in l(t) : \nabla f(x) = 0\}$ has zero $\mathcal{H}^n_g$-measure [10]. This basic observation guarantees that the integrals below are well-defined. Here $\mathcal{H}^n_g$ is the n-dimensional Hausdorff measure associated with the Riemannian metric g.

On the non-empty set level $l(t)$ for each $t \in \mathbb{R}$, we define the following energy functional if $l(t)$ is a nonempty set,

$$\omega_f(t) = \int_{l(t)} |\nabla f|^2 d\mathcal{H}^{n-1}_{g|_{l(t)}}. \tag{1.3.1}$$

We will denote $\omega(t)$ by $\omega_f(t)$ if there is no confusion in the context and $d\mathcal{H}^{n-1}_{g|_{l(t)}}$ is the Hausdorff measure associated with the Riemannian metric on $l(t)$ that is induced from the ambient metric g. Note that $\omega(t)$ is a continuous and locally Lipschitz function on $\mathbb{R}$. Hence, $\omega'(t)$ exists almost everywhere in $\mathbb{R}$.

Foremost, we obtain that

Theorem 1.3.2. *Let $(M^3, \partial M, g)$ be a complete, non-compact three-dimensional Riemannian manifold with non-negative scalar curvature $Sc(g) \geqslant 0$, and its boundary be connected and closed. If $b_1(M) = 0$ and M has one end. Then, we have differential inequalities as follows:*

- *If (M^3, g) is non-parabolic, then for any $t \in (0, 1)$,*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi t - \frac{\omega'(1) - \omega(1) - 4\pi}{2} t^2\right) \leqslant 0. \tag{1.3.2}$$

  Moreover, there exists a $T \in (0, 1)$ such that the equality holds if and only if $L(T, 1)$ is isometric to $A(\frac{1}{4\pi}, \frac{1}{4\pi T})$. Here $A(\frac{1}{4\pi}, \frac{1}{4\pi T})$ is the annulus in $\mathbb{R}^3$ with outer radius $R = \frac{1}{4\pi T}$ and inner radius $r = \frac{1}{4\pi}$;

- *If (M^3, g) is parabolic, then for any $t \in (1, \infty)$,*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi t - \frac{\omega'(1) - \omega(1) - 4\pi}{2} t^2\right) \geqslant 0. \tag{1.3.3}$$

If, in addition, $Ric(g) \geqslant 0$, we obtain the following geometric inequalities related to $\omega(t)$ and the characterization of rigidity. Moreover, Motivated by Yau's Problem(See [79, 80]), we deduce an upper bound on an integral involved with the scalar curvature, which is

Theorem 1.3.3. *Let (M^3, g) be a complete, non-compact, non-parabolic three dimensional Riemannian manifold with $Ric(g) \geqslant 0$, and its boundary be connected and closed. Then, there exists a universal constant $k \in \mathbb{Z}^+$ such that*

1. *For any $t \in (0, 1]$, we obtain that*
$$\omega(t) \leqslant 4k\pi t^2 + \frac{\omega'(1) - \omega(1) - 4k\pi}{2} t^3; \tag{1.3.4}$$
$$A(l(t)) \geqslant \frac{1}{4k\pi t^2 + \frac{\omega'(1) - \omega(1) - 4k\pi}{2} t^3}. \tag{1.3.5}$$
Moreover, $b_1(M) = 0$ and there exists a constant $T \in (0, 1)$ such that
$$A(l(T)) = \frac{1}{4\pi T^2 + \frac{\omega'(1) - \omega(1) - 4\pi}{2} T^3}.$$
if and only if M is isometric to $\mathbb{R}^3 \backslash B(\frac{1}{4\pi})$;

2. *Boundary characterization:*
$$3\omega(1) - \omega'(1) \leqslant 4k\pi.$$
In particular, $b_1(M) = 0$ and $3\omega(1) - \omega'(1) = 4\pi$ if and only if M is isometric to $\mathbb{R}^3 \backslash B(\frac{1}{4\pi})$;

3. *There exists a constant $c > 0$ such that*
$$\sup_{t \to 0} \int_{L(t,1)} Sc |\nabla f| d\mathcal{H}^n_g \leqslant c.$$

Remark 1.3.4. *Due to the estimate (2) in Theorem 1.3.3, we introduce a quantity $\mathcal{B}(M)$ (See Definition 5.1.6 below) for any n-dimensional Riemannian manifold. Indeed, by Theorem 1.3.3, $\mathcal{B}(M)$ has an upper bound and a rigidity characterization. Hence, $\mathcal{B}(M)$ carries the global geometry information of three-dimensional Riemannian manifolds with non-negative Ricci curvature.*

Remark 1.3.5. *(3) is motivated by Yau's problem: Suppose that (M^n, g) is a complete, non-compact Riemannian manifold with $Ric(g) \geqslant 0$. Then for any $p \in M$,*

$$\limsup_{R\to\infty} \frac{1}{R^{n-2}} \int_{B(p,R)} Sc(g) < \infty?$$

Here $B(p, R)$ is the geodesic ball of M with center p and radius R. Indeed, Xu [79] deduces a similar estimate by using the monotonicity formulas of Colding-Minicozzi under the assumption that (M^3, g) is non-parabolic and has maximal volume growth [15, 16]. For more results related to Yau's problem, see [11, 61, 80, 88] and literature therein. In fact, in the case of three-dimensional Riemannian manifolds, it is very promising to prove a stronger version of Yau's problem: there exists a universal constant c such that for any $p \in M$ and $R > 0$,

$$\int_{B(p,R)} Sc(g) < cR.$$

Finally, we have a corollary.

Corollary 1.3.6. *Let (M^3, g) be a complete, non-compact three-dimensional, non-parabolic Riemannian manifold with $Sc(g) \geqslant 0$ with connected and closed minimal surface boundary, $b_1(M) = 0$ and one end. Then for any $t \in (0, 1)$,*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi t\right) \leqslant -(\omega(1) + 4\pi)t < -4\pi t.$$

It implies no closed minimal surface in $\mathbb{R}^3$.

The proof this section will be organized as follows: in Section 5.1, we first introduce the basic material related to the harmonic functions on any complete, non-compact Riemannian manifolds and then obtain some curvature formulas on its level set in terms of the harmonic functions. Then, we deduce some identities in terms of $\omega(t)$ and parameter α (See Proposition 5.1.5). In comparison with the calculations in [52], our calculations cover a more generalized case, which can be applied to understand the geometry of the scalar curvature on Riemannian manifolds not modelled on Euclidean space. In Section 5.2, we obtain a comparison theorem of harmonic function on both non-parabolic and parabolic manifold with boundary. In Section 5.3, we obtain some geometric inequalities and then characterize the rigidity and finally obtain an upper bound on the integral involved with the scalar curvature.

Chapter 2

Basic concept and background

To motivate the readers and make the thesis self-contained, we will first introduce the basic concept in Riemannian geometry and then will investigate the history of studies of scalar curvature over the past fifty years, and the recent progresses on the geometry and topology of the scalar curvature. In this chapter, we assume that the readers are familiar with the theory of differential geometry of curvature and surfaces, differential manifolds and basic algebraic topology.

2.1 Riemannian manifolds and Curvature

In this section, we will introduce the basic concept in Riemannian manifold, and readers can refer to the textbooks [42, 60, 63].

Definition 2.1.1. *Suppose that M^n is a smooth topological manifold of dimension n. (M^n, g) is said to be a smooth Riemannian manifold of dimension n if the following conditions satisfy*

1. *g is a smooth $(0, 2)$ tensor on M*
2. *For any $p \in M$, $g(p)$ is an inner product on T_pM;*

We often use the word "metric" to refer to a Riemannian metric and assume that all Riemannian manifolds are smooth in our context when there is no chance of confusion. Moreover, using a partition of unity, we can prove that every smooth manifold admits a smooth Riemannian metric. Moreover, we write the Riemannian volume element $\sqrt{det(g)}dL^n$ as $d\mathcal{H}^n_g$, which is also called Hausdorff measure associated with the Riemannian metric g on M.

Now, we assume that (E, M, π) is a smooth vector bundle over a smooth manifold M and $\mathcal{E}(M)$ denote the space of smooth section of E. A connection in E is a map:

$$\nabla : \Gamma(M) \times \mathcal{E} \to \mathcal{E},$$

written as $(X, Y) \mapsto \nabla_X Y$ with the following properties:

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1+f_2X_2}Y = f\nabla_{X_1}Y + f_2\nabla_{X_2}Y, f_1, f_2 \in C^\infty(M);$$

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 b\nabla_X Y_2, a, b \in \mathbb{R};$$

3. ∇ satisfies with the following product rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y, f \in C^\infty(M).$$

Here, $\nabla_X Y$ is called the covariant derivative of Y in the direction of X.

Theorem 2.1.2. *Let (M^n, g) be a complete Riemannian manifold. Then (M^n, g) admits a connection that is called Riemannian connection, with the following properties: for any $X, Y, Z \in \Gamma(M)$,*

- $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(X, \nabla_X Z)$;
- $\nabla_X Y - \nabla_Y X = [X, Y]$.

Moreover, for any $X, Y, Z \in \Gamma(M)$, we introduce

$$R(X, Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z. \tag{2.1.1}$$

It is a $(3, 1)$-tensor field. Then, we define the associated $(4, 0)$ tensor field

$$Rm(X, Y, Z, W) = (R(X, Y)Z, W). \tag{2.1.2}$$

In local coordinate $\{x^i\}$, we have

$$R = R^l_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l.$$

Here $g_{lm}R^m_{ijk} = Rm_{ijkl}$, and Rm is called Riemann curvature tensor.

Now, we are in a position to introduce the following three curvatures as follows

Definition 2.1.3. *Let (M^n, g) be a complete manifold. Then,*

- *Sectional curvature:* $K(\partial_i, \partial_j) = \frac{Rm(\partial_i,\partial_j,\partial_i,\partial_j)}{|\partial_i \wedge \partial_j|^2}$,
- *Ricci curvature:* $Ric_{ij} = g^{kl} R_{ikjl}$;
- *Scalar curvature:* $Sc = g^{ij} Ric_{ij}$.

Since our main research topic is the geometry and topology of the scalar curvature on complete Riemannian manifolds, let's focus on the basic properties of scalar curvature in this thesis.

Let (M^n, g) be a complete Riemannian manifold, for any small $r > 0$ and any $p \in M$, we have

$$vol_g(B(p,r)) = vol_{\mathbb{R}^n}(B(r)) \left(1 - \frac{Sc(g,p)}{6(n+1)} r^2 + \cdots \right). \tag{2.1.3}$$

Here, $B(p,r)$ is the geodesic ball in M with center p and radius r. By this basic observation, we obtain the volume comparison of the small geodesic ball,

Theorem 2.1.4. *Suppose that (M^n, g) and (N^n, h) are complete Riemannian manifolds and $m \in M, n \in N$ and (M^n, g) is the space form. If $Sc(g,m) < Sc(h,n)$, then*

$$vol_g(B_g(m,r)) > vol_h(B_h(n,r)) \tag{2.1.4}$$

for sufficiently small $r > 0$.

Motivated by Theorem 2.1.4, a natural conjecture related to the scalar curvature as follows: we assume that ω_n is the volume of n-dimensional Euclidean unit ball.

Conjecture 2.1.5. [21] *Let (M^n, g) be a complete Riemannian manifold and $p \in M$. Suppose that exists $R > 0$ such that for any $r < R$,*

$$vol_g(B(p, r)) = \omega_n r^n$$

Then, (M^n, g) is flat.

Note that Conjecture 2.1.5 has been confirmed as $n = 3, 4$ by a further calculating the Taylor expansion of the volume of the geodesic ball on (M^n, g). However, it remains open as $n \geqslant 5$ in general case. Indeed, from the perspective of the volume comparison, it is not very successful to understand the geometry and the topology of the scalar curvature. But Conjecture 2.1.5 does deeply connect the local geometry with the global geometry on Riemannian manifolds. For more details about Conjecture 2.1.5, refer to the textbook [21] and the references therein. We do believe that any progress of the naive conjecture could lead to a much better understanding of the scalar curvature globally. Many other basic introductions of the scalar curvature, you may refer to [25, 33].

2.2 Scalar curvature and Yamabe invariant

Materials in the sections are partly from the recent survey [40] and they are also connected with the minimal surface techniques in the study of the scalar curvature. Besides, the integral of the scalar curvature also reflects the differential topology of four manifolds. Suppose that (M^n, g) is a complete, closed Riemannian manifold, we first introduce the normalized Einstein-Hilbert action

$$\mathcal{E}(M^n, g) = \frac{\int_M Sc(g) d\mathcal{H}^n}{(\int_M d\mathcal{H}^n_g)^{1-\frac{2}{n}}}. \tag{2.2.1}$$

Note that $\mathcal{E}(M^n, g)$ is a scalar invariant. Moreover, the action $\mathcal{E}(M^n, g)$ still depends quite sensitively on the metric. In fact, $\mathcal{E}(M^n, g)$ is neither be bounded above nor below and its critical points turn out to exactly be the Einstein metrics. However, Yamabe discovered that its restriction to any conformal class of metrics is always bounded below. To see this, we set $p = \frac{2n}{n-2} > 0$ and

$$\tilde{g} = u^{p-2} g.$$

By the conformal change of metric, we obtain that [63]

$$Sc(\tilde{g}) = u^{2-p}[(p+2)\Delta + Sc(g)],$$

where $\Delta = -\nabla^2$. Hence, equation 2.2.1 can be rewritten as

$$\mathcal{E}(\tilde{g}) = \frac{\int_M (p+2)|\nabla u|^2 + Sc(g)u^2 d\mathcal{H}^n_g}{\|u\|^2_{L^p}}. \tag{2.2.2}$$

Hence, g is a critical point of $\mathcal{E}_{[g]}$ if and only if its scalar curvature $Sc(g)$ is constant. Neil Trudinger observed that, whenever that Yamabe constant

$$Y(M,[g]) := \inf_{h\in[g]} \mathcal{E}(h)$$

is non-positive, the minimizers u of $\mathcal{E}$ always exits among the conformal class of g and the minimizers contributes to a smooth metric with constant scalar curvature. Aubin discovered that Trudinger's method actually work whenever

$$Y(M^n, g) \leqslant \mathcal{E}(S^n, g_0),$$

also observed that one always has

$$Y(M^n,[g]) \leqslant \mathcal{E}(S^n, g_0), \tag{2.2.3}$$

for any closed Riemannian manifold $(M^n, g), n \geqslant 3$. This observation has deduced Yamabe's problem to that of showing the equality case only occurs when $(M^n[g])$ is the standard n-sphere. Later, he proved that it is automatically true except as $n \leqslant 5$ or $[g]$ is locally conformal flat. Finally, Schoen [64] completed the proof of Yamabe's claim by using Schoen-Yau positive mass theorem to eliminate all the remaining cases.

Moreover, we introduce the Yamabe invariant on any smooth closed Riemannian manifold (M^n, g),

$$\mathcal{Y}(M) = \sup_{[g]} Y(M,[g]). \tag{2.2.4}$$

$$\mathcal{R}(g) = \int_M |Sc(g)|^{\frac{n}{2}} d\mathcal{H}^n_g. \tag{2.2.5}$$

Then,

$$\mathcal{E}(g) \leqslant (\mathcal{R}(g))^{\frac{2}{n}},$$

for any metric g with equality if and only if $Sc(g) = costant$. Also,

$$\mathcal{E}(u^{p-2}g) \geqslant -(\mathcal{R}(g))^{\frac{n}{2}}.$$

with equality if and only if $u = const$ and $Sc(g) = const \leqslant 0$. Hence, we conclude that any Yamabe metric g can alternatively be characterized as a minimizer of $\mathcal{R}$ in its conformal class $[g]$. Finally, we define

$$\mathcal{I}_s(M^n) = \inf_g \mathcal{R}(g). \tag{2.2.6}$$

Theorem 2.2.1. *Let (M^n, g) be a complete, closed Riemannian manifold. Then,*

- $\mathcal{Y}(M) > 0$ *if and only if M admits metrics g with $Sc(g) > 0$;*
- $$\mathcal{I}_s(M) = \begin{cases} 0 & if\ \mathcal{Y}(M) \geqslant 0, \\ |\mathcal{Y}(M)|^{\frac{n}{2}} & if\ \mathcal{Y}(M) \leqslant 0. \end{cases}$$

Theorem 2.2.1 implies that $\mathcal{Y}(M^n) > 0$ if and only if M admits a Riemannian metric g with positive scalar curvature. However, it is known that not every such manifold M has this property. For the existence of Riemannian metric with positive scalar curvature on a closed manifold, see Section 2.5 for details below. In fact, it's a quite essential topic to understand the obstructions to the existence of metric with positive scalar curvature on smooth manifolds.

Gromov-Lawson [27] and Schoen-Yau [82] proved that any surgery on any codimension greater than 2 can preserve the positivity of the scalar curvature, and many conjectures related to the scalar curvature are motivated by this surgery observation. First, Peteau showed that the Gromov-Lawson surgery arguments also imply that for any $\epsilon > 0$, the condition $\mathcal{Y}(M) > -\epsilon$ is preserved under the elementary surgeries in comdimension $n \geqslant 3$. Second, Peteau discovered that adjoining a well-chosen of collection of Ricci flat manifolds of special holonomy to Stolz's $\mathbb{HP}_2$-bundle shows that the spin-cobordism ring $\Omega^{Spin}(M)$ is generated by manifolds with non-negative Yamabe invariant. Hence,

Theorem 2.2.2. [59] *Any closed simply-connected Riemannian manifold M^n, $n \geqslant 5$ has Yamabe invariant $\mathcal{Y}(M) \geqslant 0$. Moreover, such a manifold has $\mathcal{Y}(M) = 0$ if and only if M is a spin manifold with $\alpha(M) \neq 0$*

For simply connected manifolds of dimension $n \neq 4$, Theorem 2.2.2 provides a complete understanding of the sign of the Yamabe invariant, but usually says nothing about its precise value. On the other hand, Equation 2.2.3 gives us a universal upper bound, while Obata provides a non-trivial lower bound for $\mathcal{Y}(M)$ whenever M admits an Einstein metric of positive scalar curvature. In conjunction with Kobayashi's inequality

$$\mathcal{Y}(M^n), \mathcal{Y}(N^n) > 0$$

implies

$$\mathcal{Y}(M\#N) \geqslant min(\mathcal{Y}(M^n), \mathcal{Y}(N^n)).$$

This confines the Yamabe invariants of many manifolds to specific ranges. In this direction, the best available analogue of Gromov-Lawson-Petean surgery result is a theorem of A-D-H which states that, for every n, there is a constant $\Lambda_n > 0$ such that, whenever $\epsilon < \Lambda_n$, the condition $\mathcal{Y}(M) > \epsilon$ is invariant under elementary surgeries in codimension $\geqslant 3$. One consequence is the following gap theorem

Theorem 2.2.3. *For any $n > 0$, there exists a constant $\delta_n > 0$ such that every closed simply connected manifold M^n with $\mathcal{Y}(M) > 0$ actually satisfies $\mathcal{Y}(M^n) > \delta_n$.*

Indeed, all theorems around the Yamabe invariant and the scalar curvature are restricted into the case that the manifolds are simply connected. It remains open to understand the existence of positive scalar curvature on smooth manifolds. In this section, we can see a direct way to relate the scalar curvature to Yamabe equation. However, in the coming section, you may see some implicit relations by the minimal surface method.

2.3 Scalar curvature and minimal hypersurface

Suppose that (M^n, g) is a complete Riemannian manifold and Σ^{n-1} is an oriented, embedded submanifold of dimension $n-1$ in M, ν is the unit normal vector field of Σ in M and $\{\nu, e_1, e_2, \cdots, e_{n-1}\}$ forms an orthonormal basis of TM in the local coordinate. Then, we introduce the second fundamental form of Σ in M with respect to ν,

$$A_{ij} = A(e_i, e_j) = (\nabla_{e_i}\nu, e_j) = (\nabla\nu)(e_i, e_j). \tag{2.3.1}$$

Then, taking the contraction, we define

$$H = tr(A) = \Sigma_{i=1}^{n-1} (\nabla_{e_i} \nu, e_j). \tag{2.3.2}$$

H is said to be the mean curvature of Σ with respect to ν. Here, we would not use the sign of the second fundamental form A and the mean curvature H in this work. Hence, we do not need to emphasize the orientation of the unit normal vector field ν, which we should choose to define the second fundamental form and the mean curvature on Σ.

Let $(\Sigma^n, h) \subset (M^{n+1}, g), n \geqslant 2$ be a closed oriented minimal hypersurface in an oriented Riemannian manifold and let $\Sigma_t \subset M$ be any smooth 1-parameter variation of $\Sigma_0 = \Sigma$ with normal variation vector field $X = \varphi\nu$ where ν is the unit normal vector of Σ and $\varphi : \Sigma \times (-\epsilon, \epsilon) \to M$. The second variation formula then asserts that the n-dimensional volume $A(t)$ of Σ_t satisfies

$$A''(t) = \int_\Sigma |\nabla\varphi|^2 - (Ric(\nu,\nu) + |A|^2)\varphi^2 d\mathcal{H}_h^n.$$

where $Ric(\nu, \nu)$ is the Ricci tensor of the ambient metric and A is the second fundamental form of $\Sigma \subset M$. However, the Gauss-Codazzi equations imply that the scalar curvature of h and g are related along Σ by

$$Sc(h) = Sc(g) - 2Ric(\nu, \nu) + H^2 - |A|^2,$$

where H is the mean curvature of Σ. Hence,

$$\int_\Sigma (2|\nabla\varphi|^2 + Sc(h)\varphi^2) d\mathcal{H}_h^n = 2A''(0) + \int_\Sigma (Sc(g) + |A|^2)\varphi^2 d\mathcal{H}_h^n. \tag{2.3.3}$$

Now, we assume that (M, g) has positive scalar curvature $Sc(g) > 0$ and $\Sigma \subset M$ is volume minimizing in its homology class, it then follows that Σ carries a positive scalar curvature metric $\hat{h}$ that is conformal to h. Indeed, since our volume minimizing hypothesis on Σ forces

$$A''(0) \geqslant 0$$

for 1-parameter variation. Plugging the positivity of $Sc(g)$, we obtain

$$\int_\Sigma (2|\nabla\varphi|^2 + Sc(h)\varphi^2) d\mathcal{H}_h^n > 0$$

Moreover, for any $n \geqslant 3$, we let $\hat{h} = u^{p-2}h$ be a Yamabe metric where $p = \frac{2n}{n-2} > 2$, we obtain

$$Y(\Sigma, [h]) = \frac{\int_\Sigma (p+1)|\nabla u|^2 + Sc(h)\varphi^2 d\mathcal{H}_h}{\|u\|_{L^2}^2}.$$

which shows that $(\Sigma, \hat{h})$ has positive scalar curvature. On the other hand, if $n = 2$, it yields $\chi(\Sigma) > 0$ by setting $\varphi = 1$. so one has $(M, [h]) \simeq (S^2, [g_0])$ by classical uniformization theory.

Schoen-Yau'arguments now proceeds by downward induction on the dimension of the manifolds. Suppose that a smooth closed oriented Riemannian (M^n, g) with $Sc(g) \geqslant 0$ and let $a \in H^1(M, \mathbb{Z})$ be a non-trivial cohomology class. Compactness results in geometric measure theory guarantee that there is a mass-minimizing rectifiable current that represents the Poincare dual homology class $a \in H_{n-1}(M, \mathbb{Z})$. Then, as $n < 8$, by the regularity theorem, we can obtain a smooth hypersurface $\Sigma^{n-1} \subset M^n$ which admits a metric of positive scalar curvature by the argument above. To continue this process, we can go downward to obtain a 2-surface with positive Gauss curvature. The arguments play an essential role in studying in existence of positive scalar curvature and the proof of positive mass theorem [64]. Forty years have passed. we still can not escape this method, and we are in hungry to investigate more programs to understand the scalar curvature.

The first deep theorem related to the scalar curvature and minimal surface technique had been obtained by Schoen-Yau [84],

Theorem 2.3.1. *Suppose that $M^n, n \leqslant 7$ is a closed oriented manifold with scalar curvature $Sc(g) > 0$. Then, there exists a minimal closed hypersurface which represents an element in $H_{n-1}(M, \mathbb{Z})$ and admits a metric with positive scalar curvature.*

Several years ago, Gromov proposed a generalization of minimal surface to understand the geometry and topology of the scalar curvature on complete Riemannian manifolds, which is called μ-bubble and actually called Brane action in physics.

Let (M^n, g) be a complete Riemannian manifold. Given any $\mathcal{H}_g^n$ measure set $\Omega \subset M$ with nonempty boundary $\partial^\star\Omega$ such that $\partial^\star\Omega$ is a $\mathcal{H}_g^{n-1}$ set. Then, we introduce

Definition 2.3.2. *Given a continuous function μ defined on M, we define*

$$\mu(\Omega) = \mathcal{H}^{n-1}(\partial^\star\Omega) - \int_\Omega \mu \, d\mathcal{H}_g^{n-1}.$$

Moreover, Σ is a called a μ bubble if Σ is a critical point of the functional μ, i.e., the first variation of μ vanishes, and Σ is called a stable μ bubble if the second variation of μ is non-negative.

Now, we take any smooth variational vector field $X = \varphi\nu$. Here, ν is the unit outer normal vector field of $\partial\Omega$. By the first variation of μ along X, we have

$$\frac{d}{dt}\mu|_{t=0} = \int_{\partial\Omega}(H-\mu)\varphi d\mathcal{H}_g^{n-1}.$$

Hence, Σ is a μ bubble if and only if $H = h$. Moreover, by the second variation of μ, we obtain

$$0 \leqslant \frac{d^2}{dt^2}\mu|_{t=0} = \int_{\partial\Omega}|\nabla\varphi|^2 - \left(|A|^2 + Ric(\nu,\nu) + \nabla_\nu\mu\right)\varphi^2 d\mathcal{H}_g^{n-1}.$$

Here, H, A is the mean curvature and second fundamental form of $\partial\Omega$ with respect to ν respectively and $H := tr(A)$, $Ric(\nu,\nu)$ is the Ricci curvature in ν. Then, we obtain that

$$\partial\Omega = \Sigma_1 \cup \cdots \cup \Sigma_k, k \geqslant 0.$$

Here, $\{\Sigma_i\}$ is a collection of are two-sided, closed, connected 2-surfaces. On each stable μ-bubble $\Sigma_i, 1 \leqslant i \leqslant k$, we have

$$H = \mu \text{ on } \Sigma_i, \text{ and } \int_{\Sigma_i}|\nabla_{\Sigma_i}\varphi|^2 - \left(|A|^2 + Ric(\nu,\nu) + \nabla_\nu\mu\right)\varphi^2 d\mathcal{H}^2 \geqslant 0.$$

Moreover, we define

$$R_- = -\frac{1}{2}(Sc(\partial\Sigma) - Sc(g) + |A|^2 - H^2),$$

where $Sc(\partial\Sigma)$ is scalar curvature of Σ. Besides, we have

$$|A|^2 \geqslant \frac{H^2}{n-1} \text{ and } |\nabla_\nu\mu| \geqslant -|d\mu|.$$

Then, we define

$$R_+ = \frac{n\nu}{n-1} - 2|d\mu| + Sc(g).$$

Hence, we reach, on stable μ-bubble

$$\int_\Sigma |\nabla\varphi|^2 + (\frac{1}{2}Sc(\Sigma) - \frac{1}{2}R_+)\varphi^2 d\mathcal{H}_g^{n-1} \geqslant 0. \tag{2.3.4}$$

This will lead to the following elliptic operator

$$L = -\Delta + \frac{1}{2}Sc(\Sigma) - \frac{1}{2}R_+, \tag{2.3.5}$$

which is non-negative operator on Σ.

Then, some basic examples will be shown as follows:

Example 2.3.3. *1. Suppose that $M = \mathbb{R}^n$ and $\mu = \frac{n-1}{r}$, we have*

$$R_+ = \frac{n(n-1)}{r} - \frac{2(n-1)}{r^2} = Sc(S^{n-1}(r)).$$

2. Suppose that $M = \mathbb{R}^{n-1} \times \mathbb{R}$ be the hyperbolic space with metric $g = dr^2 + e^{2r}g_{eucl}$, we have

$$R_+ = n(n-1) - 0 - n(n-1) = Sc(\mathbb{R}^{n-1}).$$

3. Suppose that $M = (-\frac{\pi}{n}, \frac{\pi}{n}) \times Y$ with the metric $g = dt^2 + \varphi^2 h$, where h is a metric on Y and

$$\varphi(t) = (\cos(\frac{nt}{2}))^{\frac{2}{n}}.$$

Then, we have

$$R_+ = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

We may pick h such that

- *g is flat, then*

$$R_+ = Sc(S^{n-1}(r));$$

- *$Sc(g) = 0$, then*

$$R_+ = 0.$$

Moreover, if $Sc(h) = 0$, then $Sc(M) = 0$ and $R_+ = 0$.

In the direction, we have the following important two applications related to the scalar curvature and Yamabe invariants in Section 2.2.

Theorem 2.3.4. *Let (M^n, g) be a complete Riemannian manifold and u a continuous function on M. Suppose that Σ is a smooth stable μ-bubble on M. Then,*

- *If $R_+ > 0$, then Σ admits a metric h with $Sc(h) > 0$;*

- $\Sigma \times \mathbb{R}$ *admits a warped product metric* $k = g_\Sigma + \varphi^2 dr^2$ *such that*

$$Sc(k)((y,r)) \geqslant R_+(y)$$

for any $(y,r) \in \Sigma \times \mathbb{R}$.

An immediate corollary of μ-bubble is that: any complete Riemannian manifolds (M^n, g) with positive scalar curvature $Sc(g) \geqslant n(n-1)$ can be exhausted by a family of hypersurfaces $\{\Sigma_i^{n-1}\}$ that admits a metric with positive scalar curvature.

Now let's discuss the existence and regularity of μ-bubble. Suppose that M is a connected, compact Riemannian manifold with non-empty boundary $\partial M = \partial_- \bigcup \partial_+$. Here, ∂_-, ∂_+ are disjoint compact domains in ∂M. Then, Given a continuous function μ on M with the following properties

$$\mu(x) \geqslant H(\partial_-, x) \text{ and } \mu(x) \leqslant H(\partial_+, x) \tag{2.3.6}$$

Then, by the maximum principle in geometric measure theory, we obtain

Theorem 2.3.5. *Assumption (2.3.6) implies that there exists a stable μ-bubble $Y_{min} \subset M$ which separates ∂_- from ∂_+.*

By the Federer's regularity theorem, smooth minimal μ-bubble always exists only for $n \leqslant 7$. Here, you may refer to [89]. Consequentially, this is the main reason that many applications of minimal surfaces to scalar curvature are restricted into the dimension $n \leqslant 7$.

In particular, we assume that $(M = Y \times (a,b), g)$ and Y is a compact Riemannian manifold with possibly nonempty boundary. If μ is a continuous function such that

$$\mu(x) \to \pm\infty, x \to \partial_\pm.$$

Then, M can be exhausted by compact manifolds M_i with distinguished domains $(\partial_\pm)_i \subset \partial_i M$ such that

- These $(\partial_\mp)_i$ separates $(\partial_\infty)_-$ from $(\partial_\infty)_+$ and

$$(\partial_\pm)_i \to (\partial_\pm)_\pm;$$

- The restriction of μ to $(M_i, (\partial_\pm)_i)$ satisfies with condition 2.3.6.

Then, there exists a locally minimizing μ-bubble in M which separates $(\partial_\infty)_-$ from $(\partial_\infty)_+$.

Finally, over the past years, μ-bubble technique has been used to study the geometry of the scalar curvature, which is called the torical band estimate and to prove the nonexistence of Riemannian metric with positive scalar curvature on closed aspherical manifolds of dimension $4, 5$ in [13, 32].

Remark 2.3.6. *Compared with the minimal surface techniques, the advantages of μ-bubble is to provide the flexibility in the choice of μ that can be adapted to the geometry of the manifold M. Similarly, on the Dirac operator technique, the choices of μ are parallel to the choices of unitary bundles $L \to M$ in the incoming Section 2.4. However, we still have few understandings of the deep relations between the two techniques. More efforts are need investigating to understand the geometry and topology of the scalar curvature on Riemannian manifolds. It is very possible to use the μ-bubble technique to study the geometry of curvature decay on complete Riemannian manifolds*

Remark 2.3.7. *Given $\mu = n - 1$ and $M = \mathbb{R}^n$, we have that the unit sphere $\mathbb{S}^{n-1}(1)$ is the μ-bubble. However, it is unstable.*

2.4 Scalar curvature and Dirac operator

In this section, we will introduce the basic concepts on spin manifolds [39]. In fact, many beautiful results have been on spin manifolds by the Dirac operator. For further introduction and advanced result, see the textbook [39].

Definition 2.4.1. *Let V be a real vector space with a quadratic form Q. The Clifford algebra of (V, Q) denoted by $C(V, Q)$ is the algebra over $\mathbb{R}$ generated by V with*

$$v \cdot w + w \cdot v = -2Q(v, w) \cdot 1_V.$$

for any $v, w \in V$ and 1_V is the unit of V as an algebra.

The Clifford algebra also has the following equivalent characterization:

Proposition 2.4.2. *Let A be an algebra and $c : V \to A$ be a linear map with*

$$c(v)c(w) + c(w)c(v) = -2Q(v, w) \cdot 1.$$

for any $v, w \in V$. Then, there exists a unique algebra homomorphism from $C(V, Q)$ to A extending the given map from V to A. Hence, the Clifford algebra can be written as

$$C(V, Q) = T^2(V)/\{v \otimes w + w \otimes w + 2Q(v, w)\}.$$

Hence, $C(V, Q)$ is a $\mathbb{Z}_2$-graded algebra. Moreover, as a vector space, the Clifford algebra is isomorphic to the exterior algebra ΛV of V, however, the multiplications between $C(V, Q)$ and $\Lambda(V)$ differs from each other.

By Proposition 2.4.2, we obtain that $v \to -v$ extends to an involutive automorphism χ of the Clifford algebra, which determines a $\mathbb{Z}_2$ grading,

$$Cl(Q) = Cl^0(Q) \oplus Cl^1(Q).$$

Now, let E be a $\mathbb{Z}_2$ module over $C(V, Q)$, by the Clifford actions, we obtain that

$$C^+(V, Q) \cdot E^\pm \subset E^\pm, \quad C^-(V) \cdot E^\pm \subset E^\mp.$$

Since $T(V)$ carries a natural action of group of $O(V, Q)$ of the linear maps on V that preserves the quadratic Q and above ideal, it follows that the Clifford algebra $C(V, Q)$ carries a natural action of $O(V, Q)$ as well.

Definition 2.4.3. *Let Q be a positive definite quadratic form, we say that a Clifford module E of $C(V)$ with an inner product is self-adjoint if $c(a^\star) = c(a)^\star$. This is equivalent to the operators $c(v)$ being skew-adjoint.*

Example 2.4.4 (Clifford algebra acts on exterior algebra)**.** *The exterior algebra of V is a Clifford module. Let's define the Clifford module action of $C(V)$ on $\Omega(V)$. Define*

$$\epsilon(v)\alpha = v \wedge \alpha,$$

and l_v is defined as the contraction with the co-vector $Q(v, \cdot) \in V^\star$. Now we define for any $\alpha \in \Lambda(V)$,

$$c(v)\alpha = \epsilon(v)\alpha - l(v)\alpha.$$

If Q is positive definite, $l(v)$ is the adjoint of $\epsilon(v)$. Hence, the Clifford module $\Lambda(V)$ is self-adjoint.

Remark 2.4.5. *Indeed, $Cl(V,Q)$ is a subalgebra of $End(V,A)$ be above example because $c(u)c(v) + c(v)c(u) = 2Q(u,v) \cdot 1$.*

Now, we define

$$\sigma : C(V,Q) \to \Lambda V.$$

by

$$\sigma(v) = c(v)1 \in \Omega(V).$$

Then, its inverse is

$$\mathbf{c} : \Omega(V) \to C(V).$$

by

$$\mathbf{c}(e_{i_1} \wedge \cdots \wedge e_{i_j}) = c_{i_i} \cdots c_{i_j}.$$

σ is called symbol map and $\mathbf{c}$ is called quantization map. Hence, as a vector space, $dim(C(V,Q)) = 2^n$, and σ is an isomorphism of $\mathbb{Z}_2$ graded $O(V)$ modules.

Moreover, the Clifford algebra has a natural increasing filtration structure

$$C(V) = \bigcup_i C_i(V,Q),$$

which is defined as the smallest filtration such that $C_0(V) = \mathbb{R}; C_1(V) = V \oplus \mathbb{R}$. Hence, we obtain a graded algebra $gr(C(V))$, which is naturally isomorphic to the exterior algebra, the isomorphism is given by sending

$$\sigma_i : v_1 \wedge \cdots \wedge v_i \in \Omega^i(V) \to v_1 \cdots v_i \in C_i(V,Q).$$

Note that for any $v \in V, a \in \Lambda(V)$, we obtain

$$\sigma([v,a]) = -2l(v)\sigma(a).$$

$$\Omega^2(V) \simeq so(V) = C^2(V).$$

Let (M^n, g) be a Riemannian manifold. The Clifford bundle of (M^n, g) is the total space

$$Cl(M,g) = \bigcup_{x \in M} Cl(T_xM, g_x)$$

of all the Clifford algebras of the tangent space. A bundle of Clifford modules on (M, g) is a complex vector bundle S over M with a homomorphism of bundles of algebras

$$\gamma : Cl(M, g) \to End(S),$$

i.e., for each $x \in M$, the vector space S_x is a left module over the algebra $Cl(T_xM, g_x)$. Restricted to $TM \in Cl(g)$, the map γ is a Clifford morphism, i.e., a homomorphism of vector bundles such that

$$\gamma(v)^2 = -\|v\| id_{S_x}$$

for each $x \in M$ and $v \in T_xM$. It follows from the universal property of Clifford algebras that conversely, given a vector bundle S over M and a Clifford morphism: $TM \to End(S)$, one can extend it to a homomorphism of bundles of algebras.

If $n \geqslant 3$, then the fundamental group of the special orthogonal group $SO(n)$ is $\mathbb{Z}_2$ and the simply connected universal cover is a group called $Spin(n)$. Now, we will use Clifford algebras to describe this group.

We set

$$Pin(n) = \{u : u = u_1 \cdots u_k,\ u_i \in Cl_n,\ \|u_i\| = 1, i = 1, \cdots, k.\}$$

Then, for any $u \in Pin(n)$, we define

$$\rho(u) : x \mapsto u \cdot x \cdot u^{-1},$$

where u is a unit vector and x is any vector in $\mathbb{R}^m$. It describes the reflection in the hyperplane $u^\perp$ and hence defines a representation from $Pin(n)$ to $O(n)$ that is a double cover. Since $O(n)$ has two connected components, we can restrict to the preimage of the identity component $SO(n)$ to obtain the spinor group $Spin(n)$. Therefore,

$$Spin(n) = Pin(n) \bigcap Cl_n^0.$$

It is equivalent to saying that

$$Spin(n) = \{v_1 \cdots v_{2l} \in Cl_n \mid q(v_i, v_i) = \pm 1,\ i = 1, \cdots, l\}.$$

Moreover, we can complexify the Clifford algebra $Cl_n^c = Cl_m \otimes \mathbb{C}$ and define the complex spinor group as $Spin^c(n) = Spin(n) \otimes_{\mathbb{Z}_2} S^1$. One basic property of the spinor group is that there exist representations, which do not descend to $SO(n)$. The basic representation space is called the space of spinors.

In even dimension $n = 2k$, the algebra Cl_n is a simple matrix algebra, and there is a unique faithful and irreducible Dirac presentation in a complex, 2^n dimensional vector space $\mathbb{S}$ called the spinor space such that $Cl_n \otimes \mathbb{C} = End(\mathbb{S})$. Restricted to Cl_n^0, this representation decomposes into the direct sum of two irreducible and inequivalent, half-spinor Weyl representation

$$\mathbb{S} = \mathbb{S}^+ \bigoplus \mathbb{S}^-.$$

The splitting is basically given by the eigenspaces representation of Clifford multiplication with the volume element η. In odd dimension, we can use the isomorphism: $Cl_{2n} = Cl_{2n+1}^0$ to obtain the unique irreducible complex spinor representation of dimension 2^n. There are exactly two irreducible representations of Cl_{2k+1} of complex dimension 2^k, which become isomorphic representations when restricted to $Spin(2k+1)$ since the volume form η is the interwinning map.

Let E be an oriented vector bundle of rank r with a fiber metric over a manifold M and U_α be a simple cover of M such that E has a transition function $g_{\alpha\beta} \in U_{\alpha\beta}(r)$ on $U_\alpha \bigcap U_\beta$ satisfying the cocycle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\beta}$. We say that E admits a spin structure if E is oriented, and we can define lifts $\tilde{g}_{\alpha\beta}$ of the transition functions to $Spin(r)$ such that the cocycle condition is preserved. This can be expressed in terms of the Stiefel-Whitney classes simply as $\omega_2(E) = 0$. The set of all in-equivalent spin structure is then parametrized by $H^1(M, \mathbb{Z})$. Similarly, the necessary and sufficient topological condition to define a $Spin^c$ structure on a unitary bundle E such that $\omega_2(E)$ is the mod 2 reduction of an integral cohomology class. This is always true for a Hermitian vector bundle E since $\omega_2(E) = c_1(E)$ mod 2.

Let (M^n, g) be a Riemannian manifold with Clifford bundle $Cl(M)$ and let S be any bundle of left modules over $Cl(M)$ Suppose that S admits a metric and ∇ is the connection which preserves the metric and is compatible with the Clifford module structure, i.e.,

- $\nabla(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2)$ for any $s_1, s_2 \in \Gamma(S)$;

- $\nabla(\omega \cdot s) = \nabla\omega \cdot s + \omega \cdot \nabla s$ for $s \in \Gamma(S)$ and $\omega \in Cl(M)$.

Then the Dirac operator of S is the canonical first-order differential operator defined by

$$D\sigma = e_k \cdot \nabla_{e_k}\sigma. \tag{2.4.1}$$

where $\{e_k\}$ is an orthonormal base of TM and $\sigma \in \Gamma(S)$. Note that D is globally well-defined. For even dimensional manifolds, the spinor representation has a natural splitting

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-,$$

and the Dirac operator splits as $D = D^+ + D^-$ with $D^\pm : \mathbb{S}^\pm \otimes E \to \mathbb{S}^\pm \otimes E$, and D^- is the adjoint of D^+. Since the Dirac operator on a closed compact manifold is a self-adjoint elliptic operator, it has a real discrete spectrum with finite multiplicities on a compact manifold. In particular, the index of D^+:

$$index(D^+) = dim(Ker(D^+)) - dim(Ker(D^-)) \tag{2.4.2}$$

is a topological invariant given by the famous Atiyah-Singer Index Theorem:

$$index(D^+) = \int_M \hat{A}(M) \wedge ch(E) \tag{2.4.3}$$

where $\hat{A}$ genus is in the Pontryagin classes of M and $ch(E)$ is the Chern character of the vector bundle E.

The Chern character of a complex bundle E of rank r can be defined by

$$ch(E) = \sum_{k=1}^{r} exp(x_k)$$

where the total Chern class is expressed as

$$C(E) = 1 + c_1(E) + \cdots + c_r(E) = \Pi_{k=1}^{r}(1 + x_k),$$

so that $c_k(E)$ is given by the k-th elementary symmetric function of the $x_k^{,}$. The first few terms are:

$$ch(E) = dim(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots$$

The Chern character satisfies:

$$ch(E_1 \oplus E_2) = ch(E_1) \oplus ch(E_2),\ ch(E_1 \otimes E_2) = ch(E_1)ch(E_2),$$

and hence defines a ring homomorphism $ch : K(M) \to H^{even}(M)$.

Besides, the total $\hat{A}$ genus is given by

$$\hat{A}(M) = \Pi_{k=1}^{r} \frac{x_k/2}{sinh(x_k/2)}$$

where now the total Pontryagin class of TM is formally expressed as

$$p(M) = 1 + p_1(M) + \cdots + p_r(M) = \Pi_{k=1}^{r}(1 + x_k),$$

so that p_k is given by the k-th elementary symmetric function of the (x_k^2)'s. The first terms are:

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{2^7 3^2 5}(-4p_2 + 7p_1^2) + \cdots .$$

Indeed, in terms of differential forms, we have

$$ch(E) = Tr\left(\exp(\frac{F^{\nabla}}{2\pi i})\right) \tag{2.4.4}$$

where F^{∇} is the curvature of a connection ∇ for E, which is an $End(E)$-valued two form.

$$\hat{A} = \sqrt{det}\left(\frac{R/2}{sinh(R/2)}\right) \tag{2.4.5}$$

where R is the Riemannian curvature of the metric g, which is an $End(TM)$-valued two form and $\sqrt{det}$ is the Pfaffian.

Now, we are in a position to introduce the elliptic operator of second order

$$D^2 = \nabla^{\star}\nabla + \frac{Sc}{4}, \tag{2.4.6}$$

where ∇ is the Levi-Civita connection, $\nabla^{\star}$ its adjoint and Sc is the scalar curvature. (2.4.6) is said to be Lichnerowicz formula. we obtain

Theorem 2.4.6. *Let (M^n, g) be a closed, spin Riemannian manifold with $Sc(g) > 0$. Then M admits no non-zero harmonic spinors, which implies that any closed spin manifold with nonzero $\hat{A}$ does not carry metric with positive scalar curvature.*

Moreover, let's introduce the twisted Dirac operator with values in a vector bundle E, the Lichnerowicz formula for D^2 is calculated to be

$$D^2(\sigma \otimes \varphi) = \nabla^*\nabla(\sigma \otimes \varphi) + \frac{Sc}{4}\sigma \otimes \varphi + \mathcal{R}(\sigma \otimes \varphi). \tag{2.4.7}$$

for $\sigma \otimes \varphi \in \Gamma(\mathbb{S} \otimes F)$, where $\nabla^\star\nabla$ is the rough Laplacian and R is the scalar curvature and the last term is explicitly given by:

$$\mathcal{R}(\sigma \otimes \varphi) = \frac{1}{2}\sum_{j,k=1}^{n} \gamma(e_\alpha)\sigma \otimes R^\nabla(e_\alpha)\varphi.$$

where $\{e_\alpha\}$ is an orthonormal base with respect to the metric g for $\Lambda^2(T_pM)$, R^∇ is the curvature tensor of the connection in the bundle E and γ is a Clifford multiplication for g. Twisted bundle is an essential technique to characterize the geometry and topology of manifolds with positive scalar curvature on spin manifolds. Analytically, the twisted bundle technique used in the spin manifold is parallel to that of test functions in geometric analysis.

2.5 Scalar curvature: Existence

Assume that (M^n, g) is a closed Riemannian manifold. By a simple application of the Gauss-Bonnet Theorem

$$\int_M K d\mathcal{H}^2_g = 2\pi\chi(M),$$

we obtain that M is S^2, $\mathbb{RP}^2$, T^2 or the Klein bottle or surfaces with negative Euler characteristic.

Given a manifold M, The basic question is that whether M admits a Riemannian metric with positive or non-negative scalar curvature. A remark result of Kazdan and Warner implies that it suffices to study the following three classes of manifolds:

1. Closed manifolds admitting a Riemannian metric whose scalar curvature function is non-negative and not identically zero;

2. Closed manifolds admitting a Riemannian metric with vanishing scalar curvature but not in class 1;

3. Closed manifolds not in classes 1 or 2.

Theorem 2.5.1. *Suppose that M^n is a closed manifold of $n \geqslant 3$. Then*

- *If M belongs to class 1, every smooth function can be realized as the scalar curvature function of some Riemannian metric on M;*

- *If M belongs to class 2, a function f is the scalar curvature of some metric if and only if either $f(x) < 0$ for some point $p \in M$ or else $f = 0$. Moreover, if the scalar curvature of some metric g vanishes identically, then g is flat;*

- *If M belongs to class 3, then $f \in C^\infty(M)$ is the scalar curvature if and only if $f(x) < 0$ for some point $x \in M$.*

Thus, Theorem 2.5.1 shows that class 1 is equivalent to determining whether M admits a metric with uniformly positive scalar curvature. Moreover, there exist no restrictions on the possibilities for the scalar curvature. Hence, a basic question is when can M be given a Riemannian metric for which the scalar curvature is uniformly positive. Indeed, there are three known obstruction theories:

- Dirac operator technique on a spin manifold, on a spin manifold (M^n, g) with positive scalar curvature,
$$D^2 = \nabla^\star\nabla + \frac{1}{4}Sc(g).$$
Then, the Dirac operator D can not have any kernel and this would imply some topological invariant vanish. See Section 2.4 for the details.

- Schoen-Yau minimal surface technique, which implies that if M^n is an oriented manifold of positive scalar curvature and if N^{n-1} is a closed stable minimal surface in M which is dual to a nonzero in $H^1(M, \mathbb{Z})$, then N also admits a Riemannian metric of positive scalar curvature;

- The Seiberg-Witten technique, which implies that if M^4 is a closed manifold with nonzero Seiberg-Witten invariant, then M does not admit a metric of positive scalar curvature(See section 2.6).

Each of these three techniques has its own advantages and disadvantages. The Dirac operator technique applies to manifolds of all dimension which is almost the most powerful. However, it only applies to spin manifolds. The Schoen-Yau minimal surface technique applies whether M, N are spin or not. But it requires that $H^1(M, \mathbb{Z})$ to be nonzero. Additionally, since the solutions to the minimal hypersurface equations in general dimension have singularities and hence the minimal surface technique only works up to dimension 8. Finally, Seiberg-Witten technique does not require a spin condition, but it only works in dimension 4.

The first obstruction to the existence of positive scalar curvature metric on closed manifolds was discovered by Lichnerowicz, who observed that Dirac operator $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ on a Riemannian spin manifold satisfies the so-called Weitzenb$\ddot{o}$ck formula

$$D^2 = \nabla^\star \nabla + \frac{Sc}{4}, \tag{2.5.1}$$

has trivial kernel and cokernel if the scalar curvature $Sc(g)$ is everywhere positive. However, as $n = 0$ mod 4, the full spinor bundle decomposes a Whitney sum

$$\mathbb{S} = \mathbb{S}^+ \bigoplus \mathbb{S}^-$$

of the so-called chiral spinor bundles, and the Dirac operator correspondingly decomposed as

$$D = D^+ \bigoplus D^-$$

where D^+ the chiral Dirac operator

$$D^\star : \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-),$$

is an elliptic operator whose index $\hat{A}(M)$ has previously been shown by Atiyah and Singer to be a specific linear combination of Pontryagin numbers and thus a cobordism invariant. This allows Lichnerowicz to prove that a smooth compact spin manifold M^{4n} can not admit a metric with positive scalar curvature if $\hat{A}(M) \neq 0$. Later, Hitchin generalized Lichnerowicz's result and gave an obstruction to the existence of positive scalar curvature metrics on a spin manifold of dimension $n = 1$, or 2 mod 8. In fact, in these dimension there is for each spin structure on a smooth compact manifold, a $\mathbb{Z}_2$ valued invariant α given by $dim(ker(D))$ mod 2 when $n = 1$ mod 8, or by dim $ker(D^+)$ mod $n = 2$ mod 8. Since this element of $\mathbb{Z}_2$ is independent of the choice of a Riemannian metric g on M, Hichin was able to prove that a necessary condition for the existence condition for the existence of a positive scalar curvature is that $\alpha = 0$ for every spin structure. When M is simply-connected, it can have at most two spin structure, so this discussion only involves an invariant $\alpha(M) \in \mathbb{Z}_2$ of any smooth compact simply connected manifold of dimension $n = 1$ or $2 \bmod 8$ with $\omega_2(M) = 0$. To keep the notation as simply as possible, one extends the definition of $\alpha(M)$ to the smooth closed spin manifold M of other dimension of other dimensions n by setting

$$\alpha(M) = \hat{A}(M) \in \mathbb{Z}$$

if $n = 0 \bmod 4$ and $\alpha(M) = 0$ if $n = 3, 5, 6$, or $7 \bmod 8$. Hichin's generalization of Lichnerowicz's theorem then tells us that a simply connected spin manifold M can not admit a metric of positive scalar curvature if $\alpha(M) \neq 0$. Remarkably, $\alpha(M)$ is actually invariant under spin cobordisms and so only depends on the spin-cobordism class $[M] \in \Omega_n^{Spin}(M)$. Hence, we have

Theorem 2.5.2. *Let (M^n, g) be a closed spin Riemannian manifold with positive scalar curvature. Then $\alpha(M) = 0$.*

The role of spin structure cobordism in this story means that the obstruction $\alpha(M)$ is invariant under elementary surgeries in a suitable range of dimensions. Conversely, Gromov-Lawson and Schoen-Yau independently proved that the existence of a positive scalar curvature metric on M is invariant under elementary surgeries in codimension $\geqslant 3$.

Using this, Gromov and Lawson can prove that every closed compact simply connected non-spin manifold $M^n, n \geqslant 5$ admits metrics of positive scalar curvature by proving that every such manifold is obtained by a sequence of such surgeries on products and disjoint unions of specific positive scalar curvature generators of the oriented cobordism ring $\Omega^{SO}(M)$.

For simply connected spin manifolds, they conjectured that Hitchin's obstruction

$$\alpha : \Omega_n^{Spin}(M) \to KO^{-n}(pt)$$

was the only obstruction to the existence of positive scalar curvature metrics and observed that this would follow from their surgeries result if one could show that $ker(\alpha)$ were generated by spin manifolds of positive scalar curvature. Finally, Stolz proved that this conjecture by showing that every cobordism class in $ker(\alpha)$ can actually be represented by the total space of an $\mathbb{HP}_2$-bundle over spin a manifold. Consequently, every simply connected manifold $M^n, n \geqslant 5$ satisfies exactly one of the following:

- either $\mathcal{Y}(M) > 0$; or
- M is a spin manifold with $\alpha(M) \neq 0$.

In this direction, we have the following conjecture

Conjecture 2.5.3. *Suppose that M is a connected closed spin Riemannian of dimension $n \geqslant 5$. Then,*

- *(Gromov-Lawson-Rosenberg.) M admits a metric with positive scalar curvature if and only if $\alpha^{\mathbb{R}}(M) = 0 \in KO_\star(C_{\mathbb{R}}\pi_1(M))$;*
- *$\alpha^{\mathbb{R}}(M) = 0$ if and only if $M \times B^k$ admits a metric with positive scalar curvature. Here, B is a Bott manifold. i.e., a simply connected 8-dimensional manifold spin manifold with $\hat{A}(M) = 1$.*

Indeed, Gromov-Lawson-Rosenberg conjecture holds if M is a closed connected spin manifold of dimension $n \geqslant 5$ and $\pi_1(M) = 0$. However, there exists a closed spin manifold $M^n, 5 \leqslant n \leqslant 8$ with $\alpha(M) = 0$ such that M^n admits a metric with positive scalar curvature. In fact, a weaker conjecture claims that any obstruction to the existence of the positive scalar curvature on closed spin manifold M^n with $n \geqslant 5$ which is based on index theory of Dirac operators can be read from the Rosenberg index $\alpha^{\mathbb{R}}(M) \in KO_\star(C_{\mathbb{R}}\pi_1(M))$. For the progress, readers may refer to the references [62, 72]

On the one hand, Schoen and Yau [84] obtain a topological obstruction to $Sc(g) > 0$ on a class of manifolds. However, it can not be covered by the spin method even in the case of spin manifolds.

Definition 2.5.4. *A closed oriented manifold M^n is said to be Schoen-Yau-Schick if it admits a smooth map $f : M \to \mathbb{T}^{n-2}$ such that the homology class of the pullback of a generic point $h = [f^{-1}(t)] \in H_2(M)$ is non-spherical.*

Then,

Theorem 2.5.5. *Let M^n be a Schoen-Yau-Schick manifold of dimension $n < 7$. Then M admits no metric with positive scalar curvature.*

On the other hand, Dirac operator argument also presents some obstruction to $Sc(g) > 0$, which lie beyond the range of minimal surface techniques.

Theorem 2.5.6. *Suppose that M^{2n} is an oriented, closed manifold with a closed 2-form such that $\int_M \omega^n \neq 0$ and the lift of ω to the universal cover of M is exact. Then, M admits no metric with positive scalar curvature.*

Note that this theorem can apply to even dimensional torus, to aspherical four dimensional manifolds with $H^2(M, \mathbb{R}) \neq 0$ and to products of such manifolds but not to the general SYS-manifolds. Even some obstructions to $Sc(g) > 0$ have been obtained over the past fifty years. However, the deep mystery is unexpended.

2.6 Scalar curvature: 4 Manifolds

In this section, we introduce some special result on four-dimensional manifold. Note that $SO(n)$ is simply Lie group as $n \geqslant 3$ other than $n = 4$. In the case of four-dimensional manifolds, we obtain

$$Spin(4) = Sp(1) \times Sp(1) = Spin(3) \times Spin(3),$$

the adjoint action of $SO(4)$ on the $so(4)$ is consequently reducible:

$$so(4) = so(3) \oplus so(3).$$

Notice that $so(4)$ is isomorphic to $\Lambda^2(\mathbb{R}^2)$ as $SO(n)$-modules, the decomposition has an immediate and powerful impact on the geometry of 2-form. Hence, it implies that on four-dimensional Riemannian manifolds, the rank 6 bundles of 2-forms decomposes as the Whitney sum of two rank 3 bundles

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-. \tag{2.6.1}$$

Indeed, $\Lambda^\pm$ turns out to be the ± 1 eigenspaces of the Hodge star operator $\star : \Lambda^2 \to \Lambda^2$,

On any oriented Riemannian four-dimensional Riemannian manifold (M^4, g), the bundle $\Lambda^+ \to M$ carries a natural inner product and orientation, so every fiber of its unit sphere bundle $Z = S(\Lambda^+)$ carries both a metric and orientation. This allows us to consider the twistor space Z as a bundle of complex projection $\mathbb{CP}_1$ and

$$S(\Lambda^+) = \mathbb{P}(\mathbb{V}_+)$$

as the projectivization of a rank 2 complex vector bundle $\mathbb{V}_+ \to M$. Essentially, the choice of $\mathbb{V}_+$ is equivalent to choosing a spin^c structure on M. This stems from the fact that Z can be expressed as

$$S(\Lambda^+) = F/U(2),$$

where F is the principal $SO(4)$-bundle of oriented orthonormal frames.

$Spin^c(n)$ structure plays an important role on four-dimensional Riemannian manifold, we will give more details on this structure as follows.

$$Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} U(1).$$

Since $Spin(n)$ is a double cover of $SO(n)$, $Spin^c(n)$ is a double cover of $SO(n) \times U(1)$.

Definition 2.6.1. *Let (M^n, g) be a complete Riemannian manifold, M is said to be spin^c if given the bundle $P_{SO}(M)$, there are principal bundles $P_{U(1)}(M)$ and $P_{Spin^c}(TM)$ with spin^c-equivariant structure map*

$$\xi : P_{spin^c}(TM) \to P_{SO}(TM) \times P_{U(1)}(TM)$$

Theorem 2.6.2. *Let (M^n, g) be a complete Riemannian manifold. Then the following statements are equivalent:*

- *M is spin^c;*
- *There exists a complex line bundle L on M such that $TM \bigoplus L$ has a spin structure;*
- *The second Stiefel Whitney class $\omega_2(M)$ the mod 2 reduction of an integral class.*

Hence, as a corollary, any oriented four-dimensional Riemannian manifold can be equipped with a *spin^c* structure, which is crucial in the study of the Seiberg Witten equations. Now, let's back to four dimensional Riemannian manifold.

Definition 2.6.3. *Let (M^4, g) be a complete Riemannian manifold. Then, the following two definitions of Spin^c structure are equivalent.*

1. *(Geometric Definition) A spin^c structure on M is a complex line bundle $\mathcal{L} \to Z$ on the twister space that has degree 1 on S^2 fiber of $Z \to M$;*

2. *(Standard Definition) A* $spin^c$ *structure is a circle bundle* $\hat{F} \to F$ *over the oriented orthonormal frame bundle that is also compatibly endowed with the structure of a principal* $Spin^c(4)$ *bundle, where*

$$Spin^c(4) = Spin(3) \times_{\mathbb{Z}_2} Spin(3) \times_{\mathbb{Z}_2} U(1).$$

By the standard definition of $spin^c$ structure, A $spin^c$ structure on (M^4, g) is a choice of principal $Spin^c(4)$- bundle $\hat{F} \to M$ where

$$Spin^c(4) = Sp(1) \times_{\mathbb{Z}_2} Sp(1) \times_{\mathbb{Z}_2} U(1),$$

together with a fixed isomorphism $F = F/U(1)$. Indeed, such a structure is determined by the Chern class $c \in H^2(F, \mathbb{Z})$ of the circle bundle $\hat{F} \to F$ and c can be regarded as an element of $H^2(F, \mathbb{Z})$ whose restriction to the fiber yields the non-trivial element of $H^2(SO(4), \mathbb{Z})$. On the other hand, $\mathbb{P}(\mathbb{V}_+)$ gives rise to a $\mathbb{O}(1)$ line bundle $\mathcal{L} \to Z$ and it produces a cohomology $c_1(\mathcal{L}) \in H^2(Z, \mathbb{Z})$ with the property $(c_1(\mathcal{L}), [S^2]) = 1$. Hence, given any $\mathbb{V}_+$, we obtain a unique $spin^c$ structure by setting $c = q^\star(c_1(\mathcal{L}))$ where $q : \mathcal{F} \to \mathcal{F}/U(2)$.

Conversely, we can construct $\mathbb{V}_+$ from a principal $Spin^c(4)$ bundle $\hat{F} \to M$ by applying the associated bundle construction to the representation of $Spin^c(4)$ on $\mathbb{C}^2$

$$Sp(1) \times_{\mathbb{Z}_2} Sp(1) \times_{\mathbb{Z}_2} U(1) \to Sp(1) \times_{\mathbb{Z}_2} U(1) = U(2),$$

obtained by dropping the second $Sp(1)$ factor. Instead, if we drop the first $Sp(1)$ factor, we reach

$$S(\Lambda^-) = \mathbb{P}(\mathbb{V}_-).$$

The relations between two representations implies that

$$Hom(\mathbb{V}_+, \mathbb{V}_-) = \mathbb{C} \otimes T^\star M, \tag{2.6.2}$$

and the associated Hermitian line bundle $L = \Lambda^2(\mathbb{V}_+) = \Lambda^2(\mathbb{V}_-)$.

Now we fix a $spin^c$ structure on (M^4, g) and choose some Hermitian connection θ on the associated line bundle $L \to M$. If g is a complete Riemannian metric on M and its Levi-Civita connection and θ together induce a unitary connection

$$\nabla_\theta : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_+ \otimes T^\star M).$$

On the other hand, Clifford multiplication induces a bundle homomorphism

$$\mathbb{V}_{+} \otimes T^{\star}M \to \mathbb{V}_{-}.$$

This is an elliptic first-order differential operator, and it acts like the usual operator of spin geometry. Hence, we obtain a twisted Dirac operator

$$D_{\theta} : \Gamma(\mathbb{V}_{+}) \to \Gamma(\mathbb{V}_{-}).$$

It implies the Lichnerowicz Weitzenb$\ddot{o}$ck formula

$$D_{\theta}^{\star} D_{\theta} = \nabla_{\theta}^{\star} \nabla_{\theta} + \frac{Sc(g)}{4} - \frac{1}{2} F_{\theta}^{+}. \tag{2.6.3}$$

Hence, we obtain

$$(\Psi, D_{\theta}^{\star} D_{\theta} \Psi) = \frac{1}{2} \Delta |\Psi|^2 + |\nabla_{\theta} \Psi|^2 + \frac{Sc(g)}{4} |\Psi|^2 + 2(-\sqrt{-1} F_{\theta}^{+}, \sigma(\Psi)). \tag{2.6.4}$$

for any $\Psi \in \Gamma(\mathbb{V}_{+})$, where $F_{\theta}^{+} \in \sqrt{-1}\Lambda^{+}$ is the self-dual part of the curvature of θ that acts on $\mathbb{V}_{+}$, and $\sigma : \mathbb{V}_{+} \to \Lambda^{+}$ is a natural real quadratic map satisfying

$$|\sigma(\Psi)| = \frac{1}{2\sqrt{2}} |\Psi|^2.$$

In contrast to Dirac operator on spin geometry, we can not hope to derive any interesting geometric information due to the extra term F_{A}^{+}. However, Witten consider the following equations (Seiberg-Witten equations)

$$D_{\theta} \Psi = 0, \; -\sqrt{-1} F_{\theta}^{+} = \sigma(\Psi).$$

These equations are non-linear, but they become an elliptic first-order system once one impose the gauge condition

$$d^{\star}(\theta - \theta_0) = 0$$

where θ_0 is an arbitrary connection on L.

Definition 2.6.4. *Let M^4 be a smooth compact oriented manifold with $b_{+} \geqslant 2$. An element $\alpha \in H^2(M, \mathbb{R})$ is called a monopole class of M if and only if there exists a spin^c structure c on M such that*

$$c_1^{\mathbb{R}}(L) = \alpha$$

In this case, the Seiberg-Witten equations have a solution for any Riemannian metric g on M.

Hence, These Seiberg-Witten equations can never admit a solution (Ψ, θ) with $\Psi \neq 0$ relative to a metric with $Sc(g) > 0$. Indeed, the solutions to Seiberg-Witten equations are called Seiberg-Witten invariant. However, It's known that any simply connected manifold $M^n, n \geqslant 5$ admits a metric with positive scalar curvature. The simplest example of four manifolds where nonexistence of metrics with $Sc(g) > 0$ follows from non-vanish of Seiberg-Witten invariants are complex algebraic surfaces X in $\mathbb{CP}^3$ of degree 3. Hence, we obtain a spin or nonspin four manifold $M \subset \mathbb{CP}^3$ given by

$$x_1^d + x_2^d + x_3^d + x_4^d = 0, d \geqslant 5.$$

However, we still wish that we should prove this obstruction theorem from different angles other than using the Seiberg-Witten invariant.

2.7 Scalar curvature and mean curvature

Philosophically, the scalar curvature on Riemannian manifolds theory corresponds to the mean curvature on sub-manifolds theory. i.e., Ricic flow corresponds to the mean curvature flow and similarities and classifications have been deduced. Here, we introduce their similarities from different angles. and I would like to suggest the reader should check the article [31] for many conjectures and questions, where we can see the similarities between the positive scalar curvature and positive mean curvature.

Theorem 2.7.1. [28] *Let (Y, g) be a compact manifold with non-empty boundary ∂Y. Suppose that $Sc(g) \geqslant 0, H(\partial Y) > 0$ and $X = Y +_\partial Y$. i.e., the doubling of Y. Then, X admits a smooth Riemannian metric h such that $Sc(X, h) \geqslant 0$.*

Note that if X_1^n, Y_2^n are two closed manifolds which admits metrics with positive scalar curvature, However, we can not conclude that $X_1 \#_p X_2$ admits a metric with positive scalar curvature. This indicates that connected sum of manifolds can not preserve the positivity of the scalar curvature. Moreover, Gromov-Lawson and Schoen-Yau proved independently that

Theorem 2.7.2. [27,82] *Suppose that M admits a metric with positive scalar curvature and M' is obtained from M via the surgery on codimension $\geqslant 3$. Then M admits a metric with positive scalar curvature.*

Combining Theorem 2.7.1 with Theorem 2.7.2, we can say that we obtain the positivity of the scalar curvature on manifolds by sacrificing the mean curvature on their boundaries. Besides, this can be seen from the following two important topics:

- Suppose that $(M, \partial M, g)$ is compact manifold with non-empty ∂M. We consider

$$F(M) = \int_M Sc d\mathcal{H}_g^n + 2\int_{\partial M} H d\mathcal{H}_g^{n-1}.$$

  To some extent, the functional F may indicate that we may increase scalar curvature by decreasing the mean curvature in the average sense. However, this angle is still very vague. Besides, the Yamabe problem on compact manifolds with nonempty boundary may reflect the deep relations between scalar curvature and mean curvature on their boundaries(See [18]).

- Suppose that g is a smooth metric on the unit ball $B^n \subset \mathbb{R}^n$ with the following properties:

  - The scalar curvature of g is non-negative;
  - The induced metric on the boundary ∂B^n agrees with the standard metric on ∂B^n;
  - The mean curvature of ∂B^n with respect to g is at least $n-1$.

  Then g is isometric to the standard metric on B. The result is deeply connected with positive mass theorem. For the related topic in this direction, see the survey [6]

The following two questions are trying to answer the deep relations between the scalar curvature and mean curvature [33, 50, 66].

Problem 2.7.3 (Extension problem for $Sc \geqslant \sigma$)**.** *Suppose that M is a smooth manifold with boundary $Y = \partial M$ and h is a Riemannian metric on Y and σ, μ are smooth functions on M and Y. What are necessary and what are sufficient conditions for the existence of a complete Riemannian metric g on M, which extends h. i.e., $g|_Y = h$, such that*

$$H(Y) = \mu, Sc(M) \geqslant \sigma.$$

Problem 2.7.4 (Fill-in problem for $Sc \geqslant \sigma$)**.** *Let $Y = (Y, h)$ be a Riemannian manifold and $\mu(y)$ a smooth function on Y. Under what condition does there exist, for given σ, a complete Riemannian manifold (X, g) with $Sc(g) \geqslant \sigma$ with boundary $\partial X = Y$ such that*

$$g|_Y = h, H_g(Y) = \mu.$$

and where, if Y is compact, one may require that X is also compact?

Moreover, Theorem 2.7.2 indicates that manifolds with positive scalar curvature are very flexible in the sense of codimension greater than 3. However, many examples shows that there exist some rigidity phenomena in codimension less than or equal to 2. Gromov conjectures that

Conjecture 2.7.5. [24] *Let (M^n, g) be a complete Riemannian manifold with positive scalar curvature $Sc(g) \geqslant n(n-1)$. Then, there exists a constant $c_n > 0$ such that*

$$width_{n-1}(M) \leqslant width_{n-2}(M) \leqslant c_n.$$

Conjecture 2.7.5 is equivalent to Conjecture 1.1.2. Parallel to Conjecture 2.7.5, Gromov [31] asked

Conjecture 2.7.6. *Suppose that X is strictly mean convex domain in $\mathbb{R}^n$ such that $H(\partial X) \geqslant n-1$. Then, there exists a constant c_n and f such that, for any $x \in X$*

$$d(x, f(x)) \leqslant c_n.$$

Now Conjecture 2.7.5 and 2.7.6 remains open for any $n \geqslant 4$. As $n = 3$, the readers may refer to the literatures [33, 44] and [78] respectively. Indeed, [25, 26, 36] provides the motivation for the studies related to the positive scalar curvature and size of manifolds

2.8 Scalar curvature and Novikov conjecture

In this section, we mainly introduce the Novikov conjecture, part of which is related to the existence of positive scalar curvature on manifolds. We would illustrate the relations between the positive scalar curvature and Novikov conjecture rather than dive into the Novikov conjecture itself. We may refer to references [19, 33, 38, 62, 85] if you are interested in this topic.

Roughly speaking, the Novikov conjecture claims that any closed smooth manifolds are rigid at an infinitesimal level. More precisely, the Novikov conjecture states that the higher signatures of closed smooth manifolds are invariant under orientation-preserving homotopy equivalences. In particular, if M is an aspherical manifold, the Novikov conjecture is an infinitesimal version of the Borel conjecture stated as: all closed aspherical manifolds are topologically rigid. i.e., if N is a closed manifold and homotopy equivalent to M, then N is homeomorphic to M. In fact, Novikov proved that the rational Pontryagin classes are invariant under orientation-preseving homeomorphism. Hence, Novikov conjecture would follow from Borel conjecture in the case of aspherical manifolds. We refer to Yu's recent survey on the Novikov conjecture [85].

Let M^{n+4k}, N^n be two smooth oriented manifolds and $f : M \to N$ be a proper, smooth map. Then, we define $sign(f)$ to be the signature of the pullback $M_x^{4k} = f^{-1}(x)$ of a generic point $x \in N$, that is the signature of the intersection form on the homology $H_{2k}(M_x^{4k}, \mathbb{R})$. For generic $x, y \in N$, we have $M_x^{4k} - M_y^{4k} = \partial f^{-1}([x, y])$. Hence,

$$sign(M_x^{4k}) = sign(M_y^{4k}).$$

Similarly, we can obtain that $sign(f)$ depends only on the proper homotopy class $[f]_{hom}$ of f. Given N and a proper homotopy class of maps f, $sign(f)$ is a smooth invariant, which is denoted by $sign_{[f]}(M)$.

Conjecture 2.8.1 (Novikov Conjecture). *If N is a closed aspherical manifold. Then, $sign_{[f]}(M)$ depends only on the homotopy type of M.*

Historically, in 1966, Novikov proved this as $M = Y \times \mathbb{R}^n, N = \mathbb{T}^n$ and f is the projection $Y \times \mathbb{T}^n \to \mathbb{T}^n$. In 1972, Gheorghe Lusztig found a proof for general $f : M \to \mathbb{T}^n$ based on the Atiyah-Singer index theorem. Here, let's outline the basic ideas of Lusztig since it plays an essential role in the study of scalar curvature and Dirac operator.

Let $\bar{L}_p$ be a flat complex unitary line bundle over $\mathbb{T}^n$ that is parametrized by P. Indeed, P is the n-torus of homomorphism $\mathbb{Z}^n = \pi_1(\mathbb{R}^n) \to \mathbb{Z}$. Then, we consider the pull-back line bundle $L_p = f^{\star}(\bar{L}_p), p \in P$ over M and assume that s is signature over X. Hence, we obtain a twister bundle $L_p \otimes s$. Lusztig calculated that the index of a family of differential operators is equal to $sign(f)$ and hence $sign(f)$ is a homotopy invariant.

Moreover, Lusztig's argument also implied that

Theorem 2.8.2. *Let M^{2n} be a closed oriented spin manifolds and $f : M \to \mathbb{T}^n$ with non-zero degree. Then,*

$$ind(D_{\otimes\{L_p\}}) \neq 0$$

Hence, M admits no metric with positive scalar curvature.

Remarkably, a significant portion of Lusztig's argument generalized to all discrete groups Π, where the algebra $C^\star(\Pi)$ of bounded operators on $l_2(\Pi)$ is regarded as the algebra of continuous functions on a non-commutative space dual to Π. This motivated to the Strong Novikov conjecture

Conjecture 2.8.3. *Let M^{2n} be a closed oriented spin manifold and $f : M \to B\Pi$ such that $f_\star([M]) \neq 0$ in $H_n(B\Pi, \mathbb{R})$ and $[M] \in H_n(M, \mathbb{R})$. Then, the Dirac operator D over M twisted with some flat unitary Hilbert bundle over M has non-zero kernel.*

On one hand, Conjecture 2.8.3 would imply that M admits no metric with positive scalar curvature. It is related to the following conjecture

Conjecture 2.8.4. *Let (M^n, g) be a closed aspherical manifold. Then M admits no metric with positive scalar curvature.*

On the other hand, Conjecture 2.8.3 implies the following conjecture

Conjecture 2.8.5. *Let (M^n, g) a complete, spin manifold and there exists a group action on (M, g) that is cocompact. Then the spectrum of the Dirac operator D on M contains zero.*

A geometric version of Conjecture 2.8.5 is

Conjecture 2.8.6. *Let (M, g) be a complete uniformly contractible Riemannian manifold. Then the spectrum of the Dirac operator D contain zero.*

By a direct observation, Conjecture 2.8.5 and 2.8.6 implies Conjecture 2.8.4. Some progresses have been made: Conjecture 2.8.4 have been confirmed as $n = 3, 4, 5$; However, Conjecture 2.8.5 and 2.8.6 remain open. Now, it is hard to believe in or be against these conjectures. However, these conjectures have motivated lots of interesting studies over the past forty years.

Chapter 3

Volume growth on complete manifolds

3.1 Preliminaries and Notations

In this section, let's make some preparations and prove some lemmas for the proof of the main theorems in the paper. We will start by the stability of the geodesics in a Riemannian manifold.

3.1.1 Variation of Geodesic

Suppose that (M^n, g) is a complete Riemannian manifold and $\gamma : [a, b] \to M$ is a smooth curve in M with $\|\gamma'(t)\| = 1$ and $\gamma(a) = p, \gamma(b) = q,\ p, q \in M$. Then, we consider a smooth variation of $\gamma(t)$:

$$\gamma(t, s) : [a, b] \times [-\epsilon, \epsilon] \to M.$$

and $\gamma(t, 0) = \gamma(t)$ and $\gamma(a, s) = p$ and $\gamma(b, s) = q$. We say that γ is a geodesic if γ is a critical point of the length functional

$$L(s) = \int_a^b \|\frac{\partial\gamma(t, s)}{\partial t}\| dt.$$

That is for any variation vector field $X(t)$ and $X(t) = \frac{\partial\gamma(t,s)}{\partial s}|_{s=0}$ with $X(a) = X(b) = 0$, we have

$$0 = L'(0) = \int_a^b \langle \gamma''(t), X(t) \rangle dt. \tag{3.1.1}$$

It is equivalent to saying that $\gamma''(t) = 0$ on $[a, b]$.

Moreover, we calculate the second variation of the arc length functional on geodesic γ, then

$$L''(0) = \int_a^b \|\nabla X\|^2 - \langle R(X, \gamma'(t))\gamma'(t), X \rangle dt. \tag{3.1.2}$$

Here, $\langle R(X, \gamma'(t))\gamma'(t), X \rangle$ is the Riemannian curvature and ∇ is the Levi-Civita connection on (M, g). A geodesic is said to be stable if the second variation is non-negative. i.e., $L''(0) \geqslant 0$. Since the calculations above are classical and standard on any Riemannian geometry textbook, we omitted the details (See [42]).

Then, for any fixed point $x \in M$, we introduce the exponential map as introduce

$$exp : T_xM \to M, exp(v) = \gamma(1), v \in T_xM.$$

Definition 3.1.1. *Let (M^n, g) be a complete Riemannian manifold. Then we define the injectivity radius of M as follows*

$$\mathrm{Inj}(M) := \inf_{x \in M} \sup_r \{r : exp : B(x, r) \to exp(B(x, r)) \textit{ is a diffeomorphism}\}.$$

and the conjugate radius of M as follows

$$\mathrm{conj}(M) = \inf_x \sup_r \{r : exp : B(x, r) \to exp(B(x, r)),\ exp : isalocalhomeomorphism.\}$$

Finally, (M^n, g) is said to be a manifold with a pole at p if $exp_p : T_pM \to M$ is a diffeomorphism.

By the definition of injectivity radius and conjugate radius of (M, g), we have $\mathrm{inj}(M) \leqslant \mathrm{conj}(M)$. Moreover, let (M^n, g) be a complete Riemannian manifold and $\gamma_{x,v}(t)$ the unique geodesic with initial conditions

$$\begin{cases} \gamma_{(x,v)}(0) = x; \\ \gamma'_{(x,v)}(0) = v, v \in T_xM. \end{cases}$$

The initial problem is solvable uniquely by the classical ODE problem and hence γ always exists.

Definition 3.1.2. *For a given $t \in \mathbb{R}$, we define a diffeomorphism of the tangent bundle TM*

$$\varphi_t : TM \to TM.$$

as follows

$$\varphi_t(x, v) = (\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t)).$$

In fact, the family of diffeomorphism φ_t is a flow. i.e., it satisfies with $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for any $t, s \in \mathbb{R}$ since the uniqueness of the geodesic with respect to the initial conditions. Besides, let SM be the unit tangent bundle of M, that is,

$$SM = \{(x, v) : x \in M, v \in T_xM, \|v\| = 1\}.$$

Since geodesics travel with constant speed, we have that φ_t leaves SM invariant. Given $(x, v) \in SM$, we obtain that $\varphi_t(x, v) \in SM$ for all $t \in \mathbb{R}$. It is well known that any closed, compact Riemannian manifold admits a complete geodesic flow. Finally, you may refer to the textbook [5] for the details about the geodesic flow and the following results.

Definition 3.1.3. *Let (M^n, g) be a complete Riemannian manifold. We define the Liouville measure L on SM. The measure L is given locally by the product of the Riemannian volume on M and the Lebesgue measure on the unit sphere. That is, for any subset $A = (U, A_x) \subset SM$, where $U \subset M$ is a subset of M, A_x is a subset of the unit sphere of the tangent space at $x \in U$, L is defined by*

$$L(A) = \int_{x \in U} \int_{A_x} d\mathbb{S}^{n-1} dvol(x).$$

where $d\mathbb{S}^{n-1}$ is the usual Lebesgue measure on the unit sphere.

A well known result related to the Liouville theorem is,

Lemma 3.1.4. *Let (M^n, g) be a complete manifold and φ_t the geodesic flow. Then for any Borel set B in SM and $t \in \mathbb{R}$, we have,*

$$L(\varphi_t(B)) = L(B).$$

That is, geodesic flow preserves Liouville measure.

Finally, we need the following basic integral form of the scalar curvature $Sc(g)$ in terms of the Ricci curvature $Ric(g)$.

Lemma 3.1.5. *Let (M^n, g) be a complete Riemannian manifold. Then for any $p \in M$, we obtain,*

$$Sc_p = \frac{n}{\mathrm{vol}(\mathbb{S}^{n-1})} \int_{v \in \mathbb{S}^{n-1}} Ric_p(v) dv. \tag{3.1.3}$$

Proof. Let $\{e_i\}_{i=1}^n \in T_pM$ be an orthonormal coordinate such that $Ric_p(e_i) = \lambda_i e_i$. Then, for any $v \in \mathbb{S}^{n-1}$,

$$v = x_i e_i.$$

Hence

$$\sum_{i=1}^n x_i^2 = 1,\ Sc_p = \sum_{i=1}^n \lambda_i,$$

and then

$$\int_{v \in \mathbb{S}^{n-1}} Ric_p(v) dv = \int_{x \in \mathbb{S}^{n-1}} \sum_{i=1}^n x_i^2 \lambda_i^2 dx = \sum_{i=1}^n \lambda_i^2 \int_{x \in \mathbb{S}^{n-1}} x_i^2 dx = \frac{\mathrm{vol}(\mathbb{S}^{n-1}) Sc_p}{n}.$$

□

3.1.2 Integral of Curvatures

Definition 3.1.6. *Let (M^n, g) be a complete, non-compact manifold, $\gamma(t) = \exp(tv), t \geqslant 0, v \in T_pM$ is called a ray if it is minimal on every interval*

$$d(\gamma(t), \gamma(s)) = |s - t|,\ s, t > 0,$$

and the unit vector v is called a direction of $\gamma(t)$. Assume that $\gamma_i(t) = \exp(tv_i), v_i \in T_pM$ are rays, $\{\gamma_i(t)\}$ are independent and orthogonal if their directions $\{v_i\}$ are linearly independent and mutually orthogonal at p. Moreover, we define the Busemann function $B_\gamma(x)$ associated with any ray $\gamma(t)$

$$B_\gamma(x) = \lim_{t \to \infty} (t - d(x, \gamma(t))).$$

Here, $B_\gamma(x)$ is well-defined since $f(t) = t - d(x, \gamma(t))$ is increasing in terms of t and uniformly bounded from above.

Lemma 3.1.7. *Let (M^n, g) be a complete, non-compact manifold with a pole p and $Ric(g) \geqslant 0$. Then, for any ray $\gamma(t)$ with $\gamma(0) = p$,*

$$\frac{1}{r}\int_0^r t^2 Ric(\gamma'(t), \gamma'(t))dt \leqslant n - 1. \tag{3.1.4}$$

In particular, as $n = 3$,

$$\frac{1}{r}\int_{B(p,r)} Ric(\nu, \nu)(x) d\mathcal{H}^3(x) \leqslant 8\pi. \tag{3.1.5}$$

where ν is the outer unit normal vector field of the geodesic sphere.

Proof. Assume that $\gamma(t)$ with $\gamma(0) = p$ is a ray and $V(t)$ is a smooth vector field along $\gamma(t)$, we consider the variation of $\gamma(t)$:

$$\gamma(t, s) = \exp_{\gamma(t)}(sV(t)), s \in \mathbb{R}.$$

Since $\gamma(t)$ is a minimizing geodesic, then, on any interval $[0, b]$, we obtain that the second variation of length functional is non-negative by 3.1.2. i.e.,

$$\int_a^b |\nabla V(t)|^2 - K(\gamma'(t), V(t))dt \geqslant 0.$$

Hence,

$$\int_a^b K(\gamma'(t), V(t))dt \leqslant \int_a^b |\nabla V|^2 dt.$$

Then, for any $t \in [0, b]$, we assume that $\{e_i(t)\}_{i=1}^{n-1}$ is the parallel vector field such that $\{e_i(t), \gamma(t)\}$ forms an orthonormal base in $T_{\gamma(t)}M$. Now we fix any $x \in (0, b)$ and then take

$$V_i(t) = \begin{cases} \frac{t}{x} e_i(t), & t \in [0, x]; \\ \frac{b-t}{b-x} e_i(t), & t \in [x, b]. \end{cases}$$

Then, we plug $V_i(t)$ into the inequality to obtain

$$\frac{1}{x^2}\int_0^x t^2 Ric(\gamma'(t), \gamma'(t))dt + \frac{1}{(b-x)^2}\int_0^{b-x} t^2 Ric(\gamma'(b-t), \gamma'(b-t))dt \leqslant (n-1)(\frac{1}{x} + \frac{1}{b-x}).$$

By taking $b \to \infty$ and the assumption that $Ric(g) \geqslant 0$, we obtain

$$\frac{1}{x^2}\int_0^x t^2 Ric(\gamma'(t), \gamma'(t))dt \leqslant \frac{n-1}{x}.$$

Hence, for any $r > 0$

$$\frac{1}{r}\int_0^r t^2 Ric(\gamma'(t),\gamma'(t))dt \leqslant n-1. \tag{3.1.6}$$

Since $p \in M$ is a pole, we have for any $t > 0$, the geodesic sphere $\partial B(p,t) \subset M$ is a sphere and on which the unit normal vector field is well-defined by ν. As a result, we obtain $f(x) = Ric(\nu,\nu)(x)$ is a well-defined, induced function on each point $x \in M$. Finally, for any $v \in T_pM$, $\exp(tv)$ is a ray and hence we can obtain the estimate (3.1.6) on it. Then, by the pointwise volume comparison theorem $\|Dexp\| \leqslant 1$ on $\mathbb{R}^3$ and then by change of variables, we obtain

$$\frac{1}{r}\int_{B(p,r)} Ric(\nu,\nu)d\mathcal{H}^3 = \frac{1}{r}\int_{B(0,r)} Ric(\nu,\nu)\|Dexp\|dL^3 \leqslant \frac{1}{r}\int_0^r Ric(\nu,\nu)dL^3 \leqslant 4\pi(n-1).$$

Here, $\mathcal{H}^3$ is the 3 dimensional Hausdorff measure on M^3 and $B_3(0,r)$ is the Euclidean ball in $\mathbb{R}^3$ with center at the origin and radius r. □

Then, let's introduce the following type of geometrically relative Bochner formula. Let f be a smooth function defined on a complete Riemannian manifold (M^n, g), we define the level set of f as

$$L_t^f = \{x \in M : f(x) = t\}.$$

On each level set L_a^f, if it is a smooth $n-1$ dimension, embedded submanifold in M, we define the second fundamental form and mean curvature of L_t^f by A and H respectively with respect to the outer unit normal vector field $u = \frac{\nabla f}{|\nabla f|}$. Hence,

$$A = \nabla_{L_t^f}(u), \;\; H = \mathrm{Div}(u).$$

Here, $\nabla_{L_t^f}$ is the restriction of ∇ on L_t^f. Then, we introduce $G = H^2 - |A|^2$.

Then, we have the following geometrically relative Bochner formula in [61]. Here, we give a different proof using Bochner formula of the vector field.

Lemma 3.1.8. *Suppose that (M^n, g) is a complete Riemannian manifold and there exists no critical point in $[a,b]$ for f. Then,*

$$\int_{f^{-1}[a,b]} Ric(u,u) - G = \int_{L_a^f} H - \int_{L_b^f} H. \tag{3.1.7}$$

Proof. By Bochner formula of the vector field X over M[47], we have

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \mathrm{Div}(L_X g)(X) - Ric(X,X) - \nabla_X \mathrm{Div}(X) \tag{3.1.8}$$

Now, we take $X = u = \frac{\nabla f}{|\nabla f|}$ in (3.1.8), then,

$$Ric(u,u) = |\nabla u|^2 + \mathrm{Div}(L_u g)(u) - \nabla_u \mathrm{Div}(u).$$

For any $x \in L_t^f$, we obtain that $u(x) = \frac{\nabla f(x)}{|\nabla f|(x)}$ is the unit normal vector field of L_t^f at the point of $x \in L_t^f$ and then we assume that $\{e_i, u\}_{i=1}^{n-1}$ forms an orthonormal base in $T_x M$, hence

$$|A|^2 = |\nabla u|^2,\ H = \mathrm{Div}(u).$$

Moreover, by using integration by parts over $f^{-1}([a,b])$ for $\nabla_u(\mathrm{Div}(u))$, we obtain,

$$\begin{aligned} & -\textstyle\int_{f^{-1}([a,b])} \nabla_u(\mathrm{Div}(u)) \\ = & -\textstyle\int_{f^{-1}([a,b])} \nabla_u H \\ = & -\textstyle\int_{f^{-1}([a,b])} \mathrm{Div}(Hu) - H\mathrm{Div}(u) \\ = & -\textstyle\int_{f^{-1}([a,b])} \mathrm{Div}(Hu) - H^2 \\ = & \textstyle\int_{f^{-1}([a,b])} H^2 + \int_{L_a^f} H - \int_{L_b^f} H. \end{aligned}$$

For the term $\mathrm{Div}(L_u g)(u)$,

$$\begin{aligned} & \textstyle\int_{f^{-1}([a,b])} \mathrm{Div}(L_u g)(u) \\ = & \textstyle\int_{f^{-1}([a,b])} \nabla_{e_i}(L_u g)(e_i, u) \\ = & \textstyle\int_{f^{-1}([a,b])} \nabla_{e_i}(L_u g(u, e_i)) - (L_u g)(e_i, \nabla_{e_i} u) \\ = & \textstyle\int_{f^{-1}([a,b])} \nabla_{e_i}((L_u g)u, e_i)) - (L_u g)(e_i, \nabla_{e_i} u) \\ = & -2\textstyle\int_{f^{-1}([a,b])} |A|^2. \end{aligned}$$

Hence,

$$\int_{f^{-1}([a,b])} Ric(u,u) = \int_{f^{-1}([a,b])} G - \int_{L_b^f} H + \int_{L_a^f} H$$

□

Remark 3.1.9. *From the perspective of function theory on a complete manifold: Sectional curvature would impose the condition on the Hessian of functions or the second fundamental form of level set. Ricci curvature would impose the condition on the Laplacian of functions or the mean curvature of the level set. Lemma 3.1.8 seems trivial,* *but it is deep for the author, since the integral of the second fundamental form can be expressed in terms of the mean curvature and the Ricci curvature on the ambient Riemannian manifold.*

3.1.3 Width and Positive Scalar Curvature

Let's introduce the result related to the positive scalar curvature to our article. The study of non-negative and positive scalar curvature is a very important topic in geometry analysis, which is pioneered by the works [28, 82] of Gromov-Lawson and Schoen-Yau. These works provide us two paths of the understandings of the geometry and topology of scalar curvature bounded below: spin techniques and minimal surface techniques. According to their works, it's known that $\mathbb{T}^n$ admits no complete Riemannian metric with non-negative scalar curvature unless it is flat. In recent Gromov's work [33], he introduces the μ bubble, which is detailed by Zhu in his work [89] where his result indicates that positive scalar curvature implies that 2-systole is bounded above in terms of the lower bound of the scalar curvature. After that, many applications of μ bubble have been expanded to study the existence of Riemannian metric with positive scalar curvature [13, 32]. Here, we will use the following result, which relates the size to the positive scalar curvature [30].

Suppose that $M^n = T^{n-1} \times I$ where I is an interval $[a, b], a < b$. Here, M is called a torical band. We define

$$\partial M = T^{n-1} \times \{b\} \bigcup T^{n-1} \times \{a\} =: \partial_+ M \bigcup \partial_- M.$$

then,

$$d(\partial_+ M, \partial_- M) = \inf_{x \in \partial_+ M, y \in \partial_- M} \{d(x, y)\}. \tag{3.1.9}$$

Then, the following theorem holds

Theorem 3.1.10 (G-WXY [30,33,75])**.** *Let (M, g) be a n-dimensional torical band with $Sc(g) \geqslant n(n-1)$. Then,*

$$d(\partial_+ M, \partial_- M) < \frac{2\pi}{n}. \tag{3.1.10}$$

Theorem 3.1.10 plays a vital role in the proof of Theorem 1.1.8. Combing it with the work [49] of McLeod-Topping, we avoid analyzing the singular Ricci limiting space to achieve our goal.

Theorem 3.1.11 (Spherical Lipschitz Bound Theorem [30])**.** *Let (M^n, g) be a Riemannian manifold(possibly incomplete) with $Sc(g) \geqslant n(n-1)$. Then, for all continuous maps f from M to the unit sphere $\mathbb{S}^n$ (and also to the hemisphere to $\mathbb{S}^n_+(1)$ of non-zero degrees, we have,*

$$Lip(f) > \frac{c}{\pi\sqrt{n}} \text{ for the above } c > \frac{1}{3}.$$

Finally, the following lemma is also needed for the proof of Theorem 1.1.8.

Lemma 3.1.12. *Let (M^n, g) be a complete, non-compact Riemannian manifold with $Ric(g) \geqslant 0$. If there exists a sequence of $p_i \to \infty$ and $R_i \to \infty$ such that*

$$vol(B(p_i, R_i)) = c(R_i)R_i^k, n-1 \geqslant k \geqslant 1.$$

with $c(R_i) \to \infty$ as $i \to \infty$ and $vol(B(p, 1)) \geqslant v > 0$ for all $p \in M$, then there exists a sequence $q_i \in M$ such that (M, q_i) pointedly Gromov Hausdorff converges to a length space $(X \times \mathbb{R}^l, p_\infty)$ with

$$l \geqslant k+1.$$

Hence, it implies that there exists at least $k+1$ rays $\{\gamma_l^{(i)}\}_{l=1}^{k+1}$ which are linearly independent and orthogonal at q_i in M such that for all $l = 1, 2, \cdots, k, k+1$, the length of $\gamma_l^{(i)}$, $L(\gamma_l^{(i)}) \to \infty$ as $i \to \infty$.

Proof. Since we assume that $Ric(g) \geqslant 0$, by the precompactness theorem and Cheeger-Colding theory [8], we obtain that, up to subsequence, (M, p_i, vol_i) converges to a metric measured length space $(X, x_\infty, \mu_\infty)$ with a Borel measure μ_∞ on X. Moreover, since it is assumed that $vol(B(p, 1)) \geqslant v > 0$, for any geodesic ball $B(p_i, r) \subset M$ and $B(x_\infty, r) \subset X$, we have,

$$\lim_{i\to\infty} vol(B(p_i, r)) = \mu_\infty(B(x_\infty, r)).$$

Furthermore, let $\gamma_i : [0, \infty) \to M$ be a ray with $\gamma_i(0) = p_i$, then we introduce that

$$\sigma_i(t) = \gamma_i(t + R_i) : [-R_i, \infty) \to M.$$

and we set $q_i = \sigma_i(0)$. By the assumption of the volume growth, we have,

$$vol(B(q_i, 2R_i) \geqslant c(R_i)R_i^k.$$

Hence, we can replace p_i by q_i in the precompactness theorem. So we have σ_i will converge to a line σ_∞ in X. By the splitting theorem in the Ricci limiting space [8], we obtain

$$X = X_1 \times \mathbb{R} \text{ and } \mu_\infty = \mu_\infty^1 \times \mathbb{R}.$$

If we let $q_i \to q_\infty$ and R_i large, then

$$\mu_\infty(B(q_\infty, R_i) \geqslant c(R_i)R_i^k, c(R_i) \to \infty, \text{ as } R_i \to \infty.$$

Here $c(r)$ may be different line by line. Hence, for metric ball $B_1(q_\infty^1, R_i) \subset X_1$,

$$\mu_\infty^1(B_1(q_\infty^1, R_i) \geqslant c(R_i)R_i^{k-1}, \text{and } c(R_i) \to \infty, \text{ as } R_i \to \infty.$$

Here, q_∞^1 is from $q_\infty = (q_\infty^1, x^n), x^n \in \mathbb{R}$. Hence, we obtain that (X_1, μ_∞^1) is a non-compact metric measured length space. Then, there exists a ray in X_1, we can retake our base point in M to obtain a line associated with the ray as we did above by pulling back the ray to M. Hence, we have

$$(X_1 = X_2 \times \mathbb{R}, \mu_\infty^1 = \mu_\infty^2 \times \mathbb{R}).$$

Finally, we continue this process $k-1$ times to obtain the limiting space

$$(X_{k-1} = X_k \times \mathbb{R},\ \mu^k = \mu_\infty^{k-1} \times \mathbb{R})$$

and for any metric ball $B(q_\infty^k, R_i) \subset X^k$,

$$vol(B(q_\infty^k, R_i))) \geqslant c(R_i), , c(R_i) \to \infty, \text{ as } R_i \to \infty.$$

Here $q_\infty^{k-1} = (q_\infty^k, x^{n-k+1}), x^{n-k+1} \in \mathbb{R}$. Hence, X_k is still non-compact, otherwise, its volume should be finite. Hence, by the same argument above, we obtain that X_k splits as $X_{k+1} \times \mathbb{R}$. Hence, we finally find a sequence $q_i \in M$ such that (M, q_i) pointedly Gromov Hausdorff converges to $X \times \mathbb{R}^{k+1}$. We complete the proof of the lemma. □

3.1.4 Proof of Proposition 1.1.4

Before we are going to the proof, let's first see the compact case: Setting $l = \text{conj}(M)$, we consider any geodesic ball $B(m,l), m \in M$ and any $q \in \partial B(m,l)$, there exists arc length parameter $\gamma : [0,l] \to M$ which is the shortest geodesic connecting m, q . Then, we consider the index form for any variational vector X of $\gamma(t)$ with $X(0) = X(l) = 0$ $0 \leqslant I(X,X) = \int_0^l \|\nabla X\|^2 - \langle R(X,\gamma'(t))\gamma'(t), X\rangle ds$. Hence,

$$\int_0^l \langle R(X,\gamma'(t))\gamma'(t), X\rangle ds \leqslant \int_0^l \|\nabla X\|^2 ds. \tag{3.1.11}$$

If we pick $X = sin(\frac{\pi}{l}t)\nu$ with $\nu = \nu(t)$ a parallel unit vector field along $\gamma(t)$, then

$$\int_0^l sin^2(\frac{\pi}{l}t)\langle R(\nu,\gamma'(t))\gamma'(t),\nu\rangle dt \leqslant (\frac{\pi}{l})^2 \int_0^l sin^2(\frac{\pi}{l})dt. \tag{3.1.12}$$

By a direct calculation, we obtain $(\frac{\pi}{l})^2 \int_0^l sin^2(\frac{\pi}{l}t)dt = \frac{\pi^2}{2l}$ and then

$$\int_0^l sin^2(\frac{\pi}{l}t)\langle R(\nu,\gamma'(t))\gamma'(t),\nu\rangle dt \leqslant \frac{\pi^2}{2l}.$$

$$\int_0^l sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dt \leqslant \frac{(n-1)\pi^2}{2l}.$$

Integrating over the unit tangent bundle SM, we have,

$$\int_{SM}\int_0^l sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dtdL \leqslant \int_{SM} \frac{(n-1)\pi^2}{2l} dL. \tag{3.1.13}$$

For the integral on the left, we have,

$$\begin{aligned}
&\int_{SM}\int_0^l sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dtdL \\
= \;& \int_0^l\int_{SM} sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dLdt \\
= \;& \int_0^l\int_{m\in M}\int_{v\in S_mM} sin^2(\frac{\pi}{l}t)Ric((\varphi_t)_\star v,(\varphi_t)_\star v)dv d\mathrm{vol}(m)dt \\
= \;& \int_0^l sin^2(\frac{\pi}{l}t)dt\int_{m\in M}\int_{S_mM} Ric(v)dv d\mathrm{vol}(m) \\
= \;& \frac{\mathrm{vol}(\mathbb{S}^{n-1})}{n}\int_0^l sin^2(\frac{\pi}{l}t)dt\int_{m\in M} Sc(m)d\mathrm{vol}(m) \\
\geqslant \;& \frac{(n-1)l}{2}\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{vol}(M) \;\; \text{since } Sc \geqslant n(n-1).
\end{aligned}$$

By a direct calculation, we obtain that

$$\int_{SM}\frac{(n-1)\pi^2}{2l}dL = \frac{(n-1)\pi^2}{2l}\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{vol}(M).$$

Hence,

$$l \leqslant \pi.$$

Finally, if $l = \pi$, then all above inequalities are equalities. Hence, M has constant sectional curvature $K = 1$ with Diam(M) = Inj$(M) = \pi$. Hence, by Theorem 1.1.1, we obtain that M is isometric to the round sphere $\mathbb{S}^n$.

Now, let's come back to the proof of Proposition 1.1.4: Rather than integrating over the unit vector bundle SM, we consider the geodesic ball $B = B(p,r) \subset M$ and $B_{-l} = B(p, r-l)$. Then we start from inequality (3.1.13),

$$\int_{SB}\int_0^l sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dtdL \leqslant \int_{SB}\frac{(n-1)\pi^2}{2l}dL.$$

$$\begin{aligned}
& \int_{SB}\int_0^l sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dtdL \\
= & \int_0^l\int_{SB} sin^2(\frac{\pi}{l}t)Ric(\gamma'(t),\gamma'(t))dLdt \\
\geqslant & \int_0^l sin^2(\frac{\pi}{l}t)dt\int_{m\in M}\int_{S_mB_{-l}} Ric(v)dvd\mathrm{vol}(m), Ric(g)\geqslant 0 \text{ on } B(p,R) \\
= & \frac{\mathrm{vol}(\mathbb{S}^{n-1})}{n}\int_0^l sin^2(\frac{\pi}{l}t)dt\int_{m\in B_{-l}} Sc(m)d\mathrm{vol}(m).
\end{aligned}$$

Hence,

$$\int_{B(p,r-l)} Sc \leqslant n(n-1)\frac{\pi^2}{l^2}vol(B(p,r)).$$

Moreover, if we assume that $Ric(g)\geqslant 0$ and $Sc(g)\geqslant n(n-1)$ on M, then for any $r>l$,

$$\frac{\mathrm{vol}B(p,r-l)}{\mathrm{vol}(B(p,r))}\leqslant\frac{\pi^2}{l^2}.$$

By volume comparison theorem, we obtain,

$$\lim_{r\to\infty}\frac{\mathrm{vol}B(p,r-l)}{\mathrm{vol}(B(p,r))}=1.$$

Therefore $l\leqslant\pi$. We complete the proof of Proposition 1.1.4.

As a corollary, we have

Corollary 3.1.13. *Let (M^n,g) be a complete, non-compact manifold with $Ric(g)\geqslant 0$ with $inj(M)=c>0$. Then, for any $p\in M$,*

$$\frac{1}{\mathrm{vol}(B(p,c))}\int_{B(p,c)} Sc\leqslant\frac{2^n n(n-1)}{c^2}$$

From the perspective of Cheeger-Colding theory and Anderson's C^α convergence, it is too strong that we assume that the injectivity radius has a uniformly lower bound. But, if you pay more attention to the generalized scalar curvature on Ricci limiting space, we still do not have a systematic way to introduce a useful scalar curvature on this singular space. Probably, this inequality may help study the Ricci limiting space for non-collapsing case in the future.

3.2 Proof of Theorems

In this section, we will prove Theorem 1.1.7, 1.1.8 and 1.1.12. For the proof of Theorem 1.1.7, we will use the geometrically relative Bochner formula along the distance function and the stability of ray. For the proof of Theorem 1.1.8, we will combine the Gromov-Hausdorff convergence with the estimate of torical band to obtain the volume estimate. For the proof of Theorem 1.1.12, we analyze the level set of Busemann function to obtain the existence of function required.

3.2.1 Proof of Theorem 1.1.7

Theorem 3.2.1. *Let (M^3, g) be a complete, non-compact three-dimensional Riemannian manifold with a pole p and $Ric(g) \geqslant 0$. Then*

$$\limsup_{r\to\infty} \frac{1}{r} \int_{B(p,r)} Sc \leqslant 20\pi. \tag{3.2.1}$$

Proof. Let's first define $f(x) := d(p, x)$, on each level set L_t^f, we have the following type of Gauss equation called S-Y trick on minimal surface [82]

$$2\bar{K} = Sc - 2Ric(\nu, \nu) + G.$$

Here, $\bar{K}$ is the Gauss curvature of the level set L_t^f. Hence,

$$Sc = 2\bar{K} + 2Ric(\nu, \nu) - G.$$

Then integrating it over $B(a, b) = B(p, a, b)$, we obtain

$$\int_{B(a,b)} Sc d\mathcal{H}^3 = \int_{B(a,b)} (2\bar{K} + 2Ric(\nu, \nu) - G) d\mathcal{H}^3.$$

By Lemma 3.1.8, we obtain,

$$\int_{B(a,b)} G d\mathcal{H}^3 = \int_{B(a,b)} Ric(\nu, \nu) d\mathcal{H}^3 + \int_{L_b^f} H - \int_{L_a^f} H.$$

Hence,

$$\int_{B(a,b)} Sc\ d\mathcal{H}^3 = 2\int_{B(a,b)} \bar{K} d\mathcal{H}^3 + \int_{B(a,b)} Ric(\nu, \nu) d\mathcal{H}^3 - \int_{L_b^f} H + \int_{L_a^f} H.$$

By the coarea formula and Gauss Bonnet theorem on each level set surface, we have,

$$\int_{B(a,b)} \bar{K} d\mathcal{H}^3 = \int_a^b \int_{\partial B(r)} \bar{K} d\mathcal{H}^2 dr = \int_a^b 2\pi\chi(\partial B(r)) dr = 4\pi(b-a).$$

Here, we used $\partial B(r)$ is a topological sphere for any $r \in [a,b]$. Moreover, by the proof of volume comparison theorem, we deduce that

$$\int_{L_b^f} H \leqslant 4\pi b.$$

And by Lemma 3.1.7, we obtain that

$$\int_{B(a,b)} Ric(\nu,\nu) d\mathcal{H}^3 \leqslant \int_{B(0,b)} Ric(\nu,\nu) d\mathcal{H}^3 \leqslant 8\pi b.$$

Hence, together all estimates above, we get

$$\int_{B(a,b)} Sc d\mathcal{H}^3 \leqslant 8\pi(b-a) + 8\pi b + 4\pi b - \int_{L_a^f} H.$$

By taking $b \to \infty$ and then $a \to 0$, we have,

$$\limsup_{r\to\infty} \frac{1}{r} \int_{B(p,r)} Sc \, d\mathcal{H}^3 \leqslant 20\pi.$$

□

3.2.2 Proof of Theorem 1.1.8

Theorem 3.2.2. *Let (M^3, g) be a complete, non-compact three-dimensional Riemannian manifold with $Ric(g) \geqslant 0$ and $Sc(g) \geqslant 6$. Then, for any $p \in M$, we obtain,*

$$\limsup_{R\to\infty} \frac{vol(B(p,R))}{R} < \infty, \tag{3.2.2}$$

provided that $vol(B(q,1)) \geqslant \epsilon > 0$ for all q.

Proof. Let's show that it is sufficient to prove that there exists one $p \in M$ such that

$$\limsup_{R\to\infty} \frac{\mathrm{vol}(B(p,R))}{R} < \infty. \tag{3.2.3}$$

We assume that the estimate (3.2.3) holds for p, then for any $q \in M$, we have

$$\frac{\mathrm{vol}(B(q,r))}{r} \leqslant \frac{\mathrm{vol}(B(p,r+2d(p,q)))}{r} = \frac{\mathrm{vol}(B(p,r+2d(p,q)))}{r+2d(p,q)} \frac{r+2d(p,q)}{r}.$$

Then, by taking the lim sup on both sides, we have,

$$\limsup_{R\to\infty} \frac{\mathrm{vol}(B(q,R))}{R} < \infty.$$

Now, let's prove inequality (3.2.3) by contradiction argument: Suppose that there exists a sequence of $p_i \in M, R_i \in \mathbb{R}$ such that $p_i \to \infty$ and $R_i \to \infty$ and

$$\frac{\mathrm{vol}(B(p_i,R_i))}{R_i} \to \infty, i \to \infty.$$

By the Lemma 3.1.12, there exists at least 2 rays that are independent and orthogonal at each p_i. By the main theorem in [49] and Cheeger-Colding theory [8], there exists a subsequence $\{q_i\}$ of $\{p_i\}$ such that

$$(M^3, p_i, g_i) \to (M_\infty, p_\infty, d), \tag{3.2.4}$$

in the sense of Gromov-Hausdorff convergence with the following properties ($**$):

- M_∞ is a smooth manifold. Notice that the topological regularity is only known for $n = 3$. Here, smooth manifold means that M_∞ is a smooth differential manifold topologically, and we do not know anything about the deep metric structure of the Ricci limiting space. In fact, we mainly use the topological structure in our paper;

- For any p_∞, there exists a sequence of smooth maps

$$\varphi_i : B_d(p_\infty, i) \to M_i,$$

  such that $B_d(p_\infty, i)$ is diffeomorphic onto $\varphi_i(B_d(p_\infty, p))$ and $\varphi_i(p_\infty) = q_i$. Here $B_d(p_\infty, i)$ is the metric ball in (M_∞, d) with radius i and center p_∞;

- Under the above item, for any $R > 0$,

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d(x,y),$$

  uniformly on $B_d(p_\infty, R)$ as $i \to \infty$. Hence, the convergence is at least C^0 convergence only in the sense of metric space.

Remark 3.2.3. *For the notation used in 3.2.4: actually, $g_i = g$, we write it as g_i since we want to match g_i with the base point p_i. Moreover, we do not know if the Riemannian metric g_i will C^0-converge to a smooth Riemannian metric.*

Moreover, for the limiting space M_∞, we have the following two cases by Lemma 3.1.12: $M^1 \times \mathbb{R}^2$, $\mathbb{R}^3$.

- **Case 1:** If

$$M_\infty \simeq M^1 \times \mathbb{R}^2.$$

isometrically, we will have M^1 is one dimensional, topologically, smooth manifold. This implies that $M^1 = \mathbb{S}^1$. Hence, $M_\infty \simeq \mathbb{S}^1 \times R^2$. Here, we merely obtain that the manifold is smooth in the sense of topology. However, we did not know if the metric d_∞ on $M^1 \times \mathbb{R}^2$ is induced by a smooth Riemannian metric. Now we can overcome these difficulties as follows.

On the limiting space $\mathbb{S}^1 \times \mathbb{R}^2$, we consider the set

$$T = B_d(p_\infty, 2R) - B_d(p_\infty.R).$$

As R is a large, fixed number, i.e., $R \geqslant 100$, T is $T^2 \times [R, 2R]$ topologically and $d(\partial_+ T, \partial_- T) = R$ under the metric d. Since we know that, as $i \geqslant 10R$, $T \subset B_d(p_\infty, i)$ and $B_d(p_\infty, i)$ is diffeomorphic onto $\varphi_i(B_d(p_\infty, i))$, we reach that $\varphi_i(T)$ is a torical band in M_i with a minor damage on the band distance. Moreover, we can always perturb $\varphi_i(T)$ to K such that K becomes a smooth manifold and its topology is kept fixed and

$$d_{g_i}(\partial_+ K, \partial_- K) \geqslant \frac{1}{2} R.$$

Hence, we obtain a compact Riemannian manifold with boundary

$$(K = \mathbb{T}^2 \times I, g_K = g|_K, Sc(g_K) \geqslant 6, d(\partial_+ K, \partial_- K) \geqslant 25).$$

This contradicts with the Gromov's torical band estimate Theorem 3.1.10.

- **Case 2:** Otherwise, by Cheeger-Colding theory in [8],

$$M_\infty \simeq \mathbb{R}^3,$$

isometrically. Since the limiting space is $\mathbb{R}^3$, we can always pick a big torical band in $\mathbb{R}^n$ and then proceed the same argument in case 1 to reach a contradiction. Here, we will not repeat the argument again since it is totally the same as case 1. In fact, we may also use the argument in the proof of volume non-collapse in the Corollary 1.1.9 below.

Together with all arguments above, we proved that for any $p \in M^3$,

$$\limsup_{R\to\infty} \frac{\mathrm{vol}(B(p,R))}{R} < \infty.$$

This completes our proof. □

Remark 3.2.4. *In fact, by the proof, we see that the manifold with non-negative Ricci curvature, positive scalar curvature and volume non-collapse is asymptotic to* $\mathbb{S}^2 \times [R, \infty)$ *at infinity or splits globally. In fact, this can be also seen from the perspective of the minimal surface argument.*

By the proof of Theorem 1.1.8, we have the following weaker version of volume growth in higher dimension.

3.2.3 The Proof of Corollary 1.1.9

Proof.

- **Proof of volume non-collapsed case**

  As the proof of Theorem 1.1.8, we assume that the result does not hold. Hence, there exists a sequence $p_i \in M$ such that (M, p_i) pointedly Gromov Hausdorff converges to $\mathbb{R}^n$. However, we do not know if the convergence is C^0 convergence or not in the sense of Riemannian metric convergence. Hence, we can not directly use Gromov's upper semi-continuity of scalar curvature under Riemannian metric C^0 convergence. Instead, we make use of an estimate in [30]

  First, for all $\epsilon > 0$, there exists a Lipschitz map f from $\mathbb{R}^n$ to the standard unit sphere $\mathbb{S}^n$ with $deg(f) \geqslant 1$, $Lip(f) \leqslant \epsilon$ and f is constant at infinity. Namely, f is constant on $B^c(R) \subset \mathbb{R}^n$. Then, since we have, (M_i, r_i) converges to $\mathbb{R}^n$ with respect to the Gromov Hausdorff convergence. Hence, there exists a map $\varphi_i : (M_i, r_i) \to (\mathbb{R}^n, 0)$ with Lipschitz constant $Lip(\varphi_i) \leqslant 2$ and $B(2R) \subset Im(\varphi_i(B(r_i, 3R))$ for large i. Here $B(r_i, 2R)$ is the geodesic ball in M_i centered at r_i. Finally, we construct a map $F_i = f \circ \varphi_i : M_i \to \mathbb{S}^n$ with $Lip(F) \leqslant 2\epsilon$ with $Sc(M_i) \geqslant n(n-1)$ and $deg(f_i) \geqslant 1$. If we pick ϵ small enough, then what we obtain contradicts with the Spherical Lipschitz Bound Theorem 3.1.11 (cited from [30]). Hence,

$$\limsup_{R\to\infty} \frac{vol(B(p,R))}{R^{n-1}} < \infty.$$

- **Proof of injectivity radius non-collapsed case**

  As the proof of Theorem 1.1.8, we assume that the result does not hold. Hence, there exists a sequence $p_i \in M$ such that (M, p_i) $C^\alpha, \alpha \in (0,1)$ converges to a smooth manifold $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ since we assume that the injectivity radius has a uniformly positive lower bound [4]. Then, we take a torical band $T^{n-1} \times [R, 2R]$ in $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ for large R. Since the convergence is C^α, we have that the properties $(**)$ are automatically satisfied by Anderson's result in [4]. Hence, we will reach a contradiction, since the following steps follow the same argument in the proof of Theorem 1.1.8, we have

$$\limsup_{R\to\infty} \frac{vol(B(p,R))}{R^{n-2}} < \infty.$$

□

3.2.4 Proof of Theorem 1.1.12

Proof. Since we assume that (M^3, g) has nonnegative Ricci curvature, we have (M, g) has at most 2 ends.

- If (M^3, g) has 2 ends, we have (M, g) is split. i.e.

$$M^3 = \mathbb{S}^2 \times \mathbb{R}.$$

  In this case, we take the function f as the projection $\mathbb{S}^2 \times \mathbb{R}$ to $\mathbb{R}$. Since $Sc(g) \geqslant 2$, it is trivial that for any $r \in \mathbb{R}$. $diam(f^{-1}(r)) \leqslant 4\pi$ and $f^{-1}(r)$ is a 2 sphere and hence connected;

- If (M^3, g) has 1 end, we take any ray $\gamma(t) \in M$ and obtain the associated Busemann function

$$B_\gamma(x) : M \to \mathbb{R}.$$

  Claim: $f(x) = B_\gamma(x)$ is a continuous function as required.

  Assume that there exists a sequence $r_i \to \infty$ such that, $diam(f^{-1}(r_i)) \to \infty$. By the proof of Theorem 1.1.8, there exists a subsequence $p_i \in f^{-1}(r_i)$ such that (M, p_i) pointedly Gromov Hausdorff converges to a length space(smooth manifold)

$(M_\infty = X^2_\infty \times \mathbb{R}, p_\infty, d_\infty)$ and X_∞ is a compact manifold. Then, we take a large metric ball $B(p_\infty, 10R) \subset M_\infty$ such that for large i,

$$5R \geqslant diam(f^{-1}(r_i)) \geqslant R$$

but the level set $f^{-1}(r_i)$ is contained into some neighborhood of γ:

$$N_s(\gamma) = \{x \in M, d_g(x, \gamma) \leqslant s\}.$$

Here, s only depends on $diam(X^2_\infty)$.

Since we assume that $\mathrm{diam} f^{-1}(r_i)$ diverges to ∞ and we know

$$\gamma(r_i) \in f^{-1}(r_i),$$

then, for the large i picked above, we take a point $y_i \in f^{-1}(r_i)$ such that

$$d_g(y_i, \gamma(r_i))$$

is large. Moreover, we pick $q_i \in \gamma$ such that $d(y_i, \gamma) = d(q_i, y_i)$ that is small relative to the r_i. Hence, for any large $t \geqslant r_i$, by the definition of Busemann function, we have

$$r_i = f(y_i) = B_\gamma(y_i) > t - d(y_i, \gamma(t)).$$

If we initially pick i large enough such that $d_{GH}(B(p_i, 10R), B(p_\infty, 10R)$ is small, we obtain that for large t

$$r_i \geqslant t - d(y_i, \gamma(t)) > r_i + 1.$$

Hence, we reach a contradiction. We conclude that there exists a constant c such that $diam(f^{-1}(r)) \leqslant c$.

□

Remark 3.2.5. *Geometrically, the proof is very clear. If we keep $X_\infty \times \mathbb{R}$ in mind, it would be natural to argue the level set is uniformly bounded even the proof seems indirect.*

Chapter 4

Uryson width of three-dimensional mean convex domain

4.1 Preliminaries

4.1.1 Stable free boundary minimal surfaces

In this subsection, we will prove a diameter upper bound for stable minimal surfaces in a complete, three-dimensional manifold $(M, \partial M, g)$ with non-negative Ricci curvature and strictly mean convex boundary. Note that by Proposition 1.8 in [7] and Lemma 48 in [2], the length of boundary of a two-sided stable free boundary minimal surface is bounded even if M is non-compact. Therefore, the diameter upper bound of these surfaces would follow from an upper bound on the distance to their boundaries for all interiors.

Recall that Schoen-Yau [65] proved a diameter upper bound for stable minimal surfaces in three-dimensional manifolds with strictly positive scalar curvature. By adapting their arguments, one can obtain a diameter upper bound for stable constant mean curvature (cmc) surfaces in three-dimensional manifolds with non-negative scalar curvature. We state the result here and refer to Proposition 2.2 in [45] for more details.

Proposition 4.1.1. *Let $(M^3, \partial M, g)$ be a three-dimensional Riemannian manifold with* *nonempty boundary and scalar curvature $R_M \geqslant 0$ and Σ be a connected, embedded, compact stable cmc surface with mean curvature $H > 0$. Then for any $x \in \Sigma$,*

$$\mathrm{dist}_\Sigma(x, \partial\Sigma) \leqslant \frac{4\pi}{3H}. \tag{4.1.1}$$

Observe that each surface with mean curvature bounded by 1 can be a one-sided barrier for constant mean curvature surface with $H = 1$. Thus, we can construct a stable cmc surface by a minimizing process if there is a "large" surface-with-boundary having bounded mean curvature. Then Proposition 4.1.1 implies the closeness of such a minimizer, which contradicts the non-negative Ricci curvature.

Theorem 4.1.2. *Let $(M^3, \partial M, g)$ be a compact Riemannian manifold with nonempty boundary. Suppose that* Ric $\geqslant 0$ *and $H_{\partial M} \geqslant 1$. Let Σ be an embedded surface with $|H_\Sigma| < 1$. Then,*

$$\sup_{x\in\Sigma} \mathrm{dist}_M(x, \partial\Sigma) \leqslant \frac{4\pi}{3} + 2.$$

In particular, Σ is compact if and only if its boundary is compact.

Proof. Suppose not, then there exist $p \in \Sigma$ and $\epsilon > 0$ such that

$$\mathrm{dist}_M(p, \partial\Sigma) > \frac{4\pi}{3} + 2 + 2\epsilon.$$

Let $M_1 = M \cap B(p, \frac{4\pi}{3} + 2 + 2\epsilon)$ and T denotes the closure of $\partial M_1 \cap \mathrm{Int}M$. Here we assume that T is transverse to ∂M.

Now we cut M_1 along Σ and denote by M_2 the metric completion of $M_1 \backslash \Sigma$. So long as Σ separates M_1, we choose one of the connected components of $M_1 \backslash \Sigma$, still denoted by M_2. Then we set $\Sigma' = \partial M_2 \backslash \partial M_1$, which belongs to one of the following:

- Σ' is a double cover of Σ;
- Σ' is diffeomorphic to Σ;
- Σ' has two connected components and each component is diffeomorphic to Σ.

In each case, there exists a point (still denoted by p) so that

$$\mathrm{dist}_{M_2}(p, T \cap M_2) > \frac{4\pi}{3} + 2 + 2\epsilon. \tag{4.1.2}$$

Note that by Ricci comparison theorem [60, Chapter 9, Proposition 39], $\mathrm{dist}_M(p, \partial M) \leqslant 2$ since M has non-negative Ricci curvature. Then there exists a smooth curve $\gamma : [0,1] \to M_2$ with

$$\gamma(0) = p, \quad \gamma(1) \in M_2 \cap \partial M \quad \text{and} \quad \mathrm{Length}(\gamma) \leqslant 2 + \epsilon. \tag{4.1.3}$$

Let $t_0 \in [0,1]$ so that $\gamma(t_0) \in \Sigma'$ and $\gamma \notin \Sigma'$ for all $t \in (t_0, 1)$. It follows that $\gamma' = \gamma|_{[t_0,1]}$ intersects Σ' with algebraic intersection number 1. Now we consider the minimizing problem of the following functional

$$\mathcal{A}^1(\Omega') := \mathcal{H}^2(\partial\Omega' \backslash (\Sigma' \cup T)) - \mathcal{H}^3(\Omega')$$

among all domains $\Omega' \subset M_2$ that contain Σ'. Let Ω be a minimizer of $\mathcal{A}^1$. Then by Corollary 3.8 in [51] , $\partial\Omega\backslash T$ is a smooth, embedded, stable cmc surface because $H_{\partial M} \geqslant 1$ and $|H_\Sigma| < 1$. Note that $\partial\Omega$ intersects γ' with algebraic intersection number 1. Let Γ be a connected component of $\partial\Omega\backslash T$ that intersects γ'. It follows that $\partial\Gamma \subset T$.

Now we take $q \in \gamma' \cap \Gamma$. By (4.1.2) and (4.1.3), together with triangle inequalities,

$$\mathrm{dist}_{M_2}(q, T) \geqslant \mathrm{dist}_{M_2}(p, T \cap M_2) - \mathrm{dist}_M(p, q) > \frac{4\pi}{3} + \epsilon.$$

Then applying Proposition 4.1.1, $\partial\Gamma = \varnothing$ since Γ is stable. Thus, we conclude that Γ is a closed embedded stable cmc surface, and then it contradicts $\mathrm{Ric} \geqslant 0$. Hence, this completes the proof of Theorem 4.1.2.

□

Observe that Lemma 48 in [2] gives an upper bound of length of boundary for two-sided, free boundary minimal surfaces with index 1. Moreover, the number of connected components is also uniformly bounded. Together with Theorem 4.1.2, we obtain a diameter upper bound for free boundary minimal surfaces with index less than or equal to 1.

Lemma 4.1.3 ([2, 7])**.** *Let $(M^3, \partial M, g)$ be a complete Riemannian manifold with non-empty smooth boundary and* $\mathrm{Ric}(g) \geqslant 0$*, $H_{\partial M} \geqslant 1$. Let Σ be a two-sided, embedded free boundary minimal surface in M.*

1. *If Σ is stable, then Σ is a disk and*

$$\sup_{x,y\in\Sigma} \mathrm{dist}_\Sigma(x, y) \leqslant \pi + \frac{8}{3}.$$

2. If Σ has index one, then

$$\sup_{x,y\in\Sigma} \operatorname{dist}_M(x,y) \leqslant \frac{59\pi}{3} + 28.$$

Proof. The statement (1) is given by Carlotto-Franz Proposition 1.8 in [7]. Therefore, it suffices to prove the statement (2) as follows.

Since Σ has index one, then by [2, Lemma 48],

$$|\partial\Sigma| \leqslant 2(8-r)\pi,$$

where $r \geqslant 1$ is the number of the connected components of $\partial\Sigma$. Denote by $C_1, \cdots, C_r$ the connected components of $\partial\Sigma$. Then by Theorem 4.1.2, for each C_i, there exists $C_j \neq C_i$, so that

$$\operatorname{dist}_M(C_i, C_j) \leqslant \frac{8\pi}{3} + 4.$$

Note that Theorem 4.1.2 gives that for any $x \in \Sigma$,

$$\operatorname{dist}_M(x, \partial\Sigma) \leqslant \frac{4\pi}{3} + 2.$$

Thus for any $x, y \in \Sigma$,

$$\operatorname{dist}_M(x,y) \leqslant r(\frac{8\pi}{3} + 4) + \frac{1}{2}|\partial\Sigma| \leqslant 2r(\frac{4\pi}{3} + 2) + (8-r)\pi.$$

Since $\partial\Sigma$ is non-empty, we obtain that $r \leqslant 7$. It follows that

$$\operatorname{dist}_M(x,y) \leqslant \frac{59\pi}{3} + 28.$$

The statement (2) is proved. □

4.1.2 Geometrically prime regions

In this part, we will introduce a class of manifolds obtained from manifolds by cutting along properly embedded free boundary minimal surfaces, which will be used in the next sections. The new boundary components generated from the cutting process are called *portions*. Precisely, we introduce the following definition.

Definition 4.1.4. *$(N, \partial_r N, T, g)$ is said to be a Riemannian manifold with* relative boundary $\partial_r N$ and portion T *if*

(1) $\partial_r N \cup T$ *is exactly the topological boundary of* N *and* $(N, \partial_r N \cup T, g)$ *is a Riemannian manifold with piecewise smooth boundary;*

(2) $\partial_r N$ *and* T *are smooth hypersurfaces ;*

(3) $\partial_r N \cap \mathrm{Int}(T) = \varnothing$ *and* $\partial_r N$ *is transverse to* T*.*

Recall that $(\Sigma, \partial\Sigma) \subset (N, \partial_r N)$ always denotes a surface in N with boundary $\partial\Sigma \subset \partial_r N$. Let $(\Sigma, \partial\Sigma) \subset (N, \partial_r N, T, g)$ be an embedded free boundary minimal surface and N' the metric completion of $N \backslash \Sigma$. Conventionally, we always let

$$\partial_r N' = \partial N \cap N' \quad \text{and} \quad T' = \partial N' \backslash \partial_r N.$$

Clearly, $(N', \partial_r N', T', g)$ is a Riemannian manifold with relative boundary and portion.

For Riemannian manifolds with relative boundary and portion, we generalize the concept of geometrically prime manifolds given by Liokumovich-Maximo [44].

Definition 4.1.5. *Let* $(N^3, \partial_r N, T, g)$ *be a Riemannian manifold with relative boundary* $\partial_r N$ *and portion* T*. Denote by* T_0 *the union of connected components of* T *that are unstable free boundary minimal surfaces. Then* N *is said to be* geometrically prime *if*

1. T_0 *is a connected free boundary minimal surface of Morse index 1 if* $T_0 \neq \varnothing$*;*
2. *every closed curve* γ *bounds* a surface Γ in N relative to T*, i.e.* $\partial\Gamma \backslash T = \gamma$*.*

For geometrically prime Riemannian manifolds with non-empty boundary and portion, we adapt Gromov-Lawson's trick [28] to bound the diameter of level sets of the distance functions. We state the results here and will defer the details in Section 4.4 for the sake of completeness.

Proposition 4.1.6. *Let* $(N^3, \partial_r N, T, g)$ *be a three-dimensional geometrically prime Riemannian manifold with non-empty relative boundary* $\partial_r N$*. Suppose*

- $Ric(g) \geqslant 0$*,* $H_{\partial_r N} \geqslant 1$*;*
- $T_0 \neq \varnothing$ *and* o *is the distance function to* T_0*.*

Then for any continuous curve $\gamma : [0,1] \to N$ *with*

$$o(\gamma(0)) = o(\gamma(1)) \leqslant o(\gamma(t)) \quad \textit{for all} \;\; t \in [0,1],$$

we have

$$\mathrm{dist}_N(\gamma(0), \gamma(1)) \leqslant 25\pi + 36.$$

Moreover, if T is stable, the upper bound can be improved by the following proposition. The proof is parallel to Proposition 4.1.6.

Proposition 4.1.7. *Let* $(N^3, \partial_r N, T, g)$ *be a three-dimensional geometrically prime Riemannian manifold with non-empty relative boundary* $\partial_r N$*. Suppose that*

- $\mathrm{Ric}(g) \geqslant 0$*,* $H_{\partial_r N} \geqslant 1$*;*
- T *is stable;*
- $p \in N$ *is a fixed point and* o *is the distance to* p*.*

Then for any continuous curve $\gamma : [0,1] \to N$ *with*

$$o(\gamma(0)) = o(\gamma(1)) \leqslant o(\gamma(t)) \quad \textit{for all} \;\; t \in [0,1],$$

we have

$$\mathrm{dist}_N(\gamma(0), \gamma(1)) \leqslant 8\pi + 12.$$

4.2 Decomposing manifolds into geometrically prime regions

4.2.1 Free boundary minimal surfaces with index one

In this subsection, we will consider $(N, \partial_r N, T, g)$ a Riemannian manifold with relative boundary and portion that satisfies the following assumptions:

(A) Each connected component of T is a stable free boundary minimal surface;

(B) Each compact, two-sided, properly embedded surface in $(N \backslash T, \partial_r N, g)$ separates N;

(C) For each two-sided, embedded, stable free boundary minimal surface Γ, $N\backslash\Gamma$ has a connected component whose metric completion is diffeomorphic to $\Gamma \times [0,1]$;

(D) Each one-sided, embedded free boundary minimal surface has an unstable double cover;

(E) Any two one-sided, embedded free boundary minimal surfaces intersect each other.

The goal in this section is to construct a two-sided, index one, free boundary minimal surface that separates the manifolds into two geometrically prime regions.

Let $(N, \partial_r N, T, g)$ be a Riemannian manifold with relative boundary and portion. Denote by $\mathcal{U}_\Lambda(N)$ the collection of one-sided, stable free boundary minimal surfaces whose double covers are stable and have area less than or equal to Λ.

Now we introduce a $(\mathcal{U},\Lambda)$-*process* to remove the elements in $\mathcal{U}_\Lambda(N)$: Let $p \in N$ be a fixed point. Then we take a sequence of disjoint surfaces $\{\Sigma_j\} \subset \mathcal{U}_\Lambda(N)$ satisfying

$$\mathrm{dist}_N(\Sigma_j, p) = \inf\{\mathrm{dist}_N(\Sigma', p); \Sigma' \in \mathcal{U}_\Lambda(N) \text{ does not intersect } \Sigma_i \text{ for all } i \leqslant j-1\}.$$

The existence of Σ_j is guaranteed by the compactness of $\mathcal{U}_\Lambda(N)$. It suffices to prove that $\mathrm{dist}_N(p, \Sigma_j) \to \infty$ provided that $\{\Sigma_j\}$ has infinitely many elements. Suppose not, Σ_j smoothly converges to some $S \in \mathcal{U}_\Lambda(N)$ by the compactness of stable free boundary minimal surfaces with bounded area; c.f. [34]. Since Σ_j does not intersect Σ_i for $i \neq j$, then Σ_j does not intersect S. Then in the metric completion of $N\backslash S$, Σ_j smoothly converges to the double cover of S, which contradicts that S_j are one-sided.

Denote by N' be the metric completion of $N\backslash \cup_j \Sigma_j$. By convention, $\partial_r N' = \partial_r N \cap N'$ and $T' = \partial N' \backslash \partial_r N'$. Note that each connected component of $T'\backslash T$ is a double cover of some Σ_j. Furthermore, $(N', \partial_r N', T', g)$ does not contain any elements in $\mathcal{U}_\Lambda(N)$. From the construction, it is also clear that $\{\Sigma_j\}$ contains only finitely many elements provided that N is compact.

Proposition 4.2.1. *Let $(N, \partial_r N, T, g)$ be a Riemannian manifold with relative boundary and portion satisfying Assumptions (A)–(E) and* $\mathrm{Ric}(N) \geqslant 0$*, $H_{\partial_r N} \geqslant 1$. If N is not geometrically prime, then there exists a compact, two-sided, embedded free boundary minimal surface S such that*

1. *S has index 1;*

2. *each connected component of the metric completion of $N\backslash S$ is geometrically prime.*

Proof. The proof is divided into five steps.

Step I: *Construct $(\hat{N}, \partial_r\hat{N}, \hat{T}, g)$ which does not contain any two-sided, stable free boundary minimal surfaces and satisfies Assumptions (A)–(E).*

Denote by $\{\Gamma_j\}$ the union of the connected components of T. For Γ_1, we define B_1 as the collection of two-sided, stable free boundary minimal surfaces that are homologous to Γ_1. We suppose that B_1 is non-empty; otherwise, we skip this step for Γ_1 and then consider Γ_2. By Assumption (C), for each $\Gamma' \in B_1$, the connected component of $N\backslash\Gamma'$ containing Γ_1 is diffeomorphic to $\Gamma' \times [0, 1)$.

Claim 4.2.2. *There exists $\hat{\Gamma}_1 \in B_1$ so that*

$$\mathrm{dist}_N(\Gamma_1, \hat{\Gamma}_1) = \sup_{\Gamma' \in B_1} \mathrm{dist}_N(\Gamma_1, \Gamma').$$

Proof of Claim 4.2.2. Since N is not geometrically prime, then N contains a closed curve that does not bound a surface relative to T, which implies

$$\sup_{\Gamma' \in B_1} \mathrm{dist}_N(\Gamma_1, \Gamma') < \infty.$$

Let $\{\Gamma'_j\} \subset B_1$ be a subsequence such that $\mathrm{dist}_N(\Gamma_1, \Gamma'_j)$ converges to

$$\sup_{\Gamma' \in B_1} \mathrm{dist}_N(\Gamma_1, \Gamma').$$

Together with Lemma 4.1.3, all of $\{\Gamma'_j\}$ are contained in a compact domain Ω. Recall that by [7], the length of $\partial\Gamma'_j$ are uniformly bounded. Then by [20], the area of Γ'_j is bounded by a constant depending only on Ω. Thus by the compactness theorem [34], a subsequence of $\{\Gamma'_j\}$ smoothly converges to a stable free boundary minimal surface $\hat{\Gamma}_1$ which is either two-sided, or one-sided with a stable double cover. By Assumption (D), $\hat{\Gamma}_1$ is two-sided. This completes the proof of Claim 4.2.2. □

Denote by N_1 the connected component of the metric completion of $N\backslash\hat{\Gamma}_1$ that does not contain Γ_1. Clearly, N_1 is diffeomorphic to N and there is no two-sided, stable free boundary minimal surface in $(N_1, \partial_r N \cap N_1, \hat{\Gamma}_1 \cup T\backslash\Gamma_1, g)$ that is isotopic to $\hat{\Gamma}_1$. By the same argument for each Γ_j, we obtain a region $\hat{N} \subset N$ such that $(\hat{N}, \partial_r \hat{N}, \hat{T})$ satisfies Assumptions (A)–(E) and $\hat{N} \backslash \hat{T}$ does not contain any two-sided, stable free boundary minimal surfaces.

Step II: *We approximate $\hat{N}$ by compact domains that have no one-sided, stable free boundary minimal surfaces with small area.*

Let $\gamma \subset N$ be a closed curve that does not bound a surface relative to T. By the construction of $\hat{N}$, there exists $\hat{\gamma} \subset \hat{N}$ which is isotopic to γ in N. It follows that $\hat{\gamma}$ does not bound a surface in $\hat{N}$ relative to $\hat{T}$.

Then we assume that $\{B_k\}$ is an exhausting sequence of compact domains such that $\partial B_k \backslash \partial \hat{N}$ is smooth and transverse to $\partial \hat{N}$. Without loss of generality, we assume that $\partial B_k \backslash \partial \hat{N}$ does not intersect $\hat{T}$. We choose a metric g_k on B_k such that

- $\partial B_k \backslash \partial \hat{N}$ is a stable free boundary minimal surface with respect to g_k;
- $g_k = g$ except in a $1/k$ neighborhood of $\partial B_k \backslash \partial \hat{N}$ that does not intersect $\hat{T}$.

By our convention, let $\partial_r B_k = \partial_r \hat{N} \cap B_k$ and $T_{B_k} = \partial B_k \backslash \partial_r B_k$. Hence,

$$(B_k, \partial_r B_k, T_{B_k}, g_k)$$

is a compact Riemannian manifold with relative boundary and portion.

Claim 4.2.3. *$\hat{\gamma}$ does not bound a surface in B_k relative to T_{B_k}.*

Proof of Claim 4.2.3. Suppose not, then there exists a surface Γ' with $\partial \Gamma' \backslash T_{B_k} = \hat{\gamma}$. Recall that each connected component of $\hat{T}$ is a disk. Thus, we can take Γ' satisfying $\partial \Gamma' \backslash (T_{B_k} \backslash \hat{T}) = \gamma$. Then there exists an area minimizing surface $F \subset B_k$ among all of these Γ' described as above. Recall that $\operatorname{dist}_N(\hat{\gamma}, T_{B_k} \backslash \hat{T})$ is sufficiently large. By Theorem 4.1.2, F is a compact minimal surface and does not intersect $\{g \neq g_k\}$. This contradicts that $\hat{\gamma}$ does not bound a surface in $\hat{N}$ relative to $\hat{T}$. □

By Claim 4.2.3, for some fixed large k_0, there is an area minimizing surface Σ in $(B_{k_0}, \partial_r B_{k_0}, g_{k_0})$ intersecting $\hat{\gamma}$ with algebraic intersection number 1. Note that Σ is also an embedded surface in $(\hat{N}, \partial_r N, T, g)$.

We always take $k \geqslant k_0$. Applying $(\mathcal{U}, 2\operatorname{Area}(\Sigma, g))$-process to $(B_k, \partial_r B_k, T_{B_k}, g_k)$, we obtain finitely many disjoint surfaces $\{G_k^j\}_j$ such that $(\hat{B}_k, \partial_r \hat{B}_k, T_{\hat{B}_k}, g_k)$ does not contain any one-sided stable free boundary minimal surfaces whose double covers are stable and have area less than or equal to $2\text{Area}(\Sigma, g)$. Here $\hat{B}_k$ is the metric completion of $B_k \backslash \cup_j G_k^j$ and $\partial_r \hat{B}_k = \hat{B}_k \cap \partial_r B_k$, $T_{\hat{B}_k} = \partial \hat{B}_k \backslash \partial_r \hat{B}_k$. By next claim, we conclude that $\hat{B}_k$ converges to $\hat{N}$ in the Gromov-Hausdorff topology.

Claim 4.2.4. $\inf_j \text{dist}_{g_k}(G_k^j, \hat{\gamma}) \to \infty$ *as* $k \to \infty$.

Proof of Claim 4.2.4. Assume that G_k^1 achieves $\inf_j \text{dist}_{g_k}(G_k^j, \hat{\gamma})$. Suppose on the contrary that $\text{dist}_{g_k}(G_k^1, \hat{\gamma})$ remains uniformly bounded. Then, by the compactness for stable free boundary minimal surfaces [34], G_k^1 smoothly converges to a one-sided stable free boundary minimal surface $G \subset \hat{N}$ whose double cover is stable. This contradicts the construction of $\hat{N}$. Therefore, we conclude that $\text{dist}_{g_k}(G_k^1, \hat{\gamma}) \to \infty$ as $k \to \infty$. □

Step III: *Cut along two-sided stable free boundary minimal surfaces with small area in compact domains with perturbed metrics.*

Let F_k be the collection of two-sided, stable free boundary minimal surfaces

$$(\Gamma', \partial\Gamma') \subset (\hat{B}_k, \partial_r \hat{B}_k, g_k)$$

with $\Gamma' \subset \hat{B}_k \backslash T_{\hat{B}_k}$ and $\text{Area}(\Gamma', g_k) \leqslant 2\text{Area}(\Sigma, g)$. Now we assume that F_k is nonempty; otherwise, we skip this step for $\hat{B}_k$.

Claim 4.2.5. *There exists* $\Sigma_k^1 \in F_k$ *so that*

$$\text{dist}_N(\Sigma_k^1, \hat{\gamma}) = \inf_{\Sigma' \in F_k} \text{dist}_N(\Sigma', \hat{\gamma}).$$

Proof of Claim 4.2.5. Suppose that $\{\Sigma_j'\} \subset F_k$ and $\text{dist}(\Sigma_j', \hat{\gamma})$ converges to

$$\inf_{\Sigma' \in F_k} \text{dist}_N(\Sigma', \hat{\gamma}).$$

Then, by the compactness theorem [34], Σ_j' smoothly converges to a stable free boundary minimal surface Σ_k^1 in $(\hat{B}_k, \partial_r \hat{B}_k)$. Moreover, Σ_k^1 either is two-sided or have a stable double cover. Note that $(\hat{B}_k, \partial_r \hat{B}_k)$ does not contain a stable free boundary minimal surface whose double cover is stable and has area less than or equal to $2\text{Area}(\Sigma, g)$. Thus, we conclude that Σ_k^1 is two-sided. This finishes the proof of Claim 4.2.5. □

By a similar argument as in Claim 4.2.4, we also have $\mathrm{dist}_N(\Sigma_k^1, \hat{\gamma}) \to \infty$ as $k \to \infty$.

Suppose not, then by the compactness for stable free boundary minimal surfaces [34], Σ_k^1 smoothly converges to a stable free boundary minimal surface $\widetilde{\Sigma}$ in $(N, \partial N, T, g)$, which is either two-sided or one-sided but having a stable double cover. This contradicts the construction of $\hat{N}$. Therefore, we conclude that $\mathrm{dist}_N(\Sigma_k^1, \hat{\gamma}) \to \infty$ as $k \to \infty$.

Then for any large fixed k, let $\hat{N}_k^1$ denote the connected component of the metric completion of $\hat{B}_k \backslash \Sigma_k^1$ that contains $\hat{\gamma}$. If $\hat{N}_k^1 \backslash \Sigma_k^1$ contains elements in F_k, then we take $\Sigma_k^2 \in F_k$ that is contained in $\hat{N}_k^2 \backslash \Sigma_k^1$ so that

$$\mathrm{dist}_N(\Sigma_k^2, \hat{\gamma}) = \inf \big\{ \mathrm{dist}_N(\Sigma', \hat{\gamma}); \Sigma' \in F_k \text{ is contained in } \hat{N}_k^2 \backslash \Sigma_k^1 \big\}.$$

By continuing this argument, we obtain two sequences $\{\Sigma_k^j\}_j$ and $\{\hat{N}_k^j\}_j$.

Now we are going to prove that this sequence $\{\Sigma_k^j\}_j$ consists of finitely many surfaces. Suppose not, By the compactness theorem [34], we have Σ_k^j smoothly converges to a stable free boundary minimal surface C_k, which either is two-sided or has a stable double cover. Recall that $\hat{B}_k \backslash T_{\hat{B}_k}$ does not contain an embedded one-sided free boundary minimal surface with stable double cover. Hence, C_k is two-sided. Then the stability of C_k gives that Σ_k^j lies in one side of C_k for all large j. From the construction of $\hat{N}_k^j$, $\hat{\gamma}$ and C_k lie in the same side of Σ_k^j. It follows that

$$\mathrm{dist}_N(\Sigma_k^j, \hat{\gamma}) < \mathrm{dist}_N(C_k, \hat{\gamma})$$

for all sufficiently large k. This contradicts the choice of Σ_k^j. Thus $\{\Sigma_k^j\}_j$ is finite.

For simplicity, let $\Sigma_k := \cup_j \Sigma_k^j$ and $\widetilde{N}_k$ be the connected component of the metric completion of $\hat{B}_k \backslash \Sigma_k$ that contains $\hat{\gamma}$. Note that

$$\partial_r \widetilde{N}_k = \partial_r \hat{N} \cap \widetilde{N}_k \quad \text{and} \quad \widetilde{T}_k = \partial \widetilde{N}_k \backslash \partial_r \widetilde{N}_k.$$

By Claim 4.2.4 and the fact that $\mathrm{dist}_N(\Sigma_k, \hat{\gamma}) \to \infty$, we have that $(\widetilde{N}_k, g_k)$ converges to $(\hat{N}, g)$ in the Gromov-Hausdorff topology. Hence, $\widetilde{N}_k$ contains Σ for all large k. Moreover, by the construction of Σ_k, $(\widetilde{N}_k \backslash \widetilde{T}_k, \partial_r \widetilde{N}_k, g_k)$ does not contain any stable free boundary minimal surfaces that are

- two-sided with area less than or equal to $2\mathrm{Area}(\Sigma, g)$;
- one-sided and has a stable double cover with area less than or equal to $2\mathrm{Area}(\Sigma, g)$.

Furthermore, $\widetilde{N}_k$ satisfies a $2\text{Area}(\Sigma, g)$-*Frankel property*: any two free boundary minimal surfaces intersect if they are two-sided (or one-sided) and have area less than or equal to $2\text{Area}(\Sigma, g)$ (resp. $\text{Area}(\Sigma, g)$).

Claim 4.2.6. *Every connected component of $\widetilde{T}_k$ has a* contracting neighborhood*, i.e., there exists a neighborhood of Γ' (one of the connected components of $\widetilde{T}_k$) in $\hat{N}$ that is foliated by free boundary minimal surfaces with mean curvature vector pointing towards Γ'.*

Proof. Since $\hat{\gamma}$ does not bound a surface in $\hat{N}$. There exists an area minimizing surface $(D, \partial D) \subset (\hat{N}, \partial_r \hat{N})$ intersects $\hat{\gamma}$ with algebraic intersection number 1. By Assumptions (C) and (D), D is one-sided and has unstable double cover. Let N' be the metric completion of $\widetilde{N} \backslash D$. By convention, $\partial_r N' = \partial_r \widetilde{N}$ and $T' = \partial N' \backslash \partial_r \widetilde{N}$. Then T' contains two disjoint unstable components. By a minimizing process, we obtain a two-sided, stable free boundary minimal surface separating $\widetilde{N}$, which leads to a contradiction. □

Step IV: *The first width of $(\widetilde{N}_k, g_k)$ is bounded by $2\text{Area}(\Sigma, g)$.*

Recall that $\widetilde{N}_k$ contains Σ which intersects $\hat{\gamma}$ with algebraic intersection number 1. By a minimizing process, there exists a free boundary minimal surface $E_k \subset (\widetilde{N}_k \backslash \widetilde{T}_k, \partial_r \widetilde{N}_k, g_k)$ with

$$\text{Area}(E_k, g_k) \leqslant \text{Area}(\Sigma, g_k) = \text{Area}(\Sigma, g).$$

By Steps II and III, E_k is one-sided and has unstable double cover. Then by a similar argument as in [77, Lemma 2.5], there exists a neighborhood $\mathcal{N}_k \subset \widetilde{N}_k$ of E_k that is foliated by free boundary surfaces with mean curvature vector pointing away from E_k. Let $(\Omega_k^t)_t$ denote the free boundary level set flow starting from $\widetilde{N}_k \backslash \mathcal{N}_k$. Then by [17, Theorem 1.5], each connected boundary of $\partial \Omega_k^\infty$ is

- either a two-sided stable free boundary minimal surface with area less than or equal to $2\text{Area}(\Sigma, g)$;
- or a one-sided stable free boundary minimal surface with area less than or equal to $\text{Area}(\Sigma, g)$ and its double cover is stable.

By the construction of $\widetilde{N}_k$, there are no such surfaces in $\widetilde{N}_k \backslash \widetilde{T}_k$. Thus, we conclude that $\partial \Omega_k^\infty = \widetilde{T}_k$. As a corollary, each connected component of $\widetilde{T}_k$ has *a contracting neighborhood*, i.e. for each connected component Γ' of T_k, there exists a neighborhood of Γ' in $\widetilde{N}_k$ that is foliated by free boundary surfaces with mean curvature vector pointing towards Γ'. Moreover, the above argument gives that the first width for $\widetilde{N}_k$ is bounded from above by $2\text{Area}(\Sigma, g)$.

Step V: *Apply min-max theory to N_k to find the desired surfaces.*

By min-max theory [41] (see [77, Theorem 3.10] for manifolds with relative boundaries and portions), there exist an integer m and an embedded free boundary minimal surface $(E_k, \partial E_k)$ in $(\widetilde{N}_k \backslash \widetilde{T}_k, \partial_r \widetilde{N}_k, g_k)$ such that

$$m\text{Area}(E_k; g_k) \leqslant 2\text{Area}(\Sigma; g), \quad \text{and} \quad \text{Index}(E_k) \leqslant 1.$$

Moreover, by Catenoid Estimates in [37] (see [76] for a free boundary version), E_k is two-sided. Recall that $(\hat{N}_k \backslash \hat{T}_k, \partial_r \hat{N}_k, g_k)$ does not contain any two-sided stable free boundary minimal surfaces with area less than or equal to $2\text{Area}(\Sigma)$. Thus E_k has index one. Then, by the $2\text{Area}(\Sigma, g)$-Frankel property, E_k intersects $\Sigma \cup \hat{\gamma}$ for all large k; otherwise, one can construct a one-sided free boundary minimal surface that does not intersect E_k and has an unstable double cover. Letting $k \to \infty$, one can obtain a free boundary minimal surface in $\hat{N}$ which is

- either two-sided and has index one;
- or one-sided and has a double cover with index less than or equal to one.

In both cases, the limit of E_k is compact by Lemma 4.1.3. Thus E_k is a free boundary minimal surface in $(\hat{N}, \partial_r \hat{N}, \hat{T}, g)$ for all sufficiently large k.

Now we pick $S = E_k$ and then prove that S is the desired surface. Let W_1 and W_2 be two connected components of the metric completion of $N \backslash S$. Let $\hat{W}_1$ and $\hat{W}_2$ be two connected components of the metric completion of $\hat{N} \backslash S$ respectively.

Claim 4.2.7. *W_1 and W_2 are both geometrically prime.*

Proof of Claim 4.2.7. Suppose on the contrary that W_1 is not geometrically prime. Then there exists a one-sided, compact, connected stable free boundary minimal surface $V \subset W_1$. Recall that $\hat{N}$ is obtained by cutting countably many domains which are diffeomorphic to $\Gamma' \times [0,1]$ for some disk Γ'. Thus, we can take $V \subset \hat{W}_1$.

By Assumption (D), the double cover of V is unstable. Let $\widetilde{W}_1$ be the metric completion of $\hat{W}_1 \backslash V$. Then $\widetilde{W}_1$ contains two disjoint unstable free boundary minimal surfaces: one is S and the other one is the double cover of V. Thus, by a minimizing process, $\widetilde{W}_1 \backslash T$ contains a two-sided stable free boundary minimal surface S', which contradicts the construction of $\hat{N}$. This completes the proof of Claim 4.2.7. $\square$

Therefore, S is the desired free boundary minimal surface and Proposition 4.2.1 is proved. $\square$

4.2.2 Geometrically prime decomposition

In this subsection, we will decompose a (possibly non-compact) Riemannian manifold with boundary into geometrically prime regions.

Let $\mathcal{O}_S$ (resp. $\mathcal{U}_S$) be the collection of two-sided (resp. one-sided) stable free boundary minimal surfaces. Let $\mathcal{U}_S^1$ (resp. $\mathcal{U}_S^2$) be the collection of $\Sigma \in \mathcal{U}_S$ whose double cover is stable (resp. unstable).

Observe that a sequence of surfaces in $\mathcal{O}_S$ or $\mathcal{U}_S^1$ converges subsequently if they are bounded. Indeed, Lemma 4.1.3 gives an upper bound of the length of their boundaries. Together with [20, Lemma 2.2], their areas are uniformly bounded. Then the convergence of subsequences would follow from the compactness theorem in [34].

Lemma 4.2.8. *Let $(M, \partial M, g)$ be a three-dimensional Riemannian manifold with non-empty boundary. Suppose that M satisfies that $\mathrm{Ric}(g) \geqslant 0$ and $H_{\partial M} \geqslant 1$. Then there exist countably many disjoint free boundary minimal surfaces $\{P_j\}$, $\{D_j\}$ and $\{S_j\}$ such that*

1. *$\{D_j\} \subset \mathcal{O}_S$, $\{P_j\} \subset \mathcal{U}_S^1$ and $\{S_j\}$ are two-sided free boundary minimal surfaces of index 1;*

2. *Each connected component of the metric completion of $M \backslash \mathcal{D}$ is geometrically prime. Here*
$$\mathcal{D} = \bigcup_{i,j,k} (P_i \cup D_j \cup S_k).$$

Proof. Let $p \in M$ be a fixed point.

Step I: *There exists a sequence of disjoint surfaces* $\{P_j\} \subset \mathcal{U}_S^1$ *such that every* $\Gamma' \in \mathcal{U}_S^1$ *intersects* P_j *for some* $j \geqslant 1$.

We will choose these surfaces inductively. Suppose we have chosen $\{P_j\}_{j\leqslant k}$. Then we take P_{k+1} that minimizes $\mathrm{dist}_M(p, \Gamma')$ among all Γ' satisfying the following:

- $\Gamma' \in \mathcal{U}_S^1$;
- Γ' does not intersect P_j for all $1 \leqslant j \leqslant k$.

To finish Step I, it suffices to prove that $\mathrm{dist}_M(p, P_j) \to \infty$ provided that $\{P_j\}$ is an infinite set. Suppose not, then by the compactness theorem in [34], P_j smoothly converges to an element $P \in \mathcal{U}_S^1$. Since P_j does not intersect P_i for $i \neq j$, then P_j does not intersect P. It follows that P_j smoothly converges to the double cover of P. Hence, P_j is two-sided, it contradicts the choice of P_j. This finishes the proof of Step I.

Let M_1 be the metric completion of $M \setminus \cup_j P_j$. Then, by Step I, there is no surface $\Gamma' \subset M_1$ that belongs to $\mathcal{U}_S^1$.

Step II: *There exists a sequence of disjoint surfaces* $\{C_j\} \subset \mathcal{O}_S$ *in* M_1 *such that the metric completion of* $M_1 \setminus \cup_j C_j$ *satisfies Assumption (B); see the beginning of this section.*

We use the inductive method again. Suppose we have chosen $\{C_j\}_{j\leqslant k}$. Then we take C_{k+1} that minimizes $\mathrm{dist}_M(\Gamma', p)$ among all Γ' satisfying the following:

- $\Gamma' \in \mathcal{O}_S$;
- $\Gamma' \subset M_1 \setminus \cup_{j\leqslant k} C_j$;
- Γ' does not separate $M_1 \setminus \cup_{j\leqslant k} C_j$.

We now prove that $\mathrm{dist}_M(p, C_j) \to \infty$ provided that $\{C_j\}$ consists of infinitely many elements. Suppose not, then by [34], C_j smoothly converges to a stable free boundary minimal surface C that belongs to $\mathcal{O}_S$ or $\mathcal{U}_S^1$. By Step I, $(M_1, M_1 \cap \partial M)$ does not contain surfaces in $\mathcal{U}_S^1$. Thus $C \in \mathcal{O}_S$. Then by the smooth convergence, C_k is a positive graph over C for all sufficiently large k, which contradicts that C_{k+1} does not separate $M_1 \setminus \cup_{j\leqslant k} C_j$.

Denote by M_2 the metric completion of $M_1\backslash \cup_j C_j$. Then each compact, two-sided, embedded surface in M_2 with boundary on $M_2 \cap \partial M$ separates M_2. Otherwise, by a minimizing process, one can find an $S \in \mathcal{O}_S$ that does not separate M_2 and $\mathrm{dist}_M(p, S) < \infty$, which leads to a contradiction.

Step III: *There exists a sequence of disjoint surfaces $\{E_j\} \subset \mathcal{O}_S$ such that the metric completion of $M_2\backslash \cup_j E_j$ satisfies Assumption (C).*

We construct these surfaces inductively. Suppose we have chosen $\{E_j\}_{j\leqslant k}$. Then we take E_{k+1} that minimizes $\mathrm{dist}_M(\Gamma', p)$ among all Γ' satisfying the following:

- $\Gamma' \in \mathcal{O}_S$;
- $\Gamma' \subset M_2\backslash \cup_{j\leqslant k} E_j$;
- the metric completion of $M_2\backslash \cup_{j\leqslant k+1} E_j$ does not have a connected component whose closure is diffeomorphic to $\Gamma' \times [0, 1]$.

Now we are going to prove $\mathrm{dist}_M(p, E_j) \to \infty$ provided that $\{E_j\}$ consists of infinitely many elements. Suppose not, E_j smoothly converges to an embedded surface $S \in \mathcal{O}_S$ by the same argument as that in Step II. Then, by the smooth convergence, E_j lies on one side of S for all sufficiently large j. Then the metric completion of the connected component of $M_2\backslash \cup_{i\leqslant j+1} E_i$ that contains E_j and E_{j+1}, is diffeomorphic to $E_j \times [0, 1]$ that contradicts the choice of $\{E_j\}$.

Now let M_3 be the metric completion of $M_2\backslash \cup_j E_j$. Clearly, M_3 satisfies (C).

Step IV: *There exists a sequence of two-sided, index one, free boundary minimal surfaces $\{S_j\} \subset M_3$ such that each connected component of the metric completion of $M_3\backslash \cup_j S_j$ is geometrically prime.*

Let N be a connected component of M_3 and

$$\partial_r N = \partial M \cap N \quad \text{and} \quad T = \partial N \backslash \partial_r N.$$

Now we verify that $(N, \partial_r N, T)$ satisfies Assumptions (A)–(E). Note that every connected component of T is from one of the following:

- a double cover of $P_j \in \mathcal{U}_S^1$;

- one of $C_j \in \mathcal{O}_S$;
- one of $E_j \in \mathcal{O}_S$.

Thus N satisfies (A). By Step I and II, (D) and (B) are satisfied respectively, and Step III gives (C) immediately.

Finally, it remains to verify (E). Suppose not, then there exist two disjoint surfaces $S_1, S_2 \in \mathcal{U}_S^2$ that are contained in N. Let $\widetilde{N}$ be the metric completion of $N \backslash (S_1 \cup S_2)$. Note that S_1 and S_2 have unstable double covers since N satisfies (D). Let $\widetilde{S}_1$ and $\widetilde{S}_2$ be the unstable free boundary minimal surface in $\widetilde{N}$ arising from cutting along S_1 and S_2 respectively. By taking an area minimizer of the homology class in $H_2(\widetilde{N}, \partial_r \widetilde{N}; \mathbb{Z})$ represented by $\widetilde{S}_1$, we obtain a stable free boundary minimal surface $\Gamma \in \mathcal{O}_S$. Since N satisfies (B), then Γ separates N. Moreover, S_1 and S_2 lie in two different connected components of $N \backslash \Gamma$, which contradicts the choice of $\{E_j\}$ in Step III. Hence, (E) is satisfied.

Let $\{N_j\}$ be the collection of connected components of M_3. Thus each N_j satisfies Assumptions (A)–(E). By Proposition 4.2.1, there exists a two-sided free boundary minimal surface S_j with index 1 such that the metric completion of $N_j \backslash S_j$ is geometrically prime provided that N_j is not geometrically prime. This finishes Step IV.

Therefore, Lemma 4.2.8 follows by relabelling $\{C_j\} \cup \{E_j\}$ as $\{D_j\}$.

□

4.3 Upper bounds for Uryson 1-width

In this section, we are in a position to prove an upper bound of Uryson 1-width for all three-dimensional Riemannian manifolds with non-negative Ricci curvature and strictly mean convex boundary. By the definition, it suffices to construct a continuous function such that every connected component of all the level sets has a uniformly bounded diameter.

In the first part of this section, we construct a function on each geometrically prime region. By Gromov-Lawson's tricks, the distance function to the unstable component of the portion is a good choice. However, for later reason, the desired function on each connected component in the portion is required to have the same value. Hence, we modify the distance function near the portion so that the diameter bound still holds.

Recall that T_0 always denotes the unstable component if T is a free boundary minimal surface by Definition 4.1.5.

Lemma 4.3.1. *Let $(N^3, \partial_r N, T, g)$ be a three-dimensional geometrically prime Riemannian manifold with non-empty relative boundary $\partial_r N$ and portion T. Suppose that* $\mathrm{Ric}(g) \geqslant 0$, *$H_{\partial_r N} \geqslant 1$ and T is a free boundary minimal surface. Then there exists a continuous function $f : N \to [0, \infty)$ satisfying*

1. *$f(x) = 0$ for all $x \in T_0$, where T_0 is the union of unstable components of T;*
2. *$f(x) = 1$ for all $x \in T \backslash T_0$;*
3. *$\mathrm{dist}_N(x, y) < 117$ for any t and x, y in the same connected component of $f^{-1}(t)$.*

Moreover, the upper bound in the third statement can be improved to be 49 if T is stable.

Proof. Let $\{T_j\}_{j \geqslant 1}$ be the collection of the connected components of $T \backslash T_0$. Since each T_j is compact, then there exist a positive constant $\epsilon < 100^{-1}$ and $p \in N \backslash T$ so that

$$\mathrm{dist}_N(T \backslash T_0, T_0) > 5\epsilon \text{ if } T_0 \neq \varnothing \ ; \ \mathrm{dist}_N(T, p) > 5\epsilon \text{ if } T_0 = \varnothing.$$

and for each T_j, there exists $\epsilon_j \in (0, \epsilon/2^j)$ such that $\mathrm{dist}_N(T_j, T \backslash T_j) > 5\epsilon_j$. For $j \geqslant 1$, set

$$U_j := \{x \in N; \mathrm{dist}_N(x, T_j) \leqslant 2\epsilon_j\};$$

and for $j = 0$, set

$$U_0 := \{x \in N; \mathrm{dist}_N(x, T_0) \leqslant \epsilon\} \quad \text{if } \ T_0 \neq \varnothing \ ;$$

$$U_0 := \{x \in N; \mathrm{dist}_N(x, p) \leqslant \epsilon\} \quad \text{if } \ T_0 = \varnothing.$$

Let $\eta : [0, \infty) \to [0, 1]$ be a continuous cut-off function such that

$$\eta(t) = 0 \quad \text{for} \quad t \geqslant 2; \quad \eta(t) = 1 \quad \text{for} \quad t \in [0, 1].$$

Define $h : N \to [0, \infty)$ by

$$h(x) = \sum_{j \geqslant 1} \eta(\epsilon_j^{-1} \, \mathrm{dist}_N(x, T_j)).$$

It follows that $\eta(\epsilon_j^{-1} \operatorname{dist}_N(x, T_j)) = 0$ outside U_j. Hence, h is well-defined, supported in $\cup_j U_j$ and

$$0 \leqslant h \leqslant 1; \quad h(x) = 0 \ \text{ for } \ x \in N \backslash \cup_{j \geqslant 1} U_j.$$

Now we define the desired function.

- If $T_0 \neq \varnothing$, $f : N \to [0, \infty)$ is defined by

$$f(x) := h(x) + \epsilon^{-1}(1 - h(x)) \operatorname{dist}_N(x, T_0).$$

- If $T_0 = \varnothing$, then, we take $p \in N$ with $\operatorname{dist}_N(p, T) > 5\epsilon$ and define f by

$$f(x) := h(x) + \epsilon^{-1}(1 - h(x)) \operatorname{dist}_N(x, p).$$

For $x \in U_0$, it follows that $\operatorname{dist}_N(x, T_j) \geqslant 5\epsilon_j$ for $j \geqslant 1$ and then $h(x) = 0$. Hence, for $x \in U_0$,

$$f(x) = \epsilon^{-1} \operatorname{dist}_N(x, T_0) \quad \text{for} \quad T_0 \neq \varnothing; \ f(x) = \epsilon^{-1} \operatorname{dist}_N(x, p) \quad \text{for} \quad T_0 = \varnothing. \tag{4.3.1}$$

Thus the first statement is satisfied. Also, for all $x \in T \backslash T_0$, we have $h(x) = 1$, it implies that $f(x) = 1$. This gives the second statement.

Now let's verify that f satisfies the third requirement. For $t \geqslant 0$ and any two points y, z in the same connected component of $f^{-1}(t)$, there exists a continuous curve $\gamma : [0, 1] \to N$ with $f(\gamma(s)) = t$ for all $s \in [0, 1]$ and $\gamma(0) = y$, $\gamma(1) = z$.

We now consider the case that $T_0 \neq \varnothing$. If $\operatorname{dist}_N(\gamma(s'), T_0) < \epsilon$ for some $s' \in [0, 1]$, then by (4.3.1), we conclude that $y, z \in U_0$ and

$$\begin{aligned}\operatorname{dist}_N(y, z) &\leqslant \operatorname{dist}_N(y, T_0) + \operatorname{dist}_N(z, T_0) + \sup_{x_1, x_2 \in T_0} \operatorname{dist}_N(x_1, x_2)\\ &\leqslant \frac{59\pi}{3} + 28 + 2\epsilon + 2\epsilon < 117.\end{aligned}$$

If $\operatorname{dist}_N(\gamma, T_0) \geqslant \epsilon$, it follows that

$$f(\gamma(s)) \leqslant \epsilon^{-1} \operatorname{dist}_N(x, T_0). \tag{4.3.2}$$

Then we have the following two cases.

Case 1: $h(\gamma(s)) > 0$ *for all* $s \in (0,1)$.

Note that h is supported on disjoint compact sets $\cup_j U_j$. Thus, $\gamma \subset U_j$ for some $j \geqslant 1$. It follows that

$$\mathrm{dist}_N(z,y) \leqslant \sup_{x_1,x_2 \in T_j} \mathrm{dist}_N(x_1,x_2) + 2\epsilon_j + 2\epsilon_j \leqslant \pi + \frac{8}{3} + 2\epsilon,$$

which is the desired inequality.

Case 2: $h(\gamma(s_0) = 0$ *for some* $s_0 \in (0,1)$.

Let

$$s_1 := \inf\{s \in [0,s_0]; h(\gamma(s)) = 0\}; \quad s_2 := \sup\{s \in [s_0,1]; h(\gamma(s)) = 0\}.$$

Then $\gamma|_{[0,s_0]}$ (resp. $\gamma|_{[s_1,1]}$) lies in U_j for some $j \geqslant 1$. Moreover, by the same argument in Case 1,

$$\mathrm{dist}_N(y,\gamma(s_1)) \leqslant \pi + \frac{8}{3} + 4\epsilon \quad \text{and} \quad \mathrm{dist}_N(z,\gamma(s_2)) \leqslant \pi + \frac{8}{3} + 4\epsilon.$$

Next, we bound the distance from $\gamma(s_1)$ to $\gamma(s_2)$. Indeed, by (4.3.2), for any $s \in [s_1, s_2]$ and $j \in \{1,2\}$,

$$\mathrm{dist}_N(\gamma(s),T_0) \geqslant \epsilon \cdot f(\gamma(s)) = \epsilon \cdot f(\gamma(s_j)) = \mathrm{dist}_N(\gamma(s_j),T_0).$$

Then by Proposition 4.1.6,

$$\mathrm{dist}_N(\gamma(s_1),\gamma(s_2)) \leqslant 25\pi + 36.$$

Then by the triangle inequality,

$$\begin{aligned}\mathrm{dist}_N(y,z) &\leqslant \mathrm{dist}_N(y,\gamma(s_1)) + \mathrm{dist}_N(\gamma(s_1),\gamma(s_2)) + \mathrm{dist}_N(\gamma(s_2),z)\\ &\leqslant 27\pi + 26 + \frac{16}{3} + 4\epsilon < 117.\end{aligned}$$

This finishes the proof for $T_0 \neq \varnothing$.

Now it remains to improve the upper bound as $T_0 = \varnothing$. If $\mathrm{dist}_N(\gamma(s'),p) < \epsilon$ for some $s' \in [0,1]$, the inequality is trivial. If $\mathrm{dist}_N(\gamma,T_0) \geqslant \epsilon$, we only consider the *Case 2:* $h(\gamma(s_0) = 0$ *for some* $s_0 \in (0,1)$. Then the triangle inequality gives

$$\begin{aligned}\mathrm{dist}_N(y,z) &\leqslant \mathrm{dist}_N(y,\gamma(s_1)) + \mathrm{dist}_N(\gamma(s_1),\gamma(s_2)) + \mathrm{dist}_N(\gamma(s_2),z)\\ &\leqslant 10\pi + 12 + \frac{16}{3} + 4\epsilon < 49.\end{aligned}$$

This finishes the proof of Lemma 4.3.1. □

Theorem 4.3.2. *Let $(M^3, \partial M, g)$ be a three-dimensional Riemannian manifold with non-empty boundary ∂M. Suppose that $\mathrm{Ric}(g) \geqslant 0$ and $H_{\partial M} \geqslant 1$. Then there exists a continuous function $f : M \to [0, \infty)$ satisfying that*

$$\mathrm{dist}_M(x, y) \leqslant 117$$

for all $t \geqslant 0$ and x, y in the same connected component of $f^{-1}(t)$. In particular, if M is a domain in $\mathbb{R}^3$, then the upper bound can be improved to be 49.

Proof. By Lemma 4.2.8, there exists a sequence of free boundary minimal surfaces $\{D_j\}$ such that each connected component of the metric completion of $M \backslash \cup_j D_j$ is geometrically prime.

Let $\{N_j\}_{j \geqslant 1}$ denote the union of the connected components of the metric completion of $M \backslash \cup_j D_j$. Note that, $\partial_r N_j = N_j \cap \partial M$ and $T_j = \partial N_j \backslash \partial_r N_j$. Denote by $T_{j,0}$ the unstable component of T_j. Let $\{T_{j,i}\}_{i \geqslant 1}$ denote the union of the connected components of $T_j \backslash T_{j,0}$. Then by Lemma 4.3.1, for any N_j, there exists a continuous function $f_j : N_j \to [0, \infty)$ such that

- $f_j(x) = 0$ for all $x \in T_{j,0}$, where $T_{j,0}$ is the union of the unstable component of T_j;
- $f_j(x) = 1$ for all $x \in T \backslash T_{j,0}$;
- $\mathrm{dist}_N(x, y) < 117$ if x and y lie in the same connected component of $f_j^{-1}(t)$.

Since $f_j|_{T_{j,0}} = 0$ if $T_{j,0} \neq \varnothing$, and $f|_{T_j \backslash T_{j,0}} = 1$, then the gluing of all these functions induces a continuous function f on M.

It remains to prove the diameter bound. Let K be a connected component of $f^{-1}(t)$ for any fixed $t > 0$.

Case I: There is no D_j intersecting K.

Clearly, K is contained in one of N_j's. Then the desired upper bound follows immediately.

Case II: K intersects some D_j.

Then $K \subset \{x \in M; \mathrm{dist}_M(x, D_j) \leqslant 2\epsilon\}$. It follows that for any $x, y \in K$,

$$\mathrm{dist}_M(x, y) \leqslant \sup_{x_1, x_2 \in D} \mathrm{dist}_M(x_1, x_2) + 4\epsilon < \pi + \frac{8}{3} + 4\epsilon,$$

where the second inequality is from Lemma 4.1.3.

In particular, if $M \subset \mathbb{R}^3$, then $\mathcal{U}_S = \varnothing$. By Lemma 4.3.1, every connected component of level sets of f_j has diameter upper bound 49. Therefore, such a f satisfies our requirements and Theorem 4.3.2 is proved.

□

4.4 Gromov-Lawson's tricks

In this section, by slightly modification of Corollary 10.11 in [28] (c.f. [44]), we give the proof of Proposition 4.1.6 for $T_0 \neq \varnothing$. The proof for $T_0 = \varnothing$ is parallel and we omit the details.

Proof of Proposition 4.1.6. Since $(N, \partial_r N, T, g)$ is geometrically prime, then T_0 is connected. By Lemma 4.1.3, for any $x_1, x_2 \in T_0$,

$$\mathrm{dist}_N(x_1, x_2) \leqslant \frac{59\pi}{3} + 28.$$

For simplicity, let x and y denote $\gamma(0)$ and $\gamma(1)$, respectively. Then we have the following two cases. If $o(x) \leqslant \frac{8\pi}{3} + 4$,

$$\mathrm{dist}_N(x, y) \leqslant \mathrm{dist}_N(x, T_0) + \frac{59\pi}{3} + 28 + \mathrm{dist}_N(y, T_0) \leqslant 25\pi + 36, \tag{4.4.1}$$

Now we assume that $o(x) > \frac{8\pi}{3} + 4 + 2\epsilon$ for some $\epsilon > 0$. Let p_x and p_y be the closest points in T_0 to x and y, respectively. Let γ_x (resp. γ_y) be the minimizing curve in N from p_x to x (resp. y), i.e.

$$\mathrm{Length}(\gamma_x) = o(x), \quad \text{and} \quad \mathrm{Length}(\gamma_y) = o(y).$$

Let σ denote a curve in T_0 connecting p_x and p_y. Note that $\overline{\gamma_x \gamma {\gamma_y}^{-1} \sigma^{-1}}$ is a closed curve (denoted by $\widetilde{\gamma}$) on N. Since N is geometrically prime, then $\widetilde{\gamma}$ bounds a surface $\Gamma' \subset N$ relative to T_0, i.e. $\partial \Gamma' \backslash T_0 = \widetilde{\gamma}$. Let Γ be an area minimizing surface among all those surfaces bounded by $\widetilde{\gamma}$ relative to T_0. Then $\mathrm{Int}(\Gamma)$ is a smoothly embedded stable minimal surface. By Theorem 4.1.2, for any $x' \in \Gamma$,

$$\mathrm{dist}_N(x', \partial\Gamma) \leqslant \frac{4\pi}{3} + 2. \tag{4.4.2}$$

Next, we take t' such that

1. $o(x) - \frac{4\pi}{3} - 2 - 2\epsilon \leqslant t' \leqslant o(x) - \frac{4\pi}{3} - 2 - \epsilon$;

2. $o^{-1}(t')$ is transverse to Γ, γ_x and γ_y.

Then there exists a curve $\alpha \subset o^{-1}(t') \cap \Gamma$ joining γ_x and γ_y. Recall that $o(x) = o(y) \leqslant o(\gamma(s))$. This implies that

$$\mathrm{dist}_N(\gamma, \alpha) \geqslant \inf_{x' \in \gamma, y' \in \alpha} \{o(x') - o(y')\} \geqslant \frac{4\pi}{3} + 2 + \epsilon.$$

Together with (4.4.2), we can find $z \in \alpha$ such that

$$\mathrm{dist}_\Gamma(z, \gamma_x) \leqslant \frac{4\pi}{3} + 2, \quad \text{and} \quad \mathrm{dist}_\Gamma(z, \gamma_y) \leqslant \frac{4\pi}{3} + 2.$$

Let x_1 and y_1 be the closest points to z on γ_x and γ_y respectively, by triangle inequalities, we obtain that

$$o(x_1) \geqslant t' - \frac{4\pi}{3} - 2, \quad \text{and} \quad o(x_2) \geqslant t' - \frac{4\pi}{3} - 2,$$

and then

$$\mathrm{dist}_N(x, x_1) \leqslant \frac{8\pi}{3} + 4 + 2\epsilon, \quad \text{and} \quad \mathrm{dist}_N(y, y_1) \leqslant \frac{8\pi}{3} + 4 + 2\epsilon.$$

Therefore,

$$\mathrm{dist}_N(x, y) \leqslant \mathrm{dist}_N(x, x_1) + \mathrm{dist}_N(x_1, z) + \mathrm{dist}_N(z, y_1) + \mathrm{dist}_N(y_1, y) \leqslant 8\pi + 12 + 4\epsilon.$$

Finally, combining with two cases above, we finish the proof of Proposition 4.1.6.

□

Chapter 5

Comparison theorem and integral of scalar curvature on three manifolds

5.1 Harmonic functions and its level set

Suppose that (M^n, g) is a complete Riemannian manifold and Σ^{n-1} is an oriented, embedded submanifold of dimension $n-1$ in M, ν is the unit normal vector field of Σ in M and $\{\nu, e_1, e_2, \cdots, e_{n-1}\}$ forms an orthonormal basis of TM in the local coordinate. Then, we introduce the second fundamental form of Σ in M with respect to ν,

$$A_{ij} = A(e_i, e_j) = (\nabla_{e_i}\nu, e_j) = (\nabla\nu)(e_i, e_j). \tag{5.1.1}$$

Then, taking the contraction, we define

$$H = tr(A) = \Sigma_{i=1}^{n-1}(\nabla_{e_i}\nu, e_j). \tag{5.1.2}$$

H is said to be the mean curvature of Σ with respect to ν. Here, we would not use the sign of the second fundamental form A and the mean curvature H in this work. Hence, we do not need to emphasize the orientation of the unit normal vector field ν, which we should choose to define the second fundamental form and the mean curvature on Σ. For any harmonic functions defined in Section 1 and any regular value t, $l(t)$ should be an embedded hypersurface in M if only $l(t)$ is a nonempty set. Now we pick $\nu = \frac{\nabla f}{|\nabla f|}$ for our use. Then, we deduce our lemma as follows:

Lemma 5.1.1. *[Geometry on level set $l(t)$] Suppose that (M^n, g) is a complete Riemannian manifold, f is a proper harmonic function on M and t is any a regular value of f such that $l(t)$ is a nonempty set. Then,*

1. *Let $A(t,x)$ be the second fundamental form of $l(t)$ with respect to the unit normal vector field $\frac{\nabla f}{|\nabla f|}$. Then*
$$A(t,x) = \frac{\nabla^2_{l(t)} f}{|\nabla f|}.$$
Here $\nabla^2_{l(t)} f$ stands for the restriction of $\nabla^2 f$ on $l(t)$ and ∇^2 is the Hessian of (M, g);

2. *Let $H(t,x)$ be the mean curvature of $l(t)$, i.e., $H(t,x) = tr(A(t,x))$. Then*
$$H(t,x) = -\nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|^2};$$

3. *On the level set $l(t)$, we have*
$$\frac{\Delta|\nabla f|}{|\nabla f|} = \frac{1}{2}Sc - \frac{1}{2}Sc(l(t)) + \frac{1}{2}\frac{|\nabla^2 f|^2}{|\nabla f|^2} \tag{5.1.3}$$
Here $Sc(l(t))$ is the intrinsic scalar curvature of $l(t)$ with respect to the Riemannian metric on $l(t)$ that is induced from the ambient Riemannian metric g.

Proof. Let $\{e_0 = \frac{\nabla f}{|\nabla f|}, e_1, \cdots, e_{n-1}\}$ be a normal coordinate of M at $p \in l(t)$. Then, by the definition of the second fundamental form, we have
$$A(t,x) = \nabla_{l(t)}(\frac{\nabla f}{|\nabla f|}).$$

In local coordinate, we have $A_{ij} = (\frac{f_i}{|\nabla f|})_j$ for $1 \leqslant i, j \leqslant n-1$. By a direct calculation, we have
$$A_{ij}(x,t) = \frac{\nabla^2_{l(t)} f}{|\nabla f|}.$$

Moreover, since f is a harmonic function on M, we have $f_{00}+\cdots+f_{nn}=0$. Equivalently,

$$-f_{00}=(f_{11}+\cdots f_{nn}).$$

This directly implies that

$$H(x,t)=\frac{-f_{00}}{|\nabla f|}=-\nabla|\nabla f|\cdot\frac{\nabla f}{|\nabla f|}.$$

Finally, by the Bochner formula and $\Delta f=0$, we obtain

$$|\nabla f|\Delta|\nabla f|+|\nabla|\nabla f||^2=\frac{1}{2}\Delta|\nabla f|^2=|\nabla^2 f|^2+Ric(\nabla f,\nabla f),$$

we obtain that

$$\frac{\Delta|\nabla f|}{|\nabla f|}=\frac{|\nabla^2 f|^2}{|\nabla f|^2}-\frac{|\nabla|\nabla f||^2}{|\nabla f|^2}+Ric(\frac{\nabla f}{|\nabla f|},\frac{\nabla f}{|\nabla f|}).$$

Moreover, by using the Schoen-Yau trick on minimal surface on the level set $l(t)$, we have

$$Ric(\frac{\nabla f}{|\nabla f|},\frac{\nabla f}{|\nabla f|})=\frac{1}{2}Sc-\frac{1}{2}Sc_{l(t)}+\frac{1}{2}(H^2-|A|^2).$$

Together it with the calculation above, we obtain that

$$\frac{1}{2}(H^2-|A|^2)=-\frac{1}{2}\frac{|\nabla^2 f|^2}{|\nabla f|^2}+\frac{|\nabla|\nabla f|^2}{|\nabla f|^2}.$$

Hence,

$$\frac{\Delta|\nabla f|}{|\nabla f|}=\frac{1}{2}Sc-\frac{1}{2}Sc_{l(t)}+\frac{1}{2}\frac{|\nabla^2 f|^2}{|\nabla f|^2}.$$

Here, the trick of Bochner formula used on the level set that has been written systematically is due to Stern's work [71]. □

Remark 5.1.2. *We will always assume that $l(t)$ is nonempty in this work and will not emphasize this point when we will use $l(t)$ later.*

After this basic preparation, we will calculate the derivatives of $\omega(t)$ in the incoming lemma.

Lemma 5.1.3. *Let (M^n, g) be a complete Riemannian manifold and f be a harmonic function on M. Then, for almost all $t \in \mathbb{R}$, we obtain*

$$\omega'(t) = \int_{l(t)} \frac{(\nabla|\nabla f|, \nabla f)}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}} = -\int_{l(t)} H|\nabla f| d\mathcal{H}^{n-1}_{g|_{l(t)}}, \tag{5.1.4}$$

$$\omega''(t) = \int_{l(t)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}}. \tag{5.1.5}$$

In particular if $Ric(g) \geqslant 0$, then $\omega''(t) \geqslant 0$.

Proof. For any regular value t of f, we fix one point $p \in l(t)$. Let $x = (x^1, x^2, \cdots, x^n)$ be a local natural coordinate chart of $l(t)$ around the point p. Thus, $f(x(t)) = t$. After differentiating on both sides, we could pick

$$x'(t) = \frac{\nabla f}{|\nabla f|^2} = \varphi \cdot \frac{\nabla f}{|\nabla f|} \text{ and } \varphi = \frac{1}{|\nabla f|}.$$

By direct calculation,

$$\frac{\partial}{\partial t}(d\mathcal{H}^{n-1}_{g|_{l(t)}}) = H\varphi d\mathcal{H}^{n-1}_{g|_{l(t)}}.$$

Hence,

$$\omega'(t) = \int_{l(t)} \frac{\partial}{\partial t}|\nabla f|^2 d\mathcal{H}^{n-1}_{g|_{l(t)}} + \int_{l(t)} |\nabla f|^2 \frac{\partial}{\partial t}(d\mathcal{H}^{n-1}_{g|_{l(t)}}) \tag{5.1.6}$$

$$= \int_{l(t)} 2|\nabla f|\nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|^2} + |\nabla f|^2 H \frac{1}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}}. \tag{5.1.7}$$

This directly implies that

$$\omega'(t) = \int_{l(t)} \frac{(\nabla|\nabla f|, \nabla f)}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}} = -\int_{l(t)} H|\nabla f| d\mathcal{H}^{n-1}_{g|_{l(t)}}.$$

Moreover, it is clear that $l(1)$ and $l(t)$ enclose a domain which is either

- $L(1, t)$ as $t > 1$ if M is a parabolic manifold; or
- $L(t, 1)$ as $t < 1$ if M is a non-parabolic manifold.

Hence, we will discuss the following two cases to reach $\omega''(t)$.

Case 1: M is parabolic.

By integration by part over $L(1,t)$, we obtain,

$$\int_{L(1,t)} \Delta|\nabla f| d\mathcal{H}^n_g = \int_{l(t)} \nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}} - \int_{l(1)} \nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(1)}}.$$

Hence,

$$\int_1^t \int_{l(r)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr = \omega'(t) - \omega'(1).$$

Since $h(r) = \int_{l(r)} \frac{\Delta|\nabla f|}{|\nabla f|}$ is continuous at any regular value $r \in \mathbb{R}$, we deduce that

$$\omega''(t) = \int_{l(t)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}};$$

Case 2: M is non-parabolic.

$$\int_{L(t,1)} \Delta|\nabla f| d\mathcal{H}^n_g = \int_{l(1)} \nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(1)}} - \int_{l(t)} \nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}}.$$

Hence,

$$\int_t^1 \int_{l(r)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr = \omega'(1) - \omega'(t).$$

Then,

$$\omega''(t) = \int_{l(t)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}}.$$

Combining two cases together, we finally reach

$$\omega''(t) = \int_{l(t)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(t)}}.$$

Finally, by Bochner formula and Kato's inequality, it is clear that

$$\frac{\Delta|\nabla f|}{|\nabla f|} = \frac{|\nabla^2 f|^2 - |\nabla|\nabla f||^2}{|\nabla f|^2} - Ric(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}), \ |\nabla|\nabla f||^2 \leqslant \frac{2}{3}|\nabla^2 f|^2.$$

Since $Ric(g) \geqslant 0$, we obtain that

$$\frac{\Delta|\nabla f|}{|\nabla f|} \geqslant 0.$$

It follows that $\omega''(t) \geqslant 0$. □

Now let's study an example, which is for our use later.

Example 5.1.4. *Let* $G(0,x) = \frac{1}{4\pi|x|}$ *be a Green function of Beltrami Laplace on* $\mathbb{R}^3$ *and*

$$\omega(t) = \int_{l(t)} |\nabla G|^2 d\mathcal{H}^3_g.$$

Then, we obtain $\omega(t) = 4\pi t^2$*. It directly lead to*

$$\omega'(1) = 8\pi, \omega(1) = 4\pi.$$

It implies that

$$\omega'(1) - \omega(1) - 4\pi = 0.$$

This basic example will motivate us to obtain a characterization of rigidity below.

Proposition 5.1.5. *Let* (M^n, g) *be a complete, non-compact Riemannian manifold,* f *be a harmonic function defined on section* 1 *and* $\alpha \in \mathbb{R}$*,*

1.

$$t^\alpha \omega'(t) = \alpha t^{\alpha-1}\omega(t) + (\omega'(1) - \alpha\omega(1)) - \alpha(\alpha-1)\int_1^t r^{\alpha-2}\omega(r)dr + \int_1^t r^\alpha \int_{l(r)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr. \tag{5.1.8}$$

If M *is parabolic,* $t \in [1, \infty)$*; if* M *is non-parabolic,* $t \in (0, 1]$*;*

2. *If* (M^n, g) *is non-parabolic, we obtain, for any* $k \in \mathbb{R}$ *and* $t \in (0, 1)$*,*

$$\begin{aligned} t^\alpha \omega'(t) \leqslant{}& (\alpha+3k)t^{\alpha-1}\omega(t) + (\omega'(1) - (\alpha+3k)\omega(1)) \\ &- \left(\alpha^2 + (3k-1)\alpha + 3k(k-1)\right)\int_1^t r^{\alpha-2}\omega(r)dr \\ &- \frac{1}{2}\int_1^t r^\alpha \int_{l(r)} Sc_{l(t)} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr + \frac{1}{2}\int_1^t r^\alpha \int_{l(r)} Sc \ d\mathcal{H}^{n-1}_{g|_{l(r)}} dr; \end{aligned} \tag{5.1.9}$$

3. *If* (M^n, g) *is parabolic, we obtain, for any* $k \in \mathbb{R}$ *and* $t \in (1, \infty)$

$$\begin{aligned} t^\alpha \omega'(t) \geqslant{}& (\alpha+3k)t^{\alpha-1}\omega(t) + (\omega'(1) - (\alpha+3k)\omega(1)) \\ &- (\alpha^2 + (3k-1)\alpha + 3k(k-1))\int_1^t r^{\alpha-2}\omega(r)dr \\ &- \frac{1}{2}\int_1^t r^\alpha \int_{l(r)} Sc_{l(r)} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr + \frac{1}{2}\int_1^t r^\alpha \int_{l(r)} Sc \ d\mathcal{H}^{n-1}_{g|_{l(r)}} dr; \end{aligned} \tag{5.1.10}$$

Proof. Let's first assume that M is non-parabolic. Let's start with the Green identity for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\int_{L(t,1)} (\Delta|\nabla f| f^\alpha - |\nabla f|\Delta f^\alpha)\, d\mathcal{H}^n_g \\
&= \int_{l(1)} (\nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} f^\alpha - |\nabla f| \nabla f^\alpha \cdot \frac{\nabla f}{|\nabla f|}) d\mathcal{H}^{n-1}_{g|_{l(1)}} \\
&\quad - \int_{l(t)} (\nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} f^\alpha - |\nabla f| \nabla f^\alpha \cdot \frac{\nabla f}{|\nabla f|}) d\mathcal{H}^{n-1}_{g|_{l(t)}}.
\end{aligned}$$

Then, by a direct calculation, the term on the right-hand side is actually equal to

$$\omega'(1) - \alpha\omega(1) - (t^\alpha \omega'(t) - \alpha t^{\alpha-1}\omega(t)).$$

By using coarea formula, we obtain that the term on the left-hand side is actually equal to

$$\int_t^1 r^\alpha \left(\int_{l(r)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}} \right) dr - \alpha(\alpha-1) \int_t^1 t^{\alpha-2}\omega(r) dr.$$

Here, we have used that f is a harmonic function on M. Therefore, we obtain that

$$\begin{aligned}
&\omega'(1) - \alpha\omega(1) - (t^\alpha \omega'(t) - \alpha t^{\alpha-1}\omega(t)) \\
&= \int_t^1 r^\alpha \int_{l(r)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr - \alpha(\alpha-1) \int_t^1 r^{\alpha-2}\omega(r) dr.
\end{aligned}$$

Then,

$$\begin{aligned}
t^\alpha \omega'(t) &= \alpha t^{\alpha-1}\omega(t) + (\omega'(1) - \alpha\omega(1)) \\
&+\alpha(\alpha-1) \int_t^1 r^{\alpha-2}\omega(r) dr - \int_t^1 r^\alpha \int_{l(t)} \frac{\Delta|\nabla f|}{|\nabla f|} d\mathcal{H}^{n-1}_{g|_{l(r)}} dt.
\end{aligned}$$

This is equation (5.1.8) on non-parabolic manifold.

Now, let's assume that M is parabolic, and then the arguments of (5.1.10) are similar to that of (5.1.9). we will write the details for the reader's convenience.

For any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\int_{L(1,t)} (\Delta|\nabla f| f^\alpha - |\nabla f|\Delta f^\alpha)\, d\mathcal{H}^n_g = \\
&\int_{l(t)} (\nabla|\nabla f| \cdot \frac{\nabla f}{|\nabla f|} f^\alpha - |\nabla f| \nabla f^\alpha \cdot \frac{\nabla f}{|\nabla f|}) d\mathcal{H}^{n-1}_{g|_{l(t)}}
\end{aligned}$$

$$-\int_{l(1)}(\nabla|\nabla f|\cdot\frac{\nabla f}{|\nabla f|}f^{\alpha}-|\nabla f|\nabla f^{\alpha}\cdot\frac{\nabla f}{|\nabla f|})d\mathcal{H}^{n-1}_{g|_{l(1)}}.$$

Therefore, we obtain that

$$\begin{aligned}&t^{\alpha}\omega'(t)-\alpha t^{\alpha-1}\omega(t)-(\omega'(1)-\alpha\omega(1))\\=&\int_1^t r^{\alpha}\int_{l(r)}\frac{\Delta|\nabla f|}{|\nabla f|}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr-\alpha(\alpha-1)\int_1^t r^{\alpha-2}\omega(r)dr.\end{aligned}$$

Then,

$$\begin{aligned}&t^{\alpha}\omega'(t)=\alpha t^{\alpha-1}\omega(t)+(\omega'(1)-\alpha\omega(1))\\&-\alpha(\alpha-1)\int_1^t r^{\alpha-2}\omega(r)dr+\int_1^t r^{\alpha}\int_{l(r)}\frac{\Delta|\nabla f|}{|\nabla f|}d\mathcal{H}^{n-1}_{g|_{l(r)}}dt.\end{aligned}$$

This is equation (5.1.8) on parabolic manifold. Combining two cases together, we deduce the identity in (a) on both non-parabolic and parabolic manifolds.

Now, we deal with the last term $\int_1^t r^{\alpha}\int_{l(t)}\frac{\Delta|\nabla f|}{|\nabla f|}d\mathcal{H}^{n-1}_{g|_{l(r)}}dt$ on the right-hand side if M is non-parabolic. We should keep in mind that $t\in(0,1)$ in this case.

$$\begin{aligned}&\int_t^1 r^{\alpha}\int_{l(r)}\frac{\Delta|\nabla f|}{|\nabla f|}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr\\=&\int_t^1 r^{\alpha}\int_{l(r)}\frac{1}{2}Sc-\frac{1}{2}Sc_{l(r)}+\frac{1}{2}\frac{|\nabla^2 f|^2}{|\nabla f|^2}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr\\=&\frac{1}{2}\int_t^1 r^{\alpha}\int_{l(r)}Sc\,d\mathcal{H}^{n-1}_{g|_{l(r)}}dr-\frac{1}{2}\int_t^1 r^{\alpha}\int_{l(r)}Sc_{l(r)}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr\\&+\frac{1}{2}\int_t^1 r^{\alpha}\int_{l(r)}\frac{|\nabla^2 f|^2}{|\nabla f|^2}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr.\end{aligned}$$

For the last term $\frac{1}{2}\int_t^1 r^{\alpha}\int_{l(r)}\frac{|\nabla^2 f|^2}{|\nabla f|^2}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr$, we have

$$\begin{aligned}\frac{1}{2}\int_t^1 r^{\alpha}\int_{l(r)}\frac{|\nabla^2 f|^2}{|\nabla f|^2}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr &\geqslant \frac{3}{4}\int_t^1 r^{\alpha}\int_{l(r)}\frac{|\nabla|\nabla f||^2}{|\nabla f|^2}d\mathcal{H}^{n-1}_{g|_{l(r)}}dr\\&\geqslant \frac{3}{4}\int_t^1 r^{\alpha}\frac{(\omega'(r))^2}{\omega(r)}dr.\end{aligned}$$

Now let $k\in\mathbb{R}$, we consider

$$0\leqslant\omega(t)(\frac{\omega'(r)}{\omega(r)}-\frac{2k}{r})^2\leqslant\frac{(\omega'(r))^2}{\omega(r)}-\frac{4k\omega'(r)}{r}+\frac{4k^2\omega(r)}{r^2}.$$

Hence, we obtain

$$\frac{(\omega'(r))^2}{\omega(r)} \geqslant 4k\omega'(r) - \frac{4k^2\omega(r)}{r^2}.$$

Then, by integration by parts and calculations above. we have

$$\frac{1}{2}\int_t^1 r^\alpha \int_{l(r)} \frac{|\nabla^2 f|^2}{|\nabla f|^2} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr \tag{5.1.11}$$

$$\geqslant 3k\int_t^1 r^{\alpha-1}\omega'(r)dr - 3k^2\int_t^1 r^{\alpha-2}\omega(r)dr \tag{5.1.12}$$

$$=3k\omega(1) - 3kt^{\alpha-1}\omega(t) - (3k^2 + 3k(\alpha-1))\int_t^1 r^{\alpha-2}\omega(r)dr. \tag{5.1.13}$$

Hence,

$$\frac{1}{2}\int_t^1 r^\alpha \int_{l(r)} \frac{|\nabla^2 f|^2}{|\nabla f|^2} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr$$

$$\geqslant 3k\omega(1) - 3kt^{\alpha-1}\omega(t) - (3k^2 + 3k(\alpha-1))\int_t^1 r^{\alpha-2}\omega(r)dr.$$

Now we assemble all calculations above and then reach the inequality (5.1.9).

Finally, let's prove the estimate (5.1.10). If we follow the details of the proof of estimate (5.1.10), we only need to repeat almost all calculations above. However, we need to calculate (5.1.11). In this case, we calculate

$$\frac{1}{2}\int_1^t r^\alpha \int_{l(r)} \frac{|\nabla^2 f|^2}{|\nabla f|^2} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr \tag{5.1.14}$$

$$\geqslant 3k\int_1^t r^{\alpha-1}\omega'(r)dr - 3k^2\int_1^t r^{\alpha-2}\omega(r)dr \tag{5.1.15}$$

$$=3kt^{\alpha-|}\omega(t) - 3k\omega(1) - (3k^2 + 3k(\alpha-1))\int_t^1 r^{\alpha-2}\omega(r)dr. \tag{5.1.16}$$

Hence,

$$\frac{1}{2}\int_1^t r^\alpha \int_{l(r)} \frac{|\nabla^2 f|^2}{|\nabla f|^2} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr$$

$$\geqslant 3kt^{\alpha-1}\omega(t) - 3k\omega(1) - (3k^2 + 3k(\alpha-1))\int_1^t r^{\alpha-2}\omega(r)dr.$$

Therefore, we reach (5.1.10).

□

The study of geometry at infinity on complete Riemannian manifold is a very challenging question in the field of geometry analysis. Here, we introduce a global quantity associated with the complete Riemannian manifold.

Definition 5.1.6. *Let (M^n, g) be a complete, non-compact Riemannian manifold with compact, connected boundary ∂M and f a harmonic function. Then, we define*

$$\mathcal{B}_f(M) = 3\omega_f(1) - \omega_f'(1).$$

Parallel to the definition, for a complete, non-compact Riemannian manifold (M^n, g) without boundary, we define

$$\mathcal{B}_f(M) = \sup_{\Sigma}(3\omega_f(1) - \omega_f'(1)),$$

Here, the supremum over Σ is taken from any connected, compact hypersurface in M dividing M into two connected components. Therefore, we define

$$\mathcal{B}(M) = \sup_f \mathcal{B}_f(M).$$

where supremum over f can be taken from some special categories prescribed.

In a three-dimensional, non-parabolic Riemannian manifold with compact connected boundary, we can obtain a uniformly upper bound on $\mathcal{B}_f(M^3)$ and $B(M^3)$. In fact, since harmonic function considered in our context is a global concept, theoretically, $\mathcal{B}_f(M)$ and $\mathcal{B}(M)$ would carry the information of the geometry at infinity. Further investigation on $\mathcal{B}(M)$ would be interesting. Deeply, $\mathcal{B}(M)$ is related to the Penrose inequality in three-dimensional case. Suppose that (M^3, g) be an asymptotically flat with $Sc(g) \geqslant 0$ with mass $m > 0$ and ∂M is the only minimal surface and connected, Then, hopefully, the term $4\pi - \mathcal{B}(M)$ can be expressed as a function of mass m.

5.2 Proof of Theorem 1.3.2

Theorem 5.2.1. *Let (M^3, g) be a complete, non-compact three-dimensional Riemannian manifold with non-negative scalar curvature $Sc(g) \geqslant 0$, and its boundary ∂M be connected and closed. If $b_1(M) = 0$ and M has one end. Then, we have differential inequalities as follows:*

- *If (M^3, g) is non-parabolic, then for any $t \in (0,1)$,*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi t - \frac{\omega'(1) - \omega(1) - 4\pi}{2}t^2\right) \leqslant 0. \tag{5.2.1}$$

Moreover, there exists $T \in (0,1)$ such that the equality holds if and only if

$$L(T,1) \text{ is isometric to } A(\frac{1}{4\pi}, \frac{1}{4\pi T}).$$

Here $A(\frac{1}{4\pi}, \frac{1}{4\pi T})$ is the annulus in $\mathbb{R}^3$ with outer radius $R = \frac{1}{4\pi T}$ and inner radius $r = \frac{1}{4\pi}$;

- *If (M^3, g) is parabolic, then for any $t \in (1, \infty)$,*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi t - \frac{\omega'(1) - \omega(1) - 4\pi}{2}t^2\right) \geqslant 0. \tag{5.2.2}$$

Proof. Let's prove the first differential inequality (5.2.1) in the case that (M^3, g) is non-parabolic. We take $\alpha = -2, k = 1$ in (5.1.9), then

$$t^{-2}\omega'(t) \leqslant t^{-3}\omega(t) + (\omega'(1) - \omega(1)) \tag{5.2.3}$$

$$+ \frac{1}{2}\int_t^1 r^{-2}\int_{l(r)} Sc_{l(r)} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr - \frac{1}{2}\int_t^1 r^{-2}\int_{l(r)} Sc\, d\mathcal{H}^{n-1}_{g|_{l(r)}} dr; \tag{5.2.4}$$

Since the dimension of the manifold M is 3, we have

$$K_{l(r)} = \frac{1}{2}Sc_{l(r)}.$$

Here $K_{l(r)}$ is the Gauss curvature on $l(r)$. Besides, the assumption that $b_1(M) = 0$ implies that $l(r)$ is connected for any $r \in (0,1]$. Hence, by the Gauss-Bonnet theorem on $l(r)$, we reach

$$\int_{l(r)} K_{l(r)} d\mathcal{H}^{n-1}_{g|_{l(r)}} \leqslant 4\pi.$$

It directly leads to

$$\int_t^1 r^{-2}\int_{l(r)} K_{l(r)} d\mathcal{H}^{n-1}_{g|_{l(r)}} dr \leqslant \frac{4\pi}{t} - 4\pi.$$

Hence,

$$t^{-2}\omega'(t) \leqslant t^{-3}\omega(t) + \frac{4\pi}{t} + (\omega'(1) - \omega(1) - 4\pi) - \frac{1}{2}\int_t^1 r^{-2}\int_{l(r)} Sc d\mathcal{H}^{n-1}_{g|_{l(r)}} dr. \quad (5.2.5)$$

Since $Sc(g) \geqslant 0$, we have

$$t^{-2}\omega'(t) \leqslant t^{-3}\omega(t) + \frac{4\pi}{t} + (\omega'(1) - \omega(1) - 4\pi). \quad (5.2.6)$$

This directly implies (1.3.2).

Moreover, as equality holds in (1.3.2) at $t = T$, then all inequalities in the process of proof of Proposition 5.1.5 and (1.3.2) should be equalities. Hence, we have for all $t \in [T, 1]$

$$|\nabla|\nabla f||^2 = \frac{2}{3}|\nabla^2 f|,$$
$$|\nabla|\nabla f|| = \lambda|\nabla f|^2 \text{ for some constant } \lambda \in \mathbb{R},$$
$$\frac{\omega'(t)}{\omega(t)} = \frac{2}{t},$$
$$t^{-2}\omega'(t) = t^{-3}\omega(t) + \frac{4\pi}{t} + (\omega'(1) - \omega(1) - 4\pi).$$

By solving the system of differential equations, we have

$$f_{11} = 32\pi f^3, f_{22} = f_{33} = -16\pi f^3,\ |\nabla f| = 4\pi f^2,$$

and

$$\omega'(1) - \omega(1) - 4\pi = 0,$$
$$\lambda = \frac{2}{r}.$$

Now, we introduce $F(x) = \frac{1}{4\pi f(x)}$. By a direct calculation, we obtain

$$|\nabla F| = 1, \mathrm{Hess}(\frac{1}{2}F^2(x)) = I_{3\times 3}.$$

By the relationship between Hessian and geometry of level set, we directly obtain

$$L(T, 1) \text{ is isometric to } A(\frac{1}{4\pi}, \frac{1}{4\pi T}).$$

By a similar argument due to the Proposition 5.2.1, we can prove the second differential inequality on parabolic manifold. Here, we will not write the details since all the arguments are similar.

□

Remark 5.2.2. *If we did not assume $b_1(M) = 0$, we would have obtained that $l(t)$ is connected. However, as we assume that the number of the connected components of $l(t)$ has a uniformly upper bound k, we can deduce that*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi kt - \frac{\omega'(1) - \omega(1) - 4\pi k}{2}t^2\right).$$

is either non-negative or non-positive correspondingly. Indeed, the author can not further characterize the differential inequality on parabolic manifolds since we can not find a model for our manifolds, which is core topics of the study of scalar curvature in the sense of comparison geometry. Indeed, the parabolic case is even more interesting to the study of the scalar curvature.

5.3 Proof of Theorem 1.3.3

Theorem 5.3.1. *Let (M^3, g) be a complete, non-compact, non-parabolic Riemannian manifold with $Ric(g) \geqslant 0$. Then, there exists a constant $k \in \mathbb{R}$ such that*

1. *For any $t \in (0, 1]$, we have*

$$\omega(t) \leqslant 4k\pi t^2 + \frac{\omega'(1) - \omega(1) - 4k\pi}{2}t^3; \tag{5.3.1}$$

$$A(l(t)) \geqslant \frac{1}{4k\pi t^2 + \frac{\omega'(1)-\omega(1)-4k\pi}{2}t^3}. \tag{5.3.2}$$

Moreover, $b_1(M) = 0$ and there exists a $T \in (0, 1)$ such that

$$A(l(T)) = \frac{1}{4\pi T^2 + \frac{\omega'(1)-\omega(1)-4\pi}{2}T^3}$$

if and only if M is isometric to $\mathbb{R}^3 \backslash B(\frac{1}{4\pi})$;

2.

$$3\omega(1) - \omega'(1) \leqslant 4k\pi.$$

In particular, $b_1(M) = 0$ *and* $3\omega(1) - \omega'(1) = 4\pi$ *if and only if* M *is isometric to* $\mathbb{R}^3 \backslash B(\frac{1}{4\pi})$*;*

3.

$$\limsup_{t \to 0} \int_{L(t,1)} Sc|\nabla f| d\mathcal{H}^n_g \leqslant 8k\pi + 2\omega'(1).$$

Proof. Since (M^3, g) is a complete, noncompact Riemannian manifold with nonnegative Ricci curvature. Hence, by the splitting theorem in [9], (M^3, g) has at most two ends. Moreover, since M is non-parabolic, it admits only one end. By [1], we have $b_1(M) \leqslant 3$.

Now, by Remark 5.2.2 and the lemma in [52], there exists a constant k depending on the number of ends and $b_1(M)$ [52] such that

$$b_1(M) \leqslant k.$$

Then, for any $t \in (0, 1)$, we have

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi k t - \frac{\omega'(1) - \omega(1) - 4\pi k}{2} t^2\right) \leqslant 0. \tag{5.3.3}$$

Therefore, for any $1 > t \geqslant \delta > 0$, we have

$$\frac{\omega(t)}{t} - 4\pi k t - \frac{\omega'(1) - \omega(1) - 4\pi k}{2} t^2 \leqslant \frac{\omega(\delta)}{\delta} - 4\pi k \delta - \frac{\omega'(1) - \omega(1) - 4\pi k}{2} \delta^2.$$

Since $Ric(g) \geqslant 0$ and by Theorem 6.1 in [42], we obtain

$$\lim_{\delta \to 0^+} \frac{\omega(\delta)}{\delta} = 0.$$

Hence, for any $t \in (0, 1]$,

$$\frac{\omega(t)}{t} - 4\pi k t - \frac{\omega'(1) - \omega(1) - 4\pi k}{2} t^2 \leqslant 0.$$

Then,

$$\omega(t) \leqslant 4\pi k t^2 + \frac{\omega'(1) - \omega(1) - 4\pi k}{2} t^3.$$

This confirms (5.3.1).

Moreover,
$$\int_{l(t)} |\nabla f|^2 d\mathcal{H}_g^n \leqslant 4\pi k t^2 + \frac{\omega'(1) - \omega(1) - 4\pi k}{2} t^3.$$
By [43], we have $1 = \int_{l(t)} |\nabla f| \leqslant (\omega(t))^{\frac{1}{2}} (A(t))^{\frac{1}{2}}$. It implies that
$$A(t) \geqslant \frac{1}{\omega(t)} \geqslant \frac{1}{4\pi k t^2 + \frac{\omega'(1)-\omega(1)-4\pi k}{2} t^3}.$$

In particular, if $b_1(M) = 0$ and
$$A(l(T)) = \frac{1}{4\pi T^2 + \frac{\omega'(1)-\omega(1)-4\pi}{2} T^3},$$
then, we have that T is a critical point of
$$F_k(t) = \frac{\omega(t)}{t} - 4\pi k t - \frac{\omega'(1) - \omega(1) - 4\pi k}{2} t^2, \ k = 1.$$
Hence, we have $F_1'(T) = 0$. Then by Theorem 5.2.1, we reach that that $L(T, 1)$ is isometric to $A(\frac{1}{4\pi}, \frac{1}{4\pi T})$. Besides, for $t \leqslant T$, we have
$$F(T) = 0, F(t) \leqslant 0, F'(t) \leqslant 0, \lim_{t \to 0^+} F(t) = 0.$$
Hence, $F'(t) = 0$ for all $t \in (0, T)$. It implies that $L(0, T)$ is isometric to $\mathbb{R}^3 \backslash B(\frac{1}{4\pi T})$. Finally, we proved that M is isometric to $\mathbb{R}^3 \backslash B(\frac{1}{4\pi})$. By the Example 5.1.4, the converse is trivial. Hence, we finshed the proof of (a).

Moreover, we let $t = 1$, then
$$\omega(1) \leqslant 4k\pi + \frac{\omega'(1) - \omega(1) - 4\pi k}{2}.$$
Hence,
$$3\omega(1) - \omega'(1) \leqslant 4k\pi.$$

Now we assume that $b_1(M) = 0$, then $k = 1$. Hence $3\omega(1) - \omega'(1) \leqslant 4\pi$ and if equality holds. Then, we have
$$F_1(1) = 0, \lim_{t \to 0} F_1(t) = 0, F_1(t) \leqslant 0, F'(t) \leqslant 0, t \in (0, 1].$$
Hence, we have $F'(t) = 0, t \in (0, 1]$. Hence, combining these with Example 5.1.4, we have the rigidity equivalent characterization.

Finally, by Equation (5.1.3) and (5.1.5),

$$\begin{aligned} \tfrac{1}{2}\int_{l(t)} Sc d\mathcal{H}^{n-1}_{g|_{l(t)}} &= \int_{l(t)} K_{l(t)} d\mathcal{H}^{n-1}_{g|_{l(t))}} - \tfrac{1}{2}\int_{l(t)} \tfrac{|\nabla^2 f|^2}{|\nabla f|^2} d\mathcal{H}^{n-1}_{g|_{l(t)}} + \omega''(t) \\ &\leqslant 4k\pi + \omega''(t). \end{aligned}$$

By coarea formula, we have

$$\frac{1}{2}\int_{L(t,1)} Sc|\nabla f| d\mathcal{H}^n_g \leqslant 4k\pi(1-t) + \omega'(1) - \omega'(t).$$

Since $\omega''(t) \geqslant 0$ and $\lim_{t\to 0^+}\omega(t) = 0$. Hence, $\lim_{t\to 0^+}\omega'(t) = 0$. Finally,

$$\limsup_{t\to 0}\int_{L(t,1)} Sc|\nabla f| d\mathcal{H}^n_g \leqslant 8k\pi + 2\omega'(1).$$

□

Remark 5.3.2. *All estimates in the Theorem* 5.3.1 *hold if only*

$$\liminf_{x\to\infty} Ric(x) \geqslant 0.$$

Corollary 5.3.3. *Let (M^3, g) be a complete, non-compact, non-parabolic three dimensional Riemannian manifold with $Sc(g) \geqslant 0$ with connected, closed minimal surface boundary, $b_1(M) = 0$ and one end. Then*

$$\frac{d}{dt}\left(\frac{\omega(t)}{t} - 4\pi t\right) \leqslant -(\omega(1) + 4\pi)t \leqslant -4\pi t.$$

It implies no closed minimal surface in $\mathbb{R}^3$.

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