Generalized symplecitc log Calabi-Yau divisors

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Abstract

We study the symplectic divisors corresponding to the Hamiltonian circle actions on symplectic surfaces. In [33], it is showed that counting toric actions on a fixed symplectic rational surface is equivalent to counting toric log Calabi-Yau divisors. Inspired by a formula in [15], we introduce generalized symplectic log Calabi-Yau divisors on symplectic irrational ruled surfaces. Using the language of marked divisors we prove a version of Torelli theorem, stating that the symplectic deformation classes of these divisors (with a few extra conditions) are determined by their homological information. We show that there is an one-to-one correspondence between Hamiltonian circle actions and S^1-generalied symplectic log Calabi-Yau divisors in a fixed symplectic irrational ruled surface. As an application, we give a new proof of the finiteness of inequivalent Hamiltonian circle actions.

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Chapter 1

Introduction

Let X be a smooth rational surface and $D \subset X$ an effective reduced anti-canonical divisor. (X, D) is called an anti-canonical pair and has been carefully studied since Looijenga [34]. The moduli space of anti-canonical pairs have been studied in [34] and [11], where the Torelli type theorems were proved. Friedman gives an excellent survey in [8]. The symplectic analogue of anti-canonical pairs, symplectic log Calabi-Yau pairs, was introduced in [29] and studied in [29], [32] and [33], with applications to contact geometry and symplectic fillings.

In [33] several equivalences between various moduli spaces are established. In particular, it is showed that counting toric actions on symplectic rational surfaces is equivalent to counting toric log Calabi-Yau divisors. In this thesis we introduce generalized symplectic log Calabi-Yau divisors and study their relations with Hamiltonian circle actions on symplectic irrational ruled surfaces.

1.1 Generalized symplectic log Calabi-Yau divisors

A symplectic 4-manifold (M, ω) is a smooth 4-manifold M equipped with a closed nondegenerate 2-form $\omega \in \Omega^2(M)$, called the symplectic form. A symplectic surface C in (M, ω) is a smooth surface in M such that ω restricts to a symplectic form on C.

More generally, we can study configurations of symplectic surfaces. A **symplectic divisor** is a connected configuration of finitely many closed embedded symplectic surfaces in a symplectic 4-manifold such that all intersections are positively transversal and there are no triple intersections. A **symplectic spherical divisor** is a symplectic divisor such that each component is an embedded symplectic sphere. See [31] for more details.

A **symplectic log Calabi-Yau pair** (M, ω, D) is a closed symplectic 4-manifold (M, ω) together with a nonempty symplectic divisor $D = \cup C_i$ representing the Poincare dual of $c_1(M, \omega)$. The symplectic divisor D is then called a **symplectic log Calabi-Yau divisor** in (M, ω).

An easy observation is that D is either a torus or a cycle of spheres ([30]). In the former case, (M, ω, D) is called an **elliptic log Calabi-Yau pair**; in the later case, it is called a **symplectic Looijenga pair** and can only happen when (M, ω) is rational. In [29] it is proved that two symplectic log Calabi-Yau pairs are deformation equivalent if and only if they are homological equivalent. Using this result, [33] showed that a toric symplectic log Calabi-Yau divisor is charaterized by its homological data (Torelli theorem), which can be used to reconstruct the corresponding Delzant polytope. Thus there is an one-to-one correspondence between the moduli space of (ω-orthogonal) toric symplectic Calabi-Yau divisors and that of toric actions in a symplectic rational surface.

Karshon gave classification of compact four-dimensional Hamitonian S^1-space in [18], which is the simplest case of symplectic manifolds endowed with Hamiltonian torus actions other than the symplectic toric manifolds. Naturally, we would like to know if there are certain divisors on these spaces which characterize the Hamiltonian circle actions. Inspired by a formula in [15] for the first Chern class (see Proposition 2.3.3), we give the following definition:

Definition 1.1.1. *A **generalized symplectic log Calabi-Yau pair** (M, ω, D) is a symplectic irrational ruled surface (M, ω) over $\Sigma = \Sigma_g$, together with a nonempty symplectic divisor $D = \cup_{i=1}^m C_i$ representing $\mathrm{PD}(c_1(M, \omega)) + pF$, where $p \geq 2g$ is an integer and F is the fiber class. The symplectic divisor D is called a **generalized symplectic log Calabi-Yau divisor** in (M, ω).*

The pF part corresponds to $(2g + k - 2)x_h$ in Proposition 2.3.3, and we require that $p \geq 2g$ since $k \geq 2$ and $2g + k - 2 \geq 2g$.

One issue for such definition is that it entails many divisors that we are not interested in. Hence, we introduce **good** generalized symplectic log Calabi-Yau divisors (see Definition 2.6.1), which serve the role of symplectic log Calabi-Yau divisors. In chapter 3, we prove the following:

Theorem 1.1.2. *(Torelli Theorem) Let $(M^i, \omega^i, D^i = \cup_j C_j^i)$ be good generalized symplectic log Calabi-Yau pairs for $i = 1, 2$. Then*

(1) they are (strictly) symplectic deformation equivalent if and only if they are (strictly) homological equivalent;

(2) suppose there is a lattice isomorphism

$$\gamma : H^2(M^2; \mathbb{Z}) \to H^2(M^1; \mathbb{Z})$$

such that $\gamma(\mathrm{PD}([C_j^2])) = \mathrm{PD}([C_j^1])$ for all j and $\gamma([\omega^2]) = [\omega^1]$. Then (M^1, ω^1, D^1) and (M^2, ω^2, D^2) are strictly symplectic deformation equivalent. They are symplectomorphic if *they are ω^1(resp. ω^2)-orthogonal.*

1.2 Hamiltonian circle actions

A symplectic Looijenga pair $(M, \omega, D = \cup_{i=1}^m C_i)$ is called a **toric** symplectic log Calabi-Yau pair if $q(D) = 0$, where $q(D) = 12 - m - [D]^2 = 12 - 3m - \sum_{i=1}^m [C_i]^2$. It is a simple observation that a symplectic log Calabi-Yau pair is toric if and only if it is an iterated toric blow-ups of minimal models ([33]). For a good generalized symplectic log Calabi-Yau pair $(M, \omega, D = \cup C_i)$, we call it an $\boldsymbol{S^1}$ pair if $\sum [C_i]^2 = -3k$, where M is the k points blow-up of a minimal irrational ruled surface (see Definition 4.1.2). It is an easy observation that a good generalized symplectic log Calabi-Yau pair is S^1 if and only if it is iterative S^1 blow-ups of the minimal models (Lemma 4.1.4). These S^1 divisors serve the role of toric symplectic log Calabi-Yau divisors and correspond to Hamiltonian circle actions as we desire:

Theorem 1.2.1. *Suppose (M, ω) is a symplectic irrational ruled surface, then the map of taking the skeleton divisor induces an one-to-one correspondence from the set of equivalent Hamiltonian circle actions to the set of equivalent S^1 generalized log Calabi-Yau divisors in it. Here the equivalence relation on divisors is generated by: (1) $D_1 \sim^s D_2$ if they are strictly symplectic deformation equivalent; (2) $D_1 \sim^F D_2$ if they differ by finite components which are all in the homology class F.*

For a fixed connected closed symplectic 4-manifold, there are finitely many inequivalent maximal Hamiltonian circle actions on it ([17],[15],[20]). We give another proof for the irrational ruled case in the language of symplectic divisors.

Theorem 1.2.2. *Fix a symplectic irrational ruled manifold (M, ω). There are only finitely many inequivalent Hamiltonian circle actions on it.*

The organization of the thesis is as follows.

In chapter 2 we review some materials on Hamiltonian group actions and then introduce (good) generalized symplectic log Calabi-Yau divisors. In section 2.1-2.2 we review the key ingredients of Hamiltonian torus actions and Hamiltonian circle actions on symplectic four-manifolds. In section 2.3, we review some facts on skeleton divisors. In section 2.4, we review some basic properties of symplectic ruled surfaces, and then introduce generalized symplectic log Calabi-Yau divisors. In section 2.5, we derive some homological conditions on the generalized symplectic log Calabi-Yau divisors. In section 2.6, we further introduce good generalized symplectic log Calabi-Yau divisors to focus on the divisors we are interested in.

In chapter 3 we prove Theorem 1.1.2 (Torelli Theorem). The idea is straightforward: prove the case for the minimal models first and then deal with the general case by minimal reduction. To record the information in the blow-up/down process, we use the language of marked divisors which have been carefully studied in [29].

In chapter 4 we prove Theorem 1.2.1 and Theorem 1.2.2. In section 4.1 we introduce S^1 generalized symplectic log Calabi-Yau divisors. In section 4.2 we show that up to certain equivalence relation they correspond to Hamiltonian circle actions. Using this correspondence, we give another proof of the finiteness of inequivalent Hamiltonian circle actions in the irrational ruled case. In section 4.3 we summerize the partial results we get for symplectic rational surfaces.

Chapter 2

Generalized symplectic log Calabi-Yau divisors

2.1 Hamiltonian group actions

Let (M, ω) be a symplectic manifold with a Lie group G acting on it by symplectomorphisms. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ its dual, and $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the pairing between the two. Any $\xi \in \mathfrak{g}$ induces a vector field ξ_M on M by:

$$\xi_M(x) = \frac{d}{dt}\bigg|_{t=0} \exp(t\xi) \cdot x.$$

Definition 2.1.1. *A **moment map** for a symplectic G-action on a symplectic manifold (M, ω) is a smooth map $\mu : M \to \mathfrak{g}^*$ such that*

(1) $d\mu^\xi = -\iota_{\xi_M}\omega$, where $\xi \in \mathfrak{g}$, ξ_M is the vector field generated by ξ and $\mu^\xi(x) = \langle \mu(x), \xi \rangle$;

(2) μ is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.

Definition 2.1.2. *A symplectic G-action on (M, ω) is an **Hamiltonian group action** if a moment map $\mu : M \to \mathfrak{g}^*$ exists.*

(M, ω, G, μ) is called a Hamiltonian G-space. The simplest case is when $G = \mathbb{T}^m$ is a torus. Some of the earliest results in the area are due to Atiyah, Guillemin and Sternberg :

Theorem 2.1.3 (Atiyah [2], Guillemin-Sternberg [13]). *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian $\mathbb{T}^m$-action. Suppose $\mu : M \to \mathbb{R}^n$ is the moment map. Then:*

(1) the level sets of μ are connected;

(2) the image of μ is the convex hull of the images of fixed points of the action.

If (M, ω) is a $2n$-dimensional symplectic manifold equipped with an effective Hamiltonian $\mathbb{T}^m$-action, then $m \leq n$.

Definition 2.1.4. *A $2n$-dimensional **symplectic toric manifold** is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian $\mathbb{T}^n$-action, with a moment map $\mu : M \to \mathbb{R}^n$.*

Symplectic toric manifolds are classified by Delzant using the Delzant polytopes, and they are all Kähler toric varieties.

Definition 2.1.5. *A **Delzant polytope** Δ in $\mathbb{R}^n$ is a convex polytope such that the slopes of the edges of each vertex are given by a basis of $\mathbb{Z}^n$.*

Theorem 2.1.6 (Delzant [6])**.** *The moment map induces an one-to-one correspondence between symplectic toric manifolds (up to equivariant symplectomorphisms) and Delzant polytopes (up to lattice isomorphisms).*

This classification result is generalized to compact connected contact toric manifolds by Lerman [24], toric non-compact symplectic toric manifolds by Karshon-Lerman [21], toric log symplectic manifolds by Gualtieri-Li-Pelayo-Ratiu [12] and presymplectic manifolds by Ratiu-Zung [41].

When $n = 2$, Delzant polytopes are polygons in $\mathbb{R}^2$ such that the slope of all edges are rational or infinite, and every two consecutive edges have integral outward normal vectors (k, b) and (k', b') with $kb' - k'b = 1$.

Example 2.1.7 ([18], Example 2.9)**.** *The dots mark the weight lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. It is easy to verify that they are Delzant polygons.*

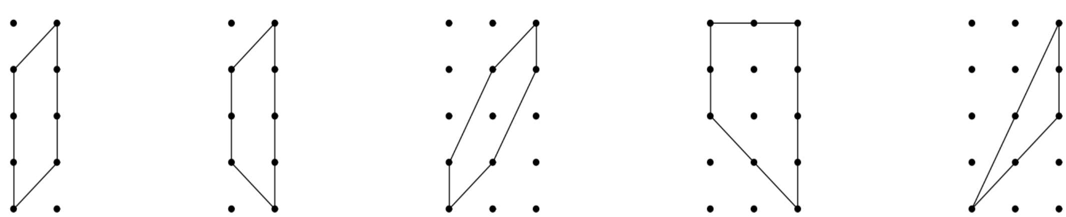

Figure 2.1: Some Delzant polygons

2.2 Hamiltonian circle actions on symplectic four-manifolds

The next breakthrough in the area of equivariant symplectic geometry is the work of Karshon [18], which classifies effective Hamiltonian circle actions on compact symplectic four manifolds. We review some basic results in this section.

2.2.1 The graph

A Hamiltonian S^1-space is a triple (M, ω, Φ) where (M, ω) is compact symplectic manifold admitting a circle action and $\Phi : M \to \mathbb{R}$ is a smooth map satisfying

$$d\Phi = -\iota(\xi_M)\omega.$$

Here ξ_M is the vector field generating the circle action, and Φ is the moment map for this action. We always assume that M is connected and the circle action is effective.

Karshon [18] gave a complete classification of compact four dimentional Hamiltonian S^1-spaces using (labelled) **graphs**. Consider a four dimensional Hamiltonian S^1-space (M, ω, Φ), each component of the fixed point set is either a single point or a symplectic surface. For each isolated fixed point, there is a vertex labelled by its moment map value. For each fixed symplectic surface Σ, there is a **fat** vertex labelled by its moment map value, its normalied symplectic area $\frac{1}{2\pi}\int_\Sigma \omega$ and its genus g.

Consider the set of points whose stabilizer is equal to the cyclic subgroup of S^1 of order $k > 1$, each connected component of the closure of this set is a closed symplectic two-sphere, on which the circle acts by rotating k times, fixing the north and south poles. This is called a $\boldsymbol{Z_k}$**-sphere**. To every Z_k-sphere, there is an edge labelled by k connecting its north and south poles in the graph.

The graph has a simple shape:

- there is a unique top vertex and a unique bottom vertex;
- there are only finitely many branchs, with moment map labels increasing along each one; a branch doesn't necessarily reach an extremal vertex;
- an extremal vertex is reached by at most two edges; a fat vertex is either the maximum component or the minimum component, and it is not reached by any edge;

We also recall the following facts regarding the labels and the isotropy weights:

- the area label of a Z_k-sphere S is $\frac{1}{k}$ times the difference of the moment map labels of its two poles;
- if there are two fat vertices, they have the same genus. If there is only one fat vertex, it must have genus 0 and the manifold is simply connected. If there are no fat vertices, the manifold is also simply connected;
- for $k > 1$, a fix point has an isotropy weight $-k$ if and only if it is the north pole of a Z_k-sphere, and a weight k if and only if it is the south pole of a Z_k-sphere;

- a fixed point has an isotropy weight 0 if and only if it lies on a fixed surface;
- two edges incident to the same vertex have relatively prime edge labels, since the action is effective.

Example 2.2.1. *Take a four-dimensional symplectic toric manifold. If we restrict the torus action to the sub-circle* $\{e\} \times S^1$ *in* $T = (S^1)^2$*, we get a compact four dimensional Hamiltonian* S^1*-space. The moment map for this* S^1*-action is the* T*-moment map composed with the projection* $\mathbb{R}^2 \to \mathbb{R}$ *to the second coordinate. The fixed surfaces are the pre-images, under the* T*-moment map, of the horizontal edges of the Delzant polygon. They have genus zero and their normalized symplectic areas are equal to the length of the corresponding horizontal edges. The isolated fixed points are the pre-images of the vertices of the polygon not lying on horizontal edges. The* Z_k*-spheres are the pre-images of edges with slope* $\pm k/b$ *in reduced form, i.e.* $(k, b) = 1$*. With this information, it is easy to construct the graphs from Delzant polygons.*

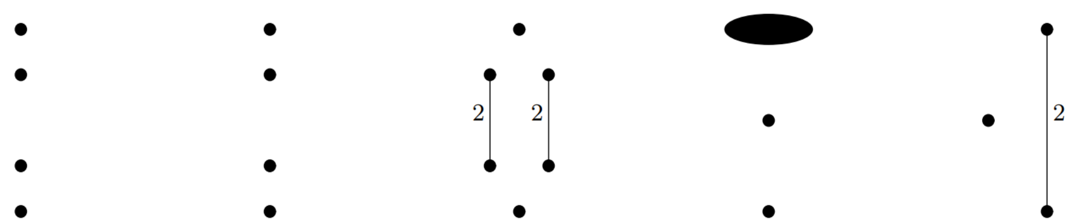

Figure 2.2: Graphs for Example 2.1.7.

2.2.2 Metrics and extended graphs

A **compatible** metric on (M, ω, Φ) is an S^1 invariant Riemann metric for which the endomorphism $J : TM \to TM$ defined by $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$ is an almost complex structure, i.e. $J^2 = -id$. Such a J is S^1 invariant. The gradient vector field of the moment map is

$$\text{grad}\Phi = -J\xi_M,$$

where J is the corresponding almost complex structure and ξ_M is the vector field generating the circle action. Since the metric is S^1 invariant, the gradient flow commutes with the circle action so they fit together into an $\mathbb{R} \times S^1 \cong \mathbb{C}^\times$ action, generated by the vector fields ξ_M and $J\xi_M$. Since these two vector fields span a symplectic subspace of T_pM for each p, each orbit of the $\mathbb{C}^\times$-action is either a fixed point or a two-dimensional symplectic sub-manifold of M. The closure of a nontrivial $\mathbb{C}^\times$ orbit is a sphere, called a **gradient sphere**. On a gradiant sphere, the circle acts by rotation with two fixed points at the north and the south poles; all other points have the same stabilizer. A gradient sphere is **free** if its stabilizer is trivial; otherwise it is **non-free**. In a compact four dimensional Hamiltonian S^1-space with a compatible metric, every Z_k-sphere is a gradient sphere and every non-free gradient sphere is a Z_k-sphere. A free gradient sphere might not be smooth at its poles (see Proposition 2.3.2).

All but finite number of gradient spheres are **trivial** gradient spheres, whose north and south poles are extrema for the moment map. The **non-trivial** ones are those with a finite non-trivial stabilizer, and those whose north or south pole is an interior fixed point. A **chain of gradient spheres** is a sequence of gradient spheres, $C_1, ..., C_l$, such that the south pole of C_1 is a minimum for the moment map, the north pole of C_{i-1} is the south pole of C_i for $1 < i \leq l$ and the north pole of C_l is a maximum of the moment map. A chain is **trivial** if it contains a single free gradient sphere; a chain is **non-trivial** if it contains more than one gradient sphere, or if it contains a non-free gradient sphere.

A compatible metric is **generic** if there exists no free gradient spheres whose north and south poles are both interior fixed points. The set of generic compatible metrics is open and dense in the space of all compatible metrics with the C^∞ topology.

The key insight of understanding four dimensional Hamiltonian S^1-spaces is due to Audin, Ahara and Hattori([1]): compact four dimentional Hamiltonian S^1-spaces are essentially determined by the arragement of gradient spheres with respect to a generic compatible Riemann metric.

Consider the graph corresponding to a compact four dimensional Hamiltonian S^1-space. Define an **extended graph** to be a graph obtained from it by adding edges with label 1, such that every interior vertex is contained in exactly two edges, the moment map labels remain monotone along each branch of edges and there are at least two chains of edges. We can get an extended graph by choosing a compatible metric (not necessarily generic) and marking a new edge with label 1 for each non-trivial free gradient sphere, and possibly for some trivial ones too. The branches in the extended graph correpond to non-trivial chains of gradient spheres. In the rest of this thesis, we always choose a generic compatible metric when we mention the extended graph.

We summerize some properties of extended graphs:

- if an edge has label 1 then it is either the first or the last one in a chain from minimum to maximum.
- for every interior fixed point that is not connected to top or bottom, there is exactly one edge from above and one edge from below, both with label > 1.
- only edges of label 1 can emanate from a fat vertex.

Example 2.2.2 ([18], Kähler toric varieties)**.** *In a Kähler toric variety, the pre-image of an edge of the Delzant polygon is a complex invariant two-sphere. Therefore, when we view the space as a Hamiltonian S^1-space, this two-sphere is either fixed by the action, or is a gradient sphere for the Kähler metric. Thus the arrangement with respect to the Kähler metric is given by the arrangement of the non-horizontal edges of the Delzant polygon. These arrangements for spaces in Example 2.1.7 are given below. Note that for the second space, the Kähler metric is not generic.*

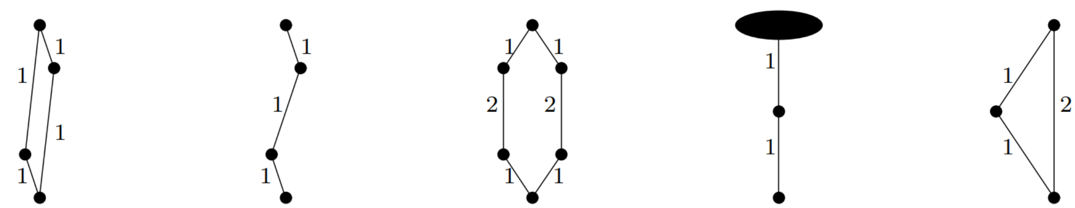

Figure 2.3: Extended graphs for Example 2.1.7.

2.3 Skeleton divisors

The easiest way to construct symplectic log Calabi-Yau divisors is by taking the boundary divisors (i.e. preimage of the boundary of a Delzant polygon) of four dimensional symplectic toric manifolds. This also works for part of compact four dimensional Hamiltonian S^1-spaces.

Definition 2.3.1. *For a four dimensional Hamiltonian S^1-space with two fixed surfaces (M, ω, Φ), we define its* ***skeleton divisor*** *to be the symplectic divisor consisting of the gradient spheres together with the fixed surfaces in the extended graph.*

The skeleton divisor depends on the choice of a generic compatible metric, and is well defined up to strict symplectic deformation equivalence (see section 3.1 and Lemma 4.1.5).

Here we follow the notations in [15]. Consider a four dimensional Hamiltonian S^1-space (M, ω, Φ). Associate the extended graphs as in section 2.2.2. The authors use the notation **extended decorated graphs** by further assuming that the minimum value of the moment map is 0.

Denote by k the number of chains of gradient spheres, l_i the number of edges in the i^{th} chain and $m_{i,j}$ the (weight) label of the j^{th} edge $\sigma_{i,j}$ from the bottom in the i^{th} chain. If there is a maximal fixed surface we order the chains such that $m_{1,1} \geq m_{2,1} \geq 1 = m_{3,1} = ... = m_{k,1}$. If there is a minimal fixed surface we order the chains such that

$m_{1,l_1} \geq m_{2,l_2} \geq 1 = m_{3,l_3} = ... = m_{k,l_k}$. If there are no fixed surfaces then $k = 2$; if there are 2 fixed surfaces, then $m_{i,1} = m_{i,l_i} = 1$ for all i so we use any order of $\{1, ..., k\}$.

Gradient spheres may not be smooth at its poles. By [1, Lemma 4.9], a gradient sphere is smooth at its poles except when it is free and the pole is an isolated minimum or maximum with both isotopy weights larger than 1. In particular, a non-free gradient sphere is smoothly embedded. An edge is called **ephemeral** if there is no embedded complex analytic sphere corresponding to it. A non-ephemeral edge corresponds to an S^1-invariant embedded symplectic sphere.

Proposition 2.3.2 ([15], Proposition 2.26). *If there are ephemeral edges in an extended decorated graph, then the number of fat vertices is exactly one. Assuming it is the maximal vertex, and order the chains so that $m_{1,1} \geq m_{2,1} \geq 1 = m_{3,1} = \cdots = m_{k,1}$. Then an edge is ephemeral if and only if it is the first edge in the i^{th}chain, $i \geq 3$, and $m_{2,1} \geq 2$.*

Using the above notations, Tara and Liat showed that the first Chern class can be given in terms of generators corresponding to edges in the extended decorated graph. They proved it by firstly showing the equivariant counterpart and then taking the map $I^* : H^*_{S^1}(M) \to H^*(M)$.

Proposition 2.3.3 ([15], Lemma 8.10). *For every Hamiltonian circle action on a symplectic four manifold (M, ω) we have*

$$c_1(M, \omega) = x_0 + x_\infty + \sum_{i=1}^{k} \sum_{j=1}^{l_i} x_{i,j} - (2g + k - 2)x_h,$$

where the classes x_0, x_∞ are Poincare duals of the fixed symplectic surfaces corresponding to the bottom and top fat vertices (set to be 0 if it is a fixed point); $x_{i,j}$ is the Poincare dual of the invariant embedded sphere corresponding to $\sigma_{i,j}$, if it is a non-ephemeral edge. In a graph with two fat vertices and zero isolated vertices, x_h is Poincare dual of a fiber class represented by an invariant embedded symplectic sphere and we have $x_h = x_{1,1} = x_{2,1}$; otherwise, denote $x_h = \sum_{j=1}^{l_1} m_{1,j} x_{1,j}$. If $\sigma_{i,j}$ is an ephemeral edge, which can only be the first edge in the i^{th} chain in a graph with one fat vertex (Proposition 2.3.2), then we define $x_{i,1} = x_h - \sum_{j=2}^{l_i} m_{i,j} x_{i,j}, i \geq 3$.

For symplectic rational surfaces, $g = 0$ and $c_1(M, \omega) = x_0 + x_\infty + \sum_{i=1}^{k} \sum_{j=1}^{l_i} x_{i,j} - (k - 2)x_h$. Note that the coefficients of the $x_{i,j}$'s are all 1 but in x_h the coeffient of $x_{i,j}$ is $m_{i,j}$. Thus, in order to get a sympletic log Calabi-Yau divisor we need that the labels $m_{i,j} = 1$ for all but 2 chains (say the first two) so that $c_1(M, \omega) = x_0 + x_\infty + \sum_{i=1}^{2} \sum_{j=1}^{l_i} x_{i,j}$.

Example 2.3.4. *A non-toric example will be $(M_4, \omega_{\epsilon_1})$, 4 points blow-ups of $(\mathbb{C}P^2, \omega_{FS})$ of equal size $\epsilon_1 < 1/3$, where you do 3 points toric blow-ups and then do 1 point S^1-equivariant* *blow-up on the minimal fixed component. By [19, Theorem 4.1], $(M_4, \omega_{\epsilon_1})$ does not admit a toric action. There will be three non-trivial chains of gradient spheres, each of length 2 and the labels of all components $m_{i,j} = 1$, so we can simply discard any one chain of the skeleton divisor and use the remaining 6 components to get a symplectic log Calabi-Yau divisor. Check Lemma 3.5 and Figure 9 of [19] for more details.*

Remark 2.3.5. *We can also derive a simple non-example from Example 2.3.4. Perform 3 S^1-equivariant blow-ups of $(M_4, \omega_{\epsilon_1})$ of small size ϵ_2 at three interior fixed points. In $(M_7, \omega_{\epsilon_1,\epsilon_2})$ we have three chains of length 3, and the weight labels along each one are 1, 2, 1. In this case we can't get a symplectic log Calabi-Yau divisor. More generally, for a four-dimensional Hamiltonian S^1-space with two fixed surfaces, we get a symplectic log Calabi-Yau divisor only when all but two chains in the extended graph have length two (so that the weight labels along them are 1,1). Then those two chains together with two fixed surfaces give us a symplectic log Calabi-Yau divisor.*

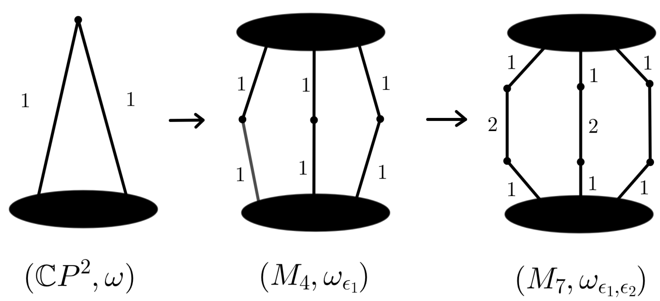

Figure 2.4: Extended graphs of $(\mathbb{C}P^2, \omega_{FS}), (M_4, \omega_{\epsilon_1}), (M_7, \omega_{\epsilon_1,\epsilon_2})$ (only with the edge labels).

2.4 Symplectic ruled surfaces

We recall some basic facts about **symplectic ruled surfaces**, following the notations in [16]. A symplectic ruled surface M is a compact symplectic four manifold M which is a S^2-bundle $\pi : M \to \Sigma$ over a compact Riemann surface Σ. Let $g = g(\Sigma)$. If $g = 0, M$ is called a **symplectic rational ruled surface**, otherwise it is called a **symplectic irrational ruled surface**. There are two S^2-bundles over Σ up to diffeomorphisms: the trivial bundle $\Sigma \times S^2$ and the non-trivial bundle M_Σ. Fix basepoints $* \in S^2$ and $* \in \Sigma$. For the trivial bundle $\Sigma \times S^2$, we denote $F = [* \times S^2], B = [\Sigma \times *]$, classes in the homology group $H_2(\Sigma \times S^2; \mathbb{Z})$. For the non-trivial S^2-bundle $\pi : M_\Sigma \to \Sigma$, we denote $F := [\pi^{-1}(*)] \in H_2(M_\Sigma; \mathbb{Z})$. For each l, the trivial bundle admits a section $\sigma_{2l} : \Sigma \to \Sigma \times S^2$ whose image $\sigma_{2l}(\Sigma)$ has even self intersection number $2l$; the non-trivial bundle admits a section $\sigma_{2l+1} : \Sigma \to M_\Sigma$ whose image $\sigma_{2l+1}(\Sigma)$ has odd self intersection number $2l+1$. We denote $B_n = [\sigma_n(\Sigma)] \in H_2$ over $\mathbb{Z}$. Then $B_n = B_{-n} + nF$.

For a non-negative integer k, denote by $(\Sigma \times S^2)_k$ the k points complex blowups of $\Sigma \times S^2$ and by $(M_\Sigma)_k$ the k points complex blowups of M_Σ. For simplicity, we abuse the notation and still call them symplectic ruled surfaces. Let $E_1, \dots, E_k$ denote the homology classes of the exceptional divisors in a fixed order.

Let $M = (\Sigma \times S^2)_k$ or $(M_\Sigma)_k$. A vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k) \in \mathbb{R}^{2+k}$ **encodes** a degree 2 cohomology class $\Omega \in H^2(M; \mathbb{R})$ if $\frac{1}{2\pi}\langle \Omega, F\rangle = \lambda_F$, $\frac{1}{2\pi}\langle \Omega, E_i\rangle = \delta_i$ for $i = 1, \dots, k$, and $\frac{1}{2\pi}\langle \Omega, B\rangle = \lambda_B$ when $M = (\Sigma \times S^2)_k$ or $\frac{1}{2\pi}\langle \Omega, B_{-1} + \frac{1}{2}F\rangle = \lambda_B$ when $M = (M_\Sigma)_k$. For $k \geq 2$, a cohomology class $\Omega \in H^2(M; \mathbb{R})$ encoded by a vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ is **in** g-**reduced form** or is g-**reduced** if

$$\delta_1 \geq \cdots \geq \delta_k, \delta_1 + \delta_2 \leq \lambda_F \tag{2.1}$$

and, if $g(\Sigma) = 0, M = (\Sigma \times S^2)_k$,

$$\lambda_F \leq \lambda_B, \tag{2.2}$$

and, if $g(\Sigma) = 0, M = (M_\Sigma)_k$,

$$\frac{1}{2}\lambda_F + \delta_1 \leq \lambda_B. \tag{2.3}$$

A symplectic form ω is in g-reduced form or is g-reduced if $[\omega]$ is.

Definition 2.4.1. *Suppose $g(\Sigma) > 0$. A **blowup form** on $(\Sigma \times S^2)_k$ or $(M_\Sigma)_k$ is a symplectic form for which there exists disjoint embedded symplectic spheres in the homology classes $F, E_1, \dots, E_k$.*

The symplectic canonical class of the blowup forms is the standard one. We have a nice characterization of the vectors encoding blowup forms on symplectic irrational ruled surfaces:

Lemma 2.4.2 ([16], Lemma 4.8)**.** *Assume that $g(\Sigma) > 0$. A vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ encodes the cohomology class of a blowup form on $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$ if and only if*

(i) $\lambda_F, \lambda_B, \delta_1, \dots, \delta_k > 0$;

(ii) $\lambda_F > \delta_i$ for all i;

(iii) the volume inequality $\lambda_F\lambda_B - \frac{1}{2}(\delta_1^2 + \cdots + \delta_k^2)$ holds.

Holm and Kessler give an algorithm that turns a blowup form into its unique g-reduced form, by a composition of permutations of the exceptional divisors and the Cremona transformation:

Proposition 2.4.3 ([16], Theorem 2.14)**.** *Let $k \geq 2$ be an integer, and let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. Assume $g(\Sigma) > 0$. Given a blowup form ω on M, there exists a unique blowup form ω' that is g-reduced such that $(M, \omega) \cong (M, \omega')$.*

Note that the symplectomorphism that turns ω to ω' preserves the fiber class F. Thus we may always assume that a blowup form ω is g-reduced when we are working on the symplectic irrational ruled surfaces. However, this is not true in the rational case (see Remark 4.3.13).

Definition 2.4.4. *Two symplectic forms ω_1 and ω_2 on M are **equivalent** if there exists a diffeomorphism f of M acting trivially on the homology $H^2(M)$, such that $f^*(\omega_2)$ and ω_1 are homotopic through symplectic forms.*

Lemma 2.4.5. *The (non-empty) set of blowup forms on M is an equivalence class of symplectic forms.*

The proof for the irrational ruled case is in [16]; the proof for the rational case is in [20].

Fix a compact symplectic four-manifold (M, ω). An almost complex structure J on M is **tamed** by ω if $\omega(u, Ju) > 0$ for all nonzero $u \in TM$. Let $\mathcal{J}_\tau(M, \omega)$ denote the set of almost complex structures J that are tamed by ω, and let $\mathcal{J}(M)$ denote the union of $\mathcal{J}_\tau(M, \omega)$ over all blowup forms ω. The first Chern class $c_1(TM, J)$ is the same for all $J \in \mathcal{J}_\tau(M)$ since $\mathcal{J}_\tau(M)$ is contractible. It follows that this first Chern class and the Gromov-Witten invariant are the same for all blowup forms on $M = (\Sigma \times S^2)_k$ or $(M_\Sigma)_k$. We denote the first Chern class and the Gromov-Witten invaraint associated to any blowup form on M by $c_1(M, \omega)$ and GW.

An **exceptional sphere** in a symplectic four-manifold (M, ω) is an embedded ω-symplectic sphere of self intersection -1. A homology class $E \in H_2(M)$ is **exceptional** if it is represented by an exceptional sphere. The set $\mathcal{E}(M)$ in $H_2(M)$ of exceptional classes is the same for all blowup forms:

Lemma 2.4.6 ([20], Lemma 2.9)**.** *Let M be a compact four-manifold, equipped with an equivalence class $\mathcal{SF}_M$ of symplectic forms . Let $E \in H_2(M)$ be a homology class. Then the following are equivalent:*

(a) There exists a symplectic form $\omega \in \mathcal{SF}_M$ such that the class E is represented by an embedded ω-symplectic sphere with self intersection -1.

(b) (i) $c_1(M, \omega)(E) = 1$;

(ii) $E \cdot E = -1$;

(iii) the genus zero Gromov-Witten invariant $GW(E) \neq 0$.

(c) For every symplectic forms $\omega' \in \mathcal{SF}_M$, the class E is represented by an embedded ω'-symplectic sphere with self intersection -1.

For symplectic irrational ruled surfaces, there is an explicit identification of the exceptional classes by Biran ([3, Corollary 5.C]):

$$\mathcal{E}(M) = \{E_1, \ldots, E_k, F - E_1, \ldots, F - E_k\}. \tag{2.4}$$

2.5 Generalized symplectic log Calabi-Yau divisors

Suppose $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$, ω is a blowup form on M that is g-reduced, and $D = \cup_{i=1}^m C_i$ is a generalized symplectic log Calabi-Yau divisor in (M, ω). Denote $g = g(\Sigma), g_i = g(C_i)$. In this section, we derive some homological conditions on the components of D. We need the following facts:

Fact 2.5.1. *(1) $\{F, B, E_1, \ldots, E_k\}$ form a basis of $H_2((\Sigma \times S^2)_k)$. $\{F, B_{-1}, E_1, \ldots, E_k\}$ form a basis of $H_2((M_\Sigma)_k)$. Recall that we have $B_n = B + \frac{n}{2}F \in H_2((\Sigma \times S^2)_k)$ for even n and $B_n = B_{-1} + \frac{n+1}{2}F \in H_2((M_\Sigma)_k)$ for odd n.*

(2) The intersections numbers are as follows, where $i \neq j$: $F \cdot F = B \cdot B = F \cdot E_i = B \cdot E_i = E_i \cdot E_j = 0, F \cdot B = F \cdot B_n = -E_i^2 = 1, B_n \cdot B = \frac{n}{2}, B_n^2 = n$.

(3) $c_1(M,\omega)(F) = 2, c_1(M,\omega)(E_i) = 1, \forall 1 \leq i \leq k$. In $(\Sigma \times S^2)_k, c_1(M,\omega)(B) = 2 - 2g$. In $(M_\Sigma)_k, c_1(M,\omega)(B_{-1}) = 1 - 2g$. Combined with (2) we have

$$\mathrm{PD}(c_1(M,\omega)) = (2-2g)F + 2B - \sum_{i=1}^k E_i$$

in $(\Sigma \times S^2)_k$,

$$\mathrm{PD}(c_1(M,\omega)) = (3-2g)F + 2B_{-1} - \sum_{i=1}^k E_i$$

in $(M_\Sigma)_k$.

We don't need to discuss $(\Sigma \times S^2)_k$ and $(M_\Sigma)_k$ seperately since they are symplectomorphic:

Lemma 2.5.2 ([16], Corollary 5.4)**.** *Let $k \geq 1$. The symplectic manifold $(M_\Sigma)_k$ (resp. $(\Sigma \times S^2)_k$) with a blowup form ω with $[\omega]$ encoded by $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ is symplectomorphic to $(\Sigma \times S^2)_k$ (resp. $(M_\Sigma)_k$) with a blowup form ω' with $[\omega']$ encoded by $(\lambda_F, \lambda_B + \frac{1}{2}\lambda_F - \delta_1; \lambda_F - \delta_1, \ldots, \delta_k)$. Moreover if $k \geq 2$ and $[\omega]$ is in g-reduced form, then so is $[\omega']$.*

By Lemma 2.5.2, we may always assume that $M = (\Sigma \times S^2)_k$ when M is not minimal. Recall that by definition we have $[D] = \sum_{i=1}^m [C_i] = \mathrm{PD}(c_1(M,\omega)) + pF$.

Proposition 2.5.3. *(1) $\sum_{i=1}^m [C_i] \cdot F = 2$.*

(2) $[C_i] \cdot F \geq 0$ *for all* i.

(3) $[C_i] \cdot (\sum_{j \neq i}[C_j]) = 2 - 2g_i + p([C_i] \cdot F)$ *for all* i.

Proof. (1) Fiber classes satisfy $F \cdot F = 0, F \cdot c_1(M, \omega) = 2$, thus we have $\sum_{i=1}^m [C_i] \cdot F = (\mathrm{PD}(c_1(M, \omega)) + pF) \cdot F = c_1(M, \omega) \cdot F + pF \cdot F = 2$.

(2) This is Proposition 3.2 of [44].

(3) By the adjunction formula, we have $[C_i] \cdot [D] = [C_i] \cdot (\mathrm{PD}(c_1(M, \omega)) + pF) = [C_i]^2 + 2 - 2g_i + p([C_i] \cdot F)$. Subtracting $[C_i]^2$ from both sides we get the desired equality. □

By Proposition 2.5.3(1)(2), we have at most two **section-type components**, i.e. $[C_i] \cdot F > 0$ and the others are **fiber-type components**, i.e. $[C_i] \cdot F = 0$. $\cup_{i=1}^m C_i$ is called a **chain** of fiber-type components if each C_i is a fiber-type component with C_i intersecting C_{i+1} for $1 \leq i \leq m-1$, and C_1, C_m each intersecting a section-type component. A chain is **trivial** if it consists of a single fiber-type component in the homology class F. Otherwise it is called **nontrivial**.

• Case (1). There exists a section-type component with $[C_i] \cdot F = 2$. After possible relabelling we assume $[C_1] \cdot F = 2$ and $[C_i] \cdot F = 0$ for $i > 1$. By Proposition 2.5.3(3) we have $[C_1] \cdot (\sum_{j>1}[C_j]) = 2 - 2g_1 + 2p, [C_i] \cdot (\sum_{j \neq i}[C_j]) = 2 - 2g_i$ for $i > 1$.

If $m = 1$ and $D = C_1$, then $2 - 2g_1 + 2p = 0$ so that $g_1 = 1 + p$. For instance, in $M = \Sigma \times S^2, [C_1] = \mathrm{PD}(c_1(M, \omega)) + pF = (2 - 2g + p)F + 2B$ is represented by an embedded symplectic surface of genus $1 + p$, where $p \geq 2g$.

If $m > 1$, then $[C_1] \cdot (\sum_{j>1}[C_j]) = 2 - 2g_1 + 2p > 0$ and $[C_i] \cdot (\sum_{j \neq i}[C_j]) = 2 - 2g_i > 0$ by connectedness of D, thus $1 - g_1 + p > 0$ and $g_i = 0$ for $i > 1$. For any fiber-type component $[C_i]$, we can extend it into a **U-shape chain** until two end components of the chain hit C_1. Each chain will contribute 2 to $[C_1] \cdot (\sum_{j>1}[C_j])$, hence there are $(1 - g_1 + p)$ U-shape chains. For instance, in $M = \Sigma \times S^2, [D] = [C_1] + [C_2] = \mathrm{PD}(c_1(M, \omega)) + pF = 2F + 2B$, where $p = 2g, [C_1] = F + 2B$ is represented by an embedded symplectic surface of genus $g_1 = 2g$, $[C_2] = F$ is the fiber class.

• Case (2). There are two section-type components. After possible relabelling we assume that $[C_1] \cdot F = [C_2] \cdot F = 1$, then by Proposition 2.5.3(3) $[C_1] \cdot (\sum_{j>1}[C_j]) = 2 - 2g_1 + 2p, [C_2] \cdot (\sum_{j \neq 2}[C_j]) = 2 - 2g_2 + 2p, [C_i] \cdot (\sum_{j \neq i}[C_j]) = 2 - 2g_i$ for $i > 2$.

If $m = 2$, then $[C_1] \cdot [C_2] = 2 - 2g_1 + 2p = 2 - 2g_2 + 2p$, thus $g_1 = g_2$. For instance in $M = \Sigma \times S^2, [D] = [C_1] + [C_2] = \mathrm{PD}(c_1(M, \omega)) + pF = 2F + 2B$, where $p = 2g, [C_1] = 2F + B$ and $[C_2] = B$ are both represented by embedded symplectic surfaces of genus g.

If $m > 2$ then $1 - g_1 + p > 0, 1 - g_2 + p > 0$ and $g_i = 0$ for $i > 2$ by connectedness of D. As in case (1) we can extend any fiber-type component into a a chain until two end components hit either C_1 or C_2. Two end components are either adjacent to one of C_1, C_2 (**U-shape chain**) or both of them (**I-shape chain**). For instance in $M = \Sigma \times S^2, [D] =$

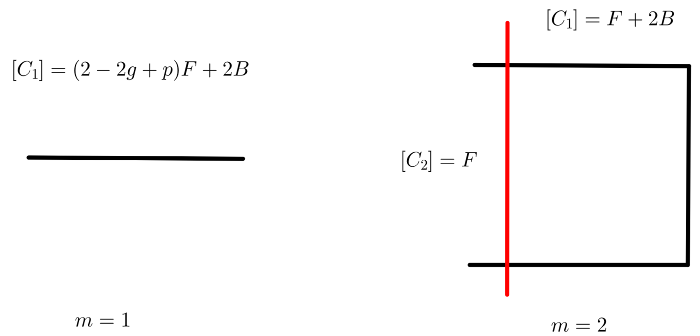

Figure 2.5: Examples of generalized symplectic log Calabi-Yau divisors in case (1).

$[C_1]+[C_2]=\mathrm{PD}(c_1(M,\omega))+2gF=2F+2B, [C_1]=F+B$ and $[C_2]=B$ are represented by embedded symplectic surfaces of genus g, $[C_3]=F$ is the fiber class.

Since in the graphs of symplectic irrational ruled surfaces, there are always two fat vertices, we will focus on case (2) in the rest of this thesis. Given any chain, two end components can not intersect a section-type component at the same point, otherwise they will have intersections larger than two with the rest of the divisor. If they intersect at two distinct points with one section-type component then we get a U-shape chain. We show that this cannot happen:

Lemma 2.5.4. *Suppose $\cup_{j=1}^{i_k} C_{i_j}$ is a chain of fiber-type components in $M=(\Sigma\times S^2)_k$, with two end components not intersecting each other. Suppose C_1 is a section-type component, then $[C_1]\cdot F=1$ implies that $[C_1]=cF+B-\sum_{i=1}^k c_iE_i$ with $c, c_i\in\mathbb{Z}$. Let $A=\sum_{j=1}^{i_k}[C_{i_j}]$. Then we have:*

(1) $g(C_1)=g(\Sigma)=g$ and $c_i\in\{0,1\}$ for all i.

(2) $A=F-\sum_{i=1}^k b_iE_i$ or $E_i-\sum_{j>i} b_jE_j$, where $1\le i\le k, b_i\in\{0,1\}$. Thus $A\cdot[C_1]<2$ and the chain can not be of U-shape (i.e. two end components both intersecting C_1).

Proof. (1) Consider the projection $f: C_1\to\Sigma$. Its mapping degree is $[C_1]\cdot F=1$. By Kneser's theorem we have $2g(C_1)-2\ge 1\cdot(2g-2)$, thus $g(C_1)\ge g$. By the adjunction formula we have $c_1(M,\omega)([C_1])=[C_1]^2+2-2g(C_1)$ which simplifies to $\sum_{i=1}^k(c_i^2-c_i)+2(g(C_1)-g)=0$. The claim follows since $c_i^2-c_i\ge 0$ for all i, and the equality holds if and only if $c_i\in\{0,1\}$.

(2) By previous discussion in case (2) we know each C_{i_j} is an embedded symplectic sphere. We can assume $i_k=1$ due to an elementary fact: if $[C_i],[C_j]$ are represented by embedded symplectic spheres with $[C_i]\cdot[C_j]=1$, then so is $[C_i]+[C_j]$. Thus we may assume that A is represented by a single fiber-type component. $A\cdot F=0$ so we have

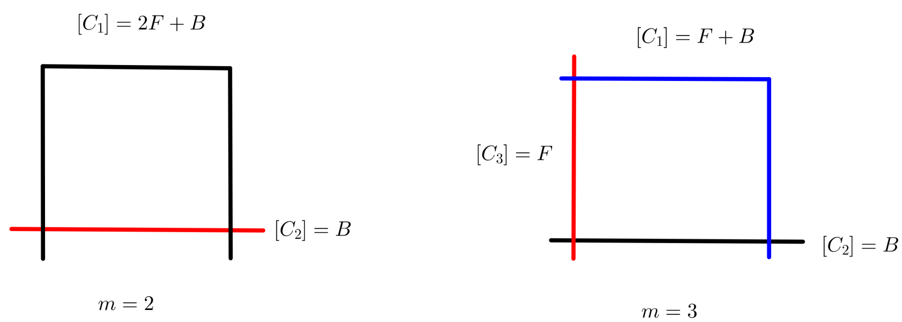

Figure 2.6: Examples of generalized symplectic log Calabi-Yau divisors in case (2).

$A = aF - \sum_{i=1}^k b_i E_i$ with $a, b_i \in \mathbb{Z}$. There are three cases:

(i) $A^2 \geq 0$. By positive intersection with $\{E_i, F - E_i, 1 \leq i \leq k\}$, we have $b_i = 0$ for all i. Then we have $a = 1$ by the adjunction formula.

(ii) $A^2 < 0, a \geq 0$. The adjunction formula for A simplifies to $2a + \sum_{i=1}^k (b_i^2 - b_i) = 2$. Since $b_i^2 - b_i \geq 0$ for all i, $a = 0$ or 1. If $a = 0$, then $\sum_{i=1}^k (b_i^2 - b_i) = 2, \omega(A) > 0$ and the reduced condition on ω implies that there exists some i with $b_i = -1$ and $b_j = 0$ for $j < i$, $b_j \in \{0, 1\}$ for $j > i$. Thus $A = E_i - \sum_{j>i} b_j E_j$. If $a = 1$, then $A = F - \sum_{i=1}^k b_i E_i$. Since $b_i^2 - b_i \geq 0$, we must have $b_i \in \{0, 1\}$ for $1 \leq i \leq k$.

(iii) $A^2 < 0, a < 0$. We will exclude this possibility. We adopt the argument in [4, Lemma 3.4]. Let $b_i^- = \max\{0, -b_i\}$ and $b_i^+ = \max\{0, b_i\}$, and consider the class $\tilde{A} = |a|F - \sum_{i=1}^k (b_i^- + b_i^+) E_i$. Note that $b_i^- = -b_i$ when $b_i < 0$ and equals 0 otherwise, $b_i^+ = b_i$ when $b_i > 0$ and equals 0 otherwise. Thus $\tilde{A}^2 = A^2$ and $\tilde{A}$ is the image of $-A$ under the action of the composition of the reflections $R(E_i)$, where i runs over the set of indices such that $b_i > 0$. Here the reflection $R(E_i)$ on $H_2(M)$ is defined by: $R(E_i)\beta = \beta + 2(\beta \cdot E_i)E_i$. Since $R(E_i)$ and $-id$ are induced by orientation-preserving diffeomorphisms of M (cf. [25]), $\tilde{A}$ is represented by a smoothly embedded sphere.

Pick a sufficiently small $\epsilon > 0$, and let $e = F + B - \epsilon \sum_{i=1}^k E_i$. We claim that PD$(e)$ lies in the symplectic cone associated to the symplectic canonical class $c_1(K_\omega) = -c_1(M, \omega)$. We need to verify that $e^2 > 0$ and $e \cdot E > 0$ for any $E \in \mathcal{E}(M)$ ([28]). $e^2 = 2 - k\epsilon > 0$ for sufficiently small ϵ; $e \cdot E_i = \epsilon > 0, e \cdot (F - E_i) = 1 - \epsilon > 0$ for sufficiently small ϵ.

We also have $e \cdot \tilde{A} = |a| - \epsilon \sum_{i=1}^k (b_i^- + b_i^+) > 0$ for sufficiently small ϵ. This together with the fact that e lies in the symplectic cone associated to $c_1(K_\omega)$ imply the following inequality on the symplectic genus $\eta(\tilde{A})$ of $\tilde{A}$([25, Definition 3.1]):

$$\eta(\tilde{A}) \geq \frac{1}{2}(\tilde{A}^2 - c_1(M,\omega)\cdot\tilde{A}) + 1. \tag{2.5}$$

On the other hand, the minimal genus is bounded from below by the symplectic genus ([25, Lemma 3.2]). Thus, by combining with the adjuction formula for A, $\eta(\tilde{A}) \leq 0 = \frac{1}{2}(A^2 - c_1(M,\omega)\cdot A) + 1$, so that $c_1(M,\omega)\cdot A \leq c_1(M,\omega)\cdot\tilde{A}$, which simplifies to

$$\sum_{i=1}^{k} b_i^- \leq 2|a|. \tag{2.6}$$

The adjunction formula for A gives

$$\sum_{i=1}^{k} b_i(b_i - 1) = 2 + 2|a|, \tag{2.7}$$

thus $b_i^-(b_i^- - 1) \leq b_i(b_i - 1) \leq 2(1+|a|)$ and $b_i^- \leq 1 + |a|$ for all i.

Suppose $b_i^- \leq |a|$ for all i. Then we can write A as $A = -(|a|F - \sum_{i=1}^k b_i^- E_i) - \sum_{i=1}^k b_i^+ E_i$, where the class $|a|F - \sum_{i=1}^k b_i^- E_i$ can be written as a sum of classes of the form $F, F - E_i, F - E_i - E_j, i \neq j$ by (2.6). Since all these classes have non-negative symplectic area by the reduced condition, it follows that $\omega(A) \leq 0$, which is a contradiction. Thus $b_i^- = |a|+1$ for some i. As a result, $b_i = -(|a|+1)$ and $b_i(b_i-1) = (|a|+1)(|a|+2) \leq 2+2|a|$ by (2.7), hence $|a| \leq 0$. Contradiction. □

Remark 2.5.5. *(1) We actually prove that the homology class of any fiber type component is $F - \sum_{i=1}^k b_i E_i$ or $E_i - \sum_{j>i} b_j E_j$, where $1 \leq i \leq k, b_i \in \{0,1\}$.*

(2) We use the reduced condition on ω when $k \geq 2$. When $k = 0, 1$, it can be easily verify that the same arguments still work. Note that the reduced condition is essential in the proof. On symplectic irrational ruled surfaces, the only irreducible enmbedded symplectic spheres with nonnegative self-intersections are in the fiber class F ([44, Corollary 3.3]). If we only assume that ω is a blowup form and A is represented by an embedded symplectic sphere with $A^2 = -l, l \geq 1$, then up to permutations of E_i, $A = bF + \sum_{i=1}^{1-b} E_i - \sum_{j=2-b}^{l} E_j$ for some integer $b \leq 1$ ([7, Chapter 6]).

2.6 Good generalized symplectic log Calabi-Yau divisors

Suppose we have a generalized symplectic log Calabi-Yau pair (M,ω,D) with two section-type components C_1, C_2. By Lemma 2.5.4, $[C_1] = cF + B - \sum_{i=1}^k c_i E_i, [C_2] = dF + B - \sum_{i=1}^k d_i E_i$, where $c_i, d_i \in \{0,1\}$. Recall that in the graphs of symplectic irrational ruled surfaces, two fixed surfaces do not intersect each other, thus we should require that $[C_1] \cdot [C_2] = 0$. But this is not enough: C_1, C_2 might intersect in the blow-downs (check section 3.3 for details of blowups/blowdowns). For instance in $M = (\Sigma \times S^2)_1$, two disjoint section-type components might intersect after we blow down E_1(see the figure below). To avoid this, we need $c + d = 0$, or equivalently $([C_1] + [C_2]) \cdot B = 0$.

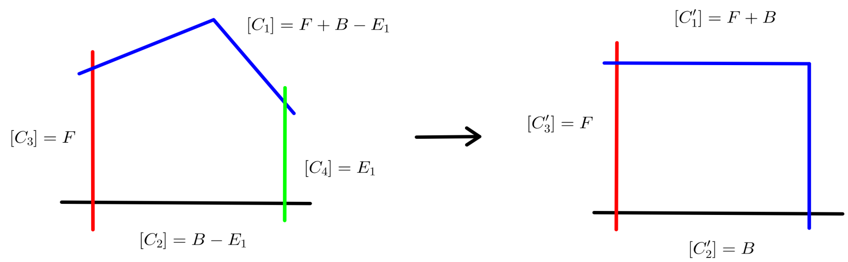

Figure 2.7: Blowing down E_1 in $(\Sigma \times S^2)_1$.

Definition 2.6.1. *Let (M, ω, D) be a generalized symplectic log Calabi-Yau pair. We call D a **good** generalized symplectic log Calabi-Yau divisor if it has two disjoint section-type components C_1, C_2, with $([C_1] + [C_2]) \cdot B' = 0$, where $B' = B_n + B_{-n}$. Here n is any even (resp. odd) integer if $M = (\Sigma \times S^2)_k$ (resp. $M = (M_\Sigma)_k$). In this case we call (M, ω, D) a good generalized symplectic log Calabi-Yau pair.*

Let $l = p + 2 - 2g$, then by definition $2 \le l \le \max\{2, k\}$, $[D] = \mathrm{PD}(c_1(M, \omega)) + pF = lF + 2B - \sum_{i=1}^k E_i$. Recall that each I-shape chain contributes 1 to the total intersetions of C_1, C_2 with the rest of the symplectic divisor, thus by Proposition 2.5.3(3) and Lemma 2.5.4(1) there are l I-shape chains. Suppose $\cup_{j=1}^{i_k} C_{i_j}$ is such a chain and let $A = \sum_{j=1}^{i_k} [C_{i_j}]$. Then the latter case in Lemma 2.5.4(2) cannot happen. Suppose $A = E_i - \sum_{j>i} b_j E_j$. $A \cdot [C_1] = A \cdot [C_2] = 1$ implies that $c_i = d_i = 1$, contradicting $[C_1] \cdot [C_2] = 0$.

Suppose $[C_1] = -nF + B - \sum_{i \in S_{-1}} E_i, [C_2] = nF + B - \sum_{i \in S_0} E_i$ and the j-th I-shape chain has homology summing up to $A_j = F - \sum_{i \in S_j} E_i$ for $1 \le j \le l$. The homological conditions $[D] = [C_1] + [C_2] + \sum_{i=1}^l A_i = lF + 2B - \sum_{i=1}^k E_i, [C_1] \cdot [C_2] = 0, A_i \cdot [C_1] = A_i \cdot [C_2] = 1, A_i \cdot A_j = 0$ for $i \ne j$, imply that $S_i \cap S_j = \emptyset$ for $-1 \le i \ne j \le l$, $\cup_{i=-1}^l S_i = \{1, \cdots, k\}$.

Chapter 3

Moduli space of good generalized symplectic log Calabi-Yau divisors

3.1 Homotopy of symplectic divisors

In this section we review several types of homotopies of symplectic divisors in [29]. They will be used to show that the deformation class of a marked symplectic divisor is stable under various operations.

- A **symplectic homotopy** (resp. **isotopy**) of (M, ω, D) is a smooth one parameter of symplectic divisors (M, ω_t, D_t) with $(M, \omega_0, D_0) = (M, \omega, D)$ (resp. such that in addition $\omega_t = \omega$ for all t).

- (M', ω', D') is **symplectic deformation equivalent** to (M, ω, D) if it is symplectomorphic to (M, ω_1, D_1) for some symplectic homotopy (M, ω_t, D_t) of (M, ω, D). The symplectic deformation equivalence is **strict** if the symplectic homotopy is a symplectic isotopy.

- Two symplectic log Calabi-Yau pairs $(M^i, \omega^i, D^i = \cup_{j=1}^k C_j^i)$ for $i = 1, 2$ are **homological equivalent** if there is a diffeomorphism $\Phi : M^1 \to M^2$ such that $\Phi_*[C_j^1] = [C_j^2]$ for $j = 1, \cdots, k$. The homological equivalence is **strict** if $\Phi^*[\omega^2] = [\omega^1]$.

We also consider the more restrictive homotopies keeping D fixed:

- D-symplectic homotopy (M, ω_t, D), and

- D-symplectic isotopy (M, ω_t, D) with constant $[\omega_t]$.

To compare these notions we have:

Definition 3.1.1. *Two symplectic homotopies are **symplectomorphic** if they are related by an one parameter family of symplectomorphisms.*

Using the Moser isotopy lemma and smooth isotopy extension theorem it can shown that they are actually equivalent:

Lemma 3.1.2 ([29], Lemma 2.2)**.** *A symplectic homotopy (resp. isotopy) of a symplectic divisor is symplectomorphic to a D-symplecitc homotopy (resp. isotopy) and vice versa.*

3.2 Blow-up/down of symplectic divisors

Throughout this thesis, we use the following notions for symplectic blow-ups/blow-downs of $D \subset (M, \omega)$.

For a generalized symplectic log Calabi-Yau divisor $D = \cup C_i$ in (M, ω), an $\boldsymbol{S^1}$ **blow-up** of D is the total transform of a symplectic blow-up centered at an intersection point of D.

A **non-$\boldsymbol{S^1}$ blow-up** of D is the proper transform of a symplectic blow-up centered at a smooth point of D.

Here for blow-ups at a smooth point p on the divisor D, we first do a C^0 small perturbation of D to D' fixing p and then we do a symplectic blow-up of a ball centered at p so that D' is a complex subspace in the local coordinate given by the ball. For blow-ups at an intersection point, a C^0 small perturbation is performed so that D' is ω-orthogonal at p and D' coincide with two complex subspaces in the local coordinates given by the ball.

An $\boldsymbol{S^1}$ **blow-down** refers to blowing down an exceptional sphere (in homology class E) contained in D that intersects exactly two other irreducible components and exactly once for each of them. Such an exceptional sphere is called an S^1 exceptional sphere.

A **non-$\boldsymbol{S^1}$ blow-down** refers to blowing down an exceptional sphere not contained in D that intersects exactly one component and exactly once with positive and tranversal intersection. Such an exceptional sphere is called a non-S^1 exceptional sphere.

Similar to blow-ups, for blow-down of a S^1 or non-S^1 exceptional sphere E, we first perturb the symplectic divisor D to D' (or perturbing E) so that the intersections of D' and E are ω-orthogonal (if E is an irreducible component of D, we require that E has ω-orthogonal intersections with all other irreducible components). Then we do the symplectic blow-down of E and D' will descend to a symplectic divisor.

Definition 3.2.1. *An exceptional class E is called **non-$\boldsymbol{S^1}$** if E has trivial intersection pairing with all but one of the homology classes of the irreducible components of D and the only non-trivial pairing is 1.*

An exceptional class E is called $\boldsymbol{S^1}$ if E is cohomologous to an irreducible component of D such that it pairs non-trivially with the homology classes of exactly two other irreducible components and the pairings are 1.

The homology class of an (non-)S^1 exceptional sphere is an (non-)S^1 exceptional class. Conversely, for an S^1 exceptional class E, the component of D with class E is an S^1 exceptional sphere in the class E; for a non-S^1 exceptional class E, we have an exceptional sphere in the class E when D is ω-orthogonal.

Lemma 3.2.2. *Let D be an ω-orthogonal symplectic divisor. There is a non-empty subspace $\mathcal{J}(D)$ of the space of ω-tamed almost complex structure making D J-holomorphic such that for any non-S^1 exceptional class E, there is a residue subset $\mathcal{J}(D,E) \subset \mathcal{J}(D)$ so that E has an embedded J-holomorphic representative for all $J \in \mathcal{J}(D,E)$.*

Proof. E is D-good in the sense of Definition 1.2.4 in [38]. Then it follows from Theorem 1.2.7 of [38]. □

3.3 Marked symplectic divisors and symplectic Torelli theorem

In [29] various operations on symplectic log Calabi-Yau pairs and marked divisors are introduced and it is showed that the (strict) symplectic deformation equivalance classes are determined by their (strict) homological classes. In this section we show that similar results hold for generalized symplectic log Calabi-Yau pairs.

Theorem 3.3.1 ([29], Theorem 1.4)**.** *Let (M^i, ω^i, D^i) be symplectic log Calabi-Yau surfaces for $i = 1, 2$. Then they are (strictly) symplectic deformation equivalent if and only if they are (strictly) homological equivalent.*

Definition 3.3.2. *A **marked symplectic divisor** is a 5-tuple*

$$\Theta = (M, \omega, D, \{p_j\}, \{I_j\}),$$

such that

(1) D is an ω-orthogonal symplectic divisor in (M, ω);

(2) $\{p_j\}$, called the centers of marking, are points on D (intersection points are allowed);

(3) $I_j : (B(\delta_j), \omega_{std}) \to (M, \omega)$, called the coordinates of marking, are symplectic embeddings sending the origin to p_j and with $I_j^{-1}(D) = \{z = 0\} \cap B(\delta_j)$ (resp. $I_j^{-1}(D) = (\{z = 0\} \cup \{w = 0\}) \cap B(\delta_j)$) if p_j is a smooth (resp. an intersection) point of D, where (z, w) is the complex coordinate for $B(\delta_j)$. Moreover, we require that the I_j's have disjoint images.

Definition 3.3.3. *Let $\Theta = (M, \omega, D, \{p_j\}, \{I_j\})$ be a marked divisor. A **D-symplectic homotopy** (resp. **D-symplectic isotopy** of Θ is a 4-tuple $(M, \omega_t, D, \{p_j\})$ such that ω_t is a smooth family of (resp. cohomologous) symplectic forms on M with $\omega_0 = \omega$ and D*

*being ω_t-symplctic for all t. Two marked symplectic divisors are **(strict) D-symplectic deformation equivalent** if they are symplectomorphic up to a D-symplectic homotopy (isotopy).*

For any symplectic divisor, we can do a C^0 small perturbation among the strict D-symplectic deformation class to make the intersection points ω-orthogonal([10]). Thus we may assume a symplectic divisor to be ω-orthogonal when necessary. Every ω-orthogonal symplectic divisor is naturally a marked divisor with $\{p_j\}$ consisting of all intersection points and I_j exsits since D is ω-orthogonal. For simplicity, we denote a marked symplectic divisor as (M, ω, D, p_j, I_j) or Θ and call it a marked divisor if there is no confusion.

An ω-orthogonal symplectic divisor can also been viewed as a marked divisor with empty markings.

Lemma 3.3.4 ([29], Lemma 2.7)**.** *Two symplectic divisors are (strictly) symplectic deformation equivalent if and only if they are (strictly) D-deformation equivalent as marked symplectic divisors (with empty markings).*

For marked divisors, both D-symplectic deformation equivalence and its strict version do not involve the symplectic embeddings I_j. There is a seemingly stronger definition of deformation:

Definition 3.3.5. *A **strong D-symplectic homotopy (isotopy)** is a 5-tuple $\Theta = (M, \omega_t, D, \{p_j\}, \{I_{j,t}\})$ such that*

(1) the 4 tuple $\Theta = (M, \omega_t, D, \{p_j\}$ is a D-symplectic homotopy (isotopy) of Θ;

(2) D is ω_t-orthogonal;

(3) $I_{j,t} : (B(\epsilon_j), \omega_{std}) \to (M, \omega)$ are symplectic embeddings sending the origin to p_j, $I_{j,0} = I_j$ and $I_{j,t}^{-1}(D) = \{z = 0\} \cap B(\epsilon_j)$ (resp. $I_{j,t}^{-1}(D) = (\{z = 0\} \cup \{w = 0\}) \cap B(\epsilon_j)$) if p_j is a smooth (resp. an intersection) point of D, for some $\epsilon_j < \delta_j$, where (z, w) is the complex coordinate for $B(\delta_j)$.

*Two marked symplectic divisors are **strong (strict) D-symplectic deformation equivalent** if they are symplectomorphic up to a strong D-symplectic homotopy (isotopy).*

These two deformations turn out to be the equivalent:

Lemma 3.3.6 ([29], Lemma 2.9)**.** *If two marked symplectic divisors are (strict) D-symplectic deformation equivalent, then they are strong (strict) D-symplectic deformation equivalent.*

This lemma combined with the symplectic ball extention theorem ([39, Theorem 3.3.1]) implies that:

Lemma 3.3.7 ([33], Lemma 2.9)**.** *Two strong D-symplectic isotopic ω-orothogonal symplectic divisors are symplectomorphic. In particular, they are Hamiltonian diffeomorphic if $H^1(M; \mathbb{R}) = 0$.*

These two lemmas above imply the following:

Proposition 3.3.8 ([33], Proposition 2.10)**.** *Two ω-orthogonal symplectic divisors D and D' in (M,ω) are strictly symplectic deformation equivalent if and only if they are symplectomorphic. Moreover, two general symplectic divisors are strictly symplectic deformation equivalent if and only if they are symplectomorphic after a small symplectic isotopy locally supported near the intersection points.*

In the holomorphic category, we have a version of Torelli Theorem roughly saying that two log Calabi-Yau pairs are isomorphic if and only if there is an admissible integral isomotry between the second integral cohomology, which maps the homology of the components of one divisor to the other's ([8, Theorem 8.5]). Here an integral isometry means automorphism preseving the intersection form.

On a closed four-manifold M, each diffeomorphism induces an automorphism of the lattice of the second integral cohomology, which induces a natural map $\mathrm{Diff}(M) \to A(M) = \mathrm{Aut}(H^2(M;\mathbb{Z}))$. This map is not always surjective. For example, for rational surfaces it is surjective only when the Euler characteristic $\chi \leq 12$ ([9],[43]). Let $\mathrm{D}(M)$ be the image of this natural map. Using Proposition 3.3.7 and a characterization of $\mathrm{D}(M)$ in [25], a much stronger version of Theorem 3.3.1 is obtained:

Proposition 3.3.9 ([33], Proposition 2.11)**.** *Let $D = \cup C_i, D' = \cup C_i'$ be two symplectic log Calabi-Yau divisors in (M,ω) and an automorphism*

$$\gamma : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$$

be such that $\gamma(\mathrm{PD}([C_i])) = \mathrm{PD}([C_i'])$ and $\gamma([\omega]) = [\omega]$. Then D and D' are strictly symplectic deformation equivalent. They are symplectomorphic if they are both ω-orthogonal.

In our context, good generalized symplectic log Calabi-Yau divisors serve the role of symplectic log Calabi-Yau divisors and we have the following version of Torelli theorem:

Theorem 3.3.10. *(=Theorem 1.1.2) Let $(M^i,\omega^i,D^i = \cup_j C_j^i)$ be good generalized symplectic log Calabi-Yau pairs for $i = 1,2$. Then*

(1) they are (strictly) symplectic deformation equivalent if and only if they are (strictly) homological equivalent;

(2) suppose there is a lattice automorphism

$$\gamma : H^2(M^2;\mathbb{Z}) \to H^2(M^1;\mathbb{Z})$$

such that $\gamma(\mathrm{PD}([C_j^2])) = \mathrm{PD}([C_j^1])$ for all j and $\gamma([\omega^2]) = [\omega^1]$. Then (M^1,ω^1,D^1) and (M^2,ω^2,D^2) are strictly symplectic deformation equivalent. They are symplectomorphic if they are ω^1(resp. ω^2)-orthogonal.

3.4 Operations on marked divisors

There are several natural operations on marked divisors.

- **Perturbation**. A perturbation of a marked divisor is a symplectic isotopy of the corresponding (unmarked) divisor.

- **Marking addition**. A marking addition of a marked divisor $(M, \omega, D, \{p_j\}_{j=1}^{l}, \{I_j\}_{j=1}^{l})$ is another marked divisor $(M, \omega, D, \{p_j\}_{j=1}^{l+1}, \{I_j\}_{j=1}^{l+1})$ with the additional marking (p_{l+1}, I_{l+1}).

- **Marking moving**. Sometimes it is useful to move an intersection point of the divisor.

- **Canonical blow-up**. Given a marked divisor with l markings, there are l canonical blow-ups we can do, i.e. blow-ups using the symplectic embeddings I_j of sizes $B(\delta_j)$. A canonical blow-up of a marked divisor is still a marked divisor with one less of the p_j's.

The stability properties of these operations are summerized as follows:

Lemma 3.4.1 ([29], Lemma 2.11-2.14)**.** *(1) Perturbations of a marked divisor preserve the strict D-symplectic deformation class.*

(2) Let $(M, \omega, D, \{p_j\}_{j=1}^{l}, \{I_j\}_{j=1}^{l})$ be a marked divisor. If two marked divisors $(M, \omega, D, \{p_j\}_{j=1}^{l} \cup \{q_1\}, \{I_j\}_{j=1}^{l} \cup \{I_{q_1}\})$ and $(M, \omega, D, \{p_j\}_{j=1}^{l} \cup \{q_2\}, \{I_j\}_{j=1}^{l} \cup \{I_{q_2}\})$ are obtained by adding markings, then they are strict D-symplectic deformation equivalent if

(i) the centers q_1, q_2 coincide (intersection points of D allowed), or

(ii) q_1, q_2 are distinct smooth points of the same irreducible component.

(3) Let $(M, \omega, D = \cup_{i=1}^{k} C_i, \{p_j\}_{j=1}^{l}, \{I_j\}_{j=1}^{l})$ be a marked divisor. Let $[C_2]^2 = -1$ and $p_1 = C_1 \cap C_2$. For any smooth point p_1' on C_2, there is a marked divisor $(M, \omega', D = C_1' \cup (\cup_{i=2}^{k} C_i), \{p_1\}' \cup \{p_j\}_{j=2}^{l}, \{I_j'\}_{j=1}^{l})$ such that $p_1' = C_1' \cup C_2$, where $\omega' = \omega$ and $C_1' = C_1$ away from a small open neighborhood of C_2. Moreover, these two marked divisors are D-symplectic deformation equivalent.

(4) If two marked divisors are D-symplectic deformation equivalent, so are the two obtained by canonical blow-ups at the corresponding marked points.

Using this lemma we have the following crucial result for the minimal reduction process:

Proposition 3.4.2. *Let $\Theta^i = (M^i, \omega^i, D^i, p_j^i, I_j^i)(i = 1, 2)$ be two good generalized log Calabi-Yau divisors viewed as marked divisors both with l marked points.*

(1) Up to moving inside the D-symplectic deformation class, we can blow down an S^1 or non-S^1 exceptional class in Θ^1 and Θ^2 to obtain marked divisors $\overline{\Theta}^1$ and $\overline{\Theta}^2$ with an extra

marked point (for S^1 exceptional class, orginal marked points on the exceptional sphere will be removed after blow-down).

(2) Moreover, if the blown-down divisors $\overline{\Theta}^1$ and $\overline{\Theta}^2$ are D-symplectic deformation equivalent such that the extra marked points correspond to each other in the equivalence, then Θ^1 and Θ^2 are D-symplectic deformation equivalent.

Proof. (1) For a non-S^1 class E, we can find by Lemma 3.2.2, a J-holomorphic representative S such that D is also J-holomorphic, after possibly doing a perturbation to make D ω-orthogonal (Lemma 3.4.1(1)). By positivity of intersection, S intersects exactly one irreducible component of D and the intersection is positively transversallly once and hence is a non-S^1 exceptional sphere. By perturbing S, we may assume that it has ω-orthogonal intersection with D. After blowing down S we get a marked divisor with a marked point corresponding to the contracted S.

For an S^1 class E, we apply Lemma 3.4.1(1) again to get an ω-orthogonal divisor in the same strict D-symplectic deformation class. The irreducible component S of D in the class E is an S^1 exceptional sphere. Hence S intersects two other irreducible components of D once. By Lemma 3.4.1(3), we can find another representative of Θ in the D-symplectic deformation class such that after blowing down the exceptional sphere S, the intersection point corresponding to the exceptional sphere is ω-orthogonal and the descended divisor is still a marked divisor.

(2) Suppose the blown-down divisors are D-symplectic deformation equivalent. We can use canonical blow-ups and marking additions to recover the original divisors. Marking additions are necessary because when one blows down a divisor which originally has markings on it, markings disappear after blow-downs. Therefore when we blow up, we need to add markings to get back the original marked divisor. We may not get back the exactly same divisor Θ^1 and Θ^2 by just canonical blow-ups and marking additions, but we can get some pair in the same D-symplectic deformation class by Lemma 3.4.1(1). Since D-symplectic deformation equivalence is stable under canonical blow-ups and marking additions by Lemma 3.4.1, we conclude that Θ^1 and Θ^2 are D-symplectic deformation equivalent. □

3.5 Minimal models

The following is straightforward.

Lemma 3.5.1. *The operations of S^1 and non-S^1 blow-up/down preserve being good generalized log Calabi-Yau.*

Definition 3.5.2. *A good generalized symplectic log Calabi-Yau pair (M, ω, D) is called a* ***minimal model*** *if M is minimal.*

Lemma 3.5.3. *Every good generalized symplectic log Calabi-Yau pair can be transformed to a minimal model via a sequence of non-S^1 blow-downs followed by a sequence of S^1 blow-downs.*

Proof. Suppose E is an exceptional class intersecting each component of D non-negatively. Then by the adjunction formula $[D]\cdot E = c_1(M,\omega)(E) + pF\cdot E = 1$ (note that $F \cdot E = 0$ for $E \in \mathcal{E}$). Thus E is a non-S^1 exceptional class.

By Lemma 3.2.2, there is an ω-compatible almost complex structure such that D is J-holomorphic and E has an embedded J-holomorphic sphere representative, which we can perform non-S^1 blow-down along.

By iterative non-S^1 blow-downs, we end up with a good generalized symplectic log Calabi-Yau surface (M_0, ω_0, D_0) such that each exceptional class pairs negatively with some component of D.

If M_0 is not minimal, then for any ω_0-compatible J_0 making D_0 J_0-holomorphic, the exceptional class with minimal ω_0-area has an embedded J_0-holomorphic representative, by Lemma 1.2 of [40]. Therefore, this embedded representative coincide with an irreducible component C of D_0.

By definition C intersects two other components of D_0 and hence is an S^1 exceptional sphere. In this case we perform S^1 blow-down along C to get another good generalized symplectic log Calabi-Yau pair (M_0', ω_0', D_0'). We claim that there is no exceptional class in M_0' that pairs all irreducible components of D_0' non-negatively. If there were one, by Lemma 3.2.2, after possible perturbing D_0' to be ω_0'-orthogonal, there would be an embedded J_0'-holomorphic representative C' intersecting exactly one irreducible component of D_0' at a smooth point. This C' can be lifted to (M_0, ω_0, D_0) since the contraction of C becomes a point of D_0', which is away from C'. Contradiction. Therefore we can continue to perform S^1 blow-downs until (M, ω) is minimal. □

We can enumerate the minimal models up to homology classes of the components:

Lemma 3.5.4. *Suppose (M, ω, D) is a minimal model.*

(1) If $M = \Sigma \times S^2$, then up to relablling, $D = \cup_{i=1}^{2+l} C_i, [C_1] = -nF + B, [C_2] = nF + B, [C_3] = \cdots = [C_{2+l}] = F$, where $0 \leq n < \frac{\lambda_B}{\lambda_F}$ is an integer, $l = 2 - 2g + p$.

(2) If $M = M_\Sigma$, then up to relablling, $D = \cup_{i=1}^{2+l} C_i, [C_1] = -nF + B_{-1}, [C_2] = (1 + n)F + B_{-1}, [C_3] = \cdots = [C_{2+l}] = F$, where $0 \leq n < \frac{\lambda_{B_{-1}}}{\lambda_F}$ is an integer, $l = 2 - 2g + p$.

Proof. (1) By definition we have $[D] = \mathrm{PD}(c_1(M,\omega)) + pF = lF + 2B, [C_1] = -nF + B, [C_2] = nF + B$, where $0 \leq n < \frac{\lambda_B}{\lambda_F}$ is an integer since $\omega(C_1) > 0$. There are l I-shape chains $C_3, \cdots, C_{2+l}$ and by Lemma 2.5.4 each of them is in homology class F. We claim that each C_i is irreducible, i.e. there is a single component in each chain. Suppose that

$[C_i] = [C_i'] + [C_i'']$ with $[C_i'] = aF + bB, [C_i''] = (1-a)F - bB, [C_i'] \cdot [C_1] = [C_i''] \cdot [C_2] = 1$. It follows that $a - bn = 1 - a - bn = 1$, which implies that $2a = 1$. Contradiction.

(2) By definition we have $[D] = \mathrm{PD}(c_1(M, \omega)) + pF = (3 - 2g + p)F + 2B_{-1}, [C_1] = -nF + B_{-1}, [C_2] = (1+n)F + B_{-1}$, where $0 \leq n < \frac{\lambda_{B_{-1}}}{\lambda_F}$ is an integer since $\omega(C_1) > 0$. There are l I-shape chains $C_3, \cdots, C_{2+l}$ and by Lemma 2.5.4 each of them is in the homology class F. We claim that each chain irreducible. Suppose that $[C_i] = [C_i'] + [C_i'']$ with $[C_i'] = aF + bB_{-1}, [C_i''] = (1-a)F - bB_{-1}, [C_i'] \cdot [C_i''] = 1$. It follows that $b^2 - ab + b(1-a) = 1$, which implies that $b(1 + b - 2a) = 1$. Thus, either $b = 1 + b - 2a = 1$ or $b = 1 + b - 2a = -1$. Contradiction. □

We firstly prove the Torelli theorem for minimal models. By pulling everything onto the same manifold and using the Moser lemma, we only need to consider the following situation:

Proposition 3.5.5. *Let $(M, \omega, D^i = \cup_{j=1}^m C_j^i)$ $(i = 1, 2)$ be two minimal models such that $[C_j^1] = [C_j^2]$ for $j = 1, \cdots, m$. Then (M, ω, D^1) is strictly symplectic deformation equivalent to (M, ω, D^2).*

Proof. We only consider the case when $M = \Sigma \times S^2$; the proof for the non-trivial bundle case is similar. There exists $\phi \in \mathrm{Symp}^h(M, \omega)$ mapping two section-type components of D^1 to those of D^2 ([14, Section 9] and [5]). Thus we can assume that D^1, D^2 have the same section-type components. By Lemma 3.2.2 we can find almost complex structures J_1, J_2 such that D^1 is J_1-holomorphic and D^2 is J_2-holomorphic. Each of J_1, J_2 gives a ruling of embedded symplectic spheres in the class F. For such J_1, J_2, we find an arbitery path $J_t \in \mathcal{J}$ interpolating them. Then for each J_t, M admits a ruling of embedded J_t-holomorphic spheres in the class F ([36, Lemma 4.1]). This gives a symplectic isotopy between a fiber-type component of D^1 and that of D^2. □

3.6 Proof of the Torelli Theorem

We are ready to prove Theorem 1.1.2/3.3.10:

Proof. (1) The only if direction is obvious. We only prove that homological equivalence implies symplectic deformation equivalence; the strict part follows from (2). Let (M^i, ω^i, D^i) be good generalized symplectic log Calabi-Yau surfaces for $i = 1, 2$ and $\Phi : M^1 \to M^2$ a strict homological equivalence.

Let $\{E_i\}_{i=1}^{\beta}$ be a maximal set of pairwisely orthogonal non-S^1 exceptional classes in M^1. After possibly deforming D^1, we may choose an almost complex structure J^1 such that D^1 is J^1-holomorphic and all E^i has embedded J^1-holomorphic representative by Lemma 3.2.2. Since Φ is a homological equivalence, $\{\Phi_*(E_i)\}$ is a maximal set of pairwisely orthogonal non-S^1 exceptional classes in M^2. We can find an ω^2-tamed almost complex structure (possibly after deforming D^2) J^2 such that D^2 is J^2-holomorphic and $\Phi_*(E_i)$ has embedded J^2-holomorphic representative. After blowing down the $J^1(J^2)$-holomorphic representatives of E_i and $\Phi_*(E_i)$ for $1 \leq i \leq \beta$, we obtain two good generalized symplectic log Calabi-Yau surfaces $(\overline{M^i}, \overline{\omega^i}, \overline{D^i})$ for $i = 1, 2$.

$(\overline{M^1}, \overline{\omega^1}, \overline{D^1})$ and $(\overline{M^2}, \overline{\omega^2}, \overline{D^2})$ are homological equivalent for some natural choice of diffeomorphism $\overline{\Phi}$. Since a component in $\overline{D^1}$ is exceptional if and only if the corresponding component in $\overline{D^2}$ is exceptional, we can pass to minimal models $(\widetilde{M^i}, \widetilde{\omega^i}, \widetilde{D^i})$ by S^1 blow-downs. By identifying $\widetilde{M^1}$ and $\widetilde{M^2}$ using a natural choice of diffeomorphism $\widetilde{\Phi}$, the homology classes of the components of $\widetilde{D^1}$ and $\widetilde{D^2}$ are the same.

By Proposition 1.2.15 of [38], up to a D-symplectic homotopy, we can assume $[\widetilde{\omega^1}] = \widetilde{\Phi}^*[\widetilde{\omega^2}]$. Therefore, $\widetilde{M^1}$ and $\widetilde{M^2}$ are symplectomorphic ([23],[42]) and we can thus choose $\widetilde{\Phi}$ to be a symplectomorphism from $(\widetilde{M^1}, \widetilde{\omega^1}, \widetilde{\Phi}^{-1}(\widetilde{D^2}))$ to $(\widetilde{M^2}, \widetilde{\omega^2}, \widetilde{D^2})$. By applying Proposition 3.5.5 to $(\widetilde{M^1}, \widetilde{\omega^1}, \widetilde{D^1})$ and $(\widetilde{M^1}, \widetilde{\omega^1}, \widetilde{\Phi}^{-1}(\widetilde{D^2}))$, $(\widetilde{M^1}, \widetilde{\omega^1}, \widetilde{D^1})$ and $(\widetilde{M^2}, \widetilde{\omega^2}, \widetilde{D^2})$ are symplectic deformation equivalent . By Lemma 3.3.4, they are D-symplectic deformation equivalent.

We can record the sequence of non-S^1 and S^1 blow-downs by marking $\widetilde{D^1}$ and $\widetilde{D^2}$. As marked divisors, they are D-symplectic deformation equivalent by Lemma 3.4.1(2). Finally by Proposition 3.4.2 (and viewing unmarked divisors as marked divisors with empty markings), (M^1, ω^1, D^1) is D-symplectic deformation equivalent to (M^2, ω^2, D^2) and hence symplectic deformation equivalent to it by Lemma 3.3.4. It is easy to verify that the symplectomorphism in the symplectic deformation equivalence between (M^1, ω^1, D^1) and (M^2, ω^2, D^2) has the same homological effect as Φ.

(2) It is clear that γ maps non-S^1 exceptional classes to non-S^1 exceptional classes. By minimal reduction, we see that certain linear combination of the homology classes of the fiber components in any I-shape chain and the non-S^1 exceptional classes intersecting this chain gives the fiber class F, thus γ maps the fiber class of M^2 to that of M^1. By Proposition 4.4 of [26], γ is realized by a homological equivalence Φ. Since $\gamma([\omega^2]) = [\omega^1]$, Φ is a strict homological equivalence.

Up to symplectic isotopies of D^1 and D^2, which preserve the strict D-symplectic deformation classes (Lemma 3.4.1), we can assume that D^i are ω^i-orthogonal for $i = 1, 2$. We have shown in (1) that there is a D-symplectic homotopy (M^1, ω^1_t, D^1) of (M^1, ω^1, D^1) and a symplectomorphism $\Psi : (M^1, \omega^1_1, D^1) \to (M^2, \omega^2, D^2)$ with the same homological effect as Φ. Therefore $[\omega^1] = \Phi^*[\omega^2] = \Psi^*[\omega^2] = [\omega^1_1]$. By Theorem 1.2.12 of [38], ω^1_t can be chosen such that $[\omega^1_t]$ is constant for all t. By Corollary 1.2.13 of [38], there is a symplectic isotopy (M^1, ω^1, D^1_t) such that $D^1_0 = D^1$ and (M^1, ω^1, D^1_1) is sympletomorphic to (M^1, ω^1_1, D^1) and hence to (M^2, ω^2, D^2). Thus, (M^1, ω^1, D^1) and (M^2, ω^2, D^2) are strictly symplectic deformation equivalent. If they are ω^1(resp. ω^2)-orthogonal in the first place, then they are symplectomorphic by Proposition 3.3.8. □

Chapter 4

S^1 generalized symplectic log Calabi-Yau divisors and Hamiltonian circle actions

4.1 S^1 generalized symplectic log Calabi-Yau divisors

Definition 4.1.1. *Suppose (M, ω, D) is a good generalized symplectic log Calabi-Yau pair such that all exceptional classes are S^1 (Definition 3.2.1). Suppose $\{C_1, \cdots, C_l\}$ forms an I-shape chain of D, with C_i intersecting C_{i+1}, $1 \le i \le l-1$, and C_1, C_l intersecting the section components. We define the* ***formal weight*** *k_i of the fiber-type component C_i as follows: $k_1 = k_l = 1, k_{i-1} + k_i[C_i]^2 + k_{i+1} = 0, 2 \le i \le l-1$.*

The formal weights serve the role of weights/labels of edges in the extended graphs, and can be calculated similarly by induction. Suppose $\{C_1, \cdots, C_l\}$ is an I-shape chain of D and C_i has formal weight k_i. If we do an S^1 blow-up at the intersection point of C_1 and a section component, then the new chain has homology class $E, [C_1] - E, [C_2], \cdots, [C_l]$ and formal weights $1, k_1, \cdots, k_l$. If we do an S^1 blow-up at the intersection point of C_i and C_{i+1}, then the new chain has homology class $[C_1], \cdots, [C_i] - E, E, [C_{i+1}] + E, \cdots, [C_l]$ and formal weights $k_1, \cdots, k_i, k_i + k_{i+1}, k_{i+1}, \cdots, k_l$.

Recall that a symplectic Looijenga pair $(M, \omega, D = \cup_{i=1}^m C_i)$ is called a **toric** symplectic log Calabi-Yau pair if $q(D) = 0$, where $q(D) = 12 - m - [D]^2 = 12 - 3m - \sum_{i=1}^m [C_i]^2$. It is a simple observation that a symplectic log Calabi-Yau pair is toric if and only if it is an iterated toric blow-up of minimal models ([33, Lemma 2.36]). For good generalized symplectic log Calabi-Yau pairs, we have a similar result.

Definition 4.1.2. *Suppose $(M = M_k, \omega, D = \cup_{i=1}^m C_i)$ is a good generalized symplectic log Calabi-Yau pair. We call it an* ***S^1*** *generalized symplectic log Calabi-Yau pair if:*

(1) $\sum_{i=1}^m s_i = -3k$*, where* $s_i = [C_i]^2$*, in which case all exceptional classes are* S^1*; and*

(2) all fiber components have formal weights larger than 1, except for two end components in each chain.

Remark 4.1.3. *The second condition is necessary since we only consider the skeleton divisors of Hamiltonian* S^1*-spaces with respect to generic compatible metrics. When a chain of gradient spheres contains a free gradient sphere in the middle, we can break the chain into two by perturbing the compatible metric (to a generic one), so that the two new gradient spheres emanate from the maximum/minimum (see the figure below which we borrow from [18]). Thus, for divisors, when we do an* S^1*-blowup at the intersection point of a section component and a fiber component not in the homology class* F*, what we really want is firstly adding a new component in the homology class* F *(corresponding to a trivial chain of gradient spheres) and then doing an* S^1*-blowup at the intersection point of this new component and the section component. This can 'perturb' a non-*S^1 *divisor into an* S^1 *divisor. See Lemma 3.6 and Example 5.23 of [18] for more details.*

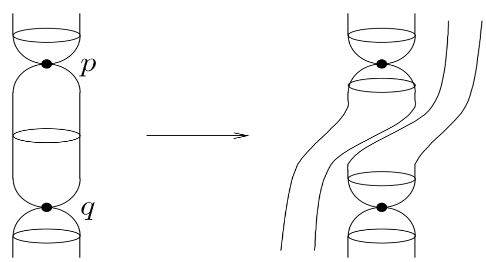

Figure 4.1: Breaking a free gradient sphere

Lemma 4.1.4. *A good generalized symplectic log Calabi-Yau pair* $(M, \omega, D = \cup_{i=1}^m C_i)$ *is* S^1 *if and only if it is an iterative* S^1 *blow-ups of minimal models, and all blow-ups don't happen at intersection points of a fiber component and a section component, except when the fiber component is in homology class* F*.*

Proof. It is an easy calculation that an S^1 blow-up decreases $\sum s_i$ by 3, a non-S^1 blow-up decreases $\sum s_i$ by 1 and $\sum s_i = 0$ for minimal models. By Lemma 3.5.3 and Lemma 3.5.4 D is an iterative S^1 blow-ups of minimal models. All blow-ups can not happen at intersection point of a fiber component and a section component, except when the fiber component is in homology class F. Otherwise some fiber component which is not at the end of a chain will have formal weight one. $\square$

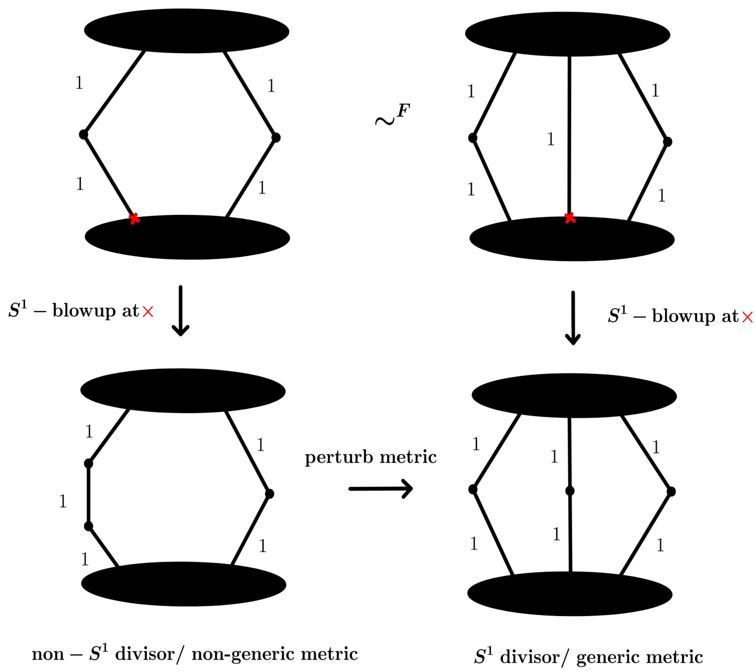

Figure 4.2: Use of $\sim^F$ to 'perturb' non-S^1 divisors

Lemma 4.1.5. *Suppose (M, ω, Φ) is a symplectic irrational ruled surface over Σ_g admitting a Hamiltonian circle action. Associate the skeleton divisor (with respect to a generaic compatible metric) as in section 2.3. Then the skeleton divisor is an S^1 generalized symplectic log Calabi-Yau divisor. Moreover, if we choose two different generic compatible metrics, the resulting skeleton divisors are strictly symplectic deformation equivalent.*

Proof. By Proposition 2.3.3 the skeleton divisor is a generalized symplectic log Calabi-Yau divisor. By [18], we know the extended graph comes from a graph with two fat vertices and no intereior fixed points (with some trivial edges of label 1 added) followed by blow-ups at two fat vertices and the interior fixed points. In particular, two section components have no intersection and the F coefficient in the sum of homology classes of them remain the same in the blowup process. Thus the skeleton divisor is good. By Remark 4.1.3 and Lemma 4.1.4, we know that the skeleton divisor is also an S^1 generalized symplectic log Calabi-Yau divisor.

Next we show that the skeleton divisor is well-defined up to strict symplectic deformation equivalence. (M, ω, Φ) is a tall complexity one space (see [22, Example 1.7]). By Proposition 1.2 of [22], the geometric quotient M/S^1 is homeomorphic to $\Sigma_g \times (\text{image } \Phi)$. By composing this homeomorphism with the projection onto the first factor, any free gradient sphere connecting an isolated fixed point p and a fixed surface (say the maximal one) is mapped to a path in Σ_g. Note that for different choices of generic compatible metrics, such paths have the same starting point (corresponding to p) but could have different ending points. In Σ_g we can find a homotopy between any two such paths since we are allowed to move the ending point. As a result, two free gradient spheres (corresponding to two generic compatible metrics) that connect p and the maximal fixed surface are in the same homological class. Thus, for two different generic compatible metrics, the resulting skeleton divisors are strictly homological equivalent and strictly symplectic deformation equivalent by Theorem 1.1.2. □

4.2 Hamiltonian circle actions

By Lemma 4.1.5, the skeleton divisor of a symplectic irrational ruled surface is an S^1 generalized symplectic log Calabi-Yau divisor. Moreover, we show that the converse is also true up to strict symplectic deformation equivalence and $\sim^F$:

Proposition 4.2.1. *Let (M, ω) be a symplectic irrational ruled surface. Given an S^1 generalized symplectic log Calabi-Yau divisor $D \subset (M, \omega)$, there is a Hamiltonian circle action on (M, ω) whose skeleton divisor is equivalent to D in the sense of Theorem 1.2.1.*

Proof. By Lemma 4.1.4, (M, ω, D) is obtained from a minimal model $(\overline{M}, \overline{\omega}, \overline{D})$ by a sequence of S^1 blowups. By Lemma 3.5.3 and classification of Karshon graphs for symplectic irrational ruled surfaces ([18, Lemma 6.15]), and deleting the extra components of $\overline{D}$ in the homology class F if necessary, there exists a Hamiltonian circle action on $(\overline{M}, \overline{\omega})$ whose skeleton divisor $\overline{D'}$ is strictly homological equivalent to $\overline{D}$. By Theorem 1.1.2 they are strictly symplectic deformation equivalent. Thus we have the desired result for the minimal model. For all S^1 blowup operated on $\overline{D}$, the symplectic areas of all components in the blowup process being positive implies that we can do corresponding S^1-equivariant symplectic blowups on $(\overline{M}, \overline{\omega})$ ([18, Proposition 7.2]), which will have the same homological effects on the skeleton divisor as do the S^1 blowups have on $\overline{D}$. Thus there is a Hamiltonian circle action on (M, ω) whose skeleton divisor D' is strictly homological equivalent to D. By Theorem 1.1.2, they are strictly symplectic deformation equivalent. □

We are ready to prove Theorem 1.2.1:

Proof of Theorem 1.2.1. Consider the natural map that takes a Hamiltonian circle action to its skeleton divisor (with respect to a generic compatible metric). By Lemma 4.1.5, the skeleton divisor is an S^1 generalized log Calabi-Yau divisor and is well-defined up to strict symplectic deformation equivalence for different choices of generic compatible metrics.

Thus the induced map in Theorem 1.2.1 is well-defined. By Proposition 4.2.1, this map is surjective.

On the other hand, suppose we have two Hamiltonian circle actions on (M, w) whose skeleton divisors D_1, D_2 are equivalent in the sense of Theorem 1.2.1. Firstly, up to strictly symplectic deformation equivalences, they can not only differ by several components which are all in the homology class F. This is because the extended graphs of symplectic irrational ruled surfaces contain no trivial gradient spheres when there are at least 3 chains, thus the skeleton divisors with at least 3 I-shape chains have no components in the homology class F. Hence, D_1, D_2 have to be strictly symplectic deformation equivalent, and also strictly homological equivalent. By minimal reduction, it is easy to see that the formal weights of fiber components of a skeleton divisor are the same as their weights in the extended graph (cf. [18, Lemma 5.2]). Thus if D_1, D_2 are strictly homological equivalent as generalized symplectic log Calabi-Yau divisors, two Hamiltonian circle actions must have isomorphic extended graphs (labels other than weights are also determined by homological information, for instance the moment map labels are determined by weight labels and homological information by [18, Lemma 2.5]). Since a Hamiltonian circle action is determined by its Karshon graph and also its extended graph, we also prove the injectiveness. □

For a fixed symplectic manifold (M, ω), there are finitely many inequivalent toric actions ([37, Proposition 3.1]). Similarly, there are finitely many inequivalent maximal Hamiltonian circle actions on any connected closed symplectic 4-manifold ([17][15][20]). We give another proof in the language of symplectic divisors.

Proof of Theorem 1.2.2. By Lemma 2.5.4 the homology class of any component of an S^1 pair is equal to $B \pm nF - \sum E_i, F - \sum E_j$ or $E_k - \sum E_l$, and there are only finitely many such classes with positive ω-area in (M, ω). Note that distinct components in a divisor D must be in distinct homolgy classes except for those components in the homology class F, otherwise their intersections with other components in D will be the same, which is impossible. Thus, up to the equivalence relation $\sim^F$, there are finitely many possible strict homological equivalence classes of S^1 divisors in (M, ω), and the result follows by Theorem 1.1.2 and Theorem 1.2.1. □

One one hand, this proof is soft in the sense that it does not give an explicit upper bound of the number of inequivalent maximal Hamiltonian circle actions. We can get the upper bound using minimal reduction along with Theorem 1.1.2 and Theorem 1.2.1, but that will be essentially the same as the methond used in [16]. On the other hand, this proof is hard since we use some pseudo-holomorphic tools. A soft proof in this sense is given in [15], and Proposition 8.9 there plays the same role as our Theorem 1.2.1.

4.3 The rational case

After taking care of the irrational ruled case, it is natural to consider the same questions for symplectic rational surfaces. However there are several difficulties which we are not able to solve for now. In this section we summarize the results that we get for the rational case.

Denote by $(M_k, \omega_{\lambda;\delta_1,\cdots,\delta_k})$ a symplectic manifold that is obtained from $(\mathbb{CP}^2, \lambda\omega_{FS})$ by symplectic blowups of sizes $\delta_1, \cdots, \delta_k$. It is well defined up to symplectormorphisms([35]). Let $H \in H_2(M_k)$ denote the the image of the homology class of a line $\mathbb{CP}^1$ in $\mathbb{CP}^2$ under the inclusion map $H_2(\mathbb{CP}^2) \to H_2(M_k)$. Let $E_1, \dots, E_k$ denote the homology classes of the exceptional divisors. Then $H, E_1, \dots, E_k$ is the standard basis of $H_2(M_k)$ with intersection numbers $H^2 = 1, E_i^2 = -1, H \cdot E_i = E_i \cdot E_j = 0, 1 \leq i \neq j \leq n$. A symplectic form ω on M_k is called a **blowup form** if there exist disjoint embedded symplectic spheres in the homololgy class $H, E_1, \cdots, E_k$. In the following we always assume that the symplectic form is a blowup form.

We need the following useful properties of symplectic rational surfaces ([20, Chapter 4], [27, Proposition 2.4]):

Lemma 4.3.1. *(1) The (nonempty) set of blowup forms $\mathcal{SF}_{M_k}$ is an equivalence class of symplectic forms (in the sense of Definition 2.4.4). Thus we can define the corresponding Gromov-Witten invariant* $\mathrm{GW}(\cdot)$ *and first Chern class* $c_1(M, \omega) = c_1(TM_k)$. *The Poincare dual of the first Chern class is* $\mathrm{PD}(c_1(M, \omega)) = 3H - \sum_{i=1}^k E_i$.

(2) The classes $E_1, \cdots, E_k$ and the classes $H - E_i - E_j$ for $1 \leq i < j \leq k$ are all in $\mathcal{E}(M_k)$ and have nonzero Gromov-Witten invariants.

(3) If $\{B_i\}$ is a collection of symplectic spherical classes with self-intersection at least -1, then for a generic almost complex structure $\mathcal{J}$ tamed by ω, there is an embedded $\mathcal{J}$-holomorphic rational curve in each class B_i. Consequencetly, by the positivity of intersections, $B_i \cdot B_j \geq 0$ if $i \neq j$.

Definition 4.3.2. *Suppose $(M = M_k, \omega)$ is a symplectic rational surface with $k \geq 1$. A* ***generalized symplectic log Calabi-Yau divisor*** *in (M, ω) is a nonempty symplectic divisor $D = \cup_{i=1}^m C_i$ representing* $\mathrm{PD}(c_1(M, \omega)) + pF$ *with $[C_i] \cdot F \geq 0$, where $p \geq 1$ is an integer and $F = H - E_1$ is a fiber class. We call (M, ω, D) a* ***generalized symplectic log Calabi-Yau pair****.*

Proposition 4.3.3. *(1)* $\sum_{i=1}^m [C_i] \cdot F = 2$.

(2) $[C_i] \cdot F \geq 0$ *for all* i.

(3) $[C_i] \cdot (\sum_{j \neq i} [C_j]) = 2 - 2g_i + p([C_i] \cdot F)$ *for all i, where $g_i = g(C_i)$.*

Proof. (1) and (3) follow in the same way as Proposition 2.5.3, and (2) is part of the definition. ∎

By Proposition 4.3.3(1)(2) there are at most two section-type components and the others are fiber-type components. As in the irrational ruled case, we are interested in the case where there are two disjoint section-type components.

Lemma 4.3.4. *Suppose (M,ω,D) is a generalized symplectic log Calabi-Yau pair. Suppose $\sum_{j=1}^{i_k} C_{i_j}$ is a chain of fiber-type components in M_k. Let $A=\sum_{j=1}^{i_k}[C_{i_j}]=aH-\sum_{i=1}^k b_iE_i$. Suppose C_1 is a section-type component intersecting the chain with $[C_1]=cH-\sum_{i=1}^k c_iE_i$. Then we have:*

(1) all components of the divisor D are embedded symplectic spheres.

(2) $c=c_1+1, c_2,\cdots,c_k\in\{0,1\}$.

(3) $a=b_1\le 1, \sum_{i\ge 2}(b_i^2-b_i)=2-2a$.

Proof. For fiber-type components, $[C_i]\cdot(\sum_{j\neq i}[C_j])=2-2g_i>0$, thus $g_i=0$. For section-type component as C_1, $[C_1]\cdot F=1$ implies that $c=c_1+1$. The adjunction formula for C_1 gives $c_1(M,\omega)([C_1])=[C_1]^2+2-2g_1$ which simplifies to $2g_1+\sum_{i\ge 2}(c_i^2-c_i)=0$. Note that $c_i^2-c_i\ge 0$ for all i, and the equality holds if and only if $c_i\in\{0,1\}$. This finishes the proof of (1)(2).

Next we prove (3). As in Lemma 2.5.4 we may assume $i_k=1$. Note that $A\cdot F=0$, thus $a=b_1$. The adjunction formula gives $c_1(M,\omega)(A)=A^2+2$, which implies that $\sum_{i\ge 2}(b_i^2-b_i)=2-2a$ and $a\le 1$. □

A vector $(\lambda;\delta_1,\dots,\delta_k)\in\mathbb{R}^{1+k}$ **encodes** a degree 2 cohomology class $\Omega\in H^2(M_k;\mathbb{R})$ if $\frac{1}{2\pi}\langle\Omega,L\rangle=\lambda$, $\frac{1}{2\pi}\langle\Omega,E_i\rangle=\delta_i$ for $i=1,\dots,k$. $\omega_{\lambda;\delta_1,\cdots,\delta_k}$ can be taken to be a blowup form on M_k whose cohomology is encoded by the vector $(\lambda;\delta_1,\dots,\delta_k)$. It is unique up to a diffeomorphism that acts trivially on the homology.

For $k\ge 3$, the vector $(\lambda;\delta_1,\dots,\delta_k)$ is **reduced** if $\delta_1\ge\cdots\ge\delta_k>0, \lambda-\delta_1-\delta_2-\delta_3>0$.

Proposition 4.3.5 ([20], Theorem 1.5)**.** *Let $k\ge 3$. Given a blowup form $\omega_{\lambda';\delta_1',\cdots,\delta_k'}$ on M_k, there exists a unique reduced vector $(\lambda;\delta_1,\dots,\delta_k)$ such that $(M_k,\omega_{\lambda';\delta_1',\cdots,\delta_k'})$ is symplectomorphic to $(M_k,\omega_{\lambda;\delta_1,\cdots,\delta_k})$.*

In this case, $F=H-E_1$ is special among fiber classes in the following sense:

Lemma 4.3.6. *If the symplectic form $\omega_{\lambda;\delta_1,\cdots,\delta_k}$ on M_k is reduced, then $H-E_1$ has the smallest ω-area among fiber classes.*

Proof. If $k=1$, then it is easy to see that $H-E_1$ is the unique fiber class by the adjuction formula; if $k=2$, it is easy to see that $H-E_1$ and $H-E_2$ are all fiber classes. In either case the lemma follows immediately.

If $k\ge 3$, by positive pairing with the symplectic spherical classes $H,E_1,\cdots,E_k$ (Lemma 4.3.1(3)), any fiber class is of the form $F=dH-\sum_{i=1}^k a_iE_i$ with $d\ge 0, a_i\ge 0$. d can not be 0 since $F\cdot F=0$. By the adjunction formula, $c_1(M,\omega)(F)=2$, hence $3d=2+\sum_{i=1}^k a_i$. By possitive pairing with the symplectic spherical classes $H-E_1$ and $H-E_2$, $a_1\leq d$ and $a_2\leq d$, thus $a_1+a_2\leq 2d$. Therefore we can write F as a sum $F=U_1+\cdots+U_{d-1}+V$ where each $U_\alpha=H-E_{\alpha_1}-E_{\alpha_2}-E_{\alpha_3}$ with no repeated E_1 or E_2, and $V=H-E_\beta$. Observe that $\omega(U_\alpha)\geq 0$ and $\omega(V)\geq\omega(H-E_1)$ by the reduced condition. □

Lemma 4.3.7. *Assume the same conditions as in Lemma 4.3.4. Suppose that the symplectic form ω is reduced. Then either $A=E_i-\sum_{j>i}b_jE_j$ with $i\geq 2, b_j\in\{0,1\}$ or $A=H-E_1-\sum_{i\geq 2}b_iE_i$, with $b_i\in\{0,1\}$.*

Proof. If $A^2\geq 0$, then by positive intersection with H (Lemma 4.3.1(3)) we have $a\geq 0$. If $A^2<0$ and $a<0$, then by Lemma 3.4 of [4], $A=aH-(|a|+1)E_1-E_{j_2}-\cdots-E_{j_s}$, contradicting Lemma 4.3.4(3). Either way we have $a\in\{0,1\}$. If $a=b_1=0$, then $\sum_{i\geq 2}(b_i^2-b_i)=2$ implies that some $b_i\in\{-1,2\}$ and the other b_j's $\in\{0,1\}$. Since ω is reduced and $\omega(A)>0$, we must have $A=E_i-\sum_{j>i}b_jE_j$ with $b_j\in\{0,1\}$. If $a=b_1=1$, then $\sum_{i\geq 2}(b_i^2-b_i)=0$ implies that $b_i\in\{0,1\}$ for $i\geq 2$, thus $A=H-E_1-\sum_{i\geq 2}b_iE_i$, with $b_i\in\{0,1\}$. □

Recall that a chain of fiber-type components is trivial if it contains a single fiber-type component in the homology class F. Otherwise it is called non-trivial.

Definition 4.3.8. *Let D be a generalized log Calabi-Yau divisor in the symplectic rational surface (M,ω). We call D **good** if it has two disjoint section-type components C_1,C_2 with $([C_1]+[C_2])\cdot H=1$. Furthermore, we require that there are at least three non-trivial chains of fiber-type components. We call (M,ω,D) a good generalized log Calabi-Yau pair.*

The condition $([C_1]+[C_2])\cdot H=1$ serves the same role as $([C_1]+[C_2])\cdot B'=0$ in Definition 2.6.1. We need the extra condition on the number of non-trivial chains since we are only interested in divisors corresponding to maximal Hamiltonian circle actions ([18, Proposition 5.21]).

Definition 4.3.9. *A good generalized symplectic log Calabi-Yau pair (M,ω,D) is called a **minimal model** if M is diffeomorphic to $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$.*

The reason for such definition is that every Hamiltonian circle action on M_k extends to a toric action if $k\leq 3$.

Lemma 4.3.10. *Suppose (M,ω,D) is a minimal model. Let $E_{i_1\cdots i_j}$ denote $E_{i_1}+\cdots+E_{i_j}$ and p be the positive integer in Definition 4.3.2. Then up to relabeling one of the following happens:*

(1) D has two section-type components C_1,C_2 with $[C_1]=(c_1+1)H-c_1E_1,[C_2]=-c_1H+(c_1+1)E_1$, three fiber-type components C_3,C_4,C_5 in homology classes $F-E_2,F-$

$E_3, F - E_4$ respectively, and $(p-1)$ fiber-type components which are all in homology class F.

(2) D has two section-type components C_1, C_2 with $[C_1] = (c_1+1)H - c_1E_1, [C_2] = -c_1H + (c_1+1)E_1 - E_2$, and four fiber-type components C_3', C_3'', C_4, C_5 (the notation C_3', C_3'' indicates that they are in the same chain) in homology classes $F - E_2, E_2, F - E_3, F - E_4$ respectively, and $(p-1)$ fiber-type components which are all in homology class F.

(3) D has two section-type components C_1, C_2 with $[C_1] = (c_1+1)H - c_1E_1, [C_2] = -c_1H + (c_1+1)E_1 - E_{23}$, and five fiber-type components $C_3', C_3'', C_4', C_4'', C_5$ in homology classes $F - E_2, E_2, F - E_3, E_3, F - E_4$ respectively, and $(p-1)$ fiber-type components which are all in homology class F.

(4) D has two section-type components C_1, C_2 with $[C_1] = (c_1+1)H - c_1E_1, [C_2] = -c_1H + (c_1+1)E_1 - E_{234}$, and six fiber-type components $C_3', C_3'', C_4', C_4'', C_5', C_5''$ in homology classes $F - E_2, E_2, F - E_3, E_3, F - E_4, E_4$ respectively, and $(p-1)$ fiber-type components which are all in homology class F.

(5) D has two section-type components C_1, C_2 with $[C_1] = (c_1+1)H - c_1E_1 - E_2, [C_2] = -c_1H + (c_1+1)E_1 - E_3$, and five fiber-type components $C_3', C_3'', C_4', C_4'', C_5$ in homology classes $E_2, F - E_2, F - E_3, E_3, F - E_4$ respectively, and $(p-1)$ fiber-type components which are all in homology class F.

(6) D has two section-type components C_1, C_2 with $[C_1] = (c_1+1)H - c_1E_1 - E_i, [C_2] = -c_1H + (c_1+1)E_1 - E_{jk}$, and six fiber-type components $C_3', C_3'', C_4', C_4'', C_5', C_5''$ in homology classes $E_i, F - E_i, F - E_j, E_j, F - E_k, E_k$ respectively, and $(p-1)$ fiber-type components which are all in homology class F, where $\{i, j, k\} = \{2, 3, 4\}$.

Proof. By Lemma 4.3.4 and Definition 4.3.8 homology classes of the two section-type components lie in one of (1)-(6) (up to relabeling). By Lemma 4.3.7 homology class of each I-shape chain is $A = E_i - \sum_{j>i} b_jE_j$ with $i \geq 2, b_j \in \{0, 1\}$ or $A = H - E_1 - \sum_{i\geq 2} b_iE_i = F - \sum_{i\geq 2} b_iE_i$, with $b_i \in \{0, 1\}$. But the former one can't happen in any of (1)-(6) since it contradicts that $A \cdot [C_1] = A \cdot [C_2] = 1$.

Suppose we are in case (1) and there are m I-shape chains with homology $A_1, \cdots, A_m$. $[D] = [C_1] + [C_2] + \sum_i A_i = 3H - E_1 - E_{234} + pF$, thus $\sum_i A_i = (p+2)F - E_{234}$. Since each A_i contains one copy of F, we must have $m = p+2$. Up to relabeling we have (i) $A_1 = F - E_{234}, A_2 = \cdots = A_{p+2} = F$ or (ii) $A_1 = F - E_{23}, A_2 = F - E_4, A_3 = \cdots = A_{p+2} = F$ or (iii) $A_1 = F - E_2, A_2 = F - E_3, A_3 = F - E_4, A_4 = \cdots = A_{p+2} = F$. For case (i), the first chain in homology class A_1 is irreducible, i.e. it consists of a single component. Otherwise, we can write $A_1 = A_1' + A_1''$ with $A_1' \cdot A_1'' = 1$ and both of them are either $E_i - \sum_{j>i} b_jE_j$ or $F - \sum_{i\geq 2} b_iE_i$, but this is impossible. The rest chains are also irreducible, otherwise assume for instance that $A_2 = A_2' + A_2''$ with $A_2 = E_i - \sum_{j>i} b_jE_j$ or $F - \sum_{i\geq 2} b_iE_i$. In either case $A_1 \cdot A_2' > 0$, contradiction. Similarly we can show that in case (ii)(iii), all three I-shape chains are irreducible. By Definition 4.3.8 only case (iii) gives us a good generalized divisor.

The proof for case (2)-(6) is similar. □

Using the tools developed in chapter 3, we can prove the Torelli theorem for good generalized log Calabi-Yau pairs in symplectic rational surfaces, although the proof for minimal models is more technical. Next we consider divisors corresponding to maximal Hamiltonian circle actions.

Definition 4.3.11. *Let $D = \cup C_i$ be a good generalized log Calabi-Yau divisor in the symplectic rational surface (M_k, ω) where $k \geq 4$. We call it an $\boldsymbol{S^1}$ generalized symplectic log Calabi-Yau divisor if it can be blown down to a minimal model (see section 3.2) and $\sum s_i = -3(k-1)$, where $s_i = [C_i]^2$.*

Lemma 4.3.12. *A good generalized symplectic log Calabi-Yau divisor D in (M, ω) is S^1 if and only if it is an iterative S^1 blow-ups of case (4) or (6) in Lemma 4.3.10.*

Proof. It is an easy calculation that an S^1 blow-up decreases $\sum s_i$ by 3 and a non-S^1 blow-up decreases $\sum s_i$ by 1. Also note that $\sum s_i = -9$ for minimal models in case (4)(6) and $\sum s_i$ is larger than -9 for other minimal models. The claim then follows immediately. □

Using a similar argument as in section 4.2, we can show that up to $\sim^s$ and $\sim^F$, S^1 generalized symplectic log Calabi-Yau divisors in symplectic rational surfaces can be realized as skeleton divisors in the sense of section 2.3. However, in the rational case, not all skeleton divisors are naturally S^1 generalized symplectic log Calabi-Yau divisors (cf. Lemma 4.1.5).

Remark 4.3.13. *(1) In the rational case, an extended graph corresponding to a maximal Hamiltonian circle action might only have one fat vertex and contain ephemeral edges (see Proposition 2.3.2), in which case the skeleton divisor is not even a symplectic divisor. Moreover, x_h is not a fiber class in general and we don't know what it is a priori . See Example 4.3.14 below.*

(2) Even if we only consider those maximal Hamiltonian circle actions whose corresponding extend graphs contain two fat vertices (hence no ephemeral edges), in general we can not assume that D represents $\mathrm{PD}(c_1(M, \omega)) + pF$ and the symplectic form ω is reduced at the same time. This is because the symplectomorphism in Proposition 4.3.5 could take $F = H - E_1$ to another fiber class F'.

Example 4.3.14 ([18], Example 6.2 and 7.4)**.** *Consider the Delzant polygon in the figure below on the left. The edge vectors, starting from the low left vertex and proceeding counterclockwise, are $(1, 0), (1, 1), (7, 14), (-8, -12), (-1, -2)$, and $(0, -1)$. Take the corresponding symplectic toric manifold, and perform an S^1-equivariant blow-up of size 1 at a point on the minimal surface of the moment map. The resulting space has three fixed points on the* *same level set of the moment map, and so it will have three non-trivial chains of gradient spheres for any compatible metric (generic or not). Hence the Hamiltonian circle action can not be extended to a toric action ([18, Proposition 5.21]).*

The free gradient sphere that reaches the maximum is not smooth at its north-pole and corresponds to an ephemeral edge in the extended graph. The homology classes of the other smooth gradient spheres in each chain (from left to right, top to bottom) are $H - E_1, E_1 - E_2, E_2 - E_3; E_4; H, H - E_{123}$. *The homology class of the minimal fixed sphere is* $E_3 - E_4$. $\mathrm{PD}(x_h) = 3H - E_{123}$ *and* $\mathrm{PD}(c_1(M, \omega)) = 3H - E_{1234}$. *Note that* $x_{2,2} = \mathrm{PD}(x_h) - x_{2,1} = 3H - E_{1234}$ *is not representable by an embedded symplectic sphere.*

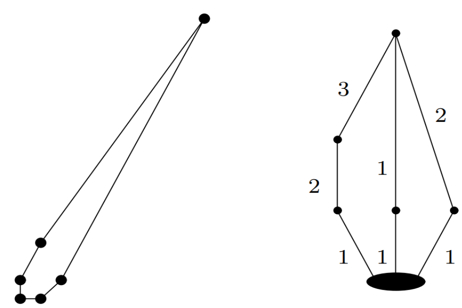

Figure 4.3: S^1-equivariant blow-up at the minimum

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