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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455–0436
Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

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C. Foias[1], M.S. Jolly[1] I. G. Kevrekidis[2], E. S. Titi[3,4]

[1]*Department of Mathematics, Indiana University, Bloomington, Indiana 47405*

[2]*Department of Chemical Engineering, Princeton University, Princeton, New Jersey 08544*

[3]*Mathematical Sciences Institute, Cornell University, Ithaca, NY 14853*

[4]*Department of Mathematics, University of California, Irvine, CA 92717*

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Abstract

We show that the way in which finite differences are applied to the nonlinear term in certain PDEs can mean the difference between dissipation and blowup. For fixed parameter values and arbitrarily fine discretizations we construct solutions which blow up in finite time for two semi-discrete schemes. We also show the existence of spurious steady states whose unstable manifolds, in some cases, contain solutions which explode. This connection between the blowup phenomenon and spurious steady states is also explored for Galerkin and nonlinear Galerkin semi-discrete approximations. Two fully discrete finite difference schemes derived from a third semi-discrete scheme, shown in Foias and Titi (1990) to be dissipative, are analyzed. Both latter schemes are shown to have a stability condition which is independent of the initial data. A similar result is obtained for a fully discrete Galerkin scheme. While the results are stated for the Kuramoto-Sivashinsky equation,

most naturally carry over to other dissipative PDEs.

1. Introduction

In this paper we compare how well various numerical schemes capture the property of dissipation for dissipative partial differential equations (PDEs). We show the drastic effect the way in which finite differences are applied to a particular nonlinear term can have on the property of dissipation. With one such semi-discrete scheme (time is continuous) this property is preserved [Foias and Titi (1990)]. We show that similar semi-discrete schemes have solutions which blow up in finite time. The blowup phenomenon is linked to the existence of spurious steady state solutions. Similar spurious solutions have been observed [Jolly *et al.* (1990b)] for Galerkin and certain nonlinear Galerkin methods (also known as approximate inertial manifold methods). This connection has led in turn to numerical evidence that these particular nonlinear Galerkin methods also have solutions which blow up in finite time. This is to be contrasted with other nonlinear Galerkin methods which have been shown to be dissipative. [Jolly *et al.* (1990b), see also Marion and Temam (1988)]. Though we will illustrate the behavior of the schemes as applied to the Kuramoto-Sivashinsky equation (KSE), much of the results carry over directly to other dissipative PDEs [see e.g. Constantin *et al.* (1988) for a number of such equations].

We also study the long time stability of fully discrete versions of the dissipative finite difference scheme of Foias and Titi (1990), and the Galerkin approximation. One scheme is implicit, the other semi-implicit as it is also linear. These schemes are shown to have stability conditions which are independent of the initial data and the solution itself. This independence represents a strong correspondence between the dissipation of the approximating dynamical system and that generated by the dissipative PDE. While for the implicit scheme we are only able to derive a uniqueness condition that depends on the initial data, we encounter no such difficulty with the semi-implicit scheme. However the implicit scheme yields a significantly better estimate for the size of the absorbing ball than the semi-implicit scheme. A semi-implicit fully discretized Galerkin scheme is also shown to have a stability condition which is independent of the initial data. Shen (1990) has derived stability conditions for certain fully discrete nonlinear Galerkin schemes applied to the Navier-Stokes equation. In that paper, however, the conditions depend on the initial data.

The KSE has been the subject of much analytic and computational study (see e.g. Jolly *et al.* (1990a) for references). In addition, the KSE is known to have an inertial manifold, i.e. a finite dimensional, positively invariant Lipschitz manifold which exponentially attracts all trajectories, and thus contains the global attractor. In Foias and Titi (1990) an interpretation is given for the finite difference scheme in terms of approximate inertial manifolds.

One can combine the results of Demengel and Ghidaglia (1988), Eden *et al.* (1990), and Shen (1990) to show that the fully discrete schemes presented here generate discrete dynamical systems which have global attractors, and in fact inertial manifolds [see Foias and Titi (1990) for the semi-discrete case and also Yan (1990)]. Moreover, as indicated in the works above, the inertial manifolds for the discrete systems converge to those of the of the PDE, as the discretization parameters tend to zero.

This paper is structured as follows. We construct solutions which blow up in finite time for certain arbitrarily fine spatial discretizations of the KSE in section 2. In section 3 we construct spurious steady states whose unstable manifolds, in some cases, contain solutions which explode. We also carry out a similar construction for the Galerkin approximation and present numerical evidence that there is blowup for some nonlinear Galerkin methods. We analyze the two fully discrete finite difference schemes in section 4, and a fully discrete

Galerkin scheme in section 5.

1.1 The PDE

We consider the renormalized Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad (x,t) \in \mathrm{R} \times \mathrm{R}^+ \tag{KS}$$

$$u(x,0) = u_0(x), \qquad x \in \mathrm{R},$$

subject to periodic boundary conditions

$$u(x,t) = u(x+L,t), \qquad L > 0.$$

Let $H^m_{\mathrm{per}}((0,L))$ denote the subspace of the Sobolev space $H^m((0,L))$ consisting of functions which, along with all their derivatives up to order $m-1$, are periodic with period L. It is known that for every $u_0 \in L^2((0,L))$ which is periodic with period L, there exists a unique solution $u(x,t) \in H^4_{\mathrm{per}}((0,L))$ for all $t > 0$ (see Nicolaenko and Scheurer (1984) and Tadmor (1986)). Moreover, if we assume in addition that u_0 is an odd function (i.e. $u_0(x) = -u_0(L-x)$, a.e. in R), then the corresponding solution $u(x,t)$ is also an odd function ($u(x,t) = -u(L-x,t)$, a.e. in R).

For (KS) restricted to the invariant subspace of odd functions, it has been shown by Nicolaenko *et al.* (1985) (see also Foias *et al.*) that for every $\rho > 0$ there exists a time $T^*(\rho) > 0$ such that whenever $\|u_0\|_{L^2} \leq \rho$, then

$$\|u(t)\|_{L^2} \leq \rho_0 \equiv c_0 L^{5/2}, \tag{1.1}$$

$$\|\nabla u(t)\|_{L^2} \leq \rho_1 \equiv c_1 L^{7/2}, \tag{1.2}$$

for all $t \geq T^*$. The constants c_0 and c_1 (as well as $c_2, c_3, \ldots$ to follow) are universal in that they are independent of the problem parameter L and the initial value u_0.

To date rigorous studies of the long time dynamics of the Kuramoto-Sivashinsky equation have been limited to the invariant subspace of odd functions, as the inequalities (1.1), (1.2) are essential in showing the existence of a compact global attractor and subsequently in estimating the dimensions of the global attractor and inertial manifolds (see e.g. Nicolaenko *et al.* (1985), Foias *et al.* (1988c), Constantin *et al.*(1988,1989)). Il'yashenko (1990) has recently proved the existence of an absorbing ball for the general periodic case. This means that results concerning the existence of a global attractor and inertial manifold for the odd case could also enjoy similar extensions, for the general case. Nevertheless, *the results in this paper are for the odd case.*

2. Semi-Discrete Finite Difference Schemes

Given a positive, *even* integer N, we denote the space discretization step by $h = L/N$ and the nodal values by $\xi_j(t) = u(x_j, t)$, where $x_j = jh$ for $j = 1, 2, \ldots, N$. Since u is periodic with period L, it is convenient to extend to a double infinite sequence $\xi_j, j = 0, \pm 1, \pm 2, \ldots$, which satisfies

$$\xi_j = \xi_{j+N} . \tag{2.1}$$

Note that the oddness condition

$$\xi_j = -\xi_{N-j} , \quad j \in \mathbf{Z}, \tag{2.2}$$

together with (2.1) implies

$$\xi_{N/2+j} = \xi_{N+(j-N/2)} = \xi_{j-N/2} = -\xi_{N/2-j}, \quad j \in \mathbf{Z} . \tag{2.3}$$

In addition we have $\xi_0 = \xi_{N/2} = 0$.

Direct application of the centered difference formula to the second and fourth order terms in (KS) yields

$$\left[\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial x^2}\right](x_j) \approx \frac{\xi_{j+2}-4\xi_{j+1}+6\xi_j-4\xi_{j-1}+\xi_{j-2}}{h^4}+\frac{\xi_{j+1}-2\xi_j+\xi_{j-1}}{h^2} \equiv \mathcal{L}(\xi)\,. \quad (2.4)$$

There are however, several ways to treat the bilinear term. The most direct approach, is to take

$$\left[u\frac{\partial u}{\partial x}\right](x_j) \approx \frac{\xi_j(\xi_{j+1}-\xi_{j-1})}{2h.} \tag{2.5}$$

An alternative is to rewrite the bilinear term so that

$$\left[u\frac{\partial u}{\partial x}\right](x_j) = \frac{1}{2}\left[\frac{\partial(u^2)}{\partial x}\right](x_j) \approx \frac{\xi_{j+1}^2-\xi_{j-1}^2}{4h}\,. \tag{2.6}$$

Finally, one may consider linear combinations of approximations (2.5) and (2.6), in particular the combination

$$\left[u\frac{\partial u}{\partial x}\right](x_j) \approx \frac{1}{3}\left[\frac{\xi_j(\xi_{j+1}-\xi_{j-1})}{2h}\right]+\frac{2}{3}\left[\frac{\xi_{j+1}^2-\xi_{j-1}^2}{4h}\right], \tag{2.7}$$

considered in Foias and Titi (1990). Since the solutions of (KS) can be shown to belong to a Gevrey analytic class of regularity (Foias and Temam (1989)) it follows that they are in C^∞. Hence the approximations of the bilinear term in (2.5), (2.6), and (2.7) are, like those of the linear terms in (2.4), $O(h^2)$.

The three treatments above provide three different semi-discrete systems:

$$\frac{d\xi_j}{dt}+\mathcal{L}(\xi)+\begin{cases} \frac{\xi_j(\xi_{j+1}-\xi_{j-1})}{2h}=0 & \text{(a);}\\ \frac{\xi_{j+1}^2-\xi_{j-1}^2}{4h}=0 & \text{(b);}\\ \frac{\xi_j(\xi_{j+1}-\xi_{j-1})+\xi_{j+1}^2-\xi_{j-1}^2}{6h}=0 & \text{(c),}\end{cases} \tag{2.8}$$

to approximate the evolution of the nodal values. We emphasize that these are semi-finite difference schemes as time is not discretized. It is easily verified that for each of the systems (2.8a-c), condition (2.1) defines an invariant N-dimensional subspace of the infinite dimensional system. Moreover, assuming N is even, condition (2.2) reduces (2.8a-c) to an equivalent system of dimension $N/2$. While all three schemes share the same overall quantitative error estimate of $O(h^{3/2})$ (a factor of $\sqrt{N}$ is introduced when computing the ℓ^2 norm of the error vector), they produce substantially different long time qualitative behavior. We recall that system (2.8c) correctly inherits the property of dissipation from the full KS equation, provided the parameters h, L are suitably chosen.

Theorem 2.1. *(Foias and Titi (1990)) Suppose N is large enough so that*

$$(3 - h^2)^2 - 24h^3 L \geq 0. \tag{2.9}$$

Then system (2.8c) has a global solution for all $t \in \mathrm{R}_+$. Moreover, there exists $r_0 > 0$, such that for every solution $\vec{\xi}(t)$ of (2.8c) there exists $t^*(|\vec{\xi}(0)|)$ such that

$$|\vec{\xi}(t)| \leq r_0, \text{ for all } t \geq t^*,$$

where $r_0 = 2\frac{|\vec{f}|}{\mu_1^2}$, with $\vec{f}$, $|\cdot|$, and μ_1 as defined in section 4.

We will show in section 3 below that the system (2.8c) is not dissipative for all possible values of h and L.

2.1 Blowup

We now show that the result in Theorem 2.1 does not hold for system (2.8a) and (2.8b). In fact for arbitrarily fine discretizations, both systems (2.8a) and (2.8b) have particular solutions which blow up in finite time.

Theorem 2.2. *For fixed $N = 6k$, $k \in \mathbf{Z}^+$, and fixed $h = L/N < \sqrt{3}$, there exists a particular solution ξ of (2.8a) and a finite time t^*, such that $\xi(t) \to \infty$ as $t \to t^*$. The same statement holds for system (2.8b).*

The proof will be given in terms of system (2.8a) only, as the proof for system (2.8b) is completely analogous.

Proof for System (2.8a):

We first concentrate on the case $k = 1$, where (2.1), (2.2) and (2.3) imply

$$\xi_0 = \xi_3 = \xi_6 = 0 \ , \ \xi_{-1} = \xi_5 = -\xi_1 \ , \ \xi_4 = -\xi_2 \ .$$

It follows that (2.6a) can be reduced to the pair of equations

$$\frac{d\xi_1}{dt} + \frac{5\xi_1 - 4\xi_2}{h^4} + \frac{\xi_2 - 2\xi_1}{h^2} + \frac{\xi_1\xi_2}{2h} = 0 \tag{2.10}$$

$$\frac{d\xi_2}{dt} + \frac{5\xi_2 - 4\xi_1}{h^4} + \frac{\xi_1 - 2\xi_2}{h^2} - \frac{\xi_1\xi_2}{2h} = 0. \tag{2.11}$$

Let

$$\sigma = \xi_1 + \xi_2 \text{ and } \delta = \xi_1 - \xi_2. \tag{2.12}$$

By first adding (2.10) and (2.11), and then subtracting (2.11) from (2.10), one obtains

$$\begin{aligned} &\frac{d\sigma}{dt} + a\sigma = 0 \\ &\frac{d\delta}{dt} + b\delta + \frac{\sigma^2 - \delta^2}{4h} = 0 \ , \end{aligned} \tag{2.13}$$

where

$$a = a(h) = h^{-4} - h^{-2} \text{ and } b = b(h) = 9h^{-4} - 3h^{-2}. \tag{2.14}$$

Taking $\sigma(0) = 0$ so that $\sigma(t) \equiv 0$ is the first component of a particular solution to (2.13), we have that the δ-component must satisfy the Ricatti equation

$$\frac{d\delta}{dt} + b\delta - \frac{\delta^2}{4h} = 0 \ . \tag{2.15}$$

One may solve (2.15) by elementary methods to obtain the solution

$$\delta(t) = \frac{4hb\delta_0}{\delta_0 - (\delta_0 - 4hb)e^{bt}} \ ,$$

which blows up at time

$$t^* = \frac{1}{b} \ln\left(\frac{\delta_0}{\delta_0 - 4hb}\right) \tag{2.16}$$

for initial data satisfying

$$\delta_0 > 4hb \tag{2.17}$$

in the case when $b > 0$, i.e. when $h < \sqrt{3}$. The extension by (2.1), (2.2), and (2.3) to a particular solution of (2.8a) generated by

$$\xi_1(t) = -\xi_2(t) = \frac{\delta(t)}{2} , \tag{2.18}$$

completes the proof for $k = 1$.

For $k > 1$ we consider system (2.8a) once again with only 6 nodes, but this time over the interval of length $l = \frac{L}{k}$. It follows that the particular solution constructed in (2.18) over the interval of length l, can be extended by (2.1) to a solution of (2.8a) for $N = 6k$ over the interval of length L, which blows up at t^* given in (2.16).

3. Spurious Stationary Solutions

We now turn to the issue of spurious stationary solutions for various space discretizations of the KS equation. We observe some similarities between the spurious solutions for finite difference and Galerkin methods, and make a connection with the blowup phenomenon described in section 2.

3.1. Finite differences

For the semi-discrete systems (2.8a) and (2.8b), and $N = 6$, we again let σ and δ be as in (2.12), and look in the invariant subspace defined by (2.2) and (2.3). The condition for a stationary solution can then be written

$$a\sigma = 0 \quad , \quad b\delta + \frac{\sigma^2 - \delta^2}{4h} = 0 \quad \text{for } (2.8a)$$
$$a\sigma - \sigma\delta = 0 \quad , \quad b\delta + \frac{\sigma^2 + \delta^2}{4h} = 0 \quad \text{for } (2.8b),$$

where a and b are as in (2.14). In either case, $\sigma = 0$, $\delta = 4bh$ for (2.8a), $\delta = -4bh$ for (2.8b) define a one parameter families (or branches) of stationary solutions. Most significant is that the stationary solution for (2.8a,b) given by

$$\xi_1 = -\xi_2 = 2bh = \frac{6(3N^3 - L^2 N2)}{L^3}, \tag{3.1}$$

along with conditions (2.2) and (2.3), grows without bound as $L \to 0$, when in fact for the PDE, it is easy to show by energy estimates that the global attractor for $L < 2\pi$ is simply the trivial solution $u \equiv 0$. This branch represents a significant deviation from the true dynamics of the PDE and is therefore spurious. It also follows from (2.17) that some of the solutions in the unstable manifold blow up in finite time. These branches are shown in figure 3.1 for $N = 6$ (reduced dynamical system of dimension two) after rescaling time and space so that (KS) becomes

$$\frac{\partial u}{\partial t} + 4\frac{\partial^4 u}{\partial x^4} + \alpha\left[\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}\right] = 0, \quad 0 \le x \le 2\pi, \tag{$KS\alpha$}$$

where $\alpha = L^2/\pi^2$. All computations presented in this paper are done for (KSα). The bifurcation diagrams are produced using the package AUTO [Doedel (1981)].

For $N = 6$ and σ, δ, a, and b defined as above, the stationary solution equations for (2.8c) reduce to

$$a\sigma - \sigma\delta/6h = 0, \quad b\delta + \sigma^2/6h = 0. \tag{3.2}$$

There are then two nontrivial stationary solutions derived from (3.2), one of which is given by

$$\sigma = 6h\sqrt{-ab},\ \delta = 6ha,\ \text{for } 1 \le h \le \sqrt{3}. \tag{3.3}$$

The other occurs when $b = 9h^{-4} - 3h^{-2} = 0$, i.e. when $h = L/N = \sqrt{3}$, and has $\sigma = 0$ with δ unspecified (see figure 3.2). With h fixed, the existence of these latter type of steady states with arbitrarily large δ component implies that (2.8c) is not dissipative for all values of L. However the solutions in (3.2) and (3.3) do not exist in the region delimited by Theorem 2.1 since condition (2.9) is equivalent to

$$h^2 \le \left(\frac{3}{\sqrt{24N}+1}\right) < 1.$$

The branch in (3.3) is eliminated by condition (2.9) it is bounded, and in fact for certain fixed N, may play the role of what is considered to be a true branch of stationary solutions, i.e. one observed to persist under finer and finer discretizations. For example in figure 3.3 for $N = 12$ (so the reduced system is of dimension five) this branch which exists between $\alpha \approx 16$ and $\alpha \approx 43$ displays the same secondary bifurcations as the 'bimodal' branch commonly observed in computational studies of the KSE. Note however, that the branch in (3.3) will migrate with increasing L as h decreases, rather than converge to a fixed branch. The diagram in figure 3.3 is quite similar to the diagram for the five mode Galerkin approximation while that for $N = 24$ (reduced system has dimension 11, in figure 3.4) is virtually identical to that for the 12 mode Galerkin scheme [see Jolly *et al.* (1990a) to compare].

3.2. Galerkin and Nonlinear Galerkin Approximations

The appearance of a vertical, dissipation destroying branch of stationary solutions has also been observed for the Galerkin approximation of the KSE [Jolly *et al.*(1990a)]. The success in analyzing this phenomenon for the finite difference approximation has led to the following rigorous treatment of the Galerkin case. Let P_M be the projection of the space of odd, periodic functions onto the span of $\{\sin(x), \ldots, \sin(Mx)\}$. The M-mode Galerkin approximation amounts to substituting

$$p_M(x,t) = \sum_{j=1}^{M} a_j(t)\sin(jx)$$

into $(KS\alpha)$ and applying the projection P_M. Thus the condition for a stationary solution to this approximation reads

$$P_M\left[\frac{\partial^4 p_M}{\partial x^4} + \alpha\frac{\partial^2 p_M}{\partial x^2} + \alpha p_M\frac{\partial p_M}{\partial x}\right] = 0. \tag{3.4}$$

The vertical branch will occur when the linear part of (3.4) vanishes for the j^{th} mode, for large enough j. More precisely we have

Proposition 3.1. *The functions $p_M(x) = a_j\sin(jx)$, $a_j \in \mathbf{R}$, satisfy (3.4) provided $\alpha = 4j^2$ and $M/2 < j \leq M$.*

Proof:

By the choice of α we have that

$$\frac{\partial^4 p_M}{\partial x^4} + \alpha\frac{\partial^2 p_M}{\partial x^2} = 0.$$

Also, since

$$p_m\frac{\partial p_m}{\partial x} = \frac{a_j^2}{2}\sin(2jx),$$

and $M < 2j$, we have that

$$P_M[\alpha p_m\frac{\partial p_m}{\partial x}] = 0.$$

Unfortunately such a simple explanation does not seem available for the presence of spurious stationary solution branches for certain nonlinear Galerkin approximations of $(KS\alpha)$ [see Jolly *et al.* (1990a,b)]. These methods express the higher modes represented by

$$q_M(x) = \sum_{j \geq M+1} a_j \sin(jx)$$

in terms of the lower modes represented by p_M. Consider for example the nonlinear Galerkin approximation

$$\frac{dp_M}{dt} + Ap_M + P_M F(p_M + q_M) = 0 \qquad q_M = -A^{-1} Q_M F(p_M)\,, \tag{3.5}$$

where

$$A = \frac{\partial^4}{\partial x^4}, \quad F(u) = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha u \frac{\partial u}{\partial x}\ , \text{and} \quad Q_M = I - P_M\,.$$

In this case there is a spurious branch of stationary solutions which behaves similarly to those for the finite difference discretizations (2.8a,b) in that apparently it grows without bound as $L \to 0$. The blowup of (2.8a,b) under these circumstances suggests that the same phenomenon may accompany the spurious branch for the nonlinear Galerkin system. Indeed, certain numerical trajectories for the latter system, starting with initial condition data near the spurious stationary solution, but with slightly larger norm, also appear to blow up in finite time (see figure 3.5).

4. Fully Discrete Finite Difference Schemes

In this section we present stability results for two fully discrete finite difference schemes for KS. It is convenient to adopt the following notation. Let $S^N_{\text{odd,per}} = \{$ all double infinite sequences satisfying (2.1) and (2.2)$\}$. We will represent the elements of $S^N_{\text{odd,per}}$ by N-dimensional vectors $\vec{\xi} = (\xi_i)_{i=0}^{N-1}$ with the understanding that $\vec{\xi}$ satisfies (2.2) and can be extended by (2.1). Throughout this section, we denote by $\langle \cdot, \cdot \rangle$ the inner product

$$\langle \vec{\xi}, \vec{\eta} \rangle \equiv h \sum_{k=0}^{N-1} \xi_k \eta_k,$$

and by $|\cdot|$, the induced norm. We recall the following results.

Proposition 4.1. *(Foias and Titi (1990)) Let $B^h : S^N_{\text{odd,per}} \times S^N_{\text{odd,per}} \longrightarrow S^N_{\text{odd,per}}$ be defined as follows: for every $\vec{\xi}, \vec{\eta} \in S^N_{\text{odd,per}}$*

$$B^h_k(\vec{\xi}, \vec{\eta}) = \frac{\xi_k(\eta_{k+1} - \eta_{k-1}) + \xi_{k+1}\eta_{k+1} - \xi_{k-1}\eta_{k-1}}{6h} \tag{4.1}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Then

$$h \sum_{k=0}^{N-1} B^h_k(\vec{\xi}, \vec{\eta})\eta_k = \langle B^h(\vec{\xi}, \vec{\eta}), \vec{\eta} \rangle = 0. \tag{4.2}$$

Remark 4.2. From Proposition 4.1 it follows easily that

$$\langle B^h(\vec{\xi}, \vec{\eta}_1), \vec{\eta}_2 \rangle = -\langle B^h(\vec{\xi}, \vec{\eta}_2), \vec{\eta}_1 \rangle.$$

The next result is well known.

Proposition 4.3. *Let $\Delta_h : \mathrm{R}^N \times \mathrm{R}^N$ be the matrix*

$$\Delta_h = \frac{-1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & -1 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & & -1 & 2 & -1 \\ -1 & 0 & & 0 & -1 & 2 \end{pmatrix}.$$

Set $\omega = e^{\frac{2\pi i}{N}}$, *then the real and the imaginary parts of the vector* $(1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k})$ *are eigenvectors of* $-\Delta_h$ *with corresponding eigenvalue* $\mu_k = \frac{2}{h^2}\left(1 - \cos\left(\frac{2\pi}{N}k\right)\right)$, *for* $k = 0, 1, \ldots N-1$.

Corollary 4.4. *(Foias and Titi (1990)) The matrix* $(-\Delta_h)$ *is a symmetric nonnegative definite. Moreover, for every* $\vec{\xi} \in S^N_{\mathbf{odd,per}}$ *we have*

$$\langle(-\Delta_h)\vec{\xi}, \vec{\xi}\rangle = h \sum_{k=0}^{N-1} \left[(-\Delta_h)\vec{\xi}_k \xi_k\right] \geq \mu_1 |\vec{\xi}|^2 \tag{4.3}$$

where $\mu_1 = \frac{2}{h^2}\left(1 - \cos\left(\frac{2\pi}{N}\right)\right) = \frac{4}{h^2}\sin^2\left(\frac{\pi}{N}\right)$. *Notice that for* $h \ll 1, \mu_1 \geq \frac{2\pi^2}{L^2}$.

We may now express system (2.8c) as

$$\begin{cases} \frac{d\vec{\xi}}{dt} + \Delta_h^2\vec{\xi} + \Delta_h\vec{\xi} + B^h(\vec{\xi}, \vec{\xi}) = 0 \\ \vec{\xi}(0) = \vec{\xi}_0 \in S^N_{\text{odd,per}} \end{cases}. \tag{4.4}$$

4.1 An Implicit Scheme

Consider the following implicit fully discrete scheme for (KS)

$$\vec{\xi}^{n+1} - \vec{\xi}^{n} + \tau\Delta_h^2\vec{\xi}^{n+1} + \tau\Delta_h\vec{\xi}^{n+1} + \tau B^h(\vec{\xi}^{n+1}, \vec{\xi}^{n+1}) = \vec{0}. \tag{4.5}$$

We will show below that the discrete dynamical system in (4.5) is dissipative. Moreover, we will derive a stability condition involving the discretization parameters h and τ, which is independent of initial condition. The proof requires the translation by a fixed vector defined in the following lemma.

Lemma 4.5. *(Foias and Titi (1990)) Let* $\vec{\eta} \in S^N_{\text{odd,per}}$ *be such that*

$$\begin{cases} \eta_j &= jh - \frac{L}{2} \quad j = 1, \ldots, N-1, \\ \eta_0 &= \eta_N = 0 \end{cases} \tag{4.6}$$

and let h *be small enough so that*

$$(3 - h^2)^2 - 24h^3 L \geq 0. \tag{4.7}$$

Then for every $\vec{\xi} \in S^N_{\text{odd,per}}$ we have

$$\langle \Delta_h^2 \vec{\xi}, \vec{\xi} \rangle + \langle \Delta_h \vec{\xi}, \vec{\xi} \rangle + \langle B^h(\vec{\xi}, \vec{\eta}), \vec{\xi} \rangle \geq \frac{1}{4} |(-\Delta_h)\vec{\xi}|^2 : \tag{4.8}$$

which implies that the linear operator

$$\Delta_h^2 + \Delta_h + B^h(\cdot, \vec{\eta})$$

is coercive.

Replacing $\vec{\xi}$ by $\vec{\xi} + \vec{\eta}$, we rewrite (4.5) as

$$\begin{aligned} \vec{\xi}^{n+1} - \vec{\xi}^{n} + \tau \Delta_h^2 \vec{\xi}^{n+1} + \tau \Delta_h \vec{\xi}^{n+1} & \\ + \tau \left[B^h(\vec{\xi}^{n+1}, \vec{\eta}) + B^h(\vec{\eta}, \vec{\xi}^{n+1}) + B^h(\vec{\xi}^{n+1}, \vec{\xi}^{n+1}) \right] &= \vec{f}, \end{aligned} \tag{4.9}$$

$$\text{where } \vec{f} = -B^h(\vec{\eta}, \vec{\eta}) - \Delta_h^2 \vec{\eta} - \Delta_h \vec{\eta}. \tag{4.9a}$$

We now state the main result of this subsection.

Proposition 4.6. *Suppose $\vec{\xi}^n$, $n = 0, 1, 2, \ldots$ is a solution to (4.9). For all $n \geq 0$ we have*

$$|\vec{\xi}^{n+1}|^2 \leq |\vec{\xi}^0|^2 (1+\delta)^{-(n+1)} + \frac{L^4}{64} |\vec{f}|^2 (1 - (\delta+1)^{-(n+1)}), \tag{4.10}$$

where $\delta = \tau \frac{64}{L^4}$, and $\vec{f}$ is as in (4.9a). Thus the ball of radius $|\vec{f}|^2 \frac{L^4}{64}$ is invariant and absorbing.

Proof:

Taking the inner product of (4.9) with $\vec{\xi}^{n+1}$, and using Proposition 4.1 we find that

$$\begin{aligned} 2\langle \vec{\xi}^{n+1} - \vec{\xi}^{n}, \vec{\xi}^{n+1} \rangle + 2\tau \left[\langle \Delta_h^2 \vec{\xi}^{n+1}, \vec{\xi}^{n+1} \rangle + \langle \Delta_h \vec{\xi}^{n+1}, \vec{\xi}^{n+1} \rangle + \langle B^h(\vec{\xi}^{n+1}, \vec{\eta}), \vec{\xi}^{n+1} \rangle \right] & \\ = 2\tau \langle \vec{f}, \vec{\xi}^{n+1} \rangle. & \end{aligned}$$

Now by Lemma 4.5 we have

$$2\langle \vec{\xi}^{n+1} - \vec{\xi}^{n}, \vec{\xi}^{n+1} \rangle + 2\tau \frac{1}{4} |(-\Delta_h)\vec{\xi}^{n+1}|^2 \leq 2\tau |\vec{f}| |\vec{\xi}^{n+1}|. \tag{4.11}$$

Using the Hilbert space identity

$$2\langle v-w,v\rangle \equiv |v|^2+|v-w|^2-|w|^2 \tag{4.12}$$

and (4.11) we obtain

$$|\vec{\xi}^{n+1}|^2+|\vec{\xi}^{n+1}-\vec{\xi}^{n}|^2-|\vec{\xi}^{n}|^2+\frac{\tau}{2}|(-\Delta_h)\vec{\xi}^{n+1}|^2 \leq 2\tau|\vec{f}||\vec{\xi}^{n+1}|. \tag{4.13}$$

¿From Corollary 4.4 and the fact that $\sin(x)\geq \frac{2}{\pi}x$ for $0\leq x\leq \pi/2$ it follows that

$$\begin{aligned}|(-\Delta_h)\vec{\xi}^{n+1}|^2 \geq \mu_1^2|\vec{\xi}^{n+1}|^2 = \left[\frac{16}{h^4}\sin^4(\pi/N)\right]|\vec{\xi}^{n+1}|^2 &\geq \frac{16}{h^4}\left[\frac{2}{\pi}\left(\frac{\pi}{N}\right)\right]^4|\vec{\xi}^{n+1}|^2\\ &= \left(\frac{16}{L^2}\right)^2|\vec{\xi}^{n+1}|^2\end{aligned}$$

for $N>2$. Hence, from (4.13) we have

$$|\vec{\xi}^{n+1}|^2+|\vec{\xi}^{n+1}-\vec{\xi}^{n}|^2-|\vec{\xi}^{n}|^2+\frac{\tau}{2}\left(\frac{16}{L^2}\right)^2|\vec{\xi}^{n+1}|^2 \leq 2\tau|\vec{f}||\vec{\xi}^{n+1}|.$$

By Young's inequality it follows that

$$|\vec{\xi}^{n+1}|^2+|\vec{\xi}^{n+1}-\vec{\xi}^{n}|^2-|\vec{\xi}^{n}|^2+\frac{\tau}{4}\left(\frac{16}{L^2}\right)^2|\vec{\xi}^{n+1}|^2 \leq 4\tau|\vec{f}|^2\frac{L^4}{(16)^2}$$

and hence

$$|\vec{\xi}^{n+1}|^2\left[1+\tau\left(\frac{8}{L^2}\right)^2\right] \leq |\vec{\xi}^{n}|^2+\left(\frac{L^2}{8}\right)^2\tau|\vec{f}|^2. \tag{4.14}$$

Define $\delta\equiv\tau\left(\frac{8}{L^2}\right)^2$. We may then rewrite the inequality (4.13) as

$$|\vec{\xi}^{n+1}|^2 \leq (1+\delta)^{-1}|\vec{\xi}^{n}|^2+\left(\frac{\tau^2}{\delta}\right)|\vec{f}|^2(1+\delta)^{-1} \tag{4.15}$$

for $n=0,1,2,\ldots$. Thus we have that

$$|\vec{\xi}^{n+1}|^2 \leq (1+\delta)^{-2}|\vec{\xi}^{n-1}|^2+\left(\frac{\tau^2}{\delta}\right)|\vec{f}|^2\left[(1+\delta)^{-1}+(1+\delta)^{-2}\right],$$

and in general that

$$
\begin{aligned}
|\vec{\xi}^{n+1}|^2 &\le (1+\delta)^{-(n+1)}|\vec{\xi}^0|^2 + (\frac{\tau^2}{\delta})|\vec{f}|^2 [(1+\delta)^{-1} + \ldots + (1+\delta)^{-(n+1)}] \\
&= (1+\delta)^{-(n+1)}|\vec{\xi}^0|^2 + (\frac{\tau^2}{\delta})\frac{|\vec{f}|^2}{(1+\delta)}\left[\frac{1-(1+\delta)^{-(n+1)}}{1-(1+\delta)^{-1}}\right] \\
&= (1+\delta)^{-(n+1)}|\vec{\xi}^0|^2 + (\frac{\tau^2}{\delta^2})|\vec{f}|^2 [1-(1+\delta)^{-(n+1)}],
\end{aligned}
$$

from which (4.10) follows immediately.

Remark 4.7. Notice that a similar result holds for the untranslated scheme in (4.5), with the radius of the absorbing ball replaced by $|\vec{f}|^2 \frac{L^4}{64} + |\vec{\eta}|$. We also remark that the last estimate in the proof above is similar to the discrete Gronwall inequality which appeared in Eden *et al.* (1990) and Shen (1989).

Next we show that the scheme in (4.5) has a solution. Let us recall the following classical proposition of Poincaré from Constantin and Foias (1988), page 58.

Proposition 4.8. *Let $\mathcal{B}$ be a closed ball in R^n. Suppose that $\Phi :\rightarrow \mathrm{R}^n$ is continuous and $(\Phi(v), v) < 0$ for all $v \in \partial\mathcal{B}$. Then there exists $v \in \mathcal{B}$ such that $\Phi(v) = 0$.*

We now apply this result to establish the existence of a solution to (4.5). Assume that $\vec{\xi}^n$ is given. We want to show that we can find $\vec{\xi}^{n+1}$ satisfying (4.9). Let $\mathcal{B}$ be the ball of radius R, where

$$
R^2 = 2(1+\delta)^{-1}[|\vec{\xi}^n|^2 + \frac{\tau^2}{\delta}|\vec{f}|^2], \tag{4.16}
$$

and δ is defined as in the preceding proof. ¿From (4.15) we have that all solutions $\vec{\xi}^{n+1}$ of (4.9) satisfy

$$
|\vec{\xi}^{n+1}|^2 < \frac{3R^2}{4}. \tag{4.17}
$$

Define

$$
-\Phi(\vec{\xi}) = 2(\vec{\xi} - \vec{\xi}^n) + 2\tau[\Delta_h^2\vec{\xi} + \Delta_h\vec{\xi} + B^h(\vec{\xi},\vec{\eta}) + B^h(\vec{\eta},\vec{\xi}) + B^h(\vec{\xi},\vec{\xi}) - \vec{f}].
$$

Take the inner product with $\vec{\xi}$. Then, as in the proof of Proposition 4.6, we have

$$\begin{aligned}
-\langle \Phi(\vec{\xi}), \vec{\xi} \rangle &\geq |\vec{\xi}|^2 + |\vec{\xi} - \vec{\xi}^n|^2 - |\vec{\xi}^n|^2 + \frac{\tau}{2}|(-\Delta_h)\vec{\xi}|^2 - 2\tau|\vec{f}||\vec{\xi}| \\
&\geq |\vec{\xi}|^2 - |\vec{\xi}^n|^2 + \delta|\vec{\xi}|^2 - \frac{\tau}{\delta}|\vec{f}|^2 \\
&\geq (1+\delta)\left[|\vec{\xi}|^2 - (1+\delta)^{-1}(|\vec{\xi}^n|^2 + \frac{\tau}{\delta}|\vec{f}|^2)\right]
\end{aligned}$$

for $\vec{\xi} \in \partial B$ we have $|\vec{\xi}|^2 = R^2$, so that

$$-\langle \Phi(\vec{\xi}), \vec{\xi} \rangle \geq (1+\delta)\left[R^2 - \frac{1}{2}R^2\right] > 0.$$

So by Proposition 4.8 we have a solution.

As for uniqueness, we can show that it holds up to a time step that depends on the initial data.

Proposition 4.9. *There exists a time step $\tau_0 = \tau_0(|\vec{\xi^0}|, \vec{f})$ such that if $0 < \tau < \tau_0$, the solution to (4.5) with initial condition $\vec{\xi^0}$ is unique.*

The proof of Proposition 4.9 will use the following lemma which is proved in the Appendix.

Lemma 4.10. *The following hold for all $\vec{\xi}$, $\vec{\omega} \in S^N_{\mathrm{odd,per}}$*

(A) $$|\langle B^h(\vec{\omega}, \vec{\omega}), \vec{\xi} \rangle| \leq 2|(-\Delta)^{1/2}\vec{\omega}||\vec{\xi}| \max_{0 \leq k \leq N-1} |\omega_k|$$

(B) $$\max_{0 \leq k \leq N-1} |\omega_k| \leq \sqrt{2}|\vec{\omega}|^{1/2}|(-\Delta)^{1/2}\vec{\omega}|^{1/2} .$$

Proof of Proposition 4.9:

Suppose $\vec{\xi^n}$ is given and consider two solutions $\vec{\xi}_1^{n+1}$ and $\vec{\xi}_2^{n+1}$ of (4.9). Let $\vec{\omega} = \vec{\xi}_1^{n+1} - \vec{\xi}_2^{n+1}$, so that

$$\vec{\omega} + \tau\Delta_h^2\vec{\omega} + \tau\Delta_h\vec{\omega} + \tau B^h(\vec{\omega}, \vec{\eta}) + \tau B^h(\vec{\eta}, \vec{\omega}) + \tau B^h(\vec{\omega}, \vec{\xi}_1^{n+1}) + \tau B^h(\vec{\xi}_2^{n+1}, \vec{\omega}) = \vec{0} .$$

We take the inner product with $\vec{\omega}$ and apply Proposition 4.1 and Lemma 4.5 to get

$$|\vec{\omega}|^2 + \frac{\tau}{4}|(-\Delta_h)\vec{\omega}|^2 \leq \tau|\langle B^h(\vec{\omega}, \vec{\xi}_1^{n+1}), \vec{\omega}\rangle| \,.$$

Applying in order, Remark 4.2, Lemma 4.10 parts A and B, and the inequalities of Cauchy-Schwarz and Young, we have

$$\begin{aligned}
|\vec{\omega}|^2 + \frac{\tau}{4}|(-\Delta_h)\vec{\omega}|^2 &\leq \tau|\langle B^h(\vec{\omega}, \vec{\omega}), \vec{\xi}_1^{n+1}\rangle| \\
&\leq 2\tau|(-\Delta_h)^{1/2}\vec{\omega}||\vec{\xi}_1^{n+1}| \max_{0\leq k\leq N-1} |\omega_k| \\
&\leq 2\tau|(-\Delta_h)^{1/2}\vec{\omega}|^{3/2}\sqrt{2}|\vec{\omega}|^{1/2}|\vec{\xi}_1^{n+1}| \\
&\leq 2^{3/2}\tau|(-\Delta_h)\vec{\omega}|^{3/2}|\vec{\omega}|^{5/4}|\vec{\xi}_1^{n+1}| \\
&\leq \frac{\tau}{4}|(-\Delta_h)\vec{\omega}|^2 + \left[5\tau\left(\frac{3}{4}\right)^{3/5}|\vec{\xi}_1^{n+1}|\right]|\vec{\omega}|^2 \,.
\end{aligned}$$

¿From (4.17) we have that for all n

$$|\vec{\xi}_1^{n+1}|^2 < \frac{3R^2}{4},$$

where $R = R(|\vec{f}|, |\vec{\xi}^0|)$ is defined in (4.16). Therefore, if

$$5\tau\left(\frac{3}{4}\right)^{3/5}\left(\frac{3R^2}{4}\right)^{8/5} < 1,$$

it follows that $\vec{\omega} = \vec{0}$.

4.2 A semi-implicit Scheme

We now analyze the scheme

$$\vec{\xi}^{n+1} - \vec{\xi}^n + \tau\Delta_h^2\vec{\xi}^{n+1} + \tau\Delta_h\vec{\xi}^{n+1} + \tau B^h(\vec{\xi}^n, \vec{\xi}^{n+1}) = \vec{0}. \tag{4.18}$$

Since (4.18) is a linear system, it is enough to show uniqueness to infer existence. This will also establish the invertibility of the operator defined by

$$\vec{\xi} \mapsto \vec{\xi} + \tau\Delta_h^2\vec{\xi} + \tau\Delta_h\vec{\xi} + \tau B^h(\vec{\xi}^n, \vec{\xi})$$

so that the scheme in (4.18) is indeed semi-implicit.

To this end let $\vec{\xi}^n$ be given, and suppose $\vec{\xi}_1^{n+1}$ and $\vec{\xi}_2^{n+1}$ are two solutions of (4.18). Then $\vec{\omega} = \vec{\xi}_1^{n+1} - \vec{\xi}_2^{n+1}$ satisfies

$$\vec{\omega} + \tau\Delta_h^2\vec{\omega} + \tau\Delta_h\vec{\omega} + \tau B^h(\vec{\xi}^n, \vec{\omega}) = \vec{0}$$

Taking the inner product with $\vec{\omega}$, observing (4.2) and Young's inequality we have

$$|\vec{\omega}|^2 + \tau|\Delta_h\vec{\omega}|^2 \leq \tau|\langle\Delta_h\vec{\omega}, \vec{\omega}\rangle| \leq \tau|\Delta_h\vec{\omega}||\vec{\omega}| \leq \frac{\tau}{4}|\Delta_h\vec{\omega}|^2 + |\vec{\omega}|^2.$$

Thus a solution to (4.18) exists and is unique provided $\tau < 4$.

To carry out the stability analysis, we proceed as above and translate by $\vec{\eta}$, as defined in Lemma 4.5, to obtain

$$\begin{aligned} &\vec{\xi}^{n+1} - \vec{\xi}^n + \tau\Delta_h^2\vec{\xi}^{n+1} + \tau\Delta_h\vec{\xi}^{n+1} \\ &\qquad + \tau\big[B^h(\vec{\xi}^n, \vec{\eta}) + B^h(\vec{\eta}, \vec{\xi}^{n+1}) + B^h(\vec{\xi}^n, \vec{\xi}^{n+1})\big] = \tau\vec{f}, \end{aligned} \tag{4.19}$$

where $\vec{f}$ is as in (4.9a).

Proposition 4.11. *Let τ, h satisfy*

$$0 \leq \tau \leq \frac{4\cdot 18\pi}{L^4}\Big[\frac{16\pi^2h^2}{L^2 + 16\pi^2h^2}\Big]. \tag{4.20}$$

Suppose $\vec{\xi}^n$, $n = 0, 1, 2, \ldots$ is a solution to (4.19). For all $n \geq 0$ we have

$$|\vec{\xi}^{n+1}|^2 \leq |\vec{\xi}^0|^2(1+\delta)^{-(n+1)} + \Big(\frac{L}{\pi}\Big)^4|\vec{f}|^2(1 - (\delta+1)^{-(n+1)}), \tag{4.21}$$

where $\delta = \frac{\tau}{8}\mu_1^2$, and $\vec{\eta}$ is as in Lemma 4.5. Thus the ball of radius $|\vec{f}|^2(\frac{L}{\pi})^4$ is invariant and absorbing.

We show the contrast of using the bilinear term in (2.8b) as opposed to using that in (2.8c) in a fully discrete scheme in figures 4.1a,b. The proof of Proposition 4.11 will require the following.

Lemma 4.12. *For all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^N one has that*

$$|\langle B^h(\vec{u},\vec{v}),\vec{w}\rangle| \le \frac{1}{6}|\vec{u}||\vec{w}| \max_{0\le k\le N-1} \frac{|v_{k+1}-v_{k-1}|}{h} + \frac{1}{3}|\vec{u}||(-\Delta_h)^{1/2}\vec{w}| \max_{0\le k\le N-1} |v_k| . \tag{4.22}$$

The proof of Lemma 4.12 is in the Appendix.

Proof of Proposition 4.11:

As in the proof of Proposition 4.6, we start by taking the inner product of equation (4.19) and observing Proposition 4.1 to obtain

$$2\langle\vec{\xi}^{n+1}-\vec{\xi}^{n},\vec{\xi}^{n+1}\rangle + 2\tau\big[\langle\Delta_h^2\vec{\xi}^{n+1},\vec{\xi}^{n+1}\rangle + \langle\Delta_h\vec{\xi}^{n+1},\vec{\xi}^{n+1}\rangle + \langle B^h(\vec{\xi}^{n+1},\vec{\eta}),\vec{\xi}^{n+1}\rangle + \langle B^h(\vec{\xi}^{n}-\vec{\xi}^{n+1},\vec{\eta}),\vec{\xi}^{n+1}\rangle\big] = 2\tau\langle\vec{f},\vec{\xi}^{n+1}\rangle.$$

Applying once again Lemma 4.5 and (4.12) we have

$$|\vec{\xi}^{n+1}|^2 + |\vec{\xi}^{n+1}-\vec{\xi}^{n}|^2 - |\vec{\xi}^{n}|^2 + \frac{\tau}{2}|(-\Delta_h)\vec{\xi}^{n+1}|^2 + 2\langle B^h(\vec{\xi}^{n}-\vec{\xi}^{n+1},\vec{\eta}),\vec{\xi}^{n+1}\rangle \le 2\tau|\vec{f}||\vec{\xi}^{n+1}|. \tag{4.23}$$

It follows from (4.6) that

$$\max_{0\le k\le N-1}|\eta_k| \le \frac{L}{2}, \qquad \max_{0\le k\le N-1}\left|\frac{\eta_{k+1}-\eta_{k-1}}{h}\right| \le \frac{L}{2h}. \tag{4.24}$$

By Lemma 4.12, (4.24), and Young's inequality we now have that

$$\begin{aligned} &2\tau|\langle B^h(\vec{\xi}^{n}-\vec{\xi}^{n+1},\vec{\eta}),\vec{\xi}^{n+1}\rangle| \\ &\le \frac{\tau}{3}|\vec{\xi}^{n+1}-\vec{\xi}^{n}||\vec{\xi}^{n+1}|\frac{L}{2h} + \frac{2\tau}{3}|\vec{\xi}^{n+1}-\vec{\xi}^{n}||(-\Delta_h)^{1/2}\vec{\xi}^{n+1}|\frac{L}{2} \\ &\le \frac{|\vec{\xi}^{n+1}-\vec{\xi}^{n}|^2}{2} + \frac{\tau^2}{18}|\vec{\xi}^{n+1}|^2\frac{L^2}{4h^2} + \frac{|\vec{\xi}^{n+1}-\vec{\xi}^{n}|^2}{2} + \frac{4\tau^2}{9\cdot 2}|(-\Delta_h)^{1/2}\vec{\xi}^{n+1}|^2\frac{L^2}{4}. \end{aligned}$$

¿From (4.23) and Young's inequality we now have that

$$\begin{aligned} &|\vec{\xi}^{n+1}|^2 - |\vec{\xi}^{n}|^2 + \frac{\tau}{2}|(-\Delta_h)\vec{\xi}^{n+1}|^2 \\ &\le \frac{\tau^2L^2}{18\cdot 4h^2}|\vec{\xi}^{n+1}|^2 + \frac{\tau^2L^2}{18}|(-\Delta_h)\vec{\xi}^{n+1}|^2 + 2\tau|\vec{f}||\vec{\xi}^{n+1}| \\ &\le \frac{\tau^2L^2}{18\cdot 4h^2\mu_1^2}|(-\Delta_h)\vec{\xi}^{n+1}|^2 + \frac{\tau^2L^2}{18\mu_1}|(-\Delta_h)\vec{\xi}^{n+1}|^2 + \frac{2\tau}{\mu_1}|\vec{f}||(-\Delta_h)\vec{\xi}^{n+1}| \\ &\le |(-\Delta_h)\vec{\xi}^{n+1}|^2\left(\frac{\tau^2L^2}{18\cdot 4h^2\mu_1^2} + \frac{\tau^2L^2}{18\mu_1}\right) + \frac{4\tau}{\mu_1^2}|\vec{f}|^2 + \frac{\tau}{4}|(-\Delta_h)\vec{\xi}^{n+1}|^2. \end{aligned}$$

It follows that

$$|\vec{\xi}^{n+1}|^2 - |\vec{\xi}^n|^2 + \frac{\tau}{4}|(-\Delta_h)\vec{\xi}^{n+1}|^2 \le |(-\Delta_h)\vec{\xi}^{n+1}|^2\Big(\frac{\tau^2 L^2}{18\cdot 4h^2\mu_1^2} + \frac{\tau^2 L^2}{18\mu_1}\Big) + \frac{4\tau}{\mu_1^2}|\vec{f}|^2. \quad (4.25)$$

Using the fact that

$$\mu_1 = \frac{4}{h^2}\sin^2\Big(\frac{\pi}{N}\Big) \le \frac{4}{h^2}\Big(\frac{\pi}{N}\Big)^2 = \frac{4}{h^2}\Big(\frac{\pi h}{L}\Big)^2 = \Big(\frac{2\pi}{L}\Big)^2$$

we have

$$\Big[\frac{4\cdot 18\pi}{L^4}\Big(\frac{16\pi^2h^2}{L^2+16\pi^2h^2}\Big)\Big]^{-1} = \Big(\frac{L^2}{18}\Big)\Big(\frac{L}{2\pi}\Big)^2\Big[\frac{1}{4h^2}\Big(\frac{L}{2\pi}\Big)+1\Big]$$
$$\le \frac{L^2}{18\mu_1}\Big[\frac{1}{4h^2\mu_1}+1\Big].$$

Thus whenever condition (4.20) holds, we will satisfy the quadratic inequality

$$\frac{\tau}{8} \ge \frac{\tau^2L^2}{18\cdot 4h^2\mu_1^2} + \frac{\tau^2L^2}{18\mu_1}. \quad (4.26)$$

Combining (4.25) and (4.26) with the lower bound for μ_1 in Corollary 4.4 we have that

$$|\vec{\xi}^{n+1}|^2 - |\vec{\xi}^n|^2 + \frac{\tau}{8}\mu_1^2|\vec{\xi}^{n+1}|^2 \le |\vec{\xi}^{n+1}|^2 - |\vec{\xi}^n|^2 + \frac{\tau}{8}|(-\Delta_h)\vec{\xi}^{n+1}|^2$$
$$\le \frac{4\tau}{\mu_1^2}|\vec{f}|^2 \le \Big(\frac{L}{\pi}\Big)^4|\vec{f}|^2.$$

The rest of the proof is as in that of Proposition 4.6, starting from (4.14).

5. A fully discrete Galerkin scheme

Using a similar technique as in the last section, we will show that a certain fully discrete, semi-implicit Galerkin scheme has a stability condition which is independent of initial condition. To define our scheme we first write the KSE in functional form. It is well known (see e.g. Foias *et al.* (1988), Temam (1988)) that (KS) is equivalent to

$$\frac{du}{dt} + Au - A^{1/2}u + B(u,u) = 0, \quad u \in H \tag{5.1}$$

where $A = \partial^4/\partial x^4$ with domain $D(A) = H^4_{\text{per}}((0,L))$, B is the bilinear operator defined by

$$B(u,v) = u\frac{\partial v}{\partial x}, \quad \text{for all } u,v \in H^1_{\text{per}}((0,L)), \tag{5.2}$$

and

$$H = \{u \in L^2((0,L)) \big| u(x,t) = u(x+L,t),\ u(x,t) = -u(L-x,t),\ a.e.\ x \in \mathrm{R}\},$$

the subspace of $L^2((0,L))$ of periodic odd functions. The space H has a complete orthonormal basis consisting of eigenfunctions of A, $\{w_j\}_{j=1}^{\infty}$, corresponding to eigenvalues λ_j for $j = 1,2,\ldots$. We define $P = P_M$ to be the orthogonal projection from H onto $\text{span}\{w_1, w_2, \ldots, w_m\}$. The Hilbert space H is endowed with the L^2 inner product denoted by $(\cdot,\cdot)$. Its corresponding norm will be denoted by $|\cdot|$ throughout this section.

The scheme requires a special decomposition of the bilinear function used in Jolly *et al.* (1990b). Let

$$\tilde{B}(u,v) = \frac{1}{3}v\frac{\partial u}{\partial x} + \frac{2}{3}u\frac{\partial v}{\partial x}, \quad \text{for all } u,v \in H^1_{\text{per}}((0,L)). \tag{5.3}$$

It is clear that

$$(\tilde{B}(u,v),w) = -(\tilde{B}(u,w),v), \quad \text{for all } u,v,w \in H^1_{\text{per}}((0,L)), \tag{5.4}$$

that

$$(\tilde{B}(u,v),v) = 0, \quad \text{for all } u,v \in H^1_{\text{per}}((0,L)), \tag{5.5}$$

and also that

$$\tilde{B}(u,u) = B(u,u), \quad \text{for all } u \in H^1_{\text{per}}((0,L)). \tag{5.6}$$

Thus the semi-discrete Galerkin approximation to (KS) can be written as

$$\frac{dp}{dt} + Ap - A^{1/2}p + P\tilde{B}(p,p) = 0.$$

We now define our fully discrete Galerkin scheme to be

$$p^{n+1} - p^n + \tau A p^{n+1} - \tau A^{1/2} p^{n+1} + \tau P\tilde{B}(p^n, p^{n+1}) = 0\ . \tag{5.7}$$

The derivation of the stability condition for (5.7) will require an analog of the translation used in section 4. Here we replace $\vec{\eta}$ defined in (4.5) with the gauge function ϕ introduced in Nicolaenko *et al.* (1985) and recalled below.

Proposition 5.1. *(Nicolaenko, Scheurer and Temam) For every $L > 0$ there exists an $M(L) \sim (\frac{L}{2\pi})^{8/3}$ and an odd trigonometric polynomial $\phi \in P_M H$ such that*

$$|A^{1/2}u|^2 - |A^{1/4}u|^2 - (B(u,u),\phi) \geq \frac{1}{4}|A^{1/2}u|^2 + \lambda_1|u|^2, \tag{5.8}$$

for all $u \in H^2_{\text{per}}((0,L)) \cap H$.

Note that by (5.6) the above result holds for $\tilde{B}$.

Let $p = \tilde{p} + \phi$, where $\tilde{p} \in P_M H$. Then equation (5.7) becomes:

$$\tilde{p}^{n+1} - \tilde{p}^n + \tau A\tilde{p}^{n+1} - \tau A^{1/2}\tilde{p}^{n+1} + \tau P\big[\tilde{B}(\tilde{p}^n,\phi) + \tilde{B}(\tilde{p}^n + \phi, \tilde{p}^{n+1})\big] = \tau g(\phi), \tag{5.9}$$

where $g(\phi) = -A\phi + A^{1/2}\phi - P\tilde{B}(\phi,\phi)$. For simplicity, we will henceforth drop the $\sim$ over p.

Proposition 5.2. *Let τ be small enough so that both*

$$\delta \equiv \tau\lambda_1 - \frac{2\tau^2}{9}\|\phi_x\|_\infty > 0 \ , \ \text{and} \ \frac{\tau\lambda_1^{1/2}}{2} - \frac{2\tau^2}{9}\|\phi\|_\infty > 0 \tag{5.10}$$

hold. Suppose p^n, $n = 0, 1, 2, \ldots$ is a solution to (5.9). For all $n \geq 0$ we have

$$|p^{n+1}|^2 \leq |p^0|^2(1+\delta)^{-(n+1)} + \frac{1}{\lambda_1}|g(\phi)|^2(1-(\delta+1)^{-(n+1)}).$$

Thus the ball of radius $\frac{1}{\lambda_1}|g(\phi)|^2$ is absorbing and invariant for (5.9).

The proof of Proposition 5.2 will require the following lemma which is proved in the appendix.

Lemma 5.3. *For all u, v, w in $H^1_{\text{per}}((0, L))$ we have that*

$$(\tilde{B}(u,v), w) \leq \frac{1}{3}\|v_x\|_\infty |u||w| + \frac{1}{3}\|v\|_\infty |A^{1/4}w||u|$$

proof of Proposition 5.2:

By adding and subtracting p^{n+1} within $\tilde{B}$, taking the inner product of (5.9) with p^{n+1}, and observing (5.5) we obtain

$$\begin{aligned} 2(p^{n+1} - p^n, p^{n+1}) + 2\tau\left[|A^{1/2}p^{n+1}|^2 - |A^{1/4}p^{n+1}|^2 + (\tilde{B}(p^{n+1}, \phi), p^{n+1})\right] \\ + 2\tau(\tilde{B}(p^n - p^{n+1}, \phi), p^{n+1}) = 2\tau(g(\phi), p^{n+1}) \ . \end{aligned} \tag{5.11}$$

Applying (4.12), (5.4), and then Proposition 5.1 to (5.11), we have

$$\begin{aligned} |p^{n+1}|^2 + |p^{n+1} - p^n|^2 - |p^n|^2 + 2\tau\left[\frac{1}{4}|A^{1/2}p^{n+1}|^2 + \lambda_1|p^{n+1}|^2\right] \\ + 2\tau(\tilde{B}(p^n - p^{n+1}, \phi), p^{n+1}) \leq 2\tau|g||p^{n+1}| \ . \end{aligned} \tag{5.12}$$

Note that by Lemma 5.3 and Young's inequality

$$\begin{aligned} &|2\tau(\tilde{B}(p^n - p^{n+1}, \phi), p^{n+1})| \\ &\leq \frac{2\tau}{3}\|\phi_x\|_\infty |p^n - p^{n+1}||p^{n+1}| + \frac{2\tau}{3}|A^{1/4}p^{n+1}||p^n - p^{n+1}|\|\phi\|_\infty \\ &\leq |p^n - p^{n+1}|^2 + \left(\frac{2\tau}{3}\right)^2 \frac{1}{2}\|\phi_x\|_\infty^2 |p^{n+1}|^2 + \left(\frac{2\tau}{3}\right)^2 \frac{1}{2}\|\phi\|_\infty^2 |A^{1/4}p^{n+1}|^2 \ . \end{aligned} \tag{5.13}$$

Also by Young's inequality

$$2\tau|g||p^{n+1}| \leq \frac{\tau}{\lambda_1}|g|^2 + \tau\lambda_1|p^{n+1}|^2 \,. \tag{5.14}$$

Combining (5.12), (5.13) and (5.14) we obtain

$$\begin{aligned}|p^{n+1}|^2 - |p^n|^2 &+ \frac{\lambda_1^{1/2}\tau}{2}|A^{1/4}p^{n+1}|^2 + 2\tau\lambda_1|p^{n+1}|^2 \\ &\leq \frac{2\tau^2}{9}\|\phi_x\|_\infty|p^{n+1}|^2 + \frac{2\tau^2}{9}\|\phi\|_\infty|A^{1/4}p^{n+1}|^2 + \frac{\tau}{\lambda_1}|g|^2 + \tau\lambda_1|p^{n+1}|^2\end{aligned}$$

It follows from (5.10) that

$$|p^{n+1}|^2[1+\delta] \leq |p^n|^2 + \frac{\tau}{\lambda_1}|g|^2 \,.$$

The rest of the proof is again as in Proposition 4.6 starting at (4.14).

Existence and uniqueness for (5.7) can be established in a similar fashion as for (4.18).

A. Appendix

In the next three proofs, all for statements from section 4, the norm denoted $|\cdot|$ corresponds to the inner product $\langle\cdot,\cdot\rangle$. Using (2.2), one can easily verify the following identity to be used repeatedly below.

$$\langle(-\Delta_h)\vec{\omega}, \vec{\omega}\rangle = h\sum_{k=0}^{N-1}\Big(\frac{\omega_{k+1}-\omega_k}{h}\Big)^2. \tag{A.1}$$

proof of Lemma 4.10, part A

Applying the Cauchy-Schwarz inequality to the definitions of $\langle\cdot,\cdot\rangle$ and B^h we have

$$\begin{aligned}|\langle B^h(\vec{\omega},\vec{\omega}),\vec{\xi}\rangle| &\leq |\vec{\xi}|\Big[h\sum_{k=0}^{N-1}\Big(\frac{\omega_k(\omega_{k+1}-\omega_{k-1}) + (\omega_{k+1}-\omega_{k-1})(\omega_{k+1}+\omega_{k-1})}{6h}\Big)^2\Big]^{1/2} \\ &\leq \frac{1}{2}|\vec{\xi}|\max_{0\leq k\leq N-1}|\omega_k|\Big[h\sum_{k=0}^{N-1}\Big(\frac{(\omega_{k+1}-\omega_{k-1})}{h}\Big)^2\Big]^{1/2}\end{aligned}$$

To complete the proof note that by Young's inequality, (2.1), and (A.1) we have

$$
\begin{aligned}
\sum_{k=0}^{N-1}(\omega_{k+1}-\omega_{k-1})^2 &= \sum_{k=0}^{N-1}(\omega_{k+1}-\omega_k+\omega_k-\omega_{k-1})^2 \\
&= \sum_{k=0}^{N-1}(\omega_{k+1}-\omega_k)^2+(\omega_k-\omega_{k-1})^2+2(\omega_{k+1}-\omega_k)(\omega_k-\omega_{k-1}) \\
&\leq 4\sum_{k=0}^{N-1}(\omega_{k+1}-\omega_k)^2 \\
&= 4h\langle\vec{\omega},\vec{\omega}\rangle\ .
\end{aligned}
$$

proof of Lemma 4.10, part B

Writing ω_k^2 as a telescoping sum, then using the Cauchy-Schwarz inequality and (A.1), we obtain

$$
\omega_k^2 = \left|\sum_{j=1}^{k}(\omega_j^2-\omega_{j-1}^2)\right| \leq \sum_{j=1}^{N}\left|\frac{\omega_j-\omega_{j-1}}{h}\right|\,|\omega_j+\omega_{j-1}|h \leq 2|(-\Delta_h)^{1/2}\vec{\omega}|\ |\vec{\omega}|.
$$

proof of Lemma 4.12

From the definitions of $\langle\cdot,\cdot\rangle$ and B^h we have

$$
\begin{aligned}
|\langle B^h(\vec{u},\vec{v}),\vec{w}\rangle| &= \frac{h}{6}\left|\sum_{k=0}^{N-1}\left[\frac{u_k(v_{k+1}-v_{k-1})w_k}{h}+\frac{(u_{k+1}v_{k+1}-u_{k-1}v_{k-1})w_k}{h}\right]\right| \\
&\leq \frac{1}{6}|\vec{u}||\vec{w}|\max_{0\leq k\leq N-1}\left(\frac{|v_{k+1}-v_{k-1}|}{h}\right)+\frac{h}{6}\left|\sum_{k=0}^{N-1}\frac{(u_{k+1}v_{k+1}-u_{k-1}v_{k-1})w_k}{h}\right]\Bigg|.
\end{aligned}
$$

Using (A.1) and integration by parts we obtain

$$
\begin{aligned}
\left|\sum_{k=0}^{N-1}(u_{k+1}v_{k+1}-u_{k-1}v_{k-1})w_k\right| &= \left|\sum_{k=0}^{N-1}u_kv_kw_{k-1}-u_kv_kw_{k+1}\right| \\
&= \left|\sum_{k=0}^{N-1}u_kv_k\frac{(w_{k-1}-w_{k+1})}{h}h\right| \\
&\leq \max_{0\leq k\leq N-1}|v_k||u|\left[\sum_{k=0}^{N-1}\left(\frac{w_{k-1}-w_{k+1}}{h}\right)^2h\right]^{1/2} \\
&\leq 2|\vec{u}|\max_{0\leq k\leq N-1}|v_k|\left[\langle(-\Delta_h)\vec{w},\vec{w}\rangle\right]^{1/2}.
\end{aligned}
$$

Now (4.19) follows by substitution.

proof of Lemma 5.3

The norm denoted $|\cdot|$ now corresponds to the inner product $(\cdot,\cdot)$.

Integrating by parts we obtain

$$
\begin{aligned}
\left|(\tilde{B}(u,v),w)\right| &= \left|\frac{1}{3}\int v u_x w + \frac{2}{3}\int u v_x w\right| \\
&= \left|-\frac{1}{3}\int v_x u w - \frac{1}{3}\int v w_x u + \frac{2}{3}\int u v_x w\right| \\
&\leq \frac{1}{3}\|v_x\|_\infty |u||w| + \frac{1}{3}|w_x|\|v\|_\infty |u| \\
&= \frac{1}{3}\|v_x\|_\infty |u||w| + \frac{1}{3}|A^{1/4}w|\|v\|_\infty |u| \,.
\end{aligned}
$$

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Figure Captions

Figure 3.1 Bifurcation diagram showing spurious stationary solutions for (2.8a,b) plotted as Euclidean norm versus parameter α.

Figure 3.2 Spurious stationary solutions associated with (2.8c).

Figure 3.3 Bifurcation diagram for (2.8c) with N=12 (i.e. a five-dimensional reduced system).

Figure 3.4 Bifurcation diagram for (2.8c) with N=24 (i.e. a reduced system of dimension eleven).

Figure 3.5 Time series (Euclidean norm vs. time) of a trajectory starting near a spurious steady state for the nonlinear Galerkin method in (3.5).

Figure 4.1

(a) Time series of a trajectory for the scheme in (4.18). The initial condition is near a spurious steady state for (2.8a,b).

(b) Time series of a trajectory starting with the same initial condition as in figure 4.1a using the scheme in (4.18), but with the bilinear term discretized as in (2.8a).

ENTER COMMAND

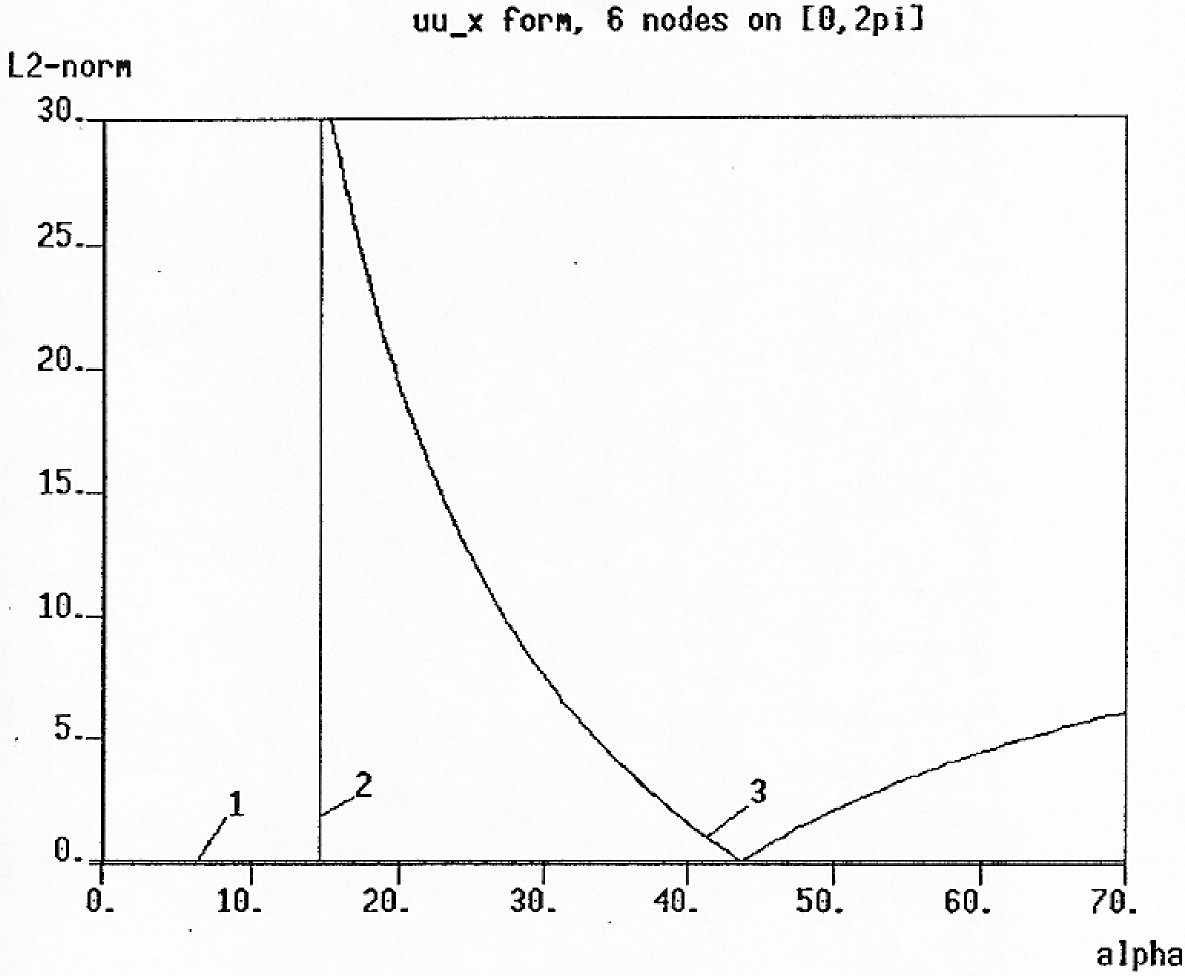


Fig 3.1

ENTER COMMAND

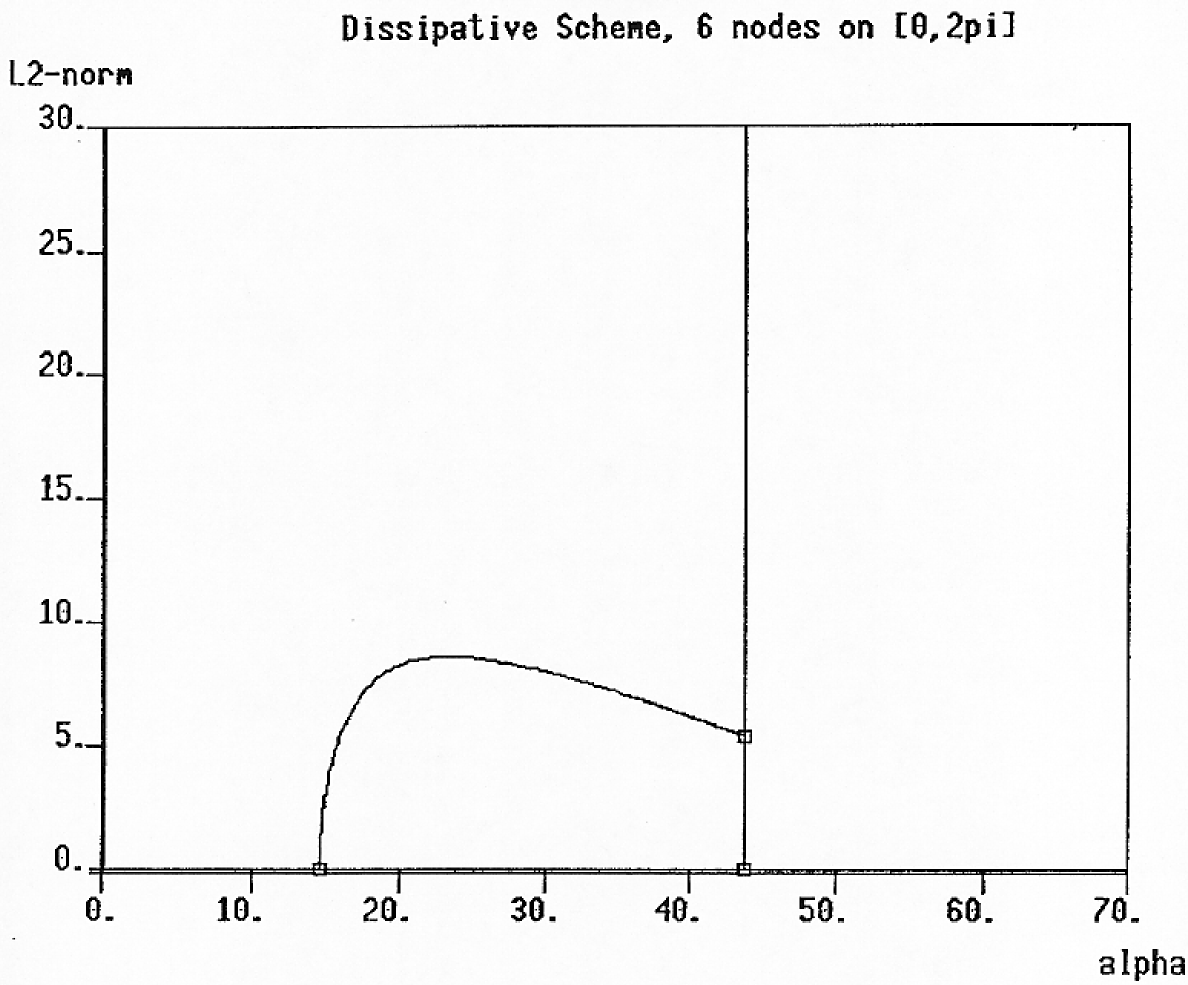
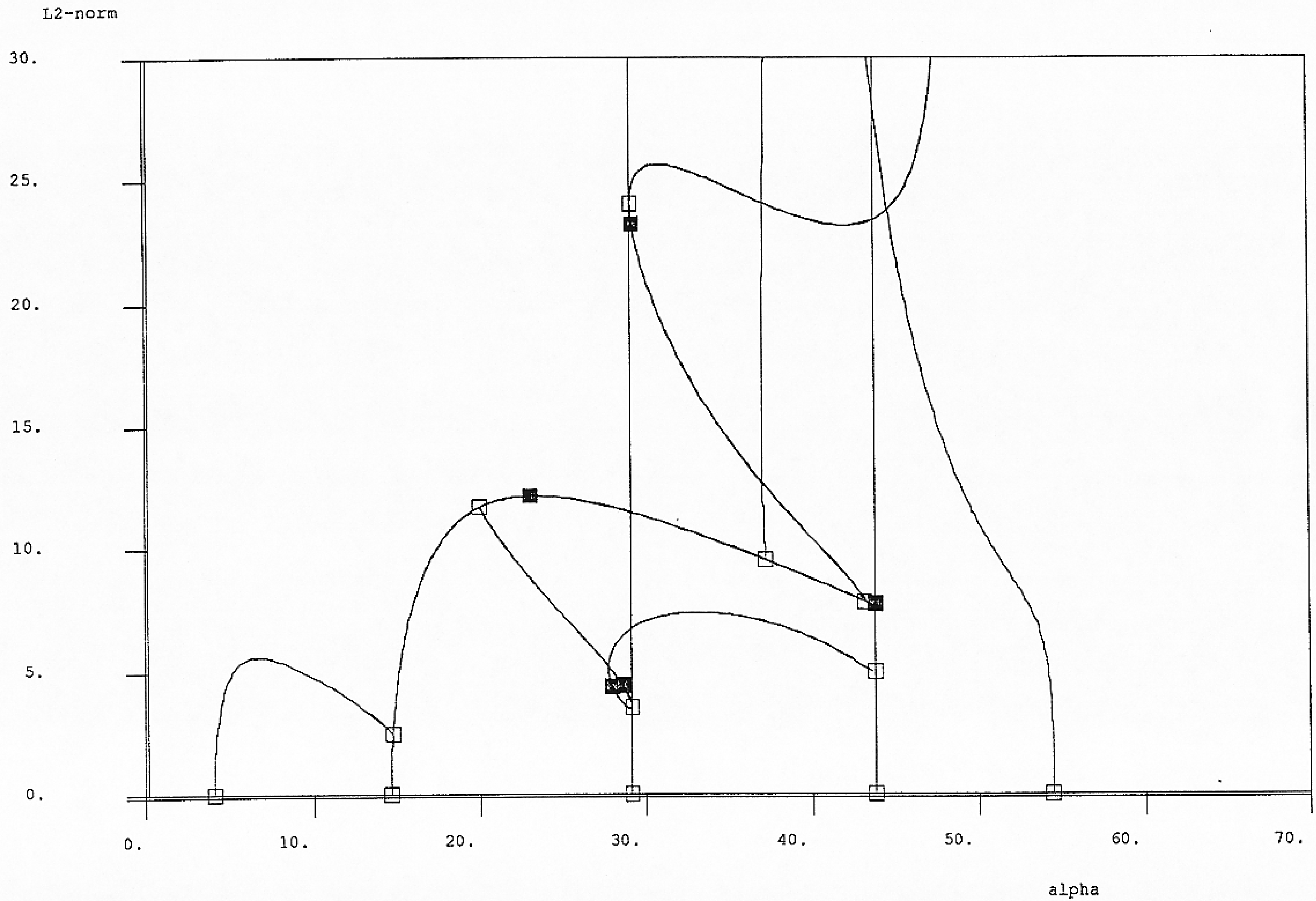


Fig 3.2

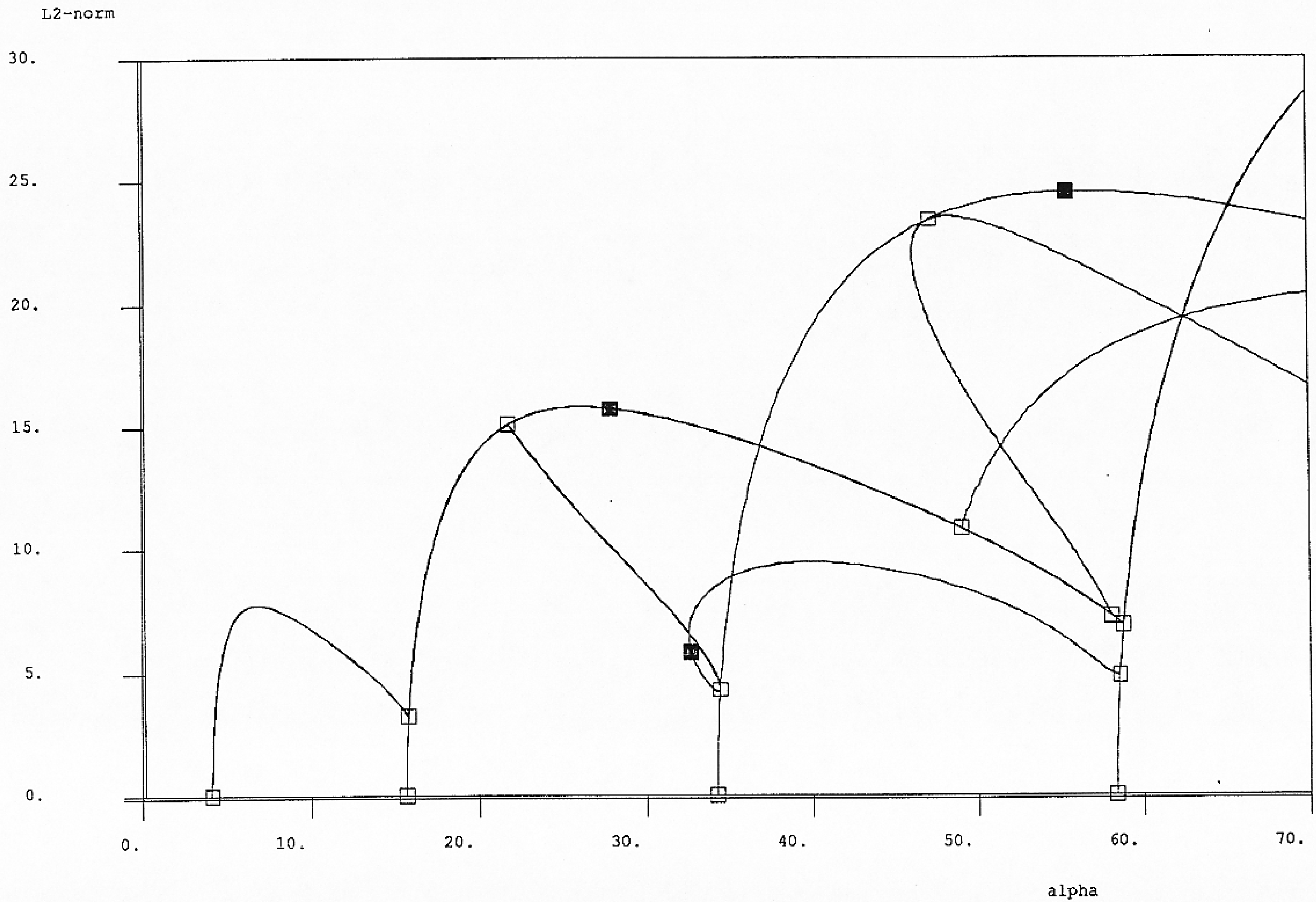
Kuramoto-Sivashinsky diss. fin. diff. scheme



12 nodes on [0,2pi], reduced dimension=5

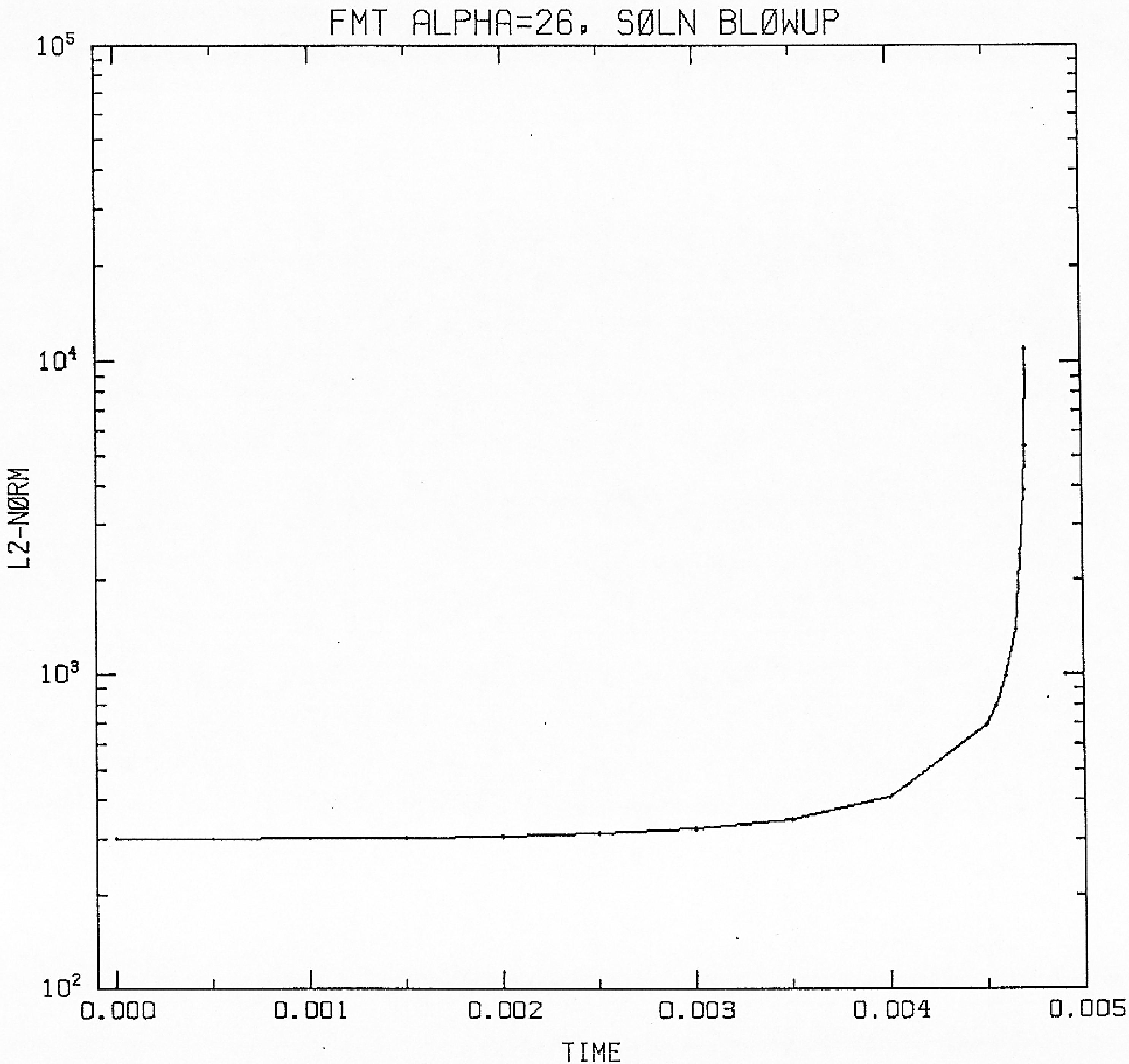
Fig 3.3

Kuramoto-Sivashinsky diss. fin. diff. scheme



24 nodes on [0,2pi], reduced dimension=11

Figure 3.4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L2-NØRM
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10^4
10^3
10^2
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0.001
0.002
0.003
0.004
0.005

Fig 3.5

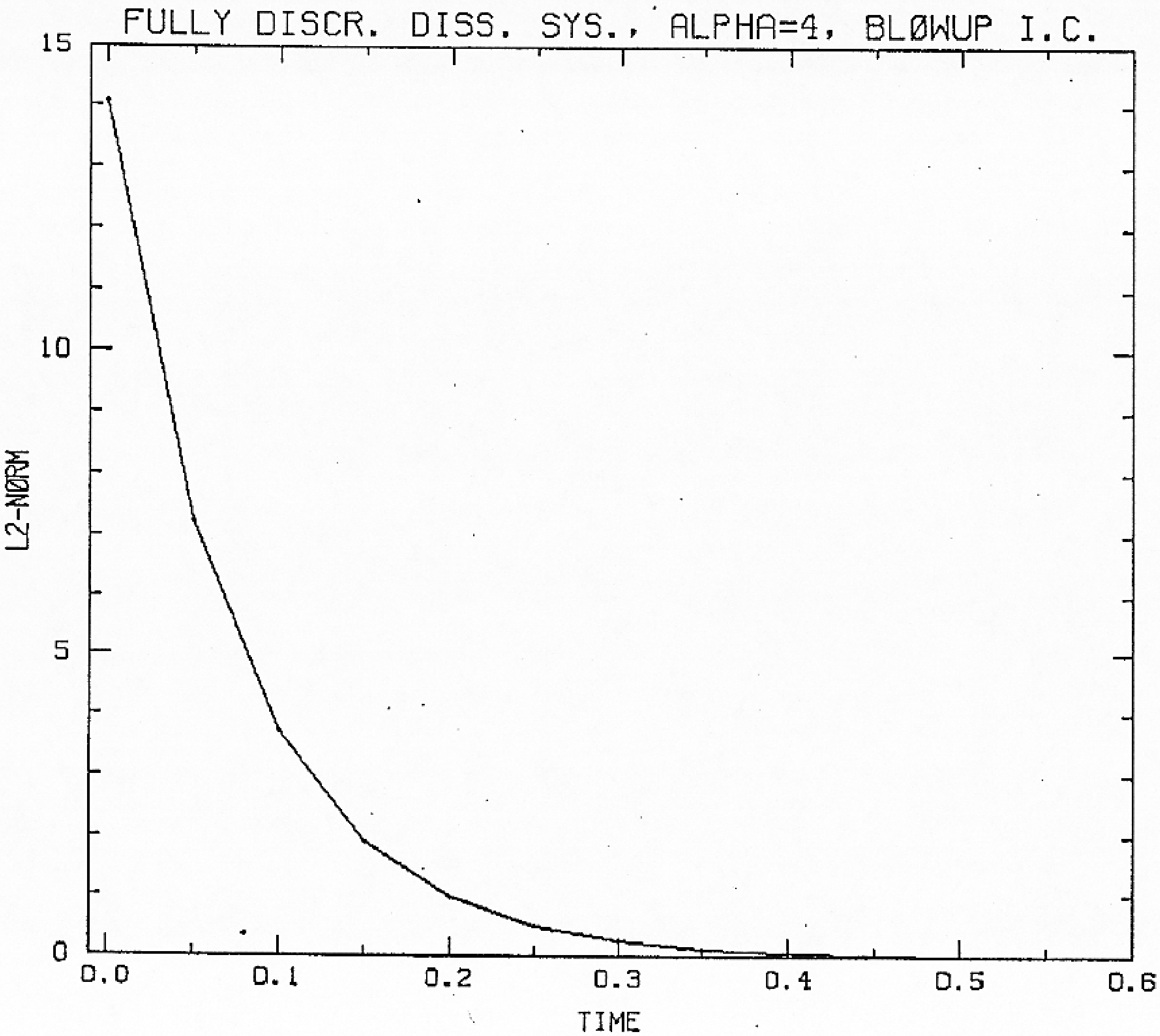
FULLY DISCR. DISS. SYS., ALPHA=4, BLØWUP I.C.
15
10
5
0
L2-NØRM
0.0
0.1
0.2
0.3
0.4
0.5
0.6
TIME

fig. 4.1a

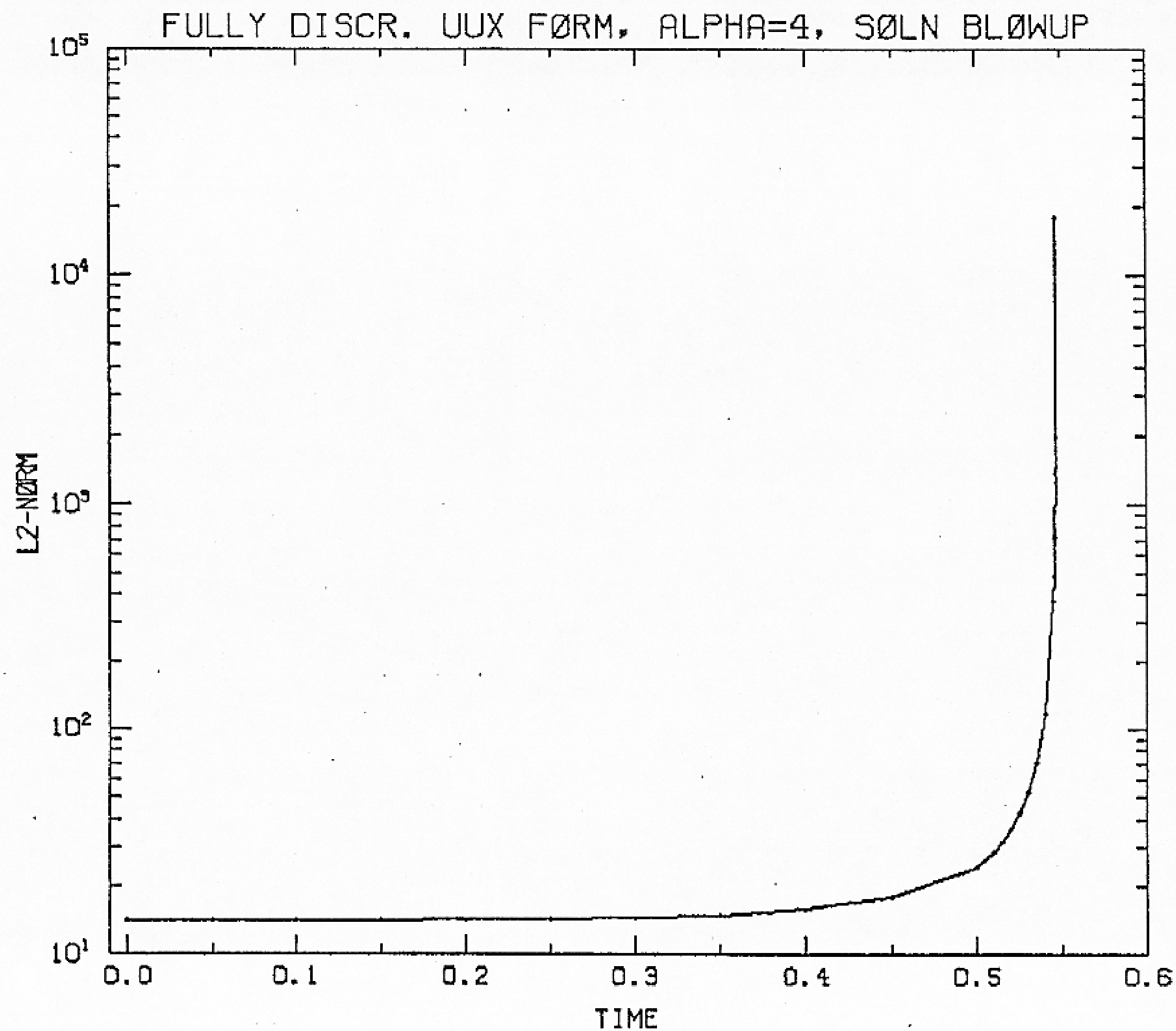
FULLY DISCR. UUX FØRM, ALPHA=4, SØLN BLØWUP
L2-NØRM
10^5
10^4
10^3
10^2
10^1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
TIME

fig. 4.1b