Global Stabilization of a Dynamic von Kármán Plate With Nonlinear Boundary Feedback

Mary Ann Horn*
Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, Minnesota 55455

Irena Lasiecka
Department of Applied Mathematics
University of Virginia
Charlottesville, Virginia 22903

Abstract

We consider a fully nonlinear von Kármán system with, in addition to the nonlinearity which appears in the equation, nonlinear feedback controls acting through the boundary as moments and torques. Under the assumptions that the nonlinear controls are continuous, monotone, and satisfy appropriate growth conditions (however, no growth conditions are imposed at the origin), uniform decay rates for the solution are established. In this fully nonlinear case, we do not have, in general, smooth solutions even if the initial data are assumed to be very regular. However, rigorous derivation of the estimates needed to solve the stabilization problem requires a certain amount of regularity of the solutions which is not guaranteed. To deal with this problem, we introduce a regularization/approximation procedure which leads to an "approximating" problem for which partial differential equation calculus can be rigorously justified. Passage to the limit on the approximation reconstructs the needed estimates for the original nonlinear problem.

1 Introduction

1.1 Statement of the Problem

Let Ω be an open bounded domain in R^2 with a sufficiently smooth (e.g., C^∞) boundary, Γ. In Ω, we consider the following von Kármán system in the variables $w(t,x)$ and $\chi(w(t,x))$ with nonlinear feedback controls, g, f_1, and f_2:

$$w_{tt} - \gamma^2 \Delta w_{tt} + \Delta^2 w + b(x) w_t = [w, \chi(w)] \quad \text{in } Q_\infty = (0,\infty)\times\Omega \tag{1.1.a}$$

$$\left.\begin{aligned} w(0,\cdot) &= w_0 \\ w_t(0,\cdot) &= w_1 \end{aligned}\right\} \quad \text{in } \Omega \tag{1.1.b}$$

*This material is based upon work partially supported under a National Science Foundation Mathematical Sciences Post-doctoral Research Fellowship.

$$\Delta w + (1-\mu)B_1 w = -f_1(\frac{\partial}{\partial \nu} w_t) \quad \text{on } \Sigma_\infty = (0,\infty)\times\Gamma \tag{1.1.c}$$

$$\Delta w + (1-\mu)B_2 w - \gamma^2 \frac{\partial}{\partial \nu} w_{tt} - w = g(w_t) - \frac{\partial}{\partial \tau} f_2(\frac{\partial}{\partial \tau} w_t) \quad \text{on } \Sigma_\infty = (0,\infty)\times\Gamma, \tag{1.1.d}$$

where $b(x) \in L^\infty(\Omega)$ satisfies $b(x) > 0$ $a.e.$ in Ω, $0 < \mu < \frac{1}{2}$ is Poisson's ratio, the operators B_1 and B_2 are given by

$$\begin{aligned} B_1 w &= 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\ B_2 w &= \tfrac{\partial}{\partial \tau}[(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})], \end{aligned} \tag{1.2}$$

and the controls, g and f_i are continuous, monotone functions and are subject to the following constraints:

$$\left.\begin{aligned} &g(s)s > 0 && for\ s \neq 0 \\ &f_i(s)s > 0 && for\ s \neq 0 \\ &m|s| \leq |f_i(s)| \leq M|s| && for\ |s| > 1,\ i = 1,2 \\ &m|s|^2 \leq g(s)s \leq M|s|^{r+1} && for\ |s| > 1, \end{aligned}\right\} (H-1)$$

where r is any positive constant. The parameter, γ, in (1.1.a) is proportional to the thickness of the plate and is therefore assumed to be small.

Remark 1.1: No assumptions are made on the behavior of g and f_i, $i = 1,2$, at the origin.

In (1.1), $\chi(w)$ satisfies the system of equations

$$\left.\begin{aligned} \Delta^2 \chi &= -[w,w] && in\ \Omega \\ \chi &= \tfrac{\partial}{\partial \nu}\chi = 0 && on\ \Gamma, \end{aligned}\right\} \tag{1.3}$$

where

$$[\phi,\psi] = \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 \psi}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 \psi}{\partial x \partial y}. \tag{1.4}$$

Define the bilinear form

$$a(w,v) = \int_\Omega (\Delta w \Delta v + (1-\mu)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}))d\Omega. \tag{1.5}$$

We define the energy functional by

$$E_w(t) = \frac{1}{2}\int_\Omega \{|w_t|^2 + \gamma^2|\nabla w_t|^2 + |\Delta\chi(w)|^2\}d\Omega + \frac{1}{2}\int_\Gamma w^2 d\Gamma + \frac{1}{2}a(w,w) \equiv E_{w,1}(t) + E_{w,2}(t), \tag{1.6}$$

where $E_{w,2}(t)$ is defined by

$$E_{w,2}(t) \equiv \frac{1}{2} \int_{\Omega} |\Delta\chi(w)|^2 d\Omega. \tag{1.7}$$

In view of this, the associated space of finite energy is $\mathcal{H} \equiv H^2(\Omega) \times H^1(\Omega)$, with the norm

$$||(w, w_t)||^2_{\mathcal{H}} \equiv ||w||^2_{H^2(\Omega)} + ||w_t||^2_{L_2(\Omega)} + \gamma^2 ||\nabla w_t||^2_{L_2(\Omega)}. \tag{1.8}$$

The following well-posedness theorem for problem (1.1)-(1.3) is a very special case of the result in [?].

Theorem 1.1 *(See [?].) For any $w_0 \in H^2(\Omega)$, $w_1 \in H^1(\Omega)$, and $T > 0$, there exists a unique solution to (1.1), $w \in C(0,T; H^2(\Omega)) \cap C^1(0,T; H^1(\Omega))$, such that*

$$\nabla w_t|_{\Gamma} \in L_2(0,T; L_2(\Gamma)). \tag{1.9}$$

Remark 1.2: Notice that the regularity property in (1.9) does not follow from a priori interior regularity of w (i.e., $w_t \in H^1(\Omega)$). It is an independent regularity result.

Our goal is to show that the boundary controls, f_1, f_2, and g, cause the energy of our system, (1.6), to decay uniformly with respect to the initial energy as time increases.

Remark 1.3: We note that the "light internal damping" represented by the term, $b(x)w_t$, alone (i.e., without boundary dissipation) will not cause uniform decay rates for the energy function. In order to obtain *uniform* decay, the presence of boundary dissipation is essential.

1.2 Literature

The problem of boundary stabilization has attracted considerable attention in recent years (see [?], [?], [?], [?], [?] and references therein). We shall concentrate on the results pertinent to model (1.1).

In the context of control theory and, in particular, stabilization theory, the von Kármán model was introduced for the first time in [?]. In fact, in [?], the exponential decay rates for the solutions to (1.1) with $\gamma = 0$ (rotational forces neglected) and with *linear* feedbacks, f_i and g, were established. This result of [?] was derived under the geometric conditions on Γ which required that Ω be "star-shaped." Subsequently, in [?], the results of [?] were extended to the case when:

(i.) $\gamma \neq 0$, i.e., the rotational forces are taken into account;

(ii.) no geometric conditions are imposed on Γ.

These generalizations required techniques different from those in [?] and were based on microlocal estimates combined with a nonlinear compactness/uniqueness argument (see [?]) (rather than the Liapunov function techniques used in [?]). Other works relevant to the stabilization of the von Kármán system are: [?], where the one dimensional problem was treated; and [?] (see also [?]), where a model accounting for in plane accelerations is considered and the exponential decay rates are justified for annular plates with linear feedbacks.

The main goal of this paper is to treat a fully nonlinear case for general two dimensional domains without any geometric restrictions imposed on Γ. This is to say that in addition to the nonlinearity appearing in the equation, the feedback controls, f_i and g, are also nonlinear. Additionally, we do not assume any growth conditions hold at the origin, which is in contrast with most of the literature related to the subject (see [?], [?], etc.). Our aim is to show that the energy function, $E_w(t)$, given by (1.6), decays to zero at a uniform rate which is *independent* of the value of the parameter γ in (1.1). Several reasons motivate the interest in studying sensitivity of feedback controls with respect to the parameter $\gamma > 0$ (see [?], [?] for the analysis in the linear case). Among them is the fact that the value of the parameter $\gamma > 0$ is usually very small. Thus it would be highly undesirable if the feedback control "loses" its properties in the limit process.

The presence of nonlinear feedbacks introduces genuine new difficulties to the problem. For example, in this fully nonlinear case, we do not have, in general, smooth solutions (even if the initial data are assumed to be very regular). On the other hand, a rigorous derivation of partial differential equation estimates (needed for the solution to the stabilization problem), including even the most fundamental "dissipation energy" estimate (see Lemma 3.1) requires a carefully selected amount of regularity of the solutions which, typically, is not guaranteed by the existence theorem. To cope with this difficulty, we shall introduce a certain regularization/approximation procedure which will lead to an "approximating" problem for which partial differential equation calculus can be rigorously justified (similar ideas were used in the context of the wave equation in [?]). Passage to the limit on the approximation (nontrivial, sue to the presence of boundary traces) will reconstruct the needed estimates for the original nonlinear problem. It should be noted that this procedure requires an existence theorem (see Theorem 1.1) which supplies certain "a priori" regularity of the traces of the solutions (not only the usual interior regularity).

1.3 Statement of Main Results

To state our stability result, we will need the following notation. Let the function $h(x)$ be defined by:

$$h(x) \equiv h_0(x) + h_1(x) + h_2(x), \tag{1.10}$$

where $h_i(x)$ are concave, strictly increasing functions with $h_i(0) = 0$ such that

$$\begin{aligned} &h_0(sg(s)) \geq s^2 + g^2(s) \quad |s| \leq 1 \\ &h_i(sf_i(s)) \geq s^2 + f_i^2(s) \quad |s| \leq 1 \quad i = 1, 2. \end{aligned} \tag{1.11}$$

(Such functions can be easily constructed. See [?].) Then $h(x)$ enjoys the same properties, i.e., it is concave, strictly increasing, and $h(0) = 0$. Define

$$\tilde{h}(x) \equiv h(\frac{x}{mes\ \Sigma_T}). \tag{1.12}$$

Since $\tilde{h}$ is monotone increasing, for every $c \geq 0$, $cI + \tilde{h}$ is invertible. Setting

$$p(x) \equiv (cI + \tilde{h})^{-1}(Kx), \tag{1.13}$$

where K is a positive constant, we see that p is a positive, continuous, strictly increasing function with $p(0) = 0$.

We are now in a position to state our result.

Theorem 1.2 *Assume hypothesis (H-1) holds. Let w be the solution to system (1.1). Then for some $T_0 > 0$,*

$$E_w(t) \leq \mathcal{S}(\frac{t}{T_0} - 1) \ for\ t > T_0, \tag{1.14}$$

where $\mathcal{S}(t) \to 0$ as $t \to \infty$ and is the solution (contraction semigroup) of the differential equation

$$\begin{cases} \frac{d}{dt}\mathcal{S}(t) + q(\mathcal{S}(t)) = 0 \\ \mathcal{S}(0) = E_w(0), \end{cases} \tag{1.15}$$

and $q(x)$ is given by

$$q(x) \equiv x - (I + p)^{-1}(x) \ for \ x > 0. \tag{1.16}$$

In this case, the constant K will generally depend on $E_w(0)$ and the constant $c = \frac{1}{mes\Sigma_T}(m^{-1} + M)$, but they will not depend on the parameter γ.

Remark 1.4: One could also consider feedback controls acting on a portion of the boundary only. In this case, the appropriate geometric conditions imposed on the *uncontrolled* part of the boundary are needed (see [?]).

In the special case when the growth at the origin of the nonlinear boundary feedbacks is specified, one can compute explicitly the governing decay rates for the energy function, $E_w(t)$. Indeed, this can be easily accomplished by constructing the appropriate function q and solving the nonlinear ODE problem (1.15). For instance, if the nonlinear functions f_i and g have a linear growth at the origin, then (1.14) specializes to

$$E_w(t) \leq C(E_w(0))e^{-\omega t} \text{ for some } \omega > 0. \tag{1.17}$$

If, instead, these nonlinearities are of polynomial growth at the origin (e.g. $\sim x^p$, $p > 1$), then $E_w(t) \leq C(E_w(0))t^{\frac{2}{1-p}}$.

Remark 1.5: As mentioned before, it is interesting to note the role played by the "light" internal damping, $b(x)w_t$. It is obvious that this damping alone (i.e., without dissipation on the boundary) would not suffice to uniformly stabilize the plate. ("Strong" internal damping which would cause the energy to decay exponentially would be of the form $b(x)\Delta w_t$, where $b(x) \geq b_0 > 0$.) On the other hand, the presence of this mild damping term seems to be essential in proving the effectiveness of the boundary dissipation. Thus, the combination of both the light interior damping and the boundary dissipation provides the desired energy decay. We note that the presence of light interior damping is physically motivated, since most vibrating materials possess some degree of interior damping.

2 Preliminary Energy Estimate

Our goal is to prove energy decay rates for problem (1.1). In order to do this, one needs to perform certain partial differential equation calculations on the problem. These calculations require regularity of the solutions higher than is available from Theorem 1.1. Since our nonlinear problem may not have a sufficiently regular solution (even if the initial data are smooth), we resort to an approximation argument (this argument was used in the context of wave equations in [?]). In fact, the idea here is to approximate solutions to the nonlinear problem (1.1) by solutions to different (linear) problems. Since this linear problem admits regular solutions for smooth initial data, the partial differential equation calculations can be performed on this problem. Final passage to the limit on the approximation problem allows us to obtain needed energy identities for the original nonlinear problem.

To follow our program, we start by defining the following approximations. To do this, we need the following corollary of Theorem 1.1.

Corollary 2.1 *Let w be a solution to (1.1). Then*

$$f_1(\frac{\partial}{\partial\nu}w_t) \in L_2(0,T;L_2(\Gamma)) \tag{2.1}$$

and

$$g(w_t) - \frac{\partial}{\partial\tau}f_2(\frac{\partial}{\partial\tau}w_t) \in L_2(0,T;H^{-1}(\Gamma)). \tag{2.2}$$

Proof of Corollary 2.1: Hypothesis (H-1) together with (1.9) of Theorem 1.1 imply

$$\begin{aligned} &f_1(\tfrac{\partial}{\partial\nu}w_t) \in L_2(\Sigma_T) \\ &f_2(\tfrac{\partial}{\partial\tau}w_t) \in L_2(\Sigma_T). \end{aligned} \tag{2.3}$$

Hence,

$$\frac{\partial}{\partial\tau}f_2(\frac{\partial}{\partial\tau}w_t) \in L_2(0,T;H^{-1}(\Gamma)). \tag{2.4}$$

On the other hand, with $\phi \in L_2(0,T;H^1(\Gamma))$,

$$\begin{aligned}
\int_0^T \int_\Gamma |g(w_t(t,x))\phi(t,x)|dxdt &\le \int_0^T \int_\Gamma |w_t(t,x)|^r |\phi(t,x)|dxdt \\
&\le C\int_0^T \|\phi(t)\|_{H^1(\Gamma)} \int_\Gamma |w_t(t,x)|^r dxdt \\
&\le C\int_0^T \|\phi(t)\|_{H^1(\Gamma)} \|w_t(t)\|^r_{L_p(\Gamma)} dt \\
&\le C\int_0^T \|\phi(t)\|_{H^1(\Gamma)} \|w_t(t)\|^r_{H^{1/2}(\Gamma)} dt \\
&\le C\int_0^T \|\phi(t)\|_{H^1(\Gamma)} \|w_t(t)\|^r_{H^1(\Omega)} dt \\
&\le C\int_0^T \|\phi(t)\|_{H^1(\Gamma)} E_w(t)^{r/2} dt \le CE_w(0)^{r/2}\int_0^T \|\phi(t)\|_{H^1(\Gamma)} dt,
\end{aligned} \tag{2.5}$$

where the first inequality follows from hypothesis (H-1), the second and third follow from Sobolev Imbeddings and the boundedness of Γ, and the fourth from trace theory. Hence,

$$g(w_t) \in L_2(0,T;H^{-1}(\Gamma)), \tag{2.6}$$

which, together with (2.3) and (2.4) prove (2.1) and (2.2). □

Let w be the solution of the original problem (1.1). By using the regularity properties in (1.9), (2.1), and (2.2), along with density of approximate (see below) Sobolev spaces, we are in a position to define

$$f_n \in H^{1,1}(Q_T); \qquad \|f_n - [w,\chi(w)]\|_{L_2(0,T;H^{-1}(\Omega))} \longrightarrow 0 \tag{2.7}$$

$$f_{1n} \in H^{1,1}(\Sigma_T); \qquad \|f_{1n} - f_1(\tfrac{\partial}{\partial\nu} w_t)\|_{L_2(\Sigma_T)} \longrightarrow 0 \tag{2.8}$$

$$f_{2n} \in H^{1,1}(\Sigma_T); \quad \|f_{2n} - [g(w_t) - \tfrac{\partial}{\partial\tau} f_2(\tfrac{\partial}{\partial\tau} w_t)]\|_{L_2(0,T;H^{-1}(\Gamma))} \longrightarrow 0 \tag{2.9}$$

$$\alpha_n \in H^{1,1}(\Sigma_T); \qquad \|\alpha_n - \tfrac{\partial}{\partial\nu} w_t\|_{L_2(\Sigma_T)} \longrightarrow 0 \tag{2.10}$$

$$\beta_n \in H^{1,1}(\Sigma_T); \qquad \|\beta_n - (w_t - \tfrac{\partial^2}{\partial\tau^2} w_t)\|_{L_2(0,T;H^{-1}(\Gamma))} \longrightarrow 0, \tag{2.11}$$

where $Q_T \equiv \Omega \times (0,T)$ and $\Sigma_T \equiv \Gamma \times (0,T)$. We consider the following approximating problem:

$$\begin{cases}
w_{n,tt} - \gamma^2 \Delta w_{n,tt} + \Delta^2 w_n + b w_{n,t} = f_n \\
\quad w_n(0) = w_{n,0}; \quad w_{n,t}(0) = w_{n,1} \\
\Delta w_n + (1-\mu)B_1 w_n + \frac{\partial}{\partial\nu} w_{n,t}|_\Gamma = -f_{1n} + \alpha_n \\
\frac{\partial}{\partial\nu}\Delta w_n + (1-\mu)B_2 w_n - \gamma^2 \frac{\partial}{\partial\nu} w_{n,tt} - w_n - w_{n,t} + \frac{\partial^2}{\partial\tau^2} w_{n,t}|_\Gamma = f_{2n} - \beta_n,
\end{cases} \tag{2.12}$$

where

$$\|w_{n,0} - w_0\|_{H^2(\Omega)} \to 0; \quad \|w_{n,1} - w_1\|_{H^1(\Omega)} \to 0, \tag{2.13}$$

and $(w_{n,0}, w_{n,1}) \in \mathcal{D}$, where $\mathcal{D}$, as dense set of $\mathcal{H}$, consists of $w_{n,0} \in H^4(\Omega)$, $w_{n,1} \in H^3(\Omega)$, where $w_{n,0}, w_{n,1}$ satisfy the appropriate compatibility conditions on the boundary. By standard linear semigroup methods, one easily shows that the linear problem, (2.12), admits a classical solution,

$$w_n \in C(0,T; H^4(\Omega)) \cap C^1(0,T; H^3(\Omega)). \tag{2.14}$$

The following proposition plays a critical role in our development.

Proposition 2.1 *Let w_n (respectively, w) be a solution of (2.12) (respectively, (1.1)). Then as $n \to \infty$, the following convergence holds.*

$$w_n \to w \ \textit{in} \ C(0,T; H^2(\Omega)) \cap C^1(0,T; H^1(\Omega)) \tag{2.15}$$

$$\nabla w_{n,t}|_\Gamma \to \nabla w_t \ \textit{in} \ L_2(\Sigma_T). \tag{2.16}$$

Proof: Consider the equation satisfied by the difference $w_n - w_m$. Taking the inner product of this equation with $w_{n,t} - w_{m,t}$ and integrating the result from 0 to T yields

$$\begin{aligned} E_{w_n - w_m,1}(T) \quad & + \int_0^T \int_\Gamma [\hat{w}_t^2 + |\nabla \hat{w}_t|^2] d\Gamma dt + \int_0^T \int_\Omega b \hat{w}_t^2 d\Omega dt \\ & = \int_0^T \int_\Omega (f_n - f_m) \hat{w}_t d\Omega dt + \int_0^T \int_\Gamma (f_{1n} + \alpha_n - f_{1m} - \alpha_m) \frac{\partial}{\partial \nu} \hat{w}_t d\Gamma dt \\ & \quad + \int_0^T \int_\Gamma (f_{2n} + \beta_n - f_{2m} - \beta_m) \hat{w}_t d\Gamma dt + E_{w_n - w_m,1}(0), \end{aligned} \tag{2.17}$$

where $\hat{w} \equiv w_n - w_m$. Hence,

$$\begin{aligned} C_0 \|\hat{w}(T)\|^2_{H^2(\Omega)} \quad & + \|\hat{w}_t(T)\|^2_{H^1(\Omega)} + \|\nabla \hat{w}_t\|^2_{L_2(\Sigma_T)} + \|\hat{w}_t\|^2_{L_2(\Sigma_T)} \\ & \le \tfrac{1}{2} \|f_n - f_m\|^2_{L_2(0,T;H^{-1}(\Omega))} + \tfrac{1}{2} \int_0^T \|\hat{w}_t(t)\|^2_{H^1(\Omega)} dt \\ & \quad + \tfrac{1}{2} \|f_{1n} - \alpha_n - f_{1m} + \alpha_m\|^2_{L_2(\Sigma_T)} + \tfrac{1}{2} \|\tfrac{\partial}{\partial \nu} \hat{w}_t\|^2_{L_2(\Sigma_T)} \\ & \quad + \tfrac{1}{2} \|f_{2n} - \beta_n - f_{2m} + \beta_m\|^2_{L_2(0,T;H^{-1}(\Omega))} + \tfrac{1}{2} \|\hat{w}_t\|^2_{L_2(0,T;H^1(\Omega))} + E_{w_n - w_m,1}(0), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \|\hat{w}(T)\|^2_{H^2(\Omega)} & +\|\hat{w}_t(T)\|^2_{H^1(\Omega)} + \|\nabla \hat{w}_t\|^2_{L_2(\Sigma_T)} \\ & \leq C[\|f_n - f_m\|^2_{L_2(0,\,T;\,H^{-1}(\Omega))} + \|f_{1n} - f_{1m}\|^2_{L_2(\Sigma_T)} + \|\alpha_n - \alpha_m\|^2_{L_2(\Sigma_T)} \\ & \quad +\|f_{2n} - f_{2m}\|^2_{L_2(0,\,T;\,H^{-1}(\Omega))} + \|\beta_n - \beta_m\|^2_{L_2(0,\,T;\,H^{-1}(\Omega))} + E_{w_n - w_m,1}(0)] \\ & \longrightarrow 0 \ as \ n \to \infty, \end{aligned} \tag{2.19}$$

where the limit follows by using (2.7)-(2.11). Thus, by (2.13) and Corollary 2.1,

$$\begin{gathered} w_n \to w^* \ in \ C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega)) \\ \nabla w_{n,t}|_\Gamma \to \nabla w_t^*|_\Gamma \ in \ L_2(\Sigma_T). \end{gathered} \tag{2.20}$$

This allows us to pass with the limit on the linear equation, (2.12). We obtain

$$\begin{cases} w^*_{tt} - \gamma^2 \Delta w^*_{tt} + \Delta^2 w^* + b w^*_t = f = [w, \chi(w)] \\ \quad w^*(0) = w_0; \quad w^*_t(0) = w_1 \in H^2(\Omega) \\ \Delta w^* + (1-\mu) B_1 w^* + \frac{\partial}{\partial \nu} w^*_t|_{\Gamma_1} = -f_1(\frac{\partial}{\partial \nu} w_t) + \frac{\partial}{\partial \nu} w_t \\ \frac{\partial}{\partial \nu} \Delta w^* + (1-\mu) B_2 w^* - \gamma^2 \frac{\partial}{\partial \nu} w^*_{tt} - w^* - w^*_t + \frac{\partial^2}{\partial \tau^2} w^*_t|_{\Gamma_1} = g(w_t) - \frac{\partial}{\partial \tau} f_2(\frac{\partial}{\partial \tau} w_t) - w_t + \frac{\partial^2}{\partial \tau^2} w_t. \end{cases} \tag{2.21}$$

Since w satisfies (2.21) and the solution to (2.21) is unique, we infer that $w \equiv w^*$ and

$$\begin{gathered} w_n \to w \ in \ C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega)) \\ \nabla w_{n,t}|_\Gamma \to \nabla w_t|_\Gamma \ in \ L_2(\Sigma_T), \end{gathered} \tag{2.22}$$

as desired. □

Now we are in a position to prove the fundamental energy relation for problem (1.1).

Lemma 2.1 *(Energy Identity) Let w be the solution to (1.1). Then the following energy identity holds.*

$$E_w(T) - E_w(0) + \int_{\Sigma_T} [g(w_t) w_t + f_1(\frac{\partial}{\partial \nu} w_t) \frac{\partial}{\partial \nu} w_t + f_2(\frac{\partial}{\partial \tau} w_t) \frac{\partial}{\partial \tau} w_t] d\Gamma dt = 0. \tag{2.23}$$

Proof: We first prove this energy identity for the solution, w_n, of the approximation problem, (2.12). Indeed, by applying a standard energy argument to (2.12), we obtain

$$\begin{aligned} E_{w_n,1}(T) - E_{w_n,1}(0) & + \int_{\Sigma_T} |\tfrac{\partial}{\partial \nu} w_{n,t}|^2 d\Gamma dt + \int_{\Sigma_T} (w_{n,t})^2 d\Gamma dt + \int_{\Sigma_T} |\tfrac{\partial}{\partial \tau} w_{n,t}|^2 d\Gamma dt \\ & = \int_0^T \int_\Omega f_n w_{n,t} d\Omega dt + \int_{\Sigma_T} (f_{1n} + \alpha_n) \tfrac{\partial}{\partial \nu} w_{n,t} d\Gamma dt - \int_{\Sigma_T} (f_{2n} + \beta_n) w_{n,t} d\Gamma dt. \end{aligned} \tag{2.24}$$

Using convergence properties (2.7)-(2.11) and the result of Proposition 2.1, we obtain

$$\begin{aligned} E_{w,1}(T) - E_{w,1}(0) \quad & + \int_{\Sigma_T} |\tfrac{\partial}{\partial \nu} w_t|^2 d\Gamma dt + \int_{\Sigma_T} (w_t)^2 d\Gamma dt + \int_{\Sigma_T} |\tfrac{\partial}{\partial \tau} w_t|^2 d\Gamma dt \\ & = \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt + \int_{\Sigma_T} [-f_1(\tfrac{\partial}{\partial \nu} w_t) + \tfrac{\partial}{\partial \nu} w_t] \tfrac{\partial}{\partial \nu} w_t d\Gamma dt \\ & \quad - \int_{\Sigma_T} [g(w_t) - \tfrac{\partial}{\partial \tau} f_2(\tfrac{\partial}{\partial \tau} w_t)] w_t d\Gamma dt + \int_{\Sigma_T} (w_t^2 + |\tfrac{\partial}{\partial \tau} w_t|^2) d\Gamma dt. \end{aligned} \tag{2.25}$$

After canceling boundary terms, we have

$$\begin{aligned} E_{w,1}(T) - E_{w,1}(0) \quad & = \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt - \int_{\Sigma_T} f_1(\tfrac{\partial}{\partial \nu} w_t) \tfrac{\partial}{\partial \nu} w_t d\Gamma dt \\ & \quad - \int_{\Sigma_T} [g(w_t) - \tfrac{\partial}{\partial \tau} f_2(\tfrac{\partial}{\partial \tau} w_t)] w_t d\Gamma dt. \end{aligned} \tag{2.26}$$

Since

$$\begin{aligned} & \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt = \int_\Omega [w, \chi(w)] w d\Omega |_0^T - \int_0^T \int_\Omega [w_t, \chi(w)] w d\Omega dt \\ & \quad \Longrightarrow \int_0^T \int_\Omega [w, \chi(w)] w_t d\Omega dt = E_{w,2}(0) - E_{w,2}(T), \end{aligned} \tag{2.27}$$

by the symmetricity of the trilinear form ([?], Lemma 5.2.1), our desired result follows directly from (2.26) and (2.27). □

3 A Priori Estimates

To proof Theorem 1.2, we first show the following inequality holds.

Lemma 3.1 *Let w be the solution to (1.1), $0 < \alpha < T/2$ and $\epsilon > 0$ be arbitrary. Then there exist constants, C, $C_{T,\alpha,\epsilon}$, and $C(E_w(0))$ such that the following inequality holds:*

$$\begin{aligned} \int_\alpha^{T-\alpha} E_w(t) dt - C E_w(0) \quad & \leq C_{T,\alpha,\epsilon} \{ \int_{\Sigma_T} (|w_t|^2 + \gamma^2 |\nabla w_t|^2) d\Gamma dt + ||f_1(\tfrac{\partial}{\partial \nu} w_t)||^2_{L_2(\Sigma_T)} + ||f_2(\tfrac{\partial}{\partial \tau} w_t)||^2_{L_2(\Sigma_T)} \\ & \quad + C(E_w(0)) \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt + \int_{Q_T} b(x) w_t^2 d\Omega dt \}, \end{aligned} \tag{3.1}$$

where $C(E_w(0))$ is an increasing function of $E_w(0)$ and $\Sigma_A \equiv \{(t,x) \in \Sigma_T : |w_t(t,x)| < 1\}$.

3.1 Multiplier Methods

To prove Lemma 3.1, we begin by using a multiplier method on the approximation problem (2.12) to prove the following preliminary estimate.

Proposition 3.1 *Let $(w_0, w_1) \in \mathcal{D}$. Then the energy of system (2.12) as given by (1.6) satisfies the following estimate:*

$$
\begin{aligned}
\frac{1}{2}\int_0^T E_{w_n,1}(t)dt \quad & -C_1(1+\gamma^2)E_{w_n,1}(T) - C_2(1+\gamma^2)E_{w_n,1}(0) \\
& \leq |(\frac{\partial}{\partial\nu}w_n, f_{1n})_{L_2(\Sigma_T)}| + |(w_n, f_{2n})_{L_2(\Sigma_T)}| \\
& \quad +|(\frac{\partial}{\partial\nu}(\vec{h}\cdot\nabla w_n), f_{1n})_{L_2(\Sigma_T)}| + |(\vec{h}\cdot\nabla w_n, f_{2n})_{L_2(\Sigma_T)}| \\
& \quad \int_{Q_T} f_n \vec{h}\cdot\nabla w_n d\Omega dt - \frac{1}{4}\int_{Q_T} f_n w_n d\Omega dt \\
& \quad +C_3\int_{\Sigma_T}\vec{h}\cdot\nu(w_{n,t}^2 + \gamma^2|\nabla w_{n,t}|^2)d\Gamma dt \\
& \quad +C_4\int_{\Sigma_T}(|\frac{\partial^2 w_n}{\partial\tau^2}|^2 + (|\frac{\partial^2 w_n}{\partial\nu^2}|^2 + (|\frac{\partial^2 w_n}{\partial\nu\partial\tau}|^2)d\Gamma dt \\
& \quad +C_5\int_{Q_T} b(x)w_{n,t}^2 d\Omega dt + C_7 l.o.(w_n),
\end{aligned}
\tag{3.2}
$$

where

$$
l.o.(w_n) \equiv \|w_n\|^2_{L_2(0,T;H^{2-\epsilon}(\Omega))}, \tag{3.3}
$$

$0 < \epsilon < 1/2$, and $\vec{h} \equiv x - x_0$ for some $x_0 \in R^2$.

Proof of Proposition 3.1: *Step 1: Identities.* ¿From [?] (p. 84, (4.5.17)), with adjustments to take both the nonhomogeneous right-hand side of (2.12) and the boundary conditions into account, we have

$$
\begin{aligned}
\int_0^T E_{w_n,1}(t)dt \quad & +\int_{Q_T} w_{n,t}^2 d\Omega dt + \frac{1}{2}[(w_{n,t}, \vec{h}\cdot\nabla w_n)_{L_2(\Omega)} + \gamma^2(\nabla w_{n,t}, \nabla(\vec{h}\cdot\nabla w_n))_{L_2(\Omega)}]_0^T \\
& -\frac{1}{2}[(w_{n,t}, w_n)_{L_2(\Omega)} + \gamma^2(\nabla w_{n,t}, \nabla w_n)_{L_2(\Omega)}]_0^T \\
& +\frac{1}{2}(\frac{\partial}{\partial\nu}w_n, f_{1n})_{L_2(\Sigma_T)} - \frac{1}{2}(w_n, f_{2n})_{L_2(\Sigma_T)} \\
= & -\int_{\Sigma_T}[\frac{\partial}{\partial\nu}(\vec{h}\cdot\nabla w_n)f_{1n} - (\vec{h}\cdot\nabla w_n)f_{2n}]d\Gamma dt - \int_{\Sigma_T}(\vec{h}\cdot\nabla w_n)w_n d\Gamma dt \\
& +\frac{1}{2}\int_{\Sigma_T}\vec{h}\cdot\nu(w_{n,t}^2 + \gamma^2|\nabla w_{n,t}|^2)d\Gamma dt \\
& -\frac{1}{2}\int_{\Sigma_T}\vec{h}\cdot\nu[w_{n,xx}^2 + w_{n,yy}^2 + 2\mu w_{n,xx}w_{n,yy} + 2(1-\mu)w_{n,xy}^2]d\Gamma dt \\
& -\int_{Q_T} b(x)w_{n,t}\vec{h}\cdot\nabla w_n d\Omega dt + \int_{Q_T} f_n\vec{h}\cdot\nabla w_n d\Omega dt \\
& -\frac{1}{2}\int_{Q_T} f_n w_n d\Omega dt + \frac{1}{2}(b(x)w_n, w_n)_{L_2(\Omega)}|_0^T.
\end{aligned}
\tag{3.4}
$$

Notice that the regularity of the solution given by (2.14) allows us to justify the calculations in [?].

To set up the inequality so that the appropriate energy term for the original system will appear on the left-hand side when we let $n \to \infty$, we need the following identity for the multiplier w.

$$\begin{aligned}\int_{Q_T} w_{n,t}^2 d\Omega dt &= [(w_{n,t}, w_n)_{L_2(\Omega)} + \gamma^2(\nabla w_{n,t}, \nabla w_n)_{L_2(\Omega)}]_0^T \\ &\quad +\tfrac{1}{2}(\tfrac{\partial}{\partial\nu} w_n, f_{1n})_{L_2(\Sigma_T)} - \tfrac{1}{2}(w_n, f_{2n})_{L_2(\Sigma_T)} + \tfrac{1}{2}(w_n, w_n)_{L_2(\Sigma_T)} + (b(x)w_n, w_n)_{L_2(\Omega)}|_0^T \\ &\quad -\gamma^2 \int_{Q_T} |\nabla w_{n,t}|^2 d\Omega dt + \int_0^T a(w_n, w_n)dt - \int_{Q_T} f_n w_n d\Omega dt.\end{aligned} \tag{3.5}$$

Multiplying the above identity by $\frac{1}{4}$, substituting into (3.4) and noting (2.1), we find

$$\begin{aligned}\tfrac{1}{2}\int_0^T E_{w_n,1}(t)dt &\le -\tfrac{1}{2}[(w_{n,t}, \vec{h}\cdot\nabla w_n)_{L_2(\Omega)} + \gamma^2(\nabla w_{n,t}, \nabla(\vec{h}\cdot\nabla w_n))_{L_2(\Omega)}]_0^T \\ &\quad +\tfrac{1}{4}[(w_{n,t}, w_n)_{L_2(\Omega)} + \gamma^2(\nabla w_{n,t}, \nabla w_n)_{L_2(\Omega)}]_0^T \\ &\quad -\tfrac{1}{4}(\tfrac{\partial}{\partial\nu} w_n, f_{1n})_{L_2(\Sigma_T)} + \tfrac{1}{4}(w_n, f_{2n})_{L_2(\Sigma_T)} \\ &\quad -(\tfrac{\partial}{\partial\nu}(\vec{h}\cdot\nabla w_n), f_{1n})_{L_2(\Sigma_T)} + (\vec{h}\cdot\nabla w_n, f_{2n})_{L_2(\Sigma_T)} \\ &\quad +\tfrac{1}{2}\int_{\Sigma_T} \vec{h}\cdot\nu(w_{n,t}^2 + \gamma^2|\nabla w_{n,t}|^2)d\Gamma dt - (\vec{h}\cdot\nabla w_n, w_n)_{L_2(\Sigma_T)} \\ &\quad -\tfrac{1}{2}\int_{\Sigma_T} \vec{h}\cdot\nu[w_{n,xx}^2 + w_{n,yy}^2 + 2\mu w_{n,xx}w_{n,yy} + 2(1-\mu)w_{n,xy}^2]d\Gamma dt \\ &\quad -\int_{Q_T} b(x)w_{n,t}\vec{h}\cdot\nabla w_n d\Omega dt + \int_{Q_T} f_n\vec{h}\cdot\nabla w_n d\Omega dt \\ &\quad -\tfrac{1}{4}\int_{Q_T} f_n w_n d\Omega dt - (b(x)w_n, w_n)_{L_2(\Omega)}|_0^T,\end{aligned} \tag{3.6}$$

Step 2: Bounding Linear Terms. To bound the first term on the last line of (3.6), we use equation (2.7) from [?] which gives us

$$-\int_{Q_T} b(x)w_{n,t}\vec{h}\cdot\nabla w_n d\Omega dt \le C_1\int_{Q_T} b(x)w_{n,t}^2 d\Omega dt + C_2 l.o.(w_n). \tag{3.7}$$

All terms which need to be evaluated at 0 and T, including the first and second lines and the last term on the right-hand side of (3.6), can be bounded by

$$C_1(1+\gamma^2)E_{w_n,1}(T) + C_2(1+\gamma^2)E_{w_n,1}(0). \tag{3.8}$$

Finally, note that the sixth line and the last term on the fifth line of the right-hand side of (3.6) can be bounded by second order traces of the solution w_n and $l.o.(w_n)$.

With the above three estimates, we obtain our desired result, (3.2). □

Recalling Proposition 2.1, we take the limit of (3.2) as $n \to \infty$ to obtain a similar inequality for the solution to (1.1), which we state in the following proposition.

Proposition 3.2 *Let $(w_0, w_1) \in \mathcal{H}$. Then the energy of system (1.1) as given by (1.6) satisfies the following estimate:*

$$\begin{aligned} \int_0^T E_w(t)dt \quad & -C_1(1+\gamma^2)E_w(T) - C_2(1+\gamma^2)E_w(0) \\ & \le C_3 \int_{\Sigma_T} (|w_t|^2 + \gamma^2 |\nabla w_t|^2) d\Gamma dt + C_4 E_w^2(0) \int_{\Sigma_T} |\Delta \chi(w)| d\Gamma dt \\ & \quad + C_5 \int_{\Sigma_T} (|\tfrac{\partial^2 w}{\partial \tau^2}|^2 + |\tfrac{\partial^2 w}{\partial \nu^2}|^2 + |\tfrac{\partial^2 w}{\partial \nu \partial \tau}|^2) d\Gamma dt \\ & \quad + C(E_w(0)) \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + C_6 \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt \\ & \quad + C_7 \int_{Q_T} b(x) w_t^2 d\Omega dt + C_8 l.o.(w), \end{aligned} \tag{3.9}$$

where $\Sigma_A \equiv \{(t,x) \in \Sigma_T : |w_t(t,x)| < 1\}$.

Proof: *Step 1: Approximation Results.* Taking the limit as $n \to \infty$ in (3.2), by virtue of (2.7)-(2.11) and Proposition 2.1, we find

$$\begin{aligned} \frac{1}{2}\int_0^T E_{w,1}(t)dt \quad & -C_1(1+\gamma^2)E_{w,1}(T) - C_2(1+\gamma^2)E_{w,1}(0) \\ & \le |(\tfrac{\partial}{\partial \nu} w, \mathcal{F}_1(w_t))_{L_2(\Sigma_T)}| + |(w, \mathcal{F}_2(w_t))_{L_2(\Sigma_T)}| \\ & \quad + |(\tfrac{\partial}{\partial \nu}(\vec{h} \cdot \nabla w), \mathcal{F}_1(w_t))_{L_2(\Sigma_T)}| + |(\vec{h} \cdot \nabla w, \mathcal{F}_2(w_t))_{L_2(\Sigma_T)}| \\ & \quad + C_3 \int_{\Sigma_T} \vec{h} \cdot \nu (w_t^2 + \gamma^2 |\nabla w_t|^2) d\Gamma dt \\ & \quad + C_4 \int_{\Sigma_T} (|\tfrac{\partial^2 w}{\partial \tau^2}|^2 + (|\tfrac{\partial^2 w}{\partial \nu^2}|^2 + (|\tfrac{\partial^2 w}{\partial \nu \partial \tau}|^2) d\Gamma dt \\ & \quad + \int_{Q_T} [w, \chi(w)] \vec{h} \cdot \nabla w d\Omega dt - \frac{1}{4} \int_{\Sigma_T} [w, \chi(w)] w d\Omega dt \\ & \quad + C_5 \int_{Q_T} b(x) w_t^2 d\Omega dt + C_6 l.o.(w), \end{aligned} \tag{3.10}$$

where $\mathcal{F}_i$ are the control terms, defined as follows:

$$\begin{aligned} \mathcal{F}_1(w_t) &\equiv -f_1(\tfrac{\partial}{\partial \nu} w_t) \\ \mathcal{F}_2(w_t) &\equiv g(w_t) - \tfrac{\partial}{\partial \tau} f_2(\tfrac{\partial}{\partial \tau} w_t). \end{aligned} \tag{3.11}$$

Step 2: Bounding Nonlinear Terms. ¿From the regularity of the solution, w, given in Theorem 1.1 and from elliptic regularity (see [?]), we infer

$$\chi \in H^{3-\epsilon}(\Omega) \cap H_0^2(\Omega) \qquad \forall \epsilon > 0. \tag{3.12}$$

Hence, $\Delta^2\chi \in H^{-1-\epsilon}(\Omega)$, which, combined with $\chi \in H_0^{1+\epsilon}(\Omega)$, allows us to define

$$(\Delta^2\chi, \chi)_{L_2(\Omega)} \tag{3.13}$$

as a duality pairing between $H_0^{1+\epsilon}(\Omega)$ and $H^{-1-\epsilon}(\Omega)$. Application of Green's formula together with regularity in (3.12) gives

$$(\Delta^2\chi, \chi)_{L_2(\Omega)} = \|\Delta\chi\|^2_{L_2(\Omega)}. \tag{3.14}$$

Similarly, from (3.12), the regularity $w \in H^2(\Omega)$, and properties of the von Kármán nonlinearity, [,], we also obtain that

$$[w, \chi(w)] \in H^{-\epsilon}(\Omega). \tag{3.15}$$

Hence,

$$\int_\Omega [w, \chi(w)]\vec{h} \cdot \nabla w d\Omega \tag{3.16}$$

can be defined as a duality pairing between $H^\epsilon(\Omega)$ and $H^{-\epsilon}(\Omega)$, $\epsilon < \frac{1}{2}$. This, in turn, allows us to justify the computation in [?] (pg. 114), based on the application of the Divergence Theorem, which then leads to

$$([w, \chi(w)], \vec{h} \cdot \nabla w)_{L_2(\Omega)} = -\frac{1}{2}\|\Delta\chi\|^2_{L_2(\Omega)} - \frac{1}{2}\int_\Gamma \vec{h} \cdot \nu(\Delta\chi)^2 d\Gamma. \tag{3.17}$$

¿From [?], we find

$$\int_{\Sigma_T} |\Delta\chi(w)|^2 d\Gamma dt \leq \frac{1}{4C_4}\int_0^T E_w(t)dt + CE_w^2(0)\int_{\Sigma_T} |\Delta\chi(w)| d\Gamma dt. \tag{3.18}$$

Recalling the definitions of $\mathcal{F}_i$, we bound the remaining nonlinear terms in (3.2) as follows:

$$\begin{aligned} |(\tfrac{\partial}{\partial\nu}w, \mathcal{F}_1(w_t))_{L_2(\Sigma_T)}| &\leq C(\|\tfrac{\partial}{\partial\nu}w\|^2_{L_2(\Sigma_T)} + \|f_1(\tfrac{\partial}{\partial\nu}w_t)\|^2_{L_2(\Sigma_T)}) \\ &\leq C(l.o.(w) + \|f_1(\tfrac{\partial}{\partial\nu}w_t)\|^2_{L_2(\Sigma_T)}), \end{aligned} \tag{3.19}$$

$$|(w, \mathcal{F}_2(w_t))_{L_2(\Sigma_T)}| \leq C(l.o.(w) + \|f_1(\frac{\partial}{\partial\tau}w_t)\|^2_{L_2(\Sigma_T)} + |(w, g(w_t))_{L_2(\Sigma_T)}|), \tag{3.20}$$

$$|(\frac{\partial}{\partial\nu}(\vec{h} \cdot \nabla w), \mathcal{F}_1(w_t))_{L_2(\Sigma_T)}| \leq C(\|\frac{\partial^2 w}{\partial\nu^2}\|^2_{L_2(\Sigma_T)} + \|\frac{\partial^2 w}{\partial\nu\partial\tau}\|^2_{L_2(\Sigma_T)} + \|f_1(\frac{\partial}{\partial\nu}w_t)\|^2_{L_2(\Sigma_T)}), \tag{3.21}$$

$$\begin{aligned} |(\vec{h} \cdot \nabla w, \mathcal{F}_2(w_t))_{L_2(\Sigma_T)}| &\leq C(\|\tfrac{\partial^2 w}{\partial\tau^2}\|^2_{L_2(\Sigma_T)} + \|\tfrac{\partial^2 w}{\partial\nu\partial\tau}\|^2_{L_2(\Sigma_T)} \\ &\quad + \|f_2(\tfrac{\partial}{\partial\tau}w_t)\|^2_{L_2(\Sigma_T)} + |(\vec{h} \cdot \nabla w, g(w_t))_{L_2(\Sigma_T)}|). \end{aligned} \tag{3.22}$$

To bound the terms involving $g(w_t)$, we proceed as follows. Define $\mathcal{B}_i$ to be

$$\begin{aligned} \mathcal{B}_1 &\equiv |(w, g(w_t))_{L_2(\Sigma_T)}| \\ \mathcal{B}_2 &\equiv |(\vec{h}\cdot\nabla w, g(w_t))_{L_2(\Sigma_T)}|. \end{aligned} \tag{3.23}$$

Estimates for $\mathcal{B}_2$. Denote $\Sigma_A \equiv \{x \in \Sigma_T; |w_t(t,x)| < 1\}$ and $\Sigma_B \equiv \Sigma_T \backslash \Sigma_A$. Then

$$\int_{\Sigma_A} |g(w_t)\vec{h}\cdot\nabla w| d\Gamma dt \leq C(\int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt + \int_{\Sigma_A} |\nabla w|^2 d\Gamma dt) \tag{3.24}$$

and

$$\int_{\Sigma_B} |g(w_t)\vec{h}\cdot\nabla w| d\Gamma dt \leq \frac{C_1}{\epsilon}\int_{\Sigma_B} |g(w_t)|^\beta d\Gamma dt + \epsilon C_2 \int_{\Sigma_B} |\nabla w|^\alpha d\Gamma dt, \tag{3.25}$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Recall hypothesis (H-1). Selecting $r(\beta - 1) = 1$, i.e., $\beta = \frac{1+r}{r} > 1$, $\alpha = r + 1$, and using (H-1), we find

$$\begin{aligned} \textstyle\int_{\Sigma_B} |g(w_t)|^\beta d\Gamma dt &= \textstyle\int_{\Sigma_B} |g(w_t)|^{\beta-1}|g(w_t)| d\Gamma dt \\ &\leq C \textstyle\int_{\Sigma_B} |w_t^{r(\beta-1)} g(w_t)| d\Gamma dt = C\int_{\Sigma_B} |w_t g(w_t)| d\Gamma dt. \end{aligned} \tag{3.26}$$

Collecting (3.25) and (3.26), using trace theory, Sobolev's imbeddings, and the following inequality,

$$\|\nabla w(t)\|^\alpha_{L_2(\Gamma)} \leq E_w(t)^{\alpha/2} = E_w(t)E_w(t)^{\alpha/2-1} \leq E_w(0)^{\alpha/2-1}E_w(t), \tag{3.27}$$

yields

$$\int_{\Sigma_B} |g(w_t)\vec{h}\cdot\nabla w| d\Gamma dt \leq \frac{C}{\epsilon}\int_{\Sigma_B} |w_t g(w_t)| d\Gamma dt + \epsilon C(E_w(0)) \int_0^T E_w(t) dt. \tag{3.28}$$

Estimates for $\mathcal{B}_1$. Splitting the integral as before, we find

$$\int_{\Sigma_A} |g(w_t) w| d\Gamma dt \leq C(\int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt + \int_{\Sigma_A} |w|^2 d\Gamma dt) \tag{3.29}$$

and

$$\int_{\Sigma_B} |g(w_t) w| d\Gamma dt \leq \frac{C}{\epsilon}\int_{\Sigma_B} |w_t g(w_t)| d\Gamma dt + \epsilon C(E_w(0)) \int_0^T E_w(t) dt. \tag{3.30}$$

Taking ϵ to be appropriately small, using assumption (H-1), then (3.24)-(3.30), completes the proof of Proposition 3.2. □

Next, we bound the second-order traces by using the following proposition.

Proposition 3.3 *Let w be the solution to (1.1). Then for any $\alpha > 0$ and $\epsilon > 0$, w satisfies the following inequality:*

$$\begin{aligned}\int_\alpha^{T-\alpha} \int_\Gamma (|\frac{\partial^2 w}{\partial \tau^2}|^2 & + |\frac{\partial^2 w}{\partial \nu^2}|^2 + |\frac{\partial^2 w}{\partial \nu \partial \tau}|^2) d\Gamma_1 dt \\ & \leq C_{T,\alpha,\epsilon}\{\|w_t\|^2_{L_2(\Sigma_T)} + \|\nabla w_t\|^2_{L_2(\Sigma_T)} + E_w^2(0)\|\chi(w)\|_{L_1(0,\,T;\,H^{3-\epsilon}(\Omega))} + l.o.(w) \\ & + \|f_1(\frac{\partial}{\partial \nu} w_t)\|^2_{L_2(\Sigma_T)} + \|f_2(\frac{\partial}{\partial \tau} w_t)\|^2_{L_2(\Sigma_T)} \\ & + C(E_w(0)) \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt\}.\end{aligned} \tag{3.31}$$

To prove this proposition, we require the following result which was proven in [?] (see also [?]) using microlocal analysis methods.

Proposition 3.4 ([?], Theorem 1.1) *Let $p(t,x)$ be a solution to the following linear problem (in the sense of distributions)*

$$\left.\begin{aligned} & p_{tt} - \gamma^2 \Delta p_{tt} + \Delta^2 p = f && \text{in } Q_T \\ & p(0,\cdot) = p_0;\ p_t(0,\cdot) = p_1 && \text{in } \Omega \\ & \Delta p + (1-\mu) B_1 p = g_1 && \text{on } \Sigma_T \\ & \frac{\partial}{\partial \nu} \Delta p + (1-\mu) B_2 p - \gamma^2 \frac{\partial}{\partial \nu} p_{tt} - p = g_2 && \text{on } \Sigma_T. \end{aligned}\right\} \tag{3.32}$$

For every $T > \alpha > 0$ and $\frac{1}{2} > \epsilon > 0$, the following estimate holds:

$$\begin{aligned}\int_\alpha^{T-\alpha} \int_\Gamma (|\frac{\partial^2 p}{\partial \tau^2}|^2 & + |\frac{\partial^2 p}{\partial \nu^2}|^2 + |\frac{\partial^2 p}{\partial \nu \partial \tau}|^2) d\Gamma dt \\ & \leq C_{T,\alpha}\{\|f\|^2_{L_2([0,\,T];\,H^{-\epsilon}(\Omega))} + \|g_1\|^2_{L_2(\Sigma_T)} + \|g_2\|^2_{L_2([0,\,T];\,H^{-1}(\Omega))} \\ & + \|p_t\|^2_{L_2(\Sigma_T)} + \|\nabla p_t\|^2_{L_2(\Sigma_T)} + \|p\|^2_{L_2([0,\,T];\,H^{3/2+\epsilon}(\Omega))}\},\end{aligned} \tag{3.33}$$

where the constant $C_{T,\alpha}$ does not depend on γ.

Proof of Proposition 3.3: We apply the result of Proposition 3.4 to system (1.1) with

$$\begin{aligned} f & \equiv -b(x) w_t + [\chi(w), w] \\ g_1 & \equiv -f_1(\frac{\partial}{\partial \nu} w_t) \\ g_2 & \equiv g(w_t) - \frac{\partial}{\partial \tau} f_2(\frac{\partial}{\partial \tau} w_t). \end{aligned} \tag{3.34}$$

¿From [?], we know that

$$\int_0^T \|f\|^2_{H^{-\epsilon_0}(\Omega)} dt \leq C\{l.o.(w) + E_w^2(0) \int_0^T \|\chi(w)\|_{H^{3-\epsilon_1}(\Omega)} dt\}. \tag{3.35}$$

Splitting the integral of the $g(w_t)$ term as in the third step of the proof of Proposition 3.2 and using (3.26) with $\beta = 1$, we find

$$\|g_2\|^2_{L_2([0,T];H^{-1}(\Omega))} \le C(E_w(0)) \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + C_1 \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt + C_2 \|f_2(\frac{\partial}{\partial \tau} w_t)\|^2_{L_2(\Sigma_T)}. \qquad (3.36)$$

Combining (3.34)-(3.36) with (3.33) and choosing $\epsilon = \epsilon_0$, we arrive at our desired result. □

At this point, we are able to prove the following energy estimate.

Lemma 3.2 *Let w be the solution to (1.1), $0 < \alpha < T$ and $\epsilon > 0$ be arbitrary. Then*

$$\begin{aligned} \int_\alpha^{T-\alpha} E_w(t)dt - C(1+\gamma^2)E_w(0) \quad & \le C_{T,\alpha,\epsilon}\{\int_{\Sigma_T} (|w_t|^2 + \gamma^2 |\nabla w_t|^2) d\Gamma dt + E_w^2(0) \int_{\Sigma_T} |\Delta \chi(w)| d\Gamma dt \\ & + E_w^2(0) \int_{\Sigma_T} \|\chi(w)\|_{H^{3-\epsilon}(\Omega)} dt + \|f_1(\tfrac{\partial}{\partial \nu} w_t)\|^2_{L_2(\Sigma_T)} + \|f_2(\tfrac{\partial}{\partial \tau} w_t)\|^2_{L_2(\Sigma_T)} \\ & + C(E_w(0)) \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt \\ & + \int_{Q_T} b(x) w_t^2 d\Omega dt + l.o.(w)\}. \end{aligned} \qquad (3.37)$$

Proof of Lemma 3.2: Applying the result of Proposition 3.2 on the interval $[\alpha, T-\alpha]$ yields

$$\begin{aligned} \int_\alpha^{T-\alpha} E_w(t)dt \quad & -C_1(1+\gamma^2)E_w(T-\alpha) - C_2(1+\gamma^2)E_w(\alpha) \\ & \le C_3 \int_{\Sigma_T} (|w_t|^2 + \gamma^2|\nabla w_t|^2) d\Gamma dt + C_4 E_w^2(\alpha) \int_{\Sigma_T} |\Delta\chi(w)| d\Gamma dt \\ & + C_5 \int_{\Sigma_{T_\alpha}} (|\tfrac{\partial^2 w}{\partial \tau^2}|^2 + |\tfrac{\partial^2 w}{\partial \nu^2}|^2 + |\tfrac{\partial^2 w}{\partial \nu \partial \tau}|^2) d\Gamma dt \\ & + C(E_w(\alpha)) \int_{\Sigma_T} g(w_t) w_t d\Gamma dt + C_6 \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt \\ & + C_7 \int_{Q_T} b(x) w_t^2 d\Omega dt + C_8 l.o.(w), \end{aligned} \qquad (3.38)$$

where $\Sigma_{T_\alpha} \equiv \Gamma \times [\alpha, T-\alpha]$. Since the energy of the system is nonincreasing, $E_w(T) \le E_w(T-\alpha) \le E_w(\alpha) \le E_w(0)$. Also, $E_w^2(\alpha) \le E_w^2(0)$. Applying Proposition 3.3 to the second-order traces on the right-hand side of (3.38) and recalling Lemma 2.1 then gives us our desired result. □

3.2 Compactness-Uniqueness Argument

¿From [?] (eq. (2.3.1)), we have

$$\int_{\Sigma_T} |\Delta\chi(w)| d\Gamma dt \le C \int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)} dt. \qquad (3.39)$$

Thus, to remove terms involving $\chi(w)$ and lower-order terms, we need the following result.

Proposition 3.5 *Let w be the solution to (1.1). Then w satisfies the following inequality:*

$$\begin{aligned}\int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)}dt + l.o.(w) \quad &\leq C(E_w(0))\{\|w_t\|^2_{L_2(\Sigma_T)} + \|\nabla w_t\|^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x)w_t^2 d\Omega dt \\ &+\|f_1(\tfrac{\partial}{\partial\nu}w_t)\|^2_{L_2(\Sigma_T)} + \|f_2(\tfrac{\partial}{\partial\tau}w_t)\|^2_{L_2(\Sigma_T)} \\ &+\int_{\Sigma_T} g(w_t)w_t d\Gamma dt + \int_{\Sigma_A} |g(w_t)|^2 d\Gamma dt\},\end{aligned} \tag{3.40}$$

where $C(E_w(0))$ is an increasing function of $E_w(0)$ and is independent of γ.

Proof of Proposition 3.5: *Step 1: Allow $C(E_w(0))$ to depend on γ.* Assume (3.40) does not hold. Then there exists a sequence of functions, $\{w_n(t)\} \in \mathcal{H}$ such that each $w_n(t)$ satisfies the system

$$\begin{aligned}
&w_{n,tt} - \gamma^2\Delta w_{n,tt} + \Delta^2 w_n + bw_{n,t} = [w_n, \chi(w_n)] && \text{in } Q_T \\
&w_n(0,\cdot) = w_{n,0};\ w_{n,t}(0,\cdot) = w_{n,1} && \text{in } \Omega \\
&\Delta w_n + (1-\mu)B_1 w_n = -f_1(\tfrac{\partial}{\partial\nu}w_{n,t}) && \text{on } \Sigma_T \\
&\tfrac{\partial}{\partial\nu}\Delta w_n + (1-\mu)B_2 w_n - \gamma^2\tfrac{\partial}{\partial\nu}w_{n,tt} - w_n = g(w_{n,t}) - \tfrac{\partial}{\partial\tau}f_2(\tfrac{\partial}{\partial\tau}w_{n,t}) && \text{on } \Sigma_T
\end{aligned} \tag{3.41}$$

and such that

$$\lim_{n\to\infty} \frac{\int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)}dt + l.o.(w_n)}{\mathcal{P}(w_n)} = \infty, \tag{3.42}$$

where

$$\begin{aligned}\mathcal{P}(w_n) \quad &\equiv \|w_{n,t}\|^2_{L_2(\Sigma_T)} + \|\nabla w_{n,t}\|^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x)w_{n,t}^2 d\Omega dt + \|f_1(\tfrac{\partial}{\partial\nu}w_{n,t})\|^2_{L_2(\Sigma_T)} \\ &+\|f_2(\tfrac{\partial}{\partial\tau}w_{n,t})\|^2_{L_2(\Sigma_T)} + \int_{\Sigma_T} g(w_{n,t})w_{n,t} d\Gamma dt + \int_{\Sigma_A} |g(w_{n,t})|^2 d\Gamma dt,\end{aligned} \tag{3.43}$$

and where the initial energy of (3.41) is uniformly bounded in n. Hence, by Lemma 2.1 and the same arguments as in Proposition 2.5 of [?], we know the following convergence properties hold:

$$\left.\begin{aligned} w_n &\xrightarrow{w} w \ in\ L_2([0,T]; H^2(\Omega)) \\ w_{n,t} &\xrightarrow{w} w_t \ in\ L_2([0,T]; H^1(\Omega))\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} &l.o.(w_n) \longrightarrow l.o.(w) \\ &\int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)}dt \longrightarrow \int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)}dt.\end{aligned}\right. \tag{3.44}$$

Case 1: Assume $\int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)}dt + l.o.(w) \neq 0$. Then

$$\mathcal{P}(w_n) \longrightarrow 0, \tag{3.45}$$

which in turn implies that

$$\begin{aligned} w_{n,t} &\longrightarrow 0 \; in \; L_2(Q_T) \\ f_1(\tfrac{\partial}{\partial \nu} w_{n,t}) &\longrightarrow 0 \; in \; L_2(\Sigma_T) \\ f_2(\tfrac{\partial}{\partial \tau} w_{n,t}) &\longrightarrow 0 \; in \; L_2(\Sigma_T), \end{aligned} \tag{3.46}$$

and, since by hypothesis (H-1),

$$\int_{\Sigma_B} g(w_{n,t}) w_{n,t} d\Gamma dt \geq \frac{1}{M} \int_{\Sigma_B} |g(w_{n,t})|^{1+1/r} d\Gamma dt, \tag{3.47}$$

we have

$$g(w_{n,t}) \longrightarrow 0 \; in \; L_1(\Sigma_T). \tag{3.48}$$

Thus, by passing with the limit as $n \longrightarrow \infty$ on (3.41), we obtain the limit system

$$\begin{aligned} &\Delta^2 w = [w, \chi(w)] && \text{in } Q_T \\ &w(0,\cdot) = w_0; \;\; w_t(0,\cdot) = w_1 && \text{in } \Omega \\ &\Delta w + (1-\mu) B_1 w = 0 && \text{on } \Sigma_T \\ &\tfrac{\partial}{\partial \nu} \Delta w + (1-\mu) B_2 w - \gamma^2 \tfrac{\partial}{\partial \nu} w_{tt} - w = 0 && \text{on } \Sigma_T. \end{aligned} \tag{3.49}$$

By the uniqueness result in [?], the solution w to (3.49) is identically zero. Hence

$$\int_0^T ||\chi(w_n)||_{H^{3-\epsilon}(\Omega)} dt + l.o.(w_n) \longrightarrow 0, \tag{3.50}$$

which contradicts our assumption. Thus Proposition 3.5 holds for Case 1.

Case 2: Assume $\int_0^T ||\chi(w)||_{H^{3-\epsilon}(\Omega)} dt + l.o.(w) = 0$. Define

$$c_n \equiv \{ \int_0^T ||\chi(w_n)||_{H^{3-\epsilon}(\Omega)} dt + l.o.(w_n) \}, \tag{3.51}$$

and

$$v_n \equiv \frac{w_n}{c_n} \; where \; c_n \longrightarrow 0. \tag{3.52}$$

Then v_n satisfies the system

$$\begin{aligned} & v_{n,tt} - \gamma^2 \Delta v_{n,tt} + \Delta^2 v_n + b v_{n,t} = [v_n, \chi(w_n)] && \text{in } Q_T \\ & v_n(0,\cdot) = v_{n,0};\ v_{n,t}(0,\cdot) = v_{n,1} && \text{in } \Omega \\ & \Delta v_n + (1-\mu) B_1 v_n = -\tfrac{1}{c_n} f_1(\tfrac{\partial}{\partial \nu} w_{n,t}) && \text{on } \Sigma_T \\ & \tfrac{\partial}{\partial \nu} \Delta v_n + (1-\mu) B_2 v_n - \gamma^2 \tfrac{\partial}{\partial \nu} v_{n,tt} - v_n = \tfrac{1}{c_n} g(w_{n,t}) - \tfrac{1}{c_n} \tfrac{\partial}{\partial \tau} f_2(\tfrac{\partial}{\partial \tau} w_{n,t}) && \text{on } \Sigma_T \end{aligned} \tag{3.53}$$

By the quadratic dependence of χ on w_n, v_n satisfies

$$\int_0^T \|\chi(v_n)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(v_n) \equiv 1. \tag{3.54}$$

By dividing both sides of (3.42) by c_n^2 and using (3.43) and (3.44), we obtain

$$\begin{aligned} & \|v_{n,t}\|^2_{L_2(\Sigma_T)} + \|\nabla v_{n,t}\|^2_{L_2(\Sigma_T)} + \int_{Q_T} b(x) v_{n,t}^2 d\Omega dt + \tfrac{1}{c_n^2} \|f_1(\tfrac{\partial}{\partial \nu} w_{n,t})\|^2_{L_2(\Sigma_T)} \\ & + \tfrac{1}{c_n^2} \|f_2(\tfrac{\partial}{\partial \tau} w_{n,t})\|^2_{L_2(\Sigma_T)} + \tfrac{1}{c_n^2} \int_{\Sigma_T} g(w_{n,t}) w_{n,t} d\Gamma dt + \tfrac{1}{c_n^2} \int_{\Sigma_A} |g(w_{n,t})|^2 d\Gamma dt \quad \longrightarrow 0. \end{aligned} \tag{3.55}$$

¿From (3.37) after division by c_n^2 and from (3.54) and (3.55), we have

$$\|v_n\|^2_{C(0,\,T;\,H^2(\Omega))} + \|v_{n,t}\|^2_{C(0,\,T;\,H^1(\Omega))} \le C(E_{w_n}(0)). \tag{3.56}$$

Since $E_{w_n}(0)$ are uniformly bounded for all n, $C(E_{w_n}(0))$ is uniformly bounded for all n as well.

We now pass with the limit as $n \longrightarrow \infty$ on system (3.53). To determine the convergence of $g(w_{n,t})$, we will show that

$$\frac{g(w_{n,t})}{c_n} \longrightarrow 0 \qquad in\ L_1(\Sigma_T). \tag{3.57}$$

Indeed,

$$\int_{\Sigma_T} \frac{|g(w_{n,t})|}{c_n} d\Gamma dt = \int_{\Sigma_A} \frac{|g(w_{n,t})|}{c_n} d\Gamma dt + \int_{\Sigma_B} \frac{|g(w_{n,t})|}{c_n} d\Gamma dt. \tag{3.58}$$

The first term in (3.58) goes to zero by virtue of (3.55), since

$$\int_{\Sigma_A} \frac{|g(w_{n,t})|^2}{c_n^2} d\Gamma dt \longrightarrow 0. \tag{3.59}$$

As for the second term, using hypothesis (H-1) and the definition of Σ_B, we write

$$\int_{\Sigma_B} \frac{|g(w_{n,t})|}{c_n} d\Gamma dt \le \int_{\Sigma_B} \frac{|g(w_{n,t}) w_{n,t}|}{c_n} d\Gamma dt \le \frac{1}{c_n} \int_{\Sigma_T} g(w_{n,t}) w_{n,t} d\Gamma dt \longrightarrow 0, \tag{3.60}$$

by (3.55) again. For the remaining terms, (3.55) implies

$$
\begin{aligned}
\tfrac{1}{c_n} f_1(\tfrac{\partial}{\partial\nu} w_{n,t}) &\longrightarrow 0 \ in \ L_2(\Sigma_T) \\
\tfrac{1}{c_n} f_2(\tfrac{\partial}{\partial\tau} w_{n,t}) &\longrightarrow 0 \ in \ L_2(\Sigma_T) \\
v_{n,t} &\longrightarrow 0 \ in \ L_2(Q_T).
\end{aligned} \tag{3.61}
$$

Thus, we obtain the limit system

$$
\begin{aligned}
&\Delta^2 v = [v, \chi(w)] && \text{in } Q_T \\
&v(0,\cdot) = v_0; \ v_t(0,\cdot) = v_1 && \text{in } \Omega \\
&\Delta v + (1-\mu) B_1 v = 0 && \text{on } \Sigma_T \\
&\tfrac{\partial}{\partial\nu} \Delta v + (1-\mu) B_2 v - \gamma^2 \tfrac{\partial}{\partial\nu} v_{tt} - v = 0 && \text{on } \Sigma_T.
\end{aligned} \tag{3.62}
$$

Again, by the result of [?], $v \equiv 0$, implying

$$
\int_0^T \|\chi(v_n)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(v_n) \longrightarrow 0, \tag{3.63}
$$

which contradicts (3.42). Thus, Proposition 3.5 holds in Case 2 and our proof is complete.

Step 2: Show $C(E_w(0))$ is uniformly bounded $\forall\gamma$. Assume not. Then there exists a sequence of functions, $\{w_\gamma(t)\} \in \mathcal{H}$ such that each $w_\gamma(t)$ satisfies the system

$$
\begin{aligned}
&w_{\gamma,tt} - \gamma^2 \Delta w_{\gamma,tt} + \Delta^2 w_\gamma + b w_{\gamma,t} = [w_\gamma, \chi(w_\gamma)] && \text{in } Q_T \\
&w_\gamma(0,\cdot) = w_{\gamma,0}; \ w_{\gamma,t}(0,\cdot) = w_{\gamma,1} && \text{in } \Omega \\
&\Delta w_\gamma + (1-\mu) B_1 w_\gamma = -f_1(\tfrac{\partial}{\partial\nu} w_{\gamma,t}) && \text{on } \Sigma_T \\
&\tfrac{\partial}{\partial\nu} \Delta w_\gamma + (1-\mu) B_2 w_\gamma - \gamma^2 \tfrac{\partial}{\partial\nu} w_{\gamma,tt} - w_\gamma = g(w_{\gamma,t}) - \tfrac{\partial}{\partial\tau} f_2(\tfrac{\partial}{\partial\tau} w_{\gamma,t}) && \text{on } \Sigma_T
\end{aligned} \tag{3.64}
$$

and such that

$$
\lim_{\gamma\to 0} \frac{\int_0^T \|\chi(w_\gamma)\|_{H^{3-\epsilon}(\Omega)} dt + l.o.(w_\gamma)}{\mathcal{P}(w_\gamma)} = \infty, \tag{3.65}
$$

where $\mathcal{P}(w_\gamma)$ is defined as in (3.43) and where the initial energy of (3.64) is uniformly bounded forall γ. By Lemma 2.1 and Lemma 2.7 of [?], the following convergence properties hold:

$$
\left.
\begin{aligned}
w_n &\xrightarrow{w} w \ in \ L_2([0,T]; H^2(\Omega)) \\
w_{n,t} &\xrightarrow{w} w_t \ in \ L_2([0,T]; L_2(\Omega))
\end{aligned}
\right\} \Longrightarrow
\left\{
\begin{aligned}
& l.o.(w_n) \longrightarrow l.o.(w) \\
& \textstyle\int_0^T \|\chi(w_n)\|_{H^{3-\epsilon}(\Omega)} dt \longrightarrow \int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)} dt.
\end{aligned}
\right. \tag{3.66}
$$

The remainder of the proof is identical to Case 1 and Case 2 in Step 1 (see also [?] where a similar argument was used in the context of a Kirchhoff plate). Hence, we obtain our desired result, (3.40). □

Completion of the Proof of Lemma 3.1: Since γ is proportional to the thickness of the plate and is therefore assumed to be small, without loss of generality, we may assume $\gamma^2 \leq M$ for some $M > 0$. Hence, by combining the results of Lemma 3.2 and Proposition 3.5, we obtain the desired result of Lemma 3.1, (3.1).

□

4 Final Estimates

Let the functions $h(x)$, $h_i(x)$, $i = 0, 1, 2$, and $\tilde{h}(x)$ be defined as in (1.10), (1.11), and (1.12), respectively.

$$h(x) \equiv h_0(x) + h_1(x) + h_2(x), \tag{4.1}$$

where $h_i(x)$ are concave, strictly increasing functions with $h_i(0) = 0$. Then $h(x)$ enjoys the same properties. Moreover, we assume that

$$\begin{aligned} h_0(sg(s)) &\geq s^2 + g^2(s) \quad |s| \leq 1 \\ h_i(sf_i(s)) &\geq s^2 + f_i^2(s) \quad |s| \leq 1 \quad i = 1, 2. \end{aligned} \tag{4.2}$$

By the hypotheses imposed on functions $h_i(x)$, we obtain

$$\int_{\Sigma_T} |f_1(\frac{\partial}{\partial \nu} w_t)|^2 d\Gamma dt = \int_{\Sigma_{A_1}} |f_1(\frac{\partial}{\partial \nu} w_t)|^2 d\Gamma dt + \int_{\Sigma_{B_1}} |f_1(\frac{\partial}{\partial \nu} w_t)|^2 d\Gamma dt, \tag{4.3}$$

where $\Sigma_{A_1} \equiv \{(t,x) \in \Sigma_T : |\frac{\partial}{\partial \nu} w_t| \leq 1\}$ and $\Sigma_{B_1} \equiv \Sigma_T \backslash \Sigma_{A_1}$. Hence, using hypothesis (H-1) on Σ_{B_1}, we find

$$\begin{aligned} \int_{\Sigma_T} |\tfrac{\partial}{\partial \nu} w_t|^2 d\Gamma dt \quad &+ \int_{\Sigma_T} |f_1(\tfrac{\partial}{\partial \nu} w_t)|^2 d\Gamma dt \\ &\leq \int_{\Sigma_{A_1}} [|\tfrac{\partial}{\partial \nu} w_t|^2 + |f_1(\tfrac{\partial}{\partial \nu} w_t)|^2] d\Gamma dt + (M + \tfrac{1}{m}) \int_{\Sigma_{B_1}} f_1(\tfrac{\partial}{\partial \nu} w_t) \tfrac{\partial}{\partial \nu} w_t d\Gamma dt \\ &\leq \int_{\Sigma_{A_1}} h_1(\tfrac{\partial}{\partial \nu} w_t f_1(\tfrac{\partial}{\partial \nu} w_t)) d\Gamma dt + (M + \tfrac{1}{m}) \int_{\Sigma_T} f_1(\tfrac{\partial}{\partial \nu} w_t) \tfrac{\partial}{\partial \nu} w_t d\Gamma dt. \end{aligned} \tag{4.4}$$

Similarly, the same argument applied to f_2 yields

$$\begin{aligned} \int_{\Sigma_T} |\tfrac{\partial}{\partial \tau} w_t|^2 d\Gamma dt \quad &+ \int_{\Sigma_T} |f_2(\tfrac{\partial}{\partial \tau} w_t)|^2 d\Gamma dt \\ &\leq \int_{\Sigma_{A_2}} h_2(\tfrac{\partial}{\partial \tau} w_t f_2(\tfrac{\partial}{\partial \tau} w_t)) d\Gamma dt + (M + \tfrac{1}{m}) \int_{\Sigma_T} f_2(\tfrac{\partial}{\partial \tau} w_t) \tfrac{\partial}{\partial \tau} w_t d\Gamma dt \end{aligned} \tag{4.5}$$

and, finally, for g, we have

$$\begin{aligned}\int_{\Sigma_A}|g(w_t)|^2 d\Gamma dt+\int_{\Sigma_T}|w_t|^2 d\Gamma dt &\le \int_{\Sigma_A}[|g(w_t)|^2+|w_t|^2 d\Gamma dt+\int_{\Sigma_B}|w_t|^2 d\Gamma dt\\ &\le \int_{\Sigma_A} h_0(w_t g(w_t))d\Gamma dt+\frac{1}{m}\int_{\Sigma_T} g(w_t)w_t d\Gamma dt.\end{aligned} \tag{4.6}$$

Define

$$\tilde{h}_i(x)\equiv h_i(\frac{x}{mes\ \Sigma_T}). \tag{4.7}$$

Then, by Jensen's inequality,

$$\begin{aligned}\int_{\Sigma_T}\{|w_t|^2 &+|\nabla w_t|^2+|f_1(\tfrac{\partial}{\partial\nu}w_t)|^2+|f_2(\tfrac{\partial}{\partial\tau}w_t)|^2+g(w_t)w_t\}d\Gamma dt+\int_{\Sigma_A}|g(w_t)|^2 d\Gamma dt\\ &\le C_1\int_{\Sigma_T}\{g(w_t)w_t+f_1(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t+f_2(\tfrac{\partial}{\partial\tau}w_t)\tfrac{\partial}{\partial\tau}w_t\}d\Gamma dt\\ &\quad +C_2\left[\tilde{h}_0(\int_{\Sigma_T} g(w_t)w_t d\Gamma dt)+\tilde{h}_1(\int_{\Sigma_T} f_1(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t d\Gamma dt)+\tilde{h}_2(\int_{\Sigma_T} f_2(\tfrac{\partial}{\partial\tau}w_t)\tfrac{\partial}{\partial\tau}w_t d\Gamma dt)\right]\\ &\le C_1\int_{\Sigma_T}\{g(w_t)w_t+f_1(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t+f_2(\tfrac{\partial}{\partial\tau}w_t)\tfrac{\partial}{\partial\tau}w_t\}d\Gamma dt\\ &\quad +C_2\sum_{i=0}^2\tilde{h}_i(\int_{\Sigma_T}\{g(w_t)w_t+f_1(\tfrac{\partial}{\partial\nu}w_t)\tfrac{\partial}{\partial\nu}w_t+f_2(\tfrac{\partial}{\partial\tau}w_t)\tfrac{\partial}{\partial\tau}w_t\}d\Gamma dt+\int_{Q_T} b(x)w_t^2 d\Omega dt),\end{aligned} \tag{4.8}$$

where the last inequality follows from the monotonicity of the functions $\tilde{h}_i$.

Denoting $\mathcal{F}\equiv\int_{\Sigma_T}\{g(w_t)w_t+f_1(\frac{\partial}{\partial\nu}w_t)\frac{\partial}{\partial\nu}w_t+f_2(\frac{\partial}{\partial\tau}w_t)\frac{\partial}{\partial\tau}w_t\}d\Gamma dt+\int_{Q_T} b(x)w_t^2 d\Omega dt$, we obtain from Lemma 3.1 and (4.8), using the monotonicity of $\tilde{h}$ once more to include $\int_{Q_T} b(x)w_t^2 d\Omega dt$,

$$\int_\alpha^{T-\alpha} E_w(t)dt - C_1E_w(0)\le C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}(\mathcal{F})]. \tag{4.9}$$

Since

$$\int_0^\alpha E_w(t)dt+\int_{T-\alpha}^T E_w(t)dt\le 2\alpha E_w(0), \tag{4.10}$$

we find

$$\int_0^T E_w(t)dt - C_{1,\alpha}E_w(0)\le C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}\mathcal{F}], \tag{4.11}$$

and by Lemma 2.1,

$$\begin{aligned}&\int_0^T E_w(t)dt\le C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}\mathcal{F}]+C_{1,\alpha}E_w(0)\\ &\Longrightarrow (T-C_{1,\alpha})E_w(T)\le C_{T,\alpha,\epsilon}(E_w(0))[\mathcal{F}+\tilde{h}(\mathcal{F})]\\ &\Longrightarrow E_w(T)\le C_T(E_w(0))[\mathcal{F}+\tilde{h}(\mathcal{F})].\end{aligned} \tag{4.12}$$

Hence, recalling (2.1),

$$(I+\tilde{h})^{-1}(\frac{E_w(T)}{C_T(E_w(0))}) \leq \mathcal{F} = E_w(0) - E_w(T). \tag{4.13}$$

Setting

$$p(s) \equiv (I+\tilde{h})^{-1}(\frac{s}{C_T(E_w(0))}), \tag{4.14}$$

we have proven the following proposition.

Proposition 4.1 *Let w be the solution to (1.1) and $E_w(t)$ be the corresponding energy at time t. If T is sufficiently large, then there exists a monotone increasing function, p, such that*

$$p(E_w(T)) + E_w(T) \leq E_w(0). \tag{4.15}$$

To arrive at the conclusion of Theorem 1.2, we need to apply the result of Lemma 3.3 in [?].

Lemma 4.1 ([?], Lemma 3.3) *Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing, we can define a function q such that $q(x) = x - (I+p)^{-1}(x)$. Notice that q is also an increasing function. Consider a sequence s_n of positive numbers which satisfy:*

$$s_{m+1} + p(s_{m+1}) \leq s_m. \tag{4.16}$$

Then $s_m \leq \mathcal{S}(m)$, where $\mathcal{S}(t)$ is a solution of a differential equation

$$\begin{cases} \frac{d}{dt}\mathcal{S}(t) + q(\mathcal{S}(t)) = 0 \\ \mathcal{S}(0) = s_0. \end{cases} \tag{4.17}$$

Moreover, if $p(x) > 0$ for $x > 0$, then $\lim_{t\to\infty} \mathcal{S}(t) = 0$.

Applying the result of Proposition 4.1, we obtain

$$E_w(m(T+1)) + p(E_w(m(T+1))) \leq E_w(mT), \tag{4.18}$$

for $m = 0, 1, ...$ Thus, applying Lemma 4.1 with

$$s_m \equiv E_w(mT), \tag{4.19}$$

yields

$$E_w(mT) \leq \mathcal{S}(m), \ \ m = 0, 1, ... \tag{4.20}$$

Setting $t = mT + \tau$, $0 \leq \tau < T$, and recalling the evolution property gives

$$E_w(t) \leq E_w(mT) \leq \mathcal{S}(m) \leq \mathcal{S}(\frac{t-\tau}{T}) \leq \mathcal{S}(\frac{t}{T} - 1) \ for \ t > T, \tag{4.21}$$

which completes the proof of Theorem 1.2. $\square$