$\mathcal{L}^{2,\mu}(Q)$-Estimates for Parabolic Equations and Applications

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Abstract

In this paper we derive $\mathcal{L}^{2,\mu}(Q_T)$-estimates for the first order derivatives of solutions to the following parabolic equation

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i u) + b_i u_{x_i} + cu = \frac{\partial}{\partial x_i} f_i + f_0,$$

where $\{a_{ij}(x,t)\}$ are assumed to be measurable and satisfied the ellipticity condition. The main idea is based on De Giorgi-Nash's estimates and Moser's iteration technique. These estimates are very useful in study of the regularity of solutions for some nonlinear problems. As a concrete example, we obtain the classical solvability for a strongly coupled parabolic system arising from the thermistor problem.

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1. Introduction

Let Ω be a bounded domain in R^n with boundary $S = \partial\Omega$ in C^1 and $Q_T = \Omega \times (0, T]$ with $T > 0$. Consider the following parabolic equation:

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i u) + b_i u_{x_i} + cu = \frac{\partial}{\partial x_i} f_i + f_0, \tag{1.1}$$

where a_{ij} satisfies the ellipticity condition:

$$a_0|\xi|^2 \le a_{ij}\xi_i\xi_j \le A_0|\xi|^2, \text{ for } \xi \in R^n \qquad (0 < a_0 \le A_0).$$

It is well known that the De Giorgi-Nash type estimate plays an essential role in the study of solvability for nonlinear parabolic equations. However, this estimate is often not enough in dealing with regularity of solutions. On the other hand, the $\mathcal{L}^{2,\mu}$-theory is powerful for investigating regularity of solutions for elliptic equations and systems (cf. [8]). In the present work we would like to derive the $\mathcal{L}^{2,\mu}(Q)$-estimates for the derivative of weak solutions of equation (1.1), where $\mu = n + \delta$ with $\delta \in (0, 1)$. It will been seen the results are very useful in applications. The core of the proof is based on accurate De Giorgi-Nash's estimates. For elliptic equations, the theory can be found in [10]. Much of the work is a modification of the proofs for elliptic equations.

For convenience we introduce some standard notations: A point (x, t) in Q_T will be denoted by z. The distance between two points $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$ is equal to

$$max\{|x_1 - x_2|, |t_1 - t_2|^{\frac{1}{2}}\}.$$

For $r > 0$,

$$B_r(x_0) = \{x \in R^n : |x - x_0| < r\} \text{ and } Q_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0].$$

For a measurable set $A \subset R^n \times [0, T]$ with a finite measure $|A| < \infty$,

$$\oint_A udz = \frac{1}{|A|}\int_A udz.$$

In particular, when $A = Q_r(z_0)$,

$$u_{z_0,r} = \oint_{Q_r(z_0)} udz.$$

For $\mu > 0$, let

$$[u]_{2,\mu,Q} = \sup_{z_0 \in Q,\ r>0} r^{-\mu} \int_{Q_r(z_0)} |u - u_{z_0,r}|^2 dz.$$

The space $\mathcal{L}^{2,\mu}(Q)$ consists of all functions in $L^2(Q)$ such that

$$[u]_{2,\mu,Q} < \infty.$$

$\mathcal{L}^{2,\mu}(Q)$ is a Banach space with the norm

$$||u||_{2,\mu,Q} = \{||u||^2_{L^2(Q)} + [u]^2_{2,\mu,Q}\}^{\frac{1}{2}}.$$

We need the following proposition:

Proposition A: The space $\mathcal{L}^{2,n+2+2\mu}(Q_T)$ and $C^{\mu,\frac{\mu}{2}}(\bar{Q}_T)$, where $\mu \in (0,1)$, are topologically and algebraically isomorphic.

If one replaces Q_T by a subset $Q_r(z_0)$, then $u(z)$ is Hölder continuous in a neighborhood of z_0. The proof of this proposition can be found in [5].

2. $\mathcal{L}^{2,\mu}(Q_T)$-theory for Parabolic Equations

We begin with the interior estimates. Let $w(x,t)$ be a weak solution of the following parabolic equation:

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j}) = 0, \tag{2.1}$$

that is

$$\int\int_{Q_T} [-wv_t + a_{ij}w_{x_j}v_{x_i}]dxdt = 0, \tag{2.2}$$

for any $v(x,t) \in H^1(0,T;H^1_0(\Omega))$ with $v(x,0) = v(x,T) = 0$.

In what follows, a constant which depends only on $||a_{ij}||_{L^\infty}$ and a_0 will be denoted by C. It may be different from one line to the next. The following two lemmas are fundamental in order to prove Hölder continuity. Their proofs can be found in [7] (Lemma 1 and Lemma 3).

Lemma 2.1: Let $w(x,t)$ be a non-negative subsolution of (2.1) and $z_0 \in Q_T$, $Q_{2r}(z_0) \subset Q_T$ $(r > 0)$, then

$$ess \sup_{Q_r(z_0)} w(x,t) \leq C\rho^{-(\frac{n}{2}+1)}||w||_{2,Q_{r+\rho}(z_0)},$$

where $0 < \rho \leq r$ and $||\cdot||_{2,A}$ represents the norm of $L^2(A)$.

Lemma 2.2 : Let $w(x,t)$ be a non-negative weak solution of (2.1) and $Q_r(z_0) \subset Q_T$. If $E = \{z \in Q_r(z_0):\ w(x,t) \geq 1\}$ has measure $\geq K|Q_r(z_0)|$, where $K \in (0,1)$ is a constant, then

$$ess \inf_{Q_{r/2}(z_0)} w(x,t) \geq C(K) > 0,$$

where the constant $C(K)$ depends on K and $||a_{ij}||_{L^\infty(Q_T)}$ and a_0, but not on z_0 nor r.

With the above results on hand, one can derive

Lemma 2.3: If $w(x,t)$ is a weak solution, then there exists a constant $\delta_0 \in (0,1)$ such that

$$\max_{\bar{Q}_\rho(z_0)} w - \min_{\bar{Q}_\rho(z_0)} w \leq C r^{-(\frac{n}{2}+1)} (\frac{\rho}{r})^{\delta_0} ||w||_{2,Q_{2r}(z_0)},$$

where $0 < \rho \leq r$ and $Q_{2r}(z_0) \subset Q_T$ while δ_0 has the same dependence as C.

The proof is almost identical to the case for elliptic equations, see [10] Theorem 2.14 on page 115 (also [2], Theorem 4 with $k = 0$).

Lemma 2.4: Let $w(x,t)$ be a weak solution of (2.1). Then for any $\rho \in [0,r]$,

$$||\nabla w||^2_{2,Q_\rho(z_0)} \leq C(\frac{\rho}{r})^{\mu_0} ||\nabla w||^2_{2,Q_r(z_0)},$$

where $\mu_0 = n + 2\delta_0$.

Proof: First of all, Poincare's inequality implies

$$||w||^2_{2,Q_r(z_0)} \leq C r^2 ||\nabla w||^2_{2,Q_r(z_0)}.$$

We may assume that z_0 is the origin. By Lemma 2.3, for $z = (x,t) \in Q_{\frac{r}{2}}(0)$,

$$\begin{aligned} |w(z) - w(0)|^2 &\leq C r^{-n-2-2\delta_0} |x|^{2\delta_0} ||w||^2_{2,Q_r(0)} \\ &\leq C r^{-n-2\delta_0} |x|^{2\delta_0} ||\nabla w||^2_{2,Q_r(0)}. \end{aligned} \tag{2.3}$$

Let $0 < \rho < \frac{r}{4}$. Introduce a cutoff function $g(x,t)$ as follows:

$$g(x,t) \in C^{1,1}(\bar{Q}_r) \text{ and satisfies:}$$

$$0 \leq g(x,t) \leq 1, \ supp\ g \subset Q_{2\rho};\ g(x,t) = 1 \text{ on } \bar{Q}_\rho.$$

Moreover,

$$|\nabla g| \leq \frac{2}{\rho},\ |g_t| \leq \frac{4}{\rho^2}.$$

Let $v(x,t) = g^2(x,t)[w(z) - w(0)]$. We can use $v(x,t)$ as a test function in the integral identity (2.2), although w_t is not necessary in $L^2(Q_r)$. Indeed, otherwise, we can always use the Steklov averaging

$$v_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,\tau)d\tau$$

to approximate $v(x,t)$ and then take the limit. Now

$$\begin{aligned}
&\int\int_{Q_r} wv_t\,dxdt = \int\int_{Q_r}[w(z)-w(0)]v_t\,dxdt + \int\int_{Q_r} w(0)v_t\,dxdt \\
&= \int\int_{Q_r}[w(z)-w(0)]^2 gg_t\,dxdt + \frac{1}{2}\int_{B_r} g^2[w(z)-w(0)]^2dx + \int_{B_r} w(0)g^2[w(z)-w(0)]dx \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

By (2.3) and the construction of g, one has

$$\begin{aligned}
|I_1| &\le Cr^{-n-2\delta_0}\rho^{2\delta_0}\rho^{-2}||\nabla w||^2_{L^2(Q_r)}|Q_{2\rho}| \\
&\le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)},
\end{aligned}$$

where $\mu_0 = n + 2\delta_0$.

Similarly,

$$|I_2| \le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)}.$$

To estimate I_3, we use Lemma 2.1 and (2.3) to obtain

$$\begin{aligned}
|I_3| &\le ||w(0)||_{L^\infty(B_\rho(0))}\cdot||w-w(0)||_{L^\infty(B_{2\rho}(0))}\cdot C\rho^n \\
&\le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)}.
\end{aligned}$$

It follows that

$$|\int\int_{Q_r} wv_t\,dxdt| \le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)}. \tag{2.4}$$

On the other hand,

$$\begin{aligned}
&\int\int_{Q_r} a_{ij}w_{x_i}v_{x_j}\,dxdt \\
&= \int\int_{Q_r} a_{ij}w_{x_i}\{g^2 w_{x_j} + 2gg_{x_j}[w-w(0)]\}dxdt \\
&\ge \frac{a_0}{2}\int\int_{Q_r} g^2|\nabla w|^2dxdt - C\max_{Q_{2\rho}}|w-w(0)|^2\int\int_{Q_r}|\nabla g|^2dxdt
\end{aligned}$$

Combining the above inequality and (2.3)-(2.4), we have

$$\int\int_{Q_\rho}|\nabla w|^2dxdt \le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)}.$$

When $\frac{r}{4} < \rho < r$, clearly we always have

$$\int\int_{Q_\rho}|\nabla w|^2dxdt \le 4^{\mu_0}(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)}.$$

This completes the proof. Q.E.D.

Lemma 2.5 Let $\Omega = B_r(0)$ and the functions $f_0(x,t) \in L^2(Q_T)$ and $f_i(x,t) \in L^{2,\mu}(Q_T)$ $(i = 1, 2, \cdots, n)$. Let $u(x,t)$ be a weak solution of the following equation:

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j}) = \frac{\partial}{\partial x_i} f_i + f_0,$$

then

$$||\nabla u||^2_{2,Q_\rho} \le C[(\frac{\rho}{r})^{\mu_0}||\nabla u||^2_{2,Q_r} + r^2||f_0||^2_{2,Q_r} + \sum_{i=1}^n ||f_i||^2_{2,Q_r}].$$

where $\mu_0 = n + 2\delta_0$.

Proof: Let $h(x,t) \in L^2(0,T; H^1_0(B_r(0))$ solve the equation

$$h_t - (a_{ij}h_{x_i})_{x_j} = (f_i)_{x_i} + f_0$$

in the weak sense with $h(x,0) = 0$. Then it is easy to see that the following energy inequality holds:

$$\int_{B_r} h^2 dx + \int\int_{Q_r} h_x^2 dx dt \le ||f_0||_{L^2(Q_r)}||h||_{L^2(Q_r)} + \sum_{i=1}^n ||f_i||_{L^2(Q_r)}||h_{x_i}||_{L^2(Q_r)}.$$

By ε-Cauchy's inequality as well as Poincare's inequality, we obtain

$$\int_{B_r} h^2 dx + \int\int_{Q_r} h_x^2 dx dt \le Cr^2||f_0||^2_{L^2(Q_r)} + \sum_{i=1}^n ||f_i||^2_{L^2(Q_r)}.$$

Now let $w(x,t) = u(x,t) - h(x,t)$ on Q_r. Then $w(x,t)$ satisfies

$$\int\int_{Q_r} [-wv_t + a_{ij}w_{x_i}v_{x_j}]dxdt = 0$$

for any $v(x,t) \in H^1(0,T; H^1_0(\Omega))$ with $v(x,T) = v(x,0) = 0$. Lemma 2.4 implies

$$||\nabla w||^2_{L^2(Q_\rho)} \le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2(Q_r)}.$$

Hence,

$$\begin{aligned} ||\nabla u||^2_{L^2(Q_\rho)} &\le 2[||\nabla w||^2_{L^2(Q_\rho)} + ||\nabla h||^2_{L^2(Q_\rho)}] \\ &\le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2 Q_r} + 2||\nabla h||^2_{L^2(Q_\rho)} \\ &\le C(\frac{\rho}{r})^{\mu_0}||\nabla w||^2_{L^2 Q_r} + C[r^2||f_0||^2_{L^2(Q_r)} + \sum_{i=1}^n ||f_i||^2_{L^2(Q_r)}]. \end{aligned}$$

Q.E.D.

Now we can show

Theorem 1: Let $u(x,t)$ be a weak solution of Equation (2.1), then for any $0 < \mu < \mu_0$,

$$||\nabla u||^2_{2,\mu,Q_T} \le C[||f_0||^2_{2,(\mu-2)^+,Q_T} + \sum_{i=1}^{n} ||f_i||^2_{2,\mu,Q_T} + ||u||^2_{L^2(0,T;(Q_T))}],$$

in particular, if $\mu > n$, then

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(Q_T), \text{ where } \alpha = \frac{\mu - n}{2},$$

where C depends only on $a_0, A_0, ||b_i||_{L^\infty}, ||a_i||_{L^\infty}, ||c||_{L^\infty}$ and Ω.

Proof: The proof is similar to that for elliptic equations. Indeed, we write the integral equality into the following form:

$$\int\int_{Q_r(z_0)} [-uv_t + a_{ij}u_{x_i}u_{x_j}]dxdt = \int\int_{Q_r(z_0)} [f_0^* v - f_i^* v_{x_i}]dxdt$$

for any $v(x,t) \in H^1(0,T;H_0^1(B_r(z_0))$ with $v(x,T) = v(x,0) = 0$. where

$$f_0^* = f_0 - b_i u_{x_i} - cu;$$
$$f_i^* = f_i - a_i u.$$

Using Lemma 2.5, we have

$$||\nabla u||^2_{L^2(Q_\rho)} \le C(\frac{\rho}{r})^{\mu_0} ||\nabla u||^2_{2,Q_r} + r^2 ||f_0^*||^2_{2,Q_r} + \sum_{i=1}^{n} ||f_i^*||^2_{2,Q_r}].$$

Note that

$$r^2 ||f_0^*||^2_{2,r} \le C[r^\mu ||f_0||^2_{2,(\mu-2)^+,Q_r} + r^{\mu+2}||u||^2_{2,\mu,Q_r} + r^2 ||\nabla u||^2_{2,Q_r(z_0)}].$$

Furthermore,

$$\sum_{i=1}^{n} ||f_i^*||^2_{2,Q_r(z_0)} \le Cr^\mu [\sum_{i=0}^{n} ||f_i||^2_{2,\mu,Q_r(z_0)} + ||u||^2_{2,\mu,Q_r(z_0)}].$$

Assume that exists a number μ such that

$$||u||^2_{2,\mu,Q_r} \le CS_\mu, \tag{2.5}$$

where

$$S_\mu = ||f_0||^2_{2,(\mu-2)^+,Q_T} + \sum_{i=1}^{n} ||f_i||^2_{2,\mu,Q_T} + ||u||^2_{L^2(0,T;H^1(\Omega))}.$$

The existence of such a μ is obvious since

$$\mathcal{L}^{2,2}(Q_T) \hookrightarrow L^2(0,T;H^1(\Omega)).$$

Hence we have

$$||\nabla u||^2_{2,Q_\rho(z_0)} \le C[(\frac{\rho}{r})^{\mu_0} + r^2)||\nabla u||^2_{2,Q_r(z_0)} + r^\mu S_\mu].$$

Applying the iteration lemma 1.18 in [10], we have the estimate

$$||\nabla u||^2_{2,\mu,Q_\rho(z_0)} \le CS_\mu,$$

where $2 \le \mu < \mu_0$.

The imbedding lemma in [5] yields $u \in \mathcal{L}^{2,\mu+2}(Q_\rho(z_0))$ and the estimate (2.5) holds for $\mu + 2$. By repeating the above process, we have the desired the estimate for any μ which satisfies $0 < \mu < \mu_0$. Q.E.D.

Now we are going to derive the global $\mathcal{L}^{2,\mu}$-estimates. Let $S = \partial\Omega$ be of class C^1.

Theorem 2: Let $u(x,t) \in L^2(0,T;H^1(\Omega))$ be a weak solution of the equation (2.1) subject to the following initial and boundary conditions:

$$u(x,t) = g(x,t), \qquad on\ S_T,$$
$$u(x,0) = u_0(x), \qquad on\ \Omega.$$

Moreover, assume there exists a function $\psi(x,t)$ such that

$$\psi(x,t) = g(x,t) \text{ on } S_T, \text{ and } \psi(x,0) = u_0(x) \text{ on } \Omega.$$

If $F(x,t) = (f_1, \cdots, f_n; f_0)$ and $D\psi(x,t) = (\psi_t, \nabla\psi) \in \mathcal{L}^{2,\mu}(Q_T)^{n+1}$ then for any $0 < \mu < \mu_0$,

$$\begin{aligned}||\nabla u||_{2,\mu,Q_T} \le \quad & C[\sum_{i=1}^n [||f_i||_{2,\mu,Q_T} + ||\psi_{x_i}||_{2,\mu,Q_T} + ||f_0||_{2,(\mu-2)^+,Q_T}] \\ & +C[||\psi_t||_{2,(\mu-2)^+,Q_T} + ||u||_{L^2(0,T;H^1(\Omega))}].\end{aligned}$$

In particular, if $n < \mu < \mu_0$,

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T),$$

where $\alpha = \frac{\mu-n}{2} \in (0, \delta_0)$.

Proof: Let $w(x,t) = u(x,t) - \psi(x,t)$ on Q_T. Then $w(x,t)$ solves equation (2.1) weakly, where $f_i, (1 \le i \le n)$ and f_0 are replaced, respectively, by

$$f_i^* = f_i - a_{ij}\psi_{x_j};$$

and

$$f_0^* = f_0 + \psi_t - b_i\psi_{x_i} - c\psi.$$

Hence we need only to show the result for homogeneous initial and boundary data. In this case, the test function will not be required to be zero on the parabolic boundary, the integral equality (2.2) still holds. As $S = \partial\Omega$ is of class C^1, we know that Lemma 2.3 holds if Q_{2r} is replaced by $Q_{2r} \bigcap Q_T$ (see [11]). All of the rest can be carried over if one uses $Q_r \bigcap Q_T$ to replace Q_r. We omit the details here.

Remark: If the boundary condition on S_T is replaced by the following:

$$[a_{ij}u_{x_j} + a_i u]cos(\vec{n}, x_i) = g(x,t),$$

where $\vec{n}$ is the outward normal on S, the result of Theorem 2 still holds.

Indeed, we may assume $g(x,t) = 0$ (otherwise, we choose a function $G(x,t)$ such that

$$[a_{ij}G_{x_j} + a_i G]cos(\vec{n}, x_i) = g(x,t)$$

and set $v(x,t) = u(x,t) - G(x,t)$). As $S \in C^1$, in a neighborhood Q_r of $z_0 \in S_T$, we can introduce a transformation to flat the lateral boundary and then extend all of the coefficients as well as the inhomogeneous terms in (1.1) into Q_r^* (the image of Q_T) by a simple reflection. Then the desired result follows from the interior estimate.

3. Applications

To illustrate some applications of the preceding theory, we consider the following strongly coupled parabolic system arising from the study of a thermistor (cf. [1], [3]):

$$\psi_t - \nabla[\sigma(u)\nabla\psi] = 0, \qquad (x,t) \in Q_T, \tag{3.1}$$

$$u_t - \nabla[k(u)\nabla u] = \sigma(u)|\nabla\psi|^2, \quad (x,t) \in Q_T, \tag{3.2}$$

$$\psi(x,t) = g(x,t),\ u(x,t) = f(x,t), \qquad on\ S_T, \tag{3.3}$$

$$\psi(x,0) = \psi_0(x),\ u(x,0) = u_0(x). \qquad on\ \Omega, \tag{3.4}$$

where $0 < \sigma_0 \leq \sigma(u) \leq \sigma_1$ and $0 < k_0 \leq k(u) \leq k_1$.

The above system can also be used as a model for an incompressible, unidirectional flow with temperature-dependent viscosity (cf. [12]). There are two major difficulties for the system. The first one is that the system is coupled in the coefficient of the leading term. The second is that the growth order with respect to the gradient of the solution is critical. Therefore, the general regularity theory is not applicable. We start with the following definition of a weak solution.

Definition: A pair of functions $(u(x,t),\ \psi(x,t))$ defined on Q_T is said to be a weak solution to the problem (3.1)-(3.4), if

$$\psi(x,t)-g(x,t)\in L^2(0,T;H_0^1(\Omega));\ u(x,t)-f(x,t)\in L^2(0,T;H_0^1(\Omega)),$$

and (u,ψ) satisfy

$$\int\int_{Q_T}[-\psi w_t+\sigma(u)\nabla\psi\dot{\nabla}w]dxdt=\int_\Omega w(x,0)\psi_0(x)dx, \tag{3.5}$$

$$\int\int_{Q_T}[-uw_t+k(u)\nabla u\dot{\nabla}w]dxdt=\int\int_{Q_T}\sigma(u)|\nabla\psi|^2dxdt\int_\Omega w(x,0)u_0(x)dx, \tag{3.6}$$

where $w(x,t)\in H^1(0,T;H_0^1(\Omega))$ is arbitrary with $w(x,T)=0$.

It has been shown in [12] that under suitable assumption on the known data, the problem (3.1)-(3.4) has at least one weak solution. By applying the $\mathcal{L}^{2,\mu}$-estimate, we can show that a weak solution is also classical.

Theorem 3: There exists a $\alpha\in(0,1)$ such that

$$u\in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T),\qquad \psi\in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T).$$

Moreover, if $u_0(x),\psi_0(x)\in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ and $g(x,t),\ f(x,t)\in C^{2+\alpha,1+\frac{\alpha}{2}}(S_T)$, and satisfy the consistency conditions on $S\times\{t=0\}$:

$$u_0(x)=f(x,0),\qquad \psi_0(x)=g(x,0);$$

$$f_t(x,0)-\nabla[k(u_0)\nabla u_0]=\sigma(u_0)|\nabla\psi_0|^2;$$

$$g_t(x,0)=\nabla[\sigma(u_0)\nabla\psi_0],$$

then, $u(x,t)\in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$ and $\psi(x,t)\in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$.

Proof: To show the local regularity, we first note that for any region Q with $dist(Q,S_T)>0$, by Lemma 2.4

$$\nabla\psi(x,t)\in\mathcal{L}^{2,\mu}(Q),$$

where $\mu=n+\delta$ for some $\delta\in(0,1)$.

Let $U(x,t)=u+\frac{1}{2}\psi^2$. Now we rewrite the equation (3.2) as follows:

$$U_t-\triangle U=\nabla[(\sigma(u)-1)\psi\nabla\psi].$$

By the maximum principle, we know that ψ is uniformly bounded. Hence

$$f_i(x,t)=(\sigma(u)-1)\psi(x,t)_{x_i}\in\mathcal{L}^{2,\mu}(Q),$$

for $i = 1, 2, \cdots, n$. It follows by Theorem 1 that

$$\nabla u \in \mathcal{L}^{2,\mu}(Q).$$

By Proposition A,

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(Q_T).$$

As $u \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$, we apply the result obtained in [6] to have

$$\psi(x,t) \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T).$$

Thus, by $W_p^{2,1}(Q)$-estimate (cf. [4]), we have

$$u(x,t) \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T).$$

Finally, applying the Schauder theory, we obtain the desired regularity. By the same procedure, we can apply the global $\mathcal{L}^{2,\mu}$-estimates to prove

$$u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T);\ \psi(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T).$$

As an immediate consequence, we have

Corollary: The solution of (3.1)-(3.4) is unique.

Remark: In practice, the electric potential $\psi(x,t)$ is often assumed to be time-independent (cf. [1], [3]). In this case the equation (3.1) is reduced into

$$-\nabla[\sigma(u)\nabla\psi] = 0.$$

Using the $\mathcal{L}^{2,\mu}$-estimates for elliptic equations (cf. [10]) and Theorem 1, we have the same regularity. This result was proved for $n = 2$ in [1].

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