SYMMETRIES OF DIFFERENTIAL SYSTEMS*

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1 Introduction.

In the classical theory of dynamical systems, symmetry has always played a central role. However it is only in the last 15 years it has also emerged as an issue in systems and control theory. Using the setting established in the behavioral approach to systems [9], [10], [11], a systematic analysis of symmetries for linear systems has been undertaken by the authors in [1], [2], [3] where the reader is also referred for a list of earlier system theoretic references on the subject.

In this paper we will focus on finite group symmetries of linear differential systems present and a rather complete theory on the subject. Together with new results, we also present, in a new light, a unified picture which encompasses many of the results in [2] and [3].

In section 1 we briefly recall some basic facts on the theory of linear differential behaviors which are defined as kernels in $C^\infty(\mathbb{R}, W)$ of linear constant coefficients ordinary differential operators, where W is a k-vector space. In section 2 we will introduce our definition of symmetry as a representation of a finite group G on the infinite-dimensional space $C^\infty(\mathbb{R}, W)$, which preserves the class $\mathbb{D}[W]$ of linear differential behaviors. Our main goal is to study the structure and the representation of behaviors in $\mathbb{D}[W]$ which are symmetric, namely those which are fixed by the action of G. In section 4 we prove a result (Theorem 1) on the classification of the symmetries on $\mathbb{D}[W]$. This result is then used in section 5 to show (Theorem 3) how the action of G on $C^\infty(\mathbb{R}, W)$ can be lifted to a dual action on the free $k[z]$-module $W^*[z]$. These can be thought as the set of the differential operators from $C^\infty(\mathbb{R}, W)$ to $C^\infty(\mathbb{R}, \mathbb{R})$. This permits to shift the analysis to a pure algebraic level and in section 6 we prove Theorems 4 and 5 which exploit the structure of finite group actions on free $k[z]$-modules. Finally, in section 7 the algebraic results are used to establish canonical differential representations of symmetric systems. We close with a number of examples involving static symmetrics and time-reversibility

The important interplay between symmetries and control problems will be studied in a later paper.

It is a pleasure to dedicate this paper to Larry Markus on his 70-th birthday. His remarkable style of exposition and his talent for combining advanced mathematical ideas

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with engineering relevance which I (JCW) was privileged to experience as a beginning researcher had a great influence on my later scientific work.

2 Differential behaviors. Preliminary facts.

Let k be equal $\mathbb{R}$ or $\mathbb{C}$ and denote by k^* the multiplicative group $k \setminus \{0\}$. Throughout this paper W and E will always denote finite dimensional vector spaces over k. Denote by $C^\infty_W = C^\infty(\mathbb{R}, W)$ the k-vector space of infinitely differentiable functions from $\mathbb{R}$ to W equipped with the canonical Frechet topology of uniform convergence on compact subsets of $\mathbb{R}$. Denote $R = k[z]$ and $E[z] = R \otimes_k E$ the R-module of polynomials with coefficients in E.

If $D \in \mathrm{Hom_k(W, E)[z]}$ we can define the differential operator

$$D\left(\frac{d}{dt}\right) : C^\infty_W \to C^\infty_E$$

as follows: if $D = \sum_{i=0}^n D_i z^i$ where $D_i \in \mathrm{Hom(W, E)}$ and $w \in C^\infty_W$ then,

$$\left(D\left(\frac{d}{dt}\right) w\right)(t) := \sum_{i=0}^{n} D_i \left(\frac{d^i w}{dt^i}(t)\right)$$

Following [10] we define a *differential behavior* over W (called the *signal space*), a subspace $\mathcal{B}$ of C^∞_W which is the kernel of a differential operator $D\left(\frac{d}{dt}\right)$. D is called a *polynomial matrix representation* of $\mathcal{B}$ and E is called the *equation space* of D. The class of differential behaviors over W is denoted by $\mathbb{D}[W]$.

Let $\mathcal{B} \in \mathbb{D}[W]$. Consider the annihilators of $\mathcal{B}$, defined by

$$\mathcal{B}^\perp := \left\{ p \in W^*[z] \mid p\left(\frac{d}{dt}\right) w = 0 \ \ \forall w \in \mathcal{B} \right\}$$

where $W^* := \mathrm{Hom_k(W,}\ k)$. Clearly $\mathcal{B}^\perp$ is an R-submodule of the R-free module $W^*[z]$, hence it is finitely generated and free. On the other hand, if M is an R-submodule of $W^*[z]$, one can consider

$${}^\perp M := \left\{ f \in C^\infty_W \mid p\left(\frac{d}{dt}\right) f = 0 \ \ \forall p \in M \right\}$$

It is a standard fact [6], [10] that

$${}^\perp(\mathcal{B}^\perp) = \mathcal{B} \quad , \quad ({}^\perp M)^\perp = M$$

for all $\mathcal{B} \in \mathbb{D}[W]$ and for all submodules M of $W^*[z]$. This yields a bijection between $\mathbb{D}[W]$ and $\mathbb{S}[W^*[z]]$, the set of all submodules of $W^*[z]$. For $\mathcal{B} \in \mathbb{D}[W]$, denote by $p(\mathcal{B})$ the rank of the free module $\mathcal{B}^{\perp}$. Clearly $p(\mathcal{B}) \leq \dim_k W$ and we can find $D \in \mathrm{Hom(W, k^{p(\mathcal{B})})[z]}$ such that $\ker D\left(\frac{d}{dt}\right) = \mathcal{B}$. This simply shows that $\mathcal{B}$ can be described by $p(\mathcal{B})$ differential equations and no less. Any polynomial matrix representation of $\mathcal{B}$ with equation space of dimension $p(\mathcal{B})$ will therefore be called *minimal*.

If M is a submodule of $E[z]$, consider

$$M_i := \{m \in M \mid \deg(m) \leq i\} \tag{1}$$

where $\deg(m)$ denotes the degree of the polynomial $m \in E[z]$. Clearly the M_i's are k-vector spaces and there holds

$$M_i + zM_i \subseteq M_{i+1} \tag{2}$$

It is a standard fact (see [9] for more details) that equality holds in (2) except for at most finitely many i's. Denote by h the largest of them. We can then construct an R-basis of M in the following inductive way: let $B_0 = \{r_1^0, \ldots, r_{\gamma_0}^0\}$ be a k-basis of M_0 and let $B_i = \{r_1^i, \ldots, r_{\gamma_i}^i\}$ be a k-basis of a complementary subspace of $M_{i-1} + zM_{i-1}$ inside M_i. It can be proven that $\cup_i B_i$ is an R-basis for M. It will be called a *canonical basis*. For $\mathcal{B} \in \mathbb{D}[W]$, let $\{r_j^i \mid j = 1, \ldots, \gamma_i, i = 1, \ldots, h\}$ be such a canonical basis for $\mathcal{B}^{\perp}$. Consider

$$D(z) = \begin{bmatrix} r_1^0(z) \\ \vdots \\ r_{\gamma_h}^h(z) \end{bmatrix} \in \mathrm{Hom[W, k^h][z]}$$

D is evidently a minimal representation of $\mathcal{B}$ and it will be called *canonical*. Define $n(\mathcal{B}) = \sum_{i=0}^{h} i\gamma_i$. These two integers $p(\mathcal{B})$ and $n(\mathcal{B})$ each have an important system theoretic interpretation: $p(\mathcal{B})$ is the number of output variables in any input/output representation of $\mathcal{B}$, while $n(\mathcal{B})$ is the Mc-Millan degree of $\mathcal{B}$, namely the dimension of the state space in any minimal state space representation of $\mathcal{B}$. See [9], [10], and [11] for precise statements and details.

The choice of working in C^{∞} is mostly done for the matter of simplicity. More general settings can indeed be chosen, see [11] and [1]. In particular, all the results we present in this paper are still true if we replace C^{∞} with the space of distributions $\mathcal{D}'$.

3 Symmetries of differential behaviors.

Denote by $GL_k(C_W^{\infty})$ the group of all the topological vector space isomorphism of C_W^{∞}. Let G be a group and let $T : G \to GL_k(C_W^{\infty})$ be a representation of G. Clearly, T induces an action of G on the class of all (closed) subspaces of C_W^{∞}:

$$g \cdot \mathcal{B} := \{T_g w \mid w \in \mathcal{B}\}$$

We will say that (G,T) is a *symmetry on* $\mathbb{D}[W]$ if the set $\mathbb{D}[W]$ is invariant by the action of G. A behavior $\mathcal{B} \in \mathbb{D}[W]$ is said to be *symmetric* if it is fixed by the action of G (i.e., $T_g w \in \mathcal{B}$ for every $w \in \mathcal{B}$ and $g \in G$). The subset of symmetric behaviors in $\mathbb{D}[W]$ will be denoted by $\mathbb{D}[W]^G$.

Because of the bijective correspondence between $\mathbb{D}[W]$ and $\mathbb{S}[W^*[z]]$, G also acts in a natural way on $\mathbb{S}[W^*[z]]$: if $M \in \mathbb{S}[W^*[z]]$ and $g \in G$, define

$$g \cdot M := (g(^{\perp}M))^{\perp}$$

Denote by $\mathbb{S}[W^*[z]]^G$ the subset of the submodules in $\mathbb{S}[W^*[z]]$ which are fixed by this action of G. It follows that $\mathbb{D}[W]^G$ and $\mathbb{S}[W^*[z]]^G$ are also in bijective correspondence through the bijection $\mathcal{B} \leftrightarrow \mathcal{B}^{\perp}$. It is not a priori evident that the action of G on $\mathbb{S}[W^*[z]]$ is induced by some action of G on $W^*[z]$. This is in fact true and will be proven in this section. First let us present some examples.

A symmetry (G,T) on $\mathbb{D}[W]$ is called *static* if there exists a representation ρ of G on W such that $(T_g w)(t) = \rho_g(w(t))$ for all $t \in \mathbb{R}$ and $g \in G$. In the sequel we will identify T and ρ. A typical example of a static symmetry is the following: let $W = \mathbb{R}^q$, $G = S_q$ the permutation group on q elements and T the permutation representation on $\mathbb{R}^q$ defined by

$$T_\sigma \, {}^t(w_1, \ldots, w_q) := \, {}^t(w_{\sigma(1)}, \ldots, w_{\sigma(q)})$$

where $w_i \in \mathbb{R}$ and where ^{t} denotes transpose. $\mathcal{B} \in \mathbb{D}[\mathbb{R}^q]$ is symmetric if and only if

$${}^t(w_1, \ldots, w_q) \in \mathcal{B} \Rightarrow \, {}^t(w_{\sigma(1)}, \ldots, w_{\sigma(q)}) \in \mathcal{B} \quad \forall \sigma \in S_q$$

where $w_i \in C^\infty$. We can think of this static symmetry as occurring when we model the position of q identical particles in $\mathbb{R}^q$.

If (G,T) is a static symmetry on $\mathbb{D}[W]$, it is easy to see that the associated action of G on $\mathbb{S}[W^*[z]]$ is given by

$$g \cdot M = \{T_g^* \cdot m \mid m \in M\}$$

where T^* is the dual representation of T and where if $m = \sum m_i z^i$, $T_g^* m := \sum (T_g m_i) z^i$ Thus, in this case, the action of G on $\mathbb{S}[W^*[z]]$ is induced by the R-linear action of G on $W^*[z]$ given by T^*. Static symmetries have been studied in much detail in [1] and [3].

An important example of non-static symmetry is *time-reversibility*. In this case $G = \mathbb{Z}_2 = (\{-1,+1\}, \cdot)$, $(T_{-1}w)(t) = w(-t)$. Differential behaviors which are symmetric with respect to this symmetry are called *time-reversible*. In this case the G-action on $W^*[z]$ which induces the G-action on $\mathbb{S}[W^*[z]]$ is the following: if $p(z) \in W^*[z]$ and $g \in G$, $(gp)(z) := p(gz)$. See [2] for more details about time-reversibility and some generalizations.

4 A classification result.

We will now set up some more notation and then prove a preliminary result of independent interest. Aside from the differential operators there are other maps acting between C^∞-spaces which will be of interest to us. If $x \in \mathbb{R}$ denote by $\sigma^x : C^\infty \to C^\infty$ the shift operator $\sigma^x w(t) := w(t+x)$. If $\xi \in k$ denote by $M_\xi : C^\infty \to C^\infty$ the multiplicative operator given by $M_\xi w(t) := e^{\xi t} w(t)$. If $\eta \in \mathbb{R}$ denote by $S_\eta : C^\infty \to C^\infty$ the scaling operator given by $S_\eta w(t) := w(\eta t)$. Clearly, σ^x, M_ξ and S_η can be extended to operators on C^∞_W by taking their tensor product with the identity on W. We will use the same symbol to denote both. If $\lambda \in k$ define $e_\lambda : \mathbb{R} \to k$ by $e_\lambda(t) = e^{\lambda t}$.

The following commutation rules are easily verified:

$$\begin{aligned} M_\xi \circ \sigma^x &= e^{-\xi x} \cdot \sigma^x \circ M_\xi \\ S_\eta \circ \sigma^x &= \sigma^{\eta^{-1} x} \circ S_\eta \\ M_\xi \circ R\left(\frac{d}{dt}\right) &= R\left(\frac{d}{dt} - \xi\right) \circ M_\xi \\ S_\eta \circ R\left(\frac{d}{dt}\right) &= R\left(\eta^{-1}\frac{d}{dt}\right) \circ S_\eta \\ M_\xi \circ S_\eta &= S_\eta \circ M_{\xi\eta^{-1}} \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}$, $\xi \in k$, $\eta \in \mathbb{R}^*$, $R \in GL_R(W[z])$.

Theorem 1: *Let $\Lambda \in GL(C^\infty_W)$. The following conditions are equivalent*

(i) *Λ leaves the class $\mathbb{D}[W]$ invariant.*

(ii) *There exist (unique) $x \in \mathbb{R}$, $\eta \in \mathbb{R}^*$, $\xi \in k$, and $R \in GL_R(W[z])$ such that*

$$\Lambda = \sigma^x \circ R\left(\frac{d}{dt}\right) \circ \mathcal{M}_\xi \circ \mathcal{S}_\eta$$

Proof: The implication (ii)⇒(i) can be straightforwardly deduced by applying the above commutation rules.

(i)⇒(ii). Notice that the only one dimensional subspaces in $\mathbb{D}[W]$ are $\mathrm{span}_k\{e_\lambda w\}$ where $\lambda \in k$ and $w \in W$. Therefore, there exist maps

$$\alpha : k \to GL_k(W) \quad \tilde{\alpha} : k \to GL_k(W)$$

$$\beta : k \to k \quad \tilde{\beta} : k \to k$$

which satisfy

$$\alpha(\lambda)\tilde{\alpha}(\lambda) = \tilde{\alpha}(\lambda)\alpha(\lambda) = I_q \quad \forall \lambda \in k \tag{4}$$

$$\tilde{\beta} \circ \beta = \beta \circ \tilde{\beta} = Id \tag{5}$$

such that

$$\begin{aligned} \Lambda(e_\lambda w) &= e_{\beta(\lambda)} \alpha(\lambda) w \quad \forall \lambda \in k \\ \Lambda^{-1}(e_\lambda w) &= e_{\tilde{\beta}(\lambda)} \tilde{\alpha}(\lambda) w \quad \forall \lambda \in k \end{aligned} \tag{6}$$

The map $\lambda \mapsto e_\lambda w$ from k to C^∞_W is entire (in the sense that it admits a global power series expansion), and since Λ is continuous, we have that also the map from k to C^∞_W given by $\lambda \mapsto e_{\beta(\lambda)} \alpha(\lambda) w$ is entire for all $w \in W$. This implies that α and $\beta\alpha$ are entire, and consequently also $\det \alpha$ and $\beta \det \alpha$. For the same reason $\det \tilde{\alpha}$ and $\tilde{\beta} \det \tilde{\alpha}$ are entire. It follows from (4) that β and $\tilde{\beta}$ are also. From (5) it immediately follows that the extension of β to $\mathbb{C}$ is an entire automorphism of $\mathbb{C}$. Consequently there exist $\eta \in k^*$ and $\xi \in k$ such that $\beta(\lambda) = \eta\lambda - \xi$. A simple argument based on the continuity of Λ shows that we must have $\eta \in \mathbb{R}^*$.

Consider now $\Omega := \Lambda \circ S_{\eta^{-1}} \circ M_{-\xi}$. The result now follows if we prove that Ω is the composition of a differential operator and a shift. It immediately follows from (6) that

$$(\Omega \circ \sigma^x) e_\lambda w = (\sigma^x \circ \Omega) e_\lambda w \quad \forall \lambda \in k \;\; \forall w \in W \;\; \forall x \in \mathbb{R}$$

Since $\{e_\lambda w \mid \lambda \in k,\; w \in W\}$ is dense in C^∞_W it follows that

$$\Omega \circ \sigma^x = \sigma^x \circ \Omega \quad \forall x \in \mathbb{R}. \tag{7}$$

This equality implies that there exists a compact support distribution $\Gamma \in (C^\infty)' \otimes_k \mathrm{Hom}_k[W, W]$ such that

$$\Omega w = \Gamma * w.$$

Now fix a basis of W and write Γ in matrix form (Γ_{ij}) where $\Gamma_{ij} \in (C^\infty)'$ for $i, j = 1, \dots, q$. Consider the differential behavior

$$\mathcal{B}_j = \{0\} \times \cdots \times \{0\} \times C^\infty \times \{0\} \times \cdots \times \{0\}$$

with C^∞ in the j-th place.

Then $\mathcal{A} = \Omega(\mathcal{B}_j)$ has to be a differential behavior, as well as all its projections [11]

$$\mathcal{A}_{hk} = \left\{ \begin{pmatrix} \Gamma_{hj} * w \\ \Gamma_{kj} * w \end{pmatrix} \mid w \in C^\infty \right\}$$

where $h \neq k$. We claim that $\mathcal{A}_{hk} \neq C^\infty \oplus C^\infty$ for all pairs (h, k). Indeed, assume to the contrary that $\mathcal{A}_{hk} = C^\infty \oplus C^\infty$ for some pair (h, k). Consider then the differential behavior

$\mathcal{A}' := \{w \in \mathcal{A} \mid w_h = 0\}$. Clearly $\{0\} \subsetneq \mathcal{A}' \subsetneq \mathcal{A}$ which yields $\{0\} \subsetneq \Omega^{-1}\mathcal{A}' \subsetneq \mathcal{B}_i$. From standard results on differential systems it follows that $\Omega^{-1}\mathcal{A}'$ has to be finite dimensional. Then $\mathcal{A}'$ also has to be finite dimensional, but this is impossible since it contains a subspace isomorphic to C^∞. Therefore $\mathcal{A}_{hk} \neq C^\infty \oplus C^\infty$ for all pairs (h, k). Consider now a pair (h, k) such that Γ_{hj} and Γ_{kj} are both not zero. Then there exist distributions r_1 and r_2 with support in 0, both different from zero, such that

$$r_1 * \Gamma_{hj} + r_2 * \Gamma_{kj} = 0$$

Passing to Fourier transforms, we obtain

$$\hat{r}_1 \cdot \hat{\Gamma}_{hj} + \hat{r}_2 \cdot \hat{\Gamma}_{kj} = 0 \tag{8}$$

All the Fourier transforms we are considering are in the algebra $\mathrm{Hol}_{\exp}(\mathbb{C})$ of the exponential holomorphic functions slowly increasing along the real axis. Distributions with support in zero correspond to the subalgebra of polynomials $\mathbb{C}[z]$. We can evidently assume that $\hat{r}_1$ and $\hat{r}_2$ are coprime in $\mathbb{C}[z]$. From (8) and from standard results on holomorphic exponential functions [8] it immediately follows that $\hat{\Gamma}_{kj} = \hat{Y}\hat{r}_2$ and $\hat{\Gamma}_{hj} = -\hat{Y}\hat{r}_1$ for suitable $\hat{Y} \in \mathrm{Hol}_{\exp}(\mathbb{C})$. By examining now all the $\hat{\Gamma}_{hj}$ for $h = 1, \ldots, q$ we see that there exist $\hat{Y}_j \in \mathrm{Hol}_{\exp}(\mathbb{C})$ and $\hat{p}_{hj} \in \mathbb{C}[z]$ such that $\hat{\Gamma}_{hj} = \hat{Y}_j \hat{p}_{hj}$. We now return to the time domain. We then obtain $\Gamma_{hj} = Y_j * p_{hj}$, where the p_{ij} are distributions with support in 0 and $Y_1, \ldots, Y_q$ are distributions with compact support. Applying now Ω to the differential behaviors

$$\mathcal{B}_{ij} = \{{}^t(0, \ldots, 0, w, 0, \ldots, 0, w, 0, \ldots, 0) \mid w \in C^\infty\}$$

with w in the i-th and j-th place, and arguing as before, we finally deduce that we can assume that $Y_1 = Y_2 = \cdots = Y_q = Y$. Since Y is necessarily invertible in the convolution algebra $(C^\infty)'$ we obtain, by a classic result of Lions [5], that $Y = \alpha\delta_x$ for some $\alpha \in k^*$ and $x \in \mathbb{R}$. This concludes the proof of existence.

Finally uniqueness is straightforward.

□

5 Group actions on polynomial modules.

Throughout the remainder of this paper we will assume that G is a finite group. Consider a representation $T : G \to GL_k(C^\infty_W)$ such that (G, T) is a symmetry on $\mathbb{D}[W]$. Then, by virtue of Theorem 1, there exist maps

$$x : G \to \mathbb{R}, \quad \xi : G \to k, \quad \eta : G \to \mathbb{R}^*$$

$$R : G \to GL_R(W[z])$$

such that

$$T_g = \sigma^{x_g} \circ R_g \left(\frac{d}{dt}\right) \circ M_{\xi_g} \circ S_{\eta_g} \tag{9}$$

By using the commutation rules (3), we easily obtain the following relations

$$\begin{aligned} x_{gg'} &= x_g + \eta_g^{-1} x_{g'} \\ \xi_{gg'} &= \xi_g + \eta_g \xi_{g'} \\ \eta_{gg'} &= \eta_g \eta_{g'} \\ R_{gg'}(z) &= \exp(-\eta_{g^{-1}} \xi_g x_{g'}) R_g(z) R_{g'}(\eta_{g^{-1}} z - \eta_{g^{-1}} \xi_g) \end{aligned} \tag{10}$$

for all g and g' in G. In particular η is a homomorphism and, since G is finite, we have that $\eta(G) \subseteq \{-1, +1\}$. Let $N = \ker \eta$. It is clear from (10) that $x_{|N} : N \to \mathbb{R}$ and $\xi_{|N} : N \to \mathbb{R}$ are homomorphism, and, since G is finite, it follows that they are both identically zero. This yields

$$\begin{aligned} 0 = \xi_e &= \xi_{g^{-1}} + \eta_{g^{-1}} \xi_g \\ 0 = x_e &= x_{g^{-1}} + \eta_g x_g \end{aligned} \qquad \forall g \in G \tag{11}$$

where e is the identity of the group G. Denote by $\mathrm{Aut}(R)$ the group of ring automorphisms of $R = k[z]$. Define the group homomorphism

$$\tau : G \to \mathrm{Aut}(R) \tag{12}$$

by

$$(\tau_g p)(z) = (p^g)(z) := p(\eta_{g^{-1}} z - \eta_{g^{-1}} \xi_g) \tag{13}$$

The action of G on R above introduced, can be extended to any tensor product of the type $R \otimes_k W$. It will still be denoted by τ or by right superscript. Using (11) and (12), the last equation of (10) can be rewritten as

$$R_{gg'} = \exp(\xi_{g^{-1}} x_{g'}) R_g R^g_{g'} \tag{14}$$

Consider now $D \in \mathrm{Hom_k(W, E)[z]}$. Then we see, using (3) and (11), that

$$D\left(\frac{d}{dt}\right) \circ T_{g^{-1}} = \sigma^{x_{g^{-1}}} \circ M_{\xi_{g^{-1}}} \circ S_{\eta_{g^{-1}}} \circ (D^g R^g_{g^{-1}}) \left(\frac{d}{dt}\right)$$

which shows that if $\mathcal{B} = \ker D\left(\frac{d}{dt}\right)$, then

$$g \cdot \mathcal{B} = \ker(D^g R^g_{g^{-1}})\left(\frac{d}{dt}\right) \tag{15}$$

Equation (15) indicates how G should act on $W^*[z]$. Define

$$U : G \to GL_k(W^*[z]) \tag{16}$$

by

$$U_g = \alpha_g(\tau_g \circ R^*_g) \tag{17}$$

where $\alpha : G \to \mathbb{R}^*$ is a normalizing factor which we will determine later and where R^* is defined as follows. If $R_g = \sum R_{i,g} z^i$, then $R^*_g := \sum R^*_{i,g^{-1}} z^i$. We have the following

Proposition 2: U is a homomorphism if and only if α is of type

$$\alpha_g = \exp\left(\frac{\xi_{g^{-1}} x_{g^{-1}}}{2}\right)\beta_g$$

where $\beta : G \to k^*$ is a homomorphism.

Proof: From (14) it easily follows that

$$R^*_{gg'} = \exp(\xi_{g'} x_{g^{-1}})(R^*_g)^{g'^{-1}} R^*_{g'}$$

Therefore,

$$U_{gg'} = \alpha_{gg'} \cdot \tau_{gg'} \circ R^*_{gg'} = \alpha_{gg'} \cdot \exp(\xi_{g'} x_{g^{-1}}) \cdot \tau_{gg'} \circ (R^*_g)^{g'^{-1}} R^*_{g'}$$

On the other hand

$$U_g U_{g'} = \alpha_g \alpha_{g'} \cdot \tau_g \circ R^*_g \circ \tau_{g'} \circ R^*_{g'} = \alpha_g \alpha_{g'} \cdot \tau_{gg'} \circ (R^*_g)^{g'^{-1}} R^*_{g'}$$

This shows that U is a homomorphism if and only if

$$\alpha_{gg'} = \exp(-\xi_{g'} x_{g^{-1}})\alpha_g \alpha_{g'} \quad \forall g, g' \in G \tag{18}$$

We claim that if $\beta : G \to k^*$ is a homomorphism, then $\alpha_g = \exp\left[\frac{1}{2}\xi_{g^{-1}} x_{g^{-1}}\right]\beta_g$ satisfies (18). Indeed with this choice we have that

$$\alpha_{gg'} = \exp\left[\frac{1}{2}(\xi_{g'^{-1}} + \eta_{g'^{-1}}\xi_{g^{-1}})(x_{g'^{-1}} + \eta_{g'} x_{g^{-1}})\right]\beta_{gg'} = \alpha_g \alpha_{g'} \exp\left[\frac{1}{2}(-\xi_{g^{-1}} x_{g'} - \xi_{g'} x_{g^{-1}})\right]$$

In order to prove the claim it clearly suffices to prove that

$$\xi_{g'} x_g = \xi_g x_{g'} \quad \forall g, g' \in G \tag{19}$$

Equation (19) is true if g or g' is in N. Since N is a subgroup at G of index 2, we can then assume that $g' = gg_0$ for some $g_0 \in N$. It then follows from (10) that $x'_g = x_g$ and $\xi'_g = \xi_g$, and so (19) is proven. On the other hand, if α and α' are two solutions of (18) it immediately follows that $\alpha' = \alpha\beta$ where $\beta : G \to k^*$ is a homomorphism. This completes the proof.

□

Consider a homomorphism $\tau : G \to \mathrm{Aut}(R)$. Notice that it will necessarily be of the form (13). Let M be an R-free module of finite rank q. Consider a representation $U : G \to GL_k(M)$. U is called a *τ-linear representation* (or also a *quasi-linear representation*) if

$$U_g(pm) = (\tau_g p) U_g(m) \quad \forall p \in R,\ m \in M,\ g \in G$$

If M is an R-graded module (e.g., $E[z]$), then U is said to be *degree-preserving* if $\deg(U_g m) = \deg(m)$ for all $m \in M$ and $g \in G$.

We have thus proven the following result.

Theorem 3: *Let (G, T) be a symmetry on $\mathbb{D}[W]$ with G finite group. Then there exists a quasi-linear representation $U : G \to GL_k(W^*[z])$ such that,*

$$(T_g \mathcal{B})^\perp = U_g(\mathcal{B}^\perp) \quad \forall g \in G \text{ and } \forall \mathcal{B} \in \mathbb{D}[W] \tag{20}$$

Moreover, U is unique up to the multiplication by a homomorphisms from G to k^*.

Proof: If we choose $\alpha_g = \exp\left(\frac{1}{2}\xi_{g^{-1}} x_{g^{-1}}\right)$, Proposition 2 shows that the corresponding U in (17) is a τ-linear representation, where τ is given in (13). So, (20) follows from (15). Finally, uniqueness follows from (15) and from Proposition 2.

□

We will refer to U as the quasi-linear representation associated with the symmetry (G, T).

Remark: Notice that in the case of static symmetries and time-reversibility discussed in section 3, the corresponding U is degree-preserving. Moreover, U is linear ($\tau = 1$) if and only if for all $g \in G$, T_g is a differential operator followed by a shift.

6 Quasilinear representation on polynomial modules.

A simple and, as we will see, canonical way to construct quasi-linear representation is the following. Let E be a k-vector space, $\rho : G \to GL_k(E)$ a linear representation, and $\tau : G \to \mathrm{Aut}(R)$ a homomorphism. Consider the R-module $E[z]$ and the representation $U : G \to GL_k(E[z])$ given by $U_g(\sum e_i z^i) = \sum \rho_g(e_i)(\tau_g z)^i$. It is evident that U is a τ-linear representation on $E[z]$. U will be called a *quasi-linear split representation.* Notice that in this case U is always degree-preserving. Let us now prove the following interesting result.

Theorem 4: *Let E be a k-vector space and assume that $E[z]$ is equipped with a τ-linear split degree-preserving representation U. Let $M \subseteq E[z]$ be a R-submodule which is G-invariant. Then there exists a k-vector space F equipped with a filtration F_i, a τ-linear split representation on $F[z]$ such that the F_i's are G-invariant, and a G-equivariant R-isomorphism $\psi : F[z] \to M$, such that $\psi(F_i) = M_i$.*

Proof: Consider the filtration of k-vector spaces M_i associated with M as in (1). Clearly the M_i's are G-invariant. Let $N_0 = M_0$ and let N_i be a complementary G-invariant k-subspace of $M_{i-1} + zM_{i-1}$ inside M_i. Let $F = N_0 \oplus N_1 \oplus \cdots \oplus N_h$ where h is the largest integer for which $M_{i-1} + zM_{i-1} \neq M_i$. Clearly F is in a natural way equipped with a k-linear representation ρ of G. Consider $F[z]$ with the quasi-linear split representation induced by τ and ρ. Define $F_i = N_0 \oplus N_1 \oplus \cdots \oplus N_i$. Define $\psi : F[z] \to M$ as the R-linear extension of the natural inclusion $F \hookrightarrow M$. It is straightforward to check that ψ satisfies all the required properties.

□

We now prove the following general fact.

Theorem 5: *Let M be a free R-module of rank p and let $U : G \to GL_k(M)$ be a τ-linear representation. Then there exist a k-vector space E of dimension p, a τ-linear split representation on $E[z]$, and a R-isomorphism $\psi : M \to E[z]$ which is G-equivariant.*

Proof: First consider the case U is an R-linear representation. In this case the theorem had already been proven in [2]. For the sake of completeness though we briefly show how to do it. Consider the specialization of M at 0, namely $M_0 = M/zM$. M_0 is a finite dimensional k-vector space on which G acts k-linearly. Consider the two $k(z)$-vector spaces $M \otimes_{k[z]} k(z)$ and $M_0 \otimes_k k(z)$. It is easy to check that the representations of G naturally induced by them have the same character, therefore there exists a G-equivariant $k(z)$-isomorphism between them. By multiplying this isomorphism by a suitable $p \in k[z]$

we then get an injective G-equivariant R-map $\psi : M \to M_0 \otimes_k k[z]$. The theorem now follows from Theorem 4.

Consider now the general case. Let M be a free R-module equipped with a τ-linear representation U where $\tau : G \to \mathrm{Aut}(R)$ is a homomorphism. Let $N = \ker \tau$. Since $U_{|N}$ is an R-linear representation, we can assume, from the previous case, that $M = E[z]$ for some k-vector space E equipped with an N-representation ρ, and that $U_{|N} = \rho$. It is easy to see, using (3), that there exists a $t_0 \in \mathbb{R}$ such that $(\tau_g p)(t_0) = p(t_0)$ for all $g \in G$ and for all $p \in R$. Consider $M_0 := M/(z - t_0)M$, equipped with the induced k-linear representation U_0. Let $j : M \to M_0[z]$ be given by $j(\sum_{i=0}^{l} m_i z^i) := \sum_{i=0}^{l} \tilde{m}_i z^i$, where $\tilde{m}_i$ is the projection of m_i in M_0. Consider

$$R_h : M \to M_0[z]$$

given by

$$R_h = j \circ \tau_h \circ U_h + U_{0h} \circ j$$

Using the fact that $h^2 \in N$ for all $h \in G$, it is easy to see that every R_h is a R-linear injective map and

$$R_h \circ U_h = (\tau_h \circ U_{0h}) \circ R_h \quad \forall h \in G \tag{21}$$

Fix now any $\tilde{h} \in G \setminus N$. If $g_0 \in N$ it follows that

$$R_{g_0 \tilde{h}} = U_{0 g_0} \circ R_{\tilde{h}} \quad R_{\tilde{h} g_0} = R_{\tilde{h}} \circ U_{g_0} \tag{22}$$

Now, if $g \in G \setminus N$, there exists $g_0 \in N$ such that $g = g_0 \tilde{h}$. By (21) and (22) it follows that

$$R_{\tilde{h}} \circ U_g = U_{0 g_0^{-1}} \circ \tau_g \circ U_{0g} \circ R_g = (\tau_g \circ U_{0 \tilde{h} g \tilde{h}^{-1}}) \circ R_{\tilde{h}} \tag{23}$$

On the other hand, for $g \in N$, it follows from (22) that

$$R_{\tilde{h}} \circ U_g = R_{\tilde{h} g} = R_{(\tilde{h} g \tilde{h}^{-1}) \tilde{h}} = U_{0 \tilde{h} g \tilde{h}^{-1}} \circ R_{\tilde{h}} \tag{24}$$

Equations (23) and (24) show that if we consider on $M_0[z]$ the quasi-linear split representation induced by $\tilde{\rho}_g = U_{0 \tilde{h} g \tilde{h}^{-1}}$ and τ, then the R-map $R_{\tilde{h}}$ is G-equivariant. We can now again apply Theorem 4.

□

Remark: In [3] the case when U is R-linear is treated in slightly greater generality considering also the case G compact. Extensions to reductive groups have also been considered (see [4]). In the case $G = \mathbb{Z}_2$, Theorem 5 had already been proven in [1] and, partially, in [2].

7 Representations of symmetric differential behaviors.

Theorems 4 and 5 are very useful in constructing nice canonical differential representation of symmetric systems. Let (G, T) be a symmetry on $\mathbb{D}[W]$, with G a finite group. Let $\mathcal{B} \in \mathbb{D}[W]^G$ and apply Theorem 5 to $\mathcal{B}^\perp \in \mathbb{S}[W^*[z]]^G$. Then there exists a vector space E, a quasi-linear split representation on $E[z]$ induced by some representation ρ on E, and, by the associated homomorphism $\tau : G \to \mathrm{Aut}(R)$, a G-equivariant injective homomorphism

$$\psi : E[z] \to W^*[z]$$

such that $\mathrm{Im}(\psi) = \mathcal{B}^\perp$. Consider the dual map

$$D = \psi^* : W[z] \to E^*[z]$$

Then, clearly

$$\mathcal{B} = \ker D\left(\frac{d}{dt}\right)$$

and it is a minimal representation. D has a very special structure reflecting the symmetry on $\mathcal{B}$. Namely,

$$\alpha_{g^{-1}} D R_g = \rho_g^* D^g \quad \forall g \in G \tag{25}$$

Moreover, it follows from Theorem 4, that in the case U is degree-preserving, D can be chosen to be canonical.

Let us now see how (25) becomes when affixed to some particular cases.

Example 1: *Static symmetries.* In the notation established in paragraph 5, we have: $x = \xi = 0, \eta = 1, R$ constant (in z). In this case (25) becomes

$$D(z) R_g = \rho_g^* D(z) \quad \forall g \in G \tag{26}$$

By choosing suitable bases in W and in E^*, relative to a decomposition of R and ρ^* into irreducible components, (26) can be further structured ending up with a diagonal block structure for $D(z)$. For more details and concrete examples, see [3].

Example 2: *Time-reversibility.* In this case $T_g = S_g$ for $g \in \{-1, +1\}$, and (25) becomes

$$D(-z) = AD(z)$$

where $A \in GL_k(E)$ is such that $A^2 = I$. By choosing a suitable basis of E we can assume that A is a signature matrix and obtain the following block structure for D

$$D(z) = \begin{bmatrix} D_1(z^2) \\ \\ zD_2(z^2) \end{bmatrix}$$

Example 3: *Involutive symmetries.* A class of involutive symmetries (i.e $G = \mathbb{Z}_2$) are given by the following: $T_{-1} = R\left(\frac{d}{dt}\right) \circ M_\xi \circ S_{-1}$ where ξ is any element of k and where

$$P(z)P(\eta^{-1}z - \xi) = I$$

This is clearly a generalization of Example 2. In this case (25) is equivalent to

$$D(-z-\xi)R(-z-\xi) = AD(z)$$

where A is a signature matrix.

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