GLOBAL ATTRACTORS FOR SEMILINEAR WAVE EQUATIONS WITH LOCALLY DISTRIBUTED NONLINEAR DAMPING AND CRITICAL EXPONENT

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GLOBAL ATTRACTORS FOR SEMILINEAR WAVE EQUATIONS WITH LOCALLY DISTRIBUTED NONLINEAR DAMPING AND CRITICAL EXPONENT

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The main objective of this paper is to prove the existence of global attractors for nonlinear wave equations with locally distributed damping. More specifically, for a bounded regular domain $\Omega \subseteq \mathbb{R}^3$ we consider the problem

(E) $u_{tt} + d(x)\, g(u_t) - \Delta u + f(u) = 0, \qquad u = u(x,t),\ x \in \Omega,\ t \in \mathbb{R}^+$

(B) $u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+$

(I) $u(.,0) = u^0 , \qquad u_t(.,0) = u^1 .$

The functions $f \in C^2(\mathbb{R}^1)$, $g \in C^1(\mathbb{R}^1)$ satisfy the conditions

(G1) $\quad g(0) = 0$, $\qquad g$ strictly increasing , (G2) $$0 < g_0 \leqq \liminf_{|z|\to\infty} g'(z) \leqq \limsup_{|z|\to\infty} g'(z) \leqq g_1 < \infty ,$$

(F1) $$| f''(z)| \leqq c (1 + | z |^p) , \qquad p \leqq 1 ,$$

(F2) $$\liminf_{|z|\to\infty} \frac{f(z)}{z} > - \lambda_1$$

where λ_1 is the optimal constant in the Poincaré inequality

$$\| \nabla u \|_2^2 \geqq \lambda_1 \| u \|_2^2 \qquad \text{for all} \quad u \in H_0^1(\Omega) .$$

We denote by $\| \; \|_q$ the standard norm on the space $L^q(\Omega)$.

We assume that the damping mechanism is effective along the boundary $\partial\Omega$, i.e. $d \in C^1(\bar{\Omega})$,

(D1) $d_1 \geqq d(x) \geqq 0$, $x \in \Omega$, $d(x) \geqq d_0 > 0$ for $x \in \omega$

where $\omega = \mathcal{O} \cap \Omega$, $\mathcal{O}$ a neighbourhood of $\partial\Omega$ in $\mathbb{R}^3$.

It is well known that the solution operator $S_t(u^0,u^1) = (u(t),u_t(t))$, $t \geqq 0$ generates a C^0-semigroup on the associated energy space $X = H_0^1(\Omega) \times L^2(\Omega)$. Moreover, if $d = \text{const} > 0$, $p < 1$ and g linear, there is a *global attractor* $\mathcal{A}$ for S_t on X , i.e. $\mathcal{A}$ is compact, invariant ($S_t(\mathcal{A}) = \mathcal{A}$, $t \geqq 0$) and $\text{dist}(S_t(B), \mathcal{A}) \to 0$ as $t \to \infty$ for any bounded set $B \subseteq X$ (cf. HALE [4], HARAUX [6], and the monographs BABIN-VISHIK [2] , HALE [5], TEMAM [12]). For nonlinear damping, the method of CERON-LOPES [3] (cf. also HALE [5]) provides the same result on condition that

(0.1) $0 < g_0 \leqq g'(z) \leqq g_1 < \infty$ for all $z \in \mathbb{R}^1$.

The critical case d = const, $p = 1$, g linear is treated in ARRIETA-CARVALHO-HALE [1] and BABIN-VISHIK [2], the former approach being based on the nonlinear variation-of-constants formula , the latter taking advantage of regularity results for the linearized equation. For g nonlinear, RAUGEL [10] proves the existence of a global attractor provided that (0.1) holds with g_0 large enough.

On the other hand, the asymptotic behavior of S_t for d localized on a subset of Ω and f nonlinear seems to be a more delicate problem. In the one-dimensional case, LOPES [9] showed that any solution approaches an equilibrium as time goes to infinity. The exponential decay to zero was proved by ZUAZUA [13], [14] under some geometrical constraints imposed on the nonlinearity f .

The main result of this paper may be stated as follows.

THEOREM 1 *Under the conditions* (G1),(G2),(F1),(F2), *and* (D1), *there exists a global attractor* $\mathcal{A} \subseteq X$ *for the problem* (E),(B),(I).

REMARK : Theorem 1 generalizes the results of ARRIETA-CARVALHO-HALE [1] and BABIN-VISHIK [2] to the case of nonlinear locally distributed damping. In particular, we remove the hypothesis concerning the constant g_0 postulated in RAUGEL [10].

The proof of Theorem 1 is based on the unique continuation property proved recently by RUIZ [11] (cf. Section 1) combined with a decomposition technique presented in Section 3.

The first step is to show the existence of a bounded absorbing set for S_t (Section 2). The standard theory (see HALE[5], TEMAM [12]) suggests to treat the problem as a *compact perturbation* of a *contraction semigroup* (cf. also ARRIETA-CARVALHO-HALE [1], CERON-LOPES [3] , RAUGEL [10] etc.). This approach in the present setting, however, seems to lead to severe restrictions concerning the nonlinearity f (cf. ZUAZUA [13]) as well as the constant g_0 (cf. RAUGEL [10]).

. For this reason, it is more convenient to deal with a *contractive perturbation* of a *compact mapping* (Section 3). The proof of Theorem 1 is then completed by means of a priori estimates in the Sobolev scales (Section 4).

1. PRELIMINARIES

PROPOSITION 1 (the unique continuation principle - see RUIZ [11])

Let ω *be as above* , $m = m(x,t) \in L^3(\Omega \times (0,T))$ *and* $V \in L^2(\Omega \times (0,T))$ *a weak solution to the problem*

(1.1) $V_{tt} - \Delta V + m V = 0 \quad , \quad V \,|_{\partial\Omega \times (0,T)} = 0$

that vanishes on $\omega \times (0,T)$ *with* $T > T_0(\Omega,\omega) = \text{diam}\,(\Omega \setminus \omega)$.

Then $V = 0$ *almost everywhere on* $\Omega \times (0,T)$.

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2. THE EXISTENCE OF AN ABSORBING SET

We start with a priori estimates for the equation with a general damping term.

LEMMA 1 *Let* $h \in C^2(\mathbb{R}^1)$ *be a function satisfying* (F1),(F2). *For a fixed* $\delta > 0$ *small, denote*

(2.1) $$L(\delta,H) = -\inf_{z \in \mathbb{R}^1} \left\{ \frac{\lambda_1 - \delta}{2} z^2 + H(z) \right\}$$

where $H(z) = \int_0^z h(s)\,ds$. *Observe that* $L(\delta,H)$ *is finite for* δ *sufficiently small in view of* (F2). *Let* $D = D(x,t,W_t)$ *be a (damping) function such that*

(2.2) $$D(x,t,W_t)W_t \geqq 0 .$$

Then any (strong) solution W *of the problem*

(2.3) $$\begin{cases} W_{tt} + D(x,t,W_t) - \Delta W + h(W) = 0 & \text{on } \Omega \times [0,T], \\ W|_{\partial\Omega\times[0,T]} = 0 \end{cases}$$

satisfies the estimate

(2.4) $$\| W_t(T) \|_2^2 + \| \nabla W(T) \|_2^2 \leqq c \left(\int_0^T \int_\Omega D^2 + D W_t + |W| + |W|^4 \, dx\, dt + \int_0^T \int_\omega W_t^2 \, dx\, dt + L(\delta, H) \right)$$

for any $T > 0$ *sufficiently large* .

Proof of Lemma 1: The proof follows the arguments of the linear theory (cf. LIONS [7], ZUAZUA [13]).

We start with two useful identities. The former is obtained by multiplying (2.3) by $q(x).\nabla W$ with $q \in [W^{1,\infty}(\Omega)]^3$:

(2.5) $$\left[\int_\Omega W_t \, q.\nabla W \, dx \right]_0^T + \frac{1}{2} \int_0^T \int_\Omega \text{div}\, q \left(|W_t|^2 - |\nabla W|^2 - 2H(W) + \sum_{k,j} \frac{\partial q_k}{\partial x_j} \frac{\partial W}{\partial x_k} \frac{\partial W}{\partial x_j} + D \; q.\nabla W \right) dx\, dt = \frac{1}{2} \int_0^T \int_{\partial\Omega} (q.\nu) \left|\frac{\partial W}{\partial \nu}\right|^2 d\Gamma\, dt$$

where ν is the outer normal vector to $\partial\Omega$.

Now, we multiply by $\xi(x)\, W$, $\xi \in W^{1,\infty}(\Omega)$ to obtain

$$(2.6)\quad \Big[\int_\Omega \xi\, W_t\, W \,dx \Big]_0^T + \int_0^T \int_\Omega \xi\, D\, W \;dx\,dt =$$

$$\int_0^T \int_\Omega \xi\,(\, |W_t|^2 - |\nabla W|^2 - h(W)\, W\,) + W\, \nabla\xi.\nabla W \;dx\;dt\;.$$

We set $q = m(x) = x - x^0$, $x^0 \in \mathbb{R}^3$ in (2.5) and $\xi = 1$ in (2.6) to deduce the estimate

$$(2.7)\quad \int_0^T E_W[H](t)\,dt \leqq - \Big[\int_\Omega W_t(\, m.\nabla W + W)\,dx \Big]_0^T - \int_0^T \int_\Omega D\,(m.\nabla W + W\,) +$$

$$4\,H(W) - h(W)\,W\,dx\,dt + \frac{1}{2}\int_0^T \int_{\Sigma(x_0)} (m.\nu)\,\Big|\frac{\partial W}{\partial \nu}\Big|^2\,d\Gamma\,dt$$

with $\Sigma(x_0) = \{\, x \in \partial\Omega \mid (x - x_0).\nu > 0 \,\}$,

where we have denoted by $E_W[H]$ the energy functional

$$(2.8)\quad E_W[H](t) = \frac{1}{2}\,[\; \| W_t(t) \|_2^2 + \| \nabla W(t) \|_2^2 \;] + \int_\Omega H(W(t))\;dx\;.$$

Now we have

$$\Big| \int_0^T \int_\Omega D\,(m.\nabla W + W)\,dx\,dt \Big| \leqq \varepsilon\; \|m\|_\infty\; \|\nabla W\|_2^2 + c(\varepsilon) \int_0^T \int_\Omega D^2\;dx\,dt\;,\;\varepsilon > 0$$

along with

$$\left| \int_0^T \int_\Omega H(W)\, dx\, dt \right| + \left| \int_0^T \int_\Omega h(W)W\, dx\, dt \right| \leqq c \int_0^T \int_\Omega |W| + |W|^4\, dx\, dt$$

and thus

$$(2.9) \quad \int_0^T E_W[H]\,(t)\, dt \leqq c \left\{ \left[\int_\Omega W_t(m.\nabla W + W)\, dx \right]_0^T + \int_0^T \int_\Omega D^2 + |W| + |W|^4\, dx\, dt + \int_0^T \int_{\Sigma(x_0)} (m.\nu) \left| \frac{\partial W}{d\nu} \right|^2 d\Gamma\, dt \right.$$

As the next step, we have to dominate the surface integral in (2.9). Following LIONS [7] we construct a neighbourhood ω' of $cl(\Sigma(x_0))$ such that

$$cl(\omega') \cap \Omega \subset \omega$$

and apply the identity (2.5) with $q = \zeta \in [W^{1,\infty}(\Omega)]^3$ where

$$(2.10) \quad \begin{cases} \zeta = \nu \text{ on } \Sigma(x_0), \quad \zeta.\nu \geqq 0 \quad \text{a.e. on } \partial\Omega \\ \zeta = 0 \text{ on } \Omega - \omega' \end{cases} :$$

$$(2.11) \quad \int_0^T \int_{\Sigma(x_0)} \left|\frac{\partial W}{\partial \nu}\right|^2 d\Gamma\, dt \leqq \int_0^T \int_{\partial\Omega} (\zeta.\nu) \left|\frac{\partial W}{\partial\nu}\right|^2 d\Gamma\, dt \leqq c \left\{ \int_0^T \int_{\omega'} W_t^2 + D^2 + |\nabla W|^2 + |W| + |W|^4\, dx\, dt \right\} + 2\left[\int_\Omega W_t\, \zeta.\nabla W\, dx \right]_0^T .$$

Thus, finally, we have to dominate the term $\int_0^T \int_{\omega'} |\nabla W|^2 \, dx \, dt$ in (2.11). To this end, we construct a function $\eta \in W^{1,\infty}(\Omega)$,

$$
(2.12) \qquad \begin{cases} \eta \in [0,1] \quad , \quad \eta = 1 \quad \text{on} \quad \omega' , \\ \eta = 0 \quad \text{on} \quad \Omega - \omega , \\ |\nabla\eta|^2 / \eta \in L^\infty(\omega) \end{cases}
$$

(cf. LIONS [7]). Applying the identity (2.6) with $\xi = \eta$ we get

$$
(2.13) \quad \int_0^T \int_\Omega \eta \, |\nabla W|^2 \, dx \, dt \leqq c \, \{ \int_0^T \int_\Omega D^2 + |W| + |W|^4 \, dx \, dt \; + \int_0^T \int_\omega W_t^2 \, dx \, dt + [\int_\Omega \eta \; W_t \, W \, dx \,]_0^T \, \}.
$$

Combining (2.9) with (2.11),(2.13) we obtain the estimate

$$
(2.14) \quad \int_0^T E_W[H]\,(t)\,dt \leqq c \, \{ \int_0^T \int_\Omega D^2 + |W| + |W|^4 \, dx \, dt + \int_0^T \int_\omega W_t^2 \, dx \, dt + [\int_\Omega W_t((m + 2\zeta).\nabla W + (1+\eta)\,W\,)\;dx\,]_0^T \, \}.
$$

In accordance with the energy equality

$$
(2.15) \quad E_W[H](T) - E_W[H](0) = - \int_0^T \int_\Omega D \, W_t \, dx \, dt \leqq 0
$$

and by virtue of (F2), we get

$$
(2.16) \quad E_W[H](t) \geqq c[\, \| W_t(t)\|_2^2 + \|\nabla W(t)\|_2^2] - L(\delta,H)\,\mu(\Omega) \quad , \quad t = 0,\, T.
$$

Thus the last term in (2.14) may be estimated as follows

$$(2.17)\qquad \Big[\int_\Omega W_t((m + 2\zeta).\nabla W + (1 + \eta)\, W\,)\, dx\,\Big]_0^T \leqq$$

$$c(\, \| W_t(T)\|_2^2 + \|\nabla W(T)\|_2^2 + \|W_t(0)\|_2^2 + \|\nabla W(0)\|_2^2\,) \leqq$$

$$c(\, E_W[H](T) \;+\; \int_0^T\int_\Omega D\;\; W_t \;+ L(\delta, H)\; dx\; dt\;)\;.$$

Finally, (2.15) yields $T\, E_W[H](T) \leqq \int_0^T E_W[H](t)\, dt$ and, consequently, (2.14),(2.17) give rise to (2.4) for T large enough.

Q.E.D.

From now on, the value of T will be fixed so large that the conclusions of Lemma 1, Proposition 1 hold. At this stage, we are ready to prove the main result of this section.

PROPOSITION 2 (the ultimate dissipativity)

The solution semigroup S_t *is ultimate dissipative, i.e. there exists a bounded set* $B_0 \subseteq X$, $S_t(B_0) \subseteq B_0$ *for* $t \geqq 0$, *and for any* B *bounded there is a time* $t_0(B)$ *such that*

$$(2.18)\qquad S_t(B) \subseteq B_0 \quad \textit{for all}\quad t \geqq t_0\,.$$

Proof of Proposition 2 : As the problem is autonomous and the energy is bounded from below (cf. (F2)) , nonincreasing , the images of bounded sets by S_t remain bounded for all $t \geqq 0$ and Proposition 2 results from the following lemma :

LEMMA 2 *There is a fixed bounded set* B_{01} *such that for any fixed* $B \subset X$ *bounded there exits a strictly positive constant* $K(B)$ *such that*

(2.19) $$E_u[F](T) - E_u[F](0) < -K(B) \quad , \quad F(z) = \int_0^z f(s)\,ds$$

whenever the initial data (u^0, u^1) *belong to* B *and*

(2.20) $$(u(T), u_t(T)) \in B \setminus B_{01} \quad .$$

The desired absorbing is then given as

$$B_0 = \bigcup_{t \geqq 0} S_t(B_{01}).$$

Proof of Lemma 2 : We argue by contradiction, i.e. we suppose that for arbitrary $M > 0$ there is a bounded set B and a sequence of solutions (u^n, u^n_t) such that

(2.21) $$(u^n(0), u^n_t(0)) \in B \quad ,$$

(2.22) $$M \leqq \| u^n_t(T) \|_2^2 + \| \nabla u^n(T) \|_2^2 \quad ,$$

and $E_{u^n}[F](T) - E_{u^n}[F](0) \to 0$, i.e.

(2.23) $$\int_0^T \int_\Omega d(x)\ g(u_t^n)\ u_t^n\ dx\ dt \to 0 \quad , \quad n \to \infty .$$

Passing to subsequences if necessary we may assume that

(2.24) $$u^n \to U \quad \text{weakly star in } L^\infty(0,T,\ H_0^1(\Omega)\ ) ,$$

(2.25) $$u_t^n \to U_t \quad \text{weakly star in } L^\infty(0,T,\ L^2(\Omega)\ ) ,$$

and, by virtue of the embedding $H_0^1(\Omega) \subset L^6(\Omega)$,

(2.26) $$f(u^n) \to f(U) \quad \text{weakly in } L^2(0,T,\ L^2(\Omega)) .$$

Consequently, the limit function U solves the problem

(2.27) $$U_{tt} - \Delta U + f(U) = 0 \quad , \ U|_{\partial\Omega\times[0,T]} = 0 \ , \ U_t|_{\omega\times[0,T]} = 0 .$$

The unique continuation principle (Proposition 1) applied to $V = U_t$ yields

(2.28) $$U_t = 0 \quad \text{identically on } \Omega \times (0,T)$$

and thus

(2.29) $U = \bar{U}$ is a stationary solution of (E),(B).

On the other hand, we can apply the conclusion of Lemma 1 to (E), (B) to obtain the relation

$$(2.30) \qquad M \leqq \| u_t^n(T) \|_2^2 + \| \nabla u^n(T) \|_2^2 \leqq$$

$$c\{ \int_0^T \int_\Omega d^2 g(u_t^n)^2 + dg(u_t^n)\, u_t^n + |u^n| + |u^n|^4 \quad dx\, dt + \int_0^T \int_\omega (u_t^n)^2\, dx\, dt + L(\delta, F)\}$$

By virtue of (G1),(G2) , we have the inequalities

$$(2.31) \qquad g(z)^2 \leqq c\, g(z)\, z \quad ,$$

$$(2.32) \qquad z^2 \leqq \varepsilon + c(\varepsilon)\, g(z)\, z \qquad \text{for all } \varepsilon > 0.$$

Combining (D1) , (2.23) together with (2.30)- (2.32) we get

$$(2.33) \qquad M \leqq c\{\lim_{n\to\infty} \int_0^T \int_\Omega |u^n| + |u^n|^4 + dx\, dt + L(\delta, F)\} =$$

$$c\{T \int_\Omega |\bar{U}| + |\bar{U}|^4 \quad dx + L(\delta, F)\}$$

in view of (2.24),(2.25) and compactness of the embedding $H_0^1(\Omega) \subset L^4(\Omega)$.

As the set of stationary solutions to (E),(B) is bounded and M was arbitrary, we have arrived at contradiction.

Q.E.D.

3. THE DECOMPOSITION

We start with the decomposition of the nonlinearity f .

LEMMA 3 : (ARRIETA-CARVALHO-HALE [1]).

Under the assumptions (F1),(F2) , *the nonlinearity* f *can be decomposed* $f = f^1 + f^2$ *where* $f^1 \in C^2(\mathbb{R}^1)$, $f^2 \in C^1(\mathbb{R}^1)$ *and*

(3.1) $z\, f^1(z) \geqq 0$ *for all* z

(3.2) $|(f^1)''(z)| \leqq c(1 + |z|)$ *for all* z,

(3.3) $|(f^2)'(z)| \leqq c(1 + |z|^{2-\delta})$ *with* $\delta > 0$ *for all* z ,

(3.4) $\liminf_{|z| \to \infty} \frac{f^2(z)}{z} > -\lambda_1$.

The key to the proof of Theorem 1 is to decompose the solution $u = v + w$ where

(E1) $v_{tt} + d(x)\,(\,g(u_t) - g(u_t - v_t)\,) - \Delta v + f^1(v) = 0$ on $\Omega \times \mathbb{R}^+$

$v|_{\partial\Omega\times\mathbb{R}^+} = 0$, $v(0) = u^0$, $v_t(0) = u^1$,

and

(E2) $w_{tt} + d(x)\, g(w_t) - \Delta w + f^1(v + w) - f^1(v) = -f^2(u)$ on $\Omega \times \mathbb{R}^+$,

$w|_{\partial\Omega\times\mathbb{R}^+} = 0$, $w(0) = w_t(0) = 0$.

The behavior of v is determined by the following lemma.

LEMMA 4 : *There is a function* $\beta : \mathbb{R}^+ \to \mathbb{R}^+$,

(3.5) $\beta(t) \to 0$ as $t \to \infty$,

such that

(3.6) $\| v_t(t) \|_2 + \| \nabla v(t) \|_2 \leqq \beta(t)$ for all $t \geqq 0$

for any solution v *of* (E1) *starting from* B_0 . *In particular,* β *is independent of* u_t .

Proof of Lemma 4 : The idea of the proof is the same as in Proposition 2. Applying the energy equality to (E1) we obtain

$$(3.7)\quad \frac{d}{dt}\ \frac{1}{2}\left[\ \| v_t\|_2^2 + \| \nabla v \|_2^2 + \int_\Omega F^1(v)\, dx\ \right] +$$

$$+ \int_0^T \int_\Omega d(x)\ (\ g(u_t) - g(u_t - v_t))\, v_t\ dx\, dt = 0 \quad ,\ F^1(z) = \int_0^z f^1(s)\, ds$$

It suffices to show that

(3.8) $E_v[F^1](T) - E_v[F^1](0) \leqq -K(M) < 0$

whenever $(u^0, u^1) \in B_0$ and

(3.9) $0 < M \leqq \| v_t(T) \|_2^2 + \| \nabla v(T) \|_2^2 \leqq c(B_0)$

for all $M > 0$.

Reasoning by contradiction we obtain a bounded sequence (v^n, v_t^n) such that

(3.10) $M \leqq \| v_t^n(T) \|_2^2 + \| \nabla v_t^n(T) \|_2^2$ for a certain $M > 0$

(3.11) $\int_0^T \int_\Omega d(x) \, (g(u_t^n) - g(u_t^n - v_t^n)) \, v_t^n \, dx \, dt \to 0$ as $n \to \infty$

Now we can repeat the arguments of the proof of Lemma 2, specifically the steps (2.24)-(2.29) to conclude that

(3.12) $v^n \to 0$ weakly star in $L^\infty(0,T, H_0^1(\Omega))$,

(3.13) $v_t^n \to 0$ weakly star in $L^\infty(0,T, L^2(\Omega))$.

We have used the fact that due to the structural properties of f^1 (cf. (3.1)) the only stationary solution to the equation (2.27) (with f replaced by f^1) is zero.

Applying Lemma 1 to (E1) and using (3.10)- (3.13) along with compactness of $H_0^1(\Omega) \subset L^4(\Omega)$ we obtain an inequality

(3.14) $M \leqq \lim_{n\to\infty} c\{ \int_0^T \int_\Omega d^2(g(u_t^n) - g(u_t^n - v_t^n))^2 + \int_0^T \int_\omega (v_t^n)^2 \, dx \, dt\}.$

Observe that $L(\delta, F^1) = 0$ for δ small .

Our ultimate goal is to show that (3.11) implies that the right-hand side of (3.14) is zero. In view of (G1),(G2), we obtain

$$(3.15) \qquad | g(s) - g(s-z) | \leqq c \, | z | \qquad \text{for all } s, z \in \mathbb{R}^1$$

which yields the convergence to zero for the first term.

By virtue of the hypothesis (D1) , the second term in (3.14) will tend to zero (provided (3.11) holds) as soon as we are able to estimate the quantity z^2 in terms of $(g(s) - g(s-z))z$. We claim that

$$(3.16) \qquad z^2 \leqq \varepsilon + c(\varepsilon)(g(s) - g(s-z)) z \qquad \text{for all } \varepsilon > 0 \text{ and } s, z \in \mathbb{R}^1.$$

Indeed, (3.16) holds for all $|z| \leqq \sqrt{\varepsilon}$ and for $|z| > \sqrt{\varepsilon}$ we have to show

$$(3.17) \qquad \inf_{\text{length}(J) \geqq \sqrt{\varepsilon}} \int_J g'(z)\, dz > 0$$

which is true due to (G1), (G2).

Thus (3.14) implies $M = 0$ in contrast with (3.10).

Q.E.D.

4. COMPACTNESS OF THE TRAJECTORIES OF (E2)

To complete the proof of Theorem 1, we want to show that the the solution couple $(w(t), \dot{w}_t(t))$ of (E2) belongs to a compact subset of X for any fixed t as soon as $(u^0, u^1) \in B_0$

Following ARRIETA-CARVALHO-HALE [1] we introduce the intermediate Sobolev spaces $H^\alpha(\Omega)$, $\alpha \in (0,1)$ with the norm

$$(4.1) \qquad [[\, v\,]]_\alpha^2 = \{ \; \| v \|_2^2 \; + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{3 + 2\alpha}} \, dx\, dy \; \}$$

(cf. LIONS-MAGENES [8]). We recall the embedding relations

$$(4.2) \qquad H^\alpha(\Omega) \subset H^\beta(\Omega) \quad \text{compactly if} \quad \alpha > \beta$$

and

$$(4.3) \qquad H^\alpha(\Omega) \subset L^q(\Omega) \quad \text{if} \quad \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{3} \; .$$

Finally , we make use of the spaces $D(A^\beta)$, $\beta \geqq 0$ where A is the L^2-realisation of the operator

$$A\, v = -\, \Delta\, v \quad \text{on} \quad \Omega \; , \quad v|_{\partial\Omega} = 0.$$

We have the relation

$$(4.4) \qquad H_0^\alpha(\Omega) \subset D(\, A^{\frac{\alpha}{2}}\,) \subset H^\alpha(\Omega)$$

where $H_0^\alpha(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^\alpha(\Omega)$.

LEMMA 5 : *For any* $t \geqq 0$, *the solution* $(w(t), w_t(t))$ *of the problem* (E2) *belong to a fixed bounded set in*

$$(4.5) \qquad H^{\beta+1}(\Omega) \times H^{\beta}(\Omega) \quad , \quad \beta > 0 \quad (\textit{small enough})$$

whenever the functions u , v *are determined by* (E),(E1) *with the data* $(u^0, u^1) \in B_0$.

Proof of Lemma 5 : To begin with, observe that the couple $w = u - v$, $w_t = u_t - v_t$ is uniformly bounded in $H_0^1(\Omega) \times L^2(\Omega)$ by virtue of Proposition 2 and Lemma 4.

As the next step, we multiply the equation (E2) by the expression $A^{2\alpha} w_t$, $\alpha > 0$ and integrate by parts to obtain

$$(4.6) \qquad \| A^{\alpha} w_t(t) \|_2^2 + \| A^{\alpha+\frac{1}{2}} w(t) \|_2^2 \leqq$$

$$c \; \{ \int_{\Omega} | (f^1(v(t) + w(t)) - f^1(v(t))) \, A^{2\alpha} w(t) | \, dx \; +$$

$$\int_{\Omega} | \; f^2(u(t)) \, A^{2\alpha} w(t) \; | \, dx \; +$$

$$\int_0^t \int_{\Omega} | \; ((f^1)'(v + w) - (f^1)'(v)) \, v_t \, A^{\alpha-\frac{1}{2}} A^{\alpha+\frac{1}{2}} w \; | \; +$$

$$| \ (f^1)'(v + w) \ w_t \ A^{\alpha-\frac{1}{2}} \ A^{\alpha+\frac{1}{2}} \ w \ | \ + \ | \ (f^2)'(u) \ u_t \ A^{\alpha-\frac{1}{2}} \ A^{\alpha+\frac{1}{2}} \ w \ | \ +$$

$$| \ A^{\alpha}(d(x)g(w_t)) \ A^{\alpha}w_t | \quad dx \ dt \ \}.$$

First of all, observe that

$$(4.7) \qquad \int_\Omega |(\ f^1(v + w) - f^1(v) \ ) \ A^{2\alpha}w \ (t)| \ dx \ \leqq \ c \ (B_0) \ ,$$

$$(4.8) \qquad \int_\Omega | \ f^2(u(t)) \ A^{2\alpha} \ w(t) \ | \ dx \ \leqq c(B_0)$$

if

$$(4.9) \qquad \alpha < \frac{1}{4} \ .$$

Next, we use the embedding relation (4.3),(4.4) along with (F1) to estimate

$$(4.10) \qquad \| \ (f^1)'(v + w) - (f^1)'(v)) \ v_t \ \|_q \leqq c(B_0) \ \| \ A^{\alpha+\frac{1}{2}} \ w \ \|_2 \quad ,$$

(4.11) $$\| (f^1)'(v+w)\, w_t \|_q \leqq c(B_0) \, \| A^{\alpha} w_t \|_2 \,,$$

and

(4.12) $$\| (f^2)'(u)\, u_t \|_s \leqq c(B_0)$$

where

(4.13) $$\frac{1}{q} = \frac{1}{6} + \frac{1}{2} + \left(\frac{1}{2} - \frac{2\alpha + 1}{3} \right) = \frac{1}{3} + \left(\frac{1}{2} - \frac{2\alpha}{3} \right), \quad s > \frac{6}{5} .$$

Moreover,

(4.14) $$\| A^{\alpha - \frac{1}{2}} A^{\alpha + \frac{1}{2}} w \|_r \leqq c \, \| A^{\alpha + \frac{1}{2}} w \|_2 \,, \quad \frac{1}{r} = \frac{1}{2} - \frac{1 - 2\alpha}{3} = 1 - \frac{1}{q} .$$

Finally, for α satisfying (4.9) , we have $H^{2\alpha} = H_0^{2\alpha}$ hence

(4.15) $$\| A^{\alpha}(d(x) g(w_t)) \|_2^2 \leqq c(d) \, [[\, g(w_t) \,]]_{2\alpha}^2 =$$

$$c\{ \, \| g(w_t) \|_2^2 + \int_\Omega \int_\Omega \frac{|\, g(w_t(x,.)) - g(w_t(y,.)) |^2}{|\, x - y \,|^{3+4\alpha}} \, dx\, dy \, \} =$$

$$c(d,g)\ [[\ w_t\ ]]^2_{2\alpha} \leq c(d,g)\ \| A^\alpha w_t \|^2_2$$

Applying the Gronwall lemma to (4.6) and using the above estimates we get the conclusion of the lemma for $\alpha > 0$ small enough.

Q.E.D.

The proof of Theorem 1 follows from Proposition 2 and Lemmas 4 , 5 by means of the standard theory of the dissipative systems (cf. HALE [5]).

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