THE WEIERSTRASS CONDITION FOR A SPECIAL CLASS OF ELASTIC MATERIALS

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IMA Preprint Series # 1050

November 1992

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August, 1992

1. INTRODUCTION

A nonlinear hyperelastic material is characterized by a stored energy function $W(\cdot):\mathcal{D}\to\mathbb{R}$, where $\mathcal{D}$ is an open subset of the set of second order tensors on $\mathbb{R}^3$ with positive determinant*. It is required that $W(\cdot)$ be *frame indifferent* or *objective*, *i.e.*,

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{1}$$

for all $\mathbf{F}\in\mathcal{D}$ and for all proper, orthogonal tensors $\mathbf{Q}$. Since (1) requires that $\mathbf{Q}\mathcal{D}\subseteq\mathcal{D}$, objectivity is a restriction on both $W(\cdot)$ and its domain $\mathcal{D}$. We will usually assume that $W(\cdot)$ is twice continuously differentiable on $\mathcal{D}$.

Let Ω be an open, bounded, and connected region of $\mathbb{R}^3$ with boundary $\partial\Omega$. Let $\mathbf{y}^*(\cdot):\partial\Omega\to\mathbb{R}^3$ be pre-assigned and consider the set $\Im$ of one-to-one functions $\mathbf{y}(\cdot):\Omega\to\mathbb{R}^3$ which are continuous, piecewise smooth, meet $\nabla\mathbf{y}(\Omega)\subseteq\mathcal{D}$, and have a continuous extension which agrees with the boundary data $\mathbf{y}^*(\cdot)$ on $\partial\Omega$. In nonlinear elasticity theory a fundamental role is played by functions $\hat{\mathbf{y}}(\cdot)\in\Im$ which

$$\text{minimize} \int_\Omega W(\nabla\mathbf{y})\,dv \quad \text{on } \Im. \tag{2}$$

As is well-known, wherever such minimizers $\hat{\mathbf{y}}(\cdot)$ are smooth enough, it is necessary that the *Euler equation* hold

$$\operatorname{div} W_{\mathbf{F}}(\nabla\hat{\mathbf{y}}(\mathbf{x})) = 0, \tag{3}$$

and that the *Weierstrass condition*

$$W(\nabla\hat{\mathbf{y}}(\mathbf{x})) + (\mathbf{F}-\nabla\hat{\mathbf{y}}(\mathbf{x}))\cdot W_{\mathbf{F}}(\nabla\hat{\mathbf{y}}(\mathbf{x})) \le W(\mathbf{F}), \tag{4}$$

be satisfied for all $\mathbf{F}\in\mathcal{D}$ such that $\mathbf{F}-\nabla\hat{\mathbf{y}}(\mathbf{x})$ is rank one. From (4) it follows easily that the *Legendre-Hadamard condition* must hold, *i.e.*,

$$W_{\mathbf{FF}}(\nabla\hat{\mathbf{y}}(\mathbf{x}))[\mathbf{u}\otimes\mathbf{v}]\cdot(\mathbf{u}\otimes\mathbf{v}) \ge 0 \tag{5}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. Strict inequality in (5) whenever $\mathbf{u}\otimes\mathbf{v}\neq 0$ of course ensures that the Euler equation (3) is *strongly elliptic* at $\mathbf{x}\in\Omega$, and, accordingly, we then say that $W(\cdot)$ meets the S–E *inequality* at $\nabla\hat{\mathbf{y}}(\mathbf{x})$. Lastly, at surfaces S in Ω across which $\hat{\mathbf{y}}(\cdot)$ is continuous but $\nabla\hat{\mathbf{y}}(\cdot)$ is discontinuous, it is necessary that

$$[\![\nabla\hat{\mathbf{y}}(\mathbf{x})]\!] = \mathbf{a}(\mathbf{x})\otimes\mathbf{n}(\mathbf{x}), \tag{6}$$

*The restriction to positive determinants may be weakened. See our Appendix.

and

$$[\![W_{\mathbf{F}}(\nabla\hat{\mathbf{y}}(\mathbf{x}))]\!]\mathbf{n}(\mathbf{x}) = 0. \tag{7}$$

Here, $[\![\cdot]\!]$ denotes the usual difference operator across the surface S, $\mathbf{n}(\mathbf{x})$ is a unit normal to the surface at $\mathbf{x} \in S$, and $\mathbf{a}(\mathbf{x}) \in I\!R^3$.

We formalize the requirement of the Weierstrass condition by saying that $\hat{\mathbf{F}} \in \mathcal{D}$ *is a point of rank* 1 *convexity for* $W(\cdot)$ *over* $\mathcal{S} \subseteq \mathcal{D}$ (equivalently, the function $W(\cdot) : \mathcal{D} \to I\!R$ *is rank 1 convex over* $\mathcal{S} \subseteq \mathcal{D}$ *at* $\hat{\mathbf{F}} \in \mathcal{D}$) *if*

$$W(\hat{\mathbf{F}}) + (\mathbf{F} - \hat{\mathbf{F}}) \cdot W_{\mathbf{F}}(\hat{\mathbf{F}}) \leq W(\mathbf{F}) \tag{8}$$

for all $\mathbf{F} \in \mathcal{S}$ *such that* $\mathbf{F} - \hat{\mathbf{F}}$ *is rank 1.* For a given set $\mathcal{S}$ of comparison tensors, let $\mathcal{D}^{r1}(\mathcal{S}) \subseteq \mathcal{D}$ denote the set of tensors at which $W(\cdot)$ is rank 1 convex over $\mathcal{S}$. From (4), we see that the gradient of a minimizing field $\hat{\mathbf{y}}(\cdot)$ must meet $\nabla\hat{\mathbf{y}}(\Omega) \subseteq \mathcal{D}^{r1}(\mathcal{D})$, and, because of this, it is common (assuming existence and appropriate boundary data) to expect non-smooth minimizing fields whenever $\mathcal{D}^{r1}(\mathcal{D})$ is not connected[†].

Except for elastic fluids — where $W(\mathbf{F}) = w(\det(\mathbf{F}))$ for $w(\cdot) : I\!R^+ \to I\!R$, and where rank 1 convexity for $W(\cdot)$ at $\hat{\mathbf{F}}$ is easily shown to be equivalent to $\det(\hat{\mathbf{F}})$ being a point of

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[†]On the other hand, suppose there exists a non-smooth minimizer $\hat{\mathbf{y}}(\cdot)$ to (2). By (5)–(7), it is then necessary that there exist distinct $\hat{\mathbf{F}}_1$ and $\hat{\mathbf{F}}_2$ in $\mathcal{D}$ such that $\hat{\mathbf{F}}_2 = \hat{\mathbf{F}}_1 + \mathbf{a}\otimes\mathbf{n}$, $W_{\mathbf{F}}(\hat{\mathbf{F}}_2)\mathbf{n} = W_{\mathbf{F}}(\hat{\mathbf{F}}_1)\mathbf{n}$, and

$$W_{\mathbf{FF}}(\hat{\mathbf{F}}_2)[\mathbf{u}\otimes\mathbf{v}]\cdot\mathbf{u}\otimes\mathbf{v} \geq 0 \quad\text{and}\quad W_{\mathbf{FF}}(\hat{\mathbf{F}}_1)[\mathbf{u}\otimes\mathbf{v}]\cdot\mathbf{u}\otimes\mathbf{v} \geq 0$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $I\!R^3$. Suppose that in fact at least one of these last two inequalities is strict for $\mathbf{u}\otimes\mathbf{v} = \mathbf{a}\otimes\mathbf{n}$. (It thus suffices for $W(\cdot)$ to meet the S–E inequality at either $\hat{\mathbf{F}}_1$ or $\hat{\mathbf{F}}_2$.) In this case we may modify an argument of Knowles & Sternberg [1] to show that, even if $\mathcal{D}$ is convex, $\mathcal{D}^{r1}(\mathcal{D})$ *must fail to be convex*, and, in particular, $\mathcal{D}^{r1}(\mathcal{D})$ *cannot contain all of the line segment between* $\hat{\mathbf{F}}_1$ *and* $\hat{\mathbf{F}}_2$.

Indeed, with $\phi(s) \equiv W_{\mathbf{F}}(\hat{\mathbf{F}}_1 + s\mathbf{a}\otimes\mathbf{n})\cdot\mathbf{a}\otimes\mathbf{n}$ — and here we have assumed that $\mathcal{D}$ contains the line segment between $\hat{\mathbf{F}}_1$ and $\hat{\mathbf{F}}_2$, we have that

$$0 = \phi(1) - \phi(0) = \int_0^1 \phi'(s)\,ds = \int_0^1 W_{\mathbf{FF}}(\hat{\mathbf{F}}_1 + s\mathbf{a}\otimes\mathbf{n})[\mathbf{a}\otimes\mathbf{n}]\cdot\mathbf{a}\otimes\mathbf{n}\,ds,$$

where the integrand $\phi'(s)$ is positive at either $s = 0$ or $s = 1$. Clearly, $\phi'(s)$ must then be negative for some $s^\star \in (0,1)$, *i.e.*, for some $s^\star \in (0,1)$ we must have that

$$W_{\mathbf{FF}}(\hat{\mathbf{F}}_1 + s^\star\mathbf{a}\otimes\mathbf{n})[\mathbf{a}\otimes\mathbf{n}]\cdot\mathbf{a}\otimes\mathbf{n} < 0.$$

Thus, the Legendre–Hadamard condition (5) fails at $\hat{\mathbf{F}}_1 + s^\star\mathbf{a}\otimes\mathbf{n}$. A fortiori, $\hat{\mathbf{F}}_1 + s^\star\mathbf{a}\otimes\mathbf{n}$ cannot be a point of rank 1 convexity for $W(\cdot)$ over $\mathcal{D}$.

(It is worth noting that the above discussion actually establishes the stronger result that, irrespective of whether the Legendre–Hadamard condition is strict at either $\hat{\mathbf{F}}_1$ or $\hat{\mathbf{F}}_2$ for $\mathbf{u}\otimes\mathbf{v} = \mathbf{a}\otimes\mathbf{n}$, it is impossible for the Legendre–Hadamard condition to hold everywhere on the line segment between $\hat{\mathbf{F}}_1$ and $\hat{\mathbf{F}}_2$ *unless* $W(\hat{\mathbf{F}}_1 + (\cdot)\mathbf{a}\otimes\mathbf{n})$ *is affine on* [0,1].)

convexity for $w(\cdot)$ — except for these special materials, the implications of rank 1 convexity and the structure of $\mathcal{D}^{r1}(\mathcal{D})$ remain important open problems that require working through. In the present paper, for a second very special class of elastic materials, we present necessary and sufficient conditions for $W(\cdot)$ to be rank 1 convex over $\mathcal{D}$ at a point $\hat{\mathbf{F}} \in \mathcal{D}$. Our Theorem 1 shows that for these materials the requirement of rank 1 convexity is at once simple and explicit, and our Theorems 2 and 3 relate rank 1 convexity to certain other, perhaps more familiar, types of convexity. Inasmuch as we do not need the notion of $W(\cdot)$ being rank 1 convex at $\hat{\mathbf{F}}$ only over certain *subsets* of $\mathcal{D}$ until the discussion of our Appendix, we will henceforth simply write $\mathcal{D}^{r1}$ for $\mathcal{D}^{r1}(\mathcal{D})$. For the same reason, we will usually omit in our statements of theorems any reference to the set (for now $\mathcal{D}$ itself) over which $W(\cdot)$ is rank 1 convex.

The materials that we analyze here, although too special to be of general applicability, reveal clearly that subtle but important consequences flow from both the structure of $\mathcal{D}$ and the structure of $\mathcal{D}^{r1}$. In this the materials of our study are rather more interesting than the too degenerate case of elastic fluids, and, indeed, one implication of our analysis is that certain results of Dacorogna [2] require modification and/or refinement if they are to fully represent the richness of response inherent in nonlinear elasticity. In Section 4, we give a brief discussion of this observation.

2. A SPECIAL ELASTIC MATERIAL

We shall assume here that the material is *isotropic* so that, in addition to (1),

$$W(\mathbf{FR}) = W(\mathbf{F}) \tag{9}$$

for all $\mathbf{F} \in \mathcal{D}$ and all proper orthogonal $\mathbf{R}$. Like (1), the condition (9) imposes a restriction on both $W(\cdot)$ and on the domain $\mathcal{D}$, and, in particular, it may be shown that (1) and (9) together imply that $W(\mathbf{F})$ has the representation

$$W(\mathbf{F}) = \varphi(\mathrm{I},\mathrm{II},\mathrm{III}), \tag{10}$$

where I=I($\mathbf{B}$), II=II($\mathbf{B}$), and III= III($\mathbf{B}$) are the three principal invariants of the left Cauchy-Green strain tensor $\mathbf{B} \equiv \mathbf{FF}^T$, *i.e.*,

$$\mathrm{I} = \mathrm{tr}(\mathbf{B}) = |\mathbf{F}|^2, \quad \mathrm{II} = \frac{1}{2}(\mathrm{I}^2 - \mathrm{tr}(\mathbf{B}^2)), \quad \mathrm{III} = \det(\mathbf{B}).$$

In the remainder of this work we consider the special class of materials where $\varphi(\mathrm{I},\mathrm{II},\mathrm{III})$ is independent of II and III, and we therefore write

$$W(\mathbf{F}) = \psi(\mathrm{I}). \tag{11}$$

We shall assume that $\psi(\cdot) : \mathcal{I} \to I\!R$ is twice continuously differentiable on $\mathcal{I} \equiv \{\mathrm{I} | \mathrm{I} = |\mathbf{F}|^2, \mathbf{F} \in \mathcal{D}\}$, and we seek to explore for these special elastic materials the implications of rank 1 convexity. First however we note a few simple properties that flow from the constitutive assumption (11).

Recall that the Cauchy stress tensor in any hyperelastic material is given by[‡]

$$\mathbf{T} = \mathbf{T}(\mathbf{F}) = \frac{1}{|\det(\mathbf{F})|} W_{\mathbf{F}}(\mathbf{F})\mathbf{F}^T, \tag{12}$$

and that, if the material is isotropic, then in fact we have the general representation

$$\mathbf{T} = \mathbf{T}(\mathbf{B}) = f_0 \mathbf{1} + f_1 \mathbf{B} + f_{-1}\mathbf{B}^{-1}$$

for certain isotropic functions $f_i = f_i(\mathbf{B})$, $i = -1, 0, 1$. Here, for the special elastic materials of (11) we easily calculate that

$$\mathbf{T} = \frac{2}{|\det(\mathbf{F})|}\psi'(\mathrm{I})\mathbf{B} = \frac{2}{\mathrm{III}^{1/2}}\psi'(\mathrm{I})\mathbf{B}. \tag{13}$$

It follows therefore that the elastic materials of (11) are a bit peculiar: *at each strain* $\mathbf{B}$ *the principal stresses* t_i, $i = 1, 2, 3$, *of* $\mathbf{T} = \mathbf{T}(\mathbf{B})$ *are all of the same sign*, positive if and only if $\psi'(\mathrm{I}) > 0$, negative if and only if $\psi'(\mathrm{I}) < 0$, and zero if and only if $\psi'(\mathrm{I}) = 0$. Stress states of combined tension and compression, *i.e.*, stress states where the principal stresses t_i are *not* all of the same sign, are impossible in the materials of (11).

In spite of the peculiar nature of these materials, we note that over certain portions of their domain they obey several of the usual *a priori* inequalities of finite elasticity theory. Specifically, let β_i denote the eigenvalues of the positive definite strain tensor $\mathbf{B}$ and recall that the numbers $\nu_i \equiv \sqrt{\beta_i}$ are then the *principal stretches* of the deformation. At a given strain tensor $\mathbf{B}$ a material of (11) meets the

(i) E (for "empirical") inequalities if and only if

$$f_0 \le 0, \quad f_1 > 0, \quad f_{-1} \le 0 \iff \psi'(\mathrm{I}) > 0;$$

(ii) B–E (for Baker-Ericksen) inequalities if and only if the greater principal stress at $\mathbf{B}$ is associated with the greater principal stretch, *i.e.*,

$$(t_i - t_j)(\nu_i - \nu_j) > 0 \text{ if } \nu_i \ne \nu_j \iff \psi'(\mathrm{I}) > 0;$$

[‡]While for now, of course, $\mathbf{F} \in \mathcal{D} \Rightarrow |\det(\mathbf{F})| = \det(\mathbf{F})$, our use of $|\det(\mathbf{F})|$ in (12) anticipates the discussion of our Appendix.

(iii) O–F (for "ordered forces") inequalities if and only if the greater principal force $T_i \equiv t_i \nu_j \nu_k$, $i \neq j \neq k \neq i$, at $\mathbf{B}$ is associated with the greater principal stretch, *i.e.*,

$$(T_i - T_j)(\nu_i - \nu_j) > 0 \text{ if } \nu_i \neq \nu_j \iff \psi'(\mathrm{I}) > 0;$$

(iv) T–E$^+$ (for "strengthened tension-extension") inequalities if and only if at $\mathbf{B}$ each principal stress is a strongly monotone increasing function of its associated principal stretch, *i.e.*,

$$\frac{\partial t_i}{\partial \nu_i} > 0, \quad i = 1,2,3 \iff \psi'(\mathrm{I}) + 2\nu_i{}^2\psi''(\mathrm{I}) > 0, \quad i = 1,2,3;$$

(v) S–E inequality if and only if for all vectors $\mathbf{u} \neq 0 \neq \mathbf{v}$

$$\mathbf{T_B}(\mathbf{B})[\mathbf{Bv} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{vB}] \cdot \mathbf{u} \otimes \mathbf{v} > 0,$$

$$\iff \psi'(\mathrm{I})|\mathbf{u}|^2(\mathbf{v} \cdot \mathbf{Bv}) + 2\psi''(\mathrm{I})(\mathbf{u} \cdot \mathbf{Bv})^2 > 0,$$

$$\iff \psi'(\mathrm{I}) > 0 \text{ and } \psi'(\mathrm{I}) + 2\nu_i{}^2\psi''(\mathrm{I}) > 0, \quad i = 1,2,3.$$

Thus, for the special elastic materials we study here the E, B–E, and O–F inequalities are all equivalent. Additionally, these three inequalities and the T–E$^+$ inequality are together equivalent to the S–E inequality.[§] Perhaps more striking, however, is the fact that the condition $\psi'(\mathrm{I}) > 0$, when put alongside (13) means that

$$\left\{ \begin{array}{c} \text{the E, the B–E, and the O–F} \\ \text{inequalities hold at a strain } \mathbf{B} \end{array} \right\} \iff \left\{ \begin{array}{c} \text{the principal stresses} \\ \text{are all positive.} \end{array} \right\}$$

Equivalently, no state of compression meets the E, the B–E, or the O–F inequalities in this material. To the extent then that these inequalities are associated with ideas of stability, we thus see that the materials of (11) may be expected to always give way to compressive loadings; they are resistant only to tensile loadings. While this might seem to argue that the constitutive equation (11) is twice-over peculiar, this need not be so: if we take (11) and the consequences we herein adduce for it as illustrations of what happens when the more

[§] As is discussed in [3], it is known that in a general isotropic elastic material

$$(i) \text{ S–E} \Rightarrow \text{both B–E and T–E}^+ \text{ and } (ii) \text{ E} \Rightarrow \text{both B–E and O–F}.$$

For the materials of (11) we see that we have the stronger relations

$$(i') \text{ S–E} \Leftrightarrow \text{both B–E and T–E}^+ \text{ and } (ii') \text{ E} \Leftrightarrow \text{B–E} \Leftrightarrow \text{O–F}.$$

general form $\varphi(\cdot,\cdot,\cdot)$ of (10) looses sensitivity to its last two arguments on some subdomain, then results like we find here may be taken as illustrating a particular mechanism of material instability.

3. RANK 1 CONVEXITY

The following theorem gives an equivalent characterization of rank 1 convexity for $W(\cdot)$ at $\hat{\mathbf{F}} \in \mathcal{D}$ in terms of the function $\psi(\cdot)$. We shall consider the consequences of this characterization after its proof.

Theorem 1. *Suppose* (11) *holds. For* $\hat{\mathbf{F}} \in \mathcal{D}$, *set* $\hat{\mathbf{B}} \equiv \hat{\mathbf{F}}\hat{\mathbf{F}}^T$, $\hat{\mathrm{I}} \equiv \mathrm{tr}(\hat{\mathbf{B}}) = |\hat{\mathbf{F}}|^2$, *and let* $\hat{\beta}_{max}$ *denote the maximum eigenvalue of* $\hat{\mathbf{B}}$. *Then, the stored energy function* $W(\cdot) : \mathcal{D} \to I\!R$ *is rank 1 convex at* $\hat{\mathbf{F}} \in \mathcal{D}$ *if and only if*

$$\psi'(\hat{\mathrm{I}}) \geq 0, \qquad (14)_1$$

and

$$\psi(\hat{\mathrm{I}}) + 2\hat{\beta}_{max}\left(\sqrt{1 + (\mathrm{I} - \hat{\mathrm{I}})/\hat{\beta}_{max}} - 1\right)\psi'(\hat{\mathrm{I}}) \leq \psi(\mathrm{I}) \qquad (14)_2$$

for all $\mathrm{I} \in \mathcal{I}$ *such that* $\mathrm{I} \geq \hat{\mathrm{I}} - \hat{\beta}_{max}$.

Proof. First, note that (8) may be rewritten as

$$W(\hat{\mathbf{F}}) + (\mathbf{a} \otimes \mathbf{b}) \cdot W_{\mathbf{F}}(\hat{\mathbf{F}}) \leq W(\hat{\mathbf{F}} + \mathbf{a} \otimes \mathbf{b}) \qquad (15)$$

for all $\mathbf{a}$ and $\mathbf{b}$ in $I\!R^3$ such that $\hat{\mathbf{F}} + \mathbf{a} \otimes \mathbf{b} \in \mathcal{D}$. Thus, since (11) yields

$$W_{\mathbf{F}} = 2\psi'\mathbf{F}, \qquad (16)$$

we may use this and $(12)_1$ to put (15) in the equivalent form

$$\psi(\hat{\mathrm{I}}) + 2\mathbf{a} \cdot \hat{\mathbf{F}}\mathbf{b}\psi'(\hat{\mathrm{I}}) \leq \psi(\hat{\mathrm{I}} + 2\mathbf{a} \cdot \hat{\mathbf{F}}\mathbf{b} + |\mathbf{a}|^2|\mathbf{b}|^2)$$

for all $\mathbf{a}$ and $\mathbf{b}$ in $I\!R^3$ such that $\hat{\mathrm{I}} + 2\mathbf{a} \cdot \hat{\mathbf{F}}\mathbf{b} + |\mathbf{a}|^2|\mathbf{b}|^2 \in \mathcal{I}$. It is convenient to introduce

$$\mathbf{c} \equiv \hat{\mathbf{F}}^{-T}\mathbf{b}$$

and write this last inequality as

$$\psi(\hat{\mathrm{I}}) + 2\mathbf{a} \cdot \hat{\mathbf{B}}\mathbf{c}\psi'(\hat{\mathrm{I}}) \leq \psi(\hat{\mathrm{I}} + 2\mathbf{a} \cdot \hat{\mathbf{B}}\mathbf{c} + |\mathbf{a}|^2\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c})$$

for all $\mathbf{a}$ and $\mathbf{c}$ in $I\!R^3$ such that $\hat{\mathrm{I}} + 2\mathbf{a} \cdot \hat{\mathbf{B}}\mathbf{c} + |\mathbf{a}|^2\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c} \in \mathcal{I}$.

Now, since $\hat{\mathbf{B}}$ is positive definite, there is no loss of generality in expressing $\mathbf{a}$ in the form

$$\mathbf{a} = \lambda\frac{\hat{\mathbf{B}}\mathbf{c}}{|\hat{\mathbf{B}}\mathbf{c}|} + \mathbf{p}, \quad \lambda \in I\!R,\ \mathbf{p} \in I\!R^3,$$

where

$$\mathbf{p} \cdot \hat{\mathbf{B}}\mathbf{c} = 0.$$

This means that we may express the inequality (15) in the equivalent form

$$\psi(\hat{\mathrm{I}}) + 2\lambda|\hat{\mathbf{B}}\mathbf{c}|\psi'(\hat{\mathrm{I}}) \leq \psi(\hat{\mathrm{I}} + 2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2 + |\mathbf{p}|^2)\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}) \tag{17}$$

for all $\lambda \in I\!R$ and all $\mathbf{p}$ and $\mathbf{c}$ in $I\!R^3$ such that

$$\mathbf{p}\cdot\hat{\mathbf{B}}\mathbf{c} = 0 \ \text{ and } \ \hat{\mathrm{I}} + 2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2 + |\mathbf{p}|^2)\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c} \in \mathcal{I}.$$

The choice $\lambda = 0$ in (17) lets us write that

$$\psi(\hat{\mathrm{I}}) \leq \psi(\hat{\mathrm{I}} + |\mathbf{p}|^2\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c})$$

for all vectors $\mathbf{c}$ and $\mathbf{p}$ such that $\mathbf{p}\cdot\hat{\mathbf{B}}\mathbf{c} = 0$ and $\hat{\mathrm{I}} + |\mathbf{p}|^2\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c} \in \mathcal{I}$. We see at once that the graph of $\psi(\cdot)$ must be such that, to the right of $\hat{\mathrm{I}}$, $\psi(\cdot)$ must never fall below $\psi(\hat{\mathrm{I}})$. In particular then

$$\psi'(\hat{\mathrm{I}}) \geq 0,$$

which confirms that $(14)_1$ is a necessary consequence of the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}} \in \mathcal{D}$.

Now return to the general inequality (17), set $\mathbf{p} = 0$, and, for the moment, view $\mathbf{c}$ as fixed. For $\lambda \in I\!R$, the range of the quadratic expression

$$\mathrm{I}(\lambda) \equiv \lambda^2\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c} + 2\lambda|\hat{\mathbf{B}}\mathbf{c}| + \hat{\mathrm{I}}$$

is the interval

$$\hat{\mathcal{I}}(\mathbf{c}) \equiv \left[\hat{\mathrm{I}} - \frac{|\hat{\mathbf{B}}\mathbf{c}|^2}{\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}}, \infty\right).$$

Select any number $\mathrm{I} \in \hat{\mathcal{I}}(\mathbf{c}) \cap \mathcal{I}$, and solve for the two roots of $\mathrm{I}(\lambda) = \mathrm{I}$. Entering the largest[¶] of these two roots into (17), we arrive at

$$\psi(\hat{\mathrm{I}}) + 2\frac{|\hat{\mathbf{B}}\mathbf{c}|^2}{\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}}\left\{\sqrt{1 + \frac{\hat{\mathbf{B}}\mathbf{c}\cdot\mathbf{c}}{|\hat{\mathbf{B}}\mathbf{c}|^2}(\mathrm{I} - \hat{\mathrm{I}})} - 1\right\}\psi'(\hat{\mathrm{I}}) \leq \psi(\mathrm{I}) \tag{18}$$

[¶]Since $\psi'(\hat{\mathrm{I}}) \geq 0$, the smallest root of $\mathrm{I}(\lambda) = \mathrm{I}$ is of no interest to us: except in the case $\mathrm{I} = \hat{\mathrm{I}}$, it entered into (17) will always give a weaker inequality than (18).

for all $\mathbf{c}$ in $\mathbb{R}^3$ and all $\mathrm{I} \in \hat{\mathcal{I}}(\mathbf{c}) \cap \mathcal{I}$.

To show the necessity of $(14)_2$ and so complete the "only if" portion of the theorem now requires only two observations. First, we note that the interval $[\hat{\mathrm{I}} - x, \infty)$ is monotone increasing in x as is also the expression

$$x\left(\sqrt{1 + \mathrm{H}/x} - 1\right),$$

for all values of H for which it is defined. Second, we observe that the function $x = x(\mathbf{c}) \equiv |\hat{\mathbf{B}}\mathbf{c}|^2/\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c}, \quad \mathbf{c} \neq 0$, meets

$$\hat{\beta}_{min} \leq x(\mathbf{c}) \leq \hat{\beta}_{max},$$

where $\hat{\beta}_{min}$ and $\hat{\beta}_{max}$ are, respectively, the smallest and the largest eigenvalues of $\hat{\mathbf{B}}$, and where equality holds whenever $\mathbf{c}$ is parallel to the appropriate eigenvector of $\hat{\mathbf{B}}$. Because $\psi'(\hat{\mathrm{I}}) \geq 0$, our first observation means that, of the family of inequalities (18), the optimal inequality occurs when we take $\mathbf{c}$ so that $|\hat{\mathbf{B}}\mathbf{c}|^2/\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c}$ is as large as possible. Here, by the optimal of the inequalities (18), we mean the one that allows both the largest range of I values to be entered into it and which, for any given value of I, has the largest possible left hand side. Our second observation means that this optimal inequality is just

$$\psi(\hat{\mathrm{I}}) + 2\hat{\beta}_{max}\left(\sqrt{1 + (\mathrm{I} - \hat{\mathrm{I}})/\hat{\beta}_{max}} - 1\right)\psi'(\hat{\mathrm{I}}) \leq \psi(\mathrm{I}),$$

which holds for all $\mathrm{I} \in [\hat{\mathrm{I}} - \hat{\beta}_{max}, \infty) \cap \mathcal{I}$ and is exactly $(14)_2$.

To show that $(14)_1$ and $(14)_2$ are together sufficient for the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}} \in \mathcal{D}$, we need only establish that (17) holds. To do this, let $\lambda \in \mathbb{R}$ and $\mathbf{p}$ and $\mathbf{c}$ in $\mathbb{R}^3$ be given such that

$$\mathbf{p} \cdot \hat{\mathbf{B}}\mathbf{c} = 0 \quad \text{and} \quad \mathrm{I} \equiv \hat{\mathrm{I}} + 2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2 + |\mathbf{p}|^2)\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c} \in \mathcal{I}.$$

Since this means that $\mathrm{I} = \mathrm{I}(\lambda) + |\mathbf{p}|^2\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c}$, it follows that

$$\mathrm{I} \geq \hat{\mathrm{I}} - \frac{|\hat{\mathbf{B}}\mathbf{c}|^2}{\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c}} + |\mathbf{p}|^2\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c} \geq \mathrm{I} - \hat{\beta}_{max} + |\mathbf{p}|^2\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c} \geq \mathrm{I} - \hat{\beta}_{max},$$

and, as a result, we may enter I into $(14)_2$ to conclude that

$$\psi(\hat{\mathrm{I}}) + 2\hat{\beta}_{max}\left(\sqrt{1 + \frac{2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2 + |\mathbf{p}|^2)\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c}}{\hat{\beta}_{max}}} - 1\right)\psi'(\hat{\mathrm{I}}) \leq \psi(\hat{\mathrm{I}} + 2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2 + |\mathbf{p}|^2)\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c}). \tag{19}$$

Now, as we have seen, $|\hat{\mathbf{B}}\mathbf{c}|^2/\mathbf{c} \cdot \hat{\mathbf{B}}\mathbf{c} \leq \hat{\beta}_{max}$. It follows then that

$$\lambda^2 \frac{|\hat{\mathbf{B}}\mathbf{c}|^2}{\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}} \frac{1}{\hat{\beta}_{max}} \le \lambda^2 \iff \lambda^2 \frac{|\hat{\mathbf{B}}\mathbf{c}|^2}{\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}} \frac{1}{\hat{\beta}_{max}} \le \lambda^2 + |\mathbf{p}|^2,$$

$$\iff \left(\frac{\lambda|\hat{\mathbf{B}}\mathbf{c}|}{\hat{\beta}_{max}}\right)^2 \le \frac{(\lambda^2+|\mathbf{p}|^2)\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}}{\hat{\beta}_{max}},$$

$$\iff 1 + \left(\frac{\lambda|\hat{\mathbf{B}}\mathbf{c}|}{\hat{\beta}_{max}}\right)^2 + \frac{2\lambda|\hat{\mathbf{B}}\mathbf{c}|}{\hat{\beta}_{max}} \le 1 + \frac{2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2+|\mathbf{p}|^2)\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}}{\hat{\beta}_{max}},$$

and therefore

$$1 + \frac{\lambda|\hat{\mathbf{B}}\mathbf{c}|}{\hat{\beta}_{max}} \le \sqrt{1 + \frac{2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2+|\mathbf{p}|^2)\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}}{\hat{\beta}_{max}}}.$$

As a result

$$\lambda|\hat{\mathbf{B}}\mathbf{c}| \le \hat{\beta}_{max}\left\{\sqrt{1 + \frac{2\lambda|\hat{\mathbf{B}}\mathbf{c}| + (\lambda^2+|\mathbf{p}|^2)\mathbf{c}\cdot\hat{\mathbf{B}}\mathbf{c}}{\hat{\beta}_{max}}} - 1\right\},$$

which, with (19) and $(14)_1$, lets us reproduce (17). ∎

Theorem 1 shows that $W(\cdot)$ is rank 1 convex at $\hat{\mathbf{F}} \in \mathcal{D}$ if and only if both

$$\psi'(\hat{\mathrm{I}}) \ge 0$$

and, see Figure 1, $\psi(\cdot)$ is bounded below on the interval $[\hat{\mathrm{I}} - \hat{\beta}_{max}, \infty) \cap \mathcal{I}$ by the parabolic curve given by

$$f(\mathrm{I}) \equiv \psi(\hat{\mathrm{I}}) + 2\hat{\beta}_{max}\left(\sqrt{1 + (\mathrm{I} - \hat{\mathrm{I}})/\hat{\beta}_{max}} - 1\right)\psi'(\hat{\mathrm{I}}). \tag{20}$$

As Figure 1 indicates, if $\psi'(\hat{\mathrm{I}}) > 0$, then $f(\cdot)$ is monotone increasing and concave with $f'(\hat{\mathrm{I}} - \hat{\beta}_{max}) = \infty$ and $f'(\infty) = 0$. Moreover, since the difference $f(\cdot) - \psi(\cdot)$ is non-positive on $[\hat{\mathrm{I}} - \hat{\beta}_{max}, \infty) \cap \mathcal{I}$ and vanishes at $\hat{\mathrm{I}}$, it is straightforward to show that $(14)_2$ implies that

$$\psi'(\hat{\mathrm{I}}) + 2\hat{\beta}_{max}\psi''(\hat{\mathrm{I}}) \ge 0,$$

which, together with $\psi'(\hat{\mathrm{I}}) \ge 0$, should be compared to the previously noted implications of the S–E inequality.

Now, in fact, for the materials of (11), the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}}$ necessitates the rank 1 convexity of $W(\cdot)$ on a much larger set of deformations. Specifically, the fact that the interval $[\hat{\mathrm{I}} - x, \infty)$ and the expression

$$x\left(\sqrt{1 + \mathrm{H}/x} - 1\right)$$

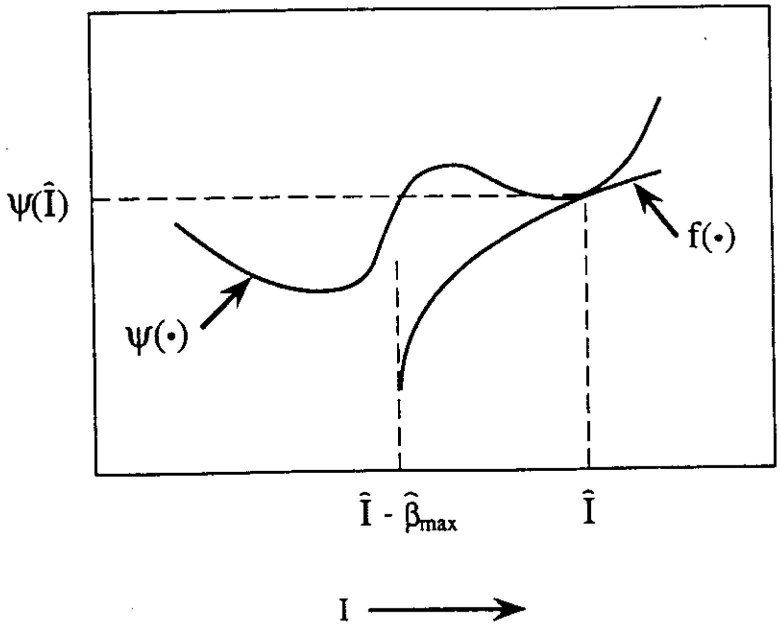

Figure 1: Illustration of the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}}$ in terms of the function $\psi(\cdot)$ of (11). The minorizing parabolic function $f(\cdot)$ is defined in (20).

are increasing functions of x allows us to establish the following

Corollary. *For the materials of* (11), *the stored energy function* $W(\cdot)$ *is rank 1 convex at* $\hat{\mathbf{F}} \in \mathcal{D}$ *if and only if* $W(\cdot)$ *is rank 1 convex at every* $\mathbf{F} \in \mathcal{D}$ that meets

$$|\mathbf{F}| = |\hat{\mathbf{F}}|,$$

and

$$\beta_{max} \equiv \text{the maximum eigenvalue of } \mathbf{F}\mathbf{F}^T \leq \hat{\beta}_{max}.$$

Proof. The "if" part of the corollary is clear. To prove the "only if" portion, let $\mathbf{F} \in \mathcal{D}$ be as described and note that $\mathrm{I} \in [\hat{\mathrm{I}} - \beta_{max}, \infty) \cap \mathcal{I} \Rightarrow \mathrm{I} \in [\hat{\mathrm{I}} - \hat{\beta}_{max}, \infty) \cap \mathcal{I}$. This means that we can apply $(14)_2$ to $\mathrm{I} \in [\hat{\mathrm{I}} - \beta_{max}, \infty) \cap \mathcal{I}$. Moreover, since $\beta_{max} \leq \hat{\beta}_{max}$,

$$\beta_{max}\left(\sqrt{1 + (\mathrm{I} - \hat{\mathrm{I}})/\beta_{max}} - 1\right) \leq \hat{\beta}_{max}\left(\sqrt{1 + (\mathrm{I} - \hat{\mathrm{I}})/\hat{\beta}_{max}} - 1\right).$$

This, with $(14)_2$ and $\psi'(\hat{\mathrm{I}}) \geq 0$, allows us to write

$$\psi(\hat{\mathrm{I}}) + 2\beta_{max}\left(\sqrt{1 + (\mathrm{I} - \hat{\mathrm{I}})/\beta_{max}} - 1\right)\psi'(\hat{\mathrm{I}}) \leq \psi(\mathrm{I}),$$

for all $\mathrm{I} \in [\hat{\mathrm{I}} - \beta_{max}, \infty) \cap \mathcal{I}$. Clearly then $W(\cdot)$ is rank 1 convex at $\mathbf{F}$. ■

The above Corollary involves no hypothesis on either the structure of the domain $\mathcal{D}$ of the stored energy or the density of points where $W(\cdot)$ is rank 1 convex. This contrasts sharply with our next result which requires not only a strong assumption about the boundary of $\mathcal{D}$ but also rests on our assuming the existence of an appropriate *sequence* $\{\hat{\mathbf{F}}_n\}$ in $\mathcal{D}$ at each element of which $W(\cdot)$ is rank 1 convex. To motivate these hypotheses, we note that if we introduce $\omega(\cdot) : \mathcal{J} \to I\!R$ by setting

$$\mathcal{J} \equiv \{z | z^2 \in \mathcal{I}\} \quad \text{and} \quad \omega(z) = \psi(z^2), \tag{21}$$

then, with $\hat{\mathrm{J}} \equiv \hat{\mathrm{I}}^{1/2}$, we can write $(14)_1$ as $\omega'(\hat{\mathrm{J}}) \geq 0$ and $(14)_2$ as

$$\omega(\hat{\mathrm{J}}) + \frac{\hat{\beta}_{max}}{\hat{\mathrm{J}}}\left(\sqrt{1 + (\mathrm{J}^2 - \hat{\mathrm{J}}^2)/\hat{\beta}_{max}} - 1\right)\omega'(\hat{\mathrm{J}}) \leq \omega(\mathrm{J}), \tag{22}$$

for all $\mathrm{J} \in \mathcal{J}$ with $\mathrm{J}^2 \geq \hat{\mathrm{J}}^2 - \hat{\beta}_{max}$, and where, of course, $\hat{\beta}_{max} < \hat{\mathrm{J}}^2$. Exploiting once again the fact that the interval $[\hat{\mathrm{J}}^2 - x, \infty)$ and the expression $x\left(\sqrt{1 + \mathrm{H}/x} - 1\right)$ are increasing functions of x, and also exploiting the fact that $\omega'(\hat{\mathrm{J}}) \geq 0$, we see that *if* we could let $\hat{\beta}_{max} \nearrow \hat{\mathrm{J}}^2$ in (22) then $(14)_2$ could be replaced by the stronger condition that

$$\omega(\hat{\mathrm{J}}) + (\mathrm{J} - \hat{\mathrm{J}})\omega'(\hat{\mathrm{J}}) \leq \omega(\mathrm{J}), \tag{23}$$

for all $\mathrm{J} \in \mathcal{J}$, *i.e.*, we would have shown that $\hat{\mathrm{J}}$ *is a point of convexity for* $\omega(\cdot)$, or, in words, that $\omega(\cdot)$ never falls below its tangent line through $(\hat{\mathrm{J}}, \omega(\hat{\mathrm{J}}))$.

The following theorem merely formalizes and slightly extends the above discussion.

Theorem 2. *For the materials of* (11), *suppose that* $W(\cdot) : \mathcal{D} \to I\!R$ *is rank 1 convex at each element of a sequence* $\{\hat{\mathbf{F}}_n\}$ *of deformations that meet*

$$|\hat{\mathbf{F}}_n|^2 = |\hat{\mathbf{F}}_1|^2 \equiv \hat{\mathrm{I}}, \quad \text{and} \quad \lim_{n \to \infty} \hat{\beta}(n)_{max} = \hat{\mathrm{I}},$$

where $\hat{\beta}(n)_{max}$ *is the maximum eigenvalue of* $\hat{\mathbf{F}}_n \hat{\mathbf{F}}_n^T$. *Let* $\hat{\mathrm{J}} \equiv \hat{\mathrm{I}}^{1/2}$, *then the function* $\omega(\cdot) : \mathcal{J} \to I\!R$ *of* (21) *has* $\hat{\mathrm{J}}$ *as a point of convexity, and, moreover,* $\omega'(\hat{\mathrm{J}}) \geq 0$.

Conversely, if $\hat{\mathrm{J}} \in \mathcal{J}$ *is a point of convexity for* $\omega(\cdot)$ with $\omega'(\hat{\mathrm{J}}) \geq 0$, *then* $W(\cdot) \equiv \omega(|\cdot|)$ *is rank 1 convex at every deformation* $\hat{\mathbf{F}} \in \mathcal{D}$ *that meets*

$$|\hat{\mathbf{F}}| = \hat{\mathrm{J}}.$$

Proof. Theorem 1 and our hypotheses let us write that at $\hat{\mathrm{J}}$ both $\omega'(\hat{\mathrm{J}}) \geq 0$ and (22) holds for each value of $\hat{\beta}(n)_{max}$. To establish the first half of the theorem, we now need only let $n \to \infty$.

To prove the second portion of the theorem, we select any $\hat{\mathbf{F}} \in \mathcal{D}$ with $|\hat{\mathbf{F}}| = \hat{\mathrm{J}}$. We then compute $\hat{\beta}_{max}$, the maximum eigenvalue of $\hat{\mathbf{F}}\hat{\mathbf{F}}^T$. Since $\hat{\beta}_{max} < \hat{\mathrm{J}}^2$, it is not hard to show that

$$\frac{\hat{\beta}_{max}}{\hat{\mathrm{J}}}\left(\sqrt{1 + (\mathrm{J}^2 - \hat{\mathrm{J}}^2)/\hat{\beta}_{max}} - 1\right) < \mathrm{J} - \hat{\mathrm{J}}$$

for any $\mathrm{J} \in \mathcal{J}$ with $\mathrm{J}^2 \geq \hat{\mathrm{J}}^2 - \hat{\beta}_{max}$. Since $\omega'(\hat{\mathrm{J}}) \geq 0$, it is now straightforward to go from (23) to (22), and, as we have seen, this is exactly $(14)_2$. ■

Now as remarked, the hypotheses of the first half of Theorem 2 are quite stong. Not only do we require a sequence of deformations at each element of which $W(\cdot)$ must be rank 1 convex, we also require that this sequence meet

$$\lim_{n\to\infty} \hat{\beta}(n)_{max} = \hat{\mathrm{I}}\,(= \hat{\beta}(n)_{min} + \hat{\beta}(n)_{mid} + \hat{\beta}(n)_{max}).$$

Clearly this is possible only if the minimum and intermediate eigenvalues of $\hat{\mathbf{F}}_n\hat{\mathbf{F}}_n^T$ both go to zero as n tends to infinity, *i.e., the sequence* $\{\hat{\mathbf{F}}_n\}$ *must necessarily be such that* $\lim_{n\to\infty}\det(\hat{\mathbf{F}}_n) = 0$. Thus, $\{\hat{\mathbf{F}}_n\}$ must necessarily approach the boundary of the set of invertible tensors, and so there is a tacit assumption here that the subdomain $\mathcal{D}^{r1} \subseteq \mathcal{D}$ be large enough to permit this.

The hypothesis of the second part of Theorem 2, that $\hat{\mathrm{J}}$ be a point of convexity for $\omega(\cdot)$, is also rather strong. Indeed, we have that $W(\cdot)$ is not just rank 1 convex at all $\hat{\mathbf{F}} \in \mathcal{D}$ that meets $|\hat{\mathbf{F}}| = \hat{\mathrm{J}}$, but also

Theorem 3. *If* $\hat{\mathrm{J}} \in \mathcal{J}$ *is a point of convexity for* $\omega(\cdot)$ *with* $\omega'(\hat{\mathrm{J}}) \geq 0$, *then every deformation* $\hat{\mathbf{F}} \in \mathcal{D}$ *that meets* $|\hat{\mathbf{F}}| = \hat{\mathrm{J}}$ *is a point of convexity for* $W(\cdot)$, *i.e., for each* $\hat{\mathbf{F}} \in \mathcal{D}$ *with* $|\hat{\mathbf{F}}| = \hat{\mathrm{J}}$

$$W(\hat{\mathbf{F}}) + (\mathbf{F} - \hat{\mathbf{F}}) \cdot W_{\mathbf{F}}(\hat{\mathbf{F}}) \leq W(\mathbf{F})$$

for all $\mathbf{F} \in \mathcal{D}$.

Conversely, let $\mathcal{D}_{\mathrm{J}} \equiv \{\mathbf{F} \in \mathcal{D} \mid |\mathbf{F}| = \mathrm{J}\}$ *and suppose that* $\hat{\mathrm{J}} \in \mathcal{J}$ *is such that the set*

$$\mathcal{D}^c_{\hat{\mathrm{J}}} \equiv \{\mathbf{F} \in \mathcal{D}_{\hat{\mathrm{J}}} \mid \mathbf{F} \text{ is a point of convexity for } W(\cdot)\}$$

is non-empty and radially generates some portion of every non-empty level set $\mathcal{D}_{\mathrm{J}}$ *in the sense that*

$$\mathcal{D}_{\mathrm{J}} \cap \left(\frac{\mathrm{J}}{\hat{\mathrm{J}}}\mathcal{D}^c_{\hat{\mathrm{J}}}\right) \neq \emptyset$$

for $\mathrm{J} \in \mathcal{J}$. *Then,* $\hat{\mathrm{J}}$ *is a point of convexity for* $\omega(\cdot)$, $\omega'(\hat{\mathrm{J}}) \geq 0$, *and, a fortiori,* $\mathcal{D}^c_{\hat{\mathrm{J}}} = \mathcal{D}_{\hat{\mathrm{J}}}$.

Recall that a set K in a linear space is said to be a *cone* (with vertex at 0) if $x \in K \Rightarrow \lambda x \in K$ for all $\lambda > 0$. In the case that the domain $\mathcal{D}$ is a cone it is easy to see that $\mathcal{D}^c_{\mathrm{J}}$ non-empty will automatically radially generate a portion of every level set $\mathcal{D}_{\mathrm{J}}$.

Proof. With $\hat{\mathbf{F}} \in \mathcal{D}$ such that $|\hat{\mathbf{F}}| = \hat{\mathrm{J}}$, we may use (21) and (11) to express the inequality (23) in the form

$$(\mathrm{J} - \hat{\mathrm{J}})\omega'(\hat{\mathrm{J}}) \leq W(\mathbf{F}) - W(\hat{\mathbf{F}})$$

for all $\mathbf{F} \in \mathcal{D}$, where $\mathrm{J} = |\mathbf{F}|$. But, by (16), the Cauchy-Schwarz inequality, the fact that $\psi'(\hat{\mathrm{I}}) \geq 0$, and (21)

$$\begin{aligned}
(\mathbf{F} - \hat{\mathbf{F}}) \cdot W_{\mathbf{F}}(\hat{\mathbf{F}}) &= 2(\mathbf{F} - \hat{\mathbf{F}}) \cdot \hat{\mathbf{F}}\psi'(\hat{\mathrm{I}}), \\
&= 2(\mathbf{F} \cdot \hat{\mathbf{F}} - |\hat{\mathbf{F}}|^2)\psi'(\hat{\mathrm{I}}), \\
&\leq 2(|\mathbf{F}||\hat{\mathbf{F}}| - |\hat{\mathbf{F}}|^2)\psi'(\hat{\mathrm{I}}), \\
&= (|\mathbf{F}| - |\hat{\mathbf{F}}|)\omega'(\hat{\mathrm{J}}).
\end{aligned}$$

Thus, as claimed, $\hat{\mathbf{F}}$ is point of convexity for $W(\cdot)$.

To prove the second half of the theorem, select any $\hat{\mathbf{F}} \in \mathcal{D}^c_{\hat{\mathrm{J}}}$. For any $\mathbf{F} \in \mathcal{D}$ we have then that

$$\begin{aligned}
\left(\frac{\mathbf{F} \cdot \hat{\mathbf{F}}}{|\hat{\mathbf{F}}|} - |\hat{\mathbf{F}}|\right)\omega'(|\hat{\mathbf{F}}|) &= (\mathbf{F} - \hat{\mathbf{F}}) \cdot \frac{\hat{\mathbf{F}}}{|\hat{\mathbf{F}}|}\omega'(|\hat{\mathbf{F}}|), \\
&= (\mathbf{F} - \hat{\mathbf{F}}) \cdot W(\hat{\mathbf{F}})_{\mathbf{F}}, \\
&\leq W(\mathbf{F}) - W(\hat{\mathbf{F}}), \\
&= \omega(|\mathbf{F}|) - \omega(|\hat{\mathbf{F}}|).
\end{aligned}$$

To complete the proof, we let $\mathrm{J} \in \mathcal{J}$ be given and select $\mathbf{A} \in \mathcal{D}_{\mathrm{J}} \cap \left(\frac{\mathrm{J}}{\hat{\mathrm{J}}}\mathcal{D}^c_{\hat{\mathrm{J}}}\right)$. Then $\mathbf{F} \equiv \mathbf{A} \in \mathcal{D}_{\mathrm{J}}$, $\hat{\mathbf{F}} \equiv \frac{\hat{\mathrm{J}}}{\mathrm{J}}\mathbf{A} \in \mathcal{D}^c_{\hat{\mathrm{J}}}$, and we may enter these choices for $\mathbf{F}$ and $\hat{\mathbf{F}}$ into the above inequality to reach exactly the condition (23) for $\hat{\mathrm{J}}$ to be a point of convexity for $\omega(\cdot)$.

Next, we use the above inequality in another way by selecting $\mathbf{F} \in \mathcal{D}_{\hat{\mathrm{J}}}$ but $\mathbf{F} \neq \hat{\mathbf{F}}$. Since $\mathcal{D}$ is open, we can always do this, and application of our inequality gives

$$(\mathbf{F} \cdot \hat{\mathbf{F}} - |\mathbf{F}||\hat{\mathbf{F}}|)\omega'(|\hat{\mathbf{F}}|) \leq 0.$$

Since the Cauchy-Schwarz inequality requires here that $\mathbf{F}\cdot\hat{\mathbf{F}} < |\mathbf{F}||\hat{\mathbf{F}}|$, we see that as claimed $\omega'(|\hat{\mathbf{F}}|) \geq 0$. Finally, from the first half of the theorem it now follows that $\mathcal{D}^c_{\mathrm{J}} = \mathcal{D}_{\mathrm{J}}$. ■

Let $\mathcal{D}^c$ denote the set of points in $\mathcal{D}$ which are points of convexity for $W(\cdot)$. As is well-known, if $\mathcal{D}^c$ is large enough then it is easy to show that the necessary condition (3) of Euler, *along with* $\nabla\hat{\mathbf{y}}(\Omega) \subseteq \mathcal{D}^c$, in fact provides *sufficient* conditions for the field $\hat{\mathbf{y}}(\cdot)$ to minimize

$$\int_\Omega W(\nabla\mathbf{y})\,dv$$

among all smooth competing fields that satisfy the boundary condition (2). Indeed, let $\mathbf{y}(\cdot)$ and $\hat{\mathbf{y}}(\cdot)$ both be smooth with $\nabla\hat{\mathbf{y}}(\mathbf{x}) \in \mathcal{D}^c$ and $\operatorname{div} W_{\mathbf{F}}(\nabla\hat{\mathbf{y}}(\mathbf{x})) = 0$. Then, from

$$W(\nabla\hat{\mathbf{y}}(\mathbf{x})) + \{\nabla\mathbf{y}(\mathbf{x}) - \nabla\hat{\mathbf{y}}(\mathbf{x})\}\cdot W_{\mathbf{F}}(\nabla\hat{\mathbf{y}}(\mathbf{x})) \leq W(\nabla\mathbf{y}(\mathbf{x})), \tag{24$_1$}$$

we find that

$$W(\nabla\hat{\mathbf{y}}(\mathbf{x})) + \operatorname{div}\left(W^T_{\mathbf{F}}(\nabla\hat{\mathbf{y}}(\mathbf{x}))\{\mathbf{y}(\mathbf{x}) - \hat{\mathbf{y}}(\mathbf{x})\}\right) \leq W(\nabla\mathbf{y}(\mathbf{x})). \tag{24$_2$}$$

Integrating this last over Ω and using the fact that $\mathbf{y}(\cdot)$ and $\hat{\mathbf{y}}(\cdot)$ both meet (2), we easily prove that

$$\int_\Omega W(\nabla\hat{\mathbf{y}})\,dv \leq \int_\Omega W(\nabla\mathbf{y})\,dv, \tag{24$_3$}$$

i.e., as claimed, the field $\hat{\mathbf{y}}(\cdot)$ is minimizing. While this result is interesting, it does not seem particularly relevant to finite elasticity since, in place of the strong condition[‖] $\nabla\hat{\mathbf{y}}(\Omega) \subseteq \mathcal{D}^c$, a minimizing field $\hat{\mathbf{y}}(\cdot)$ need only meet the Weierstrass condition (4), *i.e.*, need only meet $\nabla\hat{\mathbf{y}}(\Omega) \subseteq \mathcal{D}^{r1}$. Unfortunately, although the milder condition of rank 1 convexity is physically more reasonable than outright convexity, it seems clear that the necessary and sufficient conditions for rank 1 convexity given in $(14)_{1,2}$ allow for no easy parallel to the steps outlined above in $(24)_{1,2,3}$ that enabled us to show that $\hat{\mathbf{y}}(\cdot)$ was minimizing.

4. A FINAL REMARK

In closing it is useful to compare our results with those presented by Dacorogna [2]. In particular, in part (iii) of his Theorem 1.10 on page 133, Dacorogna claims (among other things) that

$$W(\cdot)\ \text{convex} \Leftrightarrow W(\cdot)\ \text{rank 1 convex} \Leftrightarrow \omega(\cdot)\ \text{convex and}\ \ \omega(0) = \inf\{\omega(\mathrm{J}) \mid \mathrm{J} > 0\}. \quad (\star)$$

‖The observation that to require $W(\cdot)$ to be (strictly) convex on $\mathcal{D}$ is to require too much for applications in finite elasticity goes back to Hill [4]. See also the discussion in Sections 52 and 87 of Truesdell and Noll [3].

This of course is rather different than what we have proven in our Theorems 1, 2, and 3, and, at first glance, might appear to be much sharper than our results. In fact, this is not so, and on inspection one discovers that Dacorogna's apparently stronger results trace to hypotheses far stronger than ours. In particular, to use our notation, Dacorogna both assumes that $\mathcal{D}$ is the set of *all* second order tensors on $\mathbb{R}^3$, invertible or not, and that $\mathcal{D}^{r1} = \mathcal{D}^{**}$. The first of these hypotheses is clearly inappropriate for finite elasticity, and the second, even when $\mathcal{D}$ is reasonable, rules out much rich mathematics and physics. For these reasons, in spite of common language it seems to share with our analyses, Dacorogna's assertion ($\star$) does not seem to us to bear with much force on issues in nonlinear elasticity.

**See especially the proof Dacorogna gives of his (iii) on page 138. There the assumption that $\mathcal{D}^{r1}$ is large enough to intersect the boundary of the set of invertible tensors is used in an essential way.

APPENDIX

Some workers in the field of finite elasticity would have it that the domain $\mathcal{D}$ of $W(\cdot)$ be the union, $\mathcal{D}^+ \cup \mathcal{D}^-$, of two disjoint components, the elements of $\mathcal{D}^+$ all having positive determinant, the elements of $\mathcal{D}^-$ all possessing a negative determinant. In this case, it is common to see either the condition (1) for objectivity or the condition (9) for isotropy or both of these conditions broadened to include improper orthogonal tensors. While such an approach is easily accomodated within our analysis, it does require that we take some care in formulating the minimization problem (2) for elasticity theory. This will then lead to a more restrictive formulation of the Weierstrass condition (4) and a more refined use of the definition (8) of rank 1 convexity.

To see what is at stake, assume that $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$, $\mathcal{D}^+ \neq \emptyset \neq \mathcal{D}^-$, and that (1) is required to hold for all orthogonal $\mathbf{Q}$, proper or not. As a consequence, we see straightway that we must have $\mathcal{D}^- = (-1)\mathcal{D}^+$, and $\mathbf{Q}\mathcal{D}^+ \subseteq \mathcal{D}^+$ for all proper orthogonal $\mathbf{Q}$. But now consider (8), the condition for the rank 1 convexity of $W(\cdot)$ over $\mathcal{S}$ at $\hat{\mathbf{F}}$. Since the tensor $\mathbf{F} = \hat{\mathbf{F}} - 2\mathbf{e} \otimes \hat{\mathbf{F}}^T\mathbf{e}$, where $\mathbf{e}$ is an arbitrary unit vector, clearly differs from $\hat{\mathbf{F}}$ by a rank 1 tensor, *if* we apply (8) we get

$$W(\hat{\mathbf{F}}) - 2\mathbf{e} \otimes \hat{\mathbf{F}}^T\mathbf{e} \cdot W_{\mathbf{F}}(\hat{\mathbf{F}})\hat{\mathbf{F}}^T \leq W(\hat{\mathbf{F}} - 2\mathbf{e} \otimes \hat{\mathbf{F}}^T\mathbf{e}).$$

But $\hat{\mathbf{F}} - 2\mathbf{e} \otimes \hat{\mathbf{F}}^T\mathbf{e} = (\mathbf{1} - 2\mathbf{e} \otimes \mathbf{e})\hat{\mathbf{F}}$, where $\mathbf{1} - 2\mathbf{e} \otimes \mathbf{e}$ is the (improper) orthogonal tensor corresponding to a reflection in the plane perpendicular to $\mathbf{e}$. Therefore, by (1), $W(\hat{\mathbf{F}}) = W(\hat{\mathbf{F}} - 2\mathbf{e} \otimes \hat{\mathbf{F}}^T\mathbf{e})$, and the above inequality takes the form

$$\mathbf{e} \otimes \mathbf{e} \cdot W_{\mathbf{F}}(\hat{\mathbf{F}})\hat{\mathbf{F}}^T \geq 0,$$

i.e., by (12), the principal stresses t_i at $\hat{\mathbf{F}}$, $i = 1, 2, 3$, must all be non-negative. Thus, in a general, hyperelastic material, the requirement of frame indifference forbids compressive states of stress from satisfying the condition of rank 1 convexity, *if this latter condition be applied across the components $\mathcal{D}^+$ and $\mathcal{D}^-$ of $\mathcal{D}$.* A restriction that forbids minimizers $\hat{\mathbf{y}}(\cdot)$ from having compressive states of stress associated with them is clearly too strong from the point of view of elasticity theory. To avoid it and still maintain (1) for both proper and improper orthogonal tensors requires that we amend the minimization problem (2) by requiring that, instead of $\nabla\mathbf{y}(\Omega) \subseteq \mathcal{D}$, the elements of $\Im$ all meet either $\nabla\mathbf{y}(\Omega) \subseteq \mathcal{D}^+$ or $\nabla\mathbf{y}(\Omega) \subseteq \mathcal{D}^-$. As a consequence of so limiting $\Im$, the Weierstrass condition (4) will now only be able to be established for comparison tensors $\mathbf{F}$ which are a rank 1 difference from $\nabla\hat{\mathbf{y}}(\mathbf{x})$ *and* lie in the *same* component of $\mathcal{D}$ as does $\nabla\hat{\mathbf{y}}(\mathbf{x})$. This in turn will mean that we must replace the requirement of the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}}$ over $\mathcal{D}$ by the slightly more subtle requirement of the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}}$ *over $\mathcal{D}^+$ or $\mathcal{D}^-$.*

In the same way, we see that if $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$, $\mathcal{D}^+ \neq \emptyset \neq \mathcal{D}^-$, and (9) holds for, say, even a single (improper) orthogonal tensor $\mathbf{R}$ of the form $\mathbf{1} - 2\mathbf{e} \otimes \mathbf{e}$, $\mathbf{e}$ a unit vector, then (8), with the choice $\mathbf{F} = \hat{\mathbf{F}} - 2\hat{\mathbf{F}}\mathbf{e} \otimes \mathbf{e}$, gives that

$$\mathbf{e} \otimes \mathbf{e} \cdot \hat{\mathbf{F}}^T W_{\mathbf{F}}(\hat{\mathbf{F}}) \geq 0.$$

Since $\hat{\mathbf{F}}^T W_{\mathbf{F}}(\hat{\mathbf{F}}) = |\det(\hat{\mathbf{F}})| \hat{\mathbf{F}}^T \mathbf{T}(\hat{\mathbf{F}}) \hat{\mathbf{F}}^{-T}$, we see that $\hat{\mathbf{F}}^T W_{\mathbf{F}}(\hat{\mathbf{F}})$ is equal to $|\det(\hat{\mathbf{F}})| \mathbf{T}(\hat{\mathbf{F}})$ up to a similarity transformation. As a result, $\hat{\mathbf{F}}^T W_{\mathbf{F}}(\hat{\mathbf{F}})$ and $|\det(\hat{\mathbf{F}})| \mathbf{T}(\hat{\mathbf{F}})$ have the same eigenvalues, and so the above inequality means that the maximum eigenvalue of $\mathbf{T}(\hat{\mathbf{F}})$ must be non-negative. Again we have reached far too strong a restriction, this time on all hyperelastic materials that have a symmetry group[††] containing the tensor $\mathbf{1} - 2\mathbf{e} \otimes \mathbf{e}$. To avoid it requires that we again amend the minimization problem (2): instead of asking only that $\nabla \mathbf{y}(\Omega) \subseteq \mathcal{D}$, we now must require that the elements of $\Im$ all meet either $\nabla \mathbf{y}(\Omega) \subseteq \mathcal{D}^+$ or $\nabla \mathbf{y}(\Omega) \subseteq \mathcal{D}^-$. As a consequence of so limiting $\Im$, the Weierstrass condition (4) will again now only be able to be established for comparison tensors $\mathbf{F}$ which are a rank 1 difference from $\nabla \hat{\mathbf{y}}(\mathbf{x})$ and which lie in the same component of $\mathcal{D}$ as does $\nabla \hat{\mathbf{y}}(\mathbf{x})$. Thus, again we will end up replacing the requirement of the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}}$ over $\mathcal{D}$ by the more subtle requirement of the rank 1 convexity of $W(\cdot)$ at $\hat{\mathbf{F}}$ *over* $\mathcal{D}^+$ *or* $\mathcal{D}^-$.

[††] See [3] for a detailed presentation of this idea.

ACKNOWLEDGEMENTS

We gratefully acknowledge the NSF and the U. S. Army for their partial support of this research under grants MSS-9024637 and DAAL03-91-G-0184, respectively.

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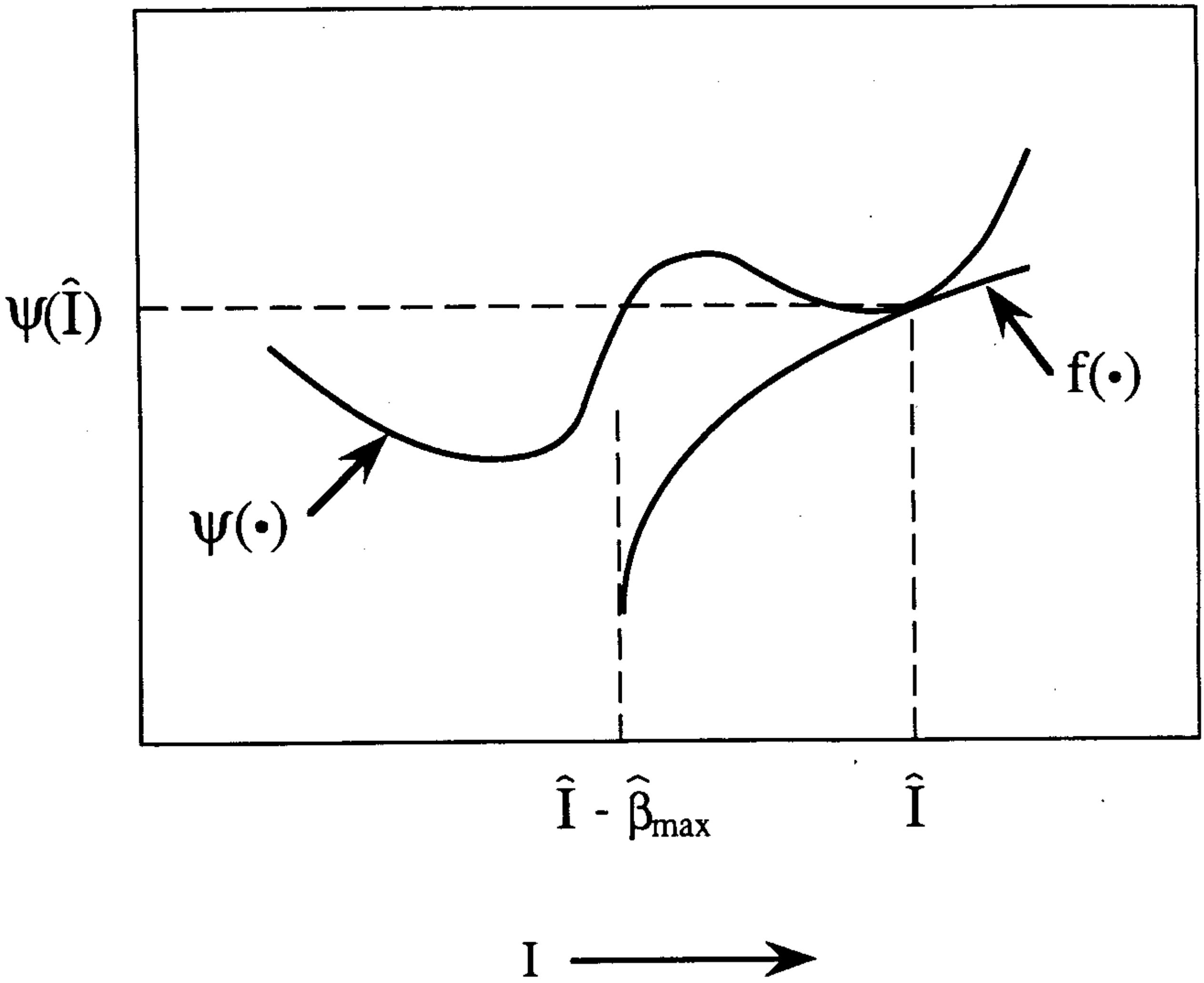

$\psi(\hat{I})$
$\psi(\bullet)$
$f(\bullet)$
$\hat{I} - \hat{\beta}_{max}$
$\hat{I}$
I