Structural Properties of Reciprocal Generalized Cluster Algebras

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Abstract

Ordinary cluster algebras were first introduced by Fomin and Zelevinsky in 2002 [24] in order to provide a concrete combinatorial framework for studying dual canonical bases and total positivity in semisimple groups. Ordinary cluster algebras have since found applications in a wide array of areas, including: the representation theory of quivers, algebraic geometry and mirror symmetry, discrete integrable systems, Poisson geometry, Teichmüller theory, and mathematical physics. This unexpected ubiquity has made ordinary cluster algebras a natural object of interest for many mathematicians. In particular, there has been great interest in understanding their structural properties.

One natural generalization of an ordinary cluster algebra is the *generalized cluster algebra*, introduced by Chekhov and Shapiro in 2013 [13]. In such algebras, the hallmark binomial exchange relations are replaced by polynomials of arbitrary degree. Given that there is a significant existing body of work about the structural properties of ordinary cluster algebras, it is natural to ask the same questions in the context of generalized cluster algebras. In particular, it is natural to ask if generalized cluster algebras exhibit positivity and if they have bases which are analogous to the various known bases for ordinary cluster algebras. In this thesis, we seek to understand these structural properties.

We begin with the construction of *generalized snake graphs*, which extend the ordinary snake graphs of Musiker, Schiffler, and Williams [58] to the setting of triangulated unpunctured orbifolds. We then use generalized snake graphs to establish cluster expansion formulas which associate cluster algebra elements to ordinary arcs, generalized arcs, and closed curves on triangulated orbifolds. As an immediate consequence, we obtain an alternate and explicitly combinatorial proof of positivity for such generalized cluster algebras. We also establish the notion of a *universal snake graph*, which can be used to recover both ordinary and generalized snake graphs.

We then turn to cluster scattering diagrams and extend the cluster scattering diagram construction of Gross, Hacking, Keel, and Kontsevich to reciprocal generalized cluster algebras. We define *generalized cluster varieties* and verify that the definitions of useful objects such as broken lines and theta functions remain the same in the generalized setting. We then show that when the upper generalized cluster algebra and generalized cluster algebra coincide, the collection of theta functions $\{\vartheta_m\}_{m\in\Theta}$ forms a basis for the generalized cluster algebra. Finally, we explicitly give the fixed data for the companion algebras associated to a particular generalized cluster algebra and explore properties of that data and the resulting cluster scattering diagrams.

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Chapter 1

Introduction

This thesis primarily concerns structural properties of *generalized cluster algebras*, a natural generalization of the ordinary cluster algebras introduced by Fomin and Zelevinsky in 2002 [24]. Ordinary cluster algebras were introduced as a concrete combinatorial framework for studying dual canonical bases and total positivity in semisimple groups [24]. They are a type of commutative algebra whose generators are related by binomial *exchange relations*. In a generalized cluster algebra, introduced by Chekhov and Shapiro in 2013 [13], these hallmark binomial exchange relations are replaced by polynomial exchange relations of a specific form. In this thesis, we consider two particular subclasses of generalized cluster algebras: *generalized cluster algebras from orbifolds* and the more broad *reciprocal generalized cluster algebras*.

1.1 Ordinary cluster algebras

An *ordinary cluster algebra* $\mathcal{A}$ *of rank* n is a commutative subring of an ambient field $\mathcal{F}$ of rational functions in n variables. One of the hallmark structural properties of an ordinary cluster algebra is that it can be presented without enumerating its entire set of generators and relations. Instead, an ordinary cluster algebra can be presented by specifying the data of a *cluster seed*: a collection of n distinguished generators $\mathbf{x} = (x_1, \ldots, x_n)$, where the x_i are referred to as *cluster variables* and $\mathbf{x}$ is referred to as a *cluster*; a collection of *coefficient variables* $\mathbf{y} = (y_1, \ldots, y_n)$; and an $n \times n$ skew-symmetrizable *exchange matrix* B with integer entries that encodes the *exchange relations* between cluster variables.

From the seed data, one can generate the remainder of the cluster variables and coefficients via an involutive process called *mutation*, which replaces a single cluster variable x_k with a uniquely determined cluster variable x'_k which was not present in the original cluster. Mutation of cluster variables occurs via binomial exchange relations encoded by B, which have the general form

$$x_k x'_k = \text{monomial} \ + \ \text{monomial}.$$

The full set of cluster variables generates $\mathcal{A}$ as a subring of $\mathcal{F}$.

Mutation also replaces the collection of coefficient variables $\mathbf{y}$ with a new collection $\mathbf{y}' = (y'_1, \ldots, y'_k, \ldots, y'_n)$ and replaces B with a modified exchange matrix B'. Details about these mutation relations are given in Section 2.1.

For any choice of initial cluster, mutation sequences can be used to obtain expressions for the remaining cluster variables in terms of that initial cluster. Remarkably, in a phenomenon known as the *Laurent phenomenon*, these expressions are always Laurent polynomials. Moreover, in an accompanying phenomenon known as *positivity*, these Laurent polynomials always have strictly non-negative coefficients.

Although Fomin and Zelevinsky gave a proof for the Laurent phenomenon in their original paper [24], positivity for arbitrary ordinary cluster algebras remained conjectural until the work of Gross, Hacking, Keel, and Kontsevich in 2018 [41]. Before this general proof, positivity had been independently verified for many subclasses of cluster algebras, including: skew-symmetric cluster algebras [52], cluster algebras of surface type [58, 70, 71], acyclic (quantum) cluster algebras [7, 16, 46, 65], and bipartite cluster algebras [60].

An important structural question in the study of ordinary cluster algebras concerns the existence of bases. Because the original definition of cluster algebras arose from a desire to understand dual canonical bases, it is natural to wonder if "desirable" bases for cluster algebras exist. In this context, a "desirable" basis should include the *cluster monomials* (i.e., monomials in the variables of any choice of cluster) and should have basis elements whose expansions in terms of any choice of cluster have strictly non-negative coefficients.

Many subclasses of ordinary cluster algebras have known bases, including: the cluster monomial basis for finite type, the generic basis for affine type [6], the generic basis for acyclic type [29, 30], the greedy basis for rank 2 (quantum) cluster algebras [51, 53], and the bangle and bracelet bases for cluster algebras of surface type [57]. In their 2018 paper, Gross, Hacking, Keel, and Kontsevich proved the existence of the *theta basis* for ordinary cluster algebras of geometric type [41].

For technical definitions, more precise statements of the Laurent phenomenon and positivity, and examples, we refer the reader to Section 2.1.

1.2 Generalized cluster algebras

When considering the definition of an ordinary cluster algebra, one natural question is to ask what happens when the hallmark binomial exchange relations are replaced by other types of polynomials. When the exchange relations are replaced by polynomials of a particular form, this question leads to the definition of *generalized cluster algebras* (sometimes also referred to in the literature as *Chekhov-Shapiro algebras*).

The introduction of generalized cluster algebras was originally motivated by the study of Teichmüller spaces of Riemann surfaces with holes and orbifold points of arbitrary order [11, 12]. Generalized cluster algebra structures have since been discovered in the representation theory of quantum affine algebras [35, 36], the representation theory of quantum loop algebras [37], exact WKB analysis [45], the cyclic symmetry of Grassmannians [28], the study of the Drinfeld double of GL_n [31, 32, 33, 34], and in certain Caldero-Chapoton algebras of quivers with relations [49].

A generalized cluster algebra is presented by specifying a slightly larger set of *generalized cluster seed* data: a collection of n distinguished generators $\mathbf{x} = (x_1, \dots, x_n)$ where the x_i are still referred to as cluster variables and the entire subset is still referred to as a cluster; a collection of coefficient variables $\mathbf{y} = (y_1, \dots, y_n)$; an $n \times n$ skew-symmetrizable exchange matrix B with integer entries; an $n \times n$ diagonal *exchange degree matrix* R with positive integer entries; and a collection $(\mathbf{a}_1, \dots, \mathbf{a}_n)$, where $\mathbf{a}_i = (a_{i,j})_{j\in[r_i]}$ is the set of *exchange polynomial coefficients* for the cluster variable x_i.

In the same manner as for ordinary cluster algebras, the generalized seed data can be used to generate the remainder of the cluster variables and coefficient variables via mutation, which remains involutive. The exchange relations now have the general form

$$x_k x_k' = 1 + a_{k,1}u + \cdots + a_{k,r_k-1}u^{r_k-1} + u^{r_k},$$

where u is specialized to be a particular product of cluster and coefficient variables.

As before, mutation replaces the collection of coefficient variables $\mathbf{y}$ with a new collection $\mathbf{y}' = (y_1', \ldots, y_k', \ldots, y_n')$ and the exchange matrix B with a modified exchange matrix B'. It also replaces the collection of exchange polynomial coefficients $(\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_n)$ with the modified collection $(\mathbf{a}_1, \ldots, \mathbf{a}_k', \ldots, \mathbf{a}_\mathbf{n})$, where $\mathbf{a}_k'$ is obtained by setting $a_{k,j}' = a_{k,r_k-j}$. The exchange degree matrix R is fixed under mutation. Further details about these mutation relations, as well as examples, are given in Section 2.5.

All of the generalized cluster algebras considered in this thesis belong to the subclass of *reciprocal generalized cluster algebras*. This subclass has the additional constraint that every exchange polynomial must be a *reciprocal polynomial* - i.e., that $a_{i,j} = a_{i,r_i-j}$ for all $i \in [n]$. As a consequence, the exchange polynomials of such generalized cluster algebras are fixed under mutation. Within this subclass, we also specifically consider *generalized cluster algebras from orbifolds*. Such algebras have exchange polynomials that are either binomial or of the form $1 + \lambda_p u + u^2$, where λ_p is a particular constant associated to an orbifold point of order p (for the definition of λ_p, see Section 2.4). Note that both of these types of exchange polynomials are necessarily reciprocal so generalized cluster algebras from orbifolds are a subclass of reciprocal generalized cluster algebras.

Generalized cluster algebras exhibit many of the same structural properties as ordinary cluster algebras. In their original paper, Chekhov and Shapiro prove that all generalized cluster algebras exhibit the Laurent phenomenon, that positivity holds for generalized cluster algebras of rank two, and that generalized cluster algebras admit the same finite-type classification as ordinary cluster algebras [13]. Positivity for arbitrary generalized cluster algebras remains conjectural. Chekhov and Shapiro also show that triangulations of orbifolds provide a geometric model for a certain subclass of generalized cluster algebras, drawing on the work of Felikson, Shapiro, and Tumarkin [17, 18].

As for ordinary cluster algebras, an important structural question in the study of generalized cluster algebras is the existence of bases. A small number of subclasses of generalized cluster algebras have known bases, including: the greedy basis for rank 2 generalized cluster algebras [68] and the monomial basis for acyclic and coprime generalized cluster algebras [1].

Much of the structural information of a given generalized cluster algebra can be encoded in a pair of ordinary cluster algebras called *companion algebras*, defined by Nakanishi and Rupel [62]. More detail about companion algebras, precise technical definitions for generalized cluster algebras, and examples of both types of algebras can be found in Section 2.5.

1.3 Overview

In this thesis, we give two constructions which can be used to study structural properties of generalized cluster algebras: *generalized snake graphs* and *generalized cluster scattering diagrams*. These constructions extend the ordinary snake graphs of Musiker, Schiffler, and Williams [58] and the cluster scattering diagrams of Gross, Hacking, Keel, and Kontsevich [41].

In Chapter 2, we review relevant background material. This includes basic definitions and examples of ordinary cluster algebras, orbifolds, generalized cluster algebras, and companion cluster algebras. We review the construction of ordinary snake graphs in Section 2.2 and that of cluster scattering diagrams in Section 2.3.

In Chapter 3, we introduce our construction for generalized snake graphs. Using this construction, we give an explicit combinatorial formula for the Laurent expansion of any arc or closed curve on a triangulated orbifold. This gives an alternate and explicitly combinatorial proof of positivity for generalized cluster algebras from orbifolds, using the geometric model introduced by Chekhov and Shapiro. We also introduce the notion of *universal snake graphs*, which can be used to recover both ordinary and generalized snake graphs and allow us to simplify the calculations and arguments of Musiker, Schiffler, and Williams [58]. We describe the poset of perfect matchings of these universal snake graphs and highlight some interesting properties, including that this poset is isomorphic to the Boolean lattice B_n. Finally, we describe the relationship between punctures and orbifold points and show that some of the results of [58] and [59] can be recovered by treating punctures as orbifold points of infinite order.

In Chapter 4, we introduce *generalized cluster scattering diagrams* for reciprocal generalized cluster algebras, building on the work of Gross, Hacking, Keel, and Kontsevich [41]. We also define *generalized cluster varieties* and the *middle generalized cluster algebras*. We then use generalized cluster scattering diagrams to define the theta basis in the context of reciprocal generalized cluster algebras. We show that when the generalized cluster algebra and upper generalized cluster algebra coincide, a particular collection of theta functions forms a basis for the generalized cluster algebra.

Chapter 4 also contains material about companion algebras. We explicitly give the fixed data for the companion algebras of a particular generalized cluster algebra and show that the fixed data of the left and right companion algebras are *Langlands dual*. We also discuss the relationship between the cluster scattering diagram of a particular generalized cluster algebra and the cluster scattering diagrams of its associated companion algebras.

Chapter 2

Background

2.1 Ordinary cluster algebras

Cluster algebras were introduced in 2002 by Fomin and Zelevinsky [24] to provide a concrete combinatorial framework for studying dual canonical bases and total positivity in semisimple groups. Subsequently, there have been many generalizations of cluster algebras. To avoid any potential for confusion, we therefore use the term *ordinary cluster algebra* whenever we refer to a cluster algebra in the original sense defined by Fomin and Zelevinsky.

Before defining an ordinary cluster algebra, $\mathcal{A}$, we must first describe its ground ring. Let $(\mathbb{P}, \oplus, \cdot)$ be an arbitrary semifield. The group ring $\mathbb{ZP}$ will serve as the ground ring for $\mathcal{A}$. Let $\mathcal{F}$ be isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{QP}$. The field $\mathcal{F}$ is referred to as the *ambient field* of $\mathcal{A}$.

Frequently, the semifield $\mathbb{P}$ is chosen to be the *tropical semifield*. There are two conventions for the tropical semifield: the min-plus convention and the max-plus convention. In the min-plus convention, the auxiliary addition $\oplus$ is defined as $x \oplus y = \min(x, y)$, whereas in the max-plus convention it is defined as $x \oplus y = \max(x, y)$. In both cases, the multiplication operation $\cdot$ is the usual addition. Regardless of the choice of convention, the resulting ordinary cluster algebra is said to be of *geometric type*.

We are now prepared to build up the definition of an ordinary cluster algebra.

Definition 2.1.1 (Definition 2.3 of [26]). *A* labeled cluster seed *is a triple* $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ *such that*

- $\mathbf{x} = (x_1, \ldots, x_n)$ *is a free generating set for* $\mathcal{F}$*,*
- $\mathbf{y} = (y_1, \ldots, y_n)$ *is an* n*-tuple with elements in* $\mathbb{P}$*,*
- *and* $B = [b_{ij}]$ *is an* $n \times n$ *skew-symmetrizable matrix with entries in* $\mathbb{Z}$*.*

We refer to $\mathbf{x}$ *as the* cluster *of* Σ*,* $\mathbf{y}$ *as the* coefficient tuple *of* Σ*, and to* B *as the* exchange matrix*. We refer to the elements* $x_1, \ldots, x_n$ *as the* cluster variables *of* Σ *and to the elements* $y_1, \ldots, y_n$ *as the* coefficient variables *of* Σ*.*

Definition 2.1.2 (Definition 2.4 of [26]). *For a cluster seed $\Sigma = (\mathbf{x}, \mathbf{y}, B)$, mutation in direction k, μ_k, is defined by the following exchange relations:*

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + ([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+) & i, j \neq k \end{cases}$$
$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{[b_{ik}]_+} (1 \oplus y_k)^{-b_{ik}} & i \neq k \end{cases}$$
$$x'_i = \begin{cases} x_k^{-1} \left(\frac{y_k \prod_{j=1}^n x_j^{[b_{kj}]_+} + \prod_{j=1}^n x_j^{[-b_{kj}]_+}}{1 \oplus y_k} \right) & i = k \\ x_i & i \neq k \end{cases}$$

where $[\cdot]_+ = max(\cdot, 0)$.

Definition 2.1.3. *Let $\mathfrak{T}_n$ be the n-regular tree whose edges are labeled by $1, \ldots, n$ such that each edge incident to a given vertex has a different label. A* cluster pattern *is an assignment of labeled seeds $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B)$ to each vertex $t \in \mathfrak{T}_n$ such that seeds assigned to adjacent vertices $t - t'$, whose mutually incident edge has label k, are related by seed mutation in direction k. We use the notation*

$$\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t}), \qquad \mathbf{y}_t = (y_{1;t}, \ldots, y_{n;t}), \qquad B_t = [b^t_{ij}].$$

Definition 2.1.4. *For a fixed choice of cluster pattern, let*

$$\mathcal{X} := \bigcup_{t \in \mathfrak{T}_n} \mathbf{x}_t = \{x_{i;t} : t \in \mathfrak{T}_n, i \in [n]\}$$

be the union of the clusters of each seed in the cluster pattern. We refer to the elements $x_{i;t} \in \mathcal{X}$ as cluster variables*. The* cluster algebra *$\mathcal{A}$ associated to the cluster pattern is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the cluster variables, $\mathcal{A} := \mathbb{ZP}[\mathcal{X}]$. We often write $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ to indicate the cluster algebra associated to the cluster pattern containing the seed $(\mathbf{x}, \mathbf{y}, B)$.*

One of the most celebrated properties of ordinary cluster algebras is the *Laurent phenomenon*:

Theorem 2.1.5 (Theorem 3.1 of [24]). *Let $\mathcal{A} = \mathcal{A}(\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y}, B)$ be an arbitrary cluster algebra. Every element of $\mathcal{A}$ can be expressed in terms of the cluster variables $x_1, \ldots, x_n$ as Laurent polynomials with coefficients in $\mathbb{ZP}$.*

Note that because the cluster algebra $\mathcal{A}$ can be defined by any choice of seed from the corresponding cluster pattern, the above theorem means that the elements of the cluster algebra can be expressed as Laurent polynomials in terms of *any* choice of initial cluster.

The Laurent phenomenon becomes even more compelling with the addition of the *positivity property*.

Conjecture 2.1.6 (c.f. Section 3 of [24]). *The coefficients of these Laurent polynomials are strictly* non-negative.

Positivity was conjectured in Fomin and Zelevinsky's original paper [24] and later verified in a variety of cases, including: skew-symmetric cluster algebras [52], cluster algebras of surface type [58, 70, 71], acyclic (quantum) cluster algebras [7, 16, 46, 65], bipartite cluster algebras [60], and cluster algebras of geometric type [41]. Note that this last case encompasses all of the prior cases and is the most general setting in which a proof of positivity is known.

2.1.1 Principal coefficients

A particularly important type of cluster algebra is one with *principal coefficients*.

Definition 2.1.7 (Definition 3.1 of [26]). *A cluster pattern is said to have* principal coefficients at vertex t *if* $\mathbb{P} = \mathit{Trop}(y_1, \ldots, y_n)$ *and* $\mathbf{y}_t = (y_1, \ldots, y_n)$*. We refer to the corresponding cluster algebra* $\mathcal{A}$ *as a* cluster algebra with principal coefficients.

This definition can be equivalently stated in terms of the *extended exchange matrix*, $\widetilde{B}$. An ordinary cluster algebra $\mathcal{A}$ is said to have *principal coefficients at vertex* t if $\mathcal{A}$ is of geometric type and is associated to the $2n \times n$ *extended exchange matrix*

$$\widetilde{B}_t := \begin{bmatrix} B_t \\ I \end{bmatrix},$$

where I is the $n \times n$ identity matrix. Note that the extended exchange matrix $\widetilde{B}_t$ could also be written as a skew-symmetric $2n \times 2n$ block matrix of the form

$$\widetilde{B}_t := \begin{bmatrix} B_t & -I \\ I & \mathbf{0} \end{bmatrix},$$

where I is the $n \times n$ identity matrix and $\mathbf{0}$ is the $n \times n$ matrix whose entries are all zero. When we discuss cluster scattering diagrams, we will prefer the $2n \times 2n$ form. Otherwise, the $2n \times n$ form is typically used for the sake of concision.

Remark 2.1.8. *The cluster algebra literature contains two conventions for defining extended exchange matrices - the* tall *convention, where extra rows are added, and the* wide convention*, where extra columns are added instead. The preceding definition for an ordinary cluster algebra with principal coefficients is given using the tall convention, consistent with the original definition of Fomin and Zelevinsky [26]. In the wide convention, we would instead have the* $n \times 2n$ *matrix*

$$\widetilde{B}_t := \begin{bmatrix} B_t & I \end{bmatrix}$$

or the $2n \times 2n$ *skew-symmetric matrix*

$$\widetilde{B}_t := \begin{bmatrix} B_t & I \\ -I & \mathbf{0} \end{bmatrix},$$

where I *and* $\mathbf{0}$ *are defined as before. In order to be consistent with the major papers which define snake graphs and cluster scattering diagrams, this thesis will actually use both conventions. We will use the tall convention when discussing generalized snake graphs in Chapter 3 and the associated background material in Sections 2.2 and 2.4. We will use the wide convention when discussing generalized cluster scattering diagrams in Chapter 4 and the associated background material in Section 2.3. We will also* *use the wide convention in Section 2.5, which provides background for both Chapter 3 and Chapter 4. Whenever there is potential for confusion or ambiguity, we will explicitly specify which convention is being used.*

The Laurent phenomenon and positivity hold for ordinary cluster algebras with principal coefficients. For such algebras, we can also define the notion of an *F-polynomial*.

Definition 2.1.9 (Definition 3.3 of [26])**.** *Let $\mathcal{A}$ be the ordinary cluster algebra with principal coefficients at vertex t_0 defined by the initial cluster seed $\Sigma_t = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$, where*

$$\mathbf{x}_{t_0} = (x_1, \ldots, x_n),\ \mathbf{y}_{t_0} = (y_1, \ldots, y_n),\ \textit{and } B_{t_0} = B^0 = [b^0_{ij}].$$

By definition, $\mathbb{P} = \mathit{Trop}(x_1, \ldots, x_n)$ and the exchange relation coefficients are monomials in the variables $y_1, \ldots, y_n$. The Laurent phenomenon allows us to express any cluster variable $x_{\ell,t}$ as a unique subtraction-free rational function in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. We denote this rational function as

$$X_{\ell,t} = X_{\ell,t}^{B^0,t_0} \in \mathbb{Q}_{sf}(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Let $F_{\ell,t} = F_{\ell,t}^{B^0,t_0}$ denote the polynomial obtained from $X_{\ell,t}$ via the specialization

$$F_{\ell,t}(y_1, \ldots, y_n) := X_{\ell,t}(1, \ldots, 1, y_1, \ldots, y_n).$$

*We refer to $F_{\ell,t}$ as a F-*polynomial.

Example 2.1.10. *Let $\mathcal{A}$ be the ordinary cluster algebra with principal coefficients defined by the initial cluster seed*

$$\Sigma_{t_0} = \left(\mathbf{x}_{t_0} = (x_1, x_2), \mathbf{y}_{t_0} = (y_1, y_2), B_{t_0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right).$$

The following table shows the extended exchange matrix $\widetilde{B}_t$, the coefficient variables $y_{1,t}$ and $y_{2,t}$, *the cluster variables* $\mathbf{x}_t = (X_{1,t}$ *and* $X_{2,t})$*, and the* F*-polynomials* $F_{1,t}$ *and* $F_{2,t}$ *at vertex* $t \in \mathfrak{T}_n$*.*

t	$\widetilde{B}_t$	$y_{1,t}$	$y_{2,t}$	$X_{1,t}$	$X_{2,t}$	$F_{1,t}$	$F_{2,t}$
0	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	y_1	y_2	x_1	x_2	1	1
1	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$	y_1	$\frac{1}{y_2}$	x_1	$\frac{1+y_2x_1}{x_2}$	1	$1+y_2$
2	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{1}{y_1}$	$\frac{1}{y_2}$	$\frac{x_2+y_1+y_1y_2x_1}{x_1x_2}$	$\frac{1+y_2x_1}{x_2}$	$1+y_1+y_1y_2$	$1+y_2$
3	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$	$\frac{1}{y_1y_2}$	y_2	$\frac{x_2+y_1+y_1y_2x_1}{x_1x_2}$	$\frac{x_2+y_1}{x_1}$	$1+y_1+y_1y_2$	$1+y_1$
4	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$	y_1y_2	$\frac{1}{y_1}$	x_2	$\frac{x_2+y_1}{x_1}$	1	$1+y_1$
5	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	y_2	y_1	x_2	x_1	1	1

Table 2.1: Data for a cluster algebra of type A_2 with principal coefficients.

Let F be a subtraction-free rational expression over $\mathbb{Q}$ in several variables, $\mathbb{P}$ be an arbitrary semifield, and $u_1, \dots, u_\ell$ be elements of $\mathbb{P}$. Then we use $F|_{\mathbb{P}}(u_1, \dots, u_\ell)$ to denote the evaluation of F at $u_1, \dots, u_\ell$. Using this notation, we can explicitly state the relationship between the Laurent expansions for cluster variables in an arbitrary ordinary cluster algebra and in the corresponding ordinary cluster algebra with principal coefficients.

Theorem 2.1.11 (Theorem 3.7 of [26])**.** *Let* $\mathcal{A}$ *be an ordinary cluster algebra over an arbitrary semifield* $\mathbb{P}$ *defined by associating the cluster seed* $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ *to the initial vertex* $t_0 \in \mathfrak{T}_n$*. Any cluster variable in* $\mathcal{A}$ *can be expressed as*

$$x_{\ell,t} = \frac{X_{\ell,t}^{B^0,t_0}\Big|_{\mathcal{F}}(x_1, \dots, x_n, y_1, \dots, y_n)}{F_{\ell,t}^{B^0,t_0}\Big|_{\mathbb{P}}(y_1, \dots, y_n)}.$$

Note that if $\mathbb{P}$ is a tropical semifield, then the denominator of the above expression for $x_{\ell,t}$ is a monomial. Hence, we have the following immediate corollary.

Corollary 2.1.12 (Corollary 2.14 of [26])**.** *Let* $\mathcal{A}$ *be the ordinary cluster algebra with principal coefficients at vertex* $t_0 \in \mathfrak{T}_n$ *defined by the initial cluster seed* $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$*. Let* $\widehat{\mathcal{A}}$ *be any ordinary cluster algebra of geometric type defined by the same initial exchange matrix* B_{t_0}*. If positivity holds for* $\mathcal{A}$*, then it holds for* $\widehat{\mathcal{A}}$*.*

An important consequence of this corollary is that for ordinary cluster algebras of geometric type, it is sufficient to show positivity in the principal coefficient case. Hence, many proofs of positivity primarily treat ordinary cluster algebras with principal coefficients.

2.2 Ordinary snake graphs

Snake graphs were defined by Musiker, Schiffler, and Williams as a tool for finding explicit combinatorial formulas for the cluster variables in any cluster algebra of surface type [58]. Via this construction, they offered the first known proof of positivity for cluster algebras from surfaces.

2.2.1 Ordinary cluster algebras from surfaces

In 2008, Fomin, Shapiro, and D. Thurston showed that a subset of ordinary cluster algebras can be modeled by triangulations of bordered surfaces with marked points [22]. Marked points may appear either on the boundary or within the interior of the surface; those that appear within the interior are called *punctures*. In this thesis, we will primarily deal with unpunctured surfaces, although we note some connections between orbifold points of infinite order and punctures in Section 3.14.

In this section, we establish some nomenclature and briefly highlight relevant features of Fomin, Shapiro, and D. Thurston's construction for unpunctured surfaces. For a much more detailed exposition, we refer the reader to Section 2 of [22].

Definition 2.2.1. *An* ordinary arc γ *on a surface* (S, M) *is a non-self-intersecting curve on* S *with endpoints in* M *that is otherwise disjoint from* M *and* ∂S*. Curves that are contractible onto* ∂S *or that cut out an unpunctured monogon or bigon are not considered ordinary arcs. Ordinary arcs are considered up to isotopy class.*

Ordinary arcs are a special type of *generalized arc*.

Definition 2.2.2. *A* generalized arc γ *on a surface* (S, M) *is a curve on* S *which may contain self-intersections or be a non-contractible closed curve with no endpoints in* M*. Curves that are contractible onto* ∂S*, that cut out an unpunctured monogon or bigon, or that are contractible to a point are not considered generalized arcs. As before, generalized arcs are considered up to isotopy.*

Let $[\gamma]$ denote the isotopy class of the arc γ. Because arcs are considered up to isotopy, we must be somewhat careful when defining intersections of arcs. Let γ and γ' be arbitrary arcs on S and α and α' denote arbitrary representatives of their isotopy classes. Then the *crossing number* $e(\gamma, \gamma')$ is the minimal number of crossings of each possible choice of α and α'. Two arcs γ and γ' are considered *compatible* if $e(\gamma, \gamma') = 0$, i.e. if it is possible to draw representatives of the isotopy classes of γ and γ' which are non-crossing. Similarly, if $T = \{\tau_1, \ldots, \tau_n\}$ is a triangulation of a surface, we define $e(\gamma, T) = \sum_{i=1}^{n} e(\gamma, \tau_i)$.

Definition 2.2.3. *An* ideal triangulation T *of a surface is a maximal collection of pairwise compatible arcs (and boundary arcs).*

Surfaces with ideal triangulations provide a useful combinatorial tool for studying certain cluster algebras, via the correspondence described in the following theorem.

Theorem 2.2.4 (c.f. Section 6 of [23])**.** *Given a surface with marked points, (S, M), there exists a unique cluster algebra $\mathcal{A} = \mathcal{A}(S, M)$ such that:*

1. *The seeds are in bijection with tagged triangulations of (S, M).*
2. *The cluster variables are in bijection with tagged arcs in (S, M).*
3. *The cluster variable x_γ corresponding to arc γ is given by the lambda length of γ, in terms of some initial triangulation.*

Under this correspondence, mutation of cluster variables in $\mathcal{A}$ is equivalent to "flipping" arcs in the triangulation T. To understand what we mean by "flipping", observe that each arc τ in T looks locally like the diagonal of a quadrilateral and that this quadrilateral has a unique diagonal, τ', which is not in T. To "flip" τ, we replace it with τ'.

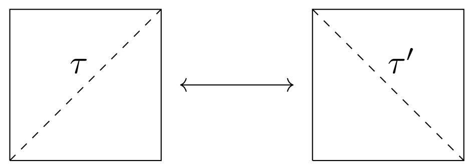


Figure 2.1: Flipping an arc on a triangulated surface.

Hence, the result of "flipping" τ in T is the new triangulation $T' = (T - \{\tau\}) \cup \{\tau'\}$. Note that this procedure is both well-defined and involutive, as expected.

2.2.2 Laminations

In the context of cluster algebras, laminations were used by Fomin and D. Thurston [23] as a tool for tracking the coefficients of a cluster algebra from a surface using W. Thurston's [72] shear coordinates and theory of measured laminations. We will review only the relevant portion of their work (for unpunctured surfaces), but refer the reader either to Chapter 12 of their work for further details about laminations in this context, or to the work of Fock-Goncharov [21] or W. Thurston [72] for more details about measured laminations and their relationship to matrix mutations.

Definition 2.2.5 (Definition 12.1 of [23])**.** *Let (S, M) be an unpunctured bordered surface. An* integral unbounded measured lamination *(henceforth referred to as just a* lamination*) on S is a finite collection of non-self-intersecting and pairwise non-intersecting curves on S such that:*

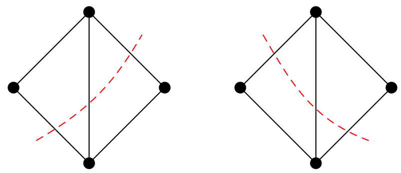
- *each curve is either a closed curve or a non-closed curve with endpoints on umarked points on ∂S,*
- *no curve bounds an unpunctured disk,*
- *and no curve with endpoints on ∂S is isotopic to a portion of the boundary containing either no or one marked point(s).*

A *multi-lamination* is a finite family of such laminations. W. Thurston's shear coordinates [72] provide a coordinate system for laminations.

Definition 2.2.6 (Definition 12.2 of [23]). *Let S be a surface with triangulation T and L be a lamination on S. For each arc $\gamma \in T$, the* shear coordinate *of L with respect to T is*

$$b_\gamma(T, L) = \sum_i b_\gamma(T, L_i)$$

where the summation runs over all individual curves in L. The shear coordinates $b_\gamma(T, L_i)$ are defined as:



Tracking principal coefficients requires the notion of an *elementary lamination.*

Definition 2.2.7. *The* elementary lamination L_i *associated to an arc τ_i in triangulation T is the lamination such that $b_{\tau_i}(T, L_i) = 1$ for $\tau_i \in T$ and $b_\tau(T, L_i) = 0$ for $\tau \notin T$.*

An ordinary cluster algebra of surface type with principal coefficients corresponds to a triangulated surface with a multi-lamination composed of all possible elementary laminations.

2.2.3 Snake graph construction

Let γ be a fixed arc and T be a fixed triangulation of some surface (S, M), where M is a set of *marked points*. Musiker, Schiffler, and Williams [58] construct a *snake graph* $G_{T,\gamma}$ by gluing together *tiles* that encode the local geometry at each intersection between γ and arcs of the triangulation. The formula for the expansion of x_γ with respect to the cluster corresponding to T is given in terms of perfect matchings of $G_{T,\gamma}$. We briefly review their construction for unpunctured surfaces, but refer the interested reader to Section 4 of [58] for the complete construction and many examples.

Let (S, M) be a bordered surface with triangulation T and γ be an ordinary arc (i.e., a non-self-intersecting arc) on S which is not in T. Fix an orientation of γ and let s and t denote, respectively, the start and end points of γ. Denote the intersection points of γ and T as $s = p_0, p_1, \ldots, p_{d+1} = t$, in order, and let τ_{i_j} denote the arc in T which contains intersection point p_j. Let Δ_{j-1} denote the ideal triangle that γ passes through just before τ_j and Δ_j denote the ideal triangle it passes through just after.

Each intersection p_j is associated with a square tile G_j formed by gluing copies of Δ_{j-1} and Δ_j along the edge labeled τ_{i_j}. This can be done in two ways: such that both triangles have orientation matching the orientations of Δ_{j-1} and Δ_j on S, or such that both triangles have the opposite orientation. Hence, there are two valid planar embeddings of G_j. We say that the tile G_j has *relative orientation* $\mathrm{rel}(G_j) = +1$ if the orientation of its triangles matches the orientations of Δ_{j-1} and Δ_j on S and $\mathrm{rel}(G_j) = -1$ otherwise. Two of the edges of the triangle Δ_j are labeled τ_{i_j} and $\tau_{i_{j+1}}$; label the remaining edge as $\tau_{[\gamma_j]}$.

The graph $G_{T,\gamma}$ is then formed by gluing together subsequent tiles $G_1, \ldots, G_d$ in the order by the corresponding intersection points. After choosing planar embeddings $\widetilde{G}_j$ and $\widetilde{G}_{j+1}$ such that

$\text{rel}(\widetilde{G}_j) \neq \text{rel}(\widetilde{G}_{j+1})$, the embedded tiles $\widetilde{G}_j$ and $\widetilde{G}_{j+1}$ are glued along the edges labeled $\tau_{[\gamma_j]}$. Gluing together all d tiles yields a graph $\bar{G}_{T,\gamma}$. The graph $G_{T,\gamma}$ can then be obtained from $\bar{G}_{T,\gamma}$ by removing the diagonal edge from each tile.

Example 2.2.8. *Consider the triangulated surface* (S, M) *corresponding to the ordinary cluster algebra*

$$\mathcal{A}\left((x_1, x_2, x_3), (y_1, y_2, y_3), \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right).$$

Below, we show two examples of ordinary arcs on (S, M) *and the corresponding snake graphs.*

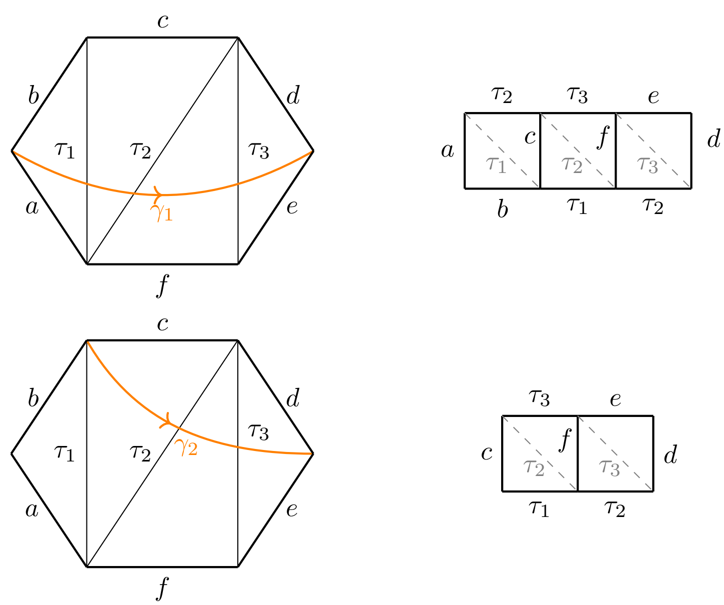


Figure 2.2: Examples of snake graphs from ordinary arcs on a triangulated surface.

The statement of Musiker, Schiffler, and William's expansion formula requires several additional definitions, beginning with the *crossing monomial*.

Definition 2.2.9 (Definition 4.4 of [58]). *For an ordinary arc* γ *crossing the sequence of arcs* $\tau_{i_1}, \dots, \tau_{i_d}$ *in* T*, the* crossing monomial *of* γ *with respect to* T *is defined as*

$$cross(T, \gamma) = \prod_{j=1}^{d} x_{\tau_{i_j}}.$$

The other monomial required to state Musiker, Schiffler, and William's expansion formula is the *height monomial*. Defining this monomial requires first establishing some notions related to *perfect matchings* of graphs. A *perfect matching* P of a graph G is a subset of the edges of G such that each vertex of G is incident to exactly one edge of P.

Definition 2.2.10 (Definition 4.5 of [58]). *A perfect matching* P *of a snake graph* G *which uses edges labeled* $\tau_{i_1}, \dots, \tau_{i_k}$ *has* weight $x(P) = x_{\tau_{i_1}} \cdots x_{\tau_{i_k}}$.

Definition 2.2.11 (Definition 4.6 of [58]). $G_{T,\gamma}$ *has exactly two perfect matchings that include only boundary edges; these are referred to as the* minimal *and* maximal *matchings of* $G_{T,\gamma}$*. The* *distinction between the two depends on the relative orientation of* $G_{T,\gamma}$*. If* $rel(G_{T,\gamma}) = 1$ *(respectively,* -1*), define* e_1 *and* e_2 *to be the edges that are immediately counterclockwise (respectively, clockwise) from the diagonal. The* minimal matching*,* P_-*, is defined to be the unique perfect matching that includes only boundary edges and does not include* e_1 *or* e_2*. The* maximal matching P_+ *is the complementary perfect matching on boundary edges that includes* e_1 *and* e_2*.*

Let $P \ominus P_- := (P \cup P_-)\backslash(P \cap P_-)$ denote the symmetric difference of an arbitrary perfect matching P with the minimal perfect matching P_-. The edges of $P \ominus P_-$ are always the set of boundary edges of a (potentially disconnected) subgraph of $G_{T,\gamma}$ which is composed of a union of cycles. These cycles enclose a finite set of tiles, $\{G_{i_j}\}_{j\in J}$.

Definition 2.2.12 (c.f. Definition 4.8 of [58])**.** *Let* $T = \{\tau_1, \ldots, \tau_n\}$ *be an ideal triangulation of an unpunctured surface* (S, M) *and* γ *be an ordinary arc on* (S, M)*. Let* P *be a perfect matching of* $G_{T,\gamma}$ *such that* $P \ominus P_-$ *encloses the set of tiles* $\{G_{i_j}\}_{j\in J}$*. The* height monomial *of* P *is*

$$y(P) = \prod_{k=1}^{n} y_{\tau_k}^{m_k},$$

where m_k *is the number of tiles in* $\{G_{i_j}\}_{j\in J}$ *with diagonal labeled* τ_{i_j}*.*

Now, we are prepared to state the expansion formula.

Theorem 2.2.13 (Theorem 4.9 of [58])**.** *Let* (S, M) *be a bordered surface with triangulation* T*,* $\mathcal{A}$ *be the corresponding cluster algebra with principal coefficients, and* γ *be an ordinary arc on* S*. Then* x_γ *can be written as a Laurent expansion in terms of the initial cluster variables as*

$$x_\gamma = \frac{1}{cross(T, \gamma)} \sum_P x(P)y(P),$$

where the sum ranges across all perfect matchings P *of the snake graph* $G_{T,\gamma}$*.*

Example 2.2.14. *Consider the snake graph for the ordinary arc* γ_1 *from Example 2.2.8. All of the perfect matchings of this snake graph are shown below, with the maximal and minimal matchings shown, respectively, on rows three and five. For each perfect matching* P_i*, the tiles enclosed by the symmetric difference* $P_i \ominus P_-$ *are shaded. The corresponding weight and height monomials are given on the right.*

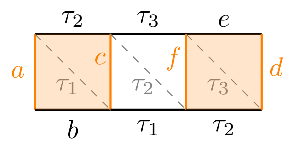
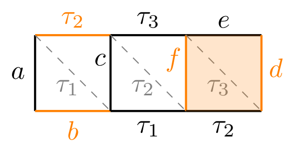
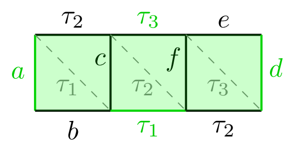
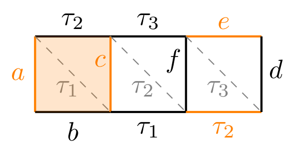
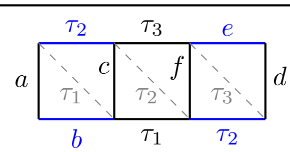
	$x(P_1) = 1$	$y(P_1) = y_{\tau_1} y_{\tau_3}$
	$x(P_2) = x_{\tau_2}$	$y(P_2) = y_{\tau_3}$
	$x(P_3) = x_{\tau_1} x_{\tau_3}$	$y(P_3) = y_{\tau_1} y_{\tau_2} y_{\tau_3}$
	$x(P_4) = x_{\tau_2}$	$y(P_4) = y_{\tau_1}$
	$x(P_5) = x_{\tau_2}^2$	$y(P_5) = 1$

Table 2.2: An example of a complete set of perfect matchings of a snake graph, along with the associated weight and height monomials.

Musiker, Schiffler, and William's expansion formula then gives us the Laurent expansion

$$x_{\gamma_1} = \frac{1}{x_{\tau_1} x_{\tau_2} x_{\tau_3}} \left(y_{\tau_1} y_{\tau_3} + x_{\tau_2} y_{\tau_3} + x_{\tau_1} x_{\tau_3} y_{\tau_1} y_{\tau_2} y_{\tau_3} + x_{\tau_2} y_{\tau_1} + x_{\tau_2}^2 \right)$$

Subsequently, Musiker and Williams extended the snake graph construction to handle *generalized arcs*, which may contain self-intersections, and closed curves [59]. Suppose γ is now a generalized arc. If γ is a contractible loop, define $x_\gamma := -2$. If γ contains a contractible kink, then let $\overline{\gamma}$ denote the corresponding arc with the kink removed and define $x_\gamma := (-1)x_{\overline{\gamma}}$. Musiker and Williams then show that Theorem 2.2.13 holds for generalized arcs. For closed curves, they define a corresponding cluster algebra element using a slight modification of snake graphs called *band graphs*, where the first and last tiles are glued to form a non-planar graph. For details, see Section 3 of [59].

The set of perfect matchings of a snake graph has a natural poset structure. Describing this structure makes use of *twists*, which are local moves on a perfect matching P where the horizontal edges of a single tile are replaced with the vertical edges of that tile, or vice versa. Building on previous work by Propp on the poset structure of perfect matchings of bipartite planar graphs [64], Musiker, Schiffler, and Williams [57] establish the following result.

Theorem 2.2.15 (Theorem 5.2 of [57]). *Consider the set of all perfect matchings of a snake graph G and construct a graph whose vertices are labeled by these perfect matchings and which has an edge between two vertices if and only if the matchings corresponding to those vertices are obtainable from each other by a single twist. An edge corresponding to twisting a tile with diagonal edge τ_j is labeled y_j. This graph is the Hasse diagram of a distributive lattice, with minimal element P_-, which is* *graded by the degree of the height monomials associated with each matching.*

2.3 Cluster scattering diagrams

Later, in Chapter 4, we will construct cluster scattering diagrams for reciprocal generalized cluster algebras. In order to set the stage for that construction, this section will review the construction of cluster scattering diagrams for ordinary cluster algebras, largely following the exposition of [20], [39], and [41]. When we extend these definitions and constructions in Chapter 4, we will explain how the ordinary definitions and constructions can be recovered as specializations of those extensions.

Scattering diagrams first appear in the literature in two dimensions, in work by Kontsevich and Soibelman [47], and then in arbitrary dimension in the work of Gross and Siebert [40]. Our discussion of cluster scattering diagrams will loosely follow Section 2 of [39] and Section 1 of [41].

Basic data

We begin with definitions of *fixed data* and *torus seed data*, which together encode the information of a cluster seed.

Definition 2.3.1 (c.f. Section 2 of [39]). *The following collection of data is referred to as the* fixed data *and is denoted by* Γ*:*

- *The* cocharacter lattice N *with skew-symmetric bilinear form* $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$.
- *A saturated sublattice* $N_{uf} \subseteq N$ *called the* unfrozen sublattice.
- *An index set* I *with* $|I| = rank(N)$ *and subset* $I_{uf} \subseteq I$ *such that* $|I_{unf}| = rank(N_{uf})$
- *A set of positive integers* $\{d_i\}_{i\in I}$ *such that* $\gcd(d_i) = 1$
- *A sublattice* $N^\circ \subseteq N$ *of finite index such that* $\{N_{uf}, N^\circ\} \subseteq \mathbb{Z}$ *and* $\{N, N_{uf} \cap N^\circ\} \subseteq \mathbb{Z}$
- *A lattice* $M = Hom(N,\mathbb{Z})$ *called the* character lattice *and sublattice* $M^\circ = Hom(N^\circ,\mathbb{Z})$.

The name 'fixed data' refers to the fact that this data is fixed under mutation.

Definition 2.3.2 (c.f. Section 2 of [39]). *Given a set of fixed data, the associated* torus seed data *is a collection* $\mathbf{s} = \{e_i\}_{i\in I}$ *such that* $\{e_i\}_{i\in I}$ *is a basis for* N*,* $\{e_i\}_i \in I_{uf}$ *is a basis for* N_{uf}*, and* $\{d_ie_i\}_{i\in I}$ *is a basis for* N°*. The torus seed data defines a new bilinear form*

$$[\cdot,\cdot]_{\mathbf{s}} : N \times N \to \mathbb{Q}$$
$$[e_i, e_j]_{\mathbf{s}} = \epsilon_{ij} = \{e_i, e_j\}d_j$$

which is not necessarily skew-symmetric.

Remark 2.3.3. *Under the wide convention for the exchange matrices of cluster algebras, the matrices* B *and* $\epsilon = [\epsilon_{ij}]_{i,j\in I}$ *coincide. Under the tall convention, used by Fomin and Zelevinsky [24], the matrices are instead related by a transpose, i.e.* $\epsilon = B^T$*.*

A choice of torus seed $\mathbf{s} = \{e_i\}_{i\in I}$ defines a dual basis $\{e_i^*\}_{i\in I}$ for M and a basis $\{f_i = d_i^{-1}e_i^*\}_{i\in I}$ for M°. It also defines two associated algebraic tori:

$$\mathcal{X}_{\mathbf{s}} = T_M = \text{Spec } \Bbbk[N],$$
$$\mathcal{A}_{\mathbf{s}} = T_{N^\circ} = \text{Spec } \Bbbk[M^\circ].$$

The torus $\mathcal{X}_{\mathbf{s}}$ has coordinates $y_1, \ldots, y_n$ and the torus $\mathcal{A}_{\mathbf{s}}$ has coordinates $x_1, \ldots, x_n$. It is also common in the literature to use the notation $A_1, \ldots, A_n$ for the coordinates of $\mathcal{A}_{\mathbf{s}}$ and $X_1, \ldots, X_n$ for the coordinates of $\mathcal{X}_{\mathbf{s}}$.

The bilinear form $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$ naturally defines maps $p_1^* : N_{\text{uf}} \to M^\circ$ and $p_2^* : N \to M^\circ/N_{\text{uf}}^\perp$ as

$$p_1^*\left(n \in N_{\text{uf}}\right) = \left(n' \in N^\circ \mapsto \{n, n'\}\right),$$
$$p_2^*\left(n \in N\right) = \left(n' \in N_{\text{uf}} \cap N^\circ \mapsto \{n, n'\}\right).$$

Based on these maps, we can then choose a map $p^* : N \to M^\circ$ such that $p^*|_{N_{\text{uf}}} = p_1^*$ and the composition of p^* with the quotient map $M^\circ \to M^\circ/N_{\text{uf}}^\perp$ agrees with p_2^*. It is important to note that the choice of p^* is not unique because there is more than one possible choice of map $N/N_{\text{uf}} \to N_{\text{uf}}^\perp$. It is also important to note that for an arbitrary choice of fixed data, the map $p_1^* : N_{\text{uf}} \to M^\circ$ is not necessarily injective. It is, however, always injective for the principal coefficient case, which will be discussed later in this section. The assumption that p_1^* is injective is sometimes referred to as the *injectivity assumption.*

The injectivity assumption is, in fact, a crucial ingredient in many of the arguments given by Gross, Hacking, Keel, and Kontsevich and therefore many of their results are proved via the principal coefficient case. For the same reason, we will also later work via the principal coefficient case.

Because the fixed data and torus seed data encode information from a cluster seed, we naturally expect that there should also be a notion of torus seed mutation.

Definition 2.3.4. *Given torus seed data* $\mathbf{s}$ *and some* $k \in I_{uf}$*, a* mutation in direction k *of the torus seed data is defined by the following transformations of basis vectors:*

$$e_i' := \begin{cases} e_i + [\epsilon_{ik}]_+ e_k & i \neq k \\ -e_k & i = k \end{cases}$$
$$f_i' := \begin{cases} -f_k + \sum_{j\in I_{uf}} [-\epsilon_{kj}]_+ f_j & i = k \\ f_i & i \neq k \end{cases}$$

The basis mutation induces the following mutation of the matrix $\epsilon = [\epsilon_{ij}]_{i,j\in I}$*:*

$$\epsilon_{ij}' := \{e_i', e_j'\}d_j = \begin{cases} -\epsilon_{ij} & k = i \text{ or } k = j \\ \epsilon_{ij} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \leq 0 \\ \epsilon_{ij} + |\epsilon_{ik}|\epsilon_{kj} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \geq 0 \end{cases}$$

Mutation of torus seed data $\mathbf{s}$ in direction k defines birational maps $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_{\mathbf{k}}(\mathbf{s})}$ and

$\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_{\mathbf{k}}(\mathbf{s})}$ via the pull-backs

$$\mu_k^* z^m = z^m (1 + z^{v_k})^{-\langle d_k e_k, m \rangle} \text{ for } m \in M^\circ, \tag{2.1}$$
$$\mu_k^* z^n = z^n (1 + z^{e_k})^{-[n, e_k]} \text{ for } n \in N, \tag{2.2}$$

where $v_k := p_1^*(e_k)$. Explictly, using dual bases, one can compute

$$v_k = e_k \left[\epsilon_{ij}\right] = \sum_{j \in I_{uf}} \epsilon_{kj} f_j.$$

Some of the most iconic equations in the study of cluster algebras are the mutation relations for the cluster variables and coefficients. We can explicitly see the familiar forms of the mutation relations, given in Definition 2.1.2, by applying μ_k^* to the cluster variables $x_i = z^{f_i}$ and $y_i = z^{e_i}$:

$$\mu_k^* x_i' = \begin{cases} x_k^{-1} \left(\prod_{\epsilon_{kj} > 0} x_j^{\epsilon_{kj}} + \prod_{\epsilon_{kj} < 0} x_j^{-\epsilon_{kj}} \right) & i = k \\ x_i & i \neq k \end{cases} \tag{2.3}$$

$$\mu_k^* y_i' = \begin{cases} y_i \left(1 + y_k^{-\operatorname{sgn}(\epsilon_{ik})} \right)^{-\epsilon_{ik}} & i \neq k \\ y_k^{-1} & i = k \end{cases} \tag{2.4}$$

Remark 2.3.5. *Equation (2.3) and Equation (2.4) can be obtained from Equation (2.1) and Equation (2.2) by setting $n = e_i$ and $m = f_i$. For example, consider the mutation of $x_i = z^{f_i}$ and $y_i = z^{e_i}$ in direction k. If $i = k$, then*

$$\mu_k^*(y_k') = \mu_k^* \left(z^{e_k'} \right) = \mu_k^* \left(z^{-e_k} \right) = z^{-e_k} \left(1 + z^{e_k} \right)^{-[-e_k, e_k]} = z^{-e_k} = y_k^{-1}$$

and

$$\begin{aligned}
\mu_k^*(x_k') = \mu_k^* \left(z^{f_k'} \right) &= \mu_k^* \left(z^{-f_k + \sum_{j \in I_{uf}} [-\epsilon_{kj}]_+ f_j} \right) \\
&= z^{-f_k + \sum_{j \in I_{uf}} [-\epsilon_{kj}]_+ f_j} \left(1 + z^{v_k} \right)^{-\langle d_k e_k, -f_k + \sum_{j \in I_{uf}} [-\epsilon_{kj}]_+ f_j \rangle} \\
&= z^{-f_k} \left(\prod_{j \in I_{uf}} z^{[-\epsilon_{kj}]_+ f_j} \right) \left(1 + z^{v_k} \right)^{\langle d_k e_k, f_k \rangle} \\
&= z^{-f_k} \left(\prod_{j \in I_{uf}} z^{[-\epsilon_{kj}]_+ f_j} \right) \left(1 + \prod_{j \in I_{uf}} z^{\epsilon_{kj} f_j} \right)^1 \\
&= z^{-f_k} \left(\prod_{j \in I_{uf},\ \epsilon_{kj} < 0} z^{-\epsilon_{kj} f_j} + \prod_{j \in I_{uf},\ \epsilon_{kj} > 0} z^{\epsilon_{kj} f_j} \right) \\
&= x_k^{-1} \left(\prod_{\epsilon_{kj} < 0} x_j^{-\epsilon_{kj}} + \prod_{\epsilon_{kj} > 0} x_j^{\epsilon_{kj}} \right)
\end{aligned}$$

If $i \neq k$, then

$$\begin{aligned}\mu_k^*(y_i') = \mu_k^*\left(z^{e_i'}\right) &= \mu_k^*\left(z^{e_i+[\epsilon_{ik}]_+e_k}\right)\\ &= z^{e_i+[\epsilon_{ik}]_+e_k}\left(1+z^{e_k}\right)^{-[e_i+[\epsilon_{ik}]_+e_k,e_k]}\\ &= z^{e_i}z^{[\epsilon_{ik}]_+e_k}\left(1+z^{e_k}\right)^{-[e_i,e_k]}\\ &= z^{e_i}z^{[\epsilon_{ik}]_+e_k}\left(1+z^{e_k}\right)^{-\epsilon_{ik}}\end{aligned}$$

If $\epsilon_{ik} > 0$, then

$$z^{e_i}z^{[\epsilon_{ik}]_+e_k}\left(1+z^{e_k}\right)^{-\epsilon_{ik}} = z^{e_i}\left(1+z^{-e_k}\right)^{-\epsilon_{ik}} = z^{e_i}\left(1+z^{-sgn(\epsilon_{ik})e_k}\right)^{-\epsilon_{ik}}.$$

If $\epsilon_{ik} < 0$, then

$$z^{e_i}z^{[\epsilon_{ik}]_+e_k}\left(1+z^{e_k}\right)^{-\epsilon_{ik}} = z^{e_i}\left(1+z^{-sgn(\epsilon_{ik})e_k}\right)^{-\epsilon_{ik}}.$$

Hence, in both cases we have

$$\mu_k^*\left(y_i'\right) = y_i\left(1+y_k^{-sgn(\epsilon_{ik})}\right)^{-\epsilon_{ik}}.$$

Finally, $\mu_k^*(x_i') = \mu_k^*\left(z^{f_i'}\right) = \mu_k^*\left(z^{f_i}\right) = z^{f_i}(1+z^{v_k})^{-\langle d_ke_k,f_i\rangle} = z^{f_i} = x_i$.

Proposition 2.4 of [39] then allows the collection $\{\mathcal{A}_{\mathbf{s}}\}$, where $\mathbf{s}$ ranges over all valid choices of torus seed data for some fixed cluster algebra, to be glued along the open pieces where the μ_k given in Equation (2.1) are defined. This produces a scheme $\mathcal{A}$, known as the $\mathcal{A}$ *cluster variety*. Similarly, the collection $\{\mathcal{X}_{\mathbf{s}}\}$ can be glued using the μ_k given in Equation (2.2) to obtain a scheme $\mathcal{X}$, known as the $\mathcal{X}$ *cluster variety*.

Each choice of torus seed $\mathbf{s}$ has an associated $\mathcal{A}$-cluster algebra $\Gamma(\mathcal{A},\mathcal{O}_{\mathcal{A}})$ and $\mathcal{X}$-cluster algebra $\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$. The $\mathcal{A}$-cluster algebra is generally referred to as the *upper cluster algebra* [4] and consists of the set of *universal Laurent polynomials*, i.e. Laurent polynomials in $\Bbbk[M^\circ]$ which remain Laurent polynomials under all mutation sequences. The *ordinary cluster algebra* itself is actually the subalgebra of the field of fractions $\Bbbk(\mathcal{A}_{\mathbf{s}}) = \Bbbk(x_1,\ldots,x_n)$ of $\mathcal{A}_{\mathbf{s}}$ generated by functions of the form $\{\mu^*(x_i')\}$ where $x_{i'}$ is a coordinate on $\mathcal{A}_{\mathbf{s}'}$ for some mutation equivalent torus seed $\mathbf{s}'$ and μ^* is the appropriate composition of pull-backs specified by the mutation sequence relating $\mathbf{s}$ and $\mathbf{s}'$.

The cluster algebras which arise via this construction are specifically those of *geometric type*.

Principal coefficients

Many of the important results of [41] were obtained via the principal coefficient case. In this section, we will use the *wide* convention for extended exchange matrices. Recall from Section 2.1.1 that an ordinary cluster algebra with principal coefficients has extended exchange matrix $\widetilde{B}$, which has the form

$$\widetilde{B} = \begin{bmatrix} B & I \\ -I & \mathbf{0} \end{bmatrix},$$

where I is the $n \times n$ identity matrix and $\mathbf{0}$ is the $n \times n$ matrix whose entries are all zero.

In this section, we review the construction of ordinary cluster scattering diagrams in the principal coefficient case, loosely following the exposition of Section 3 of [39]. Including the additional information of principal coefficients requires the following modifications to the fixed and torus seed data:

Definition 2.3.6 (Construction 2.11 of [39])**.** *Given fixed data Γ, the fixed data for the cluster variety with principal coefficients, Γ_{prin}, is defined by:*

- *The* double *of the lattice N, $\widetilde{N} := N \oplus M^\circ$, with skew-symmetric bilinear form given by*

$$\{(n_1, m_1), (n_2, m_2)\} = \{n_1, n_2\} + \langle n_1, m_2\rangle - \langle n_2, m_1\rangle.$$

  Here, $\langle\cdot,\cdot\rangle : N \times M^\circ \to \mathbb{Q}$ denotes the canonical pairing given by evaluation, $\langle n, m\rangle \mapsto m(n)$.

- *The unfrozen sublattice $\widetilde{N}_{uf} := N_{uf} \oplus 0 \cong N_{uf}$.*
- *The sublattice $\widetilde{N}^\circ := N^\circ \oplus M$ of $\widetilde{N}$.*
- *The lattice $\widetilde{M} = Hom(\widetilde{N}, \mathbb{Z}) = M \oplus N^\circ$.*
- *The lattice $\widetilde{M}^\circ = M^\circ \oplus N$, which has sublattice $\widetilde{M}$.*
- *The index set $\widetilde{I}$ given by the disjoint union of two copies of I.*
- *The unfrozen index set, $\widetilde{I}_{uf}$ given by thinking of the original I_{uf} as a subset of the first copy of I.*
- *A collection of integers $\{d_i\}_{i\in\widetilde{I}}$ taken such that within each disjoint copy of I, the d_i agree with the original torus seed $\mathbf{s}$.*

Definition 2.3.7 (Construction 2.11 of [39])**.** *Given a torus seed $\mathbf{s}$, the torus seed with principal coefficients $\mathbf{s}_{prin}$ is defined as*

$$\mathbf{s}_{prin} := \{(e_i, 0), (0, f_i)\}_{i\in\widetilde{I}}$$

For ease of notation, we will use i and j to denote indices corresponding to basis elements of the form $(e_i, 0)$ and α and β to denote indices corresponding to basis elements of the form $(0, f_\alpha)$. Because of the way the collection $\{d_i\}$ is chosen, the entries of the matrix $\widetilde{\epsilon}$ defined by the principal fixed data are determined by the following relationships:

$$\widetilde{\epsilon}_{ij} = \epsilon_{ij}, \qquad \widetilde{\epsilon}_{i\beta} = \delta_{i\beta}, \quad \text{and} \quad \widetilde{\epsilon}_{\alpha j} = -\delta_{\alpha j}.$$

That is, $\widetilde{\epsilon}$ is a block matrix of the form

$$\widetilde{\epsilon} = \left[\begin{array}{c|c} \epsilon & I \\ \hline -I & 0 \end{array}\right].$$

As before, the choice of $\mathbf{s}_{\text{prin}}$ defines dual bases for $\widetilde{M}$ and $\widetilde{M}^\circ$. The previous choice of a map $p^* : N \to M^\circ$ allows us to define a map $p^* : \widetilde{N} \to \widetilde{M}^\circ$ as

$$p^*(e_i, 0) = (p^*(e_i), e_i),$$
$$p^*(0, f_\alpha) = (-f_\alpha, 0).$$

The new map $p^* : \widetilde{N} \to \widetilde{M}^\circ$ is now necessarily injective. In fact, $p^* : \widetilde{N} \to \widetilde{M}^\circ$ is actually an isomorphism. The choice of $\mathbf{s}_{\text{prin}}$ also defines the associated algebraic tori

$$\mathcal{X}_{\mathbf{s}_{\text{prin}}} := T_{\widetilde{M}} = \text{Spec } \Bbbk[\widetilde{N}],$$
$$\mathcal{A}_{\mathbf{s}_{\text{prin}}} := T_{\widetilde{N}^\circ} = \text{Spec } \Bbbk[\widetilde{M}^\circ].$$

The principal cluster varieties $\mathcal{X}_{\text{prin}}$ and $\mathcal{A}_{\text{prin}}$ are then obtained by gluing along the birational mutation maps $\mu_k : \mathcal{X}_{\mathbf{s}_{\text{prin}}} \to \mathcal{X}_{\mu_k(\mathbf{s}_{\text{prin}})}$ and $\mu_k : \mathcal{A}_{\mathbf{s}_{\text{prin}}} \to \mathcal{A}_{\mu_k(\mathbf{s}_{\text{prin}})}$, as previously. The principal cluster varieties $\mathcal{X}_{\text{prin}}$ and $\mathcal{A}_{\text{prin}}$ depend solely on the mutation class of $\mathbf{s}$. For more details, we refer the reader to Apendix B of [41].

There are several important observations to make about the principal cluster varieties. First, the ring of global functions on $\mathcal{A}_{\text{prin}}$ is the *upper cluster algebra with principal coefficients at the seed* $\mathbf{s}$. Second, $\mathcal{A}_{\text{prin}}$ has useful relationships with the cluster varieties $\mathcal{X}$ and $\mathcal{A}$ which arise from the natural inclusions

$$\begin{aligned} \widetilde{p}^* : N &\to \widetilde{M}^\circ, \\ n &\mapsto (p^*(n), n) \end{aligned}$$

and

$$\begin{aligned} \pi^* : N &\to \widetilde{M}^\circ, \\ n &\mapsto (0, n). \end{aligned}$$

For any torus seed $\mathbf{s}$, the map $\widetilde{p}^*$ induces the exact sequence of algebraic tori:

$$1 \to T_{N^\circ} \to \mathcal{A}_{\mathbf{s}_{\text{prin}}} \xrightarrow{\widetilde{p}} \mathcal{X}_{\mathbf{s}} \to 1$$

The map $\widetilde{p} : \mathcal{A}_{\mathbf{s}_{\text{prin}}} \to \mathcal{X}_{\mathbf{s}}$ defined by this exact sequence commutes with the mutations μ_k on $\mathcal{A}_{\mathbf{s}_{\text{prin}}}$ and $\mathcal{X}_{\mathbf{s}}$, yielding a morphism $\widetilde{p} : \mathcal{A}_{\text{prin}} \to \mathcal{X}$. Similarly, the T_{N° action on $\mathcal{A}_{\mathbf{s}_{\text{prin}}}$ yields a T_{N° action on $\mathcal{A}_{\text{prin}}$. The quotient $\mathcal{A}_{\text{prin}}/T_{N^\circ}$ is the $\mathcal{X}$-variety. The map π^* induces the projection $\pi : \mathcal{A}_{\text{prin}} \to T_M$. Let $\mathcal{A}_t := \pi^{-1}(t)$. Then the fiber $\mathcal{A}_e$, where $e \in T_M$ is the identity element, is the $\mathcal{A}$-variety.

Cluster scattering diagram construction

To construct a cluster scattering diagram, we begin with a choice of fixed data Γ and initial seed data $\mathbf{s}$ and let $\Bbbk$ be a field of characteristic zero. Let $\sigma \subseteq M_{\mathbb{R}}$ be a strictly convex top-dimensional cone and define an associated monoid $P := \sigma \cap M^\circ$ such that $p_1^*(e_i) \in J := P \backslash P^\times$ for all $i \in I_{\text{uf}}$. Here, $P^\times = \{0\}$ is the group of units of P and J is a monomial ideal in the polynomial ring $\Bbbk[P]$. Let $\widehat{\Bbbk[P]}$ denote the completion of $\Bbbk[P]$ with respect to J.

The construction also requires the assumption that $p_1^* : N_{\text{uf}} \to M^\circ$ is an injective map. It's important to note that this assumption does not hold for all choices of fixed data, but does hold for fixed data corresponding to the principal coefficient case. Because arbitrary cluster algebras can be considered as specializations of the principal coefficient case, it's sufficient for the injectivity assumption to hold for that case.

Set

$$N^+ := N^+_{\mathbf{s}} := \left\{ \sum_{i\in I_{\mathrm{uf}}} a_i e_i \,\middle|\, a_i \geq 0, \sum a_i > 0 \right\}$$

and choose a linear function $d : N \to \mathbb{Z}$ such that $d(n) > 0$ for $n \in N^+$.

Definition 2.3.8. *(Definition 1.4 of [41]) A* wall *in $M_{\mathbb{R}}$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}}) \in (N^+, \widehat{\Bbbk[P]})$ such that for some primitive $n_0 \in N^+$,*

1. *$f_{\mathfrak{d}} \in \widehat{\Bbbk[P]}$ has the form $1 + \sum_{j=1}^{\infty} c_j z^{jp_1^*(n_0)}$ with $c_j \in \Bbbk$*

2. *$\mathfrak{d} \subset n_0^{\perp} \subset M_{\mathbb{R}}$ is a convex rational polyhedral cone with dimension* rank $M - 1$.

We refer to $\mathfrak{d} \subset M_{\mathbb{R}}$ as the support *of the wall $(\mathfrak{d}, g_{\mathfrak{d}})$.*

Let $\mathfrak{m}$ denote the ideal in $\widehat{\Bbbk[P]}$ which consists of formal power series with constant term zero.

Definition 2.3.9 (Definition 1.6 of [41])**.** *A* scattering diagram $\mathfrak{D}$ *for N^+ and $\mathbf{s}$ is a set of walls $\{(\mathfrak{d}, f_{\mathfrak{d}})\}$ such that for every degree $k > 0$, there are a finite number of walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ with $f_{\mathfrak{d}} \neq 1$* mod $\mathfrak{m}^{k+1}$.

For a scattering diagram $\mathfrak{D}$,

$$\mathrm{Supp}(\mathfrak{D}) := \bigcup_{\mathfrak{d}\in\mathfrak{D}} \mathfrak{d},$$

$$\mathrm{Sing}(\mathfrak{D}) := \left(\bigcup_{\mathfrak{d}\in\mathfrak{D}} \partial\mathfrak{d}\right) \cup \left(\bigcup_{\substack{\mathfrak{d}_1,\mathfrak{d}_2\in\mathfrak{D} \\ \dim(\mathfrak{d}_1\cap\mathfrak{d}_2)=n-2}} \mathfrak{d}_1 \cap \mathfrak{d}_2\right)$$

are defined as the support and singular locus of the scattering diagram. When $\mathfrak{D}$ is finite, its support is a finite polyhedral cone complex. A $(n-2)$-dimensional cell of this complex is referred to as a *joint*. In this case, $\mathrm{Sing}(\mathfrak{D})$ is simply the union of the set of all joints of $\mathfrak{D}$. A wall $\mathfrak{d} \subset n_0^{\perp}$ is called *incoming* if $p_1^*(n_0) \in \mathfrak{d}$. Otherwise, $\mathfrak{d}$ is called *outgoing*.

Each wall $\mathfrak{d} \in \mathfrak{D}$ has an associated wall-crossing automorphism.

Definition 2.3.10 (Definition 1.2 of [41])**.** *For $n_0 \in N^+$, let $m_0 := p_1^*(n_0)$ and $f = 1 + \sum_{k=1}^{\infty} c_k z^{km_0}$. Then $\mathfrak{p}_f \in \widehat{\Bbbk[P]}$ denotes the automorphism*

$$\mathfrak{p}_f(z^m) = z^m f^{\langle n_0', m\rangle}$$

where n_0' generates the monoid $\mathbb{R}_{\geq 0} n_0 \cap N^{\circ}$.

These wall-crossing automorphisms can be composed in order to compute automorphisms associated to paths on the scattering diagram that pass through multiple walls. Such compositions are called *path-ordered products*.

Definition 2.3.11. *Let $\gamma : [0,1] \to M_{\mathbb{R}} \backslash Sing(\mathfrak{D})$ be a smooth immersion which crosses walls transversely and whose endpoints aren't in the support of $\mathfrak{D}$. Let $0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$ be a sequence such that at time t_i the path γ crosses the wall $\mathfrak{d}_i$ such that $f_i \neq 1 \mod M^{k+1}$. Definition 2.3.9* *ensures that this is a finite sequence. For each* $i \in \{1, \ldots, s\}$*, set* $\epsilon_i := -sgn(\langle n_i, \gamma'(t_i)\rangle)$ *where* $n_i \in N^+$ *is the primitive vector normal to* $\mathfrak{d}_i$*. For each degree* $k > 0$*, define*

$$\mathfrak{p}^k_{\gamma,\mathfrak{D}} := \mathfrak{p}^{\epsilon_s}_{f_{\mathfrak{d}_{t_s}}} \circ \cdots \circ \mathfrak{p}^{\epsilon_1}_{f_{\mathfrak{d}_{t_1}}},$$

where $\mathfrak{p}_{f_{\mathfrak{d}_{t_i}}}$ *is defined as in Definition 2.3.10. Then,*

$$\mathfrak{p}_{\gamma,\mathfrak{D}} := \lim_{k\to\infty} \mathfrak{p}^k_{\gamma,\mathfrak{D}}.$$

We refer to $\mathfrak{p}_{\gamma,\mathfrak{D}}$ *as a* path-ordered product.

A scattering diagram $\mathfrak{D}$ is *consistent* if $\mathfrak{p}_{\gamma,\mathfrak{D}}$ depends only on the endpoints of γ. Two scattering diagrams, $\mathfrak{D}$ and $\mathfrak{D}'$, are considered *equivalent* if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for all paths γ for which both path-ordered products are defined.

Gross, Hacking, Keel, and Kontsevich consider a particular scattering diagram, referred to as the *cluster scattering diagram*. This diagram is defined by the fixed and torus seed data as follows.

Definition 2.3.12. *Given a set of fixed data* Γ *and torus seed* $\mathbf{s}$*, let* $v_i = p_1^*(e_i)$ *for all* $i \in I_{uf}$*. The* initial scattering diagram $\mathfrak{D}_{in,\mathbf{s}}$ *is defined as*

$$\mathfrak{D}_{in,\mathbf{s}} := \{(e_i^\perp, 1 + z^{v_i}) : i \in I_{uf}\}$$

Example 2.3.13. *Consider the ordinary cluster algebra* $\mathcal{A}\left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$ *with seed data* $\mathbf{s} = ((1,0),(0,1))$*. This algebra has initial scattering diagram*

$$\mathfrak{D}_{in,\mathbf{s}} = \left\{ \left((0,1)^\perp, 1 + z^{(-1,0)}\right), \left((1,0)^\perp, 1 + z^{(0,1)}\right) \right\},$$

which can be drawn as

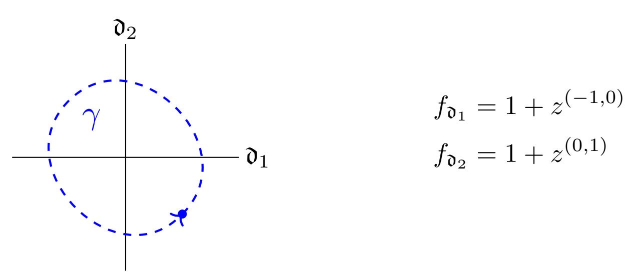


Figure 2.3: The initial cluster scattering diagram for an ordinary cluster algebra of type $A2$, with a loop γ that can be used to check for consistency.

We can see that $\mathfrak{D}_{in,\mathbf{s}}$ is not consistent by computing $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}}(z^{(0,1)})$ as:

$$
\begin{aligned}
z^{(0,1)} &\xmapsto{\mathfrak{d}_1} z^{(0,1)}\left(1+z^{(-1,0)}\right)^{\langle(0,1),(0,-1)\rangle} \\
&= \frac{z^{(0,1)}}{1+z^{(-1,0)}} \\
&\xmapsto{\mathfrak{d}_2} \frac{z^{(0,1)}\left(1+z^{(0,1)}\right)^{\langle(0,1),(1,0)\rangle}}{1+z^{(-1,0)}\left(1+z^{(0,1)}\right)^{\langle(-1,0),(1,0)\rangle}} \\
&= \frac{z^{(0,1)}}{1+\frac{z^{(-1,0)}}{1+z^{(0,1)}}} \\
&= \frac{z^{(0,1)}\left(1+z^{(0,1)}\right)}{1+z^{(0,1)}+z^{(-1,0)}} \\
&\xmapsto{\mathfrak{d}_1} \frac{z^{(0,1)}\left(1+z^{(-1,0)}\right)^{\langle(0,1),(0,1)\rangle}\left(1+z^{(0,1)}\left(1+z^{(-1,0)}\right)^{\langle(0,1),(0,1)\rangle}\right)}{1+z^{(0,1)}\left(1+z^{(-1,0)}\right)^{\langle(0,1),(0,1)\rangle}+z^{(-1,0)}} \\
&= \frac{z^{(0,1)}\left(1+z^{(-1,0)}\right)\left(1+z^{(0,1)}\left(1+z^{(-1,0)}\right)\right)}{1+z^{(0,1)}\left(1+z^{(-1,0)}+z^{(-1,0)}\right)} \\
&= \frac{z^{(0,1)}\left(1+z^{(0,1)}+z^{(-1,1)}\right)}{1+z^{(0,1)}} \\
&\xmapsto{\mathfrak{d}_2} \frac{z^{(0,1)}\left(1+z^{(0,1)}\right)^{\langle(0,1),(-1,0)\rangle}\left(1+z^{(0,1)}+z^{(-1,1)}\left(1+z^{(0,1)}\right)^{\langle(-1,1),(-1,0)\rangle}\right)}{1+z^{(0,1)}} \\
&= \frac{z^{(0,1)}\left(1+z^{(0,1)}+z^{(-1,1)}\left(1+z^{(0,1)}\right)\right)}{1+z^{(0,1)}} \\
&= \frac{z^{(0,1)}\left(1+z^{(0,1)}\right)\left(1+z^{(-1,1)}\right)}{1+z^{(0,1)}} \\
&= z^{(0,1)}\left(1+z^{(-1,1)}\right)
\end{aligned}
$$

In a consistent diagram, we would expect that $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}}(z^m) = z^m$ for any loop γ. Whereas, here $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}}\left(z^{(0,1)}\right) \neq z^{(0,1)}$. To make $\mathfrak{D}_{in,\mathbf{s}}$ consistent, we add the wall $\mathfrak{d}_3 = \left(\mathbb{R}_{\geq 0}(1,-1), 1+z^{(-1,1)}\right)$. Let $\mathfrak{D}_{\mathbf{s}}$ denote this new scattering diagram. Following the same loop γ as above, our calculation now has the additional step

$$
\begin{aligned}
z^{(0,1)}\left(1+z^{(-1,1)}\right) &\xmapsto{\mathfrak{d}_3} z^{(0,1)}\left(1+z^{(-1,1)}\right)^{\langle(0,1),(-1,-1)\rangle}\left(1+z^{(-1,1)}\left(1+z^{(-1,1)}\right)^{\langle(-1,1),(-1,-1)\rangle}\right) \\
&= \frac{z^{(0,1)}}{1+z^{(-1,1)}}\left(1+z^{(-1,1)}\right) \\
&= z^{(0,1)}
\end{aligned}
$$

and we therefore have $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}\left(z^{(0,1)}\right) = z^{(0,1)}$. To finish verifying that this additional wall makes $\mathfrak{D}_{in,\mathbf{s}}$ consistent, we would need to also check that $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}(z^{(1,0)}) = z^{(1,0)}$, as we need to verify consistency for a complete set of basis vectors of $M_{\mathbb{R}} \simeq \mathbb{R}^2$. The computation is omitted from this example, however, because it is very similar to the computation for $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}(z^{(0,1)})$.

Here, the necessary wall-crossing automorphism can be seen by inspection of $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}}\left(z^{(0,1)}\right)$. In cluster scattering diagrams with more than one outgoing wall, it is difficult to determine the support and associated wall-crossing automorphisms of those walls by inspection. Instead, there is a simple *algorithm to produce $\mathfrak{D}_{\mathbf{s}}$ from $\mathfrak{D}_{in,\mathbf{s}}$ which was introduced in two dimensions by Kontsevich and Soibelman [47] and then extended to arbitrary dimension by Gross and Siebert [40].*

The following theorems are two of the major results of [41].

Theorem 2.3.14. *(Theorem 1.12 of [41]) There is a scattering diagram $\mathfrak{D}_{\mathbf{s}}$ such that*

1. *$\mathfrak{D}_{\mathbf{s}}$ is consistent*
2. *$\mathfrak{D}_{\mathbf{s}} \supset \mathfrak{D}_{in,\mathbf{s}}$*
3. *$\mathfrak{D}_{\mathbf{s}} \backslash \mathfrak{D}_{in,\mathbf{s}}$ consists of only outgoing walls*

The diagram $\mathfrak{D}_{\mathbf{s}}$ is unique up to equivalence.

Theorem 2.3.15 (Theorem 1.13 of [41])**.** *The scattering diagram $\mathfrak{D}_{\mathbf{s}}$ is equivalent to a scattering diagram whose walls $(\mathfrak{d}, f_{\mathfrak{d}})$ all have wall-crossing automorphisms of the form $f_{\mathfrak{d}} = (1 + z^m)^c$ for some $m = p^*(n)$, $n \in N^+$, and positive integer c. In particular, all the nonzero coefficients of $f_{\mathfrak{d}}$ are positive integers.*

To illustrate Theorem 2.3.14, we can briefly return to the previous example.

Example 2.3.16. *Consider the ordinary cluster algebra $\mathcal{A}\left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$ with torus seed data $\mathbf{s} = ((1,0),(0,1))$. The initial scattering diagram, $\mathfrak{D}_{in,\mathbf{s}}$, was shown in Example 2.3.13. A cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$ which meets the conditions of Theorem 2.3.14 is shown below:*

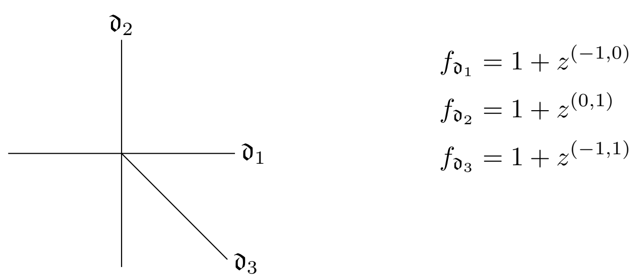


Figure 2.4: A consistent cluster scattering diagram for an ordinary cluster algebra of type $A2$.

To see that $\mathfrak{D}_{\mathbf{s}}$ meets the conditions of Theorem 2.3.14, observe that $\mathfrak{D}_{\mathbf{s}} \supset \mathfrak{D}_{in,\mathbf{s}}$, that $\mathfrak{D}_{\mathbf{s}} \backslash \mathfrak{D}_{in,\mathbf{s}}$ consists of a single outgoing wall, and that the consistency of $\mathfrak{D}_{\mathbf{s}}$ was explicitly verified in Example 2.3.13.

Mutation Invariance

Recall that an ordinary cluster algebra can be equivalently specified by any possible choice of initial cluster - there is no particular canonical choice. In the language of scattering diagrams, this means there should be no special choice of torus seed data. If two torus seeds, $\mathbf{s}$ and $\mathbf{s}'$ are mutation equivalent, we should therefore expect that the corresponding cluster scattering diagrams $\mathfrak{D}_{\mathbf{s}}$ and

$\mathfrak{D}_{\mathbf{s}'}$ are also equivalent. This expectation reflects the fact that $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$ encode information about the same ordinary cluster algebra.

In order to make the notion of mutation invariance precise, we must define the following half-spaces and piecewise linear transformation:

Definition 2.3.17 (Definition 1.22 of [41])**.** *For $k \in I_{uf}$, define*

$$\mathcal{H}_{k,+} := \{m \in M_{\mathbb{R}} : \langle e_k, m\rangle \geq 0\},$$
$$\mathcal{H}_{k,-} := \{m \in M_{\mathbb{R}} : \langle e_k, m\rangle \leq 0\}.$$

The piecewise linear transformation $T_k : M^\circ \to M^\circ$ is defined as

$$T_k(m) := \begin{cases} m + v_k\langle d_k e_k, m\rangle & m \in \mathcal{H}_{k,+} \\ m & m \in \mathcal{H}_{k,-} \end{cases}$$

The shorthand notation $T_{k,-}$ and $T_{k,+}$ is sometimes used to refer to T_k in the respective regions $\mathcal{H}_{k,-}$ and $\mathcal{H}_{k,+}$.

Intuitively, the map T_k gives us a way to "mutate" cluster scattering diagrams. By applying the map T_k to $\mathfrak{D}_{\mathbf{s}}$, we obtain a new cluster scattering diagram $T_k(\mathfrak{D}_{\mathbf{s}})$ via the following algorithm:

1. The wall $\mathfrak{d}_k = (e_k^\perp, 1 + z^{v_k})$ is replaced by $\mathfrak{d}_k' := (e_k^\perp, 1 + z^{-v_k})$,
2. For each wall $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_{\mathbf{s}} \backslash \{\mathfrak{d}_k\}$, there are either one or two walls in $T_k(\mathfrak{D}_{\mathbf{s}})$. The potential walls are
$$(T_k(\mathfrak{d} \cap \mathcal{H}_{k,-}), T_{k,-}(f_\mathfrak{d})) \text{ and } (T_k(\mathfrak{d} \cap \mathcal{H}_{k,+}), T_{k,+}(f_\mathfrak{d})),$$
where $T_{k,\pm}(f_\mathfrak{d})$ denotes the formal power series obtained by applying $T_{k,\pm}$ to each exponent in $f_\mathfrak{d}$. The first wall is dropped if $\dim(\mathfrak{d}) \cap \mathcal{H}_{k,-} < \operatorname{rank}(M) - 1$ and the second wall is dropped if $\dim(\mathfrak{d}) \cap \mathcal{H}_{k,+} < \operatorname{rank}(M) - 1$.

The following theorem justifies why we should think of the action of T_k as mutation of the cluster scattering diagram.

Theorem 2.3.18 (Theorem 1.24 of [41])**.** *If the injectivity assumption holds, then $T_k(\mathfrak{D}_{\mathbf{s}})$ is a consistent scattering diagram and the diagrams $\mathfrak{D}_{\mu_k(\mathbf{s})}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ are equivalent.*

Applying Theorem 2.3.18 multiple times gives the equivalence of $T_{k_\ell} \circ \cdots \circ T_{k_1}(\mathfrak{D}_{\mathbf{s}})$ and $\mathfrak{D}_{\mathbf{s}'}$ where $\mathbf{s}$ and $\mathbf{s}'$ are related by the sequence of mutations $\mu_{k_1}, \ldots, \mu_{k_\ell}$, i.e. $\mathbf{s}' = \mu_{k_\ell} \circ \cdots \circ \mu_{k_1}(\mathbf{s})$.

Chamber Structure

The map T_k also gives rise to a chamber structure on the cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$. Within $\mathfrak{D}_{\mathbf{s}}$, there are two important named chambers.

Definition 2.3.19. *Given a torus seed $\mathbf{s}$, we define $\mathcal{C}_{\mathbf{s}}^{\pm} \subseteq M_{\mathbb{R}}$ as*

$$\mathcal{C}_{\mathbf{s}}^{+} := \{\, m \in M_{\mathbb{R}} |\, \langle e_i, m\rangle \geq 0 \text{ for } i \in I_{uf}\},$$
$$\mathcal{C}_{\mathbf{s}}^{-} := \{\, m \in M_{\mathbb{R}} |\, \langle e_i, m\rangle \leq 0 \text{ for } i \in I_{uf}\}.$$

When $\mathbf{s}$ *is clear from context, we omit the subscript and simply write* $\mathcal{C}^{\pm}$*. We refer to* $\mathcal{C}^{+}$ *as the* positive chamber *and* $\mathcal{C}^{-}$ *as the* negative chamber*.*

The chambers $\mathcal{C}^{\pm}_{\mathbf{s}}$ are closures of connected components of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$. Similarly, the chambers $\mathcal{C}^{\pm}_{\mu_k(\mathbf{s})}$ are closures of connected components of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mu_k(\mathbf{s})})$. We can observe that this means the chambers $T_k^{-1}(\mathcal{C}^{\pm}_{\mu_k(\mathbf{s})})$ are closures of connected components of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$. Further, $\mathcal{C}^{\pm}_{\mathbf{s}}$ and $T_k^{-1}(\mathcal{C}^{\pm}_{\mu_k(\mathbf{s})})$ share a codimension one face with support $e_k^{\perp}$. This creates the following chamber structure on a subset of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$.

Let $\mathfrak{T}$ be a directed infinite rooted tree where each vertex has $|I_{\mathrm{uf}}|$ outgoing edges, labeled by the elements of I_{uf}. Let v be the root of the tree and associate some initial torus seed $\mathbf{s}$ with mutation class $[\mathbf{s}]$ to v. To indicate this choice of initial seed, we write $\mathfrak{T}_v$. An edge with label $k \in I_{\mathrm{uf}}$ corresponds to mutation in direction k. Hence, any simple path beginning at vertex v determines a sequence of mutations according to the sequence of attached edge labels. These mutation sequences determine an associated torus seed $\mathbf{s}$ for each vertex v.

Let w be an arbitrary distinct vertex of $\mathfrak{T}_{\mathbf{s}}$. Then the sequence of edges between v and w determines a map $T_w = T_{k_\ell} \circ \cdots \circ T_{k_1} : M_{\mathbb{R}} \to M_{\mathbb{R}}$. By Theorem 2.3.18, we know that $T_w(\mathfrak{D}_{\mathbf{s}}) = \mathfrak{D}_{\mathbf{s}_w}$ and so the chambers $\mathcal{C}^{\pm}_w := T_w^{-1}(\mathcal{C}^{\pm}_{\mathbf{s}_w})$ are closures of connected components of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$.

Definition 2.3.20 (Definition 1.32 of [41])**.** *Let* $\Delta^{+}_{\mathbf{s}}$ *denote the set of chambers* $\{\mathcal{C}^{\pm}_w\}$ *where* w *runs over the vertices of* $\mathfrak{T}_{\mathbf{s}}$*. The elements of* $\Delta^{+}_{\mathbf{s}}$ *are referred to as* cluster chambers*.*

In fact, this chamber structure coincides with the *Fock-Goncharov cluster complex* [20]. For further details, see Construction 1.30 and Section 2 of [41].

Theta basis

One of the major results of the work of Gross, Hacking, Keel, and Kontsevich [41] is the existence of the *theta basis*, a canonical basis for ordinary cluster algebras. The *theta functions* which form this basis can be defined on scattering diagrams via combinatorial objects called *broken lines*.

Definition 2.3.21 (Definition 3.1 of [41])**.** *Let* $\mathfrak{D}$ *be a scattering diagram,* m_0 *be a point in* $M^{\circ}\backslash\{0\}$*, and* Q *be a point in* $M_{\mathbb{R}}\backslash Supp(\mathfrak{D})$*. A* broken line *with endpoint* Q *and initial slope* m_0 *is a piecewise linear path* $\gamma : (-\infty, 0] \to M_{\mathbb{R}}\backslash Sing(\mathfrak{D})$ *with finitely many domains of linearity. Each domain of linearity,* L*, has an associated monomial* $c_L z^{m_L} \in \Bbbk[M^{\circ}]$ *such that the following conditions are satisfied:*

1. $\gamma(0) = Q$

2. *If* L *is the first domain of linearity of* γ*, then* $c_L z^{m_L} = z^{m_0}$*.*

3. *Within the domain of linearity* L*, the broken line has slope* $-m_L$ *- in other words,* $\gamma'(t) = -m_L$ *on* L*.*

4. *Let* t *be a point at which* γ *is non-linear and is passing from one domain of linearity,* L*, to another,* L'*, and define*
$$\mathfrak{D}_t = \{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} : \gamma(t) \in \mathfrak{d}.\}$$
Then the power series $\mathfrak{p}_{\gamma|_{(t-\epsilon,t+\epsilon)},\mathfrak{D}_t}$ *contains the term* $c_{L'} z^{m_{L'}}$*.*

Broken lines allow for a beautifully concrete and combinatorial definition of a theta function:

Definition 2.3.22 (Definition 3.3 of [41])**.** *Suppose $\mathfrak{D}$ is a scattering diagram and consider points $m_0 \in M^\circ \backslash \{0\}$ and $Q \in M_\mathbb{R} \backslash Supp(\mathfrak{D})$. For a broken line γ with initial exponent m_0 and endpoint Q, we define $I(\gamma) = m_0$, $b(\gamma) = Q$, and $Mono(\gamma) = c(\gamma) z^{F(\gamma)}$ where $Mono(\gamma)$ is the monomial attached to the final domain of linearity of γ. We then define*

$$\vartheta_{Q,m_0} := \sum_\gamma Mono(\gamma)$$

where the summation ranges over all broken lines γ with initial exponent m_0 and endpoint Q. When $m_0 = 0$, then for any endpoint Q we define $\vartheta_{Q,0} = 1$.

One of the key steps in proving that the theta functions form a basis is to show that the cluster variables and *cluster monomials*, i.e. products of cluster variables from a particular cluster, are themselves theta functions. Although we will not reproduce the full proof, we will highlight several important intermediate properties and results. For the full set of definitions and technical details of the proof, we refer the reader to Sections 3, 4, 6, and 7 of [41].

One important property is that theta functions with the same initial slope m_0 but with distinct endpoints Q and Q' are related by a path-ordered product.

Theorem 2.3.23 (Theorem 3.5 of [41])**.** *Let $\mathfrak{D}$ be a consistent scattering diagram, m_0 be a point in $M \backslash \{0\}$, and consider a pair of points Q and Q' in $M_\mathbb{R} \backslash Supp(\mathfrak{D})$ such that Q and Q' are linearly independent over $\mathbb{Q}$. Then for any path γ with endpoints Q and Q' for which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is defined, we have*

$$\vartheta_{Q',m_0} = \mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{Q,m_0})$$

Another important property is the existence of a bijection between broken lines in the diagram $\mathfrak{D}_\mathbf{s}$ and in the diagram $\mathfrak{D}_{\mu_k(\mathbf{s})}$. Because the diagrams $\mathfrak{D}_\mathbf{s}$ and $\mathfrak{D}_{\mu_k(\mathbf{s})}$ correspond to the same cluster algebra, this is a clearly desirable property if the theta functions are going to form a basis. That is, any choice of initial cluster (i.e., cluster corresponding to the positive chamber) should produce the same basis, up to isomorphism.

Proposition 2.3.24 (Proposition 3.6 of [41])**.** *The transformation T_k gives a bijection between broken lines with endpoint Q and initial slope m_0 in $\mathfrak{D}_\mathbf{s}$ and broken lines with endpoint $T_k(Q)$ and initial slope $T_k(m_0)$ in $\mathfrak{D}_{\mu_k(\mathbf{s})}$. In particular,*

$$\vartheta^{\mu_k(\mathbf{s})}_{T_k(Q),T_k(m_0)} = \begin{cases} T_{k,+}\left(\vartheta^{\mathbf{s}}_{Q,m_0}\right) & Q \in \mathcal{H}_{k,+} \\ T_{k,-}\left(\vartheta^{\mathbf{s}}_{Q,m_0}\right) & Q \in \mathcal{H}_{k,-} \end{cases}$$

where the superscript indicates which scattering diagram is used to define the theta function and $T_{k,\pm}$ acts linearly on the exponents in $\vartheta^{\mathbf{s}}_{Q,m_0}$.

Although Proposition 2.3.24 gives a bijection between cluster scattering diagrams generated by seeds that are related by a single mutation, repeated applications of the proposition yield such bijections for any pair of cluster scattering diagrams which correspond to the same cluster algebra.

Finally, the theta functions have structure constants with an elegant combinatorial definition in terms of pairs of broken lines. Let p_1, p_2, and q be points in $\widetilde{M}^\circ_\mathbf{s}$ and z be a generic point in $\widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}$.

There are finitely many pairs of broken lines γ_1, γ_2 such that γ_i has initial slope p_i, both γ_1 and γ_2 have endpoint z, and the sum of the slopes of their final domains of linearity is q. Define

$$a_z(p_1, p_2, q) := \sum_{\substack{(\gamma_1,\gamma_2) \\ I(\gamma_i)=p_i, b(\gamma_i)=z \\ F(\gamma_1)+F(\gamma_2)=q}} c(\gamma_1)c(\gamma_2).$$

Products of theta functions can then be written as

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{q \in \widetilde{M}_{\mathbf{s}}} \alpha_{z(q)}(p_1, p_2, q)\vartheta_q.$$

Hence, the theta functions form a legitimate vector space basis.

Moving from proving results about theta functions on scattering diagrams to proving results about the ordinary cluster algebras requires formalizing the connection between cluster scattering diagrams and cluster varieties. To do so, Gross, Hacking, Keel, and Kontsevich construct a space $\mathcal{A}_{\text{scat}}$ from the cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$ by attaching a copy of the torus T_{N° to each cluster chamber of $\mathfrak{D}_{\mathbf{s}}$ and then gluing these copies according to the birational maps given by the wall-crossing automorphisms. Up to isomorphism, this space is independent of the choice of torus seed $\mathbf{s}$ within a given mutation class. Gross, Hacking, Keel, and Kontsevich then show that the space $\mathcal{A}_{\text{scat}}$ is isomorphic to the cluster variety $\mathcal{A}_{\mathbf{s}}$.

Once this identification is made, it's then possible to formalize the relationship between the theta functions and cluster monomials. Consider a set of fixed data Γ and torus seeds $\mathbf{s}$, $\mathbf{s}_w = (e'_1, \ldots, e'_n)$. In this geometric context, a *cluster monomial on* $\mathbf{s}_w$ is defined as a monomial on $T_{N^\circ,w} \subset \mathcal{A}$ of the form z^m where $m = \sum_{i=1}^n a_i f'_i$ with all a_i non-negative. Such monomials extend to regular functions on $\mathcal{A}$. A *cluster monomial on* $\mathcal{A}$ is then defined as a regular function which is a cluster monomial on some torus seed of $\mathcal{A}$. The following theorem identifies the cluster monomials on $\mathcal{A}$ with theta functions.

Theorem 2.3.25 (Theorem 4.9 of [41]). *Let Γ be a set of fixed data which satisfies the Injectivity Assumption and $\mathbf{s}$ be a choice of torus seed. Consider a point $Q \in \mathcal{C}^+_{\mathbf{s}}$ and $m \in \sigma \cap M^\circ$ for some chamber $\sigma \in \Delta^+_{\mathbf{s}}$. Then $\vartheta_{Q,m}$ is a positive Laurent polynomial which expresses a cluster monomial of $\mathcal{A}$ in terms of the initial torus seed $\mathbf{s}$. Further, all cluster monomials can be expressed in this way.*

In Section 7, Gross, Hacking, Keel, and Kontsevich define the notions of *middle* and *upper* cluster algebras in order to prove that the theta functions form a basis for the ordinary cluster algebra. Their proof works primarily with $\mathcal{A}_{\text{prin}}$ and is then extended to $\mathcal{A}$ and $\mathcal{X}$ by using the fact that these varieties appear as a fiber and quotient, respectively, of $\mathcal{A}_{\text{prin}}$.

The *middle cluster algebra* associated to $\mathcal{A}_{\text{prin}}$ is defined as

$$\operatorname{mid}(\mathcal{A}_{\text{prin}}) := \bigoplus_{q \in \Theta} \Bbbk \cdot \vartheta_q,$$

where $\Theta \subset \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ is the collection of m_0 such that for any generic point $Q \in \sigma \in \Delta^+$, there are finitely many broken lines with initial slope m_0 and endpoint Q. By design, the structure constants $\alpha_z(p_1, p_2, q)$ make $\operatorname{mid}(\mathcal{A}_{\text{prin}})$ into an associative and commutative $\Bbbk[N]$-algebra.

The *upper cluster algebra* associated to $\mathcal{A}_{\text{prin}}$ is defined as

$$\text{up}\,(\mathcal{A}_{\text{prin}}) := \Gamma(\mathcal{A}_{\text{prin}}, \mathcal{O}_{\mathcal{A}_{\text{prin}}}),$$

and the *ordinary cluster algebra* $\text{ord}(\mathcal{A}_{\text{prin}})$ is the subalgebra of $\text{up}\,(\mathcal{A}_{\text{prin}})$ generated by the set of global monomials on $\mathcal{A}_{\text{prin}}$, i.e. the set of regular functions on $\mathcal{A}_{\text{prin}}$ which restrict to a character on some torus in its atlas.

Gross, Hacking, Keel, and Kontsevich show there are canonical inclusions

$$\text{ord}\,(\mathcal{A}_{\text{prin}}) \subset \text{mid}\,(\mathcal{A}_{\text{prin}}) \subset \text{up}\,(\mathcal{A}_{\text{prin}})\,,$$

and that therefore the theta functions form a basis for the ordinary cluster algebra when the ordinary cluster algebra and upper cluster algebra coincide.

2.4 Orbifolds

An orbifold is a generalization of a manifold where the local structure is instead given by quotients of open subsets of $\mathbb{R}^n$ under finite group actions. Orbifolds were first introduced as *V-manifolds* by Satake in 1956, in the context of modular and automorphic forms [69]. In the 1970s, Thurston subsequently introduced the term *orbifold* when studying the geometry of 3-manifolds [72].

In the context of cluster algebras from orbifolds, it is simplest to think of orbifolds as surfaces with isolated singular points, referred to as *orbifold points*. An orbifold point of order p has an associated constant $\lambda_p = 2\cos(\pi/p)$. In our context, λ_p arises geometrically from the length of diagonals in an equilateral p-gon which appears in a particular covering space called the *p-fold cover* [13, 48]. In the literature, it also appears in the work of Holm and Jørgensen on non-integral frieze patterns from polygon dissections [44].

Definition 2.4.1. *An* arc γ *on an orbifold* $\mathcal{O} = (S, M, Q)$ *is a non-self-intersecting curve in* S *with endpoints in* M *that is otherwise disjoint from* M*,* Q*, and* $\partial\mathcal{O}$*. Curves that are contractible onto* $\partial\mathcal{O}$ *are not considered arcs. Arcs are considered up to isotopy class. An arc which cuts out an unpunctured monogon with exactly one point in* Q *is called a* pending arc*, while all other arcs are called* standard arcs*.*

There are two common ways to draw pending arcs, as shown below.

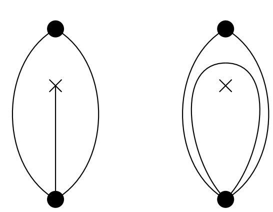


Figure 2.5: Two equivalent ways to draw pending arcs on a triangulated orbifold.

Within this thesis, we draw pending arcs as arcs cutting out unpunctured monogons that contain exactly one orbifold point, as shown on the right hand side, as this is more geometrically suggestive. In particular, this makes it clear that if a pending arc crosses another arc, it necessarily does so an even number of times. We sometimes refer to a pending arc as being *incident* to the orbifold point it encloses, in the spirit of the left hand side.

Pending arcs appear in two types of triangles: bigons with one orbifold point and monogons with two orbifold points.

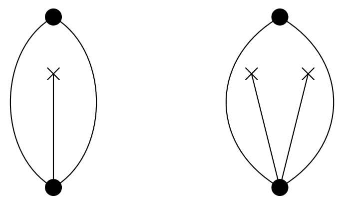


Figure 2.6: Types of triangles that contain pending arcs.

While in many pictures it appears that the pending arc is in a bigon, we can identify the two vertices to recover a monogon with two orbifold points. There is one special case - a sphere with one marked point and three orbifold points, which has exactly one triangle made up of three pending arcs, as in Table 3.5 of [17]. Our construction works in this special case as well.

We also consider generalized arcs, which may contain self-intersections. Allowing self-intersections introduces the possibility of arcs winding around orbifold points. By convention, we consider counterclockwise winding to be positive and clockwise winding to be negative. A generalized arc exhibits modular behavior when winding around an orbifold point. A winding arc can have up to $p-1$ self-intersections due to winding around an orbifold point of order p. Once the number of self-intersections reaches p, the winding arc is isotopic to an arc with no self-intersections - i.e., one that does not wind around the orbifold point at all. If a winding arc has $k > p$ self-intersections, then it is isotopic to an arc with $k \mod p$ self-intersections and $\lfloor \frac{k}{p} \rfloor$ contractible kinks. The below diagram shows examples of possible winding behavior around an orbifold point of order 4.

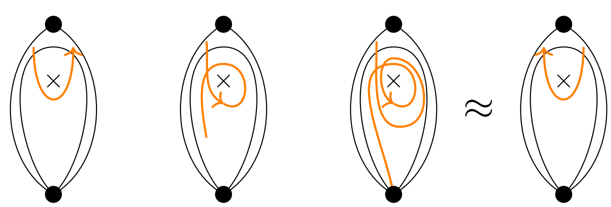


Figure 2.7: An example of the possible distinct types of winding behavior around an orbifold point of order $p = 4$.

For an orbifold point of order p, winding counter-clockwise with k self-intersections is isotopic to winding clockwise with $(p-1)-k$ self-intersections. For convenience, we use the phrasing "winding k times" to refer to winding with k self-intersections. So, for example, "winding 0 times" simply refers to crossing a pending arc twice with no self-intersections occurring between those crossings.

It is also possible to have closed curves with no self-intersections which we refer to as *loops*. A non-contractible loop is often called an *essential loop*.

Triangulated orbifolds provide geometric realizations for some ordinary cluster algebras which cannot be realized as triangulated surfaces. This realization is due to Felikson, Shapiro, and Tumarkin, who describe a correspondence between skew-symmetrizable ordinary cluster algebras and triangulated orbifolds [18]. In a later paper, Felikson and Tumarkin generalize the bracelet, bangle, and band bases to ordinary cluster algebras from unpunctured orbifolds with at least two marked points on the boundary [19]. Çanakçi and Tumarkin later showed that the assumption about the number of marked points on the boundary can be removed and extended the snake graph and band graph constructions to triangulated orbifolds which correspond to ordinary cluster algebras [9].

Triangulations of unpunctured orbifolds can also provide geometric realizations for some skew-symmetrizable quantum cluster algebras. Huang demonstrates this realization and uses it to give a proof of positivity for such quantum cluster algebras [43].

Covering spaces

When working with triangulated orbifolds, it is often useful to consider some covering space. Which particular covering space is most useful varies depending on the application, but covering spaces that appear in the literature include the *associated orbifolds* of Felikson, Shapiro, and Tumarkin [17] and the polygonal *p-fold covering* of an orbifold with a single orbifold point of order p [13, 48].

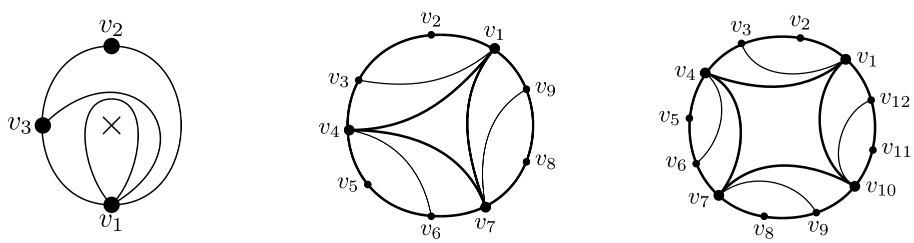


Figure 2.8: An example of a triangulated orbifold with a single orbifold point of order p (left) and the p-fold covering spaces for $p = 3$ (middle) and $p = 4$ (right).

Laminations

In Section 6 of [17], Felikson, Shapiro, and Tumarkin extend Fomin and Thurston's work on laminations [23] in order to track coefficients for cluster algebras from orbifolds. Although they define laminations on an object called an *associated orbifold* and we will work with laminations on the original orbifold, much of their work will transfer to our setting. We use their definition of a lamination on an orbifold.

Definition 2.4.2 (Definition 6.1 of [17])**.** *Let $\mathcal{O} = (S, M, Q)$ be an unpunctured orbifold. An* integral unbounded measured lamination *(henceforth, just a* lamination*) on $\mathcal{O}$ is a finite collection of non-self-intersecting and pairwise non-intersecting curves on $\mathcal{O}$ such that:*

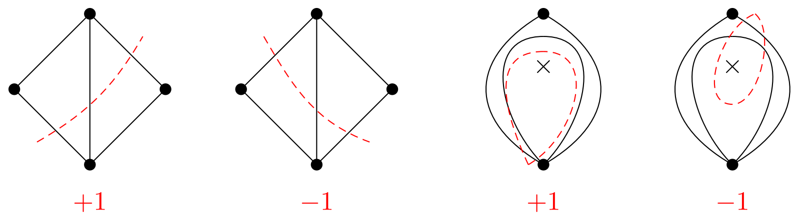
- *Each curve is either a closed curve or a non-closed curve for which each end is either an unmarked point on the boundary of $\mathcal{O}$ or an orbifold point in Q.*
- *No curve bounds an unpunctured disk or a disk containing a unique point of $M \cup Q$.*
- *No curve with both endpoints on the boundary of $\mathcal{O}$ is isotopic to a portion of the boundary containing either no or one marked point(s).*
- *No two curves begin at the same orbifold point.*

For the associated shear coordinates, however, we adopt a modified definition. The difference in our definition stems from the fact that we draw pending loops as arcs around orbifold points, rather than as having one endpoint at an orbifold point.

Definition 2.4.3. *Let $\mathcal{O}$ be an orbifold with triangulation T and L be a lamination on $\mathcal{O}$. For each arc $\gamma \in T$, the* shear coordinate *of L with respect to T is*

$$b_\gamma(T, L) = \sum_i b_\gamma(T, L_i)$$

where the summation runs over all individual curves in L. The shear coordinates $b_\gamma(T, L_i)$ are defined as:



To understand why these are the natural shear coordinate definitions for pending arcs, consider the corresponding view in the covering space. A pending curve L_i for which $b_\gamma(T, L_i) = +1$ appears in the covering space as

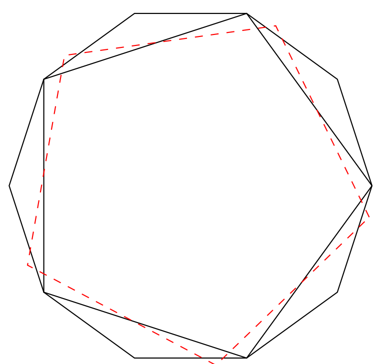


Figure 2.9: The image in the p-fold covering space of a pending curve L_i with $b_\gamma(T, L_i) = +1$.

Notice that each copy of the lamination L_i crosses a copy of the pending arc twice. One of these crossings contributes $+1$ to the shear coordinate and the other crossing contributes 0, for a net shear coordinate of $+1$. The picture for $b_\gamma(T, L_i) = -1$ is analogous.

This extended shear coordinate definition then allows us to apply the usual definition of an elementary lamination to both standard and pending arcs. Recall that if τ_i is a standard arc, the corresponding elementary lamination L_i can be found by shifting its endpoints clockwise. Similarly, if τ_i is a pending arc, then L_i can be found by shifting the singular endpoint clockwise. Examples of both are shown below.

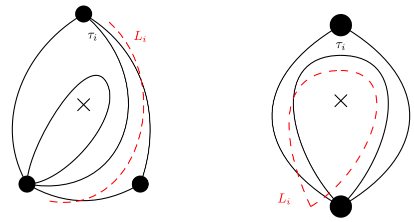


Figure 2.10: Examples of the elementary laminations corresponding to standard and pending arcs

Other types of crossings contribute 0 to the shear coordinate of the pending arc. If we look at these crossings in the cover, they resemble crossings in standard triangulations that contribute 0 to the shear coordinates.

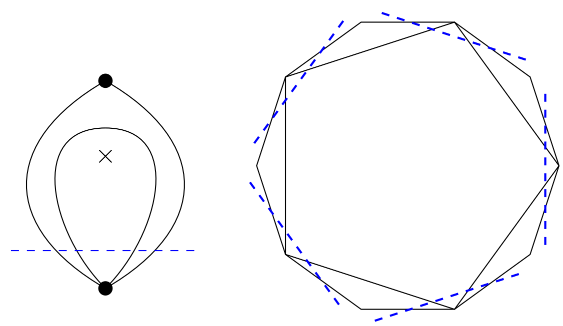
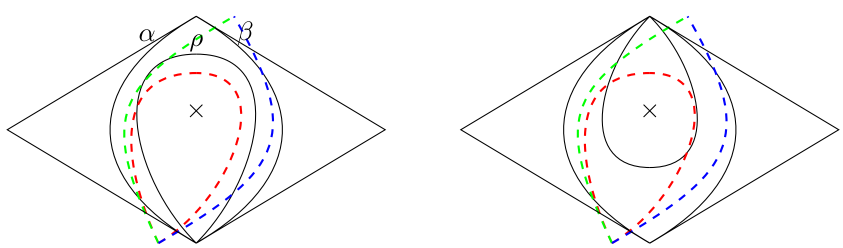


Figure 2.11: Examples of crossings which contribute 0 to the shear coordinate of a lamination.

However, these new elementary laminations associated to pending arcs will contribute 2 to a standard arc when they cross in a meaningful way. This contribution of 2 is also seen in generalized mutation rules.



$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \to \qquad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

Figure 2.12: An example of how flipping a pending arc impacts the shear coordinates of a lamination. The left-hand side of the diagram is an example of an elementary lamination from a triangulated orbifold.

On the lefthand side of Figure 2.12, we have the elementary lamination associated to this triangulation. When we flip the pending arc, then the lamination associated to the pending arc intersects α non-trivially twice. This matches the result of mutating (with generalized mutation) the extended

B-matrix associated to the left-hand picture at the index representing the pending arc. The third column and row correspond to the pending arc.

Remark 2.4.4. *The mutation of the extended portion of the B-matrix resembles the result of mutating at an orbifold point of weight 2 in the sense of Felikson, Shapiro, and Tumarkin [18]. While the dynamics of our x-variables mimic those of orbifold points of weight $\frac{1}{2}$, the y-variables more closely resemble those from orbifold points of weight 2. This seems to follow from the tropical duality between c-vectors and g-vectors, given by Nakanishi and Zelevinsky in [63].*

2.5 Generalized cluster algebras

This thesis is concerned with a particular generalization of ordinary cluster algebras which is due to Chekhov and Shapiro [13]. In a *generalized cluster algebra*, the hallmark binomial exchange relations of an ordinary cluster algebra are allowed to instead be potentially longer polynomials of a particular form. Such algebras are referred to in the literature both as *Chekhov-Shapiro algebras* and as generalized cluster algebras. We will exclusively use the later term. In this section, we will use the wide convention for exchange matrices.

To formally define generalized cluster algebras, we begin with the notion of a *generalized cluster seed*. Recall that $(\mathbb{P}, \oplus, \cdot)$ is an arbitrary semifield.

Definition 2.5.1. *A* labeled generalized cluster seed *is a quintuple $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$ such that*

- $\mathbf{x} = (x_1, \dots, x_n)$ *is a free generating set for $\mathcal{F}$,*
- $\mathbf{y}$ *is an n-tuple with elements in $\mathbb{P}$,*
- $B = [b_{ij}]$ *is an $n \times n$ skew-symmetrizable matrix with entries in $\mathbb{Z}$,*
- R *is an $n \times n$ diagonal matrix with positive integer entries whose i-th diagonal entry is denoted by r_i,*
- *and* $\mathbf{a} = (a_{i,j})_{i \in [n], j \in [r_i - 1]}$ *is a collection of formal variables, typically specialized to elements of $\mathbb{P}$.*

We refer to $\mathbf{x} = (x_1, \dots, x_n)$ *as the* cluster *of* Σ, $\mathbf{y} = (y_1, \dots, y_n)$ *as the* coefficient tuple, *B as the* generalized exchange matrix, *R as the exchange degree matrix, and $\mathbf{a}$ as the exchange coefficient collection. We refer to the elements $x_1, \dots, x_n$ as the* cluster variables *of Σ and to the elements $y_1, \dots, y_n$ as the* coefficient variables *of Σ.*

Together, the exchange degree matrix R and the exchange coefficient collection $\mathbf{a}$ determine a set of *exchange polynomials* $\rho_1, \dots, \rho_n$, where

$$\rho_i(u) = 1 + a_{i,1}u + \cdots + a_{i,r_i-1}u^{r_i-1} + u^{r_i} \in \mathbb{ZP}[u, a_{i,1}, \dots, a_{i,r_i-1}].$$

The structure of the exchange relations for mutation in direction k are determined by the k-th exchange polynomial.

Definition 2.5.2. *For a generalized cluster seed* $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$*, generalized mutation in direction* k*,* $\mu_k^{(r)}$*, is defined by the following exchange relations:*

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + r_k\left([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+\right) & i, j \neq k \end{cases}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i \left(y_k^{[b_{ik}]_+}\right)^{r_k} \left(\bigoplus_{j=0}^{r_k} a_{k,j} y_k^j\right)^{-b_{ik}} & i \neq k \end{cases}$$

$$x'_i = \begin{cases} x_k^{-1} \left(\prod_{j=1}^{n} x_j^{[-b_{kj}]_+}\right)^{r_k} \frac{\sum_{j=0}^{r_k} a_{k,j} \widehat{y}_k^j}{\bigoplus_{j=0}^{r_k} a_{k,j} y_k^j} & i = k \\ x_i & i \neq k \end{cases}$$

$$a_{k,j} = a_{k,r_k-j}$$

where $[\cdot]_+ = max(\cdot, 0)$ *and*

$$\widehat{y}_i := y_i \prod_{j=1}^{n} x_j^{b_{ij}}$$

Remark 2.5.3. *The mutation relation for* b'_{ij} *in Definition 2.5.2 is for the generalized exchange matrix,* B*. This is equivalent to writing that the matrix* BR *mutates according to the relation*

$$(br)'_{ij} = (br)_{ij} + ([-(br)_{ik}]_+ (br)_{kj} + (br)_{ik}[(br)_{kj}]_+).$$

At the matrix level, this reflects the fact that mutation commutes with right multiplication by R*: that is,* $\mu_k(BR) = \mu_k^{(r)}(B)R$*, where* μ_k *denotes ordinary matrix mutation and* $\mu_k^{(r)}$ *denotes the generalized matrix mutation given in Definition 2.5.2.*

Remark 2.5.4. *In their original paper, Chekhov and Shapiro use matrices* B *and* β *in their exchange relations [13]. Their* β *matrix is our generalized exchange matrix,* B*, and their* B *matrix is our* BR *matrix. Note that if the matrix* B *is skew-symmetrizable, then the matrix* BR *is also skew-symmetrizable.*

As in the ordinary case, defining a generalized cluster algebra first requires us to establish the notion of a *generalized cluster pattern.*

Definition 2.5.5. *Let* $\mathfrak{T}_n$ *be the* n*-regular tree whose edges are labeled by* $1, \ldots, n$ *such that each edge incident to a given vertex has a different label. A* generalized cluster pattern *is an assignment of labeled generalized cluster seeds* $\Sigma_t = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$ *to each vertex* $t \in \mathfrak{T}_n$ *such that seeds assigned to adjacent vertices* $t - t'$*, whose mutually incident edge has label* k*, are related by generalized cluster seed mutation in direction* k*. We use the notation*

$$\begin{aligned} \mathbf{x}_t &= (x_{1;t}, \ldots, x_{n;t}), \\ \mathbf{y}_t &= (y_{1;t}, \ldots, y_{n;t}), \\ B_t &= [b_{ij}^t], \\ R_t &= [r_{ij}^t], \\ \mathbf{a}_t &= (a_{i,j}^t)_{i \in [n], j \in [r_i - 1]} \end{aligned}$$

In a slight abuse of notation, we will use $\mathfrak{T}_n$ to denote both generalized cluster patterns and ordinary cluster patterns. Because it's clear from context whether we mean an ordinary or generalized cluster pattern, this should not create confusion.

Definition 2.5.6. *For a fixed choice of generalized cluster pattern $\mathfrak{T}_n$, let*

$$\mathcal{X} := \bigcup_{t \in \mathfrak{T}_n} \mathbf{x}_t = \{x_{i;t} : t \in \mathfrak{T}_n, i \in [n]\}$$

be the union of the clusters of each generalized cluster seed in the generalized cluster pattern. We refer to the elements $x_{i;t} \in \mathcal{X}$ as cluster variables*. The* generalized cluster algebra $\mathcal{A}$ *associated to the generalized cluster pattern is the $\mathbb{ZP}[a_{i,j}]$-subalgebra of $\mathcal{F}$ generated by the cluster variables, $\mathcal{A} := \mathbb{ZP}[a_{i,j}][\mathcal{X}]$. We often write $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$ to denote the generalized cluster algebra associated to the generalized cluster pattern containing the seed $(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$.*

Remark 2.5.7. *If all of the formal variables $\{a_{j,s}\}$ are specialized to elements of $\mathbb{P}$, then the generalized cluster algebra is simply a $\mathbb{ZP}$-subalgebra of $\mathcal{F}$.*

This does create some potential for notational confusion because it is also common to use $\mathcal{A}$ when referring to the $\mathcal{A}$-variety. It is typically clear from context whether $\mathcal{A}$ refers to a (generalized) cluster algebra or (generalized) cluster variety and therefore we use this notation in order to be consistent with the literature.

Finally, we can give a statement of the Laurent phenomenon for generalized cluster algebras.

Theorem 2.5.8 (Theorem 2.5 of [13])**.** *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$ be an arbitrary generalized cluster algebra. Its cluster variables can be expressed in terms of any cluster of $\mathcal{A}$ as Laurent polynomials with coefficients in $\mathbb{ZP}[a_{i,j}]$.*

Note that in the above theorem, the Laurent polynomial coefficients are in $\mathbb{ZP}$. Further, Chekhov and Shapiro prove that for a particular subclass of generalized cluster algebras, these coefficients are strictly non-negative:

Theorem 2.5.9 (c.f. Section 5 of [13])**.** *Let $\mathcal{A}$ be any generalized cluster algebra whose exchange polynomials are all reciprocal and of degree at most two. Then its cluster variables can be expressed in terms of any cluster of $\mathcal{A}$ as Laurent polynomials with non-negative coefficients in $\mathbb{ZP}[a_{i,j}]$.*

Our generalized snake graph construction gives an alternate explicit combinatorial proof of this theorem. Positivity remains conjectural, however, for arbitrary generalized cluster algebras.

We will consider two subclasses of generalized cluster algebras: reciprocal generalized cluster algebras and generalized cluster algebras from orbifolds.

Reciprocal generalized cluster algebras

When we construct generalized cluster scattering diagrams in Section 4, we will work in the same specialized setting as [61] and impose the additional requirement that $a_{i,j} = a_{i,r_i - j}$ - i.e., that all exchange polynomials are *reciprocal polynomials*. We refer to generalized cluster algebras whose exchange polynomials are all reciprocal polynomials as *reciprocal generalized cluster algebras*. Note that because reciprocal polynomials are fixed under the generalized mutation $\mu_k^{(r)}$ of Definition 2.5.2, all exchange polynomials of reciprocal generalized cluster algebras are fixed under mutation. Hence, the tuple $\mathbf{a} = (\mathbf{a_1}, \ldots, \mathbf{a_n})$ is the same in every seed associated to a given reciprocal generalized cluster algebra. Restricting our attention to reciprocal generalized cluster algebras allows us to focus on a more tractable subclass of generalized cluster algebras.

Nakanishi has shown that many useful structural properties of ordinary cluster algebras still hold in this specialized setting [61]. In particular, he gives definitions of c-vectors, F-polynomials, and g-vectors for reciprocal generalized cluster algebras.

Before stating these definitions, we will need a notion of a generalized cluster pattern with *principal coefficients*. As such, we will temporarily regard the coefficients $\mathbf{y} = (y_1, \ldots, y_n)$ as formal variables. Let $\mathrm{Trop}(\mathbf{y}, \mathbf{a})$ denote the *tropical semifield* of $\mathbf{y}$ and $\mathbf{a}$. This is the multiplicative abelian group freely generated by $\mathbf{y}$ and $\mathbf{a}$, where the tropical sum $\oplus$ is defined as

$$\left(\prod_i y_i^{d_i} \prod_{i,s} a_{i,s}^{d_{i,s}}\right) \oplus \left(\prod_i y_i^{e_i} \prod_{i,s} a_{i,s}^{e_{i,s}}\right) = \prod_i y_i^{\min(d_i, e_i)} \prod_{i,s} a_{i,s}^{\min(d_{i,s}, e_{i,s})},$$

where $d_i, d_{i,s}, e_i$, and $e_{i,s}$ are integers. Then,

Definition 2.5.10 (Definition 3.1 of [61])**.** *A* generalized cluster pattern with principal coefficients *is a generalized cluster pattern in $\mathbb{P} = \mathit{Trop}(\mathbf{y}, \mathbf{a})$ with initial seed $(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$, where $\mathbf{x}, B$, and R are arbitrary.*

We can then extend the notions of c-vectors, F-polynomials, and g-vectors to generalized cluster patterns with principal coefficients. Let $\mathbb{Q}_{\mathrm{sf}}$ be the *universal semifield* of $\mathbf{y}$ and $\mathbf{a}$, consisting of rational functions in $\mathbf{y}$ and $\mathbf{a}$ which have subtraction-free expressions. [26]

Definition 2.5.11 (Definitions 3.2 and 3.4 of [61])**.** *Consider the generalized cluster pattern Λ associated to the initial seed $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$. By the Laurent phenomenon, each x-variable x_i^t of Λ can be expressed as $X_i^t(\mathbf{x}, \mathbf{y}, \mathbf{a}) \in \mathbb{Z}\mathbb{P}[\mathbf{x}^{\pm 1}]$ where $\mathbb{P} = \mathit{Trop}(\mathbf{y}, \mathbf{a})$. Each y-variable y_i^t of Λ can be expressed as a subtraction-free rational function $Y_i^t(\mathbf{y}, \mathbf{a}) \in \mathbb{Q}_{sf}(\mathbf{y}, \mathbf{a})$. We refer to the X_i^t as* X-functions *and to the Y_i^t as* Y-functions.

For any generalized cluster pattern in $\mathbb{P}$ with a given initial exchange matrix, we have $y_i^t = Y_i^t|_{\mathbb{P}(\mathbf{y},\mathbf{a})}$ where the right hand side is understood to be the evaluation of $Y_i^t(\mathbf{y}, \mathbf{a})$ in $\mathbb{P}$ [61]. Moreover, each $y_i^t \in \mathrm{Trop}(\mathbf{y}, \mathbf{a})$ is a Laurent monomial in $\mathbf{y}$ with coefficient 1 [61]. Together, these two facts allow us to define the following useful set of matrices and vectors.

Definition 2.5.12 (Definition 3.7 of [61])**.** *Consider the generalized cluster pattern with principal coefficients Λ. Each y-variable y_j^t of Λ can be expressed as*

$$y_j^t = Y_j^t\big|_{\mathit{Trop}(\mathbf{y},\mathbf{a})(\mathbf{y},\mathbf{a})} = \prod_{i=1}^{n} y_j^{c^t_{c_{ij}}}$$

We refer to the resulting matrix $C^t = [c_{ij}^t]_{i,j=1}^n$ as the C-matrix *of Λ and to its column vectors $c_j^t = [c_{ij}^t]_{i=1}^n$ as the* c-vectors *of Λ.*

Analogously to the ordinary case, Nakanishi gives the following definition for *F-polynomials.*

Definition 2.5.13 (Definition 3.11 of [61]). *Let Λ be a generalized cluster pattern with principal coefficients. For each $t \in \mathfrak{T}_n$ and $i \in [n]$, the specialization of the X-function $X_i^t(\mathbf{x}, \mathbf{y}, \mathbf{a})$ of Λ to $x_1 = \cdots = x_n = 1$ defines a polynomial $F_i^t(\mathbf{y}, \mathbf{a})$. We refer to these polynomials as the* F-polynomials *of Λ.*

Defining g-vectors requires that we first establish a $\mathbb{Z}^n$-grading on $\mathbb{Z}[\mathbf{x}^{\pm 1}, \mathbf{y}, \mathbf{a}]$. Consider the generalized cluster pattern with principal coefficients Λ associated to the initial seed $(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$. Following the work of [26] for the ordinary case, Nakanishi [61] defines:

$$\begin{aligned} \deg(x_i) &:= \mathbf{e}_i, \\ \deg(y_i) &:= -\mathbf{b}_j, \\ \deg(a_{i,s}) &:= 0, \end{aligned}$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector and $\mathbf{b}_j = \sum_{i=1}^n b_{ij}\mathbf{e}_i$ is the j-th column of the initial exchange matrix. Note that because $\hat{y}_i = y_i \prod_{j=1}^n x_j^{b_{ji}}$, we have

$$\deg(\hat{y}_i) = \deg(y_i) + \sum_{j=1}^n b_{ji}\deg(x_j) = -\mathbf{b}_j + \mathbf{b}_j = 0$$

Another key property of this grading is that the X-functions are homogeneous with respect to it [61]. Hence, the degree vector of each X_i^t is well-defined.

Definition 2.5.14 (Definition 3.14 of [61]). *Let Λ be the generalized cluster pattern with principal coefficients associated to the initial seed $(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$. Then we can express the degree vector of each X-function X_i^t as*

$$deg\left(X_j^t\right) = \sum_{i=1}^n g_{ij}^t \mathbf{e}_i.$$

We refer to the resulting matrix $G^t = [g_{ij}^t]_{i,j=1}^n$ as the G-matrix *of Λ and to its column vectors $g_j^t = [g_{ij}^t]_{i=1}^n$ as the* g-vectors *of Λ.*

Generalized cluster algebras from orbifolds

Recall that a subset of ordinary cluster algebras have a geometric realization in terms of triangulated surfaces. Analogously, there is a subset of generalized cluster algebras which have a geometric realization in terms of triangulated orbifolds. In such generalized cluster algebras, the exchange polynomials are either binomials of the form $z_i = 1 + u$, when the cluster variable x_i is associated to a standard arc, or trinomials of the form $z_i = 1 + \lambda_p u + u^2$, when x_i is associated to a pending arc incident to an orbifold point of order p.

Triangulated orbifolds first arose in the context of cluster algebras when Felikson, Shapiro, and Tumarkin [17, 18] studied *unfoldings* of skew-symmetrizable ordinary cluster algebras. Later, Chekhov and Shapiro [13] showed that mutations for orbifold points of order $p \geq 2$ are given by trinomial exchange relations with reciprocal coefficients (when $p = 2$, $\lambda_p = 0$ and the exchange polynomial $1 + \lambda_p u + u^2$ reduces to the binomial $1 + u$). Chekhov and Shapiro also showed that both the Laurent phenomenon and positivity hold for such generalized cluster algebras using arguments similar to those given by Fomin and Zelevinsky [25, 27] for ordinary cluster algebras. Later, Labardini-Fragoso and Velasco [48] showed that the generalized cluster algebras associated to polygons with a single orbifold point of order 3 are equivalent to Caldero-Chapoton algebras of quivers with relations arising from this polygon.

When working with triangulated orbifolds, it is often useful to consider some covering space. Which particular covering space is most useful varies depending on the application, but covering spaces that appear in the literature include the *associated orbifolds* of Felikson, Shapiro, and Tumarkin [17] and the polygonal *p-fold covering* of an orbifold with a single orbifold point of order p [13, 48]. In Chapter 3, we will define and make extensive use of this covering space.

2.5.1 Companion algebras

Given a generalized cluster algebra, the associated *companion algebras* are a pair of ordinary cluster algebras which encode the data of the generalized cluster algebra. Companion algebras were first defined by Nakanishi [61] for reciprocal generalized cluster algebras which meet the *normalization condition* that $\oplus_{j=0}^{r_i} a_{i,j} = 1$ for all $i \in [n]$. In subsequent work by Nakanishi and Rupel [62], this definition was extended to the more general subclass of reciprocal generalized cluster algebras which meet the weaker *power condition* $\frac{a_{i,r_i}}{a_{i,0}} = y_i^{r_i}$.

Recall that $(\mathbb{P}, \oplus)$ is an arbitrary semifield. Let $\mathbb{QP}$ denote the field of fractions of this semifield and $\mathbb{QP}(\mathbf{x}) = \mathbb{QP}(x_1, \ldots, x_n)$ denote the field of rational functions in the algebraically independent variables $x_1, \ldots, x_n$. In earlier sections we referred to $\mathbb{QP}(\mathbf{x})$ as $\mathcal{F}$, but here it will be convenient to write $\mathbb{QP}(\mathbf{x})$ since we will also want to discuss fields of rational functions in other sets of algebraically independent variables.

Let $\mathcal{A}$ denote the generalized cluster algebra associated to the generalized cluster seed Σ. We can then state the definitions of the companion algebras as:

Definition 2.5.15 (See Section 4 of [62]). *Let* ${}^L\mathbf{x} = \mathbf{x}^{1/\mathbf{r}} = ({}^Lx_1, \ldots, {}^Lx_n) := (x_1^{1/r_1}, \ldots, x_n^{1/r_n})$ *and* ${}^L\mathbf{y} = \mathbf{y}$ *in* $\mathbb{QP}(\mathbf{x}^{1/\mathbf{r}})$ *The* left companion algebra *of* $\mathcal{A}$ *is the ordinary cluster algebra* ${}^L\mathcal{A} \subset \mathbb{QP}(\mathbf{x}^{1/\mathbf{r}})$ *with seed* $({}^L\mathbf{x}, {}^L\mathbf{y}, BR)$*. Let* ${}^L\mathbf{c}_j$*,* ${}^L\mathbf{g}_j$*, and* ${}^LF_j$ *denote the c-vectors, g-vectors, and F-polynomials of* ${}^L\mathcal{A}$*.*

Definition 2.5.16 (See Section 4 of [62]). *Let* ${}^R\mathbf{x} = \mathbf{x}$ *and* ${}^R\mathbf{y} = \mathbf{y}^{\mathbf{r}} = ({}^Ry_1, \ldots, {}^Ry_n) := (y_1^{r_1}, \ldots, y_n^{r_n})$*. The* right companion algebra *of* $\mathcal{A}$ *is the ordinary cluster algebra* ${}^R\mathcal{A} \subset \mathbb{QP}(\mathbf{x})$ *with seed* $(\mathbf{x}, \mathbf{y}^{\mathbf{r}}, RB)$*. Let* ${}^R\mathbf{c}_j$*,* ${}^R\mathbf{g}_j$*, and* ${}^RF_j$ *denote the c-vectors, g-vectors, and F-polynomials of* ${}^R\mathcal{A}$*.*

Remark 2.5.17. *Note that we use the wide convention for exchange matrices in this section, whereas Nakanishi and Rupel use the tall convention [62]. As such, our definitions of the companion algebras reverse the role of the matrices RB and BR. That is, our left companion algebra has exchange matrix BR rather than RB and our right companion algebra has exchange matrix RB rather than BR. This difference is purely notational and does not change the companion algebras associated to a given generalized cluster algebra.*

The c-vectors and g-vectors of a generalized cluster algebra and its companion algebras are related by simple transformations. Although Nakanishi [61] originally proved these relationships for the subclass of reciprocal generalized cluster algebras which meet the normalization condition, Nakanishi and Rupel [62] subsequently showed that these relationships still hold in the more general setting of reciprocal generalized cluster algebras which meet the weaker power condition.

Lemma 2.5.18 (Propositions 3.9, 3.10, 3.16, 3.17 of [61] and Corollaries 4.1, 4.2 of [62])**.** *Let $\mathcal{A}$ be the generalized cluster algebra associated to the generalized cluster seed $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$. Let $\mathbf{c}_j$ and $\mathbf{g}_j$ represent the c-vectors and g-vectors of $\mathcal{A}$.*

1. *The c-vectors of $\mathcal{A}$ agree with the c-vectors of ${}^L\mathcal{A}$ and are related to those of ${}^R\mathcal{A}$ via the transformation ${}^R\mathbf{c}_j = [r_i^{-1} c_{ij} r_j]$.*
2. *The g-vectors of $\mathcal{A}$ agree with the g-vectors of ${}^R\mathcal{A}$ and are related to those of ${}^L\mathcal{A}$ via the transformation ${}^L\mathbf{g}_j = [r_i g_{ij} r_j^{-1}]$.*

The F-polynomials of a generalized cluster algebra and its companion algebras are also related by simple transformations. For $i \in [n]$ and $j \in \{0, \ldots, r_i\}$, let $a^{\mathrm{bin}} := (a_{i,j}^{\mathrm{bin}})$ where $a_{i,j}^{\mathrm{bin}} = \binom{r_i}{j}$. Let $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ denote the set of rational functions in the variables $y_1, \ldots, y_n$ which can be written in a subtraction-free form.

Lemma 2.5.19 (Proposition 4.3, 4.6 of [62])**.** *Let $\mathcal{A}$ be the generalized cluster algebra associated to the generalized cluster seed $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$. Let F_j represent the F-polynomials of $\mathcal{A}$. For all $j \in [n]$, the following equalities hold in $\mathbb{Q}_{sf}(\mathbf{y})$:*

$$F_j\left(\mathbf{y}, \mathbf{a}^{bin}\right) = {}^L F_j\left({}^L\mathbf{y}\right)^{r_j}$$
$$F_j\left(\mathbf{y}, \mathbf{0}\right) = {}^R F_j\left({}^R\mathbf{y}\right)$$

In Section 4.12, we will discuss cluster scattering diagrams for companion algebras and show that our definitions of the fixed and torus seed data for companion algebras satisfy these structural relationships.

Chapter 3

Generalized snake graphs

This chapter describes joint work with Esther Banaian, which appears in [2] and [3]. It is motivated by, but does not use, unpublished work of Gleitz and Musiker [38].

3.1 Tiles

If $\gamma = \tau_i$ for $1 \leq i \leq n$ (recall the final $c-n$ arcs are boundary arcs), then G_γ is a single edge labeled with τ_i. Otherwise, γ must cross at least one arc in T.

Let $\tau_{i_1}, \ldots, \tau_{i_d}$ be the set of internal arcs of T that γ crosses, given a fixed orientation of γ. For each standard arc τ_{i_j} that γ crosses, we construct a square tile G_j by taking the two triangles that τ_{i_j} borders and gluing them along τ_{i_j} such that either both either the same orientation relative to $\mathcal{O}$. We say that the square tile produced has relative orientation $+1$ if the orientation of its triangles matches that of $\mathcal{O}$ and -1 otherwise. We denote this as $\mathrm{rel}(G_j) = \pm 1$.

Figure 3.1: Relative orientations of ordinary snake graph tiles.

Next, we consider the case when τ_{i_j} is a pending arc incident to an orbifold point of order p. If γ is a generalized arc who shares an endpoint with τ_{i_j}, then it could be that γ only crosses τ_{i_j} once. In this case, $j = 1$ or $j = k$, and we use a square tile as before. However, the labels of some edges will be given by normalized Chebyshev polynomials, $U_\ell(x)$, evaluated at $\lambda_p = 2\cos(\pi/p)$.

Definition 3.1.1. *We let $U_\ell(x)$ denote the ℓ-th normalized Chebyshev polynomial of the second kind, for $\ell \geq -1$. These are given by initial polynomials $U_{-1}(x) = 0, U_0(x) = 1$, and the recurrence,*

$$U_\ell(x) = xU_{\ell-1}(x) - U_{\ell-2}(x)$$

For instance, $U_1(x) = x, U_2(x) = x^2 - 1, U_3(x) = x^3 - 2x$. These polynomials are normalized as they can be recovered by evaluating the standard Chebyshev polynomials of the second kind at $x/2$.

The following lemma verifies that, up to sign, these labels are independent of increasing or decreasing the winding around an orbifold point by an integer multiple of its order.

Lemma 3.1.2. *Evaluations of these normalized Chebyshev polynomials at λ_p are periodic, in the sense that $U_{k+p}(\lambda_p) = -U_k(\lambda_p)$. In particular, $U_{p-1}(\lambda_p) = -U_{-1}(\lambda_p) = 0$.*

Lemma 3.1.2 can be readily proven using basic properties of Chebyshev polynomials. We see in Lemmas 3.12.3 and 3.12.4 that our statistics are still well-defined up up to sign.

The edge labels of these tiles contain $U_\ell(\lambda_p)$ and $U_{\ell-1}(\lambda_p)$ where ℓ is the number of self-intersections of γ around the orbifold point. For concision, U_ℓ is used as shorthand for $U_\ell(\lambda_p)$ throughout the paper. Moreover, α and β may be standard or pending arcs. If one of these arcs is pending, then this is in fact a monogon enclosing two orbifold points.

A pending arc can wind either clockwise or counterclockwise around an orbifold point. Below, we show the positive and negative orientations of tiles for each type of winding:

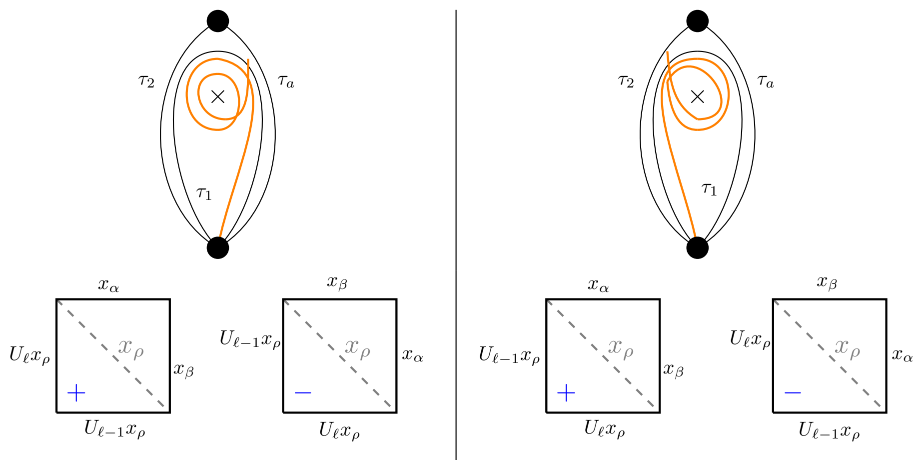


Figure 3.2: Tiles for winding pending arcs.

Above, each tile on the left has positive orientation and the tile on the right has negative orientation. We can see this, for example, by comparing the relative orientation of edges labeled x_2 and x_a with τ_2 and τ_a.

Remark 3.1.3. *Musiker and Williams discuss a similar example in [59] with a puncture rather than an orbifold point. We compare these cases in Section 3.14.*

In most cases, if γ crosses a pending arc τ_{i_j}, it crosses it twice consecutively, so that $\tau_{i_j} = \tau_{i_{j+1}}$ or $\tau_{i_j} = \tau_{i_{j-1}}$. In this case, we introduce a hexagonal tile which accounts for both intersections. These hexagonal tiles also will have edges labeled by Chebyshev polynomials evaluated at λ_p, and we again let ℓ be the number of self-intersections of γ as it winds around the orbifold point. Because these hexagonal tiles can be thought of as "containing" two square tiles, we assign them a tuple of signs. A hexagonal tile has relative orientation $(+, -)$ if the South-West triangle matches the orientation of the surface and the North-East triangle does not, as on the left hand side of the following diagram, and $(-, +)$ otherwise, as on the right hand side.

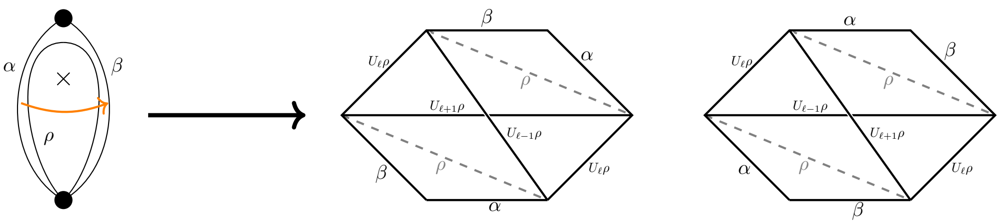


Figure 3.3: Relative orientations of hexagonal tiles.

In Section 3.10.1, we give a geometric intuition for why the edge labels $U_\ell\rho$, $U_{\ell+1}\rho$, and $U_{\ell-1}\rho$ appear in this particular arrangement on the hexagonal tiles. This geometric intuition is based on crossing diagonals in the p-fold cover. We formally justify these hexagonal tiles in Section 3.10, however, using matrix products associated to arcs, perfect matchings of abstract graphs, and Lemma 3.12.4.

Below, we give puzzle pieces that can be used to construct a generalized snake graph from such an arc. Again, α and β could be either standard or pending arcs.

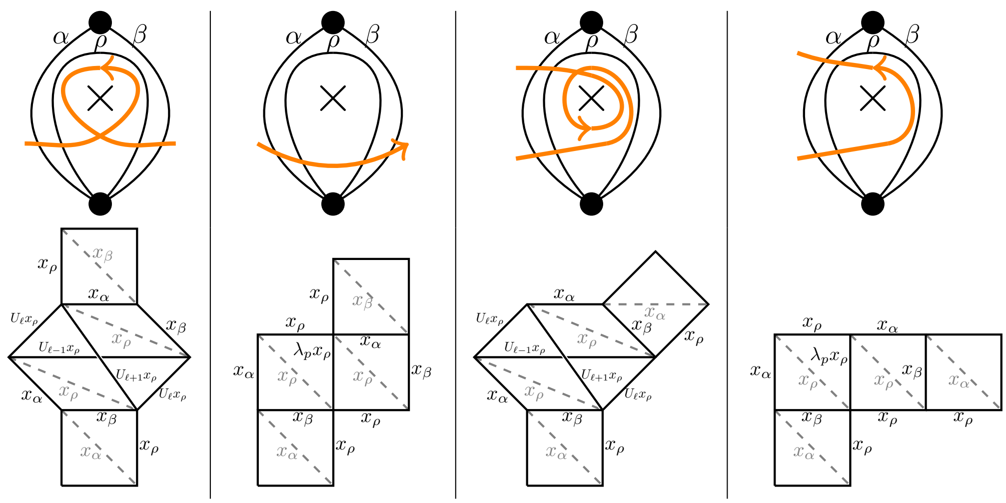


Figure 3.4: Puzzle pieces for arcs that cross pending arcs.

Note that if $\ell = 0$ (i.e., the arc does not intersect itself) then the edge labeled $U_{\ell-1}(\lambda_p)$ has weight 0. Thus, we can delete it and recover a standard snake graph with square tiles. We show side by side the general hexagonal tiles and the $k = 0$ cases. By Lemma 3.1.2, this is also true if $k = p - 2$. Moreover, if $\ell = p - 1$, then two edges have weight zero and one has negative weight. Using the symmetry of an orbifold point, this is equivalent to not crossing the pending arc at all. Thus, we will assume that γ winds less than $p-1$ times around an orbifold point of order p to avoid including absolute values in our labels.

If γ does not wind around the pending arc, then our snake graph is still composed of square tiles. However, the internal edge bordering the two tiles labeled τ_{i_j} has a potentially non-integer weight $\lambda_p\tau_{i_j}$ where $\lambda_p = 2\cos(\pi/p)$. Note that λ_p is the ratio between the length of a *2-diagonal* (an arc between marked points on the boundary which skips exactly one marked point) and the length of the sides of a regular p-gon.

3.2 Gluing puzzle pieces

To construct generalized snake graphs, we will glue together tiles corresponding to arcs crossed consecutively by γ. If γ crosses τ_i and τ_{i+1} consecutively, and τ_i and τ_{i+1} are distinct arcs, then these arcs form a triangle. Call the third arc in this triangle $\tau_{[i]}$. Then, we glue tiles G_i and G_{i+1} along the edge $\tau_{[i]}$. using the appropriate planar embeddings so $\mathrm{rel}(T, G_i) \neq \mathrm{rel}(T, G_{i+1})$. Note that this rule does not differentiate between standard and pending arcs. If G_i and G_{i+1} are either both square or both hexagonal, then the statement of the rule is clear. If G_i is square and G_{i+1} is hexagonal, then $\mathrm{rel}(T, G_{i+1})$ should be understood to mean the orientation of the South-West triangle of G_{i+1}. Likewise, if G_i is hexagonal and G_{i+1} square, then $\mathrm{rel}(T, G_i)$ should be understood to mean the orientation of the North-East triangle of G_i.

Because the choice of relative orientation for the first tile, G_1, is not fixed, there are two valid planar embeddings of $G_{T,\gamma}$ for any γ. Our cluster expansion formula produces the same result for either choice of planar embedding, so the choice is unimportant. We also make a choice to glue the tiles so that our snake graphs travel from South-West to North-East; this also will not affect any statistics related to the snake graph.

Finally, we can construct generalized band graphs using the same ideas. Band graphs calculate the length of closed curves on a surface. Choose a point p on γ such that p does not lie on any arc in T or at an intersection of γ with itself. For simplicity, we require p to not be in the interior of a pending arc. Then, construct the snake graph for γ, picking an orientation and starting and ending at p. Because the first and last tile correspond to arcs bordering the same triangle, they will always have a common edge. We glue the first and tile along this edge, producing a graph which resembles an annulus or a Mobius strip.

Band graphs have the same associated statistics as snake graphs and a version of a perfect matching on a band graph, called a *good matching*, is defined similarly. Musiker and Williams note that a good matching of a band graph can always be obtained from a perfect matching of the original, unglued snake graph used to construct the band graph. To do so, one takes a perfect matching that uses at least one of the glued edges and deletes that glued edge. For further discussion and details, see Section 3 of [59].

3.3 Cluster expansion formulas

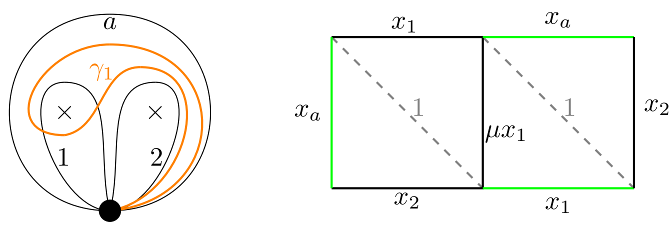
We use Musiker, Schiffler, and Williams' definitions for minimal and maximal matchings, the crossing monomial $\mathrm{cross}(T, \gamma)$, and the weight $x(P)$ and height monomial $y(P)$ associated to a perfect matching P, as stated in Section 2.2. Using this language, we can establish the following theorem for Laurent expansions of arcs (both standard and pending), a more general version of Theorem 4.9 from [58].

Theorem 3.3.1. *Let $\mathcal{O} = (S, M, Q)$ be an unpunctured orbifold with triangulation T and $\mathcal{A}$ be the corresponding generalized cluster algebra with principal coefficients with respect to $\Sigma_T = (\mathbf{x}_T, \mathbf{y}_T, B_T)$. For an ordinary arc γ with generalized snake graph $G_{T,\gamma}$, the Laurent expansion of x_γ with respect to Σ_T is*

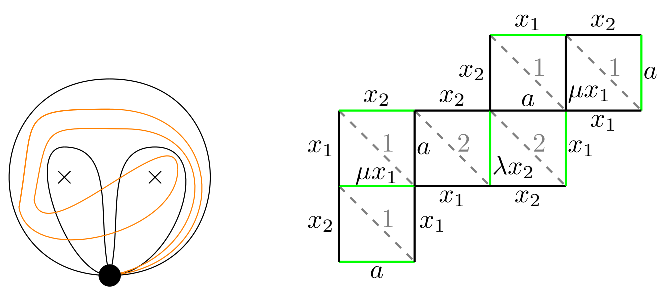
$$[x_\gamma]^{\mathcal{A}}_{\Sigma_T} = \frac{1}{cross(T, \gamma)} \sum_P x(P) y(P)$$

where the summation is indexed by perfect matchings of $G_{T,\gamma}$.

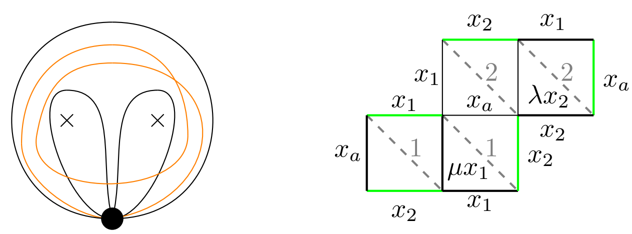
Example 3.3.2. The table below shows snake graphs for a variety of curves on the triangulated orbifold corresponding to $\mathcal{A} = \left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, (1 + \mu u + u^2, 1 + \lambda u + u^2)\right)$.



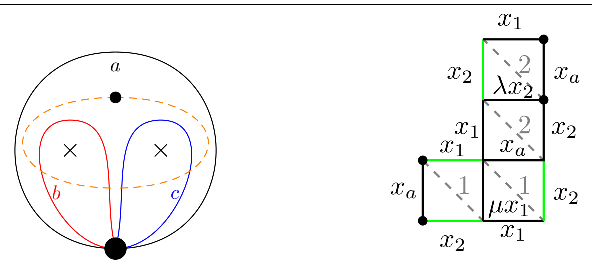
$$x_{\gamma_1} = \frac{1}{x_1}(x_a^2 + \mu y_2 x_2 x_a + y_2^2 x_2^2)$$



$$x_{\gamma_2} = \frac{1}{x_1^2 x_2}(x_a^2 x_1^2 y_1^4 y_2^2 + \mu\lambda x_a^2 x_1 x_2 y_1^3 y_2 + \lambda x_a x_1 x_2^2 y_1^2 y_2 + \lambda x_a^3 x_1 y_1^4 y_2 + \mu^2 x_a^2 x_2^2 y_1^2 + 2\mu x_a x_2^3 y_1 + 2\mu x_a^3 x_2 y_1^3 + x_2^4 + 2x_a^2 x_2^2 y_1^2 + x_a^4 y_1^4)$$



$$x_{\gamma_3} = \frac{1}{x_1 x_2}(x_a x_1^2 + \lambda y_2 x_a^2 x_1 + y_2^2 x_a^3 + \mu y_1 y_2^2 x_a^2 x_2 + y_1^2 y_2^2 x_a x_2^2)$$



$$x_{\gamma_4} = \frac{1}{x_1 x_2}(x_1^2 + \lambda y_2 x_a x_1 + y_2^2 x_a^2 + \mu y_1 y_2^2 x_a x_2 + y_1^2 y_2^2 x_2^2)$$

Figure 3.5: Examples of generalized snake graphs

Labels for arcs in the initial triangulation are only shown in the first orbifold diagram, but are consistent throughout. Snake graphs are shown for each curve γ_i, with one perfect matching and the corresponding term in the Laurent expansion highlighted. Both γ_1 and γ_2 are cluster variables of $\mathcal{A}$ which can be obtained via the respective mutation sequences μ_1 and $\mu_2\mu_1$.

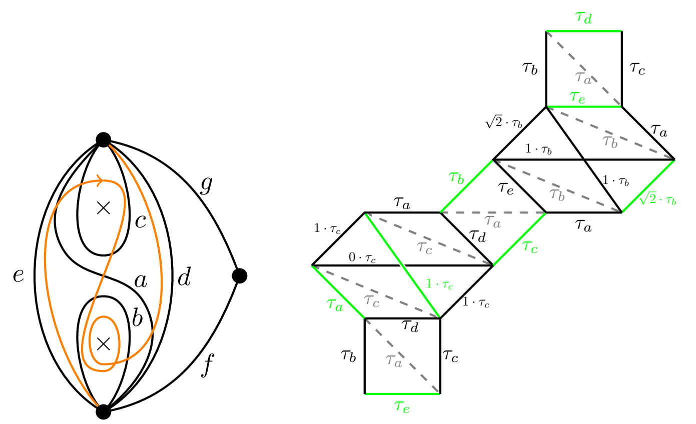
The second half of this example illustrates our results for generalized arcs and closed curves. Since γ_3 and γ_4 cross the same arcs in the same orientation, the shapes of the two associated graphs are the same. However, in the band graph associated to γ_4, we identify u with u' and v with v'. In each graph, we have highlighted the maximal matching and the corresponding term in the Laurent expansion.

Note that in each example, our expression for x_{γ_i} is given after canceling a mutual factor from the crossing monomial and the numerator. Although the exact mutual factor depends on the curve being considered, cancellation of this type occurs whenever we cross pending arcs.

In the Sections 3.4 to 3.9 we prove Theorem 3.3.1 when γ is an ordinary arc. Then, x_γ is a cluster variable in the associated generalized cluster algebra. Moreover, we are able to lift γ to a construct a triangulated polygon where expansion formulas are already known. In Section 3.10.1, we explain why this tactic does not work for generalized arcs.

3.4 The lift $\widetilde{S_\gamma}$

In the following sections, let $\mathcal{O} = (S, M, Q)$ be an orbifold with triangulation $T = \{\tau_1, \ldots, \tau_{n+c}\}$ where $\tau_1, \ldots, \tau_n$ are internal arcs and $\tau_{n+1}, \ldots, \tau_{n+c}$ are boundary arcs. Let $\gamma \notin T$ be an arc on an orbifold, and pick an orientation of γ. Let $\tau_{i_1}, \ldots, \tau_{i_d}$ be the arcs crossed by γ in order. Note that



$$x_\gamma = \frac{1}{x_a^3 x_b^2 x_c^2}(\sqrt{2} y_a y_b^2 y_c x_a x_b^2 x_c^3 x_d x_e^2 + y_a y_b y_c^2 x_a x_b^2 x_c^2 x_d^2 x_e + \cdots$$

Figure 3.6: An example of a generalized snake graph from a triangulated orbifold with one orbifold point of order 3 (top) and one orbifold point of order 4 (bottom).

it is possible to have $j \neq k$ and $\tau_{i_k} = \tau_{i_j}$, since γ may cross a given arc in T multiple times. It is even possible to have $\tau_{i_j} = \tau_{i_{j+1}}$; this occurs only when τ_{i_j} is a pending arc.

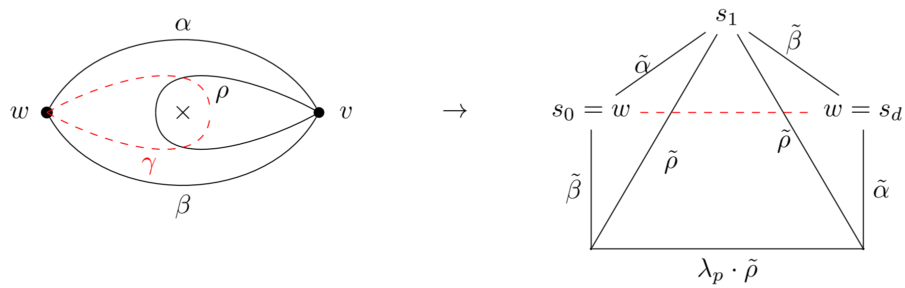
We define a polygon $\widetilde{S_\gamma}$ with triangulation $\widetilde{T_\gamma}$ which lifts the local configuration of $\mathcal{O}$ and T around γ. The triangulation $\widetilde{T_\gamma}$ consists of arcs $\sigma_1, \ldots, \sigma_d, \sigma_{d+1}, \ldots, \sigma_{2d+3}$ where $\sigma_{d+1}, \ldots, \sigma_{2d+3}$ are boundary arcs. We also construct a lift of γ in $\widetilde{S_\gamma}$, denoted as $\widetilde{\gamma}$; in short, $\widetilde{\gamma}$ will be the arc in $\widetilde{S_\gamma}$ which crosses all arcs in $\widetilde{T_\gamma}$.

Musiker, Schiffler, and Williams gave a construction of $\widetilde{S_\gamma}$ and $\widetilde{T_\gamma}$ for the case where γ is an arc on a surface [58]. We describe an extension to their construction and refer the interested reader to their paper for details of the original construction. Essentially, they keep track of when consecutive arcs in T, τ_{i_j} and $\tau_{i_{j+1}}$, share a vertex on the right or on the left of γ. We will let t_j denote the vertex shared by τ_{i_j} and $\tau_{i_{j+1}}$. The corresponding consecutive arcs σ_j and σ_{j+1} in $\widetilde{T_\gamma}$ share a vertex, s_j, on the same side of $\widetilde{\gamma}$ and $\widetilde{S_\gamma}$ is constructed by gluing together the fans formed by sets of consecutive arcs in T which share a given vertex t_j. Musiker, Schiffler, and Williams also provide a projection map $\pi : \widetilde{T_\gamma} \to T$ such that $\pi(\sigma_j) = \tau_{i_j}$. This map also can be applied to boundary arcs in $\widetilde{S_\gamma}$; we will give a full definition of π in Section 3.6.

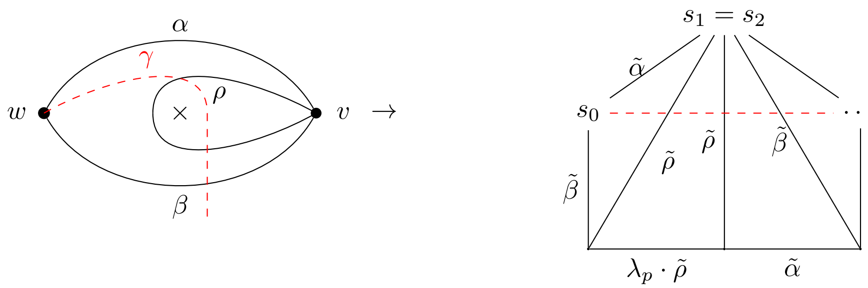
This construction can be used in the orbifold case when γ crosses consecutive standard arcs; what remains is to analyze the case when γ crosses a pending arc.

There are several possible configurations for this case. Let τ_{i_j} be a pending arc; then, τ_{i_j} is enclosed by a bigon or monogon with sides α and β. If this is a bigon, let v be the vertex shared by τ_{i_j}, α, and β and let w be the vertex only shared by α and β, as shown below. If this is a monogon, let $v = w$ be the unique vertex shared by α, β and ρ. Our configuration of s_{j-1}, s_j, and s_{j+1} will depend on how γ interacts with the bigon.

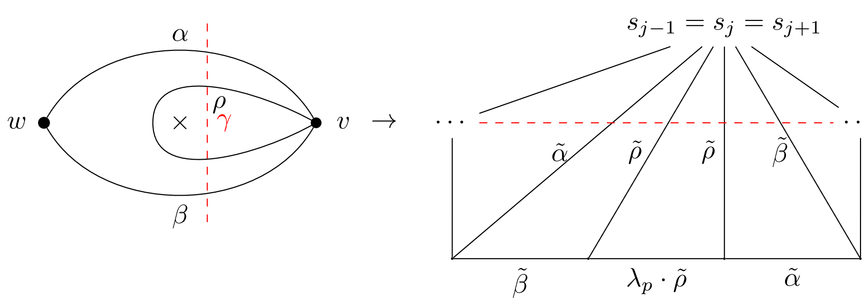
It could be that γ is the result of flipping ρ. In this case, $d = 2$, $s_1 = v$, and $\widetilde{S_\gamma}$ is a triangulated pentagon, as below. We label arcs σ_j with $\widetilde{\tau_k}$ if $\pi(\sigma_j) = \tau_k$.



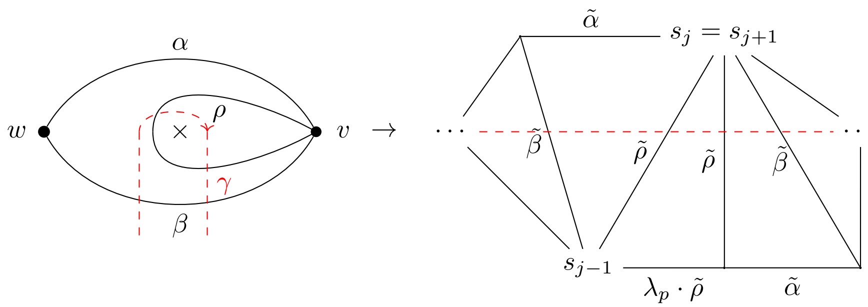
Next, consider the case where $d > 2$ and τ_{i_1} or τ_{i_d} is a pending arc. Suppose, without loss of generality, that $\tau_{i_1} = \rho$ is a pending arc. Since, as an ordinary arc, γ necessarily crosses ρ twice, $\rho = \tau_{i_1} = \tau_{i_2}$. Then regardless of whether $\tau_{i_3} = \alpha$ or $\tau_{i_3} = \beta$, we set $s_1 = s_2$. See below for the case where $\tau_{i_3} = \beta$.



Finally, we have two cases for when γ crosses the bigon twice. In this case, $d \geq 4$ and $j > 1$. If γ crosses both sides of the bigon, then $s_{j-1} = s_j = s_{j+1}$.



Alternatively, γ could cross the same side of the bigon both before and after crossing τ_{i_j}. That is to say, $\tau_{i_j} = \tau_{i_{j+1}}$ and $\tau_{i_{j-1}} = \tau_{i_{j+2}}$. If the first point of intersection between γ and $\tau_{i_{j-1}} = \tau_{i_{j+2}}$ is closer to v than the second point of intersection, then set $s_{j-1} = s_j$ and $s_{j+1} \neq s_j$; otherwise, set $s_j = s_{j+1}$ and $s_{j-1} \neq s_j$. See below for an example where $\tau_{i_{j-1}} = \tau_{i_{j+2}} = \beta$ and the second point of intersection is closer than the first.



Using these rules in addition to those in [58], we can construct $\widetilde{S_\gamma}$, a $(d+3)$-gon with triangulation $\widetilde{T_\gamma}$ consisting of d internal arcs and $d+3$ boundary arcs. The arc $\widetilde{\gamma} \in \widetilde{S_\gamma}$ crosses all arcs in $\widetilde{T_\gamma}$, and this pattern of crossings resembles the arcs that γ crosses in $\mathcal{O}$.

3.5 Quadrilateral and bigon Lemmas

The machinery of our proof that $\phi_\gamma(x_{\widetilde{\gamma}}) = x_\gamma$ will be an induction on the number of crossings between γ and T. To that end, we provide a way to express x_γ in terms of x_{ζ_i} where all arcs ζ_i have less crossings with T than γ

This was accomplished in [58] by Lemma 9.1, known as the quadrilateral lemma. The quadrilateral specified in this lemma gives slightly weaker results when pending arcs are present, but still allows us to prove our expansion formula.

Lemma 3.5.1. *Let T be a triangulation of an unpunctured orbifold $\mathcal{O}$ and γ be a standard arc not in T. Then, there exists a quadrilateral $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, of arcs in $\mathcal{O}$ such that:*

- *γ and another arc, γ', are the two diagonals of this quadrilateral,*
- *$e(\alpha_i, T) \leq e(\gamma, T)$, and*
- *$e(\gamma', T) < e(\gamma, T)$.*

Moreover, if $e(\alpha_i, T) = e(\gamma, T)$ for some i, then α_i is a pending arc and $e(\alpha_j, T) < e(\gamma, T)$ for all $j \neq i$.

If γ, instead, is a pending arc, then there exists another pending arc, ρ, and a bigon composed of arcs β_1 and β_2 such that:

- *γ and ρ are the two possible pending arcs contained within the bigon,*
- *$e(\beta_i, T) < e(\gamma, T)/2$, and*
- *$e(\rho, T) < e(\gamma, T)/2$.*

Prior to the proof, we need to establish some notation. Let γ_1 and γ_2 be two arcs which intersect at a point b. This can be an end point of the arcs or not. Let a be another point on γ_1 and let c be another point on γ_2. Then, $(a, b, c|\gamma_1, \gamma_2)$ denotes an arc which starts at a, is isotopic to γ_1 between a and b, is isotopic to γ_2 between b and c, and finally ends at c. We can generalize this notion to more arcs which consecutively intersect. We also let γ^- denote an arc that is isotopic to γ but has opposite orientation.

Proof. We will induct on $e(\gamma, T)$. We have two base cases. If $e(\gamma, T) = 1$, then γ must be a standard arc and is the result of flipping an arc $\tau \in T$, so γ is one diagonal in a quadrilateral which is entirely made up of arcs in T, and the other diagonal is τ.

The other base case is when γ is the result of flipping a pending arc ρ. Then, $e(\gamma, T) = 2$ and γ is a pending arc. It also must be that the arcs, β_1, β_2, making up the bigon about γ and ρ are in T as well.

Now, suppose first that $e(\gamma, T) = d$ and γ is a standard arc. Label the crossing points between γ and T by $1, 2, \ldots, d$. If τ, the arc that crosses γ at point $h = \lceil \frac{d}{2} \rceil$, is not a pending arc, then the construction from Lemma 9.1 in [58] holds. However, if τ is a pending arc, and γ crosses τ in spots $j_1, \ldots, j_r$ where $j_\ell = h$, then either the crossing point $h+1$ or $h-1$ is also on τ.

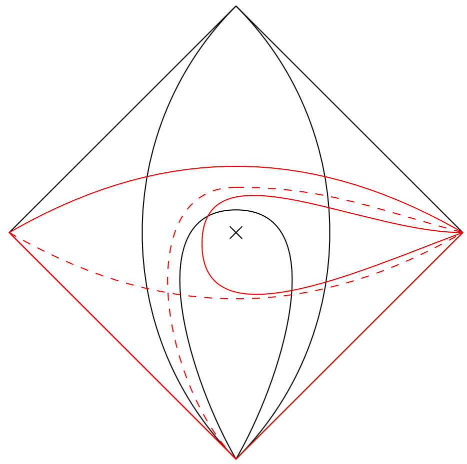
More explicitly, suppose $j_\ell + 1 = j_{\ell+1} = h + 1$ and first let d be even, so that $d = 2h$. Let $s(\gamma), t(\gamma)$ be respectively the start and end of γ once we select an orientation. Moreover, suppose that we orient τ, the pending arc containing the intersection points $j_\ell, j_{\ell+1}$ so that it visits j_ℓ before $j_{\ell+1}$. Then, assuming that $\ell > 1$, Musiker, Schiffler, and Williams [58] give the following explicit construction for the quadrilateral:

$$\begin{aligned}
\alpha_1 &= \big(s(\gamma), j_{\ell-1}, j_\ell, s(\gamma) | \gamma, \tau, \gamma^-\big)\\
\alpha_2 &= (s(\gamma), j_\ell, j_{\ell+1}, t(\gamma) | \gamma, \tau, \gamma)\\
\alpha_3 &= \big(t(\gamma), j_{\ell+1}, j_\ell, t(\gamma) | \gamma^-, \tau^-, \gamma\big)\\
\alpha_4 &= \big(t(\gamma), j_\ell, j_{\ell-1}, s(\gamma) | \gamma^-, \tau^-, \gamma^-\big)\\
\gamma' &= (s(\gamma), j_{\ell-1}, j_{\ell+1}, t(\gamma) | \gamma, \tau, \gamma)
\end{aligned}$$

From these descriptions of α_i, we can compute $e(\alpha_i, T)$, and similarly for γ'. We only highlight a few calculations as the rest are equivalent to the calculations in [58].

$$\begin{aligned}
e(\alpha_1, T) &= (j_{\ell-1} - 1) + j_\ell < (h-1) + h < d\\
e(\alpha_3, T) &= (d - j_{\ell+1}) + (d - j_\ell + 1) = (d - (h+1)) + (d - h + 1) = d
\end{aligned}$$

We can see that α_3 is a pending arc incident to the same orbifold point as τ. If instead $j_\ell - 1 = j_{\ell-1} = h - 1$ and d is still even, then we will find that $e(\alpha_1, T) = e(\gamma, T)$ and α_1 will be a pending arc. One can check that $e(\alpha_i, T) < d$ for other i and $e(\gamma', T) < d$ in both these cases.



If d is odd, then we will again have that $e(\alpha_i, T) < e(\gamma, T)$ for all i if we follow the recipe for α_i given in [58].

Now, let γ be a pending arc, and let $\rho \in T$ be the pending arc to the same orbifold point as γ. First, note that $d = e(\gamma, T)$ is necessarily even. Let $j, j+1$ be the intersections of γ and ρ. Then, $j = d/2$. Orient ρ so that, like γ, it passes j before $j+1$. Define $\beta_1 = (s(\gamma), j_\ell, s(\rho)|\gamma, \rho^-)$ and $\beta_2 = (t(\gamma), j_{\ell+1}, t(\rho)|\gamma^-, \rho)$. We can check that all of these arcs cross arcs in T fewer times than γ:

- $e(\rho, T) = 0$ as $\rho \in T$
- $e(\beta_1, T) = j - 1 < \frac{d}{2}$
- $e(\beta_2, T) = k - (j+1) < \frac{d}{2}$

□

3.6 $\widetilde{A_\gamma}$ and ϕ_γ

We first define a map $\pi : \widetilde{S_\gamma} \to \mathcal{O}$. Then, we define a morphism, ϕ_γ between the algebras from these spaces and show it is an algebra homomorphism.

We define π from $\{\sigma_1, \ldots, \sigma_{2d+3}\}$ to $\{\tau_1, \ldots, \tau_{n+c}\}$, which will also define π on the marked points of each space. Recall that $\tau_{[i_k]}$ is the third side of the triangle formed by τ_{i_k} and $\tau_{i_{k+1}}$. For completeness, we define σ_a, σ_b to be the two boundary arcs in the first triangle that $\widetilde{\gamma}$ crosses where σ_b follows σ_a in the clockwise direction. Note that $\widetilde{\gamma}$ inherits an orientation based on the orientation of γ. We define τ_a and τ_b to be analogous arc in $\mathcal{O}$; note that τ_a and tau_b are not necessarily on the boundary. Then, we define σ_w, σ_z to be the boundary arcs in the last triangle $\widetilde{\gamma}$ crosses where σ_z follows σ_w in the clockwise direction, and define τ_w and τ_z analogously in $\mathcal{O}$.

$$\pi(\sigma_j) = \begin{cases} \tau_{i_j} & 1 \leq j \leq d \\ \tau_{[\gamma_k]} & j > d \text{ and } \sigma_j \text{ incident to } \sigma_k \text{ and } \sigma_{k+1} \\ \tau_x & \sigma_j = \sigma_x \text{ for } x \in \{a, b, w, z\} \end{cases}$$

Let $\mathcal{A}$ be the generalized cluster algebra from $\mathcal{O}$, as explained in Section 2.5. Let $\widetilde{A_\gamma}$ be the cluster algebra corresponding to the polygon $\widetilde{S_\gamma}$ with initial triangulation $\widetilde{T_\gamma} = \{\sigma_1, \ldots, \sigma_d, \sigma_{d+1}, \ldots, \sigma_{2d+3}\}$

where $\sigma_1, \dots, \sigma_d$ are the arcs in the triangulation and images of the arcs that γ crosses in $\mathcal{O}$, and $\sigma_{d+1}, \dots, \sigma_{2d+3}$ are boundary arcs. In $\widetilde{A_\gamma}$, let x_{σ_i} be the variable associated to σ_i. We treat the variables from boundary arcs, $x_{\sigma_{d+1}}, \dots, x_{\sigma_{2d+3}}$, as coefficients. We also consider $\widetilde{A_\gamma}$ with principal coefficients $\{y_{\sigma_1}, \dots, y_{\sigma_d}\}$; geometrically, we place an elementary (multi)-lamination $\{\widetilde{L_1}, \dots, \widetilde{L_d}\}$ on $\widetilde{S_\gamma}$ where $\widetilde{L_i}$ is the elementary lamination from σ_i. Let $\mathbb{P} = \mathrm{Trop}(x_{\sigma_{d+1}}, \dots, x_{\sigma_{2d+3}}, y_{\sigma_1}, \dots, y_{\sigma_d})$ be the tropical semifield generated by these elements.

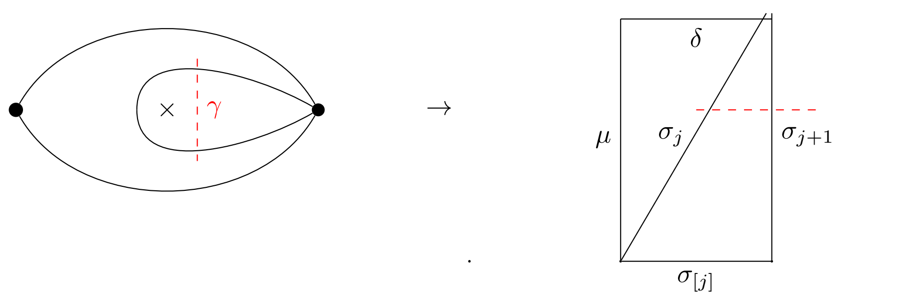
It is clear by construction that $\widetilde{A_\gamma}$ is a type A_d, acyclic cluster algebra since the triangulation $\widetilde{T_\gamma}$ has no internal triangles. Thus, we can the following proposition from Bernstein-Fomin-Zelevinsky.

Proposition 3.6.1 (Corollary 1.21 of [4]). *The algebra $\widetilde{A_\gamma}$ is the $\mathbb{ZP}$ algebra with set of generators $\{x_{\sigma_1}, \dots, x_{\sigma_d}, x'_{\sigma_1}, \dots, x'_{\sigma_d}\}$, where $x'_{\sigma_k} = \mu_k(x_{\sigma_k})$, and relations generated by those of the form $x_{\sigma_k} x'_{\sigma_k}$.*

We now construct a map, ϕ_γ, from $\widetilde{A_\gamma}$ to $\mathrm{Frac}(\mathcal{A})$. First, we will describe what ϕ_γ does to the generators of $\widetilde{A_\gamma}$, which we found in Proposition 3.6.1. Then, we will prove that this map is indeed an algebra homomorphism by showing that it sends relations in $\widetilde{A_\gamma}$ to relations in $\mathcal{A}$. We eventually will show that $\phi_\gamma(x_{\widetilde{\gamma}}) = x_\gamma$.

In most cases, we define $\phi_\gamma(x_{\sigma_j}) = x_{\pi(\sigma_j)}$; the exception will be if $\sigma_j = \sigma_{[k]}$ for some $1 \leq k < d$ and $\pi(\sigma_k) = \pi(\sigma_{k+1}) = \rho$ is a pending arc in $\mathcal{O}$. In this case, if the orbifold point incident to σ_k is order p, we set $\phi_\gamma(x_{\sigma_j}) = \lambda_p x_\rho$. Regardless of whether $\pi(\sigma_j)$ is a pending arc or standard arc, we set $\phi_\gamma(y_{\sigma_j}) = y_{\pi(\sigma_j)}$.

Next, we need to define the image of ϕ_γ on the first mutations of the mutable variables in $\widetilde{A_\gamma}$. If $\pi(\sigma_j) \in T$ is a standard arc, then we set $\phi_\gamma(x'_{\sigma_j}) = x'_{\pi(\sigma_j)}$. If $\pi(\sigma_j) = \tau_{i_j}$ is a pending arc in T, then either $\pi(\sigma_{j-1}) = \pi(\sigma_j)$ or $\pi(\sigma_j) = \pi(\sigma_{j+1})$. Without loss of generality, assume the latter. Let δ and μ be the the two other arcs in the quadrilateral in $\widetilde{T_\gamma}$ around σ_j such that δ is opposite of $\sigma_{[j]}$ in this quadrilateral.



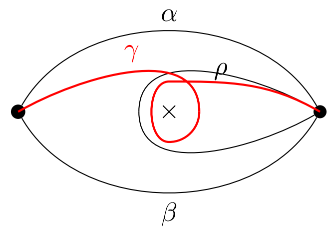
If $\sigma_{[j]}$ is counterclockwise of σ_j, as in the diagram above, then we define

$$\phi_\gamma(x_{\sigma'_j}) = \lambda_p \cdot \phi_\gamma(x_\delta) + \phi_\gamma(y_{\sigma_j}) \cdot \phi_\gamma(x_\mu) = \lambda_p \cdot x_{\pi(\delta)} + y_{\pi(\sigma_j)} \cdot x_{\pi(\mu)}.$$

Otherwise, define

$$\phi_\gamma(x_{\sigma'_j}) = \lambda_p \cdot \phi_\gamma(y_{\sigma_j}) \cdot \phi_\gamma(x_\delta) + \phi_\gamma(x_\mu) = \lambda_p \cdot y_{\pi(\sigma_j)} \cdot x_{\pi(\delta)} + x_{\pi(\mu)}.$$

Remark 3.6.2. *The expression $\lambda_p y_\rho \alpha + \beta$ is the result when you simplify the self-intersection of the arc below with the skein relation. Compare this with the arc with self-intersection we encounter when proving Proposition 3.13.4.*



Proposition 3.6.3. *The map ϕ_γ is an algebra homomorphism; that is, it maps relations in $\widetilde{A_\gamma}$ to relations in $\mathcal{A}$.*

Proof. First, let $\pi(\sigma_j)$ be a standard arc. Then, in $\widetilde{A_\gamma}$, we have a relation

$$x_{\sigma_j} x'_{\sigma_j} = y_{\sigma_j} \Pi_b x_b + \Pi_c x_c \tag{3.1}$$

where b ranges over arcs which are immediately clockwise of σ_j in $\widetilde{T_\gamma}$ and c ranges over arcs which are counterclockwise of σ_j. The image of this relation under ϕ_γ is

$$x_{\pi(\sigma_j)} x'_{\pi(\sigma_j)} = y_{\pi(\sigma_j)} \Pi_b x_{\pi(b)} + \Pi_c x_{\pi(c)}. \tag{3.2}$$

This is exactly the exchange relation for $x_{\pi(\sigma_j)}$ in $\mathcal{A}$.

Now assume $\pi(\sigma_j)$ is a pending arc in T, then x_{σ_j} has an exchange relation in $\widetilde{A_\gamma}$ akin to Equation (3.1). Using our prior notation, in the case where $\sigma_{[j]}$ is clockwise of σ_j, so that this exchange relation in $\widetilde{A_\gamma}$ is $x'_{\sigma_j} x_{\sigma_j} = y_{\sigma_j} x_\delta x_{\sigma_{[j]}} + x_{\sigma_{j+1}} x_\mu$, we have that ϕ_γ maps $x'_{\sigma_j} x_{\sigma_j}$ to the following:

$$(\lambda_p y_{\pi(\sigma_j)} x_{\pi(\delta)} + x_{\pi(\mu)}) x_{\pi(\sigma_j)} = \lambda_p y_{\pi(\sigma_j)} x_{\pi(\sigma_j)} x_{\pi(\delta)} + x_{\pi(\sigma_j)} x_{\pi(\mu)}.$$

Moreover, this is equivalent to $\phi_\gamma(y_{\sigma_j} x_\delta x_{\sigma_{[j]}} + x_{\sigma_{j+1}} x_\mu)$ since $\phi_\gamma(x_{\sigma_{[j]}}) = \lambda_p x_{\pi(\sigma_j)}$. We see a similar relation when δ is counterclockwise of σ_j. In either case, this is simply an identity in $\mathcal{A}$. Thus, all relations in $\widetilde{A_\gamma}$ are mapped to relations in $\mathcal{A}$. ☐

Remark 3.6.4. *It is reasonable that we send the exchange relation for a pre-image of a pending arc to an identity in $Frac(\mathcal{A})$ since, if both σ_j and σ_{j+1} correspond to the same arc in $\widetilde{S_\gamma}$, it does not make sense to only mutate one of them.*

3.7 Showing $\phi_\gamma(x_{\tilde{\gamma}}) = x_\gamma$

In Section 3.6, we defined $\phi_\gamma : \widetilde{A_\gamma} \to \mathcal{A}$ on the generators of $\widetilde{A_\gamma}$ and showed that it is in fact an algebra homomorphism. Now, we will show that $\phi_\gamma(x_{\tilde{\gamma}}) = x_\gamma$. In $\widetilde{S_\gamma}$, we already have expansion formulas thanks to [58] (and originally due to [56]). So, we can import the expansion formula for x_γ in $\mathcal{A}$ via our map ϕ_γ.

Proposition 3.7.1. *Let ϕ_γ be the map from the last section. Then, $\phi_\gamma(x_{\tilde{\gamma}}) = x_\gamma$*

Proof. Our proof in the orbifold case will differ from the proof of the analogous result in the surface case, Theorem 10.1 in [58], in two ways. First of all, we need to prove this for the case when γ is a pending arc. Then, we need to take account for the case when γ is an standard arc and the resulting quadrilateral from the quadrilateral lemma, $\{\alpha_i\}$, is such that $e(\alpha_i, T) = e(\gamma, T)$ for some index i.

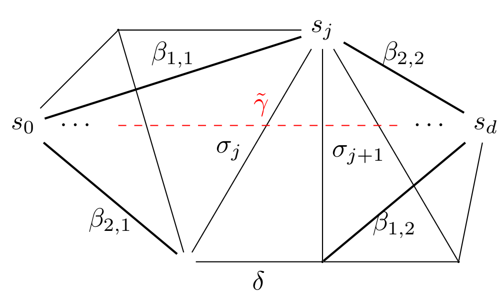
Both of these cases will utilize Lemma 3.5.1. We work through these cases simultaneously using induction on $e(\gamma, T)$.

Let $e(\gamma, T) = d$. If $d = 0$, then $\gamma \in T$, and we already have that $\phi_\gamma(x_{\widetilde{\gamma}}) = x_{\pi(\widetilde{\gamma})} = x_\gamma$. If $d = 1$, then γ is a standard arc which crosses one other standard arc, and the statement follows from Theorem 10.1 of [58].

Now, suppose that $d > 1$. First, consider the case where γ is a pending arc. Then d is necessarily even. Let $\rho \in T$ be the pending arc incident to the same orbifold point as γ. By Lemma 3.5.1, we can find β_1, β_2 such that β_1, β_2 form the bigon which contains the pending arcs ρ and γ, and $e(\beta_i, T) < \frac{d}{2}$. Suppose the orbifold point incident to γ is order p. Then, in $\mathcal{A}$, we have that $x_\gamma x_\rho = Y_1 x_{\beta_1}^2 + Y_0 \lambda_p x_{\beta_1} x_{\beta_2} + Y_{-1} x_{\beta_2}^2$, where we can compute Y_i by finding a sequence of flips from ρ to γ and performing the corresponding mutations in the cluster algebra. In Proposition 3.8.1, we will see that we can also compute these Y_i from the orientation of β_1 and β_2 and their intersections with the elementary lamination on $\mathcal{O}$.

We compare this with the scenario in the lift, $\widetilde{S_\gamma}$. Recall $\widetilde{S_\gamma}$ is a polygon triangulated by σ_i, for $1 \leq i \leq d$. For $j = \frac{d}{2}$, we have that $\pi(\sigma_j) = \pi(\sigma_{j+1}) = \rho$. Moreover, in $\mathcal{O}$, the β_i only cross arcs in $\{\pi(\sigma_i)\}_i$, implying that $\widetilde{S_\gamma}$ already contains $\widetilde{S_{\beta_i}}$ and trivially contains $\widetilde{S_\rho}$ as $\rho \in T$. Thus, we can apply ϕ_γ to β_i and ρ, as all of these are arcs in the polygon $\widetilde{S_\gamma}$.

Due to the symmetry of arcs crossed by γ, there are two lifts of β_1 and β_2 to $\widetilde{S_\gamma}$; call them $\beta_{1,i}, \beta_{2,i}$ for $i = 1, 2$. Moreover, $\beta_{1,i}, \beta_{2,i}$, and $\sigma_{[j]}$ form a pentagon in $\widetilde{S_\gamma}$, which is triangulated by σ_j and σ_{j+1}. Let $s_0 = s(\widetilde{\gamma})$ and $s_d = t(\widetilde{\gamma})$ be the start and end of the arc $\widetilde{\gamma}$. Recall we define s_j to be the vertex shared by σ_j and σ_{j+1} . Let a_j (a_{j+1}) be the other vertex of σ_j (σ_{j+1}). Note that $\pi(s_j) = \pi(a_j) = \pi(a_{j+1})$ since $\pi(\sigma_j) = \pi(\sigma_{j+1})$, and this arc is a pending arc. Then, up to changing indices, $\beta_{1,1}$ connects s_0 and s_j, and $\beta_{1,2}$ connects a_{j+1} and s_d. Similarly, $\beta_{2,1}$ connects s_0 and a_j, and $\beta_{2,2}$ connects s_j and s_d.



Using cluster algebra expansion formulas from triangulated polygons [58], in $\widetilde{A_\gamma}$ we have that $x_{\widetilde{\gamma}} x_{\sigma_j} x_{\sigma_{j+1}} = \widetilde{Y_1} x_{\beta_{1,1}} x_{\beta_{1,2}} x_{\sigma_j} + \widetilde{Y_0} x_{\beta_{1,1}} x_{\beta_{2,2}} x_\delta + \widetilde{Y_{-1}} x_{\beta_{2,1}} x_{\beta_{2,2}} x_{\sigma_{j+1}}$. The image of this relation under ϕ_γ is

$$\begin{aligned} \phi_\gamma(x_{\widetilde{\gamma}}) x_\rho^2 &= \phi_\gamma(\widetilde{Y_1}) x_{\beta_1}^2 x_\rho + \phi_\gamma(\widetilde{Y_0}) x_{\beta_1} x_{\beta_2} (\lambda_p x_\rho) + \phi_\gamma(\widetilde{Y_{-1}}) x_{\beta_2}^2 x_\rho \\ &\implies \phi_\gamma(x_{\widetilde{\gamma}}) x_\rho = \phi_\gamma(\widetilde{Y_1}) x_{\beta_1}^2 + \phi_\gamma(\widetilde{Y_0}) \lambda_p x_{\beta_1} x_{\beta_2} + \phi_\gamma(\widetilde{Y_{-1}}) x_{\beta_2}^2 \qquad (3.3) \end{aligned}$$

Comparing this with our generalized exchange relation, if we can show that $\phi_\gamma(\widetilde{Y_i}) = Y_i$, we can conclude that $\phi_\gamma(x_{\widetilde{\gamma}}) = x_\gamma$. We postpone this discussion of y-variables and laminations to Lemma 3.8.2 in the next section.

Now, let γ be an standard arc in $\mathcal{O}$ with $d = e(\gamma, T)$. Since we are in an orbifold, it may be that the quadrilateral, $\{\alpha_i\}$, which we produce from Lemma 3.5.1, has a pending arc $\alpha = \alpha_i$ for some i, such that $e(\alpha, T) = d$. In this case, $\widetilde{S_\alpha}$ is not contained in $\widetilde{S_\gamma}$, but we can glue these polygons together as the intersection of arcs crossed by α and γ is nonempty. We may also need to glue $\widetilde{S'_\gamma}$ onto this. Details about this gluing may be found in [58]. Denote this glued polygon $\hat{S}$. The advantage of this larger polygon is a preimage of our quadrilateral $\{\alpha_i\}$ with diagonals γ, γ', lives in $\hat{S}$. We already showed that $\phi_\alpha(x_{\widetilde{\alpha}}) = x_\alpha$, since α is a pending arc with $e(\alpha, T) = d$. By induction, we also know that $\phi_\gamma(x_{\widetilde{\gamma}'}) = x_{\gamma'}$ and for the other α_i, $\phi_{\alpha_i}(x_{\widetilde{\alpha_i}}) = x_{\alpha_i}$.

In $\hat{S}$, by cluster expansion formulas from surfaces, we have the exchange relation $x_{\widetilde{\gamma}} x_{\widetilde{\gamma}'} = \widetilde{Y_+} x_{\widetilde{\alpha_1}} x_{\widetilde{\alpha_3}} + \widetilde{Y_-} x_{\widetilde{\alpha_2}} x_{\widetilde{\alpha_4}}$. The image of this relation under ϕ_γ is

$$\phi_\gamma(x_{\widetilde{\gamma}}) x_{\gamma'} = \phi_\gamma(\widetilde{Y_+}) x_{\alpha_1} x_{\alpha_3} + \phi_\gamma(\widetilde{Y_-}) x_{\alpha_2} x_{\alpha_4}$$

Again, we direct our reader to the next section for discussion of laminations on an orbifold and for now assume Lemma 3.8.2. By comparing the previous discussion to the Ptolemy relation in $\mathcal{O}$ applied to the intersection of γ and γ', we conclude that $\phi_\gamma(x_{\widetilde{\gamma}}) = x_\gamma$. □

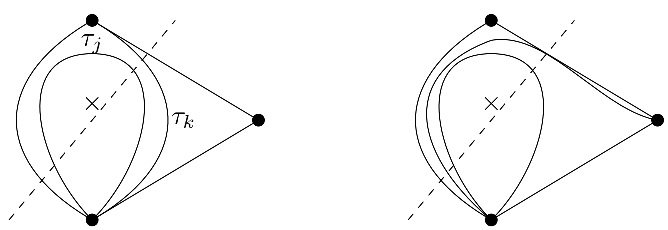
3.8 Laminations on an orbifold

We now show that the shear coordinates and elementary laminations for pending arcs defined in Section 2.4 correctly models the mutation of an extended B-matrix in a generalized cluster algebra.

Let L_{i+n} be the elementary lamination from arc $\tau_i \in T$. Recall n is the number of arcs in the triangulation T.

Proposition 3.8.1. *These shear coordinate rules for an orbifold agree with mutation of extended B-matrices in the associated generalized cluster algebra.*

Proof. First, we show that the shear coordinate associated to a pending arc, τ_j changes when we flip an standard arc, τ_k, in the same way that the bottom half of the corresponding column (call it column j) of the extended B-matrix changes when we mutate at this index, k.The entry $b_{k,j}$ is positive if and only if τ_k is counterclockwise of τ_j. For a lamination L_i, with $i > n$, the entry b_{ik} is positive if and only if L_i intersects the two arcs that are clockwise of τ_k. If both of these situations are true, then $\mu_k(b_{ij})$ will be given by $b_{ij} + b_{ik} b_{kj}$. In a picture, we can see that when we flip τ_k, it will change the bigon around τ_j, so that now L_i will intersect the bigon on the same side twice. This will increase the shear coordinate associated to L_i and τ_j. See picture below, where the shear coordinate $b_{\tau_j}(T, L)$ changes from 0 to 1. We can deal with the case where b_{ik} and b_{kj} are both negative similarly. If these entries are different signs or one is zero, it is clear from pictures that there will be no change to $b_{\tau_j}(T, L)$.



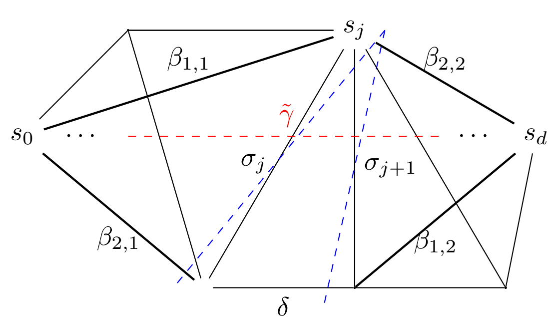
Next, we want to show that, when we flip a pending arc τ_j, all shear coordinates change according to generalized mutation rules. By set up, it is clear that the shear coordinates associated to that pending arc will flip signs. Recall other entries mutate by $\mu_j(b_{ik}) = b_{ik} + 2b_{ij}b_{jk}$ if both b_{ij} and b_{jk} are positive, $\mu_j(b_{ik}) = b_{ik} - 2b_{ij}b_{jk}$ if both b_{ij} and b_{jk} are negative, and no change otherwise. As before, b_{jk} is positive if and only if τ_j is counterclockwise of τ_k. For $i > n$, the entry b_{ij} is positive if and only if the lamination L_i intersects the side of the bigon around τ_j that is clockwise of τ_j as well as τ_j itself. Thus, both entries are positive if L_i intersects τ_k twice, both before and after intersecting τ_j. If τ_k is a pending arc, since we draw this as a loop L_i intersects τ_k four times, in two pairs. Moreover, the two intersections or pairs of intersections of L_i and τ_k could either both contribute -1, both contribute 0, or one of each contribution. We know that they cannot contribute $+1$ since L_i intersects τ_j, which is counterclockwise of τ_k. Then, when we flip τ_j, we change the quadrilateral or bigon around τ_k, depending on whether τ_k is standard or pending, which L_i intersects. Thus, we will change the shear coordinate associated to τ_k and L_i. Because of the two intersections or pairs of intersections, we will change by a multiple of two, as required in the generalized mutation rule. Figure 2.12 illustrates one example of this situation. Notice that $b_{6,1}$ changes from 0 to 2. $\square$

Lemma 3.8.2. *In the language of the previous section, $\phi_\gamma(\widetilde{Y}_i) = Y_i$.*

Proof. Recall the expressions $\phi_\gamma(\widetilde{Y}_i)$ from Equation (3.3).

First, let γ be a pending arc. Then, by the Bigon Lemma, we have a bigon β_1, β_2 around γ and the arc $\rho \in T$ at the same orbifold point, such that $e(\beta_i, T) < e(\gamma, T)$. We saw that the pre-image of this bigon in $\widetilde{S_\gamma}$ is a pentagon. We want to show that the laminations $L_{\tau_{i_k}}$ contribute the same shear coordinates in the bigon as their pre-images, L_{σ_k}, contribute in the pentagon in $\widetilde{S_\gamma}$. However, since ρ is in the triangulation T, and accordingly its images σ_j and σ_{j+1} in $\widetilde{S_\gamma}$ are in the triangulation $\widetilde{T_\gamma}$, the only elementary laminations that will contribute nontrivially to the relations in either case will be those associated to ρ in $\mathcal{O}$, or σ_j, σ_{j+1} in $\widetilde{S_\gamma}$.

In $\widetilde{S_\gamma}$, we have a pentagon with sides $\beta_{1,1}, \beta_{1,2}$, the two pre-images of $\beta_1 \in \mathcal{O}$, $\beta_{2,1}, \beta_{2,2}$, the two pre-images of β_2, and $\sigma_{[j]}$, the third arc in the triangle formed by σ_j and σ_{j+1}. This pentagon is triangulated by σ_j and σ_{j+1}, and the lift $\tilde{\gamma}$ is the arc crossing both arcs in this triangulation. By using the skein relations with y-variables from [59] in $\widetilde{S_\gamma}$ twice, on these two intersections, we have the expansion $x_{\tilde{\gamma}} = \frac{y_{\sigma_j} y_{\sigma_{j+1}} x_{\beta_{1,1}} x_{\beta_{1,2}} x_{\sigma_j} + y_{\sigma_{j+1}} x_{\beta_{1,1}} x_{\beta_{2,2}} x_{\sigma_{[j]}} + x_{\beta_{2,1}} x_{\beta_{2,2}} x_{\sigma_{j+1}}}{x_{\sigma_j} x_{\sigma_{j+1}}}$. Recalling that $\phi_\gamma(y_{\sigma_j}) = \phi_\gamma(y_{\sigma_{j+1}}) = y_\rho$, we see that our map ϕ_γ maps the y-variables as we hoped.



Next, let γ be an standard arc. From [58], we know that elementary laminations from standard arcs have the same local configuration about Q, the quadrilateral corresponding to γ and T from the quadrilateral lemma, and $\widetilde{Q}$, the lift of Q in $\widetilde{S_\gamma}$. We need to verify that the same is true for

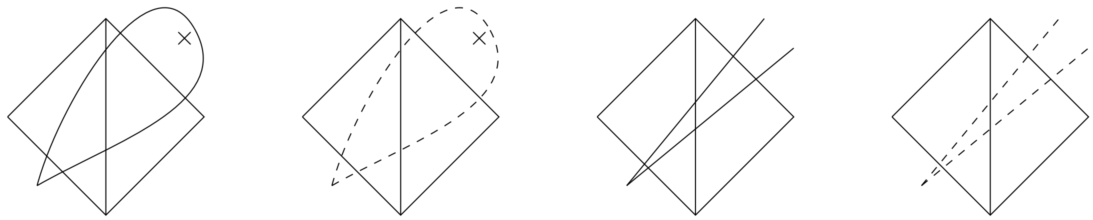


Figure 3.7: From left to right: A standard arc and a pending arc crossing, the elementary lamination from a pending arc and a standard arc crossing, and the lifts of these two scenarios to $\widetilde{S_\gamma}$.

elementary laminations from pending arcs. Suppose that $\rho \in T$ is a pending arc with elementary lamination L_ρ, and $\sigma_j, \sigma_{j+1} \in \widetilde{T_\gamma}$ are the pre-images of ρ with elementary laminations L_j, L_{j+1}. In Figure 3.7, on the left we show one example of intersections of ρ and L_ρ with Q, the quadrilateral from applying the quadrilateral lemma to γ and T. In this case, $b_\gamma(T, L_\rho) = 2$. On the right half we show first the intersections of σ_j and σ_{j+1} and then the intersections of L_j and L_{j+1} with $\widetilde{Q}$, the lift of Q to $\widetilde{S_\gamma}$. Here, $b_{\widetilde{\gamma}}(\widetilde{T_\gamma}, L_j) = b_{\widetilde{\gamma}}(\widetilde{T_\gamma}, L_{j+1}) = 1$. If y_ρ, y_j, and y_{j+1} are the y-variables associated to L_ρ, L_j and L_{j+1} respectively, then since $\phi_\gamma(y_j) = \phi_\gamma(y_{j+1}) = y_\rho$, we see that the contribution of laminations is consistent in $\mathcal{O}$ and $\widetilde{S_\gamma}$ in this case. The cases $b_\gamma(T, L_\rho) = -2$ and $b_\gamma(T, L_\rho) = 0$ are similar as, again, the local configurations around γ and $\widetilde{\gamma}$ look the same. □

3.9 Proof of cluster expansion formula

With the proof of Lemma 3.8.2, we are ready to complete our proof of Theorem 3.3.1.

Proof. In the statement of Theorem 1, we have a fixed orbifold $\mathcal{O} = (S, M, Q)$ with triangulation $T = \{\tau_1, \ldots, \tau_n, \tau_{n+1}, \ldots, \tau_{n+c}\}$ where $\tau_1, \ldots, \tau_n$ are internal arcs and $\tau_{n+1}, \ldots, \tau_{n+c}$ are boundary arcs. This determines the corresponding generalized cluster algebra $\mathcal{A}$ with principal coefficients with respect to the initial generalized seed $\Sigma_T = (\mathbf{x}_T, \mathbf{y}_T, B_T, \mathbf{z})$. For a given arc γ on $\mathcal{O}$, we defined the lifted triangulated polygon $\widetilde{S_\gamma}$, the lifted arc $\widetilde{\gamma}$, and lifted triangulation $\widetilde{T} = \{\sigma_1, \ldots, \sigma_d, \sigma_{d+1}, \ldots, \sigma_{2d+3}\}$ where $\sigma_1, \ldots, \sigma_d$ are internal arcs and $\sigma_{d+1}, \ldots, \sigma_{2d+3}$ are boundary arcs. The lift $\widetilde{S_\gamma}$ has an associated type A_d ordinary cluster algebra, $\widetilde{A_\gamma}$, where $d = e(\gamma, T)$. We then defined a projection map $\pi : \{\sigma_1, \ldots, \sigma_{2d+3}\} \to \{\tau_1, \ldots, \tau_{n+c}\}$, which in turn allowed us to define an algebra homomorphism $\phi_\gamma : \widetilde{A_\gamma} \to \mathrm{Frac}(\mathcal{A})$; in general,$\phi_\gamma$ acts by $\phi_\gamma(x_{\sigma_j}) = x_{\pi(\sigma_j)}$ and $\phi_\gamma(y_{\sigma_j}) = y_{\pi(\sigma_j)}$ for all $\sigma_j \in \{\sigma_1, \ldots, \sigma_{2d+3}\}$. We noted that when γ crosses one or multiple pending arc(s), ϕ_γ will map some variables associated to boundary arcs in $\widetilde{S_\gamma}$ to constant multiples of the variables associated to these pending arcs in $\mathcal{O}$. These multiples are determined by the orders of orbifold points. Further, we proved in Proposition 5 that $\phi_\gamma(x_{\widetilde{\gamma}}) = x_\gamma$.

Because $\widetilde{A_\gamma}$ is a type A_d ordinary cluster algebra, we know from the work of Musiker, Schiffler, and Williams [58] that we can build a snake graph $G_{\widetilde{T},\widetilde{\gamma}}$ which has the cluster expansion for $x_{\widetilde{\gamma}}$ as the generating function for its perfect matchings. This cluster expansion for $x_{\widetilde{\gamma}}$ is in terms of the variables $x_{\sigma_1}, \ldots, x_{\sigma_{2d+3}}$ and $y_{\sigma_1}, \ldots, y_{\sigma_d}$. Hence, computing the cluster expansion for x_γ in Σ_T is equivalent to specializing the variables in the generating function for perfect matchings of $G_{\widetilde{T},\widetilde{\gamma}}$ using the homomorphism ϕ_γ.

By construction, the unlabeled graphs for $G_{\widetilde{T},\widetilde{\gamma}}$ and $G_{T,\gamma}$ are identical. Because $\phi_\gamma(x_{\sigma_j}) = x_{\pi(\sigma_j)}$, applying ϕ_γ sends most edges labeled σ_j in $G_{\widetilde{T},\widetilde{\gamma}}$ to edges labeled $\pi(\sigma_j)$ in $G_{T,\gamma}$. Similarly, diagonals labeled y_{σ_j} are sent to diagonals labeled $y_{\pi(\sigma_j)}$. Hence, applying ϕ_γ to the generating function for perfect matchings of $G_{\widetilde{T},\widetilde{\gamma}}$ yields the formula in the theorem statement, which is itself the generating function for perfect matchings of $G_{T,\gamma}$, as desired. $\square$

Now we have an expansion formula for arcs without self-intersections in an unpunctured orbifold $\mathcal{O}$. These correspond to cluster variables in the associated generalized cluster algebra, $\mathcal{A}$. Arcs with self-intersections, i.e. generalized arcs - and closed curves do not correspond to cluster variables as they can never appear in a triangulation of $\mathcal{O}$. However, we can still use the rules in Sections 3.1 and 3.2 to construct snake graphs from these arcs and curves. By applying the expansion formula to these snake graphs, we associate an element of $\mathcal{A}$ to each generalized arc and closed curve. In the following sections, we will show that this association has desirable properties.

In order to study these arcs and curves, we will associate each with a product of 2×2 matrices that is developed breaking the path of the arc/curve into a sequence of "elementary steps". We can use another set of 2×2 matrices to help us compute weighted perfect matchings of graphs. We will show that these two sets of matrices are related. With these connections between arcs/curves, graphs, and matrices, we will be able to investigate properties of one object by studying another. In particular, we will use our matrix formulation to show that our expansion formula for generalized arcs and closed curves respects the skein relations. This work follows closely the work of Musiker and Williams [59], who do these calculations for cluster algebras from surfaces.

3.10 Universal snake graphs

In [59], Musiker and Williams compared their snake graph formulas to formulas arising from multiplying together strings of 2×2 matrices. These 2×2 matrices came in two types, depending on whether the matrix corresponds to adding a tile to the east or north of a snake graph. We simplify the calculations and arguments of [59] by using *universal tiles* to build *universal snake graphs*. Accordingly, we use only one type of 2×2 matrix which includes both types in [59] as specializations. We will similarly see that the universal snake graph is made up of a combination of the pieces used to build standard snake graphs.

For any positive integer n, the n-tile universal snake graph, UG_n, encodes information about the perfect matchings of all n tile ordinary snake graphs, as well as those with extra diagonals that we encounter in the orbifold setting. We will make this statement make more precise. Below is the universal snake graph with 4 tiles UG_4. The horizontal edges are labeled with a_j and the long diagonal edges, which are solid, are labeled with b_j. The dashed lines, labeled i_j, serve as labels for the individual tiles and cannot be used in a perfect matching.

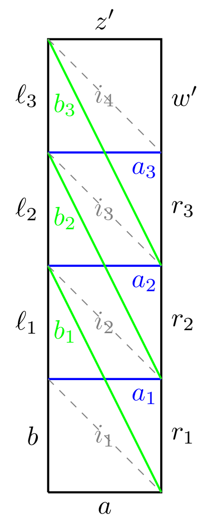


Figure 3.8: An example of a universal snake graph

Note that we can glue a or b to w' or z' to obtain a *universal band graph*. Good matchings of universal band graphs are defined analogously to good matchings of standard band graphs.

If n is even, let $w' = w$ and $z' = z$. Otherwise, $w' = z$ and $z' = w$. As a heuristic, we label the last tile so that the matching of all boundary edges that uses edge a must also include w. We call this the *minimal matching* to be consistent with the standard snake graph case. The other matching consisting of only boundary edges will include edges b and z, and we call this the *maximal matching*.

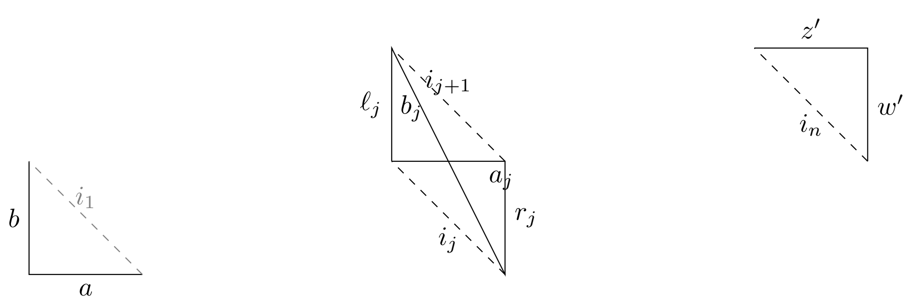
We note that we can recover any snake graph we are interested in, as well as others, from the universal snake graph of the appropriate size.

- Specializing $a_j = 0$ or $b_j = 0$ at each j will recover an ordinary snake graph. Based on the correspondence between snake graphs and sign sequences noted in [8], we know that there are 2^{n-1} snake graphs with n tiles. This is also the number of ways to choose whether $a_j = 0$ or $b_j = 0$ for $j = 1, \ldots, n-1$.

- If we do not set $a_j = 0$ or $b_j = 0$ at some j, but $a_{j-1}b_{j-1} = 0$ and $a_{j+1}b_{j+1} = 0$, we recover a hexagonal tile as in Section 3.1.

- We do not have a geometric interpretation of a graph where $a_j b_j \neq 0$ and $a_{j+1}b_{j+1} \neq 0$, or a graph where $a_j = b_j = 0$.

Remark 3.10.1. *We can think about the universal snake graph UG_n as constructed of two initial triangles and $n-1$ parallelograms with crossing diagonals,*



These parallelograms are essentially a superposition of the north-pointing and east-pointing parallelograms in [59]. If $b_j = 0$, the parallelogram is genuinely north-facing, and if $a_j = 0$, it is east-facing.

We also verify some simple properties about this graph and its perfect matchings. First, we explain how to extend the definition of a *twist* to the more complicated tiles in UG_n. As in the case of ordinary snake graphs 2.2.15, twisting induces a poset structure on the set of perfect matchings of UG_n. In Lemma 3.10.2, we see that this poset structure has a simple description.

If a perfect matching uses edges ℓ_{j-1} (set $\ell_0 = b$) and r_j (set $r_n = w'$) for $1 \leq j \leq n$, we twisting at tile j is accomplished by replacing those edges with the edges a_{j-1} (set $a_0 = a$) and a_j (set $a_n = z'$). This twist results in another valid perfect matching of UG_n. If a perfect matching instead uses edges ℓ_j (set $\ell_n = z'$) and r_{j-1} (set $r_0 = a$), for $1 \leq j \leq n$, then twisting at tile j is accomplished by replacing those edges with the edges b_{j-1} (set $b_0 = b$) and b_j (set $b_n = w'$). Both types of local move are referred to as a twist at tile j.

The poset of perfect matchings of UG_n has some of the same basic properties as the ordinary case described in Section 2.2.3 - that is, the covering relation is given by a twist at single tiles, and the poset rank function is given by the degree of the associated height monomials. As before, the height monomial for a given perfect matching P can be determined by viewing the labels of tiles enclosed by cycles in the symmetric difference $P \ominus P_-$. Note that we consider a tile to be "enclosed" by a cycle if the dashed line marking the tile is inside the cycle.

Lemma 3.10.2. *1. UG_n has 2^n perfect matchings*

2. The poset of perfect matchings of UG_n is isomorphic to the poset of subsets of $\{1, \ldots, n\}$ ordered by inclusion, B_n. This isomorphism sends a subset $\{i_1, \ldots, i_k\}$ to the matching with weight $y_{i_1} \cdots y_{i_k}$.

Proof. The first statement follows immediately from the second. We prove the second statement by induction. It is clear that the claim holds for UG_1, as this snake graph is a single tile with only a minimal and maximal matching. The maximal matching covers the minimal matching in the corresponding poset.

Now, suppose our claim holds for UG_{k-1}, and consider the poset of perfect matchings of UG_k. This contains a subposet of all matchings using the edge w; the minimal matching is in this subposet. Such matchings cannot use edges z and either cannot use ℓ_{k-1} or r_{k-1}, depending on the parity of k. If we remove z, w and either r_{k-1} or ℓ_{k-1} from UG_k, we have a graph isomorphic to UG_{k-1}; hence, the subposet of matchings using w is isomorphic to the poset of perfect matchings of UG_{k-1}.

The remaining elements of UG_k necessarily use z. The minimal element of this subposet is the perfect matching obtained by twisting the minimal matching at tile i_k. For the same reasons as for the matchings using w, this subposet is isomorphic to the poset of perfect matchings of UG_{k-1}.

Since the poset corresponding to perfect matchings of UG_{k-1} is isomorphic to B_{k-1}, and UG_k consists of exactly two disjoint subposets isomorphic to UG_{k-1} in the way described, we have that UG_k is isomorphic to B_k. Following our same induction, we can show the second statement of part 2; the subposet of matchings using w corresponds to subsets of $\{1, \ldots, k\}$ which do not include k while the subposet of matchings using z corresponds to subsets which do include k. $\square$

Along with the y_i variables from the poset structure of perfect matchings on UG_n, for each edge, η in the graph, we associate a formal variable x_η. Of course, when these graphs come from a surface,

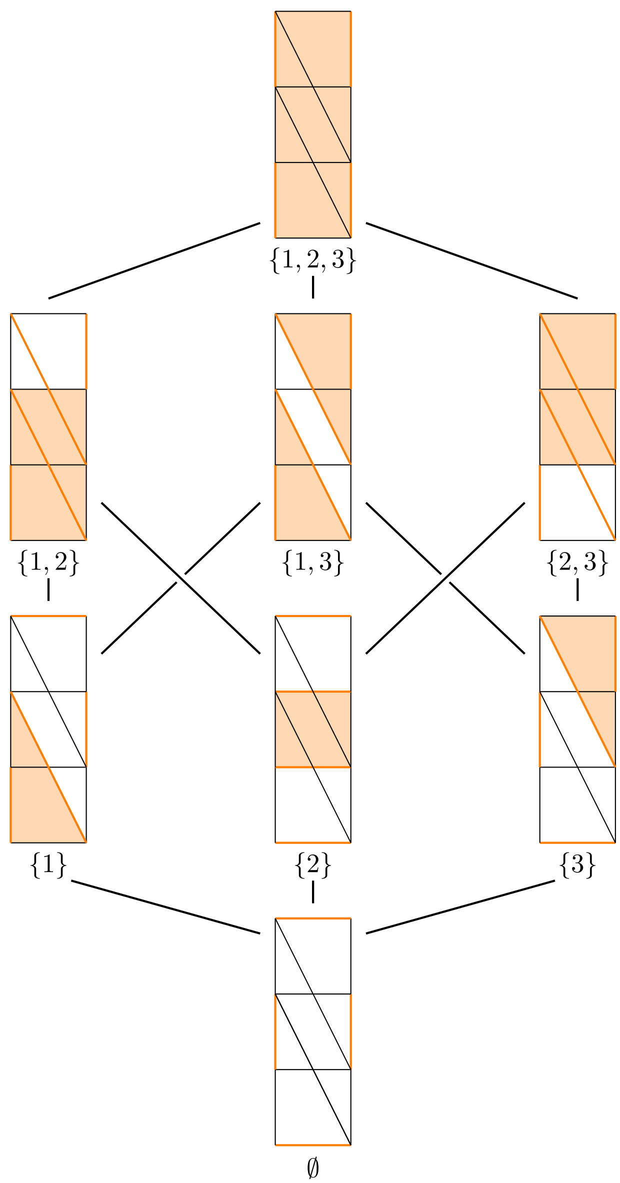


Figure 3.9: A Hasse diagram showing the poset of perfect matchings of UG_3, ranked by height monomial. For each perfect matching, the enclosed tiles are shaded.

these variables will be cluster variables. We associate a product of matrices to UG_n for each n. These products will encode all weighted perfect matchings of UG_n. Let $MG_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, for $n \geq 2$,

$$MG_n := m_{n-1} \cdots m_1 = \cdots \begin{bmatrix} \frac{x_{\ell_3}}{x_{i_3}} & y_3 x_{b_3} \\ \frac{x_{a_3}}{x_{i_3} x_{i_4}} & y_3 \frac{x_{r_3}}{x_{i_4}} \end{bmatrix} \begin{bmatrix} \frac{x_{r_2}}{x_{i_2}} & y_2 x_{a_2} \\ \frac{x_{b_2}}{x_{i_2} x_{i_3}} & y_2 \frac{x_{\ell_2}}{x_{i_3}} \end{bmatrix} \begin{bmatrix} \frac{x_{\ell_1}}{x_{i_1}} & y_1 x_{b_1} \\ \frac{x_{a_1}}{x_{i_1} x_{i_2}} & y_1 \frac{x_{r_1}}{x_{i_2}} \end{bmatrix}$$

where the last terms depend on the parity of n. Explicitly,

$$m_j := \begin{cases} \begin{bmatrix} \frac{x_{\ell_j}}{x_{i_j}} & y_j x_{b_j} \\ \frac{x_{a_j}}{x_{i_j} x_{i_{j+1}}} & y_j \frac{x_{r_j}}{x_{i_{j+1}}} \end{bmatrix} & \text{for odd } j \\ \\ \begin{bmatrix} \frac{x_{r_j}}{x_{i_j}} & y_j x_{a_j} \\ \frac{x_{b_j}}{x_{i_j} x_{i_{j+1}}} & y_j \frac{x_{\ell_j}}{x_{i_{j+1}}} \end{bmatrix} & \text{for even } j \end{cases}$$

We show that the graphs UG_n and the matrices MG_n satisfy the same relationship as Proposition 5.5 of [59].

Proposition 3.10.3. *The matrix MG_n is given by $MG_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ where*

$$A_n = \frac{\sum_{P \in S_A} x(P) y(P)}{(x_{i_1} \cdots x_{i_{n-1}}) x_a x_w} \qquad B_n = \frac{\sum_{P \in S_B} x(P) y(P)}{(x_{i_2} \cdots x_{i_{n-1}}) x_b x_w}$$

$$C_n = \frac{\sum_{P \in S_C} x(P) y(P)}{(x_{i_1} \cdots x_{i_n}) x_a x_z y_{i_n}} \qquad D_n = \frac{\sum_{P \in S_D} x(P) y(P)}{(x_{i_2} \cdots x_{i_n}) x_b x_z y_{i_n}}$$

where S_A is the set of matchings using a and w (this includes the minimal matching), S_B is the set of matchings using b and w, S_C is the set of matchings using a and z, and S_D is the set of matchings using b and z (this includes the maximal matching).

Proof. The proof proceeds by induction. The statement clearly holds for $n = 1$ or $n = 2$. Now, suppose it holds for $n-1$, and consider the graph UG_n. Suppose that n is even. Then, we have that

$$MG_n = \begin{bmatrix} \frac{x_{\ell_{n-1}}}{x_{i_{n-1}}} & y_{n-1} x_{b_{n-1}} \\ \frac{x_{a_{n-1}}}{x_{i_{n-1}} x_{i_n}} & y_{n-1} \frac{x_{r_{n-1}}}{x_{i_n}} \end{bmatrix} \begin{bmatrix} \frac{x_{r_{n-2}}}{x_{i_{n-2}}} & y_{n-2} x_{a_{n-2}} \\ \frac{x_{b_{n-2}}}{x_{i_{n-2}} x_{i_{n-1}}} & y_{n-2} \frac{x_{\ell_{n-2}}}{x_{i_{n-1}}} \end{bmatrix} \cdots \begin{bmatrix} \frac{x_{r_2}}{x_{i_2}} & y_2 x_{a_2} \\ \frac{x_{b_2}}{x_{i_2} x_{i_3}} & y_2 \frac{x_{\ell_2}}{x_{i_3}} \end{bmatrix} \begin{bmatrix} \frac{x_{\ell_1}}{x_{i_1}} & y_1 x_{b_1} \\ \frac{x_{a_1}}{x_{i_1} x_{i_2}} & y_1 \frac{x_{r_1}}{x_{i_2}} \end{bmatrix}$$

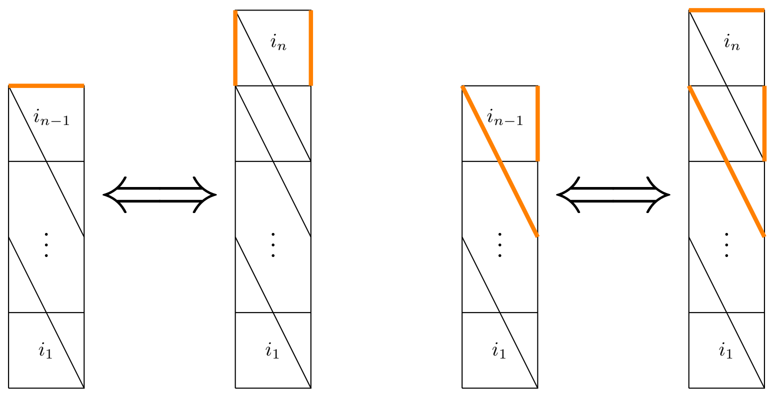
$$= \begin{bmatrix} \frac{x_{\ell_{n-1}}}{x_{i_{n-1}}} & y_{n-1} x_{b_{n-1}} \\ \frac{x_{a_{n-1}}}{x_{i_{n-1}} x_{i_n}} & y_{n-1} \frac{x_{r_{n-1}}}{x_{i_n}} \end{bmatrix} M_{n-1} = \begin{bmatrix} \frac{x_{\ell_{n-1}}}{x_{i_{n-1}}} & y_{n-1} x_{b_{n-1}} \\ \frac{x_{a_{n-1}}}{x_{i_{n-1}} x_{i_n}} & y_{n-1} \frac{x_{r_{n-1}}}{x_{i_n}} \end{bmatrix} \begin{bmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x_{\ell_{n-1}}}{x_{i_{n-1}}} A_{n-1} + y_{n-1} x_{b_{n-1}} C_{n-1} & \frac{x_{\ell_{n-1}}}{x_{i_{n-1}}} B_{n-1} + y_{n-1} x_{b_{n-1}} D_{n-1} \\ \frac{x_{a_{n-1}}}{x_{i_{n-1}} x_{i_n}} A_{n-1} + y_{n-1} \frac{x_{r_{n-1}}}{x_{i_n}} C_{n-1} & \frac{x_{a_{n-1}}}{x_{i_{n-1}} x_{i_n}} B_{n-1} + y_{n-1} \frac{x_{r_{n-1}}}{x_{i_n}} D_{n-1} \end{bmatrix}$$

Consider the subgraph consisting of tiles $i_1, \ldots, i_{n-1}$ as the graph UG_{n-1}. Since n is even, the edge that would be labeled w in this embedded copy of UG_{n-1} (which we call w_{n-1}) is labeled a_{n-1} in

UG_n. Similarly, the edge labeled r_{n-1} in UG_n would be labeled z in UG_{n-1} and so we call this edge z_{n-1}.

Let S'_A, S'_B, S'_C, S'_D be the sets of matchings satisfying the description in the proposition for the specified subgraph UG_{n-1}. Then, we have that all matchings in S_A correspond either to a matching in S'_A or in S'_C via the following correspondence. Matchings in S_A use both a and w; because n is even, they must also use either ℓ_{n-1} or b_{n-1}. If one of these matchings uses ℓ_{n-1}, it uniquely corresponds to a matching of UG_{n-1} which uses $a_{n-1} = w_{n-1}$; such a matching belongs to S'_A. If it uses b_{n-1}, then it uniquely corresponds to a matching of UG_{n-1} which uses $r_{n-1} = z_{n-1}$; this matching of UG_{n-1} belongs to S'_C.



We can then consider the weights of each matching. If a matching uses the edge ℓ_{n-1}, then its symmetric difference with the minimal matching of UG_n cannot enclose the tile labeled i_n. Hence, its weight must be equal to the weight of the corresponding perfect matching in S'_A. If the matching instead uses the edge b_{n-1}, then its symmetric difference with the minimal matching must enclose the tile labeled i_n, so its weight is given by $y_{i_{n+1}} \cdot$ (weight of corresponding matching from S'_C). Therefore, the set of matchings in S_A satisfies the relationship $A_n = \frac{x_{\ell_{n-1}}}{x_{i_{n-1}}} A_{n-1} + y_{n-1} x_{b_{n-1}} C_{n-1}$. The remaining arguments for the other matrix entries and the case where n is odd are very similar.

□

By considering several specializations, we can apply Proposition 3.10.3 to band graphs. Note that, while abstractly we can glue a or b to w or z to form a band graph, in order to get a graph which would come from a surface we must either glue a to z or b to w. In the first case, if the graph is from a closed curve on a surface, then we would also have $b = i_n$ and $w = i_1$. If we glue b to w, then $a = i_n$ and $z = i_1$. See Figure 5 in [59].

Theorem 3.10.4. *Let UG_n be a universal snake graph on n tiles. Then, we can express its sum of weighted perfect matchings by*

$$\sum_P x(P)h(P) = x_{i_1} \cdots x_{i_n} \mathrm{ur}\left(\begin{bmatrix} \frac{x_w}{x_{i_n}} & x_z y_{i_n} \\ \frac{-1}{x_z} & 0 \end{bmatrix} M_n \begin{bmatrix} 0 & x_a \\ \frac{-1}{x_a} & \frac{x_b}{x_{i_1}} \end{bmatrix} \right)$$

where ur *returns the upper right entry of a matrix.*

Now, let G be the result of gluing a and z in UG_n, and setting $b = i_n$ and $w = i_1$. Then, we can *express its sum of weighted perfect matchings by*

$$\sum_P x(P)h(P) = x_{i_1}\cdots x_{i_n}\operatorname{tr}\left(\begin{bmatrix}\frac{x_{i_1}}{x_{i_n}} & x_a y_{i_n}\\ 0 & \frac{y_{i_n}x_{i_n}}{x_{i_1}}\end{bmatrix} M_n\right)$$

Similarly, if G is the result of gluing b and w in UG_n and setting $a = i_n$ and $z = i_1$, then

$$\sum_P x(P)h(P) = x_{i_1}\cdots x_{i_n}\operatorname{tr}\left(\begin{bmatrix}\frac{x_{i_1}}{x_{i_n}} & 0\\ \frac{x_b}{x_{i_1}x_{i_n}} & \frac{y_{i_n}x_{i_n}}{x_{i_1}}\end{bmatrix} M_n\right)$$

Proof. For the case of UG_n, by using A_n, B_n, C_n, D_n as in Proposition 3.10.3, we find that

$$\operatorname{ur}\left(\begin{bmatrix}\frac{x_w}{x_{i_n}} & x_z y_{i_n}\\ \frac{-1}{x_z} & 0\end{bmatrix} M_n \begin{bmatrix}0 & x_a\\ \frac{-1}{x_a} & \frac{x_b}{x_{i_1}}\end{bmatrix}\right) = \frac{x_a x_w}{x_{i_n}}A_n + \frac{x_b x_w}{x_{i_1}x_{i_n}}B_n + x_a x_z y_{i_n}C_n + \frac{x_b x_z y_{i_n}}{x_{i_n}}D_n \tag{3.4}$$

From the definition of A_n, we see that $\frac{x_a x_w}{x_{i_n}}A_n = \frac{\sum_{P\in S_A} x(P)h(P)}{x_{i_1}\cdots x_{i_n}}$. A very similar statement is true for the terms of B_n, C_n, and D_n. Since the sets S_A, S_B, S_C, and S_D partition all perfect matchings of UG_n, the proof is complete.

Next, consider the case where we obtain G by gluing a and z in UG_n and make appropriate specializations. A good matching of this graph is one which could be extended to a perfect matching of UG_n by adding a or z. Thus, the matchings from A_n, C_n, and D_n all descend to good matchings of G. We expand the trace,

$$\operatorname{tr}\left(\begin{bmatrix}\frac{x_{i_1}}{x_{i_n}} & y_{i_n}x_a\\ 0 & \frac{y_{i_n}x_{i_n}}{x_{i_1}}\end{bmatrix} M_n\right) = \frac{x_{i_1}}{x_{i_n}}A_n + y_{i_n}x_a C_n + \frac{y_{i_n}x_{i_n}}{x_{i_1}}D_n$$

We see that the coefficients on A_n, C_n, and D_n are as in Equation (3.4), with one less factor of $x_a = x_z$. This matches the relationship between perfect matchings and good matchings. The situation is similar for a band graph obtained from gluing b and w. ☐

When a snake graph UG_n or band graph G is associated to an arc or closed curve γ on an orbifold $\mathcal{O}$ with triangulation T, we give the following cluster expansion formulas.

Definition 3.10.5. *Let $\mathcal{O} = (S, M, Q)$ be an unpunctured orbifold with triangulation T and $\mathcal{A}$ be the corresponding generalized cluster algebra with principal coefficients with respect to $\Sigma_T = (\mathbf{x}_T, \mathbf{y}_T, B_T)$. Let γ be a generalized arc with generalized snake graph $G_{T,\gamma}$.*

- *If γ has a contractible kink, then $X_{\gamma,T} = -X_{\bar{\gamma},T}$ where $\bar{\gamma}$ is γ with this kink removed.*
- *Otherwise, we define*

$$X_{\gamma,T} = \frac{1}{\mathit{cross}(T,\gamma)}\sum_P x(P)y(P)$$

Definition 3.10.6. *Let $\mathcal{O} = (S, M, Q)$ be an unpunctured orbifold with triangulation T and $\mathcal{A}$ be the corresponding generalized cluster algebra with principal coefficients with respect to $\Sigma_T = (\mathbf{x}_T, \mathbf{y}_T, B_T)$. Let γ be a closed curve with generalized band graph $G_{T,\gamma}$.*

- *If γ is a contractible loop, $X_{\gamma,T} = -2$.*

- *If γ is isotopic to a curve which bounds a disk containing a unique orbifold point, then* $X_{\gamma,T} = 2\cos(\pi/p) := \lambda_p$ *where p is the order of the orbifold ponit in this disk.*

- *Otherwise, we define*
$$X_{\gamma,T} = \frac{1}{cross(T,\gamma)} \sum_P x(P)y(P)$$
where the sum is over good matchings *of $G_{T,\gamma}$.*

These definitions also cover some special cases when it is not clear how to build a snake graph from the arc or curve.

3.10.1 Lift for generalized arcs

We give brief motivation for the crossing diagonals in generalized snake graphs from arcs which wind around orbifold points. If the order, p, of the orbifold point is greater then two, then when we lift a piece of such an arc to a p-fold cover, the lifted arc passes through a p-gon. If $p > 3$, then this is an untriangulated p-gon.

The standard snake graph construction relies on an arc passing through a triangulation. However, by using a loosened notion of T-paths, we can determine the appropriate expansion formula for such arcs which wind around orbifold points. Since the covers we consider are not triangulated but instead dissected into polygons, which is a setting not fully explored in T-path literature, we use this as a heuristic rather than a proof. Sections 3.11 - 3.13 will formally verify these formulas.

The concept of T-paths was defined originally by Schiffler and Thomas in [71] to give cluster expansion formulas in unpunctured surfaces and provide a proof of positivity for these cluster algebras as a corollary. Musiker and Gunawan expanded the T-path construction to once-punctured disks in [42].

As always, let γ be an arc on a surface (S, M) with triangulation $T = \{\tau_1, \dots, \tau_n\}$, and let $d = e(\gamma, T) > 0$. Fix an arbitrary orientation to each arc $\tau \in T$ and to γ, and let τ^- be an arc isotopic to τ with opposite orientation. Let $\tau_{i_1}, \dots, \tau_{i_d}$ be the arcs crossed by γ, with order determined by γ's orientation. Loosely, a (complete) T-path is a path $\alpha = (\alpha_1, \dots, \alpha_{2d+1})$ such that

1. Each α_i is equivalent to τ_j or τ_j^- for some $\tau_j \in T$.

2. For $1 \leq i < n$, $t(\alpha_i) = s(\alpha_{i+1})$.

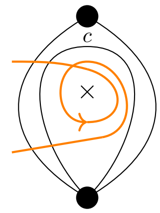
3. $s(\alpha_1) = s(\gamma)$ and $t(\alpha_{2d+1}) = t(\gamma)$.

4. (This requirement makes the T-path "complete") $\alpha_{2j} = \tau_{i_j}$ for $1 \leq j \leq d$.

From each T path we obtain a monomial where variables associated to the arcs crossed on odd steps are in the numerator and variables from arcs crossed on even steps are in the denominator. Then, we sum the monomials from all T-paths from γ to obtain x_γ. Note that for a complete T-path, each denominator is equal to cross(T, γ).

Given an arc γ, the collection of T-paths from γ are in bijection with perfect matchings of $G_{T,\gamma}$. Moreover, we can draw each complete T-path on $G_{T,\gamma}$ by using the dashed diagonals in each tile as the steps along the arcs crossed by γ, that is, the even indexed steps. The set of edges used by the odd-indexed steps (those not on dashed edges) is a perfect matching of the graph.

As an example of the sort of arcs we are describing, consider γ as below, where the orbifold point is order 5.



One lift of this configuration is shown below.

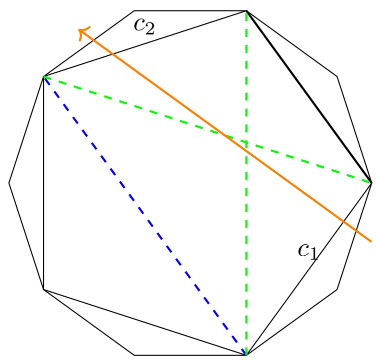


Figure 3.10: A lift of an arc that crosses a pending arc and winds around the associated orbifold point.

We consider possible sub-path $(\alpha_{2i}, \alpha_{2i+1}, \alpha_{2(i+1)})$ of a T-path from the lift of γ. As in the definition, α_{2i} will go along c_1 in some direction and $\alpha_{2(i+1)}$ will go along c_2, as required by the definition. Then, α_{2i+1} will have to connect $t(\alpha_{2i})$ and $s(\alpha_{2(i+1)})$. If we lift the requirement that each α_j is an arc in the triangulation, we see that the four diagonals highlighted (one being a side of the polygon in this case) all connect end points of c_1 and c_2.

Each of the polygons in the lifts is regular since all sides correspond to the same arc in the orbifold. Thus, we can use elementary geometry to write the lengths of these diagonals in terms of the length of the sides of the polygon.

Definition 3.10.7. *A* k-diagonal *in a polygon is one which skips $k-1$ vertices. For instance, boundary edges in a p-gon are both 1-diagonals and $(p-1)$-diagonals.*

The following lemma appears in Section 2 of [50].

Lemma 3.10.8. *A k-diagonal in a regular p-gon with sides of length s has length $U_{k-1}(\lambda_p)\cdot s$, where $U_k(x)$ denotes the k-th normalized Chebyshev polynomial as in Definition 3.1.1.*

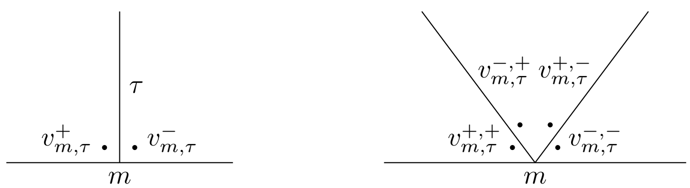
The fact that we have four options for potential steps between c_1 and c_2 leads to the hexagonal tiles discussed in Section 3.1. Note that the configuration between the two dashed lines in these tiles looks similar to the lift of the generalized arc above. In Lemma 3.12.4, we will see these Chebyshev polynomials also arise from products of matrices in $SL_2(\mathbb{R})$.

3.11 M-path from an arc in a triangulated orbifold.

In the previous section, we associated a product of 2×2 matrices to the universal snake graph. Now, following the construction of [59], we will also associate products of matrices to arbitrary arcs or curves on a triangulated orbifold; Theorem 3.12.12 will show a relationship between these two systems of matrices. This method will allow us to extend our snake graph formula to generalized arcs and closed curves.

Similar to the graph case, we break arcs or closed curves into a series of *elementary steps* and associate 2×2 matrices to each step. Arcs do not have a unique associated M-path, but in Section 3.12 we will both describe a convention for which M-path to use and show that the statistics we use do not depend on the path.

While the start and terminal point of an arc on an orbifold coincide with the set of marked points, elementary steps and the M-paths in general go between points which are near marked points but are not marked points themselves. To formalize this, draw a small circle, h_m, around each marked point m. These should be small enough that h_m does not intersect $h_{m'}$ for another distinct marked point m'. If τ is an arc incident to m, let $v_{m,\tau}$ be the intersection of τ and h_m. If τ is a standard arc, we define $v^+_{m,\tau}$ (respectively, $v^-_{m,\tau}$) to be a point on h_m that is counterclockwise (clockwise) of $v_{m,\tau}$. If τ is a pending arc, we define $v^{-,-}_{m,\tau}, v^{-,+}_{m,\tau}, v^{+,-}_{m,\tau}$, and $v^{+,+}_{m,\tau}$ to be, in counterclockwise order, four spots along h_m such that $v^{+,+}_{m,\tau}$ is clockwise of all of τ, $v^{-,-}_{m,\tau}$ is counterclockwise of all of τ, and $v^{-,+}_{m,\tau}$ and $v^{+,-}_{m,\tau}$ are contained within τ, drawn as a loop, such that $v^{-,+}_{m,\tau}$ is counterclockwise from $v^{+,-}_{m,\tau}$.



Given an arc γ, with end points $s(\gamma)$ and $t(\gamma)$, any representative M-path will go between $v^{\pm}_{s(\gamma),\tau}$ or $v^{\pm,\pm}_{s(\gamma),\tau}$ and $v^{\pm}_{t(\gamma),\tau'}$ or $v^{\pm,\pm}_{t(\gamma),\tau'}$,where τ and τ' are arcs in the triangulation incident to $s(\gamma)$ and $t(\gamma)$ respectively.

First, we recall the three types of elementary steps used in the surface case [59]:

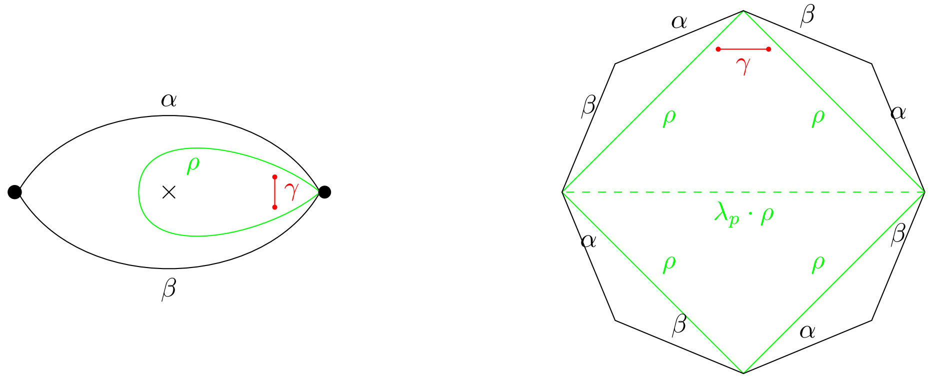
- An elementary step of type 1 goes from $v^{\pm}_{m,\tau}$ to $v^{\mp}_{m,\tau'}$ where τ and τ' share an endpoint and border the same triangle in T. If σ is the third side of the triangle, then we associate the matrix $\begin{bmatrix} 1 & 0 \\ \pm\frac{x_\sigma}{x_\tau x_{\tau'}} & 1 \end{bmatrix}$. The sign of $\frac{x_\sigma}{x_\tau x_{\tau'}}$ is positive if we travel from $v^+_{m,\tau}$ to $v^-_{m,\tau'}$ and negative otherwise.

- An elementary step of type 2 goes from $v^{\pm}_{m,\tau}$ to $v^{\mp}_{m,\tau}$; that is, this step crosses the arc τ. We associate the matrix $\begin{bmatrix} 1 & 0 \\ 0 & y_\tau \end{bmatrix}$ if we go from $v^-_{m,\tau}$ to $v^+_{m,\tau}$ and $\begin{bmatrix} y_\tau & 0 \\ 0 & 1 \end{bmatrix}$ otherwise.

- An elementary step of type 3 follows an arc τ in the triangulation. That is, if τ connects marked points m and m', then this step goes from $v^{\pm}_{m,\tau}$ to $v^{\mp}_{m',\tau}$. We associate to this the matrix $\begin{bmatrix} 0 & \pm x_\tau \\ \frac{\mp 1}{x_\tau} & 0 \end{bmatrix}$. We use $+x_\tau$ and $\frac{-1}{x_\tau}$ if this step sees τ on the right and uses the opposite signs if it sees τ on the left.

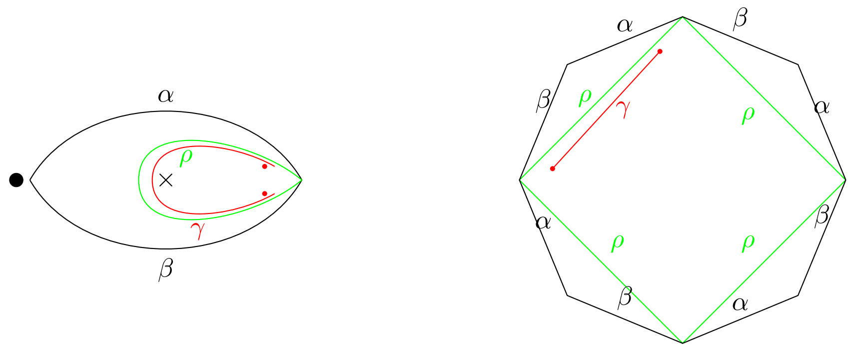
Because we're working on a triangulated orbifold rather than a triangulated surface, we update these elementary steps to handle interactions with pending arcs. In particular, we show how to decompose a portion of an arc winding around an orbifold point into a sequence of elementary steps; combining this with the above elementary steps will allow us to decompose any arc in a triangulated orbifold.

First, we can go from $v_{m,\rho}^{\pm,\mp}$ to $v_{m,\rho}^{\mp,\pm}$ where ρ is a pending arc. We also examine this local configuration in a cover.



In the cover, this resembles an elementary step of type 1 from [59]. Accordingly, we associate to this a matrix $\begin{bmatrix} 1 & 0 \\ \pm\frac{\lambda_p \rho}{\rho^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \pm\frac{\lambda_p}{\rho} & 1 \end{bmatrix}$. As for an elementary step of type 1 in a surface, we use $\frac{\lambda_p}{\rho}$ if we travel clockwise (from $v_{m,\rho}^{\mp,\pm}$ to $v_{m,\rho}^{\pm,\mp}$) and use $\frac{-\lambda_p}{\rho}$ otherwise.

If γ does not have self-intersections, this is the only sort of step we will see. But if γ winds $k \geq 1$ times around the orbifold point, we will also see an elementary step of type 3 along the pending arc, again between $v_{m,\rho}^{\pm,\mp}$ to $v_{m,\rho}^{\mp,\pm}$.



This configuration resembles an elementary step of type 3, and as a result, we associate the matrix $\begin{bmatrix} 0 & \pm\rho \\ \frac{\mp 1}{\rho} & 0 \end{bmatrix}$, with the same rule for determining the sign as before.

We can treat an elementary step of type 2 across a pending arc, that is, between $v_{m,\rho}^{\pm,\pm}$ and $v_{m,\rho}^{\mp,\mp}$, the same as for a pending arc.

Definition 3.11.1. *If κ is an M-path whose sequence of elementary steps has associated matrices $\eta_1, \ldots, \eta_n$, then we define $M(\kappa) = \eta_n \cdots \eta_1$.*

3.12 Standard M-path

We give an algorithm of assigning a M-path, κ_γ, to an arc or closed curve γ, which consists of a series of connected elementary steps. We say that this convention produces the "standard M-path" associated to γ. As an informal heuristic, we will pick an orientation of γ, then always travel along the right of γ.

First, we utilize the symmetry about an orbifold point to choose a convenient representative for γ. At each pending arc that γ crosses, we choose a representative that winds clockwise and less than p times around the incident orbifold point, with one exception. If γ crosses a pending arc which is based at a vertex to the left of γ, and if γ is isotopic to one which winds 0 times around this orbifold point, then we will use a representative of γ which winds p times around this orbifold point. The reason why we choose this will be made clear in the description of κ_γ.

As before, let γ be an arbitrary arc on an orbifold $\mathcal{O}$ with triangulation $T = \{\tau_1, \ldots, \tau_n\}$. Let $\tau_{i_1}, \ldots, \tau_{i_d}$ be the arcs which γ crosses, with order determined by an orientation on γ.

Suppose the first triangle that γ cuts through has sides $\alpha, \beta, \tau_{i_1}$, in clockwise order, so that α and β share an endpoint at $s(\gamma)$. Then, κ_γ will start at $v^-_{s(\gamma),\alpha}$, and follow α with a step of type 3, followed by a step of type 1 from a to τ_{i_1}.

Similarly, suppose the last triangle that γ cuts through has sides w, z, τ_{i_d} in clockwise order, with w and z both touching $t(\gamma)$. Then, the last few steps of κ_γ will be a step of type 2 crossing τ_{i_d}, a step of type 1 from τ_{i_d} to z, and a step of type 3 along z. Then, κ_γ will end at $v^+_{t(\gamma),z}$.

We next explain the sequence of steps we use between τ_{i_j} and $\tau_{i_{j+1}}$ for $1 \leq j \leq n-1$, where these are both standard arcs. This sequence will involve crossing τ_{i_j} but not $\tau_{i_{j+1}}$. First, we use a step of type 2 to cross τ_{i_j}. Then, if τ_{i_j} and $\tau_{i_{j+1}}$ share a vertex to the right of γ, then we use a step of type 1. Call this sequence of a step of type 2 and a step of type 1 a *compound step of type A*. If τ_{i_j} and $\tau_{i_{j+1}}$ share a vertex to the left of γ, let σ_j be the third arc in this triangle. Then we use a step of type 1 between τ_{i_j} and σ_j, a step of type 3 along σ_j, and a step of type 1 between τ_{i_j} and σ_j. We call this sequence a *compound step of type B*. A "step" will be assumed to be elementary unless otherwise specified.

Now we explain the protocol when γ crosses a pending arc $\rho = \tau_{i_\ell}$. First, we assume that ρ is not the first or last arc that γ crosses, so $\tau_{i_{\ell-1}}$ and $\tau_{i_{\ell+2}}$ are not necessarily distinct arcs in the bigon or monogon surrounding ρ. We give rules for the transition from $\tau_{i_{\ell-1}}$ to $\tau_\ell = \rho$, for the winding inside ρ, and the transition from $\tau_{\ell+1}$ to $\tau_{\ell+2}$. These depend on whether ρ is based at a vertex to the right or left of γ, and whether $\tau_{i_{\ell-1}}$ and $\tau_{i_{\ell+2}}$ are distinct or not. These will not depend on whether $\tau_{i_{\ell-1}}$ and $\tau_{i_{\ell+2}}$ are standard or pending.

First, suppose that ρ is based at a marked point w to the right of γ, and that $\tau_{i_{\ell-1}}$ and $\tau_{i_{\ell+2}}$ are distinct. Then, between $\tau_{i_{\ell-1}}$ and ρ, we use a compound step of type A. Between the two crossings of ρ, we use an elementary step of type 2 to cross ρ. If γ winds $k \geq 0$ times around the orbifold point incident to ρ, we include a step of type 1 from $v^{-;+}_{w;\rho}$ to $v^{+;-}_{w;\rho}$ followed by k iterations of a step of type 3 along ρ and a step of type 1 from $v^{-;+}_{w;\rho}$ to $v^{+;-}_{w;\rho}$. Finally, we transition from $\rho = \tau_{i_{\ell+1}}$ to $\tau_{i_{\ell+1}}$ with a compound step of type A. See the top left of Figure 3.11.

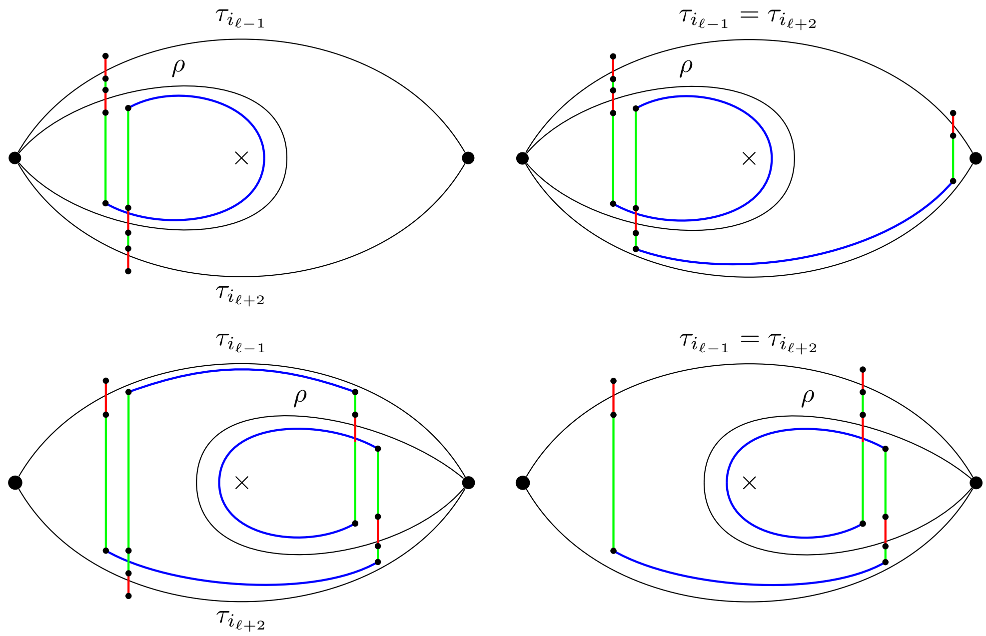


Figure 3.11: Sequences of elementary steps to use when γ crosses a pending arc twice consecutively.

Otherwise, we have that $\tau_{i_{\ell-1}} = \tau_{i_{\ell+2}}$, so that γ crosses the same arc both before and after crossing ρ. Then, at the transition from $\rho = \tau_{i_{\ell+1}}$ to $\tau_{i_{\ell+1}}$, we instead use a compound step of type B. The earlier part of the sequence remains the same. See the top right of Figure 3.11.

Now suppose that ρ is based to the left of γ, and first suppose that $\tau_{i_{\ell-1}}$ and $\tau_{i_{\ell+2}}$ are distinct. We can use a compound step of type B to transition from $\tau_{i_{\ell-1}}$ to ρ. From our choice of a representative of γ, we know that γ winds $k \geq 1$ times around the orbifold point incident to ρ. We can use the same algorithm for the sequence of steps within the pending arc ρ, but we will only include $k-1$ self-intersections. Then, we use another compound step of type B to transition from ρ to $\tau_{i_{\ell+2}}$. Note that while κ_γ only intersects itself $k-1$ times inside the pending arc γ, it intersects itself one more time outside the pending arc. Thus, κ_γ remains homotopic to γ. See the bottom left of Figure 3.11.

If $\tau_{i_{\ell-1}}$ and $\tau_{i_{\ell+2}}$ are not distinct, then we can include all k self-intersections in the pending arc ρ. In this case, we use a compound step of type A when transitioning from $\tau_{i_{\ell+1}}$ to $\tau_{i_{\ell+2}}$. See the bottom right of Figure 3.11.

Example 3.12.1. As an example, here is the corresponding expansion of matrices for the piece of γ portrayed in the case $\tau_{i_{\ell-1}} \neq \tau_{i_{\ell+2}}$ and the pending arc $\rho = \tau_{i_\ell}$ is based to the right of γ, as in the top left of Figure 3.11. For convenience, let $\alpha = \tau_{i_{\ell-1}}$ and $\beta = \tau_{i_{\ell+2}}$

$$\begin{bmatrix} 1 & 0 \\ \frac{x_\alpha}{x_\beta x_\rho} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{x_\rho} & 1 \end{bmatrix} \begin{bmatrix} 0 & x_\rho \\ \frac{-1}{x_\rho} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{x_\rho} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y_\rho \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{x_\beta}{x_\alpha x_\rho} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y_\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{x_\alpha}{x_\beta x_\rho} & y_\beta \end{bmatrix} \begin{bmatrix} \lambda_p & y_\rho x_\rho \\ \frac{(\lambda_p^2-1)}{x_\rho} & \lambda_p y_\rho \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{x_\beta}{x_\alpha x_\rho} & y_\alpha \end{bmatrix}$$

In the second line, we multiply the matrices within each compound step. Notice that these matrices resemble those we assigned to pieces of the universal snake graph. We will eventually

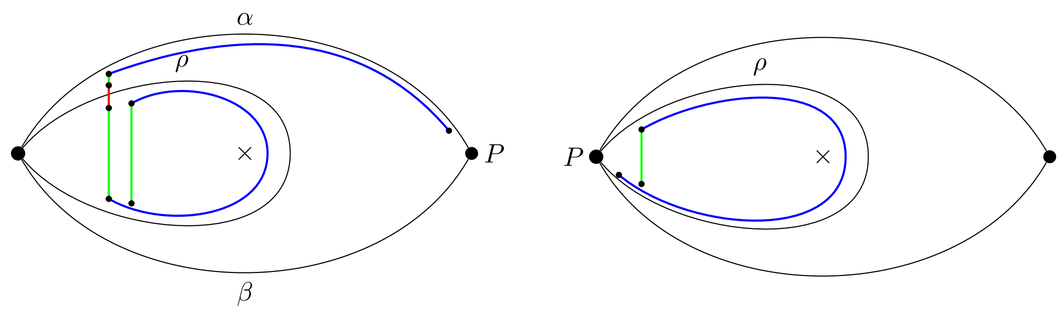


Figure 3.12: Sequences of elementary steps when a pending arc is the first arc which γ crosses.

solidify this connection.

The cases we have yet to discuss are when the first or last arc that γ crosses is a pending arc. We again will vary our procedure based on whether the pending arc is based at a vertex to the right or to the left of γ or if it is based at $s(\gamma)$.

Let $P = s(\gamma)$ be the start of γ. Recall we choose a representative of γ which winds counterclockwise around any orbifold point it encounters. If the pending arc, ρ, is based at a vertex other than P, then we use a step of type 3 along α, the boundary edge to the right of γ, and a step of type 1 from α to ρ. The following steps will depend on how many times γ winds around this orbifold point and which arc γ crosses next. These use the same compound steps as in the earlier discussion. For example, on the left-hand side of Figure 3.12 if γ crosses α after winding around the orbifold point, then after a compound step which winds around the orbifold point, as drawn, we will use a compound step of type B to transition from ρ to α.

Next, suppose $s(\gamma)$ is also the unique marked point incident to ρ and γ winds at least once around the orbifold point. Note that if γ does not wind at least once around the orbifold point, it is isotopic to an arc that does not cross the pending arc. See the right-hand side of Figure 3.12. As in the case when ρ is not based at $s(\gamma)$, the following steps depends on which arc γ crosses next.

The cases where the last arc that γ crosses is a pending arc, ρ are very similar. If $t(\gamma)$ is distinct from the unique marked point incident to ρ, the final compound step will start with a step of type 2 to cross ρ, then a step of type 1 and a step of type 3. We can see this by traveling the opposite direction along the M-path on the left of Figure 3.12.

If $t(\gamma)$ is the marked point incident to ρ, then our final compound step will be as in the case when ρ is incident to $s(\gamma)$, but again with the order reversed.

Finally, we consider the case when γ is a closed curve. Pick a triangle, Δ, such that γ consecutively crosses two of its arcs. Label these arcs τ_{i_1} and τ_{i_n} such that τ_{i_1} immediately follows τ_{i_n} in a clockwise order. Let q be the endpoint of τ_{i_1} which is not also an endpoint of τ_{i_n}. Then, the standard M-path, κ_γ, will start and stop at $v^-_{q,\tau_{i_1}}$. The M-path can start with a compound step of type A or B, depending on whether τ_{i_1} and τ_{i_2} share a vertex to the right or left of the chosen orientation of γ. Then, since by construction τ_{i_n} and τ_{i_1} share an endpoint to to the left of γ, κ_γ will end with a compound step of type B.

3.12.1 Upper right entry does not depend on choice of M-path

Lemma 4.8 in [59] shows that the upper right (trace) of matrices from M-paths associated to arcs (closed curves) on a surface does not depend on our choice of M-path. For instance, if γ is a closed curve, the trace of $M(\kappa)$ for an M-path κ from γ does not depend on $\kappa's$ start and end point since trace is invariant under cyclic permutations.

Lemma 3.12.2 (Lemma 4.8 of [59]). *Let γ_1 and γ_2 be a generalized arc and closed curve with no contractible kinks, respectively, on a triangulated surface (S, M). Then, given κ_1 and κ_2, two M-paths associated to γ_1, we have $|ur(M(\kappa_1))| = |ur(M(\kappa_2))|$. If κ_1' and κ_2' are two M-paths associated to γ_2, we have $|tr(M(\kappa_1))| = |tr(M(\kappa_2))|$.*

Since we are in an orbifold, there are more ways to adjust an M-path associated to an arc γ; in particular, if an M-path winds k times around an orbifold point of order p, we can adjust it to wind $k + mp$ times for any integer m. We show in Lemma 3.12.3 that these adjustments still do not affect the statistics of the matrices which we care about.

Lemma 3.12.3. *Let κ_1 and κ_2 be two M-paths which are identical except at one orbifold point of order p, such that at this orbifold point κ_1 winds k times and κ_2 winds $k + mp$ times where $m \in \mathbb{Z}$. Then, up to universal sign, $M(\kappa_1) = M(\kappa_2)$. In particular, $|ur(M(\kappa_1))| = |ur(M(\kappa_2))|$ and $|tr(M(\kappa_1))| = |tr(M(\kappa_2))|$.*

To prove this, we will prove a lemma about products of the elementary matrices which correspond to an M-path winding around an orbifold point of order p. It turns out products of these matrices have Chebyshev polynomials, evaluated at λ_p, as coefficients.

Lemma 3.12.4. *Let $k \geq 0$, and let $U_k(x)$ be the k-th normalized Chebyshev polynomial of the second kind. Then,*

$$\left(\begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{x_\rho} & 1 \end{bmatrix} \begin{bmatrix} 0 & x_\rho \\ \frac{-1}{x_\rho} & 0 \end{bmatrix} \right)^k = \begin{bmatrix} -U_{k-2}(\lambda_p) & U_{k-1}(\lambda_p) \cdot x_\rho \\ \frac{-U_{k-1}(\lambda_p)}{x_\rho} & U_k(\lambda_p) \end{bmatrix} \tag{3.5}$$

and

$$\left(\begin{bmatrix} 0 & -x_\rho \\ \frac{1}{x_\rho} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-\lambda_p}{x_\rho} & 1 \end{bmatrix} \right)^k = \begin{bmatrix} U_k(\lambda_p) & -U_{k-1}(\lambda_p) \cdot x_\rho \\ \frac{U_{k-1}(\lambda_p)}{x_\rho} & -U_{k-2}(\lambda_p) \end{bmatrix} \tag{3.6}$$

Proof. Recall our convention that $U_{-1}(x) = 0$, $U_0(x) = 1$, and the normalized recurrence for $\ell > 0$: $U_\ell(x) = xU_{\ell-1}(x) - U_{\ell-2}(x)$. This proof follows by induction and the recurrence for Chebyshev polynomials. □

Remark 3.12.5. *Equation 3.6 can also be thought about as making sense of what matrix should be assigned to a generalized arc which winds k times clockwise around an orbifold point.*

Remark 3.12.6. *Compare the matrices in Lemma 3.12.4 with the statement of Proposition 3.10.3 and the labels we include in hexagonal tiles from an arc with nontrivial winding about an orbifold point. In particular, note that if we consider a hexagonal tile as UG_2, then there is exactly one perfect matching in each A_n, B_n, C_n, and D_n, and each matching uses exactly one edge with label $U_\ell(\lambda_p)x_\rho$ for some ℓ and for the pending arc ρ.*

Now, we can prove Lemma 3.12.3.

Proof. If $m = 0$, this lemma is trivial. If $m > 0$, then the expansion of $M(\kappa_2)$ into elementary matrices will have a term $\left(\begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{\rho} & 1 \end{bmatrix} \begin{bmatrix} 0 & \rho \\ \frac{-1}{\rho} & 0 \end{bmatrix} \right)^{mp} = \left(\begin{bmatrix} -U_{p-2}(\lambda_p) & U_{p-1}(\lambda_p) \cdot \rho \\ \frac{-U_{p-1}(\lambda_p)}{\rho} & U_p(\lambda_p) \end{bmatrix} \right)^m$. Recall that $U_{p-2}(\lambda_p) = 1, U_{p-1}(\lambda_p) = 0$, and $U_p(\lambda_p) = -1$. Thus, this extra factor in the expansion of $M(\kappa_2)$ is simply $\pm\mathrm{Id}$, where the sign depends on the parity of m. Thus, $|\mathrm{ur}(M(\kappa_1))| = |\mathrm{ur}(M(\kappa_1))|$. The case where $m < 0$ is similar. □

From Lemma 3.12.2 and Lemma 3.12.3, we see that we can always use the standard M-path, κ_γ for any generalized arc or closed curve γ and not affect the upper right or trace, respectively, of the associated matrix.

We have that the upper right (trace) of the matrix from an M-path for most arcs (closed curves) is a well-defined statistic. However, if γ is a closed curve, there are some cases where γ does not cross any arcs on T and so the M-path may be ambiguous. Musiker and Williams deal with curves which are contractible or enclose a single puncture [59]. In an orbifold, we can also have a curve which encloses a single orbifold point.

We turn to the normalized Chebyshev polynomials *of the first kind* which Musiker, Schiffler, and Williams use to describe arcs such as we are describing.

Definition 3.12.7 (Definition 2.33 and Proposition 2.34 of [57])**.** *Let $T_\ell(x)$ denote the ℓ-th normalized Chebyshev polynomial of the first kind, for $\ell \geq -1$. These are given by initial polynomials $T_0(x) = 2, T_1(x) = x$, and the recurrence,*

$$T_\ell(x) = xT_{\ell-1}(x) - T_{\ell-2}(x)$$

While the definition in [57] keeps track of an extra variable Y, for now we set $Y = 1$. See Section 3.14 for a related discussion of y-variables.

We give the following as a corollary of Proposition 4.2 of [57].

Proposition 3.12.8. *Let γ be isotopic to a closed loop encompassing a single orbifold point with $k \geq 0$ self intersections. Then, $x_\gamma = T_{k+1}(\lambda_p)$.*

This proposition largely follows from the following relationship amongst the normalized Chebyshev polynomials discussed here.

Lemma 3.12.9. *Let $U_\ell(x)$ and $T_\ell(x)$ be normalized Chebyshev polynomials of the second and first kind, respectively, as in Definitions 3.1.1 and 3.12.7. Then, for $\ell \geq 1$,*

$$T_\ell(x) = U_\ell(x) - U_{\ell-2}(x)$$

Lemma 3.12.9 can be proved using induction and the recurrence relations for each type of Chebyshev polynomials.

Proof of Proposition 3.12.8. If γ intersects itself $k \geq 0$ times, we can build an M-path for γ, call it κ, which is a sequence of $k+1$ steps of type 1 and type 3. From Lemmas 3.12.4 and 3.12.9, we immediately see that $\mathrm{tr}M(\kappa) = U_{k+1}(\lambda_p) - U_{k-1}(\lambda_p) = T_{k+1}(\lambda_p)$. We see this follows naturally from Proposition 4.2 of [57] since if ξ is an essential loop around this single orbifold point and κ is an M-path from xi, then we have $\mathrm{tr}(M(\kappa)) = \lambda_p$. □

Now, we are prepared to state a complete definition.

Definition 3.12.10. *Let γ be a generalized arc and γ' be a closed curve on an unpunctured orbifold $\mathcal{O}$ with triangulation T. Then, $\chi_{\gamma,T} = |\mathrm{ur}(M(\kappa_\gamma))|$.*

If γ' is contractible, set $\chi_{\gamma',T} = -2$. If γ' is isotopic to a closed loop encompassing a single orbifold point of order p, with $k \geq 0$ self-intersections, let $\chi_{\gamma',T} = T_{k+1}(\lambda_p)$. Otherwise, let $\chi_{\gamma',T} = |\mathrm{tr}(M(\kappa_{\gamma'}))|$.

In Theorem 3.12.12 we will compare $\chi_{\gamma,T}$ with $X_{\gamma,T}$. Recall we found $X_{\gamma,T}$ by building a snake graph from γ.

Remark 3.12.11. *Musiker and Williams show in Section 4 of [59] that their matrices, after specializations, generalize work of Fock and Goncharov in [21] which also associated matrix products to paths in triangulated surfaces as a way to construct coordinates on the corresponding Teichmuller space.*

In their paper defining generalized cluster algebras, Chekhov and Shapiro update the matrix products which compute X-coordinates (in the sense of Fock-Goncharov) to include orbifolds [13]. They accomplish this by assigning the matrix $F_p = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda_p \end{pmatrix}$ to the piece of a path going around an orbifold point. If an arc winds k times around an orbifold point, they include $(-I_2)^{k-1}F_p^k$ where I_2 is a 2×2 identity matrix.

Notice that when $k = 1$ and when we specialize $x_\rho = 1$ in the matrix in equation 3.5, we get a matrix similar to F_p. Thus, we can interpret this matrix as recording a composition of steps of type 1 and 3. (Musiker-Williams also have matrices which differ by a sign along the diagonal from Fock-Goncharov, which does not affect the desired matrix statistics.) This is akin to the quasi-elementary steps which Musiker-Williams associate to matrices from Fock-Goncharov which correspond to a path turning left or right inside a triangle. Thus, we can interpret these new matrices from Chekhov-Shapiro as a way to record turning "inside" a pending arc (when pending arcs as visualized as loops around orbifold points, as shown in Section 2.4).

3.12.2 Connecting Arcs and Snake Graphs

So far, given an arc or closed curve γ on an orbifold $\mathcal{O}$ with corresponding generalized cluster algebra $\mathcal{A}$, we have provided two elements of $\mathcal{A}$ from γ: $X_{\gamma,T}$ and $\chi_{\gamma,T}$. We now show these are always the same element of $\mathcal{A}$

Theorem 3.12.12. *Let $\mathcal{O}$ be an unpunctured orbifold with triangulation T, and let γ be any arc or closed curve on $\mathcal{O}$. Let $e(\gamma, T) = d \geq 1$ and $G_{T,\gamma}$ be the snake graph (or band graph) constructed from γ. Then,*

$$\chi_{\gamma,T} = X_{\gamma,T} = \frac{1}{cross(T,\gamma)} \sum_P x(P)h(P) \tag{3.7}$$

where the summation ranges over all perfect matchings P of $G_{T,\gamma}$.

Proof. First, we briefly discuss the case where $d = 0$ for use in later portions of the proof. If $d = 0$, then $\gamma \in T$. In this case, the standard M-path is a step of type 3 along γ and its associated matrix is simply $\begin{bmatrix} 0 & \pm x_\gamma \\ \frac{\pm 1}{x_\gamma} & 0 \end{bmatrix}$. The snake graph $G_{T,\gamma}$ consists of two vertices connected by a single edge with label γ. Such a graph has exactly one perfect matching.

Now, we are prepared to consider $d > 0$. First, we consider the case when γ is an arc. Recall Theorem 3.10.4,

$$\frac{1}{\operatorname{cross}(T,\gamma)}\sum_P x(P)h(P) = \operatorname{ur}\left(\begin{bmatrix}\frac{x_w}{x_{i_d}} & x_z y_{i_d}\\ \frac{-1}{x_z} & 0\end{bmatrix} M_d \begin{bmatrix}0 & x_a\\ \frac{-1}{x_a} & \frac{x_b}{x_{i_1}}\end{bmatrix}\right).$$

Moreover, recall that $M_d = m_{d-1}\cdots m_1$ where for $j \geq 1$,

$$m_{2j} = \begin{bmatrix}\frac{r_{2j}}{i_{2j}} & y_{2j}a_{2j}\\ \frac{b_{2j}}{i_{2j}i_{2j+1}} & y_{2j}\frac{\ell_{2j}}{i_{2j+1}}\end{bmatrix} \qquad m_{2j-1} = \begin{bmatrix}\frac{\ell_{2j-1}}{i_{2j-1}} & y_{2j-1}b_{2j-1}\\ \frac{a_{2j-1}}{i_{2j-1}i_{2j}} & y_{2j-1}\frac{r_{2j-1}}{i_{2j}}\end{bmatrix}$$

These elements of the matrices m_i are labels of the edges of the universal snake graph UG_d. When our graph comes from an arc on a triangulated orbifold, the labels of the edges of the graph correspond to arcs in the orbifold. Here, the first triangle that γ passes through has sides a, b, and τ_{i_1} in clockwise order, and τ_{i_1} is the arc which γ crosses. Similarly, the last triangle that γ crosses through has sides w, z, and τ_{i_d} in clockwise order, and τ_{i_d} is the last arc which γ crosses.

We gave an algorithm for determining κ_γ, the standard M-path of γ, in terms of a sequence of compound steps. If γ crosses d arcs in T, we use $d-1$ compound steps, as well as initial and final sequence of elementary steps. Each compound step has an associated matrix.

In most cases, the product of matrices associated to the elementary steps before the first crossing in the standard M-path is $\begin{bmatrix}0 & x_a\\ \frac{-1}{x_a} & \frac{x_b}{x_{i_1}}\end{bmatrix}$. When γ first crosses a pending arc, ρ, and $s(\gamma)$ is also the unique marked point incident to ρ, this matrix is of the form $\begin{bmatrix}* & x_a\\ * & \frac{x_b}{x_{i_1}}\end{bmatrix}$. However, the terms in the first column will not affect the upper right entry of the product of matrices. This is similar for the product of matrices associated to the elementary steps at and after the last crossing in κ_γ.

Next, we compare the matrices m_i in the description of M_d with the matrices from each compound step of κ_γ. In our rules for κ_γ, if γ crosses two consecutive standard arcs, τ_{i_j} and $\tau_{i_{j+1}}$, which share a vertex to the right of γ, then we use a compound step of type A. Multiplying the elementary matrices these correspond to gives the matrix $\begin{bmatrix}1 & 0\\ \frac{x_{c_j}}{x_{i_j}x_{i_{j+1}}} & y_{i_j}\end{bmatrix}$ where c_j is the third edge in the triangle formed by τ_{i_j} and $\tau_{i_{j+1}}$. If γ crosses a standard arc τ_{i_j} and then a pending arc $\tau_{i_{j+1}}$ and $\tau_{i_{j+1}}$ is based to the right of γ, then we have the same form of matrix.

From our construction of snake graphs, if τ_{i_j} and $\tau_{i_{j+1}}$ share a vertex to the right of γ, and i_j is odd, then we use a north-pointing parallelogram, so $b_j = 0$. In this case, $r_j = i_{j+1}$ and $\ell_j = i_j$. If i_j is even, then we use an east-pointing parallelogram, so $a_j = 0$, $r_j = i_j$ and $\ell_j = i_{j+1}$. These specializations apply even if $\tau_{i_{j+1}}$ is a pending arc. In either case, the matrix from the j-th compound step in κ_γ matches the matrix we use for the j-th parallelogram in $G_{T,\gamma}$.

If τ_{i_j} and $\tau_{i_{j+1}}$ share a vertex to the left of γ, or if $\tau_{i_{j+1}}$ is a pending arc based to the left of γ, then we use a compound step of type B to transition between these. Multiplying the matrices in this compound step gives $\begin{bmatrix}\frac{x_{i_{j+1}}}{x_{i_j}} & y_{i_j}x_{c_j}\\ 0 & \frac{y_{i_j}x_{i_j}}{x_{i_{j+1}}}\end{bmatrix}$. When constructing $G_{T,\gamma}$, if i_j is odd, we use an east-pointing parallelogram and if i_j is even, we use a north-pointing parallelogram. When we use the relevant specializations, we see again that in either case the matrix m_j matches the matrix in the expansion of $M(\kappa)$ from this compound step.

Next, consider when $\tau_{i_j} = \tau_{i_{j+1}}$; this implies τ_{i_j} is a pending arc. Suppose that γ winds $k \geq 0$ times around the orbifold point enclosed by τ_{i_j}. Then, the product matrices from the series of elementary steps from the standard M-path are

$$\left(\begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{\rho} & 1 \end{bmatrix}\begin{bmatrix} 0 & \rho \\ \frac{-1}{\rho} & 0 \end{bmatrix}\right)^k \begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{\rho} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y_{i_j} \end{bmatrix}.$$

By Lemma 3.12.4 and the recurrence relation for Chebyshev polynomials, we have

$$\begin{bmatrix} -U_{k-2}(\lambda_p) & U_{k-1}(\lambda_p)\cdot x_\rho \\ \frac{-U_{k-1}(\lambda_p)}{x_\rho} & U_k(\lambda_p) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{\lambda_p}{x_\rho} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y_{i_j} \end{bmatrix} = \begin{bmatrix} U_k(\lambda_p) & U_{k-1}(\lambda_p)\cdot y_{i_j} x_\rho \\ \frac{U_{k+1}(\lambda_p)}{x_\rho} & U_k(\lambda_p)\cdot y_{i_j} \end{bmatrix}$$

If $k = 0$, then $U_{-1}(\lambda_p) = 0$, and this resembles the case when τ_{i_j} and $\tau_{i_{j+1}}$ are standard arcs which share a vertex to the right of γ. If $k = p-2$, then $U_{(p-2)+1}(\lambda_p) = 0$, and this resembles the case where two consecutive standard arcs share a vertex to the left of γ. If $0 < k < p-2$, then all four entries of this matrix are nonzero.

In our construction of $G_{T,\gamma}$, when an arc winds $k > 0$ times around an orbifold point, we associate a hexagonal tile. In the language of the universal snake graph, we construct this with a parallelogram where both diagonals are included. Moreover, we have $i_j = i_{j+1} = x_\rho$, and $r_j = \ell_j = U_k(\lambda_p)x_\rho$. Then, either $a_j = U_{k-1}(\lambda_p)x_\rho$ and $b_j = U_{k+1}(\lambda_p)x_\rho$ or vice versa.

When τ_{i_j} is a pending arc and $\tau_{i_{j+1}}$ is not a pending arc, we have to consider both whether τ_{i_j} is to the left or right of γ and whether $\tau_{i_{j-2}}$ is distinct from $\tau_{i_{j+1}}$. We saw these four cases in the description of κ_γ. If $\tau_{i_{j+1}}$ and $\tau_{i_{j-2}}$ are distinct arcs, then we use the same compound step between $\tau_{i_{j-2}}$ and $\tau_{i_{j-1}}$ as we do between τ_{i_j} and $\tau_{i_{j+1}}$. For example, if $\tau_{i_{j+1}}$ and $\tau_{i_{j-2}}$ are distinct arcs and τ_{i_j}, the pending arc, is based to the right of γ, then between τ_{i_j} and $\tau_{i_{j+1}}$ we use a compound step of type A, just as we use between $\tau_{i_{j-2}}$ and $\tau_{i_{j-1}}$. When constructing $G_{T,\gamma}$ in this case, at indices $j-2$ and j we either have both parallelograms facing north or both facing east. Thus, in the standard labeling of the universal snake graph, either both $a_{j-2} = a_j = 0$ or $b_{j-2} = b_j = 0$. Conversely, if $\tau_{i_{j-2}} = \tau_{i_{j+1}}$, in κ_γ we use opposite compound steps between $\tau_{i_{j-2}}$ and $\tau_{i_{j-1}}$ and between τ_{i_j} and $\tau_{i_{j+1}}$. In the construction of $G_{T,\gamma}$, we use opposite parallelograms at indices $j-2$ and j. By specializing the entries of the matrices m_i from the parallelograms at each case, we will see that the matrices from the graph and M-path agree again.

Putting all these cases together demonstrates that the matrices from the compound step decomposition of κ_γ largely match the matrices used in Theorem 3.10.3 to encode weighted perfect matchings of $G_{\gamma,T}$. The initial and final matrices will not necessarily completely match. However, we can conclude the following

$$|\mathrm{ur}(M(\kappa_\gamma))| = \left|\mathrm{ur}\left(\begin{bmatrix} \frac{x_w}{x_{i_d}} & x_z y_{i_d} \\ \frac{-1}{x_z} & 0 \end{bmatrix} M_d \begin{bmatrix} 0 & x_a \\ \frac{-1}{x_a} & \frac{x_b}{x_{i_1}} \end{bmatrix}\right)\right| = \frac{1}{\mathrm{cross}(T,\gamma)}\sum_P x(P)h(P).$$

Now, let γ be a closed curve, and let $q = v^{\pm}_{\tau_{i_1},m}$ be a point chosen on γ for κ_γ to start and end. Then, τ_{i_1} and τ_{i_d} form two sides of the triangle which q lives in; call the third side of this triangle a. We see that we can start with a compound step of type A or B since we start adjacent to the first arc which γ crosses. However, when we cross τ_{i_d}, we will need to include a compound step of type B to return to q, since τ_{i_1} follows τ_{i_d} immediately clockwise. This compound step has matrix $\begin{bmatrix} \frac{x_{i_1}}{x_{i_d}} & 0 \\ \frac{x_a}{x_{idn}x_{i_1}} & \frac{y_{i_d}x_{i_1}}{x_{i_d}} \end{bmatrix}$. Thus, $M(\kappa_\gamma) = \operatorname{tr}\left(\begin{bmatrix} \frac{x_{i_1}}{x_{i_d}} & 0 \\ \frac{x_a}{x_{i_d}x_{i_1}} & \frac{y_{i_d}x_{i_1}}{x_{i_d}} \end{bmatrix} M_d \right)$. By Theorem 3.10.4, this is equivalent to the weighted sum of perfect matchings of the band graph $G_{T,\gamma}$. □

Corollary 3.12.13. *Let γ be an arc or closed curve on $\mathcal{O}$ with no contractible kinks which winds at most $p-1$ times around any particular orbifold point of order p. Then the coefficients of the Laurent expansion for x_γ obtained from the cluster expansion formula in Theorem 3.12.12 are non-negative.*

3.13 Skein relations with y-variables

The following definition will be useful for a condensed discussion of the skein relations in [59].

Definition 3.13.1. *A* multicurve*, C, on $\mathcal{O}$ is a finite multi-set of arcs and closed curves on $\mathcal{O}$. If these arcs and curves are $\gamma_1, \ldots, \gamma_n$, then we define the monomial $\chi_{C,T}$ to be the product $\chi_{\gamma_1,T} \cdots \chi_{\gamma_n,T}$.*

Musiker and Williams [59] prove in Propositions 6.4, 6.5, and 6.6 that, in the surface case, the quantities $\chi_{\gamma,T}$ respect the skein relation. Let C be the multicurve which consists of either γ_1 and γ_2, two generalized arcs or closed curves which intersect, or γ, an arc or closed curve with points of self-intersection. At one point of intersection between γ_1 and γ_2, or one point of self-intersection on γ, we can use *smoothing*. This will create two new multicurves, call them C_1 and C_2; amongst the arcs in C_1 and C_2, there is at least one less intersection than amongst the arcs in C. See [59] for more details about the process of smoothing and the proofs of these propositions.

Theorem 3.13.2 (Propositions 6.4, 6.5, and 6.6 of [59])**.** *Let $\mathcal{A}$ be the cluster algebra associated to surface (S, M) with initial triangulation T. Let C be a multicurve consisting of two intersecting arcs/curves or one arc with self-intersection, and C_1, C_2 be the multicurves obtained by smoothing one point of intersection in C. Then in $\mathcal{A}$ we have*

$$\chi_{C,T} = \pm Y_1 \chi_{C_1,T} \pm Y_2 \chi_{C_2,T},$$

where Y_1 and Y_2 are monomials in the y-variables which can be computed by analyzing the intersections of the arcs/curves in C, C_1, and C_2 with the elementary laminations from the initial triangulation T.

These skein relations can also be applied to pending arcs, or, more generally, to arcs which wind around orbifold points. Consequently, we can extend Theorem 3.13.2 to unpunctured triangulated orbifolds.

Proposition 3.13.3. *Theorem 3.13.2 holds on an unpunctured triangulated orbifold $\mathcal{O}$. In particular, it holds for multicurves which include pending arcs or arcs that wind around orbifold points.*

Proof. The arguments used by Musiker and Williams in [59] to prove Propositions 6.4, 6.5, and 6.6 can also be applied in the orbifold setting. Smoothing a multicurve C on an orbifold works in the same manner as smoothing a multicurve C on a surface, and we have shown that the expansion of the cluster algebra element associated to an arc or closed curve can be encoded by a product of a sequence of matrices. Thus, we can use the same matrix equalities (Lemma 6.11 of [59]) which are fundamental to their proofs. Because we consider only unpunctured orbifolds, we don't require the notion of a *loosened M-path*. ☐

These skein relations show that our choice of cluster algebra element $\chi_{\gamma,T}$ associated to a generalized arc or closed curve γ is the right choice. In particular, we can decompose γ into a sum of products of ordinary arcs. By Theorem 3.3.1, we already know the correct cluster algebra elements to associate to the ordinary arcs. Proposition 3.13.3 shows that the associated cluster algebra elements satisfy the same decomposition. Since $\chi_{\gamma,T} = X_{\gamma,T}$, we can conclude our expansion formula provides the right choice of cluster algebra element for arbitrary arcs and closed curves.

Standard skein relations resemble the binomial exchange relation in an ordinary cluster algebra. When two pending arcs intersect, we can use the standard skein relation twice to recover a three-term relation which models the generalized exchanges in the generalized cluster algebras we consider.

Proposition 3.13.4. *Let γ_1, γ_2 be two distinct pending arcs to the same orbifold point in an unpunctured orbifold $\mathcal{O}$ with triangulation T. Choose an orientation for γ_1, and let $q_1, \ldots, q_{2\ell}$ be the intersections of γ_1 and γ_2, with order determined by the orientation of γ_1. Orient γ_2 so that it visits q_ℓ before $q_{\ell+1}$. Let $\beta_1 = (\gamma_1, \gamma_2^-|s(\gamma_1), q_\ell, s(\gamma_2))$ and let $\beta_2 = (\gamma_1^-, \gamma_2|t(\gamma_1), q_{\ell+1}, t(\gamma_2))$. Then,*

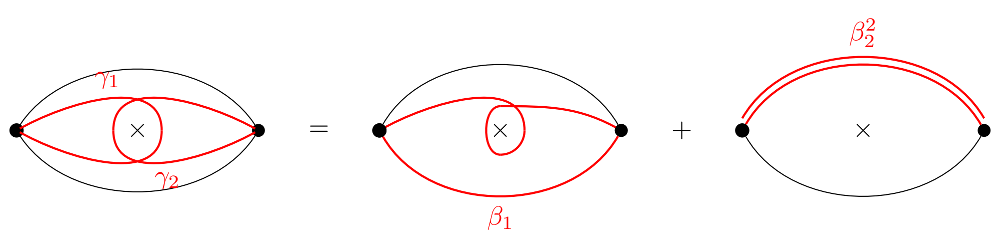
$$\chi_{\gamma_1,T}\chi_{\gamma_2,T} = Y_0\chi^2_{\beta_1,T} + Y_1\lambda_p\chi_{\beta_1,T}\chi_{\beta_2,T} + Y_2\chi^2_{\beta_2,T}$$

Proof. Note that β_1 and β_2 form a bigon around the orbifold point incident to γ_1 and γ_2 such that these are the two pending arcs inside the bigon.

First, we use Proposition 6.4 of [59] to smooth γ_1 and γ_2 at q_ℓ. In the vocabulary of Theorem 3.13.2, if $C = \{\gamma_1, \gamma_2\}$, then $C_1 = \{\beta_1, \alpha\}$ and $C_2 = \{\beta_2, \beta_2\}$ where $\alpha = (\gamma_1^-, \gamma_2|t(\gamma_1), q_\ell, t(\gamma_2))$. Note that α has one self-intersection. We can then use Proposition 6.6 of [59] to smooth α. If $\bar{C} = \{\alpha\}$, then after smoothing we get the two multicurves $\bar{C}_1 = \{\xi, \beta_2\}$ and $\bar{C}_2 = \{\beta_1\}$ where ξ is an essential loop around the orbifold point incident to γ_1 and γ_2. Then, we can decompose C_1 to $C_{1,1} = \{\xi, \beta_1, \beta_2\}$ and $C_{1,2} = \{\beta_1, \beta_1\}$. There are no crossings amongst the arcs in $C_{1,1}, C_{1,2}$, and C_2, and we have that $x_C = x_{C_{1,1}} + x_{C_{1,2}} + x_{C_2}$. By Proposition 3.12.8, $x_\xi = \lambda_p$ where p is the order of this orbifold point. This yields the desired equality

$$\chi_{\gamma_1,T}\chi_{\gamma_2,T} = \widetilde{Y}\chi_{\beta_1,T}\chi_{\alpha,T} + Y_2\chi^2_{\beta_2,T} = Y_0\chi^2_{\beta_1,T} + Y_1\chi_{\beta_1,T}\chi_{\beta_2,T} + Y_2\chi^2_{\beta_2,T}$$

A diagram of this smoothing on the orbifold is shown below.



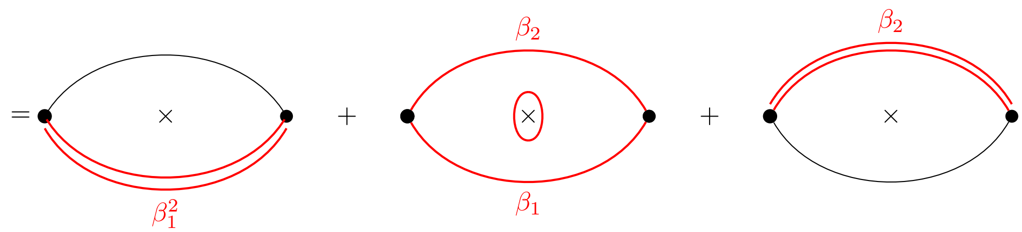


Figure 3.13: Smoothing the intersections of two distinct pending arcs incident to the same orbifold point using the standard skein relation.

□

3.14 Connection to punctures

Within this chapter, we restricted our discussion to unpunctured orbifolds. Recall that a puncture is a marked point which appears in the interior of a surface or orbifold. The original snake graph construction in [58] does handle surfaces with punctures. In this section, we give some interesting examples which illustrate how some results from [58] and [59] concerning punctures can be recovered by treating the puncture as an orbifold point with infinite order.

As motivation, recall that an arc which winds k times around an orbifold point of order p is isotopic to an arc winding $k \pm np$ times for any integer n - even if this means that the winding arc switches directions. This type of isotopy does not exist for arcs winding around punctures; thus, in some sense we could consider the puncture to have infinite order. Moreover, note that $\lambda_\infty := \lim_{p\to\infty} \lambda_p = \lim_{p\to\infty} 2\cos(\pi/p) = 2$. Thus, a loop which is contractible to an orbifold point of infinite order has the same weight as a loop contractible to a puncture.

We will specifically consider the case of a puncture inside a self-folded triangle, as this most closely resembles a pending arc, and compare the x-variables and y-variables in these situations. We note that by specializing $\lambda_\infty = 1 + y_r$, we can nearly recover the F-polynomials from these cluster algebra elements.

Previously, we discussed normalized Chebyshev polynomials of the second kind. Now, we introduce another formal variable to these Chebyshev polynomials.

Definition 3.14.1. *Let $\{U_k^Y(x)\}_k$ be a family of polynomials indexed by $k = -1, 0, 1, \dots$ such that $U_{-1}^Y(x) = 0, U_0^Y(x) = 1$, and for $k \geq 1$,*

$$U_k^Y(x) = x \cdot U_{k-1}^Y(x) - Y \cdot U_{k-1}^Y(x)$$

For example, $U_1^Y(x) = x, U_2^Y(x) = x^2 - Y$, and $U_3^Y(x) = x^3 - 2Yx$.

We then record some results about our normalized Chebyshev polynomials, with and without coefficients, for later use.

Lemma 3.14.2. *Let U_k and U_k^Y be as in definitions 3.1.1 and 3.14.1. Then for $k \geq 1$,*

1. $U_k(2) = k + 1$
2. $U_k^Y(1 + Y) = 1 + Y + \cdots + Y^k$

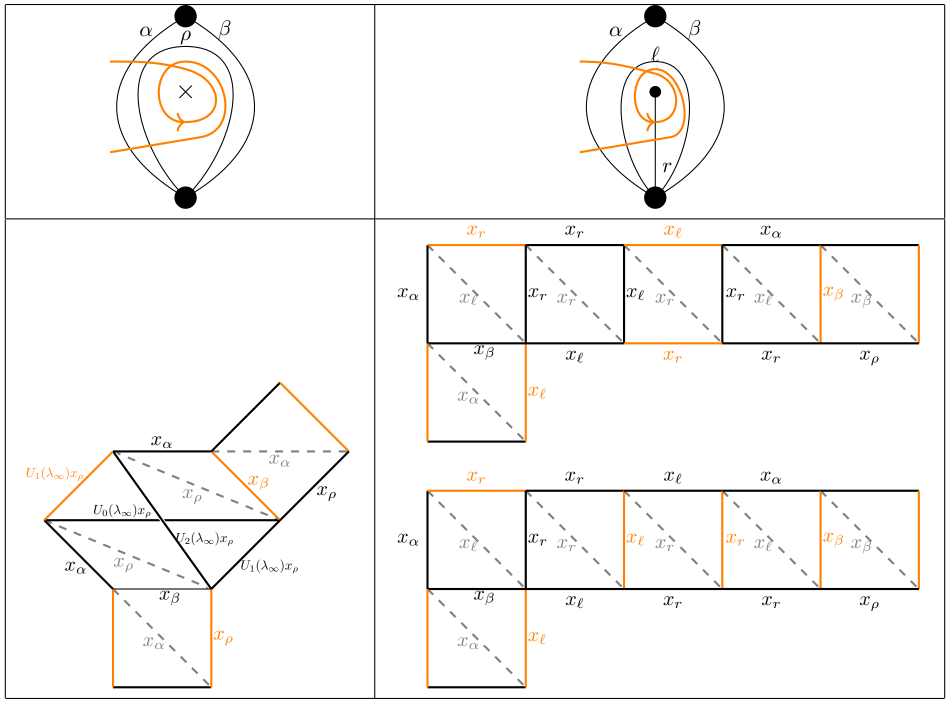


Figure 3.14: A comparison of the local snake graphs for arcs crossing pending arcs and self-folded triangles. The highlighted perfect matching of the generalized snake graph corresponds to the two highlighted perfect matchings in ordinary snake graph for the punctured surface case.

Proof. The first statement follows from the second by setting $Y = 1$, so we need only prove the second statement. We do so by induction. The statement clearly holds for $U_1^Y(x)$, and by definition $U_2^Y(1+Y) = (1+Y)^2 - Y = 1 + Y + Y^2$. Then, for $k \geq 3$,

$$\begin{aligned} U_k^Y(1+Y) &= (1+Y) \cdot U_{k-1}^Y(1+Y) - Y \cdot U_{k-2}^Y(1+Y) \\ &= (1+Y)(1+Y+\cdots+Y^{k-1}) - Y(1+Y+\cdots+Y^{k-2}) \\ &= 1+Y+\cdots+Y^k \end{aligned}$$

as desired. □

In Figure 3.14, we compare the generalized snake graph from an arc that crosses a pending arc twice and has a single self-intersection to the ordinary snake graph for an analogous arc where the orbifold point has been replaced by a puncture and the pending arc by a self-folded triangle. If we set $\lambda_\infty = 2$, so that $U_0(\lambda_\infty) = 1, U_1(\lambda_\infty) = 2$ and $U_2(\lambda_\infty) = 3$, several edge labels on the generalized snake graph become positive integer multiples of cluster variables. A perfect matching which uses one of these edges corresponds to multiple perfect matchings in the snake graph from the surface case. The highlighted perfect matchings in Figure 3.14 show an example of this.

In the generalized snake graph, the perfect matching P uses an edge labeled $U_1(\lambda_\infty)x_\rho = 2x_\rho$. Considering only the arcs drawn, $x(P) = 2x_\beta x_\rho^2$. Recall that in the denominator of the cluster expansion formula, we have the crossing monomial x_ρ^2. However, a factor of x_ρ also appears in each of the other terms in the numerator. Canceling this factor gives the reduced weight $2x_\beta x_\rho$. In the ordinary snake graph, we show two matchings, which each have weight $x_\beta x_r^2 x_\ell^2$. Similar to

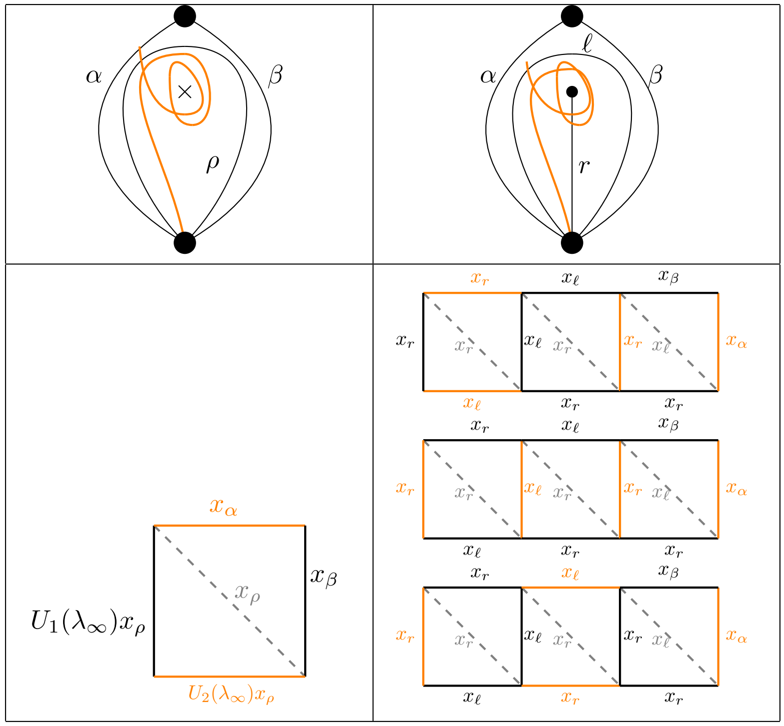


Figure 3.15: Another comparison of generalized and ordinary snake graphs, for an example of arcs crossing the pending arc or loop a single time.

the orbifold case, here the crossing monomial is $x_r^2 x_\ell^2$. By observation, we see that every perfect matching will use at least two edges labeled x_r and one edge labeled x_ℓ; thus, we can cancel a factor of $x_r^2 x_\ell$ to obtain the reduced weight $x_\beta x_\ell$ for each perfect matching. Setting $\lambda_p = 1 + y_r$, so that $U_1(\lambda_p) = 1 + y_r$, also makes sense in this example, since the two highlighted matchings differ only by a twist at a tile with diagonal label r.

Chapter 4

Generalized cluster scattering diagrams

This chapter describes joint work with Man-Wai Cheung and Gregg Musiker, which will appear as an extended abstract in the 2021 proceedings of the Formal Power Series and Algebraic Combinatorics (FSPAC) conference [15]. A full length version of that extended abstract is currently in preparation [14]. This work occurred contemporaneously to, and independently of, the related work of Lang Mou [54].

4.1 Basic definitions

We begin by updating some definitions for the generalized setting. First, we update the definition of fixed data to include data from the exchange degree matrix R.

Definition 4.1.1. *The following data is referred to as* generalized fixed data*, denoted* Γ*:*

- *A lattice* N *called the* cocharacter lattice *with skew-symmetric bilinear form* $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$.
- *A saturated sublattice* $N_{uf} \subseteq N$ *called the* unfrozen sublattice.
- *An index set* I *with* $|I| = rank(N)$ *and subset* $I_{uf} \subseteq I$ *such that* $|I_{unf}| = rank(N_{uf})$.
- *A set of positive integers* $\{d_i\}_{i\in I}$ *such that* $gcd(d_i) = 1$.
- *A sublattice* $N^\circ \subseteq N$ *of finite index such that* $\{N_{uf}, N^\circ\} \subseteq \mathbb{Z}$ *and* $\{N, N_{uf} \cap N^\circ\} \subseteq \mathbb{Z}$.
- *A lattice* $M = Hom(N, \mathbb{Z})$ *called the* character lattice *and sublattice* $M^\circ = Hom(N^\circ, \mathbb{Z})$.
- *A set of positive integers* $\{r_i\}_{i\in I}$.
- *A collection* $\{a_{i,j}\}_{i\in I_{uf}, j\in[r_i-1]}$ *of formal variables.*

The adjective 'fixed' refers to the fact that this data is fixed under mutation.

Note that the exchange polynomial coefficients $\{a_{i,j}\}_{i\in I_{\mathrm{uf}}, j\in[r_i]}$ are formal variables, rather than elements of $\Bbbk$. As such, we must work over the ground ring $R = \Bbbk[a_{i,j}]$ rather than over $\Bbbk$ as in the ordinary case. In doing so, we follow the work of [5] on cluster varieties with coefficients.

We also establish the notion of a *generalized torus seed*, also denoted $\mathbf{s}$.

Definition 4.1.2. *Given a set of generalized fixed data, we can define associated* generalized torus seed data $\mathbf{s} = \{(e_i, (a_{i,j}))\}_{i \in I, j \in [r_i - 1]}$ *such that the collection* $\{e_i\}_{i \in I}$ *satisfies the conditions for ordinary torus seed data and each* $(a_{i,j})$ *is a tuple of formal variables taken from the collection specified in the fixed data.*

Analogous to the ordinary case, this defines a dual basis $\{e_i^*\}_{i \in I}$ for M and $\{f_i = d_i^{-1} e_i^*\}_{i \in I}$ for M°. Note that when $r_i = 1$ for all i, our definitions reduce to the definitions for an ordinary torus seed.

We will confine our attention to the subclass of *reciprocal* generalized cluster algebras:

Definition 4.1.3. *A generalized torus seed* $\mathbf{s}$ *is called* reciprocal *if its scalar tuples* $(a_{i,j})$ *satisfy the reciprocity condition* $a_{i,j} = a_{i,r_i - j}$. *We refer to the associated algebra as a* reciprocal *generalized cluster algebra.*

Note that the exchange polynomial coefficients $a_{i,j}$ appear in both the generalized fixed data and generalized torus seed data. These coefficients must appear in the fixed data because we now work over the ground ring $R = \Bbbk[a_{i,j}]$. For reciprocal generalized cluster algebras, the exchange polynomial coefficients are fixed under mutation and so arguably do not need to also appear in the generalized torus seed data. These coefficients are not, however, fixed under mutation for non-reciprocal generalized cluster algebras. As such, we choose to include the exchange polynomial coefficients in the generalized torus seed data in order to leave open the possibility of extending our construction to arbitrary generalized cluster algebras without redefinition.

Example 4.1.4. *The generalized cluster algebra*

$$\mathcal{A}\left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, ((1, a, a, 1), (1, 1))\right)$$

has generalized fixed data Γ *with* $d = (1,1)$, $r = (3,1)$, $I = I_{uf} = \{1,2\}$, $N = N^\circ = \langle e_1, e_2 \rangle$, $M = M^\circ = \langle e_1^*, e_2^* \rangle$, *and skew-symmetric bilinear form* $\{\cdot, \cdot\} : N^\circ \times N^\circ \to \mathbb{Z}$ *specified by the exchange matrix. One possible choice of generalized torus seed data is*

$$\mathbf{s} = \{(e_1 = (1,0), (1, a, a, 1)), (e_2 = (0,1), (1,1))\}.$$

Definition 4.1.5. *Given generalized torus seed data* $\mathbf{s}$ *and some* $k \in I_{uf}$, *a* mutation in direction k *of the generalized torus seed data is defined by the following transformations of basis vectors and exchange polynomial coefficients:*

$$e_i' := \begin{cases} e_i + r_k [\epsilon_{ik}]_+ e_k & i \neq k \\ -e_k & i = k \end{cases}$$

$$f_i' := \begin{cases} -f_k + r_k \sum_{j \in I_{uf}} [-\epsilon_{kj}]_+ f_j & i = k \\ f_i & i \neq k \end{cases}$$

$$a_{k,j}' := a_{k, r_k - j}$$

The basis mutation induces the following mutation of the matrix $[\epsilon_{ij}]$:

$$\epsilon'_{ij} := \{e'_i, e'_j\} d_j = \begin{cases} -\epsilon_{ij} & k = i \text{ or } k = j \\ \epsilon_{ij} & k \neq i,j \text{ and } \epsilon_{ik}\epsilon_{kj} \leq 0 \\ \epsilon_{ij} + r_k|\epsilon_{ik}|\epsilon_{kj} & k \neq i,j \text{ and } \epsilon_{ik}\epsilon_{kj} \geq 0 \end{cases}$$

Given generalized torus seed data $\mathbf{s}$, we can then define associated algebraic tori $\mathcal{A}_{\mathbf{s}}$ and $\mathcal{X}_{\mathbf{s}}$. Recall that our ground ring is $R = \Bbbk[a_{i,j}]$. For a lattice L, let

$$T_L(R) := \operatorname{Spec}(\Bbbk[L^*] \otimes_{\Bbbk} R) = T_L \times_{\Bbbk} \operatorname{Spec}(R).$$

This notation then allows us to state the following definition:

Definition 4.1.6. *A choice of generalized torus seed data* $\mathbf{s}$ *defines the tori:*

$$\mathcal{X}_{\mathbf{s}} = T_M(R) = T_M \times_{\Bbbk} \mathit{Spec}(R) = \mathit{Spec}(\Bbbk[N]) \times_{\Bbbk} \mathit{Spec}(R)$$
$$\mathcal{A}_{\mathbf{s}} = T_{N^\circ}(R) = T_{N^\circ} \times_{\Bbbk} \mathit{Spec}(R) = \mathit{Spec}(\Bbbk[M^\circ]) \times_{\Bbbk} \mathit{Spec}(R)$$

There are several common notational conventions for the coordinates of these algebraic tori. We will use $y_1, \ldots, y_n$ for the coordinates of $\mathcal{X}_{\mathbf{s}}$ and $x_1, \ldots, x_n$ for the coordinates of $\mathcal{A}_{\mathbf{s}}$ in order to be consistent with the prevailing notation for ordinary and generalized cluster algebras. In the literature, however, it is common to see $X_1, \ldots, X_n$ used for the coordinates of $\mathcal{X}_{\mathbf{s}}$ and $A_1, \ldots, A_n$ for the coordinates of $\mathcal{A}_{\mathbf{s}}$.

Definition 4.1.7. *We define birational maps $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_k(\mathbf{s})}$ and $\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_k(\mathbf{s})}$ via the pull-back of functions*

$$\mu_k^* z^m = z^m \left(1 + a_{k,1} z^{v_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{-\langle d_k e_k, m\rangle} \tag{4.1}$$

$$\mu_k^* z^n = z^n \left(1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-[n, e_k]} \tag{4.2}$$

for $n \in N$ and $m \in M^\circ$.

As in the ordinary case, the exchange relations for generalized cluster algebras, stated in Definition 2.5.2, can be explicitly obtained from Definitions 4.1.5 and 4.1.7 by applying μ_k^* to the cluster variables $x_i = z^{f_i}$ and $y_i = z^{e_i}$.

Remark 4.1.8. *The form of the exchange relations for generalized cluster algebras can be recovered from Equation (4.1) and Equation (4.2) by setting $m = f_i$ and $n = e_i$. Consider the mutation of $x_i = z^{f_i}$ and $y_i = z^{e_i}$ in direction k. If $i = k$, then*

$$\mu_k^*(y'_k) = \mu_k^*\left(z^{e'_k}\right) = \mu_k^*\left(z^{-e_k}\right) = z^{-e_k}\left(1 + \cdots + z^{r_k e_k}\right)^{-[-e_k, e_k]} = z^{-e_k} = y_k^{-1}$$

and

$$\begin{aligned}
\mu_k^*(x_k') &= \mu_k^*\left(z^{f_k'}\right)\\
&= \mu_k^*\left(z^{-f_k + r_k \sum_{j\in I_{uf}}[-\epsilon_{kj}]_+ f_j}\right)\\
&= z^{-f_k + r_k \sum_{j\in I_{uf}}[-\epsilon_{kj}]_+ f_j}\left(1 + a_{k,1}z^{v_k} + \cdots + z^{r_k v_k}\right)^{-\langle d_k e_k, -f_k + r_k \sum_{j\in I_{uf}}[-\epsilon_{kj}]_+ f_j\rangle}\\
&= z^{-f_k}\left(\prod_{j\in I_{uf}} z^{[-\epsilon_{kj}]_+ f_j}\right)^{r_k}\left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{\langle d_k e_k, f_k\rangle}\\
&\qquad \cdot\left(\prod_{j\in I_{uf}}\left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{-\langle d_k e_k, [\epsilon_{kj}]_+ f_j\rangle}\right)\\
&= z^{-f_k}\left(\prod_{j\in I_{uf}} z^{[-\epsilon_{kj}]_+ f_j}\right)^{r_k}\left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{1}\\
&\qquad \cdot\left(\prod_{j\in I_{uf}}\left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{0}\right)\\
&= z^{-f_k}\left(\prod_{j\in I_{uf}} z^{[-\epsilon_{kj}]_+ f_j}\right)^{r_k}\left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)\\
&= z^{-f_k}\left(\prod_{j\in I_{uf}} z^{[-\epsilon_{kj}]_+ f_j}\right)^{r_k}\left(1 + a_{k,1}\left(\prod_{j\in I_{uf}} z^{\epsilon_{kj} f_j}\right) + \cdots + \left(\prod_{j\in I_{uf}} z^{\epsilon_{kj} f_j}\right)^{r_k}\right)\\
&= x_k^{-1}\left(\prod_{j\in I_{uf}} x_j^{[-b_{kj}]_+}\right)^{r_k}\left(1 + a_{k,1}\left(\prod_{j\in I_{uf}} x_j^{b_{kj}}\right) + \cdots + \left(\prod_{j\in I_{uf}} x_j^{b_{kj}}\right)^{r_k}\right)
\end{aligned}$$

If $i \neq k$, then

$$\begin{aligned}
\mu_k^*(y_i') &= \mu_k^*\left(z^{e_i'}\right)\\
&= \mu_k^*\left(z^{e_i + r_k[\epsilon_{ik}]_+ e_k}\right)\\
&= z^{e_i + r_k[\epsilon_{ik}]_+ e_k}\left(1 + a_{k,1}z^{e_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-[e_i + r_k[\epsilon_{ik}]_+ e_k, e_k]}\\
&= z^{e_i} z^{r_k[\epsilon_{ik}]_+ e_k}\left(1 + a_{k,1}z^{e_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-[e_i, e_k]}\\
&= z^{e_i}\left(z^{[\epsilon_{ik}]_+ e_k}\right)^{r_k}\left(1 + a_{k,1}z^{e_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-\epsilon_{ik}}\\
&= y_i\left(y_k^{[b_{ik}]_+}\right)^{r_k}\left(1 + a_{k,1}y_k + \cdots + a_{k,r_k-1}y_k^{r_k-1} + y_k^{r_k}\right)^{-b_{ik}}
\end{aligned}$$

and

$$\mu_k^*(x_i') = \mu_k^*\left(z^{f_i'}\right) = \mu\left(z^{f_i}\right) = z^{f_i}\left(1 + a_{k,1}z^{v_k} + \cdots + z^{r_k v_k}\right)^{-\langle d_k e_k, f_k\rangle} = z^{f_i} = x_i$$

4.1.1 Generalized cluster varieties

In order to define generalized cluster varieties, we need the following gluing construction from [5], which is a more general version of Proposition 2.4 of [39].

Proposition 4.1.9 (Lemma 3.10 of [5])**.** *Let $\{S_i\}$ be a collection of integral, separate schemes of finite type over a locally Noetherian ring R, with birational maps $f_{ij} : S_i \to S_j$ for all i, j, with $f_{ii} = Id$ and $f_{jk} \circ f_{ij} = f_{ik}$ as rational maps. Let $U_{ij} \subset S_i$ be the largest open subscheme such that $f_{ij} : U_{ij} \to f_{ij}(U_{ij})$ is an isomorphism. Then there exists a scheme*

$$S := \bigcup_i S_i$$

which is obtained by gluing the S_i along the open sets U_{ij} via the maps f_{ij}.

Although the statement of Proposition 2.4 in [39] specifically requires that R be a field, its proof actually holds in the more general case where R is a locally Noetherian ring, as explained in Remark 3.11 of [5]. In our setting, R is a Laurent polynomial ring and we therefore require the more general statement.

Fix a generalized cluster pattern, as in Definition 2.5.5.

Definition 4.1.10. *The* $\mathcal{A}$-generalized cluster variety *is the scheme*

$$\mathcal{A} := \bigcup_{\mathbf{s} \in \mathfrak{T}} \mathcal{A}_{\mathbf{s}}$$

obtained by using Proposition 4.1.9 to glue the collection of algebraic tori $\{\mathcal{A}_{\mathbf{s}}\}_{\mathbf{s}\in\mathfrak{T}}$ according to the birational maps $\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_k(\mathbf{s})}$ specified in Definition 4.1.7. Analogously, the $\mathcal{X}$-generalized cluster variety *is defined to be the scheme*

$$\mathcal{X} := \bigcup_{\mathbf{s} \in \mathfrak{T}} \mathcal{X}_{\mathbf{s}}$$

obtained by gluing the collection $\{\mathcal{X}_{\mathbf{s}}\}_{\mathbf{s}\in\mathfrak{T}}$ according to the birational maps $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_k(\mathbf{s})}$.

For readability, we will often refer to these schemes simply as the $\mathcal{A}$-variety and $\mathcal{X}$-variety.

Intuitively, we expect that the construction of the $\mathcal{A}$-variety and $\mathcal{X}$-variety should not depend on the choice of seed - that is, mutation equivalent seeds $\mathbf{s}$ and $\mathbf{s}'$ should yield isomorphic schemes. The two smaller commutative diagrams in the following proposition show that structures of the tori given in Definition 4.1.6 are compatible with generalized torus seed data mutation. This induces a similar compatibility for $\mathcal{A}$ and $\mathcal{X}$.

Proposition 4.1.11. *Let $K = ker(p_2^*)$ and $K^\circ = K \cap N^\circ$. For a given generalized torus seed $\mathbf{s}$ and mutation direction $k \in [n]$, the following diagrams are commutative:*

$$\begin{array}{ccccccc}
T_{K^\circ} & \longrightarrow & \mathcal{A}_{\mathbf{s}} & \xrightarrow{p} & \mathcal{X}_{\mathbf{s}} & \longrightarrow & T_{K^*} \\
\downarrow{=} & & \downarrow{\mu_k} & & \downarrow{\mu_k} & & \downarrow{=} \\
T_{K^\circ} & \longrightarrow & \mathcal{A}_{\mu_k}(\mathbf{s}) & \xrightarrow{p} & \mathcal{X}_{\mu_k(\mathbf{s})} & \longrightarrow & T_{K^*}
\end{array}$$

$$\begin{tikzcd} T_{(N/N_{uf})^*} \arrow[r] \arrow[d, "="] & \mathcal{X}_{\mathbf{s}} \arrow[d, "\mu_k"] \\ T_{(N/N_{uf})^*} \arrow[r] & \mathcal{X}_{\mu_k(\mathbf{s})} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_{\mathbf{s}} \arrow[r] \arrow[d, "\mu_k"] & T_{N^\circ/(N_{uf}\cap N^\circ)} \arrow[d, "="] \\ \mathcal{A}_{\mu_k(\mathbf{s})} \arrow[r] & T_{N^\circ/(N_{uf}\cap N^\circ)} \end{tikzcd}$$

Proof. There are several unlabeled maps in the above commutative diagrams. Those maps come from the following structures, as described in [39]:

1. The inclusion $K \subseteq N$ induces a map $\mathcal{X}_{\mathbf{s}} \to T_{K^*}$.
2. The inclusion $K^\circ \to N^\circ$ induces a map $T_{K^\circ} \to \mathcal{A}_{\mathbf{s}}$.
3. Let $N_{\mathrm{uf}}^\perp := \{m \in M^\circ : \langle m, n\rangle = 0 \text{ for all } n \in N_{\mathrm{uf}}\}$. Then the inclusion $N_{\mathrm{uf}}^\perp \subseteq M^\circ$ induces a map $\mathcal{A}_{\mathbf{s}} \to T_{N^\circ/(N_{\mathrm{uf}}\cap N^\circ)}$.
4. The choice of the map $p^* : N \to M^\circ$ defines a map $p : \mathcal{A}_{\mathbf{s}} \to \mathcal{X}_{\mathbf{s}}$. The map $p^* : N \to M^\circ$ induces maps $p^* : K \to N_{\mathrm{uf}}^\perp$ and $p^* : N/N_{\mathrm{uf}} \to (K^\circ)^*$ which define maps $p : T_{N/(N_{\mathrm{uf}}\cap N^\circ)} \to T_{K^*}$ and $p : T_{K^\circ} \to T_{(N/N_{\mathrm{uf}})^*}$.

Using the definitions of these maps, p, and μ_k, it is straightforward to check the commutativity of each square. $\square$

4.2 Generalized cluster scattering diagrams

As in the ordinary case, we will be interested in a particular scattering diagram which is defined by the generalized fixed and torus seed data. To define this diagram, which we refer to as the *generalized cluster scattering diagram*, we begin by modifying the definition of an initial scattering diagram.

Definition 4.2.1. *Let $v_i = p_1^*(e_i)$ for $i \in I_{uf}$. Then we define*

$$\mathfrak{D}_{in,s} := \{(e_i^\perp, 1 + a_{i,1}z^{v_i} + a_{i,2}z^{2v_i} + \cdots + a_{i,r_i-1}z^{(r_i-1)v_i} + z^{r_iv_i})\}_{i\in I_{uf}}$$

Generalized cluster scattering diagrams use the same notions of equivalence, uniqueness, and consistency as ordinary cluster scattering diagrams. For statements of these definitions, see Section 2.3.

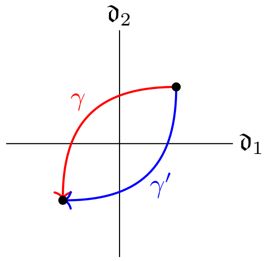
Example 4.2.2. *The generalized cluster algebra from Example 4.1.4 has birational maps μ_1, μ_2 defined by the pullbacks*

$$\begin{aligned}
\mu_1^* z^n &= z^n(1 + az^{(1,0)} + az^{(2,0)} + z^{(3,0)})^{-[n,(1,0)]}, \\
\mu_1^* z^m &= z^m(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)})^{-\langle(1,0)),m\rangle}, \\
\mu_2^* z^n &= z^n(1 + z^{(0,1)})^{-[n,(0,1)]}, \\
\mu_2^* z^m &= z^m(1 + z^{(-1,0)})^{-\langle(0,1),m\rangle}.
\end{aligned}$$

It has initial scattering diagram

$$\mathfrak{D}_{in,\mathbf{s}} = \{((0,1)^\perp, 1 + z^{(-1,0)}), ((1,0)^\perp, 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)})\},$$

which can be drawn as



$$f_{\mathfrak{d}_1} = 1 + z^{(-1,0)}$$
$$f_{\mathfrak{d}_2} = 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}$$

Consider the paths γ (traveling counterclockwise, on the left) and γ' (traveling clockwise, on the right). We can demonstrate that the diagram $\mathfrak{D}_{in,\mathbf{s}}$ is not consistent by computing $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}}$ and $\mathfrak{p}_{\gamma',\mathfrak{D}_{in,\mathbf{s}}}$. We compute $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}}$ as

$$\begin{aligned}
z^{(1,1)} &\overset{\mathfrak{d}_2}{\longmapsto} z^{(1,1)}\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}\right)^{\langle(1,1),(1,0)\rangle} \\
&= z^{(1,1)}\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}\right) \\
&\overset{\mathfrak{d}_1}{\longmapsto} z^{(1,1)}\left(1 + z^{(-1,0)}\right)^{\langle(1,1),(0,1)\rangle} \begin{pmatrix} 1 + az^{(0,1)}\left(1 + z^{(-1,0)}\right)^{\langle(0,1),(0,1)\rangle} \\ +az^{(0,2)}\left(1 + z^{(-1,0)}\right)^{\langle(0,2),(0,1)\rangle} \\ +z^{(0,3)}\left(1 + z^{(-1,0)}\right)^{\langle(0,3),(0,1)\rangle} \end{pmatrix} \\
&= z^{(1,1)}\left(1 + z^{(-1,0)}\right)\left(1 + az^{(0,1)}\left(1 + z^{(-1,0)}\right) + az^{(0,2)}\left(1 + z^{(-1,0)}\right)^2 + z^{(0,3)}\left(1 + z^{(-1,0)}\right)^3\right)
\end{aligned}$$

Similarly, we compute $\mathfrak{p}_{\gamma',\mathfrak{D}_{in,\mathbf{s}}}$ as

$$\begin{aligned}
z^{(1,1)} &\overset{\mathfrak{d}_1}{\longmapsto} z^{(1,1)}\left(1 + z^{(-1,0)}\right)^{\langle(1,1),(0,1)\rangle} \\
&= z^{(1,1)}\left(1 + z^{(-1,0)}\right) \\
&\overset{\mathfrak{d}_2}{\longmapsto} z^{(1,1)}\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}\right)^{\langle(1,1),(1,0)\rangle} \cdot \\
&\qquad \left(1 + z^{(-1,0)}\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}\right)^{\langle(-1,0),(1,0)\rangle}\right) \\
&= z^{(1,1)}\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}\right)\left(1 + \frac{z^{(-1,0)}}{\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}\right)}\right) \\
&= z^{(1,1)}\left(1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)} + z^{(-1,0)}\right)
\end{aligned}$$

Observe that $\mathfrak{p}_{\gamma,\mathfrak{D}_{in,\mathbf{s}}} \neq \mathfrak{p}_{\gamma',\mathfrak{D}_{in,\mathbf{s}}}$. Hence, $\mathfrak{D}_{in,\mathbf{s}}$ is by definition not consistent. Making the diagram *consistent requires adding four walls:*

$$\begin{aligned}
\mathfrak{d}_3 &= \left(\mathbb{R}_{\geq 0}(1,-3), 1+z^{(-1,3)}\right),\\
\mathfrak{d}_4 &= \left(\mathbb{R}_{\geq 0}(1,-2), 1+az^{(-1,2)}+az^{(-2,4)}+z^{(-3,6)}\right),\\
\mathfrak{d}_5 &= \left(\mathbb{R}_{\geq 0}(2,-3), 1+z^{(-2,3)}\right),\\
\mathfrak{d}_6 &= \left(\mathbb{R}_{\geq 0}(1,-1), 1+az^{(-1,1)}+az^{(-2,2)}+z^{(-3,3)}\right).
\end{aligned}$$

The consistent diagram is shown in Example 4.3.3.

As in the ordinary setting, the initial diagram $\mathfrak{D}_{\mathrm{in},\mathbf{s}}$ uniquely determines a consistent generalized cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$, up to equivalence.

Theorem 4.2.3 (Analogue of Theorem 1.12 of [41])**.** *Given a generalized torus seed* $\mathbf{s}$*, there exists a consistent scattering diagram* $\mathfrak{D}_{\mathbf{s}}$ *such that* $\mathfrak{D}_{in,\mathbf{s}} \subset \mathfrak{D}_{\mathbf{s}}$ *and* $\mathfrak{D}_{\mathbf{s}} \backslash \mathfrak{D}_{in,\mathbf{s}}$ *consists only of walls* $\mathfrak{d} \subset n_0^{\perp}$ *with* $p_1^*(n_0) \notin \mathfrak{d}$*. The scattering diagram* $\mathfrak{D}_{\mathbf{s}}$ *is unique up to equivalence.*

Proof. The proof given in Section 1.2 and Appendix C of [41] for ordinary cluster scattering diagrams holds in the generalized setting. That proof is a special case of results from [40] and [47] and holds in our setting because it does not require that the wall-crossing automorphisms are strictly binomial. □

4.2.1 The $r = (2,2)$ case

As an aside, we briefly explore the construction of a generalized cluster scattering diagram for the generalized cluster algebra

$$\mathcal{A} = \left((x_1,x_2),(y_1,y_2),\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right),((1,a,1),(1,b,1))$$

and offer a conjecture for the wall-crossing automorphism attached to its limiting wall. Intuitively, we can think of this generalized cluster algebra as an analogue of the ordinary 2-Kronecker quiver. As such, we will often refer to $\mathcal{A}$ as the "generalized 2-Kronecker". The associated generalized fixed data Γ has

$$\begin{aligned}
d &= (1,1),\\
r &= (2,2),\\
I &= I_{\mathrm{uf}} = \{1,2\},\\
N &= N^\circ = \langle e_1, e_2\rangle,\\
M &= M^\circ = \langle e_1^*, e_2^*\rangle,
\end{aligned}$$

and skew-symmetric bilinear form $\{\cdot,\cdot\} : N^\circ \times N^\circ \to \mathbb{Z}$ specified by the exchange matrix. One natural choice of generalized torus seed data is

$$\mathbf{s} = \{(e_1 = (1,0),(1,a,1)),(e_2 = (0,1),(1,b,1))\}.$$

For this choice of $\mathbf{s}$, we have

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} = \{((1,0)^\perp, 1 + bz^{(0,1)} + z^{(0,2)}), ((0,1)^\perp, 1 + az^{(-1,0)} + z^{(-2,0)})\}.$$

We conjecture that this initial scattering diagram gives rise to the following consistent scattering diagram $\mathfrak{D}_{\mathbf{s}}$:

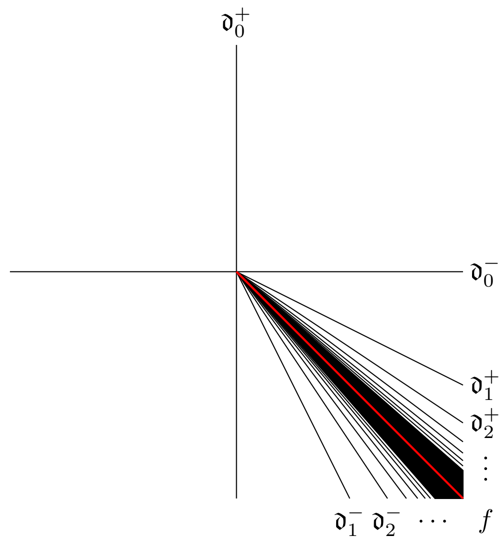


Figure 4.1: The conjectural cluster scattering diagram for the "generalized 2-Kronecker".

with associated wall-crossing automorphisms

$$\begin{aligned}
\mathfrak{d}_0^+ &= 1 + az^{(-1,0)} + z^{(-2,0)} \\
\mathfrak{d}_0^- &= 1 + bz^{(0,1)} + z^{(0,2)} \\
\mathfrak{d}_n^+ &= \begin{cases} 1 + bz^{(-(n+1),n)} + z^{(-2(n+1),2n)} & \text{for odd } n \\ 1 + az^{(-(n+1),n)} + z^{(-2(n+1),2n)} & \text{for even } n \end{cases} \\
\mathfrak{d}_n^- &= \begin{cases} 1 + az^{(-(n+1),n)} + z^{(-2(n+1),2n)} & \text{for odd } n \\ 1 + bz^{(-(n+1),n)} + z^{(-2(n+1),2n)} & \text{for even } n \end{cases}
\end{aligned}$$

and a currently unknown wall-crossing automorphism f attached to the limiting ray. Observe that when $a = b = 0$, this reduces to the known cluster scattering diagram for the ordinary 2-Kronecker with wall-crossing automorphisms, where f is known:

$$\begin{aligned}
\mathfrak{d}_0^+ &= 1 + z^{(-2,0)} \\
\mathfrak{d}_0^- &= 1 + z^{(0,2)} \\
\mathfrak{d}_n^+ &= \begin{cases} 1 + z^{(-2(n+1),2n)} & \text{for odd } n \\ 1 + z^{(-2(n+1),2n)} & \text{for even } n \end{cases} \\
\mathfrak{d}_n^- &= \begin{cases} 1 + z^{(-2(n+1),2n)} & \text{for odd } n \\ 1 + z^{(-2(n+1),2n)} & \text{for even } n \end{cases}
\end{aligned}$$

$$f = \frac{1}{\left(1 - z^{(-2,2)}\right)^2}$$

In a representation theoretic context, Reinke [67] proved that the wall-crossing automorphism on the limiting wall in the ordinary 2-Kronecker case is $\frac{1}{(1-z^{(-2,2)})^2}$. In [66], Reading offered a proof using limits of path-ordered products and ratios of powers of F-polynomials.

One can verify that appropriate path-ordered products in $\mathfrak{D}_{\mathbf{s}}$ indeed produce cluster variables of $\mathcal{A}$. For example,

$$\begin{aligned}
z^{(-1,0)} &\xmapsto{\mathfrak{d}_0^+} z^{(-1,0)}(1 + bz^{(01)} + z^{(0,2)})^{\langle(-1,0),(-1,0)\rangle} \\
&= z^{(-1,0)}(1 + bz^{(0,1)} + z^{(0,2)}) \\
z^{(0,-1)} &\xmapsto{\mathfrak{d}_0^-} z^{(0,-1)}(1 + az^{(-1,0)} + z^{(-2,0)})^{\langle(0,-1),(0,-1)\rangle} \\
&= z^{(0,-1)}(1 + az^{(-1,0)} + z^{(-2,0)}) \\
&\xmapsto{\mathfrak{d}_0^+} z^{(0,-1)}(1 + az^{(-1,0)}(1 + bz^{(0,1)} + z^{(0,2)})^{\langle(-1,0),(-1,0)\rangle} + z^{(-2,0)}(1 + bz^{(0,1)} \\
&\qquad + z^{(0,2)})^{\langle(-2,0),(-1,0)\rangle}) \\
&= z^{(0,-1)}(1 + az^{(-1,0)}(1 + bz^{(0,1)} + z^{(0,2)}) + z^{(-2,0)}(1 + bz^{(0,1)} + z^{(0,2)})) \\
&= z^{(0,-1)}(1 + az^{(-1,0)} + z^{(-2,0)} + abz^{(-1,1)} + az^{(-1,2)} + 2bz^{(-2,1)} \\
&\qquad + (2 + b^2)z^{(-2,2)} + 2bz^{(-2,3)} + z^{(-2,4)})
\end{aligned}$$

Observe that these are two of the cluster variables of the generalized 2-Kronecker, as expected.

In the ordinary 2-Kronecker, there is a formal power series which is closely related to the wall-crossing automorphism attached to the limiting ray. This formal power series appears as the limit of ratios of F-polynomials: $\lim_{i\to\infty} F_{i+1}/F_i$. The fact that this limit stabilizes to a formal power series was observed in [73]. The same ratio of F-polynomials appears to also stabilize for the generalized 2-Kronecker.

In the generalized 2-Kronecker, we conjecture that

$$\lim_{i\to\infty} \frac{F_i}{F_{i+1}} = \frac{\left(1 - z^{(-2,2)}\right)^2 + z^{(-2,2)}(a^2 + b^2) + abz^{(-1,1)}\left(1 + z^{(-1,1)}\right)^5}{\left(1 - z^{(-2,2)}\right)^4}.$$

To support our conjecture, we offer the following argument which follows the method used by Reading [66] to prove that

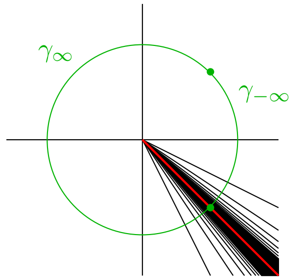
$$\lim_{i\to\infty} \frac{F_{i+1}}{F_i} = 1 + \hat{y_i} + \sum_{i,j\geq 0} (-1)^{i+j} Nar(i,j) \hat{y_1}^i \hat{y_2}^j$$

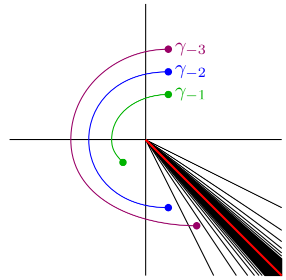
for the F-polynomials of the ordinary 2-Kronecker, where

$$Nar(i,j) = \begin{cases} 1 & i = j = 0 \\ 0 & ij = 0 \text{ but either } i \neq 0 \text{ or } j \neq 0 \\ \frac{1}{i}\binom{i}{j}\binom{i}{j-1} & i,j \geq 1 \end{cases}$$

is a *Narayana number*.

Let γ_∞ be the path which begins at the limiting ray and goes clockwise to the positive chamber. Similarly, let $\gamma_{-\infty}$ be the path which begins at the limiting ray and goes counterclockwise to the positive chamber. For $i \le -1$, we define paths γ_i as shown below on the righthand side.





Observe that crossing the limiting wall while traveling towards the north-east sends $z^{(1,1)} \mapsto z^{(1,1)} f^{\langle (1,1),(-1,-1)\rangle} = z^{(1,1)} f^{-2}$. In order for the diagram to be consistent, we must have

$$\mathfrak{p}_{-\infty}\left(z^{(1,1)}\right) = \mathfrak{p}_\infty(z^{(1,1)} f^{-2}).$$

Recall that the cluster variable x_i can be written in the form $x_i = \mathbf{x}^{\mathbf{g}_i} F_i$, where $\mathbf{g}_i$ and F_i denote, respectively, the g-vector and F-polynomial associated to x_i. Hence, for $i \le 1$, we have $x_i = x_1^i x_2^{-(i+1)} F_i$ and

$$\mathfrak{p}_{\gamma_i}\left(z^{(i,-(i+1))}\right) = z^{(i,-(i+1))} F_i,$$
$$\mathfrak{p}_{\gamma_i}\left(z^{(i+1,-(i+2))}\right) = z^{(i+1,-(i+2))} F_{i+1}.$$

The key observation is then that f is given by the anti-diagonal terms of

$$\sqrt{\frac{z^{(1,1)}}{\mathfrak{p}_{-\infty}(z^{(1,1)})}} = \lim_{i\to\infty} z^{(1,1)} F_i^{2i+3} F_{i+1}^{-2i-1}$$

Using code which implements generalized cluster algebras in *SageMath*, we then computed the F-polynomials $F_{-1}, \ldots, F_{-19}$. Below, we list the antidiagonal terms of each F-polynomial. Note that we stopped at F_{-19} due to the length of time required for computation.

$F_{-1} : 1,\ abz^{(-1,1)},\ (2+b^2)z^{(-2,2)}$

$F_{-2} : 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 4abz^{(-3,3)},\ (3+3b^2)z^{(-4,4)}$

$F_{-3} = 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+3a^2+4b^2)z^{(-4,4)},\ 9abz^{(-5,5)},$
$(4+6b^2)z^{(-6,6)}$

$F_{-4} : 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 13abz^{(-5,5)},$
$(4+6a^2+9b^2)z^{(-6,6)},\ 16abz^{(-7,7)},\ (5+10b^2)z^{(-8,8)}$

$F_{-5} : 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+9a^2+10b^2)z^{(-6,6)},\ 25abz^{(-7,7)},\ (5+10a^2+16b^2)z^{(-8,8)},\ 25abz^{(-9,9)},$
$(6+15b^2)z^{(-10,10)}$

$F_{-6} : 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 29abz^{(-7,7)},\ (5+16a^2+19b^2)z^{(-8,8)},\ 41abz^{(-9,9)},$

$(6+15a^2+25b^2)z^{(-10,10)},\ 36abz^{(-11,11)},\ (7+21b^2)z^{(-12,12)}$

$F_{-7}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+19a^2+20b^2)z^{(-8,8)},\ 50abz^{(-9,9)},$
$(6+25a^2+31b^2)z^{(-10,10)},\ 61abz^{(-11,11)},\ (7+21a^2+36b^2)z^{(-12,12)},\ 49abz^{(-13,13)},$
$(8+28b^2)z^{(-14,14)}$

$F_{-8}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 54abz^{(-9,9)},$
$(6+31a^2+34b^2)z^{(-10,10)},\ 77abz^{(-11,11)},\ (7+36a^2+46b^2)z^{(-12,12)},\ 85abz^{(-13,13)},$
$(8+28a^2+49b^2)z^{(-14,14)},\ 64abz^{(-15,15)},\ (9+36b^2)z^{(-16,16)}$

$F_{-9}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 55abz^{(-9,9)},$
$(6+34a^2+35b^2)z^{(-10,10)},\ 86abz^{(-11,11)},\ (7+46a^2+52b^2)z^{(-12,12)},\ 110ab^{(-13,13)},$
$(8+49a^2+64b^2)z^{(-14,14)},\ 113abz^{(-15,15)},\ (9+36a^2+64b^2)z^{(-16,16)},\ 81abz^{(-17,17)},$
$(10+45b^2)z^{(-18,18)}$

$F_{-10}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 55abz^{(-9,9)},$
$(6+35a^2+35b^2)z^{(-10,10)},\ 90abz^{(-11,11)},\ (7+52a^2+55b^2)z^{(-12,12)},\ 126abz^{(-13,13)},$
$(8+64a^2+74b^2)z^{(-14,14)},\ 149abz^{(-15,15)},\ (9+64a^2+85b^2)z^{(-16,16)},\ 145abz^{(-17,17)},$
$(10+45a^2+81b^2)z^{(-18,18)},\ 100abz^{(-19,19)},\ (11+55^2)z^{(-20,20)}$

$F_{-11}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 55abz^{(-9,9)},$
$(6+35a^2+35b^2)z^{(-10,10)},\ 91abz^{(-11,11)},\ (7+55a^2+56b^2)z^{(-12,12)},\ 135abz^{(-13,13)},$
$(8+74a^2+80b^2)z^{(-14,14)},\ 174abz^{(-15,15)},\ (9+85a^2+100b^2)z^{(-16,16)},\ 194abz^{(-17,17)},$
$(10+81a^2+109b^2)z^{(-18,18)},\ 181abz^{(-19,19)},\ (11+55a^2+100b^2)z^{(-20,20)},$
$121abz^{(-21,21)},\ (12+66b^2)z^{(-22,22)}$

$F_{-12}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 55abz^{(-9,9)},$
$(6+35a^2+35b^2)z^{(-10,10)},\ 91abz^{(-11,11)},\ (7+56a^2+b^2)z^{(-12,12)}+139abz^{(-13,13)},$
$(8+80a^2+83b^2)z^{(-14,14)},\ 190abz^{(-15,15)},\ (9+100a^2+110b^2)z^{(-16,16)},\ 230abz^{(-17,17)},$
$(10+109a^2+130b^2)z^{(-18,18)},\ 245abz^{(-19,19)},\ (11+100a^2+136b^2)z^{(-20,20)},$
$221abz^{(-21,21)},\ (12+66a^2+121b^2)z^{(-22,22)},\ 144abz^{(-23,23)},\ (13+78b^2)z^{(-24,24)}$

$F_{-13}: 1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},$
$(4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 55abz^{(-9,9)},$
$(6+35a^2+35b^2)z^{(-10,10)},\ 91abz^{(-11,11)},\ (7+56a^2+b^2)z^{(-12,12)},\ 140abz^{(-13,13)},$
$(8+83a^2+84b^2)z^{(-14,14)},\ 199abz^{(-15,15)},\ (9+110a^2+116b^2)z^{(-16,16)},\ 225abz^{(-17,17)},$
$(10+130a^2+145b^2)z^{(-18,18)},\ 294abz^{(-19,19)},\ (11+136a^2+164b^2)z^{(-20,20)},$

$302abz^{(-21,21)}$, $(12+121a^2+166b^2)z^{(-22,22)}$, $265abz^{(-23,23)}$, $(13+78a^2+144b^2)z^{(-24,24)}$, $169abz^{(-25,25)}$, $(14+91b^2)z^{(-26,26)}$

$\vdots$

As i increases, the sequence of anti-diagonal terms appears to stabilize to:

$$1,\ abz^{(-1,1)},\ (2+a^2+b^2)z^{(-2,2)},\ 5abz^{(-3,3)},\ (3+4a^2+4b^2)z^{(-4,4)},\ 14abz^{(-5,5)},\ (4+10a^2+10b^2)z^{(-6,6)},\ 30abz^{(-7,7)},\ (5+20a^2+20b^2)z^{(-8,8)},\ 55abz^{(-9,9)},\ (6+35a^2+35b^2)z^{(-10,10)},\ 91abz^{(-11,11)},\ (7+56a^2+56b^2)z^{(-12,12)},\ 140abz^{(-13,13)},\ (8+84a^2+84b^2)z^{(-14,14)},\ 204abz^{(-15,15)},\ (9+120a^2+120b^2)z^{(-16,16)},\ 285abz^{(-17,17)},\ (10+165a^2+165b^2)z^{(-18,18)},\ \ldots$$

We can then look at several subsequences of the coefficients of these terms. First, observe that the constant terms of the coefficients of terms with even exponents, i.e. of the form $z^{(-2n,2n)}$, form the sequence

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

These are the only non-vanishing terms if we set $a = b = 0$. We expect that setting $a = b = 0$ should recover the power series arising from the F-polynomial ratio $\lim_{i\to\infty} F_{i+1}/F_i$ in the ordinary 2-Kronecker. Indeed,

$$\frac{1}{(1-z^{(-1,1)})^2} = 1 + 2z^{(-2,2)} + 3z^{(-4,4)} + 4z^{(-6,6)} + 5z^{(-8,8)} + 6z^{(-10,10} + 7z^{(-12,12)}.$$

We can then look at the coefficients of $(a^2+b^2)z^{(-2n,2n)}$, which form the sequence

$$0, 1, 4, 10, 20, 35, 56, 84, 120, 165, \ldots,$$

which matches the tetrahedral numbers. So it appears that this portion of the series is converging to

$$\frac{(a^2+b^2)z^{(-2,2)}}{(1-z^{(-2,2)})^4} = z^{(-2,2)} + 4z^{(-4,4)} + 10z^{(-6,6)} + 20z^{(-8,8)} + 35z^{(-10,10)} + 56z^{(-12,12)} + \cdots$$

Finally, the coefficients of the terms with odd exponents, i.e. of the form $z^{(-(2n+1),2n+1)}$, form the sequence

$$1, 5, 14, 30, 55, 91, 140, 204, 285, \ldots$$

We were computationally limited in computing terms of this sequence by the runtime. From the terms that we were able to compute, this sequence could match three known sequences from OEIS: the square pyramidal numbers (A000330), the growth series for the affine Coxeter group (A266783), and A109678. All three sequences agree until the 25th term, at which point sequence A266783 has the term 823 and the other two sequences, A000330 and A109678 have the term 819. Distinguishing between A000330 and A109678 would require the ability to compute substantially more terms of the coefficient sequence.

We conjecture that this sequence is in fact the square pyramidal numbers, and so this portion of the series is converging to

$$\frac{abz^{(-1,1)}(1+z^{(-1,1)})}{(z^{(-1,1)}-1)^4} = 1 + 5z^{(-1,1)} + 14z^{(-3,3)} + 30z^{(-5,5)} + 91z^{(-7,7)} + \cdots$$

Combining the three types of terms yields the conjecture that

$$\lim_{i\to\infty} \frac{F_i}{F_{i+1}} = \frac{\left(1-z^{(-2,2)}\right)^2 + z^{(-2,2)}(a^2+b^2) + abz^{(-1,1)}\left(1+z^{(-1,1)}\right)^5}{\left(1-z^{(-2,2)}\right)^4}.$$

4.3 Mutation invariance

We can slightly tweak the mutation of ordinary scattering diagrams. We use the same definitions of $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$, but modify the definition of T_k as follows:

Definition 4.3.1. *We define the piecewise linear transformation $T_k : M^\circ \to M^\circ$ as*

$$T_k(m) := \begin{cases} m + r_k v_k \langle d_k e_k, m\rangle & m \in \mathcal{H}_{k,+} \\ m & m \in \mathcal{H}_{k,-} \end{cases}$$

As before, we sometimes use the shorthand $T_{k,-}$ and $T_{k,+}$ to refer to T_k in, respectively, the regions $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$.

The procedure for applying T_k to a generalized cluster scattering diagram remains the same. For reference, we reproduce it in the following definition.

Definition 4.3.2. *The scattering diagram $T_k(\mathfrak{D}_\mathbf{s})$ is obtained from $\mathfrak{D}_\mathbf{s}$ via the following procedure:*

1. *The wall $\mathfrak{d}_k$ in $\mathfrak{D}_\mathbf{s}$ becomes the wall $\mathfrak{d}'_k = (e_k^\perp, 1 + a_1 z^{-v_k} + \cdots + a_{r_k-1} z^{-(r_k-1)v_k} + z^{-r_k v_k})$ in $T_k(\mathfrak{D}_\mathbf{s})$.*
2. *For each wall $(\mathfrak{d}, f_\mathfrak{d})$ in $\mathfrak{D}_\mathbf{s}$ other than $\mathfrak{d}_k := (e_k^\perp, 1 + a_1 z^{v_k} + \cdots + a_{r_k-1} z^{(r_k-1)v_k} + z^{r_k v_k})$, there are either one or two corresponding walls in $T_k(\mathfrak{D}_\mathbf{s})$. If $dim(\mathfrak{d} \cap \mathcal{H}_{k,-}) \geq rank(M) - 1$, then add to $T_k(\mathfrak{D}_s)$ the wall $(T_k(\mathfrak{d} \cap \mathcal{H}_{k,-}), T_{k,-}(f_\mathfrak{d}))$ where the notation $T_{k,\pm}(f_\mathfrak{d})$ indicates the formal power series obtained by applying $T_{k,\pm}$ to the exponent of each term of $f_\mathfrak{d}$. If $dim(\mathfrak{d} \cap \mathcal{H}_{k,+}) \geq rank(M) - 1$, add the wall $(T_k(\mathfrak{d} \cap \mathcal{H}_{k,+}), T_{k,+}(f_\mathfrak{d}))$.*

Example 4.3.3. *Consider the generalized cluster algebra*

$$\mathcal{A}\left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, ((1,a,a,1),(1,1))\right)$$

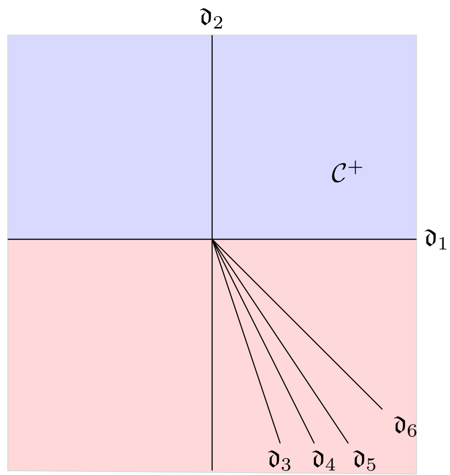
with seed data $\mathbf{s} = (((1,0),(1,a,a,1)),((0,1),(1,1)))$. For this algebra, we have

$$\begin{aligned} \epsilon_{12} &= \{e_1, e_2\} d_2 = 1 \\ \epsilon_{21} &= \{e_2, e_1\} d_1 = -1 \\ v_1 &= p_1^*((1,0)) = (0,1) \\ v_2 &= p_1^*((0,1)) = (-1,0) \end{aligned}$$

By definition, this means

$$\mathfrak{D}_{in,\mathbf{s}} = \left\{((1,0), 1 + z^{(-1,0)}), ((0,1), 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)})\right\}$$

Adding walls to make this diagram consistent, we obtain the following:



$$\begin{aligned}
f_{\mathfrak{d}_1} &= 1 + z^{(-1,0)} \\
f_{\mathfrak{d}_2} &= 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)} \\
f_{\mathfrak{d}_3} &= 1 + z^{(-1,3)} \\
f_{\mathfrak{d}_4} &= 1 + az^{(-1,2)} + az^{(-2,4)} + z^{(-3,6)} \\
f_{\mathfrak{d}_5} &= 1 + z^{(-2,3)} \\
f_{\mathfrak{d}_6} &= 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)}
\end{aligned}$$

By definition, we have the half-planes

$$\begin{aligned}
\mathcal{H}_{2,+} &= \{(0,y) : y > 0\} \\
\mathcal{H}_{2,-} &= \{(0,y) : y < 0\}
\end{aligned}$$

which are shown on $\mathfrak{D}$ in blue, for $\mathcal{H}_{2,+}$, and red, for $\mathcal{H}_{2,-}$. To mutate in direction $k = 2$, we'll use the linear transformation

$$T_2(m) = \begin{cases} m + (-1,0)\langle(0,1), m\rangle & m \in \mathcal{H}_{2,+} \\ m & m \in \mathcal{H}_{2,-} \end{cases}$$

Because T_2 fixes the walls in $\mathcal{H}_{2,-}$, the only walls that change under T_2 are $\mathfrak{d}_1$ and $\mathfrak{d}_2 \cap \mathbb{R}_{\geq 0}(0,1)$. Because the support of $\mathfrak{d}_1$ is $e_2^{\perp} = (1,0)$, it's transformed via the procedure outlined in (1) of Definition 4.3.2 and becomes

$$\mathfrak{d}_1' = (e_2^{\perp}, 1 + z^{(1,0)})$$

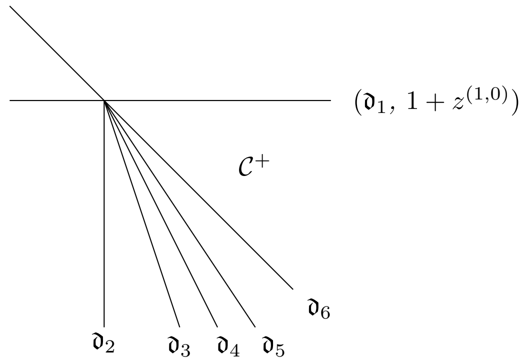
To determine the image of $\mathfrak{d}_2 \cap \mathbb{R}_{\geq 0}(0,1)$, we compute

$$T_2((0,1)) = (0,1) + (-1,0)\langle(0,1),(0,1)\rangle = (-1,1)$$

Because T_2 is a linear transformation, we know that $T_2((0,2)) = (-2,2)$ and $T_2((0,3)) = (-3,3)$. As such,

$$T_2(\mathfrak{d}_2 \cap \mathbb{R}_{\geq 0}(0,1)) = (\mathbb{R}_{\geq 0}(-1,1), 1 + a^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)})$$

and we draw $\mu_2\mathfrak{D}$ as



where $f_{\mathfrak{d}_2}, f_{\mathfrak{d}_3}, f_{\mathfrak{d}_4}, f_{\mathfrak{d}_5}$, and $f_{\mathfrak{d}_6}$ are the same automorphisms as in $\mathfrak{D}_{\mathbf{s}}$. We can also compute the new basis vectors e_1' and e_2' using Definition 4.1.5:

$$\begin{aligned} e_1' &= e_1 + r_2[\epsilon_{12}]_+ e_2 \\ &= (1,0) + (0,1) \\ &= (1,1) \\ e_2' &= -e_2 = (0,-1) \end{aligned}$$

Because $\mathcal{A}$ has exchange polynomials with reciprocal coefficients, the exchange polynomial coefficients are fixed under mutation.So we have

$$\mu_2(\mathbf{s}) = (((1,1),(1,a,a,1)),((0,-1),(1,1)))$$

Each cluster mutation μ_k can be defined by a triple $(n,m,r) \in N \times M \times \mathbb{Z}_{\geq 0}$ with $\langle n,m\rangle = 0$. Emulating the notation of [39], we denote this mutation as $\mu_{(n,m,r)}$. It is defined by the pullback

$$\mu^*_{(n,m,r)}\left(z^{m'}\right) = z^{m'} \cdot \left(1 + a_1 z^m + \cdots + a_{r-1} z^{(r-1)m} + z^{rm}\right)^{\langle n,m'\rangle},$$

where $a_1, \ldots, a_{r-1}$ are scalars and $r \in \mathbb{Z}_{\geq 0}$.

We refer to the max-plus tropicalization of a semifield $\mathbb{P}$ as the *Fock-Gonacharov tropicalization* and denote it as $\mathbb{P}^T$. Likewise, we refer to the min-plus tropicalization as the *geometric tropicalization* and denote it as $\mathbb{P}^t$. Let $\mu : T_N \to T_N$ be a positive birational map. Then $\mu^T : N \to N$ and $\mu^t : N \to N$ denote, respectively, the induced maps $T_N(\mathbb{Z}^T) \to T_N(\mathbb{Z}^T)$ and $T_N(\mathbb{Z}^t) \to T_N(\mathbb{Z}^t)$.

Proposition 4.3.4 (Analogue of Proposition 2.4 of [41])**.** *The map $T_k : M^\circ \to M^\circ$ given in Definition 4.3.1 is the Fock-Goncharov tropicalization of the map $\mu_{(v_k, d_k e_k, r_k)}$.*

Proof. The map $\mu_{(d_k e_k, v_k, r_k)} : T_{M^\circ} \to T_{M^\circ}$ is defined by the pullback

$$\mu^*_{(v_k, d_k e_k, r_k)}\left(z^m\right) = z^m \left(1 + a_1 z^{v_k} + \cdots + a_{r_k - 1} z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{\langle d_k e_k, m\rangle}$$

By definition, $\mu_{\langle d_k e_k, v_k, r_k\rangle}$ has Fock-Goncharov tropicalization

$$\begin{aligned} \mu^T_{d_k e_k, v_k, r_k} &: N \to N \\ &x \mapsto x + r_k[\langle d_k e_k, x\rangle]_+ v_k \end{aligned}$$

Observe that when $x \in \mathcal{H}_{k,-}$, then $\langle d_k e_k, x\rangle \leq 0$ and the above map reduces to $x \mapsto x$. When $x \in \mathcal{H}_{k,+}$, then $\langle d_k e_k, \rangle \geq 0$ and the map reduces to $x \mapsto x + r_k v_k \langle d_k e_k, x\rangle$. Hence, the tropicalization agrees exactly with our definition of T_k, as desired. $\square$

Theorem 4.3.5 (Analogue of Theorem 1.24 of [41])**.** *If the injectivity assumption holds, then $T_k(\mathfrak{D}_{\mathbf{s}})$ is a consistent scattering diagram for $N^+_{\mu_k(\mathbf{s})}$. Moreover, the diagrams $\mathfrak{D}_{\mu_k(\mathbf{s})}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ are equivalent.*

To prove this, we need to show that $T_k(\mathfrak{D}_{\mathbf{s}})$ is a scattering diagram for $\mathfrak{g}_{\mu_k(\mathbf{s})}$ and $N^+_{\mu_k(\mathbf{s})}$. The major technical hurdle in doing so is the fact that the wall-crossing automorphisms of $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mu_k(\mathbf{s})}$ live in different completed monoid rings. Those in $\mathfrak{D}_{\mathbf{s}}$ live in $\widehat{\Bbbk[P]}$, where P is the monoid generated by $\{v_i\}_{i\in I_{\mathrm{uf}}}$. Those in $\mathfrak{D}_{\mu_k(\mathbf{s})}$ live, instead, in $\widehat{\Bbbk[P']}$, where P' is the monoid generated by $\{v_i'\}_{i\in I_{\mathrm{uf}}}$.

To overcome this difficulty, we define an additional monoid $\overline{P}$ which contains both P and P'. Let $\sigma \subseteq M^\circ$ be a top-dimensional cone which contains the vectors $\{v_i\}_{i\in I_{\mathrm{uf}}}$ and $-v_k$, such that $\sigma \cap (-\sigma) = \mathbb{R}v_k$. For a fixed choice of σ, let $\overline{P} := \sigma \cap M^\circ$ and $J = \overline{P}\backslash(\overline{P} \cap \mathbb{R}v_k) = \overline{P}\backslash\overline{P}^\times$.

Even after choosing an appropriate monoid $\overline{P}$, we still have to deal with the fact that the wall-crossing automorphism associated to the wall

$$\mathfrak{d}_k = \left(e_k^\perp, 1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{v_k}\right) =: (e_k^\perp, f_k)$$

is an automorphism of the localization $\widehat{\Bbbk[\overline{P}]}_{f_k}$ rather than the ring $\widehat{\Bbbk[\overline{P}]}$, where the completions are with respect to the ideal J. Let $\mathfrak{p}_{\mathfrak{d}_k} \in \widehat{\Bbbk[\overline{P}]}_{f_k}$ denote the automorphism associated with crossing $\mathfrak{d}_k$ from $\mathcal{H}_{k,-}$ into $\mathcal{H}_{k,+}$. By definition,

$$\mathfrak{p}_{\mathfrak{d}_k}(z^m) = z^m(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{v_k})^{-\langle d_k e_k, m\rangle}.$$

We can then define

$$N_{\mathbf{s}}^{+,k} := \left\{ \sum_{i\in I_{\mathrm{uf}}} a_i e_i \;\middle|\; a_i \in \mathbb{Z}_{\geq 0} \text{ for } i \neq k, a_k \in \mathbb{Z}, \text{ and } \sum_{i\in I_{\mathrm{uf}}\backslash\{k\}} a_i > 0 \right\}.$$

Because $\mathbf{s}' = (\mathbf{s}\backslash\{v_k\}) \cup \{-v_k\}$, the conditions of this definition mean that $N_{\mathbf{s}}^{+,k} = N_{\mathbf{s}'}^{+,k}$. As such, we can use the abbreviated notation $N^{+,k}$ without introducing any ambiguity.

To allow us to work in $\overline{P}$, we need to slightly modify the definition of a scattering diagram:

Definition 4.3.6. *Given the monoid $\overline{P}$ and ideal J, a wall is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ such that for some primitive $n_0 \in N^{+,k}$,*

1. *$f_{\mathfrak{d}} \in \widehat{\Bbbk[\overline{P}]}$ has the form $1 + \sum_{k=1}^\infty c_k z^{kp^*(n_0)}$ and is congruent to 1 mod J,*
2. *and $\mathfrak{d} \subset n_0^\perp \subset M_{\mathbb{R}}$ is a convex rational polyhedral cone with dimension rank $N - 1$.*

For a seed $\mathbf{s}$, the slab is $\mathfrak{d}_k = (e_k^\perp, 1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{v_k})$. Because $v_k \in \overline{P}^\times$, the slab doesn't qualify as a wall under the above definition. So we extend the definition of a scattering diagram *$\mathfrak{D}$ such that:*

1. *$\mathfrak{D}$ contains a collection of walls and potentially the slab $\mathfrak{d}_k$, and*

2. *for $k > 0$, we have $f_{\mathfrak{d}} \equiv 1 mod\ J^k$ for all but finitely many walls of $\mathfrak{D}$.*

In this modified scattering diagram, crossing a wall or slab $(\mathfrak{d}, f_{\mathfrak{d}})$ induces an automorphism $\mathfrak{p}_{f_{\mathfrak{d}}}^{\pm 1} \in \widehat{\Bbbk[\overline{P}]_{f_k}}$. Note that the localization at f_k is only really required when crossing $\mathfrak{d}_k$, as otherwise $f_{\mathfrak{d}}$ lives in $\widehat{\Bbbk[\overline{P}]}$.

The proof of Theorem 4.3.5 requires the following result:

Theorem 4.3.7 (Analogue of Theorem 1.28 of [41])**.** *There exists a scattering diagram $\overline{\mathfrak{D}}_{\mathbf{s}}$ such that*

- $\overline{\mathfrak{D}_{\mathbf{s}}} \supseteq \mathfrak{D}_{in,\mathbf{s}}$*,*
- $\overline{\mathfrak{D}_{\mathbf{s}}} \backslash \mathfrak{D}_{in,\mathbf{s}}$ *consists of only outgoing walls,*
- *and the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}} \in \widehat{\Bbbk[\overline{P}]_{f_k}}$ depends only on the endpoints of γ.*

Such $\overline{\mathfrak{D}}_{\mathbf{s}}$ is unique up to equivalence. Further, because $\overline{\mathfrak{D}}_{\mathbf{s}}$ is also a scattering diagram for $N_{\mathbf{s}}^+$, it's equivalent to $\mathfrak{D}_{\mathbf{s}}$. Moreover, this implies that the only wall contained in $e_k^\perp$ is the slab $\mathfrak{d}_k$

The proof given in [41] in the ordinary setting also holds in our generalized setting. Because that proof is quite lengthy, we do not reproduce it here.

We will also need the following definition:

Definition 4.3.8. *A codimension two convex rational polyhedral cone $\mathfrak{j}$ is a* joint *of the scattering diagram $\mathfrak{D}$ if either every wall $\mathfrak{d} \subseteq n^\perp$ that contains $\mathfrak{j}$ has direction $-p^*(n) = -\{n, \cdot\}$ tangent to $\mathfrak{j}$ or direction not tangent to $\mathfrak{j}$. In the first case, where every wall is tangent to $\mathfrak{j}$, we call the joint* parallel*. In the second case, we call the joint* perpendicular*.*

We're now prepared to prove Theorem 4.3.5:

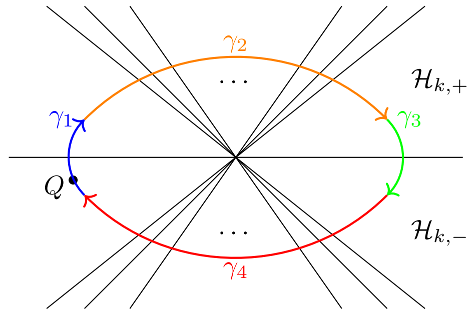
Proof. Let $\mathbf{s} = \{e_i\}_{i \in I}$ be a fixed choice of generalized torus seed and $\mathbf{s}' := \mu_k(\mathbf{s}) = \{e_i'\}_{i \in I}$. From Theorem 4.3.7, we know that the scattering diagrams for $\mathbf{s}$ and $\mathbf{s}'$ are unique up to equivalence and therefore we can choose representative scattering diagrams $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$.

Notice that if $z^m \in J^i$ for some $i > 0$, then $z^{T_{k,\pm}(m)} \in J^i$. As such, $T_k(\mathfrak{D}_{\mathbf{s}})$ is also a scattering diagram for the seed $\mathbf{s}'$ in the slightly extended sense of Definition 4.3.6. In order to use Theorem 4.3.7 to show that $\mathfrak{D}_{\mathbf{s}'}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ are equivalent, we need to (1) verify that $T_k(\mathfrak{D}_{\mathbf{s}})$ is consistent and (2) show that both diagrams are equivalent to diagrams with the same set of slabs and incoming walls.

We can begin by showing that $T_k(\mathfrak{D}_{\mathbf{s}})$ is consistent. To do so, we need to show that $\mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})} = \mathrm{id}$ for any loop γ for which the path-ordered product is defined. Because $\mathfrak{D}_{\mathbf{s}}$ is consistent and so by definition $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathrm{id}$ whenever the path-ordered product is defined, one strategy is to show that $\mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})} = \mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}$ and therefore $\mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})} = \mathrm{id}$. In areas of $\mathfrak{D}_{\mathbf{s}}$ where T_k is linear, the consistency of a loop in $T_k(\mathfrak{D}_{\mathbf{s}})$ is an immediate consequence of linearity since each wall is crossed either not at all or in both possible directions.

Therefore, we need only be concerned about when γ is a loop around a joint $\mathfrak{j}$ of $\mathfrak{D}_{\mathbf{s}}$ which is contained in the slab $\mathfrak{d}_k$. Given such γ, we can subdivide it as $\gamma = \gamma_1\gamma_2\gamma_3\gamma_4$ where γ_1 crosses $\mathfrak{d}_k$, $\gamma_2 \subseteq \mathcal{H}_{k,+}$ contains all the crossings of walls in $\mathfrak{D}_{\mathbf{s}}$ which contain $\mathfrak{j}$ and lie in $\mathcal{H}_{k,+}$, γ_3 crosses $\mathfrak{d}_4$, and γ_4 contains all the crossings of walls in $\mathfrak{D}_{\mathbf{s}}$ that contain $\mathfrak{j}$ and lie in $\mathcal{H}_{k,-}$. We can also assume that it has a basepoint Q in $\mathcal{H}_{k,-}$.

One example of a possible subdivision of γ is shown below:



Let $\mathfrak{p}_{\mathfrak{d}_k}$ denote the wall-crossing automorphism for crossing $\mathfrak{d}_k$ from $\mathcal{H}_{k,-}$ into $\mathcal{H}_{k,+}$. Similarly, let $\mathfrak{p}_{\mathfrak{d}_k}$ denote crossing $\mathfrak{d}'_k$ from $\mathcal{H}_{k,-}$ into $\mathcal{H}_{k,+}$. Explicitly, we have

$$\mathfrak{p}_{\mathfrak{d}_k}(z^m) = z^m \left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_kv_k}\right)^{-\langle d_ke_k,m\rangle}$$
$$\mathfrak{p}_{\mathfrak{d}'_k}(z^m) = z^m \left(1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_kv_k}\right)^{-\langle d_ke_k,m\rangle}$$

Because $\mathfrak{d}_k$ is the only wall contained in $e_k^\perp$, we know that $\mathfrak{p}_{\gamma_1,\mathfrak{D}_\mathbf{s}} = \mathfrak{p}_{\mathfrak{d}_k}$ and $\mathfrak{p}_{\gamma_3,\mathfrak{D}_\mathbf{s}} = \mathfrak{p}_{\mathfrak{d}_k}^{-1}$. Let $\alpha : \Bbbk[M^\circ] \to \Bbbk[M^\circ]$ be the automorphism $\alpha(z^m) = z^{m+r_kv_k\langle d_ke_k,m\rangle}$ induced by $T_{k,+}$. We can then observe the following relationships:

$$\begin{aligned}
\mathfrak{p}_{\gamma_1,T_k(\mathfrak{D}_\mathbf{s})} &= \mathfrak{p}_{\mathfrak{d}'_k}\\
\mathfrak{p}_{\gamma_2,T_k(\mathfrak{D}_\mathbf{s})} &= \alpha \circ \mathfrak{p}_{\gamma_2,\mathfrak{D}_\mathbf{s}} \circ \alpha^{-1}\\
\mathfrak{p}_{\gamma_3,T_k(\mathfrak{D}_\mathbf{s})} &= \mathfrak{p}_{\mathfrak{d}'_k}^{-1}\\
\mathfrak{p}_{\gamma_4,T_k(\mathfrak{D}_\mathbf{s})} &= \mathfrak{p}_{\gamma_4,\mathfrak{D}_\mathbf{s}}
\end{aligned}$$

So we have

$$\begin{aligned}
\mathfrak{p}_{\gamma,\mathfrak{D}_\mathbf{s}} &= \mathfrak{p}_{\gamma_4,\mathfrak{D}_\mathbf{s}} \circ \mathfrak{p}_{\gamma_3,\mathfrak{D}_\mathbf{s}} \circ \mathfrak{p}_{\gamma_2,\mathfrak{D}_\mathbf{s}} \circ \mathfrak{p}_{\gamma_1,\mathfrak{D}_\mathbf{s}}\\
&= \mathfrak{p}_{\gamma_4,\mathfrak{D}_\mathbf{s}} \circ \mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p}_{\gamma_2,\mathfrak{D}_\mathbf{s}} \circ \mathfrak{p}_{\mathfrak{d}_k},\\
\mathfrak{p}_{\gamma,T_k(\mathfrak{D}_\mathbf{s})} &= \mathfrak{p}_{\gamma_4,T_k(\mathfrak{D}_\mathbf{s})} \circ \mathfrak{p}_{\gamma_3,T_k(\mathfrak{D}_\mathbf{s})} \circ \mathfrak{p}_{\gamma_2,T_k(\mathfrak{D}_\mathbf{s})} \circ \mathfrak{p}_{\gamma_1,T_k(\mathfrak{D}_\mathbf{s})}\\
&= \mathfrak{p}_{\gamma_4,\mathfrak{D}_\mathbf{s}} \circ \mathfrak{p}_{\mathfrak{d}'_k}^{-1} \circ \alpha \circ p_{\gamma_2,\mathfrak{D}_\mathbf{s}} \circ \alpha^{-1} \circ \mathfrak{p}_{\mathfrak{d}'_k},
\end{aligned}$$

and showing that $\mathfrak{p}_{\gamma,\mathfrak{D}_\mathbf{s}} = \mathfrak{p}_{\gamma,T_k(\mathfrak{D}_\mathbf{s})}$ reduces to showing that $\alpha^{-1} \circ \mathfrak{p}_{\mathfrak{d}'_k} = \mathfrak{p}_{\mathfrak{d}_k}$. Using the fact that $a_{k,i} = a_{k,r_k-i}$, observe that

$$\begin{aligned}
\alpha^{-1}\left(\mathfrak{p}_{\mathfrak{d}'_k}(z^m)\right) &= \alpha^{-1}\left(z^m \left(1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_kv_k}\right)^{-\langle d_ke_k,m\rangle}\right)\\
&= z^{m-r_kv_k\langle d_ke_k,m\rangle}\left(1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_kv_k}\right)^{-\langle d_ke_k,m\rangle}\\
&= z^m\left(z^{r_kv_k}\left(1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_kv_k}\right)\right)^{-\langle d_ke_k,m\rangle}\\
&= z^m\left(z^{r_kv_k} + a_{k,1}z^{(r_k-1)v_k} + \cdots + a_{k,r_k-1}z^{v_k} + 1\right)^{-\langle d_ke_k,m\rangle}\\
&= z^m\left(z^{r_kv_k} + a_{k,r_k-1}z^{(r_k-1)v_k} + \cdots + a_{k,1}z^{v_k} + 1\right)^{-\langle d_ke_k,m\rangle}\\
&= \mathfrak{p}_{\mathfrak{d}_k}\left(z^m\right),
\end{aligned}$$

as desired. As such, we have that $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}} = \mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})}$ and therefore $\mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})} = \mathrm{id}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ is consistent.

Next, we want to show that $T_k(\mathfrak{D}_{\mathbf{s}})$ and $\mathfrak{D}_{\mathbf{s}'}$ have, up to equivalence, the same set of slabs and incoming walls. Recall that $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ contains only the slab and incoming walls of $\mathfrak{D}_{\mathbf{s}'}$, so it will suffice to show that the incoming walls and slab of $T_k(\mathfrak{D}_{\mathbf{s}})$ appear in $\mathfrak{D}_{\mathrm{in},\mathbf{s}}$.

First, observe that if $\mathfrak{d} \subseteq n^{\perp}$ is an outgoing wall in $\mathfrak{D}_{\mathbf{s}}$, then it's mapped to an outgoing wall in $T_k(\mathfrak{D}_{\mathbf{s}})$. This follows from the definition - recall that $\mathfrak{d}$ is outgoing if $p_1^*(n) \notin \mathfrak{d}$. Because T_k is injective, having $p_1^*(n) \notin \mathfrak{d}$ implies $T_k(p_1^*(n)) \notin T_k(\mathfrak{d})$. Hence, we need only consider the slab and incoming walls of $T_k(\mathfrak{D}_{\mathbf{s}})$. Equivalently, we consider the walls of $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$.

Let $v_i' = p^*(e_i')$. Because $e_k' = -e_k$, observe that the slab for $\mathbf{s}'$ is

$$\begin{aligned}\mathfrak{d}_k' &= \left((e_k')^{\perp}, 1 + a_{k,1}z^{v_k'} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k'} + z^{r_k v_k'}\right)\\ &= \left(e_k^{\perp}, 1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_k v_k}\right),\end{aligned}$$

which appears in both $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ and $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$ by definition.

Next, we consider the walls $\mathfrak{d}_i = (e_i^{\perp}, 1 + a_{i,1}z^{v_i} + \cdots + a_{i,r_k-1}z^{(r_i-1)v_i} + z^{r_i v_i})$ for $i \neq k$. To do so, we need to divide our argument into three cases based on whether $\langle v_i, e_k\rangle$ is positive, zero, or negative. Because $\mathfrak{d}_i$ is an incoming wall, it will necessarily lie in both $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$.

Case 1: If $\langle e_k, v_i\rangle = 0$, then the two halves of $\mathfrak{d}_i \in \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ are mapped to the walls

$$\begin{aligned}&((e_i^{\perp} \cap \mathcal{H}_{k,+}), 1 + a_{i,1}z^{T_{i,+}(v_i)} + \cdots + a_{i,r_i-1}z^{T_{k,+}((r_i-1)v_i)} + z^{T_{k,+}(r_i v_i)}),\\ &((e_i^{\perp} \cap \mathcal{H}_{k,-}), 1 + a_{i,1}z^{T_{k,-}(v_i)} + \cdots + a_{i,r_i-1}z^{T_{k,-}((r_i-1)v_i)} + z^{T_{k,-}(r_i v_i)})\end{aligned}$$

whose union is the wall

$$\left((e_i)^{\perp}, 1 + a_{i,1}z^{v_i} + \cdots + a_{i,r_i-1}z^{(r_i-1)v_i} + z^{r_i v_i}\right)$$

because having $\langle v_i, e_k\rangle = 0$ means that $v_i' = T_{k,\pm}(v_i) = v_i$. Because $e_i' = e_i$, the above wall in $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$ is the same as the wall

$$\left((e_i')^{\perp}, 1 + a_{i,1}z^{v_i'} + \cdots + a_{i,r_i-1}z^{(r_i-1)v_i'} + z^{r_i v_i'}\right),$$

which we know by definition appears in $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$.

Case 2: Suppose $\langle e_k, v_i\rangle > 0$. We must consider where $\mathfrak{d}_i \cap \mathcal{H}_{k,+}$ is mapped by T_k. This portion of $\mathfrak{d}_i$ becomes the wall

$$\mathfrak{d}_{i,+}' := \left(T_k(\mathcal{H}_{k,+} \cap e_i^{\perp}), 1 + a_{i,1}z^{T_{i,+}(v_i)} + \cdots + a_{i,r_i-1}z^{T_{k,+}((r_i-1)v_i)} + z^{T_{k,+}(r_i v_i)}\right)$$

in $T_k(\mathfrak{D}_{\mathbf{s}})$. To see that $\mathfrak{d}_{i,+}'$ is incoming in $T_k(\mathfrak{D})_{\mathrm{in},\mathbf{s}}$, observe that $p_1^*(e_i) = v_i \in (\mathcal{H}_{k,+} \cap e_i^{\perp})$ and therefore $T_k(p_1^*(e_i)) = T_k(v_i) \in \mathfrak{d}_{i,+}'$. To argue that $\mathfrak{d}_{i,+}'$ also appears as an incoming wall in $\mathfrak{D}_{\mathbf{s}'}$, we need to show that $T_k(\mathcal{H}_{k,+} \cap e_i^{\perp}) \subseteq (e_i')^{\perp}$ and that $T_{k,+}(v_i) = v_i'$. Observe that for $m \in \mathcal{H}_{k,+} \cap e_i^{\perp}$,

$$\begin{aligned}
\langle e_i', T_k(m)\rangle &= \langle e_i + r_k[\epsilon_{ik}]_+ e_k, m + r_k v_k \langle d_k e_k, m\rangle\rangle \\
&= \langle e_i, m\rangle + \langle e_i, r_k v_k \langle d_k e_k, m\rangle\rangle + \langle r_k[\epsilon_{ik}]_+ e_k, m\rangle + \langle r_k[\epsilon_{ik}]_+ e_k, r_k v_k \langle d_k e_k, m\rangle\rangle \\
&= r_k \langle d_k e_k, m\rangle \langle e_i, v_k\rangle + r_k[\epsilon_{ik}]_+ \langle e_k, m\rangle \\
&= r_k \langle d_k e_k, m\rangle \langle e_i, p_1^*(e_k)\rangle + r_k d_k \{e_i, e_k\} \langle e_k, m\rangle \\
&= r_k d_k \{e_k, e_i\} \langle e_k, m\rangle + r_k d_k \{e_i, e_k\} \langle e_k, m\rangle \\
&= r_k d_k \langle e_k, m\rangle \left(\{e_k, e_i\} + \{e_i, e_k\}\right) \\
&= 0
\end{aligned}$$

and therefore $T_k(m) \in (e_i')^{\perp}$. Next, observe that

$$\begin{aligned}
T_{k,+}(v_i) &= v_i + r_k v_k \langle d_k e_k, v_i\rangle \\
&= v_i + r_k d_k v_k \langle e_k, p_1^*(e_i)\rangle \\
&= v_i + r_k d_k v_k \{e_i, e_k\} \\
&= v_i + r_k \epsilon_{ik} v_k \\
&= p_1^*(e_i) + r_k \epsilon_{ik} p_1^*(e_k) \\
&= \{e_i, \cdot\} + r_k \epsilon_{ik} \{e_k, \cdot\} \\
&= \{e_i + r_k \epsilon_{ik} e_k, \cdot\} \\
&= \{e_i', \cdot\} \\
&= p_1^*(e_i') \\
&= v_i'
\end{aligned}$$

As such, $\mathfrak{d}_i \cap \mathcal{H}_{k,+} \in \mathfrak{D}_{\mathbf{s}}$ is mapped to the wall

$$\mathfrak{d}_i' = \left(T_k(\mathcal{H}_{k,+} \cap e_i^{\perp}), 1 + a_{i,1} z^{v_i'} + \cdots + a_{i,r_i-1} z^{(r_i-1)v_i'} + z^{v_i'}\right) \in T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}}),$$

which is half of the wall

$$\left((e_i')^{\perp}, 1 + a_{i,1} z^{v_i'} + \cdots + a_{i,r_i-1} z^{(r_i-1)v_i'} + z^{v_i'}\right) \in \mathfrak{D}_{\mathrm{in},\mathbf{s}'}.$$

Case 3: Finally, let $\langle e_k, v_i\rangle < 0$. The half of $\mathfrak{d}_i$ with support $\mathfrak{d}_i \cap \mathcal{H}_{k,-}$ is mapped by T_k to

$$\mathfrak{d}_{i,-}' := \left(T_k(\mathcal{H}_{k,-} \cap e_i^{\perp}), 1 + a_{i,1} z^{T_{k,-}(v_i)} + \cdots + a_{i,r_i-1} z^{T_{k,-}((r_i-1)v_i)} + z^{T_{k,-}(r_i v_i)}\right)$$

Since $T_{k,-}(m) = m$ for $m \in \mathcal{H}_{k,-} \cap e_i^{\perp}$ and $T_{k,-}(v_i) = v_i$, we have

$$\mathfrak{d}_{i,-}' = \left(\mathcal{H}_{k,-} \cap e_i^{\perp}, 1 + a_{i,1} z^{v_i} + \cdots + a_{i,r_i-1} z^{(r_i-1)v_i} + z^{(r_i v_i)}\right).$$

Because $\langle e_k, v_i \rangle = \{e_i, e_k\} < 0$, we know that $\epsilon_{ik} = d_k\{e_i, e_k\} < 0$. Therefore,

$$e'_i = e_i + r_k[\epsilon_{ik}]_+ e_k = e_i$$
$$v'_i = p_1^*(e'_i) = p_1^*(e_i) = v_i$$

and so $\mathfrak{d}'_{i,-}$ is simply half of the wall

$$((e'_i)^\perp, 1 + a_{i,1}z^{v'_i} + \cdots + a_{i,r_i-1}z^{(r_i-1)v'_i} + z^{r_i v'_i}) \in \mathfrak{D}_{\mathrm{in},\mathbf{s}'}.$$

Hence, we see that after dividing some of the walls of $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ into two halves, the diagrams $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$ and $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ have the same set set of incoming walls. Therefore, up to the same halving of walls, the diagrams $T_k(\mathfrak{D}_{\mathbf{s}})$ and $\mathfrak{D}_{\mathbf{s}'}$ also have the same set of incoming walls. $\square$

4.4 Principal coefficients

The principal coefficient case in the generalized setting is similar to that in the ordinary setting. We begin by making the analogous modifications to the definitions of the fixed and generalized torus seed data.

Definition 4.4.1. *Given generalized fixed data Γ, the generalized fixed data Γ_{prin} for the principal coefficient case is defined in the same way as for ordinary cluster algebras, with the additional requirement that $\widetilde{r} = (r, r)$, i.e. that $\widetilde{r}$ consists of two copies of r with $\widetilde{r}_i = r_i$ for $i \in I$.*

Definition 4.4.2. *Given a generalized torus seed $\mathbf{s}$, the generalized torus seed with principal coefficients is defined as*

$$\mathbf{s}_{prin} := \widetilde{\mathbf{s}} = \{((e_i, 0), \mathbf{a}_i), ((0, f_i), \mathbf{a}_i)\}_{i \in I}$$

We can then use these updated definitions to define the cluster varieties with principal coefficients. Recall that we work over the ring $R = \Bbbk[a_{i,j}]$.

Definition 4.4.3. *Given a generalized torus seed $\mathbf{s}$, we define the associated algebraic tori*

$$\mathcal{X}_{\mathbf{s}_{prin}} := T_{\widetilde{M}}(R) = Spec\ \Bbbk[\widetilde{N}] \times_{\mathbb{C}} Spec(R),$$
$$\mathcal{A}_{\mathbf{s}_{prin}} := T_{\widetilde{N}^\circ}(R) = Spec\ \Bbbk[\widetilde{M^\circ}] \times_{\mathbb{C}} Spec(R).$$

The $\mathcal{A}$-generalized cluster variety with principal coefficients and $\mathcal{X}$-generalized cluster variety with principal coefficients are then defined as in the ordinary case.

As before, the $\mathcal{A}$-generalized cluster variety is given by the fiber $\mathcal{A}_e$ and the $\mathcal{X}$-generalized cluster variety is given by the quotient $\mathcal{A}_{\mathrm{prin}}/T_{N^\circ}$.

The many important relationships between the various types of ordinary cluster varities also exist between the various types of generalized cluster varieties. These relationships are summarized in the following proposition:

Proposition 4.4.4 (Analogue of Proposition B.2 of [41])**.** *Given a set of generalized fixed data Γ:*

1. *There is the following commutative diagram, where the dotted arrows are only present if there are no frozen variables:*

$$\begin{tikzcd}
\mathcal{A}_t \arrow[r] \arrow[d] & \mathcal{A}_{prin} \arrow[r, "\widetilde{p}"] \arrow[d, "\pi"] \arrow[rr, bend left, "p"] & \mathcal{X} \arrow[d, "\lambda"] & \mathcal{X}_{prin} \arrow[l, "\rho"'] \arrow[d, "w"] & \mathcal{A} \arrow[l, "\xi"'] \arrow[d] \\
t \arrow[r] & T_M \arrow[r, dashed] & T_{K^*} & T_M \arrow[l, dashed] & e \arrow[l]
\end{tikzcd}$$

where t is any point in T_M, $e \in T_M$ is the identity, and p is an isomorphism which is canonical if there are no frozen variables.

2. *There is a torus action of T_{N° on $\mathcal{A}_{prin}$; T_{K° on $\mathcal{A}$; $T_{N_{uf}^\perp}$ on $\mathcal{X}$; and $T_{\widetilde{K}^\circ}$ on $\mathcal{A}_{prin}$, where $\widetilde{K}^\circ$ is the kernel of the map $N^\circ \oplus M \to N_{uf}^*$ given by $(n, m) \mapsto p_2^*(n) - m$. The action of T_{N° and $T_{\widetilde{K}^\circ}$ on T_M is such that the map $\pi : \mathcal{A}_{prin} \to T_M$ is T_{N°-equivariant and $T_{\widetilde{K}^\circ}$-equivariant. The map $\widetilde{p} : \mathcal{A}_{prin} \to \mathcal{X} = \mathcal{A}_{prin}/T_{N^\circ}$ is a T_{N°-torsor. Furthermore, there is a map $T_{\widetilde{K}^\circ} \to T_{N_{uf}^\perp}$ such that $\widetilde{p}$ is compatible with the actions of these tori on, respectively, $\mathcal{A}_{prin}$ and $\mathcal{X}$. Hence, $\tau : \mathcal{A}_{prin} \to \mathcal{X}/T_{N_{uf}^\perp}$ is a $T_{\widetilde{K}^\circ}$-torsor.*

Proof. For (1), we must first specify the named maps. At the level of the cocharacter lattices, these maps are:

$$\begin{aligned}
\pi : N^\circ \oplus M \to M, &\qquad (n, m) \mapsto m \\
\widetilde{p} : N^\circ \oplus M \to M, &\qquad (n, m) \mapsto m - p^*(n) \\
\rho : M \oplus N^\circ \to M, &\qquad (m, n) \mapsto m \\
\lambda : M \to K^*, &\qquad m \mapsto m|_K \\
w : M \oplus N^\circ \to M, &\qquad (m, n) \mapsto m - p^*(n) \\
\xi : N^\circ \to M \oplus N^*, &\qquad n \mapsto (-p^*(n), -n) \\
p : N^\circ \oplus M \to M \oplus N^\circ, &\qquad (n, m) \mapsto (m - p^*(n), n)
\end{aligned}$$

Observe that the map λ is the transpose of the inclusion $K \hookrightarrow N$. When there are no frozen variables (i.e., when $N_{\mathrm{uf}} = N$), the maps corresponding to the dotted lines are simply given on the cocharacter lattices by λ. It is straightforward to check commutativity using the above formulas for the maps. It is likewise straightforward to check that the map p is a lattice isomorphism and therefore induces an isomorphism of the associated tori.

For (2), we must specify the torus actions. The torus action of T_{N° on $\mathcal{A}_{\mathrm{prin}}$ is given at the level of cocharacter lattices by

$$\begin{aligned}
N^\circ &\to N^\circ \oplus M, \\
n &\mapsto (n, p^*(n))
\end{aligned}$$

The other torus actions are all given at the level of cocharacter lattices by inclusions. So the action of T_{K° on $\mathcal{A}$ is given by $K^\circ \hookrightarrow N^\circ$, the action of $T_{N_{\mathrm{uf}}^\perp}$ on $\mathcal{X}$ is given by $N_{\mathrm{uf}}^\perp \hookrightarrow M$, and the action of $T_{\widetilde{K}^\circ}$ on $\mathcal{A}_{\mathrm{prin}}$ is given by $\widetilde{K}^\circ \hookrightarrow N^\circ \oplus M$. It is straightforward to check that the induced actions are compatible with mutations.

The action of T_{N° on T_M is given on the level of the cocharacter lattice by the map $n \mapsto p^*(n)$. The action of $T_{\widetilde{K}^\circ}$ on T_M is given on the level of the cocharacter lattice by $(n, m) \mapsto m$. It is straightforward to check that the induced actions of T_{N° and $T_{\widetilde{K}^\circ}$ are equivariant and that $\widetilde{p} : \mathcal{A}_{\text{prin}} \to \mathcal{X}$ is a $T_{T_{N^\circ}}$-torsor.

Finally, the map $T_{\widetilde{K}^\circ} \to T_{N_{\text{uf}}^\perp}$ is given on the level of the cocharacter lattice by $(m, n) \mapsto m - p^*(n)$. It is straightforward to check that this map is compatible with the actions of $T_{\widetilde{K}^\circ}$ on $\mathcal{A}_{\text{prin}}$ and $T_{N_{\text{uf}}^\perp}$ on $\mathcal{X}$ and that the map τ is a $T_{\widetilde{K}^\circ}$. $\square$

Next, we update the definition of the initial cluster scattering diagram.

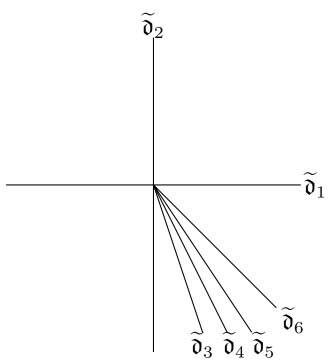
Definition 4.4.5. *Given a generalized seed* $\mathbf{s}$*, let* $\widetilde{v_i} := (v_i, e_i) = (p_1^*(e_i), e_i)$*. Then we can define*

$$\mathfrak{D}_{in,\mathbf{s}}^{\mathcal{A}_{prin}} = \left\{ \left((e_i, 0)^\perp, 1 + a_{i,1} z^{\widetilde{v_1}} + \cdots + a_{i,r_i-1} z^{(r_i-1)\widetilde{v_i}} + z^{r_i \widetilde{v_i}} \right) \right\}$$

Example 4.4.6. *Consider the generalized cluster algebra with* $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$*,* $d = (1, 1)$*, and* $R = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$*, as in Example 4.1.4. The generalized fixed data* Γ_{prin} *has index set* $\widetilde{I} = I \sqcup I$*,* $\widetilde{I}_{uf} = \{1, 2\}$*,* $\widetilde{d} = (1, 1, 1, 1)$*,* $\widetilde{r} = (3, 1, 3, 1)$ *and lattices* $\widetilde{N} = N \oplus M^\circ$*,* $\widetilde{N}^\circ = N^\circ \oplus M$*,* $\widetilde{M} = M \oplus N^\circ$*, and* $\widetilde{M}^\circ = M^\circ \oplus N$*, all of which have basis* $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$*. By definition, we have*

$$\mathfrak{D}_{in,\mathbf{s}}^{\mathcal{A}_{prin}} = \left\{ \begin{array}{c} \widetilde{\mathfrak{d}}_1 = \left((0,1,0,0)^\perp, 1 + z^{(-1,0,0,1)} \right), \\ \widetilde{\mathfrak{d}}_2 = \left((1,0,0,0)^\perp, 1 + a z^{(0,1,1,0)} + a z^{(0,2,2,0)} + z^{(0,3,3,0)} \right) \end{array} \right\}$$

which can be completed, using the definition of consistency, to give a scattering diagram for $\mathcal{A}_{prin}$*, denoted* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$*. Because this diagram is four dimensional, however, we draw scattering diagrams for these cluster varieties with principal coefficients by projecting onto* M°*. The wall* $\widetilde{\mathfrak{d}}_1 \in \mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$ *has support* $(0,1,0,0)^\perp \subset \mathbb{R}^4$*, i.e. its support is the three-dimensional hyperplane* $\langle (1,0,0,0), (0,0,1,0), (0,0,0,1) \rangle$*. When we project the diagram from* $\widetilde{M}_{\mathbb{R}}^\circ$ *onto* $M_{\mathbb{R}}^\circ$*, the wall* $\widetilde{\mathfrak{d}}_1$ *is projected onto* $\mathbb{R}(1,0) \subset \mathbb{R}^2$*, i.e. the one-dimensional hyperplane* $\langle (1,0) \rangle$*. Similarly,* $\widetilde{\mathfrak{d}}_2$ *is projected onto* $\mathbb{R}(0,1)$*, i.e. the one-dimensional hyperplane* $\langle (0,1) \rangle$*. Applying this projection to each wall of* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$*, we obtain the diagram:*



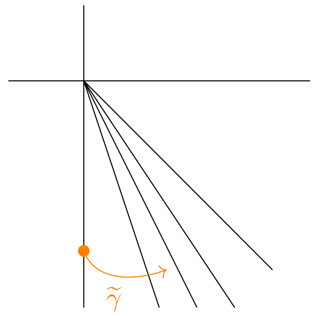
$$\begin{aligned} f_{\widetilde{\mathfrak{d}}_1} &= 1 + z^{(-1,0,0,1)} \\ f_{\widetilde{\mathfrak{d}}_2} &= 1 + a z^{(0,1,1,0)} + a z^{(0,2,2,0)} + z^{(0,3,3,0)} \\ f_{\widetilde{\mathfrak{d}}_3} &= 1 + z^{(-1,3,3,1)} \\ f_{\widetilde{\mathfrak{d}}_4} &= 1 + a z^{(-1,2,2,1)} + a z^{(-2,4,4,2)} + z^{(-3,6,6,3)} \\ f_{\widetilde{\mathfrak{d}}_5} &= 1 + z^{(-2,3,3,2)} \\ f_{\widetilde{\mathfrak{d}}_6} &= 1 + a z^{(-1,1,1,1)} + a z^{(-2,2,2,2)} + z^{(-3,3,3,3)} \end{aligned}$$

We can obtain a diagram for $\mathcal{X}_{\mathbf{s}}$*, denoted* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{X}}$*, from* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$ *by taking the slice* $\{m \in M^\circ : m = p_1^*(n)\}$ *of* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$*.*

Suppose $\mathfrak{d}_k$ *is a wall in* $\mathfrak{D}^{\mathcal{X}_{\mathbf{s}}}$ *with support given by either* $\mathbb{R} \cdot n$ *or* $\mathbb{R}_{\geq 0} \cdot n$ *for* $n \in N$*. The wall* $\mathfrak{d}_k$ *corresponds to a wall* $\widetilde{\mathfrak{d}}_k$ *in* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$ *with support either* $\mathbb{R} \cdot (p_1^*(n), n)$ *or* $\mathbb{R}_{\geq 0} \cdot (p_1^*(n), n)$*, respectively.*

To determine the wall-crossing automorphism $f_{\mathfrak{d}_k}$*, we compute a path-ordered product* $\mathfrak{p}_{\widetilde{\gamma}}$ *in* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$*, where* $\widetilde{\gamma}$ *is a path crossing only* $\widetilde{\mathfrak{d}}_k$*, and then take the slice* $\{m \in M^\circ : m = p_1^*(n)\}$*.*

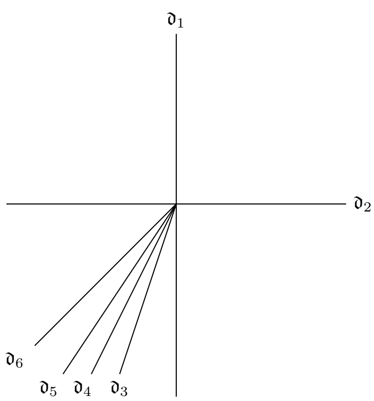
For example, consider the wall $\widetilde{\mathfrak{d}}_3$ *and the following path* $\widetilde{\gamma}$ *in* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$*:*



By definition, we compute

$$\begin{aligned}\mathfrak{p}_{\widetilde{\gamma}}\left(z^{(0,-1,-1,0)}\right) &= z^{(0,-1,-1,0)}\left(1+z^{(-1,3,3,1)}\right)^{\langle(0,-1,-1,0),(-3,-1,0,0)\rangle}\\ &= z^{(0,-1,-1,0)}\left(1+z^{(-1,3,3,1)}\right)\end{aligned}$$

The slice $\{m \in M^\circ : m = p_1^*(n)\}$ *gives us the path-ordered product* $\mathfrak{p}_\gamma(z^{(-1,0)}) = z^{(0,-1)}(1+z^{(3,1)})$ *in* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{X}}$*, from which we can read off the wall-crossing automorphism for* $\mathfrak{d}_3$ *as* $f_{\mathfrak{d}_3} = 1+z^{(3,1)}$*. Similar computations for the remaining walls allow us to draw* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{X}}$*, shown below.*



$$\begin{aligned}f_{\mathfrak{d}_1} &= 1+z^{(0,1)}\\ f_{\mathfrak{d}_2} &= 1+az^{(1,0)}+az^{(2,0)}+z^{(3,0)}\\ f_{\mathfrak{d}_3} &= 1+z^{(3,1)}\\ f_{\mathfrak{d}_4} &= 1+az^{(2,1)}+az^{(4,2)}+z^{(6,3)}\\ f_{\mathfrak{d}_5} &= 1+z^{(3,2)}\\ f_{\mathfrak{d}_6} &= 1+az^{(1,1)}+az^{(2,2)}+z^{(3,3)}\end{aligned}$$

Example 4.4.7. *Consider a generic generalized cluster algebra of rank 2 with* $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$*,* $d = (b,c)$*, and* $r = (r_1, r_2)$*. Then* $\mathfrak{D}_{in}$ *for* $\mathcal{A}_{prin}$ *is*

$$\left\{\begin{array}{l}((0,1),\ 1+a_{2,1}z^{(0,c,1,0)}+\cdots+a_{2,r_1-1}z^{(r_1-1)(0,c,1,0)}+z^{r_1(0,c,1,0)})),\\ ((1,0),\ 1+a_{1,1}z^{(-b,0,0,1)}+\cdots+a_{1,r_2-1}z^{(r_2-1)(-b,0,0,1)}+z^{r_2(-b,0,0,1)})\end{array}\right\}$$

Taking the slice $\{m \in M^\circ : m = p_1^*(n)\}$*, we find that* $\mathfrak{D}_{in}$ *for the* $\mathcal{X}$ *diagram is*

$$\left\{\begin{array}{l}((0,1),(1+a_{1,1}z^{(0,1)}+\cdots+a_{1,r_2-1}z^{(r_2-1)(0,1)}+z^{r_2(0,1)})^c),\\ ((1,0),(1+a_{2,1}z^{(1,0)}+\cdots+a_{2,r_1-1}z^{(r_1-1)(1,0)}+z^{r_1(1,0)})^b)\end{array}\right\}$$

Note that in Examples 4.4.6 and 4.4.7, the $\mathcal{X}$ scattering diagrams have the same dimension as the $\mathcal{A}$ scattering diagrams because p_1^* is injective in the principal coefficient case. Otherwise, these diagrams may not necessarily have the same dimension.

Remark 4.4.8. *Given $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{prin}}$, one can obtain the equation for mutation in direction k by computing the wall-crossing formula when crossing $\mathfrak{d}_k$ from the positive chamber. Recall that*

$$\mathfrak{d}_k = (e_k^{\perp}, 1 + a_{k,1} z^{(v_k,e_k)} + \cdots + a_{k,r_k-1} z^{(r_k-1)(v_k,e_k)} + z^{r_k(v_k,e_k)}),$$

so for $m = p_1^*(n) \in \mathcal{C}_{\mathbf{s}}^+$, crossing $\mathfrak{d}_k$ gives

$$z^{(m,n)} \mapsto z^{(m,n)} \left(1 + a_{k,1} z^{(v_k,e_k)} + \cdots + a_{k,r_k-1} z^{(r_k-1)(v_k,e_k)} + z^{r_k(v_k,e_k)}\right)^{-\langle (d_k e_k, 0),(m,n)\rangle}$$

Taking the slices $\{m \in M^\circ : m = p_1^*(n)\}$ and $\{n \in N : p_1^*(n) \in M^\circ\}$ yields the maps from Definition 4.1.7:

$$\begin{aligned}
z^m &\mapsto z^m \left(1 + a_{k,1} z^{v_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)v_k} + z^{v_k}\right)^{-\langle d_k e_k, m\rangle} \\
z^n &\mapsto z^n \left(1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-\langle d_k e_k, m\rangle} \\
&= z^n \left(1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-\{n, d_k e_k\}} \\
&= z^n \left(1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-[n, e_k]}
\end{aligned}$$

4.5 Chamber structure

In this section, we discuss the chamber structure of generalized cluster scattering diagrams. Because this structure is analogous to the chamber structure of ordinary scattering diagrams, our discussion largely reviews that of [41] for the ordinary case.

As in Section 2.3, let $\mathfrak{T}$ be a directed infinite rooted tree where each vertex has $|I_{\mathrm{uf}}|$ outgoing edges, which are labeled by the elements of I_{uf} such that each vertex has only one incident edge with a given label. Let v be the root of the tree and associate an initial generalized torus seed $\mathbf{s}$ with mutation class $[\mathbf{s}]$ to v. As before, we indicate this choice of initial seed by writing $\mathfrak{T}_v$ or $\mathfrak{T}_{\mathbf{s}}$. Let an edge with label $k \in I_{\mathrm{uf}}$ correspond to mutation in direction k, so any simple path beginning at vertex v determines a mutation sequence according to the attached edge labels. These mutation sequences then determined associated generalized torus seeds for each vertex.

Let $w \neq v$ be an arbitrary vertex in $\mathfrak{T}_{\mathbf{s}}$. Then the sequence of edge labels $k_1, \ldots, k_\ell$ on a simple path between v and w determine a map $T_w = T_{k_\ell} \circ \cdots T_{k_1} : M_{\mathbb{R}} \to M_{\mathbb{R}}$ where each T_{k_i} is defined with respect to the basis vector e_{k_i} in the mutated generalized torus seed $\mu_{k_{i-1}} \circ \cdots \circ \mu_{k_1}(\mathbf{s})$ rather than the original generalized torus seed $\mathbf{s}$. It follows from Theorem 4.3.5 that $T_w(\mathfrak{D}_{\mathbf{s}}) = \mathfrak{D}_{\mathbf{s}_w}$, where $\mathbf{s}_w$ denotes a generalized torus seed associated to the vertex w.

Let Σ be a set of generalized fixed data which satisfies the injectivity assumption and $\mathbf{s}$ be a choice of associated generalized torus seed. Although there may be multiple equivalent representatives of the associated generalized cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$, by construction every representative must include a collection of incoming walls with support $\{e_k^\perp\}_{k\in I_{\mathrm{uf}}}$. As in the ordinary case, we define

$$\mathcal{C}_{\mathbf{s}}^+ := \{m \in M_{\mathbb{R}} : \langle e_i, m\rangle \geq 0 \text{ for all } i \in I_{\mathrm{uf}}\},$$
$$\mathcal{C}_{\mathbf{s}}^- := \{m \in M_{\mathbb{R}} : \langle e_i, m\rangle \leq 0 \text{ for all } i \in I_{\mathrm{uf}}\}.$$

We refer to $\mathcal{C}_{\mathbf{s}}^+$ as the *positive chamber*. Observe that the chambers $\mathcal{C}_{\mathbf{s}}^\pm$ are closures of connected components of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$. Further, let $\mathcal{C}_{\mu_k(\mathbf{s})}^\pm$ denote the chambers where either all $\langle e_i', m\rangle \geq 0$ or $\langle e_i', m\rangle \leq 0$, respectively. Then we observe that $\mathcal{C}_{\mu_k(\mathbf{s})}^\pm$ is similarly the closure of a connected component of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mu_k(\mathbf{s})})$. Hence, $T_k^{-1}(\mathcal{C}_{\mu_k(\mathbf{s})}^\pm)$ is the closure of a connected component of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$ which shares a codimension one face, given by $e_k^\perp$, with $\mathcal{C}_{\mathbf{s}}^\pm$.

The same reasoning can be extended to generalized torus seeds which are related to $\mathbf{s}$ by longer mutation sequences. Let w be a vertex of $\mathfrak{T}_{\mathbf{s}}$ which is reachable from the root vertex via a simple path of arbitrary length. Earlier, we observed that this simple path defines a map $T_w : M_{\mathbb{R}} \to M_{\mathbb{R}}$ and that $T_w(\mathfrak{D}_{\mathbf{s}}) = \mathfrak{D}_{\mathbf{s}_w}$. It also follows from Theorem 4.3.5 and the previous paragraph that $\mathcal{C}_w^\pm := T_w^{-1}\left(\mathcal{C}_{\mathbf{s}_w}^\pm\right)$ is a closure of a connected component of $M_{\mathbb{R}}\backslash\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$.

It is important to note, however, that the collection of cones $C_v^\pm$ will not always form a dense subset of $M_{\mathbb{R}}$.

Definition 4.5.1. *Let $\mathcal{C}_v^\pm$ denote the chamber of Supp$(\mathfrak{D}_{\mathbf{s}})$ which corresponds to the vertex $v \in \mathfrak{T}_{\mathbf{s}}$ and $\Delta_{\mathbf{s}}^\pm$ denote the collection of chambers $\mathcal{C}_v^\pm$ as v runs over the vertices of $\mathfrak{T}_{\mathbf{s}}$. We refer to elements of $\Delta_{\mathbf{s}}^+$ as* cluster chambers*.*

Gross, Hacking, Keel, and Kontsevich [41] showed that the chamber structure of ordinary cluster scattering diagrams coincides with the Fock-Goncharov cluster complex. In the generalized setting, we can state the following natural analogue of the definition of the Fock-Goncharov cluster complex.

Definition 4.5.2 (Analogue of Definition 2.14 of [20], Definition 2.9 of [41])**.** *Fix a set of generalized fixed data Σ and an associated generalized torus seed $\mathbf{s}$. Then for a generalized torus seed $\mathbf{s}' = \{(e_i', (a_{i,j}'))\}$ which is reachable via a mutation sequence from $\mathbf{s}$, the* Fock-Goncharov cluster chamber associated to $\mathbf{s}'$ *is the subset*

$$\{x \in \mathcal{A}^\vee(\mathbb{R}^T) : (z^{e_i'})^T(x) \leq 0 \textit{ for all } i \in I_{uf}\},$$

which is identified with the subset

$$\{x \in \mathcal{A}^\vee(\mathbb{R}^t) : (z^{e_i'})^t(x) \leq 0 \textit{ for all } i \in I_{uf}\}\}$$

via the canonical sign-change map $i : \mathcal{A}^\vee(\mathbb{R}^T) \to \mathcal{A}^\vee(\mathbb{R}^t)$.

From this definition, we then obtain the following analogous identification.

Lemma 4.5.3 (Analogue of Lemma 2.10 of [41])**.** *Let Σ be a set of generalized fixed data which satisfies the injectivity assumption and $\mathbf{s}$ be an accompanying choice of generalized torus seed. Let $\mathbf{s}' = \{(e_i', (a_{i,j}'))\}$ be a distinct generalized torus seed which is reachable via some mutation sequence from $\mathbf{s}$. Then the positive chamber $\mathcal{C}_{\mathbf{s}'}^+ \subset M_{\mathbb{R},\mathbf{s}'}^\circ = \mathcal{A}^\vee(\mathbb{R}^T)$ (which can be identified with $\mathcal{A}^\vee(\mathbb{R}^t)$ via the sign-change map i) is the Fock-Goncharov cluster chamber associated to $\mathbf{s}'$. Therefore, the Fock-Goncharov cluster chambers are the maximal cones of a simplicial fan and Δ^+ is identified* *with $\Delta^+_{\mathbf{s}}$ for every choice of generalized torus seed* $\mathbf{s}$ *which gives an identification of $\mathcal{A}^\vee(\mathbb{R}^T)$ with $M^\circ_{\mathbb{R},\mathbf{s}}$*

Proof. The proof given by [41] holds in the generalized setting, because we showed in Proposition 4.3.4 that our modified T_k map is the Fock-Goncharov tropicalization of the generalized mutation map $\mu_{(v_k,d_ke_k,r_k)}$. □

As in the ordinary case, it follows from the previous Proposition, when the injectivity assumption holds, that:

Theorem 4.5.4 (Analogue of Theorem 2.13 of [41])**.** *For any set of initial data, the Fock-Goncharov cluster chambers in $\mathcal{A}^\vee(\mathbb{R}^T)$ are the maximal cones of a simplicial fan.*

4.6 Building $\mathcal{A}_{\text{scat}}$ from a generalized cluster scattering diagram

In this section, we parallel the exposition in Section 4 of [41], which describes how to build the space $\mathcal{A}_{\text{scat}}$ from an ordinary scattering diagram and then identifies $\mathcal{A}_{\text{scat}}$ with the $\mathcal{A}$ variety. We review relevant portions of their constructions and statements, pointing out where modifications are needed to extend the results to generalized cluster algebras with reciprocal coefficients.

Let Γ be a set of generalized initial data such that the diagram $\mathfrak{D}_{\mathbf{s}}$ yields a cluster chamber structure $\Delta^+_{\mathbf{s}}$. We will often want to discuss multiple copies of the lattices N, M, N°, and M° which arise from different choices of seed $\mathbf{s}$. To allow us to distinguish between these copies, we index both the seeds and lattices by either the vertices v of $\mathfrak{T}_v$ or chambers σ of $\Delta^+_{\mathbf{s}}$. For example, the seed $\mathbf{s}_v$ gives rise to the diagram $\mathfrak{D}_{\mathbf{s}_v}$ on the lattice $M^\circ_{\mathbb{R},\mathbf{s}_v}$. The chambers in $\mathfrak{D}_{\mathbf{s}_v}$ give the Fock-Goncharov cluster complex Δ^+ under the identification $M^\circ_{\mathbb{R},\mathbf{s}_v} = \mathcal{A}^\vee(\mathbb{R}^T)$. Because the space $\mathcal{A}^\vee(\mathbb{R}^T)$ is independent of the choice of the initial seed $\mathbf{s}$, there is a canonical bijection between the cluster chambers of $\mathfrak{D}_{\mathbf{s}_v}$ and $\mathfrak{D}_{\mathbf{s}_{v'}}$ as a consequence of this identification.

Definition 4.6.1 (Construction 4.1 of [41])**.** *Given a seed $\mathbf{s}$, we want to construct a space, $\mathcal{A}_{scat,\mathbf{s}}$ using the chambers $\sigma \in \Delta^+_{\mathbf{s}}$. For distinct $\sigma, \sigma' \in \Delta^+_{\mathbf{s}}$, there exists a path γ from σ' to σ. This path gives rise to an automorphism $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}} : \widehat{\Bbbk[P]} \to \widehat{\Bbbk[P]}$ which depends only on the choice of σ and σ' and is independent of choice of path.*

For each chamber in $\Delta^+_{\mathbf{s}}$, attach a copy of the torus $T_{N^\circ,\sigma} := T_{N^\circ}$. If γ is chosen such that it lies in the support of the cluster complex, then the wall-crossing automorphisms attached to walls crossed by γ are birational maps of T_{N°. Therefore the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}$ can be viewed as a well-defined map of fields of fractions $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}} : \Bbbk(M^\circ) \to \Bbbk(M^\circ)$ which induces a positive birational map $\mathfrak{p}_{\sigma,\sigma'} : T_{N^\circ,\sigma} \to T_{N^\circ,\sigma'}$.

The space $\mathcal{A}_{scat,\mathbf{s}}$ is constructed by gluing the collection of tori $\{T_{N^\circ,\sigma}\}_{\sigma\in\Delta^+_{\mathbf{s}}}$ using the birational maps $\{\mathfrak{p}_{\sigma,\sigma'}\}_{\sigma,\sigma'\in\Delta^+_{\mathbf{s}}}$ according to the method described in Proposition 2.4 of [39].

Proposition 4.6.2 (Analog of Proposition 4.3 of [41])**.** *Let $\mathbf{s}$ be a seed, v be the root of $\mathfrak{T}_{\mathbf{s}}$, and v' be any other vertex of $\mathfrak{T}_{\mathbf{s}}$. Let $\mu^T_{v',v} : M^\circ_{v'} \to M^\circ_v$ be the Fock-Goncharov tropicalization of $\mu_{v',v} : T_{M^\circ,v'} \to T_{M^\circ,v}$. The restriction $\mu^T_{v',v}\big|_{\sigma'} : M^\circ_{\sigma'} \to M^\circ_\sigma$ to each cluster chamber σ' of $\Delta^+_{\mathbf{s}_{v'}}$,* *gives a linear isomorphism from σ' to the corresponding cluster chamber $\sigma := \mu^T_{v',v}(\sigma')$ in $\Delta^+_{\mathbf{s}}$ and induces an isomorphism*

$$T_{v',\sigma} : T_{N^\circ,\sigma} \to T_{N^\circ,\sigma'}.$$

When σ ranges across all the cluster chambers of $\Delta^+_{\mathbf{s}_v}$, the isomorphisms $T_{v',\sigma}$ glue together to yield an isomorphism between $\mathcal{A}_{scat,\mathbf{s}_v}$ and $\mathcal{A}_{scat,\mathbf{s}_{v'}}$.

Proof. We follow the structure of the proof in [41]. In general, v and v' are related by a composition of mutations and $\mu_{v',v}$ is the inverse of that composition. To prove this statement for arbitrary v and v', it is sufficient to prove it for the special case where v and v' are related by a single mutation. In this case, $\mu^T_{v',v} = T_k^{-1}$ and the isomorphism $T_{v',\sigma} : T_{N^\circ,\sigma} \to T_{N^\circ,T_k(\sigma)}$ is induced by the restriction $T_k^{-1}\big|_{T_k(\sigma)}$. To show that gluing together these isomorphisms for all $\sigma \in \Delta^+_{\mathbf{s}_v}$ gives an isomorphism between $\mathcal{A}_{\mathrm{scat},\mathbf{s}_v}$ and $\mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}}$, we need to show commutativity of the diagram

$$\begin{tikzcd}
T_{N^\circ,\sigma} \arrow[r, "T_{v',\sigma}"] \arrow[d, "\mathfrak{p}_{\sigma,\tilde\sigma}"'] & T_{N^\circ,\sigma'} \arrow[d, "\mathfrak{p}_{\sigma',\tilde\sigma'}"] \\
T_{N^\circ,\tilde\sigma} \arrow[r, "T_{v',\tilde\sigma}"'] & T_{N^\circ,\tilde\sigma'}
\end{tikzcd}$$

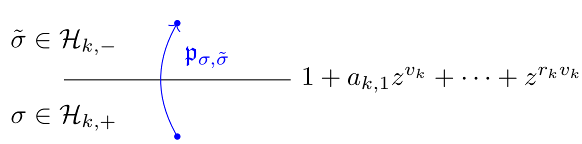
where σ and $\tilde\sigma$ are chambers in $\Delta^+_{\mathbf{s}_v}$ and $\sigma' = T_k(\sigma)$ and $\tilde\sigma' = T_k(\tilde\sigma)$ are chambers in $\Delta_{\mathbf{s}_{v'}}$. Note that the map $\mathfrak{p}_{\sigma,\tilde\sigma}$ indicates a wall-crossing in $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{p}_{\sigma',\tilde\sigma'}$ indicates a wall crossing in $\mathfrak{D}_{\mathbf{s}'}$.

If σ and $\tilde\sigma$ both fall in $\mathcal{H}_{k,-}$, then commutativity is immediate because T_k fixes the wall-crossing automorphism on the wall between σ and $\tilde\sigma$. If both chambers fall in $\mathcal{H}_{k,+}$, then commutativity follows from Theorem 4.3.5 because by definition the path-ordered product between a given pair of points is equal in equivalent diagrams. Hence, the important case to consider is when σ and $\tilde\sigma$ are on opposite sides of the wall with support $e_k^\perp$.

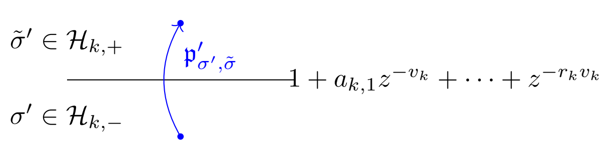
Without loss of generality, we can assume that σ is the chamber in $\mathcal{H}_{k,+}$, where e_k is non-negative. We know that the only wall in $\mathfrak{D}_{\mathbf{s}}$ contained in $e_k^\perp$ is the slab $\mathfrak{d}_k = (e_k^\perp, 1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k})$. In $\mathfrak{D}_{\mathbf{s}'}$, the slab contained in $e_k^\perp$ is

$$\mathfrak{d}'_k = (e_k^\perp, 1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_k v_k}).$$

As such, the only way for σ and $\tilde\sigma$ to be on opposite sides of $e_k^\perp$ is for the wall between them to be $\mathfrak{d}_k$ in $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{d}'_k$ in $\mathfrak{D}_{\mathbf{s}'}$. Pictorially, we can envision:



in $\mathfrak{D}_{\mathbf{s}}$ and similarly



in $\mathfrak{D}_{\mathbf{s}'}$. We can then compute that

$$
\begin{aligned}
T^*_{v',\sigma}\left(\mathfrak{p}^*_{\sigma',\tilde\sigma'}(z^m)\right) &= T^*_{v',\sigma}\left(z^m(1+a_{k,1}z^{-v_k}+\cdots+a_{k,r_k-1}z^{-(r_k-1)v_k}+z^{-r_kv_k})^{-\langle d_ke_k,m\rangle}\right)\\
&= T^*_{v',\sigma}\left(z^m\right)T^*_{v',\sigma}\left((1+a_{k,1}z^{-v_k}+\cdots+a_{k,r_k-1}z^{-(r_k-1)v_k}+z^{-r_kv_k})^{-\langle d_ke_k,m\rangle}\right)\\
&= \mu^T_{v',v}\left(z^m\right)\mu^T_{v',v}\left((1+a_{k,1}z^{-v_k}+\cdots+a_{k,r_k-1}z^{-(r_k-1)v_k}+z^{-r_kv_k})^{-\langle d_ke_k,m\rangle}\right)\\
&= T_k^{-1}\left(z^m\right)T_k^{-1}\left((1+a_{k,1}z^{-v_k}+\cdots+a_{k,r_k-1}z^{-(r_k-1)v_k}+z^{-r_kv_k})^{-\langle d_ke_k,m\rangle}\right)\\
&= z^{m-r_kv_k\langle d_ke_k,m\rangle}(1+a_{k,1}z^{-v_k}+\cdots+a_{k,r_k-1}z^{-(r_k-1)v_k}+z^{-r_kv_k})^{-\langle d_ke_k,m\rangle}\\
&= z^m\left(z^{-r_kv_k}\right)^{-\langle d_ke_k,m\rangle}(1+a_{k,1}z^{-v_k}+\cdots+z^{-r_kv_k})^{-\langle d_ke_k,m\rangle}\\
&= z^m(z^{r_kv_k}+a_{k,1}z^{(r_k-1)v_k}+\cdots+a_{k,r_k-1}z^{v_k}+1)^{-\langle d_ke_k,m\rangle}\\
&= z^m\left(1+a_{k,1}z^{v_k}+\cdots+a_{k,r_k-1}z^{(r_k-1)v_k}+z^{r_kv_k}\right)^{-\langle d_ke_k,m\rangle}\\
&= \mathfrak{p}^*_{\sigma,\tilde\sigma}\left(z^m\right)
\end{aligned}
$$

Because $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$ are reversed in $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$, our assumption that $m\in\mathcal{H}_{k,+}$ in $\mathfrak{D}_{\mathbf{s}}$ means that $m\in\mathcal{H}_{k,-}$ in $\mathfrak{D}_{\mathbf{s}'}$. As such, $z^m = T^*_{v,\tilde\sigma}(z^m)$ and we have $T^*_{v',\sigma}\left(\mathfrak{p}^*_{\sigma',\tilde\sigma'}(z^m)\right) = \mathfrak{p}^*_{\sigma,\tilde\sigma}\left(T^*_{v,\tilde\sigma}(z^m)\right)$ and the desired commutativity holds. Note that this computation relies on the reciprocity condition $a_{k,i}=a_{k,r_i-i}$. ☐

Theorem 4.6.3 (Analogue of Theorem 4.4 of [41])**.** *For a given generalized torus seed* $\mathbf{s}$*, let* v *denote the root of* $\mathfrak{T}_{\mathbf{s}}$ *and* v' *be another arbitrary vertex in* $\mathfrak{T}_{\mathbf{s}}$*. Let* $\phi^*_{v,v'}: M^\circ_{v'}\to M^\circ_{v'}$ *be the linear map* $\mu^T_{v,v'}|_{\mathcal{C}^+_{v'\in\mathbf{s}}}$ *and* $\phi_{v,v'}: T_{N^\circ,v'}\to T_{N^\circ,v'}$ *be the map between the associated tori. Then the collection* $\{\phi_{v,v'}\}_{v'}$ *glue to give an isomorphism*

$$
\mathcal{A}_{\mathbf{s}} := \bigcup_{v'} T_{N^\circ} \to \mathcal{A}_{scat,\mathbf{s}} := \bigcup_{v'} T_{N^\circ,v'}
$$

and the diagram

$$
\begin{array}{ccc}
\mathcal{A}_{\mathbf{s}} & \longrightarrow & \mathcal{A}_{scat,\mathbf{s}}\\
\downarrow & & \downarrow\\
\mathcal{A}_{\mathbf{s}_{v'}} & \longrightarrow & \mathcal{A}_{scat,\mathbf{s}_{v'}}
\end{array}
$$

commutes, where the horizontal maps are the isomorphisms that were just defined, the right-hand vertical map is the isomorphism described in Proposition 4.6.2 and the left-hand map is the natural open immersion $\mathcal{A}_{\mathbf{s}}\hookrightarrow\mathcal{A}_{\mathbf{s}_{v'}}$*.*

Proof. The proof of this theorem is identical to that of Theorem 4.4 from [41]; previous propositions check that despite the differences in wall-crossing automorphisms, we have the necessary commutativity of diagrams. ☐

This allows us to identify the rings of regular functions on $\mathcal{A}_{\text{scat}}$ and $\mathcal{A}_{\mathbf{s}}$.

Definition 4.6.4 (Definition 4.8 of [41])**.** *Let* Γ *be a set of generalized fixed data and* $\mathbf{s}$ *be an associated initial generalized torus seed. Let* $\mathbf{s}_w=(e'_1,\ldots,e'_n)$ *be a generalized torus seed with dual basis* $\{(e'_i)^*\}_i$ *and* $f'_i = d_i^{-1}(e'_i)^*$*. A* cluster monomial *in* $\mathbf{s}_w$ *is a monomial in* $T_{N^\circ,w}\subset\mathcal{A}$ *of the form* z^m *with* $m=\sum_{i=1}^n a_if'_i$ *for* $a_i\in\mathbb{Z}_{\geq 0}$*. We refer to any regular function which is a cluster monomial in some seed of* $\mathcal{A}$ *as a* cluster monomial on $\mathcal{A}$.

4.7 Broken lines and theta functions

Broken lines have a similar definition in the generalized setting as in the ordinary setting, with the caveat that we now work over the ground ring $\Bbbk[a_{i,j}]$ and so the monomials attached to the domains of linearity of a broken line lie in $\Bbbk[a_{i,j}][M^\circ]$.

Definition 4.7.1. *Let $\mathfrak{D}$ be a scattering diagram, m_0 be a point in $M^\circ \backslash \{0\}$, and Q be a point in $M_\mathbb{R} \backslash Supp(\mathfrak{D})$. A* broken line *with endpoint Q and initial slope m_0 is a piecewise linear path $\gamma : (-\infty, 0] \to M_\mathbb{R} \backslash Sing(\mathfrak{D})$ with finitely many domains of linearity. Each domain of linearity, L, has an associated monomial $c_L z^{m_L} \in k[a_{i,j}][M^\circ]$ such that the following conditions are satisfied:*

1. *$\gamma(0) = Q$*
2. *If L is the first domain of linearity of γ, then $c_L z^{m_L} = z^{m_0}$.*
3. *Within the domain of linearity L, the broken line has slope $-m_L$ - in other words, $\gamma'(t) = -m_L$ on L.*
4. *Let t be a point at which γ is non-linear and is passing from one domain of linearity, L, to another, L', and define*
$$\mathfrak{D}_t = \{(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D} : \gamma(t) \in \mathfrak{d}.\}$$
Then the power series $\mathfrak{p}_{\gamma|_{(t-\epsilon,t+\epsilon)}, \mathfrak{D}_t}$ contains the term $c_{L'} z^{m_{L'}}$.

The definition of a theta function in terms of broken lines remains the same, except that we are now thinking of broken lines in the generalized sense. For the statement of this definition, see 2.3.22. In order to justify these definitions, we can begin by verifying that broken lines and theta functions still have several crucial properties.

The Laurent phenomenon - i.e., that elements of the generalized cluster algebra can be written as Laurent polynomials in terms of any cluster - is one of the hallmark properties of generalized cluster algebras. Philosophically, the Laurent phenomenon means that a "good" basis for a generalized cluster algebra should be able to be equivalently defined in terms of any choice of initial cluster or torus seed. As such, one important property of theta functions is the following correspondence between theta functions with the same initial slope but whose endpoints lie in different chambers:

Theorem 4.7.2 (Theorem 3.5 of [41])**.** *Let $\mathfrak{D}$ be a consistent scattering diagram, m_0 be a point in $M \backslash \{0\}$, and consider a pair of points Q and Q' in $M_\mathbb{R} \backslash Supp(\mathfrak{D})$ such that Q and Q' are linearly independent over $\mathbb{Q}$. Then for any path γ with endpoints Q and Q' for which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is defined, we have*
$$\vartheta_{Q',m_0} = \mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{Q,m_0})$$

Proof. As in the ordinary setting, this is a special case of the results of Section 4 of [10]. Those results do not assume that the wall-crossing automorphisms are binomials and are therefore also applicable to our setting. ☐

In Section 4.3, we discussed the mutation invariance of generalized cluster scattering diagrams. It is also important that the theta functions exhibit this mutation invariance. Recall that the positive chamber of a cluster scattering diagram corresponds to a choice of initial torus seed $\mathbf{s}$ for the associated generalized cluster algebra. If the cluster scattering diagrams $\mathfrak{D}_\mathbf{s}$ and $\mathfrak{D}_{bfs'}$ are related by a single application of the map T_k, then the initial torus seeds are related by a single mutation, i.e. $bfs' = \mu_k(\mathbf{s})$. The following proposition exhibits a bijection between the sets of broken lines and theta functions defined on $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$.

Proposition 4.7.3 (Analog of Proposition 3.6 of [41])**.** *The transformation T_k gives a bijection between broken lines with endpoint Q and initial slope m_0 in $\mathfrak{D}_{\mathbf{s}}$ and broken lines with endpoint $T_k(Q)$ and initial slope $T_k(m_0)$ in $\mathfrak{D}_{\mu_k(\mathbf{s})}$. In particular,*

$$\vartheta^{\mu_k(\mathbf{s})}_{T_k(Q),T_k(m_0)} = \begin{cases} T_{k,+}\left(\vartheta^{\mathbf{s}}_{Q,m_0}\right) & Q \in \mathcal{H}_{k,+} \\ T_{k,-}\left(\vartheta^{\mathbf{s}}_{Q,m_0}\right) & Q \in \mathcal{H}_{k,-} \end{cases}$$

where $T_{k,\pm}$ acts linearly on the exponents in $\vartheta^{\mathbf{s}}_{Q,m_0}$.

Proof. We follow the structure of the proof of Proposition 3.6 from [41].

Let γ be a broken line in a scattering diagram $\mathfrak{D}_{\mathbf{s}}$ and $T_k(\gamma)$ denote the composite map $T_k \circ \gamma : (-\infty, 0] \to M_{\mathbb{R}}$. If any domain of linearity of γ is in both $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$, we can subdivide that domain of linearity at the point where it crosses between $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$. As such, we can assume for any domain of linearity L that $\gamma(L)$ falls either entirely inside $\mathcal{H}_{k,+}$ or entirely inside $\mathcal{H}_{k,-}$. For any domain of linearity L that's been subdivided in this way, the associated monomial $c_L z^{m_L}$ will be sent to either $c_L z^{T_{k,+}(m_L)}$ or $c_L z^{T_{k,-}(m_L)}$ depending on the portion of L being considered. We know from Theorem 4.3.5 that $\mathfrak{D}_{\mu_k(\mathbf{s})} = T_k(\mathfrak{D}_{\mathbf{s}})$, so we can think about $T_k(\mathfrak{D}_{\mathbf{s}})$ when thinking about the broken line in $\mathfrak{D}_{\mu_k(\mathbf{s})}$.

We know that $e_k^{\perp}$ lies on the boundary between $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$. So in order to understand what happens to the subdivided domains of linearity, which originally were in both $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$, we need to analyze what happens when γ crosses $e_k^{\perp}$. First, consider the original broken line γ in $\mathfrak{D}_{\mathbf{s}}$. Suppose that one domain of linearity, L, has been subdivided into L_1 and L_2 such that γ crosses $e_K^{\perp}$ at the point where it passes from the first domain of linearity, L_1, to the second domain of linearity, L_2. By definition, we know that when the monomial $c_{L_1} z^{m_{L_1}}$ passes through $e_k^{\perp}$, it's mapped to

$$c_{L_1} z^{m_{L_1}} \left(1 + a_{k,1} z^{v_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{|\langle d_k e_k, m_{L_1}\rangle|}$$

and that $c_{L_2} z^{m_{L_2}}$ must appear as a term in this polynomial.

We can then consider the image of γ in $T_k(\mathfrak{D}_{\mathbf{s}})$. If $L_1 \subseteq \mathcal{H}_{k,-}$ and $L_2 \subseteq \mathcal{H}_{k,+}$, then $c_{L_2} z^{T_{k,+}(m_{L_2})}$ must appear as a term in the polynomial

$$\begin{aligned} c_{L_1} z^{T_{k,+}(m_{L_1})} (1 + \cdots + z^{r_k v_k})^{-\langle d_k e_k, m_{L_1}\rangle} &= c_{L_1} z^{m_{L_1} + r_k v_k \langle d_k e_k, m_{L_1}\rangle} (1 + \cdots + z^{r_k v_k})^{-\langle d_k e_k, m_{L_1}\rangle} \\ &= c_{L_1} z^{m_{L_1}} \left(z^{-r_k v_k}(1 + \cdots + z^{r_k v_k})\right)^{-\langle d_k e_k, m_{L_1}\rangle} \\ &= c_{L_1} z^{m_{L_1}} \left(z^{-r_k v_k} + a_{k,1} z^{-(r_k-1)v_k} + \cdots + 1\right)^{-\langle d_k e_k, m_{L_1}\rangle} \\ &= c_{L_1} z^{T_{k,-}(m_{L_1})} \left(z^{-r_k v_k} + a_{k,1} z^{-(r_k-1)v_k} + \cdots + 1\right)^{-\langle d_k e_k, m_{L_1}\rangle} \end{aligned}$$

Due to the assumption that the exchange polynomials have reciprocal coefficients - i.e., that $a_{k,j} = a_{k,r_k-j}$ - this polynomial is equal to

$$c_{L_1} z^{T_{k,-}(m_{L_1})} \left(1 + a_{k,1} z^{-v_k} + \cdots + a_{k,r_k-1} z^{-(r_k-1)v_k} + z^{-r_k v_k}\right)^{-\langle d_k e_k, m_{L_1}\rangle}$$

and therefore $T_k(\gamma)$ satisfies the rules for bending as it crosses

$$\mathfrak{d}'_k = (e_k^{\perp}, 1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_k v_k})$$

in $T_k(\mathfrak{D}_{\mathbf{s}})$. Similarly, if $L_1 \subseteq \mathcal{H}_{k,+}$ and $L_2 \subseteq \mathcal{H}_{k,-}$, then $c_{L_2}z^{T_{k,-}(m_{L_2})} = c_{L_2}z^{m_{L_2}}$ must appear as a term in

$$\begin{aligned}
&c_{L_1}z^{T_{k,-}(m_{L_1})}(1+\cdots+z^{r_k v_k})^{\langle d_k e_k, m_{L_1}\rangle} \\
&= c_{L_1}z^{m_{L_1}}\left(1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{\langle d_k e_k, m_{L_1}\rangle} \\
&= c_{L_1}z^{m_{L_1}}\left(1 + a_{k,r_k-1}z^{v_k} + \cdots + a_{k,1}z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{\langle d_k e_k, m_{L_1}\rangle} \\
&= c_{L_1}z^{m_{L_1}}\left(z^{r_k v_k}(z^{-r_k v_k} + a_{k,r_k-1}z^{-(r_k-1)v_k}\cdots + a_{k,1}z^{-v_k} + 1)\right)^{\langle d_k e_k, m_{L_1}\rangle} \\
&= c_{L_1}z^{m_{L_1}+r_k v_k\langle d_k e_k, m_{L_1}\rangle}\left(1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_k v_k}\right)^{\langle d_k e_k, m_{L_1}\rangle} \\
&= c_{L_1}z^{T_{k,+}(m_{L_1})}\left(1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_k v_k}\right)^{\langle d_k e_k, m_{L_1}\rangle}
\end{aligned}$$

and therefore $T_k(\gamma)$ also satisfies the rules for bending at $\mathfrak{d}'_k$ in this case. As such, we've verified that for any broken line γ in $\mathfrak{D}_{\mathbf{s}}$, its image $T_k(\gamma)$ is also a broken line in $\mathfrak{D}_{\mu_k(\mathbf{s})} = T_k(\mathfrak{D}_{\mathbf{s}})$. To see that T_k is, in fact, a bijection, we must verify that $T_k^{-1}(T_k(\gamma)) = \gamma$. First, we define $T_k^{-1} : \mathfrak{D}_{\mu_k(\mathbf{s})} \to \mathfrak{D}_{\mathbf{s}}$ as

$$T_k^{-}(m) = \begin{cases} m & m \in \mathcal{H}'_{k,+} \\ m - r_k v_k\langle d_k e_k, m\rangle & m \in \mathcal{H}'_{k,-} \end{cases}$$

where $\mathcal{H}'_{k,+}$ and $\mathcal{H}'_{k,-}$ are defined relative to e'_k. Notice, however, that because mutation in direction k sends e_k to $e'_k = -e_k$, we have $\mathcal{H}'_{k,+} = \mathcal{H}_{k,-}$ and $\mathcal{H}'_{k,-} = \mathcal{H}_{k,+}$. As such, showing that $T_k^{-1}(T_k(\gamma)) = \gamma$ amounts to showing that $T_{k,+}^{-1} \circ T_{k,-} = \mathrm{id}$ and $T_{k,-}^{-1} \circ T_{k,+} = \mathrm{id}$. The first equality follows trivially from the definitions and we can verify the second by observing that

$$\begin{aligned}
T_{k,-}^{-1} \circ T_{k,+}(m) &= T_{k,-}^{-1}\left(m + r_k v_k\langle d_k e_k, m\rangle\right) \\
&= (m + r_k v_k\langle d_k e_k, m\rangle) - r_k v_k\langle d_k e_k, m + r_k v_k\langle d_k e_k, m\rangle\rangle \\
&= m + r_k v_k\langle d_k e_k, m\rangle - r_k v_k\langle d_k e_k, m\rangle - r_k v_k\langle d_k e_k, r_k v_k\langle d_k e_k, m\rangle\rangle \\
&= m - r_k v_k\langle d_k e_k, m\rangle\langle d_k e_k, r_k v_k\rangle
\end{aligned}$$

By definition, we know that $v_k = p_1^*(e_k)$ and so $\langle d_k e_k, r_k v_k\rangle = 0$ and the above expression reduces to $T_{k,-}^{-1} \circ T_{k,+}(m) = m$, as desired. $\square$

In fact, such a bijection exists for any pair of diagrams $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$ where $\mathbf{s}$ and $\mathbf{s}'$ are mutation equivalent. The explicit bijection can be obtained by simply iterating the previous proposition for each step in the mutation sequence between $\mathbf{s}$ and $\mathbf{s}'$.

The following proposition is crucial in showing that the generalized cluster variables are, in fact, theta functions.

Proposition 4.7.4 (Proposition 3.8 of [41])**.** *For a point Q in Int($\mathcal{C}_{\mathbf{s}}^{+}$) and a point m in $\mathcal{C}_{\mathbf{s}}^{+} \cap M^{\circ}$, we have*

$$\vartheta_{Q,m} = z^m$$

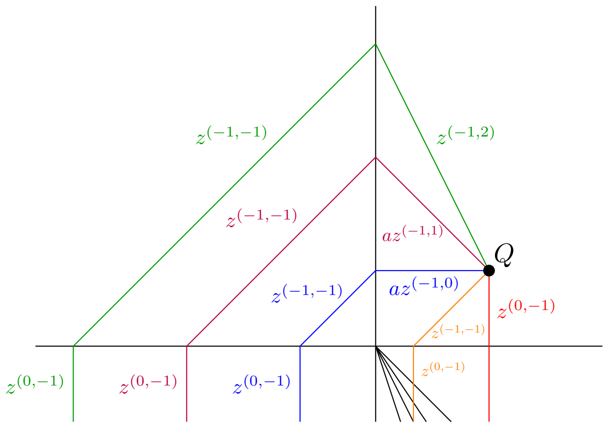


Figure 4.2: The broken line for $\vartheta_{(0,-1),Q}$ in $\mathfrak{D}_{\mathbf{s}}$ for the generalized cluster algebra and generalized torus seed from Example 4.1.4.

Proof. The proof of this proposition is identical to the proof given for the ordinary version in [41]. The fact that the wall-crossing automorphisms now contain additional terms, which offer more options for scattering, can be accounted for in the choice of the normal vectors n_i in that proof. □

One immediate corollary is that the cluster monomials are also theta functions. As with ordinary cluster algebras, this is a highly desirable property for a basis for generalized cluster algebras.

Corollary 4.7.5 (Corollary 3.9 of [41])**.** *Let $\sigma \in \Delta_{\mathbf{s}}^+$ be a cluster chamber. Then for any points $Q \in Int(\sigma)$ and $m \in \sigma \cap M^\circ$, we have $\vartheta_{Q,m} = z^m$*

Proof. The result follows from Propositions 4.7.3 and 4.7.4. □

Together, Theorem 4.7.2 and Corollary 4.7.5 give us a way to compute theta functions using path-ordered products:

Proposition 4.7.6. *Consider $m_0 \in M^\circ \backslash \{0\}$ such that there exists a path γ from m_0 to some point Q in the positive chamber $\mathcal{C}^+$ which passes through finitely many chambers. Then*

$$\vartheta_{Q,m_0} = \mathfrak{p}_{\gamma,\mathfrak{D}}(z^{m_0})$$

Proof. By assumption, we know that the path γ from m_0 to Q passes through finitely many chambers. Let σ_1 denote the first chamber through which γ passes and let Q' be a point in σ_1 which lies on γ. By Proposition 4.7.4, we know that $\vartheta_{Q',m_0} = z^{m_0}$. Let Q'' be a point in $\mathcal{C}_{\mathbf{s}}^+$ such that the coordinates of Q' and Q'' are linearly independent over $\mathbb{Q}$ and let γ' denote a path between Q' and Q'' which follows γ until within the interior of the positive chamber $\mathcal{C}_{\mathbf{s}}^+$, at which point it goes to Q'' rather than Q. By Theorem 4.7.2, we know that $\vartheta_{Q'',m_0} = \mathfrak{p}_{\gamma'}\left(\vartheta_{Q',m_0}\right) = \mathfrak{p}_{\gamma'}\left(z^{m_0}\right)$.

Because both path-ordered products and theta functions are independent of the exact location of their endpoints within the interior of a chamber, we therefore have

$$\vartheta_{Q'',m_0} = \vartheta_{Q,m_0} = \mathfrak{p}_{\gamma'}(z^{m_0}) = \mathfrak{p}_{\gamma}(z^{m_0})$$

□

We can then establish a weaker version of Theorem 4.9 of [41], without the guaranteed positivity of Laurent polynomial coefficients:

Theorem 4.7.7. *For generalized fixed data Γ, which satisfies the injectivity assumption, and a choice of initial generalized torus seed $\mathbf{s}$, consider a point $Q \in \mathcal{C}_{\mathbf{s}}^+$ and a point $m \in \sigma \cap M^\circ$ for some chamber $\sigma \in \Delta_{\mathbf{s}}^+$. Then $\vartheta_{Q,m}$ expresses a cluster monomial of $\mathcal{A}$ in $\mathbf{s}$ as a Laurent polynomial. Moreover, all cluster monomials can be expressed as $\vartheta_{Q,m}$ for some choice of Q and m.*

Proof. The proof of Theorem 4.9 from [41] holds in the generalized setting, except for the proof of positivity. The proof of positivity in [41] uses an earlier result, Theorem 1.13, for which we do not have a generalized analogue. In particular, [41, Theorem 1.13] states (in the case of ordinary cluster algebras) that the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ is equivalent to one such that all walls can be expressed as $(\mathfrak{d}, f_{\mathfrak{d}})$ where $f_{\mathfrak{d}} = (1 + z^m)^c$ with $m = p^*(n)$ for some $n \in N^+$ which is normal to $\mathfrak{d}$ and $c \in \mathbb{Z}_{>0}$. Since we are allowing polynomial exchanges that are not simply binomials, we allow ourselves to work with scattering diagrams that are not necessarily equivalent to one with walls only of this form. □

In order for the theta functions to form a viable basis, we need to understand how to decompose products of theta functions. Before we can develop and compactly state this framework, we need to introduce a few pieces of additional notation. For a broken line γ, let $\mathrm{Mono}(\gamma) = c(\gamma) z^{F(\gamma)}$ be the monomial attached to its final domain of linearity. Then $c(\gamma)$ denotes the coefficient and $F(\gamma)$ the exponent in that final domain of linearity. Let $I(\gamma)$ and $b(\gamma)$ denote the initial slope and endpoint, respectively, of γ.

With this notation, we can then define structure constants for the multiplication of theta functions.

Proposition 4.7.8 (Analogue of Definition-Lemma 6.2 in [41])**.** *Let p_1, p_2, and q be points in $\widetilde{M}_{\mathbf{s}}^\circ$ and z be a generic point in $\widetilde{M}_{\mathbb{R},\mathbf{s}}^\circ$. There are at most finitely many pairs of broken lines γ_1, γ_2 such that γ_i has initial slope p_i, both broken lines have endpoint z, and $F(\gamma_1) + F(\gamma_2) = q$. Let*

$$a_z(p_1, p_2, q) := \sum_{\substack{(\gamma_1,\gamma_2) \\ I(\gamma_i)=p_i, b(\gamma_i)=z \\ F(\gamma_1)+F(\gamma_2)=q}} c(\gamma_1)c(\gamma_2)$$

The integers $\alpha_z(p_1, p_2, q)$ are non-negative.

Proof. A major portion of this statement is a definition; the claim requiring proof is that there are finitely many such pairs of broken lines. By definition, a scattering diagram $\mathfrak{D}_{\mathbf{s}}$ has walls $(\mathfrak{d}, f_{\mathfrak{d}})$ with $f_{\mathfrak{d}} \in \widehat{\Bbbk[P_{\mathbf{s}}]}$. Recall that when a monomial z^{p_i} scatters off a wall in $\mathfrak{D}_{\mathbf{s}}$, it is mapped to a monomial of the form $a_i z^{p_i + m_i}$ with $m_i \in P_{\mathbf{s}}$, where a_i is some product of the exchange polynomial coefficients (and can simply be 1). If the monomial z^{p_i} scatters off multiple walls in $\mathfrak{D}_{\mathbf{s}}$, then it is mapped to a monomial with the same form where m_i is a sum of elements of $P_{\mathbf{s}}$. Because $P_{\mathbf{s}}$ is a strictly convex cone, we know that there are finitely many expressions of $m_i \in P_{\mathbf{s}}$ as a sum of finitely many elements of $P_{\mathbf{s}}$. Because the exchange polynomial coefficients are formal variables, the $c(\gamma_i)$ are non-negative, and therefore so is $\alpha_z(p_1, p_2, q)$.

Hence, for any broken line γ_i, we have $F(\gamma_i) = I(\gamma_i) + m_i$ for $m_i \in P_{\mathbf{s}}$ where m_i has finitely many expressions as a finite sum of elements of $P_{\mathbf{s}}$. The summation in the definition of $a_z(p_1, p_2, q)$ is indexed over pairs (γ_1, γ_2) such that $F(\gamma_1) + F(\gamma_2) = q$. This condition can be equivalently written as

$$I(\gamma_1) + I(\gamma_2) + m_1 + m_2 = q$$

There are both finitely many possible choices of m_1, m_2 and finitely many ways to express any choice of m_1 or m_2 as a sum of elements of $P_{\mathbf{s}}$. It follows that for a fixed choice of p_1, p_2 and q, there are finitely many possible ways to express q in this form. $\square$

We then obtain the following decomposition of products of theta functions:

Lemma 4.7.9 (Analogue of Proposition 6.4(3) of [41]). *Let p_1, p_2, and q be points in $\widetilde{M}^\circ_{\mathbf{s}}$ and z be a generic point in $\widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}$. Then*

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{q \in \widetilde{M}^\circ_{\mathbf{s}}} \alpha_{z(q)}(p_1, p_2, q)\vartheta_q$$

for $z(q)$ sufficiently close to q. When z is sufficiently close to q, $a_z(p_1, p_2, q)$ is independent of the choice of z and we can simply write $\alpha(p_1, p_2, q) := a_z(p_1, p_2, q)$.

Proof. The argument given in [41] for the analogous result for ordinary cluster scattering diagrams holds in our generalized setting. No portion of that argument assumes that the wall-crossing automorphisms are binomials. $\square$

4.8 Partial compactifications of generalized cluster varieties

The discussion in Sections 4.10 and 4.11 will require certain canonical partial compactifications of generalized cluster varieties. These partial compactifications are constructed in the same manner as for ordinary cluster varieties. In this section, we review the construction given in Appendix B of [41], with minor modifications when necessary to adapt the construction to our generalized setting.

We begin by recalling some constructions and notation from previous sections of this paper and from [41]. Recall from Definition 4.6.1 that for a fixed choice of generalized torus seed $\mathbf{s}$, the scattering diagram $\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\mathbf{s}}$ provides an atlas for $\mathcal{A}_{\text{prin,scat},\mathbf{s}}$. When it's clear from context that we're discussing the principal case, we will drop prin from the subscript and instead write $\mathcal{A}_{\text{scat},\mathbf{s}}$ in order to simplify the notation. Recall that the atlas for $\mathcal{A}_{\text{scat},\mathbf{s}}$ is constructed by attaching a copy of $T_{\widetilde{N}^\circ}$ to each chamber $\sigma \in \Delta^+_{\mathbf{s}}$ and gluing these copies according to the wall-crossing automorphisms. From Theorem 4.6.3, we know that $\mathcal{A}_{\text{prin},\mathbf{s}} \cong \mathcal{A}_{\text{scat},\mathbf{s}}$. Recall that $\mathcal{A}_{\text{prin}}$ has an atlas of tori $T_{\widetilde{N}^\circ,w}$ parametrized by vertices w of $\mathfrak{T}_{\mathbf{s}}$.

The choice of initial generalized torus seed $\mathbf{s}$ determines a partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}} \supset \mathcal{A}_{\text{prin}}$ as follows. Recall that the generalized torus seed $\mathbf{s}$ determines a corresponding generalized torus seed with principal coefficients $\mathbf{s}_{\text{prin}}$. In the original seed $\mathbf{s}$, the indices of the frozen variables are given by $I \backslash I_{\text{uf}}$. Recall that the index set $\widetilde{I}$ for $\mathbf{s}_{\text{prin}}$ is constructed by taking two disjoint copies of the original index set I. In this case, the frozen variables are specified by the indices $I \backslash I_{\text{uf}}$ in the first copy of I along with all of the indices in the second copy of I.

The generalized cluster variety $\mathcal{A}$ has a partial compactification $\overline{\mathcal{A}} \supset \mathcal{A}$ which is constructed by partially compactifying each of the torus charts of $\mathcal{A}$ via $T_{N^\circ,\mathbf{s}} \subset TV(\Sigma^{\mathbf{s}})$ where $\Sigma^{\mathbf{s}} = \sum_{i \notin I_{\text{uf}}} \mathbb{R}_{\geq 0} e_i \subset$

$N^{\circ}_{\mathbb{R},\mathbf{s}}$ and $TV(\Sigma^{\mathbf{s}})$ denotes the toric variety of the cone $\Sigma^{\mathbf{s}}$. Although the monomials $\{z^{f_i}\}_{i\notin I_{\mathrm{uf}}}$ are fixed under mutation, the monomials $\{z^{e_i}\}_{i\in I_{\mathrm{uf}}}$ can change under mutation despite corresponding to frozen indices (see Definition 4.1.5). Given a generalized cluster variety $\mathcal{A} = \bigcup_{\mathbf{s}\in S} T_{N^\circ,\mathbf{s}}$ and a fan $\Sigma \subset N^{\circ}_{\mathbb{R}}$ which gives a partial compactification $TV(\Sigma) \supset T_{N^\circ,\mathbf{s}'}$ for some generalized torus seed $\mathbf{s}'$ which is mutation equivalent to $\mathbf{s}$, we can canonically construct a partial compactification

$$\overline{\mathcal{A}} = \bigcup_{\mathbf{s}\in S} TV(\Sigma^{\mathbf{s}})$$

by taking $\Sigma^{\mathbf{s}'} := \Sigma$ and $\Sigma^{\mathbf{s}} := (\mu^t_{\mathbf{s},\mathbf{s}'})^{-1}(\Sigma^{\mathbf{s}'})$ where $\mu^t_{\mathbf{s},\mathbf{s}'}$ is geometric tropicalization of the birational map given by the composition $\mu_{\mathbf{s},\mathbf{s}'} : T_{N^\circ,\mathbf{s}} \subset \mathcal{A} \supset T_{N^\circ,\mathbf{s}'}$.

The seed $\mathbf{s}$ also determines a partial compactification of $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$. This partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}} \supset \mathcal{A}_{\mathrm{scat},\mathbf{s}}$ is given by the atlas of toric varieties $TV(\Sigma_{v\in\mathbf{s}}) \supset T_{N^\circ,v\in\mathbf{s}}$ where

$$\Sigma_{v'} = \begin{cases} \Sigma^{\mathbf{s}} & \text{if } v' = v \text{ is the root of } \mathfrak{T}_{\mathbf{s}} \\ \mu^t_{v,v'}(\Sigma_v) & \text{for all other vertices v' in } \mathfrak{T}_{\mathbf{s}} \end{cases}$$

Proposition 4.6.2 and Theorem 4.6.3 (and their proofs) extend to this partial compatification.

In the principal coefficient case, recall that the frozen variables are indexed by $I\backslash I_{\mathrm{uf}}$ in the first copy of I along with all the indices in the second copy of I. We obtain a partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ by taking only the second copy of I as the set of frozen indices. Then, $T_{\widetilde{N}^\circ} \subset TV(\Sigma^{\mathbf{s}})$ where $\Sigma^{\mathbf{s}}$ is now the cone generated by the basis vectors of $\mathbf{s}_{\mathrm{prin}}$ whose indices are in the second copy of I. It is important to note that although $\mathcal{A}_{\mathrm{prin}}$ depends only on the mutation class of $\mathbf{s}$, the partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ actually depends on the particular choice of generalized torus seed.

This dependence arises because the choice of $\mathbf{s}$ determines $\mathbf{s}_{\mathrm{prin}}$ and therefore the set of cluster variables $\{z^{(0,e_i)}, z^{(f_i,0)}\}$. In the partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}$, the variables $\{z^{(0,e_i)}\}_{e_i\in\mathbf{s}}$ are allowed to equal zero and the map $\pi : \mathcal{A}_{\mathrm{prin}} \to T_M$ induces a map $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$ where z^{e_i} pulls back to $z^{(0,e_i)}$. As such, the choice of $\mathbf{s}$ also determines a canonical extension of each cluster variable on a chart of $\mathcal{A}$ to a cluster variable on the corresponding chart of $\mathcal{A}_{\mathrm{prin}}$.

For any vertex v' of $\mathfrak{T}_{\mathbf{s}}$, let $\mathbf{s}_{\mathrm{prin},v'}$ denote the seed obtained by mutating $\mathbf{s}_{\mathrm{prin}}$ according to the mutation sequence determined by the path between the root v of $\mathfrak{T}_{\mathbf{s}}$ and the vertex v'. Let $\Sigma^{\mathbf{s}}_{v'}$ denote the cone generated by the basis vectors of $\mathbf{s}_{\mathrm{prin},v'}$ whose indices are in the second copy of I. The cone $\Sigma^{\mathbf{s}}_{v'}$ gives a partial compactification of the torus $T_{\widetilde{N}^\circ,v'}$. Recall that under the isomorphism given in Theorem 4.6.3, the torus $T_{\widetilde{N}^\circ,v'}$ is identified with $T_{\widetilde{N}^\circ,\mathcal{C}^+_{v'}\in\Delta^+_{\mathbf{s}}}$. Thus, the cone $\Sigma^{\mathbf{s}}_{\mathrm{scat},v'} := \phi^t_{v,v'}(\Sigma^{\mathbf{s}}_{v'})$ gives a partial compactification of $T_{\widetilde{N}^\circ,\mathcal{C}^+_{v'}\in\Delta^+_{\mathbf{s}}}$.

The cone $\Sigma^{\mathbf{s}}_{\mathrm{scat},v'}$ can be explicitly described:

Lemma 4.8.1 (Analogue of Lemma 5.2 of [41]). *The cones $\Sigma^{\mathbf{s}}_{scat,v'}$, and therefore the toric varieties in the atlas of the partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{scat,\mathbf{s}}$, are independent of the choice of vertex v' of $\mathfrak{T}_{\mathbf{s}}$. For $\mathbf{s} = (e_1,\ldots,e_n)$, each $\Sigma^{\mathbf{s}}_{scat,v'}$ is equal to the cone generated by $\{(0,e_1^*),\ldots,(0,e_n^*)\}$.*

Proof. The proof of this lemma is essentially identical to the proof of Lemma 5.2 of [41]; because the argument is short, we reproduce it here with the (minor) necessary modification.

By definition, $\Sigma^{\mathbf{s}}_{\mathrm{scat},v}$ is the cone generated by $\{(0,e_1^*,\ldots,(0,e_n^*)\}$. Each of the other fans $\Sigma^{\mathbf{s}}_{\mathrm{scat},v'}$ can be obtained from $\Sigma_{\mathrm{scat},v}$ by applying the geometric tropicalization of the birational gluing map between the corresponding tori in the atlas for $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$. Recall that these gluing maps are given by the wall-crossing automorphisms of $\mathfrak{D}_{\mathbf{s}}$. In the generalized setting, any wall between cluster chambers has an attached wall-crossing automorphism which is of the form $1+a_1 z^{p^*(n,0)}+\cdots+a_{r-1}z^{(r-1)p^*(n,0)}+z^{rp^*(n,0)}$ rather than $1+z^{p^*(n,0)}$ for some n in the convex hull of $\{e_i\}_{i\in I}$, $\widetilde{m}=p^*(n,0)$, and positive integer r. We can recognize this automorphism as the mutation $\mu_{(\widetilde{n},\widetilde{m},r)}$, for some $\widetilde{n}\in\widetilde{N}^\circ$ and $\widetilde{m}\in\widetilde{M}^\circ$. Recall that the geometric tropicalization $\mu^t_{(\widetilde{n},\widetilde{m},r)}:\widetilde{N}\to\widetilde{N}$ maps

$$x\mapsto x+r[\langle\widetilde{m},x\rangle]_-\widetilde{n},$$

where $[x]_-=\min(x,0)$. Because $\langle\widetilde{m},(0,e_i^*)\rangle=\{(n,0),(0,e_i^*)\}\geq 0$, we have $\mu^t_{(\widetilde{n},\widetilde{m},r)}(e_i^*)=e_i^*$ and therefore $\Sigma^{\mathbf{s}}_{\mathrm{scat},v'}=\Sigma^{\mathbf{s}}_{\mathrm{scat},v}$. □

We can then establish the following proposition, for use in Section 4.10:

Proposition 4.8.2 (Analogue of Corollary 5.3 of [41])**.** *Let* $\mathbf{s}$ *be a fixed choice of seed and* v *be the corresponding vertex in the tree* $\mathfrak{T}_{\mathbf{s}}$*. Then:*

1. *The map* $\pi:\overline{\mathcal{A}}^{\mathbf{s}}_{prin}\to\mathbb{A}^n_{X_1,\dots,X_n}$ *has fiber* $\pi^{-1}(0)=T_{N^\circ}$*.*
2. *The mutation maps* $TV(\Sigma^{\mathbf{s}}_w)\to TV(\Sigma^{\mathbf{s}}_{w'})$ *between tori varieties in the atlas which defines* $\overline{\mathcal{A}}^{\mathbf{s}}_{prin}$ *are isomorphisms in a neighborhood of the fiber of* $0\in\mathbb{A}^n_{X_1,\dots,X_n}$*.*
3. *The partial compactification* $\overline{\mathcal{A}}^{\mathbf{s}}_{scat,\mathbf{s}}\supset\mathcal{A}_{scat,\mathbf{s}}$ *has an atlas whose charts are indexed by chambers of* $\mathfrak{D}_{\mathbf{s}}$*. The mutation maps between two charts, which correspond to the mutation maps in (2), are isomorphisms in a neighborhood of the fiber* $0\in\mathbb{A}^n_{X_1,\dots,X_n}$ *and restrict to the identity on that fiber.*

Proof. Recall that $\mathcal{A}_{\mathrm{prin}}\cong\mathcal{A}_{\mathrm{scat}}$, as shown in Proposition 4.6.2. If the mutation maps between toric varieties in the atlas defining $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ are isomorphisms in a neighborhood of the fiber of $0\in\mathbb{A}^n_{X_1,\dots,X_n}$, then so are the mutation maps between the toric varieties in the atlas which define $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat}}$. In order for the mutation maps in (3) to restrict to the identity on that fiber, the map $\pi:\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}\cong\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat}}\to\mathbb{A}^n_{X_1,\dots,X_n}$ must have fiber $\pi^{-1}(0)=T_{N^\circ}$. As such, (3) immediately implies both (1) and (2) and it will suffice to prove (3).

Recall that each incoming wall of the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ has an attached wall-crossing function of the form $1+a_{i,1}z^{(v_i,e_i)}+\cdots+a_{i,r_i-1}z^{(r_i-1)(v_i,e_i)}+z^{r_i(v_i,e_i)}$ and that we identified $X_i=z^{e_i}$. Each of these wall-crossing functions is trivial modulo $X_1,\dots,X_n$; therefore, the diagram $\mathfrak{D}_{\mathbf{s}}$ is trivial modulo $X_1,\dots,X_n$.

Adjacent vertices v and v' in the tree $\mathfrak{T}_{\mathbf{s}}$ correspond to seeds that are related by a single mutation. As such, the gluing map $TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v})\to TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v'})$ is given by the monomial mapping $z^{\widetilde{m}}\mapsto z^{\widetilde{m}}f^{\langle\widetilde{n},\widetilde{m}\rangle}$ where f is a regular function on $TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v'})$, $\widetilde{n}\in\widetilde{N}^\circ$, and $\widetilde{m}\in\widetilde{M}^\circ$. When $X_1,\dots,X_n$ are zero, the function f is trivial. Hence, the gluing maps are the identity on the fiber $0\in\mathbb{A}^n_{X_1,\dots,X_n}$. When $X_1,\dots,X_n$ are non-zero, the gluing map $TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v})\to TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v'})$ gives an isomorphism between open subsets of $TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v})$ and $TV(\Sigma^{\mathbf{s}}_{\mathrm{scat},v'})$. The gluing maps are therefore isomorphisms in the neighborhood of the fiber $0\in\mathbb{A}^n_{X_1,\dots,X_n}$, as desired. □

4.9 g-vectors

The Laurent expansion of a cluster variable with respect to a particular initial cluster can be specified via two statistics: its F-polynomial and its g-vector. In [26], Fomin and Zelevinsky give a definition of g-vectors in terms of a particular $\mathbb{Z}^n$-grading of the ring of Laurent polynomials in $\mathbf{x}$ whose coefficients are integer polynomials in $\mathbf{y}$. Gross, Hacking, Keel, and Kontsevich [41] give a description of g-vectors in the context of cluster scattering diagrams and ordinary cluster varieties. In particular, this alternate description is useful because it allows for a definition of g-vectors on all types of ordinary cluster varieties. In this section, we state this alternate description in the context of generalized cluster scattering diagrams and generalized cluster varieties.

There is a T_{N° action on $\mathcal{A}_{\text{prin}}$ that can be specified at the level of cocharacter lattices as

$$\begin{aligned} N^\circ &\to N^\circ \oplus M, \\ n &\mapsto (n, p^*(n)). \end{aligned}$$

Under this T_{N° action, each cluster monomial on $\mathcal{A}_{\text{prin}}$ is a T_{N°-eigenfunction. Via this action, choosing a generalized torus seed $\mathbf{s}$ determines a canonical extension of each cluster monomial on $\mathcal{A}$ to a cluster monomial on $\mathcal{A}_{\text{prin}}$. This allows us to give the following definition of the g-vector of a cluster monomial of $\mathcal{A}$.

Definition 4.9.1 (Analogue of Definition 5.6 of [41])**.** *The* g-vector *with respect to the generalized torus seed* $\mathbf{s}$ *associated to a cluster monomial of* $\mathcal{A}$ *is the* T_{N°*-weight of its lift determined by* $\mathbf{s}$*.*

There is another way to characterize g-vectors which is extensible to the other types of generalized cluster varieties.

Definition 4.9.2 (Analogue of Definition 5.8 of [41])**.** *Consider the generalized cluster variety* $\mathcal{A} = \bigcup_{\mathbf{s}} T_{N^\circ,\mathbf{s}}$*. Let* $\overline{A}$ *denote a cluster monomial of the form* z^m *on a chart* $T_{N^\circ,\mathbf{s}'}$ *where* $\mathbf{s}' = \{(e_i', \mathbf{a}_i')\}$*. Identify* $\mathcal{A}^\vee(\mathbb{R}^T)$ *with* $M^\circ_{\mathbb{R},\mathbf{s}'}$ *Because* $(z^{e_i'})^T(m) \leq 0$ *for all* i*,* m *is identified with a point in the Fock-Goncharov cluster chamber* $\mathcal{C}^+_{\mathbf{s}'} \subseteq \mathcal{A}^\vee(\mathbb{R}^T)$*. Define* $\mathbf{g}(\overline{A})$ *to be this point in* $\mathcal{C}^+_{\mathbf{s}'} \subseteq \mathcal{A}^\vee(\mathbb{R}^T)$*.*

Definition 4.9.3 (Analogue of Definition 5.10 of [41])**.** *Consider a generalized cluster variety* $V = \bigcup_{\mathbf{s}} T_{L,\mathbf{s}}$*. Let* f *be a global monomial on* V *and* $\mathbf{s}$ *be a generalized torus seed such that* $f|_{T_{L,\mathbf{s}}} \subset V$ *is the character* z^m *for* $m \in Hom(L, \mathbb{Z}) = L^*$*. Then the* g-vector *of* f*, denoted* $\mathbf{g}(f)$*, is the image of* m *under the identifications* $V^\vee(\mathbb{Z}^T) = T_{L^*,\mathbf{s}}(\mathbb{Z}^T) = L^*$*.*

From Definitions 4.9.2 and 4.9.3, we obtain the following corollary.

Corollary 4.9.4 (Analogue of Corollary 5.9 of [41])**.** *Let* $\mathbf{s}$ *be a generalized torus seed and* $\overline{A}$ *be a cluster monomial on the associated generalized cluster variety* $\mathcal{A}$*. The seed* $\mathbf{s}$ *gives an identification* $\mathcal{A}^\vee(\mathbb{R}^T) = M^\circ_{\mathbb{R},\mathbf{s}}$ *under which* $\mathbf{g}(\overline{A})$ *is the* g*-vector of the cluster monomial* $\overline{A}$ *with respect to* $\mathbf{s}$*.*

Proof. The proof given in [41] for the ordinary case holds in the generalized setting, using the appropriate analogous intermediate results for □

We can then generalize the definition of a g-vector beyond the $\mathcal{A}$-variety to any type of generalized cluster variety.

Definition 4.9.5 (Analogue of Definition 5.10 of [41]). *Consider a generalized cluster variety* $V = \bigcup_{\mathbf{s}} T_{L,\mathbf{s}}$. *Let* f *be a global monomial on* V *and* $\mathbf{s}$ *be a generalized torus seed such that the restriction* $f|_{T_{L,\mathbf{s}}} \subset V$ *is the character* z^m *for some* $m \in Hom(L, \mathbb{Z}) = L^*$. *We then define the* g-vector *of* f, *denoted* $\mathbf{g}(f)$, *as the image of* m *under the identifications* $V^\vee(\mathbb{Z}^T) = T_{L^*,\mathbf{s}}(\mathbb{Z}^T) = L^*$.

Although it is not *a priori* clear from this definition, we will see in Lemma 4.11.5 that this definition of g-vector is actually independent of the choice of generalized torus seed $\mathbf{s}$. As in the ordinary case, this formulation of g-vectors allows for a very quick and elegant proof that the g-vectors are sign-coherent.

Theorem 4.9.6 (Analogue of Theorem 5.11 of [41]). *Consider an initial generalized torus seed* $\mathbf{s} = \{(e_i, (a_{i,j}))\}$, *which defines the usual set of dual vectors* $\{f_i = d_i^{-1} e_i^*\}$. *If* $\mathbf{s}'$ *is a mutation equivalent generalized torus seed, then the* i*-th coordinates of the* g*-vectors for the cluster variables in* $\mathbf{s}'$ *are either all non-negative or all non-positive when expressed in the basis* $\{f_1, \ldots, f_n\}$.

Proof. The proof for the ordinary case given in [41] holds in the generalized setting as well, using the appropriate analogues of intermediate results. □

4.10 Theta basis for $\mathcal{A}_{\text{prin}}$

The primary goal of this section is to show that when the generalized cluster algebra $\text{gen}(\mathcal{A}_{\text{prin}})$ and *upper generalized cluster algebra* $\text{up}(\mathcal{A}_{\text{prin}})$ coincide, the theta functions defined in Section 4.7 form a basis for $\mathcal{A}_{\text{prin}}$. Following the work of Gross, Hacking, Keel, and Kontsevich in the ordinary case [41], we do so by first defining the *middle generalized cluster algebra* of $\mathcal{A}_{\text{prin}}$ as a subalgebra of $\text{up}(\mathcal{A}_{\text{prin}})$. We then show that there exists a subset $\Theta \subset \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ which yields a vector space basis for the middle generalized cluster algebra. We show that the middle generalized cluster algebra necessarily contains all the cluster monomials and therefore contains the generalized cluster algebra. That is, we show the inclusions

$$\text{gen}(\mathcal{A}_{\text{prin}}) \subset \text{mid}(\mathcal{A}_{\text{prin}}) \subset \text{up}(\mathcal{A}_{\text{prin}}).$$

The upper generalized cluster algebra was first defined by Gekhtman, Shapiro, and Vainshtein [31], analogously to the definition of the ordinary upper cluster algebra. Given a generalized cluster algebra $\mathcal{A}$, the associated *upper generalized cluster algebra* is

$$\text{up}\,(\mathcal{A}) := \bigcap_{\text{clusters}\{x_1,\ldots,x_n\} \text{ of } \mathcal{A}} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \subset \mathcal{F}$$

In our work, we will use the following equivalent definition:

Definition 4.10.1. *The* upper generalized cluster algebra *up*(V) *associated to a generalized cluster variety* V *is up*$(V) := H^0(V, \mathcal{O}_V)$.

In this section, we will discuss the $\mathcal{A}_{\text{prin}}$ case. Later, in Section 4.11, we will descend to the $\mathcal{A}$ and $\mathcal{X}$ cases by using the fact that the $\mathcal{A}$-variety appears as a fiber of $\mathcal{A}_{\text{prin}} \to T_M$ and the $\mathcal{X}$-variety appears as the quotient $\mathcal{A}_{\text{prin}}/T_{N^\circ}$.

4.10.1 Expansions for universal Laurent polynomials on $\mathcal{A}_{\text{prin}}$

The essential objective of this subsection is to associate a formal summation $\sum_{q\in\mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)}\alpha(g)(q)\vartheta_q$, with coefficients $\alpha(g)(q)\in\Bbbk[a_{i,s}]$, to each universal Laurent polynomial g on $\mathcal{A}_{\text{prin}}$. In doing so, we follow the structure of Section 6 of [41] for the ordinary case, with modifications when necessary to accommodate our generalized setting. We begin by giving such a summation for a fixed choice of generalized torus seed $\mathbf{s}$, then show that the coefficients $\alpha(g)(q)$ are, in fact, independent of the choice of generalized torus seed.

Fix a choice of generalized torus seed $\mathbf{s}=\{(e_i,\{a_{i,j}\})\}$. Recall that in the generalized setting, we are working over the ground ring $R=\Bbbk[a_{i,j}]$ rather than over $\Bbbk$. Following the notation established in Section 4.8, let $X_i:=z^{e_i}$ and $I_{\mathbf{s}}=(X_1,\dots,X_n)\subset R[X_1,\dots,X_n]$. Then, set

$$\begin{aligned}\mathbb{A}^n_{(X_1,\dots,X_n),k}&:=\operatorname{Spec} R[X_1,\dots,X_n]/I_{\mathbf{s}}^{k+1},\\ \overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin},k}&:=\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}}\times_{\mathbb{A}^n_{X_1,\dots,X_n}}\mathbb{A}^n_{(X_1,\dots,X_n),k}.\end{aligned}$$

The map $\pi:\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}}\to\mathbb{A}^n_{X_1,\dots,X_n}$, which we defined in Section 4.8, then induces a map $\pi:\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin},k}\to\mathbb{A}^n_{(X_1,\dots,X_n),k}$. Let

$$\widehat{\text{up}\left(\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}}\right)}:=\varprojlim\text{up}\left(\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin},k}\right)$$

For any $g\in\text{up}(\mathcal{A}_{\text{prin}})$, we have $z^n g\in\text{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}})$ where z^n is some monomial in the X_i. This fact induces the inclusion

$$\text{up}(\mathcal{A}_{\text{prin}})\subset\widehat{\text{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}})}\otimes_{\Bbbk[N^+_{\mathbf{s}}]}\Bbbk[N]\tag{4.3}$$

where $N^+_{\mathbf{s}}\subset N$ denotes the monoid generated by $e_1,\dots,e_n$. Let $\pi_N:\widetilde{M}^\circ=M^\circ\oplus N\to N$ be the projection map and define $\widetilde{M}^{\circ,+}_{\mathbf{s}}:=\pi_N^{-1}(N^+_{\mathbf{s}})$. Let $P_{\mathbf{s}}\subset\widetilde{M}^\circ_{\mathbf{s}}$ be the monoid generated by $(v_1,e_1),\dots,(v_n,e_n)$.

We can begin by establishing the following proposition, which defines canonical functions ϑ_q on $\text{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}})\otimes_{\Bbbk[N^+]}\Bbbk[N]$ and then shows that two particular collections of such ϑ_q form bases for $\text{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin},k})$.

Proposition 4.10.2 (Analogue of Proposition 6.4(1,2,4) of [41]). *1. Given a point $q\in\widetilde{M_{\mathbf{s}}}^{\circ,+}$, the function $\vartheta_{Q_\sigma,q}$ is a regular function on $V_{\mathbf{s},\sigma,k}$. As σ varies, the $\vartheta_{Q_\sigma,q}$ glue to yield a canonically defined function $\vartheta_{q,k}\in up\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin,k}\right)$.*

2. For $q\in\mathcal{A}^\vee_{prin}$ and $k'\geq k$, $\vartheta_{q,k'}|_{\mathcal{A}^{\mathbf{s}}_{prin,k}}=\vartheta_{q,k}$. Hence, the collection $\{\vartheta_{q,k}\}_{k\geq0}$ canonically defines a function

$$\vartheta_q\in\widehat{up\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin}\right)}\otimes_{\Bbbk[N^+_{\mathbf{s}}]}\Bbbk[N].$$

Let $can(\mathcal{A}_{prin})$ denote the $\Bbbk$-vector space

$$\bigoplus_{q\in\mathcal{A}^\vee_{prin}(\mathbb{Z}^T)}\Bbbk\cdot\vartheta_q.$$

The ϑ_q are linearly independent, so there is a canonical inclusion of $\Bbbk$-vector spaces

$$can(\mathcal{A}_{prin})\subset\widehat{up\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin}\right)}\otimes_{\Bbbk[N^+_{\mathbf{s}}]}\Bbbk[N].$$

3. *The collection* $\{\vartheta_q : q \in \widetilde{M}^{\circ,+}_{\mathbf{s}} \backslash \widetilde{M}^{\circ,+}_{\mathbf{s},k+1}\}$ *restricts to a basis of up* $\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin,k}\right)$ *as a* $\Bbbk$*-vector space. Similarly, the collection* $\{\vartheta_q : q \in \pi_N^{-1}(0)\}$ *restricts to a basis of up* $\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin,k}\right)$ *as a* $\Bbbk[N^+_{\mathbf{s}}]/I^{k+1}_{\mathbf{s}}$*-module.*

Proof. The proof given in [41] for the ordinary case holds in our setting. □

In order to associate a formal summation $\sum \alpha(g)(q)\vartheta_q$ to each universal Laurent polynomial g on $\mathcal{A}_{\text{prin}}$, we will first associate a formal summation $\sum \alpha_{\mathbf{s}}(g)(q)\vartheta_q$ which depends on the choice of generalized torus seed $\mathbf{s}$. To do so, we must first define the function α_s.

Proposition 4.10.3 (Analogue of Proposition 6.5 of [41])**.** *There is a unique inclusion*

$$\alpha_{\mathbf{s}} : up\left(\widehat{\overline{\mathcal{A}}^{\mathbf{s}}_{prin}}\right) \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N] \hookrightarrow Hom_{sets}\left(\mathcal{A}^\vee_{prin}(\mathbb{Z}^T) = \widetilde{M}^\circ_{\mathbf{s}}, \Bbbk\right)$$

given by the map $g \mapsto (q \mapsto \alpha_{\mathbf{s}}(g)(q))$*. For all* $n \in N$*,* $\alpha_{\mathbf{s}}(z^n \cdot g)(q+n) = \alpha_{\mathbf{s}}(g)(q)$*.*

Proof. One consequence of Proposition 4.10.2 and Lemma 4.7.9 is that every $g \in \text{up}(\widehat{\overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}}})$ can be uniquely expressed as a convergent formal sum $\sum_{q\in\widetilde{M}^\circ_{\mathbf{s}},+} \alpha_{\mathbf{s}}(g)(q)\vartheta_q$ where the coefficients $\alpha_{\mathbf{s}}(g)(q)$ lie in $\Bbbk$. This immediately implies the desired unique inclusion. □

Definition 4.10.4 (Analogue of Definition 6.6 of [41])**.** *Let g be a universal Laurent polynomial on* $up(\mathcal{A}_{prin})$*. On the torus chart* $T_{\widetilde{N}^\circ,\mathbf{s}}$ *of* $\mathcal{A}_{prin}$*, we can write* $g = \sum_{q\in\widetilde{M}^\circ_{\mathbf{s}}} \beta_{\mathbf{s}}(g)(q)z^q$*. Because* $z^m g \in up(\widehat{\overline{\mathcal{A}}_{prin^{\mathbf{s}}}})$ *for some* $m \in \widetilde{M}^\circ_{\mathbf{s}}$*, we can also write a formal expansion* $g = \sum_{q\in\widetilde{M}^\circ_{\mathbf{s}}} \alpha_{\mathbf{s}}(g)(q)\vartheta_q$*. Let*

$$\overline{S}_{g,\mathbf{s}} := \{q \in \widetilde{M}^\circ_{\mathbf{s}} : \beta_{\mathbf{s}}(g)(q) \neq 0\}, \qquad S_{g,\mathbf{s}} := \{q \in \widetilde{M}^\circ_{\mathbf{s}} : \alpha_{\mathbf{s}}(g)(q) \neq 0\},$$

and $P_{\mathbf{s}}$ *be the monoid generated by* $\{(v_i, e_i)\}_{i\in I_{uf}}$*.*

It follows from the construction of the theta functions that $S_{g,\mathbf{s}} \subseteq \overline{S_{g,\mathbf{s}}} + P_{\mathbf{s}}$.

We are then ready to prove that on $\text{up}(\mathcal{A}_{\text{prin}})$, the function $\alpha_{\mathbf{s}}$ is actually independent of the choice of generalized torus seed $\mathbf{s}$.

Theorem 4.10.5 (Analogue of Theorem 6.8 of [41])**.** *There is a unique function* $\alpha : up\left(\mathcal{A}_{prin}\right) \to Hom_{sets}\left(\mathcal{A}^\vee_{prin}\left(\mathbb{Z}^T\right), \Bbbk\right)$ *such that:*

1. α *is compatible with the* $\Bbbk[N]$*-module structure on* $up\left(\mathcal{A}_{prin}\right)$ *and the* N*-translation action on* $\mathcal{A}^\vee_{prin}\left(\mathbb{Z}^T\right)$*, i.e.*
$$\alpha\left(z^n \cdot g\right)(x+n) = \alpha(g)(x)$$
for all $g \in up\left(\mathcal{A}_{prin}\right)$*,* $n \in N$*, and* $x \in \mathcal{A}^\vee_{prin}\left(\mathbb{Z}^T\right)$*.*

2. *For any generalized torus seed* $\mathbf{s}$*, the formal sum* $\sum_{q\in\mathcal{A}^\vee_{prin}(\mathbb{Z}^T)} \alpha(g)(q)\vartheta_q$ *converges to* g *in* $up\left(\widehat{\overline{\mathcal{A}}^{\mathbf{s}}_{prin}}\right) \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N]$*.*

3. *If* $z^n \cdot g$ *lies in up* $\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin}\right)$*, then* $\alpha\left(z^n \cdot g\right)(q) = 0$ *unless* $\pi_N(q) \in N^+_{\mathbf{s}}$*. Moreover,*
$$z^n \cdot g = \sum_{\pi_{N,\mathbf{s}}(q)\in N^+_{\mathbf{s}}\backslash(N^+_{\mathbf{s}})_{k+1}} \alpha(z^n \cdot g)(q)\vartheta_q \mod \left(I^{k+1}_{\mathbf{s}}\right)$$

and the coefficients $\alpha(z^n \cdot g)(q)$ are the coefficients for the expansion of $z^n \cdot g$ when it is viewed as an element of up $\left(\overline{\mathcal{A}}^{\mathbf{s}}_{prin}\right)$ in the basis of theta functions $\{\vartheta_q : q \in \widetilde{M}^{\circ,+}_{\mathbf{s}} \backslash M^{\circ,+}_{\mathbf{s},k+1}\}$.

4. *For any generalized torus seed $\mathbf{s}'$ reachable from $\mathbf{s}$ via a sequence of mutations, the map α is the composition of inclusions*

$$up\left(\mathcal{A}_{prin}\right) \subset up\left(\widehat{\overline{\mathcal{A}}^{\mathbf{s}'}_{prin}}\right) \otimes_{\Bbbk[N^+_{\mathbf{s}'}]} \Bbbk[N]] \subset Hom_{sets}\left(\mathcal{A}^{\vee}_{prin}(\mathbb{Z}^T) = \widetilde{M}^{\circ}_{\mathbf{s}'}, \Bbbk\right)$$

from Proposition 4.10.3 and Equation (4.3). This maps the cluster monomial $A \in up\left(\mathcal{A}_{prin}\right)$ to the delta function $\delta_{\mathbf{g}(A)}$ where $\mathbf{g}(A) \in \mathcal{A}^{\vee}_{prin}(\mathbb{Z}^T)$ is its g-vector.

Moreover, $\alpha(g)(m) = \alpha_{\mathbf{s}'}(g)(m)$ for any generalized torus seed $\mathbf{s}'$.

Proof. The proof given in [41] for the ordinary case holds in our generalized setting, with some modification. We review the proof from [41] for the ordinary case, making modifications when necessary for our setting. Note that this proof is quite long and contains two subordinate claims.

As in the ordinary case, it follows from Propositions 4.10.2 that for a fixed generalized torus seed $\mathbf{s}$, $\alpha_{\mathbf{s}}$ is the unique function satisfying statements (1)-(3). It is also clear that $\alpha_{\mathbf{s}}$ satisfies statement (4) when $\mathbf{s}' = \mathbf{s}$. As such, it is sufficient for us to show that the map $\alpha_{\mathbf{s}}$ is actually independent of the choice of generalized torus seed. To do so, we wish to show that $\alpha_{\mathbf{s}}$ expresses g as a (possibly infinite) sum of theta functions. The uniqueness of this expression then follows from the fact that the theta functions are linearly independent.

Consider a generalized torus seed $\mathbf{s} = \{(e_i, (a_{i,j}))\}_{i \in I_{\mathrm{uf}}, j \in [r_i - 1]}$. Then let $\overline{\Sigma}^{\mathbf{s}}$ denote the fan in $\widetilde{N}^{\circ} = N^{\circ} \oplus M$ whose rays are spanned by the collection $\{d_i f_i\}$. To show that $\alpha_{\mathbf{s}}$ is independent of the choice of generalized torus seed, it will suffice for us to then consider a generalized torus seed $\mathbf{s}' = \{(e'_i, (a'_{i,j}))\}$ which is related to $\mathbf{s}$ by a single mutation, i.e. $\mathbf{s}' = \mu_k(\mathbf{s})$. Without loss of generality, assume that $k = 1$.

We will then consider the union of the tori $T_{\widetilde{N}^{\circ},\mathbf{s}}$ and $T_{\widetilde{N}^{\circ},\mathbf{s}'}$ from the atlas for $\mathcal{A}_{\mathrm{prin}}$. Because $\mathbf{s}' = \mu_1(\mathbf{s})$, these tori are glued by the mutation map μ_1, which is defined by the pullback

$$\mu_1^* : z^{(m,n)} \mapsto z^{(m,n)}(1 + a_{1,1} z^{(v_1,e_1)} + \cdots + a_{1,r_1-1} z^{(r_1-1)(v_1,e_1)} + z^{r_1(v_1,e_1)})^{-\langle (d_1 e_1, 0), (m,n) \rangle}$$

where $(m,n) \in \overline{M}^{\circ} = M^{\circ} \oplus N$. We can compactify this union by gluing $\mu_1 : TV(\Sigma^{\mathbf{s}}) \to TV(\Sigma^{\mathbf{s}'})$. Let $U := TV(\Sigma^{\mathbf{s}}) \cup TV(\Sigma^{\mathbf{s}'})$ denote the partial compactification of the union under this gluing. It is important to note that U is not in the atlas for either $\overline{A}^{\mathbf{s}}_{\mathrm{prin}}$ or $\overline{A}^{\mathbf{s}'}_{\mathrm{prin}}$.

Recall that $f'_i = f_i$ for $i \neq 1$ and that $f'_1 = -f_1 + r_1 \sum_{j \in I_{\mathrm{uf}}} [-\epsilon_{kj}]_+ f_j$. As such, the cones $\overline{\Sigma}^{\mathbf{s}}$ and $\overline{\Sigma}^{\mathbf{s}'}$ share a codimension one face. Together, they form a fan $\overline{\Sigma}$. By construction, the rational maps $TV(\Sigma^{\mathbf{s}}) \to TV(\overline{\Sigma}^{\mathbf{s}})$ and $TV(\Sigma^{\mathbf{s}'}) \to TV(\overline{\Sigma}^{\mathbf{s}'})$ are regular. Because the mutation map μ_1 commutes with the projection map $\pi : T_{\widetilde{N}^{\circ}} \to T_M$, the map $\pi : U \to TV(\overline{\Sigma})$ is also regular.

The toric boundary ∂V has a unique complete one-dimensional stratum $\mathbb{P}^1$ and two zero strata $0_{\mathbf{s}}$ and $0_{\mathbf{s}'}$. We denote the complements of the zero strata in $\mathbb{P}^1$ as $\mathbb{A}^1_{\mathbf{s}'}$ and $\mathbb{A}^1_{\mathbf{s}}$, respectively. Let $\mathbb{A}^1_{\mathbf{s},k} \subset V$ denote the k-th order neighborhood of $\mathbb{P}^1$, $U_{\mathbb{A}^1_{\mathbf{s},k}}$ denote the scheme theoretic inverse image $\pi^{-1}(\mathbb{A}^1_{\mathbf{s},k}) \subset U$, and $U_{\mathbb{G}_m,k}$ denote the intersection

$$U_{\mathbb{A}^1_{\mathbf{s},k}} \cap U_{\mathbb{A}^1_{\mathbf{s}',k}} \subset U.$$

We wish to show that the theta functions form a basis of functions on the formal neighborhoods $\mathrm{up}(U_{\mathbb{A}^1_{\mathbf{s},k}})$ and $\mathrm{up}(U_{\mathbb{G}_m,k})$. In doing so, it will be useful to establish the following coordinate system.

Let $X_i := z^{(0,e_i)}$ and $X_i^{'} := z^{(0,e_i')}$. For $i \neq 1$, recall from Definition 4.1.5 that $\mu_1^*(X_i) = X_i$ and $\mu_1^*(X_i') = X_i'$. There is a map from the fan $\overline{\Sigma}$ to the fan which defines $\mathbb{P}^1$ that is defined by dividing out the subspace spanned by $\{d_i f_i\}_{i \in I \setminus \{1\}}$. Pulling back $\mathcal{O}_{\mathbb{P}^1}(1)$ to V yields a line bundle with monomial sections X and X' pulled back from X and X', where $X'/X = X_1'$. Let $A_i := z^{(f_i,0)}$ and $A_1' = z^{(f_1',0)}$. Diverging from the ordinary case, the open subset of U where $X' \neq 0$ is now given, up to codimension two, by the hypersurface

$$A_1 \cdot A_1' = \left(\prod_{i=1}^{n} A_i^{[-\epsilon_{1j}]_+}\right)^{r_1} \left(\sum_{j=0}^{r_1} a_{1,j} \left(X_1 \prod_{i=1}^{n} A_i^{\epsilon_{1i}}\right)^{j}\right).$$

Observe that the points $(f_i, 0), (0, e_i) \in (M^\circ \oplus N)_{\mathbf{s}} = \mathcal{A}_{\mathrm{prin}}^\vee(\mathbb{Z}^T)$ both lie in the chamber of $\Delta_{\mathbf{s}}^+$ corresponding to $\mathbf{s}$ and that the point $(f_1', 0) \in (M^\circ \oplus N)_{\mathbf{s}'} = \mathcal{A}_{\mathrm{prin}}^\vee(\mathbb{Z}^T)$ lies in the chamber of $\Delta_{\mathbf{s}}^+$ corresponding to $\mathbf{s}'$. Hence, we know from Proposition 4.7.4 that each of these points determines a theta function in $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$. In fact, these theta functions are the corresponding cluster monomials: $A_i = z^{(0,e_i)}$, $X_i = z^{(f_i,0)}$, and $A_1' = z^{(f_1',0)}$. There is an analogous description for the open subset where $X \neq 0$.

Consider the ideal $J = (X_i)_{i \in I \setminus \{1\}}$. As in the ordinary case, the only wall of $\mathfrak{D}_{\mathbf{s}}$ that is non-trivial modulo J is the wall $((e_1, 0)^\perp, 1 + a_{1,1} z^{(v_1,e_1)} + \cdots + a_{1,r_1-1} z^{(r_1-1)(v_1,e_1)} + z^{r_1(v_1,e_1)})$. It follows from Theorem 4.3.7 that $\mathfrak{D}_{\mathbf{s}}$ has finitely many non-trivial walls modulo J^k. Let $\pi_N : \widetilde{M}^\circ \to N$ be the projection map and let Q be a point in the positive chamber $\mathcal{C}_{\mathbf{s}}^+$. When $\pi_N(m) \in \mathrm{Span}(e_1, \ldots, e_n)$, it also follows that the theta function $\vartheta_{Q,m}$ is regular on $U_{\mathbb{A}^1_{\mathbf{s}},k}$.

As in the ordinary case, $U_{\mathbb{G}_m,k}$ is the subscheme of U defined by the ideal J^k in the open subset $XX' \neq 0 \subset U$. The open subset defined by $XX' \neq 0$ and $\prod_{i \neq 1} X_i \neq 0$ is the union of the tori $T_{\widetilde{N}^\circ,\mathbf{s}}$ and $T_{\widetilde{N}^\circ,\mathbf{s}'}$.

We are now ready to establish the first intermediate claim. First, we introduce some useful additional notation. Let $C := \sum_{k=1}^n \mathbb{N} e_i$ and $C' := \sum_{k=1}^n \mathbb{N} e_i'$. Recall from the definition of generalized torus seed mutation that $e_1' = -e_1$, so

$$\widetilde{C} := \mathbb{Z} e_1 + \sum_{k=2}^{n} \mathbb{N} e_k = \mathbb{Z} e_1' + \sum_{k=2}^{n} \mathbb{N} e_k$$

is well-defined.

Claim 4.10.6 (Analogue of Claim 6.9 of [41])**.** *The following statements hold:*

1. *The collection $\{\vartheta_{Q,m}\}$ such that $m \in \widetilde{M}^\circ$ and $\pi_N(m) \in C \setminus (\widetilde{C}_{k+1} \cap C)$ forms a $\Bbbk[a_{i,j}]$-basis of the vector space up$(U_{\mathbb{A}^1_{\mathbf{s},k}})$.*
2. *The collection $\{\vartheta_{Q,(m,0)}\}$ such that $m \in M^\circ$ forms a basis of up$(U_{\mathbb{A}^1_{\mathbf{s},k}})$ as a $H^0(\mathbb{A}^1_{\mathbf{s},k}, \mathcal{O}_{\mathbb{A}^1_{\mathbf{s},k}})$-module.*
3. *The collection $\{\vartheta_{Q,m}\}$ such that $\pi_N(m) \in \widetilde{C} \setminus \widetilde{C}_k$ forms a $\Bbbk[a_{i,j}]$-basis of up$(U_{\mathbb{G}_m,k})$.*

The proof given in [41] for Claim 6.9 also holds in the generalized setting. There is an analogous claim for $\mathbf{s}'$.

Now, we wish to show that $S_{g,\mathbf{s}} = S_{g,\mathbf{s}'}$ for all regular functions g on $\mathcal{A}_{\text{prin}}$. Because of the N-linearity of scattering diagrams, we may multiply g by any monomial from the base of $\mathcal{A}_{\text{prin}} \to T_M$. If we multiply by a monomial in $\{X_i\}_{i\neq 1}$, we may assume that g is a regular function on the open subset of U where $XX' \neq 0$. Recall that we defined $P_{\mathbf{s}} \subset \widetilde{M}^\circ_{\mathbf{s}}$ as the monoid generated by $(v_1, e_1), \ldots, (v_n, e_n)$. Analogously, $P_{\mathbf{s}'} \subset \widetilde{M}^\circ_{\mathbf{s}'}$ be the monoid generated by $(v'_1, e'_1), \ldots, (v'_n, e'_n)$.Using the notation $\overline{S}_{g,\mathbf{s}}$ and $S_{g,\mathbf{s}}$ which was established in Definition 4.10.4, we observe that $\pi_N(m) \in \widetilde{C}$ when $m \in P_{\mathbf{s}} + \overline{S}_{g,\mathbf{s}}$ or $m \in P_{\mathbf{s}'} + \overline{S}_{g,\mathbf{s}'}$. Because $\mathfrak{D}_{\mathbf{s}}$ has finitely many non-trivial walls modulo J^k for all k, it follows that for $m \in S_{g,\mathbf{s}}$ or $m \in S_{g,\mathbf{s}'}$ the theta functions $\vartheta_{Q,m}$ and $\vartheta_{Q',m}$ are finite Laurent polynomials modulo J^k, where Q and Q' are, respectively, endpoints in the chambers corresponding to $\mathbf{s}$ and $\mathbf{s}'$. We can then establish the following claim about expansions for g in terms of such theta functions.

Claim 4.10.7 (Analogue of Claim 6.10 of [41])**.** *Modulo J^k, the sums*

$$\sum_{m \in S_{g,\mathbf{s}}} \alpha_m \vartheta_{Q,m}, \qquad \sum_{m \in S'_{g,\mathbf{s}}} \alpha'_m \vartheta_{Q',m}$$

are finite and coincide with g in the charts indexed, respectively, by $\mathbf{s}$ and $\mathbf{s}'$.

The proof given in [41] for Claim 6.10 holds in the generalized setting. By Theorem 4.7.2, the theta functions $\vartheta_{Q,m}$ and $\vartheta_{Q',m}$ induces the same regular function ϑ_m on $U_{\mathbb{G}_m,k}$ when $m \in \pi_N^{-1}(\widetilde{C})$. Thus, we get equality, modulo J^k, of the expansions from Claim 4.10.7:

$$g = \sum_{m, S_{g,\mathbf{s}}} \alpha_m \vartheta_m = \sum_{m \in S_{g,\mathbf{s}'}} \alpha'_m \vartheta_m \mod J^k.$$

Varying k, it follows from (3) of Claim 4.10.6 that the coefficients α_m, α'_m are indeed equal and therefore the map $\alpha_{\mathbf{s}}$ is independent of the choice of generalized torus seed, as desired. $\square$

4.10.2 The middle generalized cluster algebra for $\mathcal{A}_{\text{prin}}$

In order to define the *middle generalized cluster algebra*, we will need to be able to discuss a particular subset of theta functions. First, we show that the theta function ϑ_{Q,m_0}, for $Q \in \sigma\Delta^+$ and $m_0 \in \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$, is a positive universal Laurent polynomial on $\Bbbk[\widetilde{M}^\circ]$.

Proposition 4.10.8 (Analogue of Proposition 7.1 of [41])**.** *Let $\mathbf{s} = \{(e_i, (a_{i,j}))\}$ be a generalized torus seed with all $a_{i,j} \geq 0$. Fix some $m_0 \in \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$. If for some generic choice of $Q \in \sigma \in \Delta^+$ there are finitely many broken lines γ in $\mathfrak{D}_{\mathbf{s}}$ with $I(\gamma) = m_0$ and $b(\gamma) = Q$, then this holds for any generic $Q' \in \sigma' \in \Delta^+$. Hence, $\vartheta_{Q,m_0} \in \Bbbk[\widetilde{M}^\circ, a_{i,j}]$ is a positive universal Laurent polynomial.*

Proof. By Theorem 4.7.2, we know that when endpoints Q and Q' lie in different chambers, the theta functions ϑ_{Q,m_0} and ϑ_{Q',m_0} are related by a composition of wall-crossings. When the endpoint varies within a chamber, the corresponding theta function does not change. Hence, it's sufficient to check that if $Q \in \sigma$ and $Q' \in \sigma'$ are in adjacent chambers with Q' close to the wall $\sigma \cap \sigma'$, then ϑ_{Q,m_0} having finitely many terms implies that ϑ_{Q',m_0} also has finitely many terms.

Fix some generalized torus seed $\mathbf{s}$. Let the wall $\sigma \cap \sigma'$ be in $n_0^\perp$ for $n_0 \in \widetilde{N}^\circ$ with $\langle n_0, Q\rangle > 0$ and denote the wall-crossing automorphism when moving from Q to Q' by $\mathfrak{p}$. Recall that a chamber of

$\mathfrak{D}_{\mathbf{s}}$ is called *reachable* if there exists a finite, transverse path between that chamber and the positive chamber $\mathcal{C}^+_{\mathbf{s}} \subset \mathfrak{D}_{\mathbf{s}}$. By Lemma 2.10 of [41], there exists a bijection between torus seeds that are mutation equivalent to the initial torus seed $\mathbf{s}$ and reachable chambers of $\mathfrak{D}_{\mathbf{s}}$. A consequence of this bijection, as described in [55], is that there exists a sequence of mutations $\mu_{k_1}, \ldots, \mu_{k_\ell}$ and corresponding piecewise linear maps $T_{k_\ell}, \ldots, T_{k_1}$ such that $\mathbf{s}' = \mu_{k_\ell} \circ \cdots \circ \mu_{k_1}(\mathbf{s})$, $\mathfrak{D}_{\mathbf{s}'} = T_{k_\ell} \circ \cdots T_{k_1}(\mathfrak{D}_{\mathbf{s}})$, and $T_{k_{i-1}} \circ \cdots \circ T_{k_1}(\sigma') \subset H_{k_i,-}$ for each $i \in \{1, \ldots, \ell\}$. In the language of green sequences, this is equivalent to saying that there exists a sequence of green mutations from $\mathbf{s}'$ to $\mathbf{s}$.

Recall that $\mathfrak{p}(z^m) = z^m f^{\langle n_0, m\rangle}$. When both chambers, σ and σ' are reachable, f has the form $1 + a_{k,1}z^q + \cdots + a_{k,r_k-1}z^{(r_k-1)q} + z^{r_k q}$ for some $q \in n_0^\perp \subset \widetilde{M}^\circ$, $r \in \mathbb{Z}_{>0}$, and $k \in I$. In particular, note that f is a positive Laurent polynomial in $a_1, \ldots, a_{r-1}$ and z. One can verify that f has this form by recalling that the wall-crossing automorphisms associated to the positive chamber $\mathcal{C}^+_{\mathbf{s}'} \subset \mathfrak{D}_{\mathbf{s}'}$ have the form $1 + a_{k,1}z^{v_k} + \cdots + z^{r_k v_k}$ for some $k \in I$ and then applying the appropriate sequence of piecewise linear maps $T_{k_1}, \ldots, T_{k_\ell}$ to obtain the wall-crossing automorphism $\mathfrak{p}$.

Monomials z^m can be classified into three groups, based on the sign of $\langle n_0, m\rangle$. The arguments for $\langle n_0, m\rangle = 0$ or $\langle n_0, m\rangle > 0$ given in [41] will work here also. Briefly, when $\langle n_0, m\rangle = 0$, the monomial is fixed by $\mathfrak{p}$ and so these terms coincide in ϑ_{Q,m_0} and ϑ_{Q',m_0}. When $\langle n_0, m\rangle > 0$, the monomial z^m is sent to $z^m f^{\langle n_0, m\rangle}$, which is by definition a polynomial. So each such z^m in ϑ_{Q,m_0} corresponds to finitely many terms in ϑ_{Q',m_0}.

The last case is when $\langle n_0, m\rangle < 0$. Consider a broken line in $\mathfrak{D}_{\mathbf{s}}$ with endpoint $Q' \in \sigma' \subset \mathfrak{D}_{\mathbf{s}}$ and a monomial of the form cz^m with $\langle n_0, m\rangle < 0$ attached to its final domain of linearity. To complete the proof, it remains to show that there are finitely many such broken lines. By way of contradiction, assume that there actually infinitely many.

The direction vector of such a broken line must be towards the wall $\sigma \cap \sigma'$, so its final domain of linearity can be extended to some point $Q'' \in \sigma$. When crossing $\sigma \cap \sigma'$ from σ' into σ, we have

$$cz^m \mapsto cz^m \left(1 + a_{k,1}z^q + \cdots + a_{k,r_k-1}z^{(r_k-1)q} + z^{r_k q}\right)^{\langle -n_0, m\rangle}$$

for some $k \in I$. Note that the primitive normal vector $-n_0$ appears in this wall-crossing computation rather than n_0 because by assumption $\langle n_0, Q\rangle > 0$, so n_0 is directed into the chamber σ rather than into σ'. The fact that $a_{k,1}, \ldots, a_{k,r_k-1}$ are formal variables means that there are no cancellations. Because ϑ_{Q,m_0} is independent of the location of Q within the chamber σ, this means there are infinitely many broken lines with initial slope m_0 and endpoint Q, a contradiction. $\square$

This then allows us to state the following definition for any $Q \in \sigma \in \Delta^+$, rather than for some particular point, because we know the cardinality of the set of broken lines with initial slope m_0 is independent of the choice of endpoint Q.

Definition 4.10.9 (Analogue of Definition 7.2 of [41]). *Let $\Theta \subset \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$ be the collection of m_0 such that for any generic point $Q \in \sigma \in \Delta^+$, there exist finitely many broken lines with initial slope m_0 and endpoint Q.*

Definition 4.10.10 (Definition 7.3 of [41]). *A subset $S \subset \mathcal{A}^\vee_{prin}$ is* intrinsically closed under addition *if $p, q \in S$ and $\alpha(p, q, r) \neq 0$ implies that $r \in S$.*

We are now prepared to state the major result of this section:

Theorem 4.10.11 (Analogue of Theorem 7.5 of [41])**.** *Let*

$$\Delta^+(\mathbb{Z}) := \bigcup_{\sigma \in \Delta^+} \sigma \cap \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$$

be the set of integral points in the chambers of the cluster complex. Then

1. $\Delta^+(\mathbb{Z}) \subset \Theta$.

2. *For* $p_1, p_2 \in \Theta$*, the product*
$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_r \alpha(p_1, p_2, r)\vartheta_r$$
is a finite sum with non-negative integer coefficients. Moreover, if $\alpha(p_1, p_2, r) \neq 0$*, then* $r \in \Theta$.

3. *The set* Θ *is intrinsically closed under addition. For any generalized torus seed* $\mathbf{s}$*, the image of* $\Theta \subset \widetilde{M}^\circ_{\mathbf{s}}$ *is a saturated monoid.*

4. *The structure constants* $\alpha(p, q, r)$ *defined in Lemma 4.7.9 make the* $\Bbbk$*-vector space*
$$mid\,(\mathcal{A}_{prin}) := \bigoplus_{q \in \Theta} \Bbbk \cdot \vartheta_q,$$
whose basis is indexed by Θ*, into an associative and commutative* $\Bbbk[N]$*-algebra. There are canonical inclusions*
$$gen(\mathcal{A}_{prin}) \subset mid(\mathcal{A}_{prin}) \subset up(\mathcal{A}_{prin}) \subset up(\widehat{\mathcal{A}_{prin,\mathbf{s}}}) \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N]$$
where each cluster monomial $Z \in gen(\mathcal{A}_{prin})$ *is identified with* $\vartheta_{\mathbf{g}(Z)} \in mid(\mathcal{A}_{prin})$ *for its g-vector* $\mathbf{g}(Z) \in \Delta^+(\mathbb{Z})$ *and each* $\vartheta_q \in mid(\mathcal{A}_{prin})$ *is identified with a universal Laurent polynomial in* $up(\mathcal{A}_{prin})$.

Proof. The majority of the proof from [41] for the ordinary case holds in our generalized setting. We review the proof here in additional detail.

The proof of (1) follows from Corollary 4.7.5 and Proposition 4.7.6. Consider $m \in \Delta^+(\mathbb{Z})$ and let Q be a generic point within the cluster complex. If m and Q lie in the same chamber of the cluster complex, then we know from Corollary 4.7.5 that $\vartheta_{Q,m} = z^m$ and so there is exactly one broken line with initial slope m and endpoint Q. Hence, $m \in \Theta$. If $Q \in \mathcal{C}^+_{\mathbf{s}}$ and m lies in some other chamber, then we know from Proposition 4.7.6 that $\vartheta_{Q,m} = \mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}(z^m)$ where γ is a path from m to Q which passes through finitely many chambers. Because each wall-crossing automorphism has finitely many terms, each wall-crossing maps a monomial to a polynomial with finitely many terms. Hence, the sequence of finitely many wall-crossings determined by γ maps the monomial z^m to a polynomial with finitely many terms, so $m \in \Theta$. This actually also handles the case where m and Q are in distinct chambers but $Q \notin \mathcal{C}^+_{\mathbf{s}}$. Let $\mathbf{s}'$ be a generalized torus seed reachable from $\mathbf{s}$ via a sequence of mutations $\mu_{k_1}, \ldots, \mu_{k_\ell}$, such that $Q \in \mathcal{C}^+_{\mathbf{s}'}$. Let $T := T_{k_\ell} \circ \cdots \circ T_{k_1}$. By the previous argument, $\vartheta_{T(Q),T(m)}$ (computed on $\mathfrak{D}_{\mathbf{s}'}$) has finitely many terms and hence there are finitely many broken lines on $\mathfrak{D}_{\mathbf{s}'}$ with initial slope $T(m)$ and endpoint $T(Q)$. From Proposition 4.7.3, we know that this set of broken lines is in bijection with broken lines on $\mathfrak{D}_{\mathbf{s}}$ with initial slope m and endpoint Q. Hence, $m \in \Theta$.

For (2), first note that the coefficients $\alpha(p_1, p_2, r)$ are non-negative by Lemma 4.7.9. Consider $p_1, p_2 \in \Theta$ and let Q be a generic endpoint in some cluster chamber. Because each ϑ_{Q,p_i} is a Laurent polynomial, we know that $\vartheta_{Q,p_1} \cdot \vartheta_{Q,p_2}$ is a product of Laurent polynomials and therefore has finitely many terms. By Lemma 4.7.9, we know that $\vartheta_{Q,p_1} \cdot \vartheta_{Q,p_2} = \sum_r \alpha(p_1, p_2, r)\vartheta_r$. By Proposition 4.10.8, we know that ϑ_r is a positive universal Laurent polynomial in $\Bbbk[\widetilde{M}^\circ, a_{i,s}]$. As such, the summation $\sum_r \alpha(p_1, p_2, r)\vartheta_r$ is a positive linear combination of series with positive coefficients and must therefore contain finitely many terms, as it cannot contain any cancellation. Hence, each ϑ_r must have finitely many terms and be a Laurent polynomial. By definition, this means that $r \in \Theta$ and therefore Θ is intrinsically closed under addition.

For (3), the fact that Θ is intrinsically closed under addition follows immediately by definition from (2), since in (2) we showed that having $p_1, p_2 \in \Theta$ and $\alpha(p_1, p_2, r) \neq 0$ implies $r \in \Theta$. To show that Θ is saturated, consider $kq \in \Theta$ for some integer $k \geq 1$ and generic endpoint Q. By definition, having $kq \in \Theta$ means there are a finite number of broken lines with initial slope kq and generic endpoint $Q \in \sigma \in \Delta^+$. Let $S(kq)$ denote the set of final monomials on these broken lines. By assumption, $S(kq)$ is finite. For every broken line γ with initial slope q and endpoint Q, there is a corresponding broken line γ' with initial slope kq and the same underlying path, such that in each domain of linearity L of γ the monomials m_L and m'_L attached, respectively, to γ and γ' satisfy the relationship $m'_L = km_L$. Hence, $q \in S(q)$ implies that $kq \in S(kq)$ and the finiteness of $S(q)$ follows from the finiteness of $S(kq)$.

For (4), we know from Proposition 4.10.8 that each $\vartheta_{Q,p}$, for $p \in \Theta$, is a universal positive Laurent polynomial in the initial cluster variables and the exchange polynomial coefficients. By Theorem 4.10.5(4), $\vartheta_p \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ is the corresponding cluster monomial for $p \in \Delta^+(\mathbb{Z})$. The associativity of multiplication on $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$ and the inclusions follow from Lemma 4.7.9 and Proposition 4.10.2. $\square$

The following corollaries are immediate consequences of Theorem 4.10.11.

Corollary 4.10.12. *Fix a set of generalized fixed data Γ and a choice of generalized torus seed $\mathbf{s}$. When the generalized cluster algebra and the upper generalized cluster algebra coincide, the collection $\{\vartheta_{Q,m}\}_{m\in\Theta}$ forms a basis for the associated generalized cluster algebra.*

Corollary 4.10.13 (Analogue of Corollary 7.6 of [41])**.** *The following hold:*

1. *There are canonically defined non-negative structure constants*

$$\alpha : \mathcal{A}^\vee_{prin}(\mathbb{Z}^T) \times \mathcal{A}^\vee_{prin}(\mathbb{Z}^T) \times \mathcal{A}^\vee_{prin}(\mathbb{Z}^T) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

   These are given by counts of broken lines.

2. *There is a canonically defined subset $\Theta \subset \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$ with $\alpha(\Theta \times \Theta \times \Theta) \subseteq \mathbb{Z}_{\geq 0}$ such that the restriction of α gives the vector subspace mid$(\mathcal{A}_{prin}) \subset$ can$(\mathcal{A}_{prin})$ with basis indexed by Θ the structure of an associative commutative $\Bbbk$-algebra.*

3. $\Delta^+(\mathbb{Z}) \subset \Theta$

4. *For the lattice structure on $\mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$ determined by any choice of seed, $\Theta \subset \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$ is closed under addition. Furthermore, $\Theta \subset \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$ is saturated: for $k > 0$ and $x \in \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$, $k \cdot x \in \Theta$ if and only if $x \in \Theta$.*

5. *There is a canonical $\Bbbk$-algebra map $\nu : mid(\mathcal{A}_{prin}) \to up(\mathcal{A}_{prin})$ which sends ϑ_q, for $q \in \Delta^+(\mathbb{Z})$, to the corresponding global monomial.*

6. *The image $\nu(\vartheta_q) \in up(\mathcal{A}_{prin})$ is a universal Laurent polynomial.*

7. *ν is injective.*

As in the ordinary case, we can verify that the theta functions are well-behaved with respect to the canonical torus action on $\mathcal{A}_{\text{prin}}$.

Proposition 4.10.14 (Analogue of Proposition 7.7 of [41])**.** *For $q \in \Theta \subset \mathcal{A}^\vee_{prin}(\mathbb{Z})$, the theta function $\vartheta_q \in up(\mathcal{A}_{prin})$ is an eigenfunction for the natural $T_{\widetilde{K}^\circ}$ action on $\mathcal{A}_{prin}$ with weight $w(q)$ given by the map $w : \widetilde{M}^\circ = (\widetilde{N}^\circ)^* \to (\widetilde{K}^\circ)^*$. Moreover, ϑ_q is an eigenfunction for the subtorus $T_{N^\circ} \subset T_{\widetilde{K}^\circ}$ with weight $w(q)$ given by the map $w : \widetilde{M}^\circ \to \widetilde{M}^\circ$ defined by mapping $(m, n) \mapsto m - p^*(n)$.*

Proof. The proof given in [41] for the ordinary case holds in the generalized case as well, since by definition we still have $w(v_i, e_i) = v_i - p^*(e_i) = 0$. $\square$

4.11 From $\mathcal{A}_{\text{prin}}$ to $\mathcal{A}_t$ and $\mathcal{X}$

As in the ordinary case, our results for $\mathcal{A}_{\text{prin}}$ induce similar results on the $\mathcal{A}$ and $\mathcal{X}$ varieties. In this section, we adapt the results of section 7.2 of [41] for our generalized setting. As such, we will closely follow the structure of their exposition.

Recall that $\mathcal{A}_t = \pi^{-1}(t)$, where π is the canonical fibration $\mathcal{A}_{\text{prin}} \to T_M$. Consider the maps $\rho : \mathcal{A}^\vee_{\text{prin}} \to \mathcal{A}^\vee$ and $\xi : \mathcal{X}^\vee \to \mathcal{A}^\vee_{\text{prin}}$ which have tropicalizations

$$\begin{aligned} \rho^T &: (m, n) \mapsto m, \\ \xi^T &: n \mapsto (-p^*(n), -n). \end{aligned}$$

The map ρ^T identifies $\mathcal{A}^\vee(\mathbb{Z}^T)$ and the quotient of $\mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ by the natural action of N. Let $w : \mathcal{A}^\vee_{\text{prin}} \to M^\circ$ be the weight map given by $w(m, n) = m - p^*(n)$. Then ξ^T identifies $\mathcal{X}^\vee(\mathbb{Z}^T)$ with $w^{-1}(0)$.

These maps allow us to define broken lines for the $\mathcal{A}$ and $\mathcal{X}$ cases. First, recall that each wall in $\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\mathbf{s}}$ has an associated wall-crossing automorphism which is a power series in $z^{(p^*(n), n)}$ for some n. Hence, $w(m, n) = w(p^*(n), n) = 0$ for every exponent which appears in one of these wall-crossing automorphisms.

First, consider the $\mathcal{X}$ case. Suppose γ is a broken line in $\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\mathbf{s}}$ with both $I(\gamma)$ and the initial domain of linearity lying in $w^{-1}(0)$. Because every exponent that appears in a wall-crossing function lies in $w^{-1}(0)$, the monomials attached to each subsequent domain of linearity must also lie in $w^{-1}(0)$. In particular, this means that $F(\gamma)$ and $b(\gamma)$ both lie in $w^{-1}(0)$. We define the set of broken lines in $\mathcal{X}^\vee(\mathbb{R}^T)$ to be the set of such broken lines.

Next, consider the $\mathcal{A}$ case. Here, the set of broken lines in $\mathcal{A}^\vee(\mathbb{R}^T)$ is defined as $\{\rho^T(\gamma)\}$ where γ ranges over the set of broken lines in $\mathcal{A}^\vee_{\text{prin}}(\mathbb{R}^T)$.

We then define

$$\Theta(\mathcal{X}) := \Theta\left(\mathcal{A}_{\text{prin}}\right) \cap w^{-1}(0),$$
$$\Theta(\mathcal{A}_t) := \rho^T\left(\Theta\left(\mathcal{A}_{\text{prin}}\right)\right).$$

Because $\mathcal{A}^\vee(\mathbb{Z}^T)$ is identified with the quotient of $\mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ under the natural N-action, it follows that $\Theta(\mathcal{A}_{\text{prin}})$ is invariant under N-translation and therefore $\Theta(\mathcal{A}_{\text{prin}}) = \left(\rho^T\right)^{-1}(\Theta(\mathcal{A}_t))$. In fact, any section $\Sigma : \mathcal{A}^\vee(\mathbb{Z}^T) \to \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ of ρ^T will induce a bijection between $\Theta(\mathcal{A}_{\text{prin}})$ and $\Theta(\mathcal{A}_t) \times N$.

Definition 4.11.1 (Analogue of Definitions 7.12 and 7.14(2) of [41]). *Define*

$$\mathit{mid}(\mathcal{X}) := \mathit{mid}(\mathcal{A}_{prin})^{T_{N^\circ}} = \bigoplus_{q \in \Theta(\mathcal{X}} \Bbbk \vartheta_q,$$
$$\mathit{mid}(\mathcal{A}_t) := \mathit{mid}(\mathcal{A}_{prin}) \otimes_{\Bbbk[N]} \Bbbk,$$

where the map $\Bbbk[N] \twoheadrightarrow \Bbbk$ is given by $t \in T_M$.

Using this definition for $\operatorname{mid}(\mathcal{X})$, we obtain the following corollary of Theorem 4.10.13 by taking T_{N°-invariants.

Corollary 4.11.2 (Analogue of Corollary 7.13 of [41]). *The results of Theorem 4.10.13 also hold for $\mathcal{X}$.*

Each choice of generalized torus seed $\mathbf{s}$ determines fans $\Sigma_{\mathbf{s},V}$ for $V = \mathcal{A}_{\text{prin}}, \mathcal{A}$, and $\mathcal{X}$. In particular,

$$\Sigma_{\mathbf{s},\mathcal{A}} := \left\{\mathbb{R}_{\geq 0} e_i : i \in I_{\text{uf}}\right\},$$
$$\Sigma_{\mathbf{s},\mathcal{X}} := \left\{-\mathbb{R}_{\geq 0} v_i : i \in I_{\text{uf}}\right\}.$$

Lemma 4.11.3 (Analogue of Lemma 7.8 of [41]). *For $m \in \mathit{Hom}(L_{\mathbf{s}}, \mathbb{Z})$, the character z^m on $T_{L,\mathbf{s}} \subset V$ is a* global monomial *if and only if z^m is regular on $TV(\Sigma_{\mathbf{s},V})$. The character z^m is regular on $TV(\Sigma_{\mathbf{s},V})$ if and only if $\langle m, n\rangle \geq 0$ for the primitive generators n of each ray in $\Sigma_{\mathbf{s},V}$. When V is an $\mathcal{A}$-type cluster variety, the set of global monomials exactly coincides with the set of cluster monomials. That is, every global monomial is a monomial in the variables of a single cluster with non-negative exponents on the non-frozen variables.*

Proof. The proof given in [41] for the ordinary case holds in the generalized setting, with one minor change: the support of Z_i is now defined by the zero locus of polynomials rather than binomials. I.e., its support is now $1 + a_{i,1} z^{v_i} + \cdots + a_{i,r_i-1} z^{(r_i-1)v_i} + z^{r_i v_i} = 0$ rather than $1 + z^{v_i} = 0$. Subsequent portions of the proof still hold after this change is made.

Note that the proof in the ordinary case uses the Laurent phenomenon. Because the Laurent phenomenon holds in the reciprocal generalized setting, see Theorem 2.5.8, this portion of the proof extends to our setting without modification. ☐

Recall that there exist canonical maps $\rho : \mathcal{A}^\vee_{\text{prin}} \to \mathcal{A}^\vee$ and $\xi : \mathcal{X}^\vee \to \mathcal{A}^\vee_{\text{prin}}$ with tropicalizations $\rho^T : (m,n) \mapsto m$ and $\xi^T : n \mapsto (-p^*(n), -n)$. Let $w : \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T) \to M^\circ$ be the weight map given by $(m,n) \mapsto m - p^*(n)$.

Observe that ρ^T identifies $\mathcal{A}^\vee(\mathbb{Z}^T)$ with the quotient of $\mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ by the natural N-action. Observe also that ξ identifies $\mathcal{X}^\vee$ with the fiber of $w : \mathcal{A}^\vee_{\text{prin}} \to T_{M^\circ}$ over the identity element e, so ξ^T identifies $\mathcal{X}^\vee(\mathbb{Z}^T)$ with $w^{-1}(0)$.

Definition 4.11.4. *For a generalized cluster variety $V = \bigcup_{\mathbf{s}} T_{L,\mathbf{s}}$, let $\mathcal{C}^+_{\mathbf{s}}(\mathbb{Z}) \subset V^\vee(\mathbb{Z}^T)$ denote the set of g-vectors of the associated generalized cluster algebra and $\Delta^+_V(\mathbb{Z}) \subset V^\vee(\mathbb{Z}^T)$ denote the union of all $\mathcal{C}^+_{\mathbf{s}}(\mathbb{Z})$.*

Lemma 4.11.5 (Analogue of 7.10 of [41])**.** *1. For an $\mathcal{A}$-type generalized cluster variety, $\mathcal{C}^+_{\mathbf{s}}$ is the set of integral points of the cone $\mathcal{C}^+_{\mathbf{s}}$ in the Fock-Goncharov cluster complex which corresponds to the seed* $\mathbf{s}$.

2. *For both $\mathcal{A}$-type generalized cluster varieties and $\mathcal{X}$, $\mathcal{C}^+_{\mathbf{s}}$ is the set of integral points of a rational convex cone $\mathcal{C}^+_{\mathbf{s}}$ and the relative interiors of $\mathcal{C}^+_{\mathbf{s}}$ as* $\mathbf{s}$ *varies are disjoint. The g-vector* $\mathbf{g}(f) \in V^\vee(\mathbb{Z}^T)$ *depends only on the function f. That is, if f restricts to a character on two distinct seed tori, the g-vectors they determine are the same.*

3. *For $m \in w^{-1}(0) \cap \Delta^+_{\mathcal{A}_{prin}}(\mathbb{Z})$, the global monomial ϑ_m on $\mathcal{A}_{prin}$ is invariant under the T_{N° action and thus gives a global function on $\mathcal{X} = \mathcal{A}_{prin}/T_{N^\circ}$. This is a global monomial and all global monomials on $\mathcal{X}$ occur in this way. Moreover, $m = \mathbf{g}(\vartheta_m)$.*

Proof. The proof given in [41] for the ordinary case holds in our generalized setting, as we have proven analogs of all the necessary previous results. We quickly review the proof given in [41] in order to point out each place where we are instead using an analogous result for the generalized setting.

First, consider (1) for an $\mathcal{A}$-type generalized cluster variety. By Lemma 4.5.3 and Lemma 4.11.3, the positive chamber $\mathcal{C}^+_{\mathbf{s}}$ is the Fock-Goncharov cluster chamber associated to $\mathbf{s}$ and a maximal cone of a simplicial fan. By Theorem 4.5.4, $\Delta^+_{\mathcal{A}}(\mathbb{Z})$ forms a simplicial fan.

The $\mathcal{A}$ case, which includes $\mathcal{A}_{\text{prin}}$, of (2) follows from Section 4.9. The $\mathcal{X}$ case follows from the $\mathcal{A}_{\text{prin}}$ case. Recall that the map $\widetilde{p} : \mathcal{A}_{\text{prin}} \to \mathcal{X}$ given by $z^{(n,m)} \mapsto z^{m-p^*(n)}$ makes $\mathcal{A}_{\text{prin}}$ into a T_{N°-torsor over $\mathcal{X}$. Hence, pulling back a monomial on $\mathcal{X}$ yields a T_{N°-invariant global monomial on $\mathcal{A}_{\text{prin}}$. By Proposition 4.10.14, we have the inclusion $\Delta^+_{\mathcal{X}}(\mathbb{Z}) \subseteq w^{-1}(0) \cap \Delta^+_{\mathcal{A}_{\text{prin}}}$. Conversely, suppose $m \in w^{-1}(0)$ and $m = \mathbf{g}(f)$ for some global monomial f on $\mathcal{A}_{\text{prin}}$. Then there exists some generalized torus seed $\mathbf{s} = \{(e_i, \mathbf{a}_i)\}$ such that f is represented by a monomial z^m on $T_{\widetilde{N},\mathbf{s}}$. Since $m \in w^{-1}(0)$, it must be of the form $m = (p^*(n), n)$ for some $n \in N$. By Lemma 4.11.3, m is non-negative on the rays $\mathbb{R}_{\geq 0}(e_i, 0)$ of $\Sigma_{\mathbf{s},\mathcal{A}_{\text{prin}}}$ and therefore n is non-negative on the rays $-\mathbb{R}_{\geq 0}v_i$ of $\Sigma_{\mathbf{s},\mathcal{X}}$ and z^n is a global monomial on $\mathcal{X}$. As such, $\Delta^+_{\mathcal{X}}(\mathbb{Z}) = w^{-1}(0) \cap \Delta^+_{\mathcal{A}_{\text{prin}}}(\mathbb{Z})$ and the cones for $\mathcal{X}$ are given by intersecting the cones for $\mathcal{A}_{\text{prin}}$ with $w^{-1}(0)$. This also gives (3). □

Lemma 4.11.6 (Analogue of 7.14(2) of [41])**.** *Given Σ, the collection $\{\vartheta_m\}_{m \in \Sigma(M^\circ)}$ forms a $\Bbbk[N]$-module basis for mid($\mathcal{A}_{prin}$) and therefore a $\Bbbk$-vector space basis for mid($\mathcal{A}_t$). For mid($\mathcal{A}_t$), this basis is independent of the choice of Σ up to scaling each basis vector. For mid($\mathcal{A}$), however, the basis is entirely independent.*

Recall that the variety $\mathcal{A}_t$ is defined as the space $\bigcup_{\mathbf{s}} T_{N^\circ,\mathbf{s}}$ where the tori are glued according to birational maps which depend on the parameter t. Because the tropicalizations of these birational maps are independent of t, we have $\mathcal{A}^\vee_t(\mathbb{Z}^T) = \mathcal{A}^\vee(\mathbb{Z}^T)$.

Theorem 4.11.7 (Analogue of 7.16 of [41]). *The following statements hold:*

1. *Given a choice of section $\Sigma : \mathcal{A}^\vee(\mathbb{Z}^T) \to \mathcal{A}^\vee_{prin}(\mathbb{Z}^T)$, there exists a map*

$$\alpha_{\mathcal{A}_t} : \mathcal{A}^\vee_t(\mathbb{Z}^T) \times \mathcal{A}^\vee_t(\mathbb{Z}^T) \times \mathcal{A}^\vee_t(\mathbb{Z}^T) \to \Bbbk \cup \{\infty\},$$

   given by the mapping

$$(p, q, r) \mapsto \sum_{n \in N} \alpha_{\mathcal{A}_{prin}}(\sigma(p), \Sigma(q), \Sigma(r) + n)\, z^n(t)$$

   if the summation is finite. Otherwise, $\alpha_{\mathcal{A}_t}(p, q, r) = \infty$. If $p, q, r \in \Theta(\mathcal{A}_t)$, then the summation will be finite.

2. *Let $\Theta = \Theta(\mathcal{A}_t) \subset \mathcal{A}^\vee_t(\mathbb{Z}^T)$. Let mid$(\mathcal{A}_t) \subset$ can$(\mathcal{A}_t)$ be the vector subspace with basis indexed by Θ. Restricting the structure constants gives mid$(\mathcal{A}_t)$ the structure of an associative and commutative $\Bbbk$-algebra.*

3. *The subset Θ contains the g-vector of each global monomial - i.e., $\Delta^+_{\mathcal{A}_t}(\mathbb{Z}^T) \subset \Theta$.*

4. *The choice of generalized torus seed determines a lattice structure on $\mathcal{A}^\vee_t(\mathbb{Z}^T)$. Within this lattice structure, the subset $\Theta \subset \mathcal{A}^\vee_t(\mathbb{Z}^T)$ is both closed under addition and saturated.*

5. *There exists a $\Bbbk$-algebra map ν : mid$(\mathcal{A}_t) \to$ up$(\mathcal{A}_t)$ which maps ϑ_p, for $p \in \Delta^+_{\mathcal{A}_t}(\mathbb{Z}^T)$, to a multiple of the corresponding global monomial.*

6. *If the vectors $\{v_i\}_{i \in I_{uf}}$ lie in a strictly convex cone, then the map ν is injective. When ν is injective, there exist canonical inclusions*

$$\mathrm{gen}(\mathcal{A}_t) \subset \mathrm{mid}(\mathcal{A}_t) \subset \mathrm{up}(\mathcal{A}_t).$$

By taking $t = e$, we obtain all of the above statements for $\mathcal{A}$.

Proof. The proof given in [41] for the ordinary case holds in the generalized setting, with our Proposition 4.10.14 replacing the use of Proposition 7.7 and our Proposition 4.4.4 replacing the use of Proposition B.2. □

4.12 Companion algebras

In Section 2.5.1, we reviewed the definitions and important properties of the *companion algebras* associated to a generalized cluster algebra. In this section, we will discuss companion algebras in the context of cluster scattering diagrams. For a given generalized cluster algebra, we will explicitly state the fixed data of the associated companion algebras and show that it satisfies the c-vectors, g-vectors, and F-polynomial relationships from Section 2.5.1. After reviewing the notions of *tropical duality* and *Langlands duality*, we will show that the fixed data of the left and right companions are Langlands dual, up to isomorphism.

4.12.1 Langlands duality and tropical duality

In this subsection, we restrict our attention to ordinary cluster algebras in order to briefly review the definitions of Langlands duality and tropical duality. In subsection 4.12.2, we will explore how these notions of duality appear in the context of companion algebras to generalized cluster algebras.

Langlands duality

In [20], Fock and Goncharov give the following definitions for the *Langlands dual* of a set of fixed data and a torus seed:

Definition 4.12.1. *Given fixed data* Γ *and torus seed* $\mathbf{s}$*, let* $D := lcm(d_1, \dots, d_n)$*. The* Langlands dual *of* Γ *is the fixed data* $\Gamma^\vee$ *defined by*

$$\begin{aligned} N^\vee &:= N^\circ, \\ I^\vee &:= I, \\ I^\vee_{uf} &:= I_{uf}, \\ d_i^\vee &:= d_i^{-1} D, \\ \{\cdot,\cdot\}^\vee &:= D^{-1}\{\cdot,\cdot\}. \end{aligned}$$

The Langlands dual *of* $\mathbf{s}$ *is the torus seed* $\mathbf{s}^\vee := (d_1 e_1, \dots, d_n e_n)$.

From these definitions, we can observe the following relationship between the bilinear forms of Γ and $\Gamma^\vee$.

$$\begin{aligned} \{e_i, e_j\} d_j &= \frac{1}{d_i}\{d_i e_i, d_j e_j\} \\ &= -D^{-1}\{d_j e_j, d_i e_i\}\left(\frac{D}{d_i}\right) \\ &= -\{e_j^\vee, e_i^\vee\}^\vee (d_i^\vee) \end{aligned}$$

It follows that $\epsilon_{ij} = -\epsilon^\vee_{ji}$ and that, on the matrix level, $\epsilon = -(\epsilon^\vee)^T$. Recall from Definition 2.3.2 that the exchange matrix B (in the Fomin-Zelevinsky sense) can be represented as

$$b_{ij} = \epsilon_{ij} = \{e_i, e_j\} d_j$$

for a given choice of torus seed $\mathbf{s}$. Hence, the Langlands dual of an ordinary cluster algebra defined by the exchange matrix $B = [\epsilon_{ij}]$ is simply the ordinary cluster algebra defined by $-B^T = [-\epsilon^T_{ij}]$.

Tropical duality

The study of cluster algebras often involves a different type of duality, called *tropical duality.* In [26], Fomin and Zelevinsky define c-vectors and g-vectors as the tropicalizations of the A and X cluster variables. From another perspective, Gross, Hacking, Keel, and Kontsevich showed in Lemma 2.10 of [41] that given a choice of seed data $\mathbf{s}$, $M^\circ_{\mathbb{R},\mathbf{s}} \cong \mathcal{A}^\vee(\mathbb{R}^T)$, where $\mathcal{A}^\vee(\mathbb{R}^T)$ denotes the tropicalization of the Fock-Goncharov dual variety. By explaining the connection between the lattices used in [41] and the c-vectors and g vectors, we can connect these seemingly disparate statements.

Consider the fixed data Γ and choice of initial torus seed data $\mathbf{s}_{\text{in}} = (e_{\text{in},i})$. By definition, the basis vectors of $N_{\mathbf{s}_{\text{in}}}$ are $\{e_{\text{in},i}\}_{i\in I}$ and the basis vectors of $M^\circ_{\mathbf{s}_{\text{in}}}$ are $\{f_{\text{in},i}\}_{i\in I}$, where $f_{\text{in},i} = \frac{1}{d_i}e^*_{\text{in},i}$. Recall that $\mathcal{A}$ cluster variables have the form z^m with $m \in M^\circ_{\mathbf{s}_{\text{in}}}$ and that the $\mathcal{X}$ cluster variables have the form z^n with $n \in N_{\mathbf{s}_{\text{in}}}$. Because the c-vectors and g-vectors are the tropicalizations of these cluster variables, we can express them as $\mathbf{c}_{\mathbf{s}_{\text{in}},i} = \sum_k c_{ik}e_{\text{in},k}$ and $\mathbf{g}_{\mathbf{s}_{\text{in}},i} = \sum_k g_{ik}f_{\text{in},k}$.

Now, consider an arbitrary seed $\mathbf{s} = (e_{\mathbf{s},i})$. Now, the lattice $N_{\mathbf{s}}$ has basis vectors $\{e_{\mathbf{s},i}\}_{i\in I}$ and the lattice $M_{\mathbf{s}}$ has basis vectors $\{f_{\mathbf{s},i}\}_{i\in I}$. The corresponding $\mathcal{A}$ and $\mathcal{X}$ cluster variables have g-vectors and c-vectors which we can write as $g_{\mathbf{s},i} = f_{\mathbf{s},i} = \sum_k g_{ik}f_{\text{in},k}$ and $c_{\mathbf{s},i} = e_{\mathbf{s},i} = \sum_k c_{ik}e_{\text{in},k}$. We denote by $C^\epsilon_{\mathbf{s}}$ (respectively, $G^\epsilon_{\mathbf{s}}$) the integer matrix with columns $\mathbf{c}_{1;\mathbf{s}},\dots,\mathbf{c}_{n;\mathbf{s}}$ (respectively, with columns $\mathbf{g}_{1;\mathbf{s}},\dots,\mathbf{g}_{n;\mathbf{s}}$), where $\epsilon = [\epsilon_{ij}]$ is the matrix defined by $\mathbf{s}_{\text{in}}$.

Nakanishi and Zelevinsky proved the following identity, referred to as a *tropical duality*, between the c-vectors and g-vectors.

Theorem 4.12.2 (Theorem 1.2 of [63]). *For any torus seed* $\mathbf{s}$ *and the associated matrix* $\epsilon = [\epsilon_{ij}] = [\{e_i, e_j\}d_j]$, *take* $\epsilon_{\text{in}} = \epsilon$ *and let* $\epsilon_{\mathbf{s}}$ *denote the matrix associated to the torus seed* $\mathbf{s}$. *Then,*

$$(G^\epsilon_{\mathbf{s}})^T = (C^{-\epsilon^T}_{\mathbf{s}})^{-1}, \tag{4.4}$$

We can also understand this tropical duality in the language of cluster scattering diagrams. In the previous subsection, we saw that replacing ϵ with $-\epsilon^T$ in the fixed data Γ is equivalent to considering its Langlands dual, $\Gamma^\vee$. Consider some choice of torus seed $\mathbf{s} = \{e_{\mathbf{s},i}\}_{i\in I}$ associated to Γ. As explained above, the associated collections of g-vectors and c-vectors are, respectively, $\{f_{\mathbf{s},i}\}$ and $\{e_{\mathbf{s},i}\}$. The associated collections of g-vectors and c-vectors of the Langlands dual torus seed data $\mathbf{s}^\vee$ are therefore $\{f^\vee_{\mathbf{s},i} = d_iD^{-1}f_{\mathbf{s},i}\}$ and $\{e^\vee_{\mathbf{s},i} = d_ie_{s,i}\}$, respectively. One can immediately see that the bases $\{f_{\mathbf{s},i}\}$ and $\{e^\vee_{\mathbf{s},i} = d_ie_{\mathbf{s},i}\}$ are dual, because

$$\langle e^\vee_{\mathbf{s},i}, f_{\mathbf{s},i}\rangle = \langle d_ie_{\mathbf{s},i}, d_i^{-1}e^*_{\mathbf{s},i}\rangle = \langle e_{\mathbf{s},i}, e^*_{\mathbf{s},i}\rangle = 1,$$

which implies the tropical duality in the language of seed basis vectors.

4.12.2 Fixed data

Fix a generalized cluster seed $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$, which defines the generalized cluster algebra $\mathcal{A}$. We will assume that $\mathcal{A}$ is a reciprocal generalized cluster algebra, so we can use our construction of cluster scattering diagrams for reciprocal generalized cluster algebras. The corresponding generalized fixed data Γ has lattices N, M, M°, N°; index sets I and I_{uf}; collections of scalars $\{d_i\}$ and $\{r_i\}$; and the collection of formal variables $\{a_{i,j}\}_{i\in I, j\in[r_i-1]}$, as specified in Definition 4.1.1. Consider the initial generalized torus seed data $\mathbf{s} = \{(e_i, (a_{i,j}))\}_{i\in I}$, where $\{e_i\}_{i\in I}$ forms a basis for N, $\{d_ie_i\}_{i\in I}$ for N°, $\{e^*_i\}_{i\in I}$ for M, and $\{f_i = \frac{1}{d_i}e^*_i\}_{i\in I}$ for M°. As usual, we let $v_i = p^*_1(e_i) \in M^\circ$ for $i \in I_{\text{uf}}$.

As explained in Section 2.5.1, the generalized cluster algebra $\mathcal{A}$ has an associated pair of companion algebras, ${}^{\text{L}}\mathcal{A}$ and ${}^{\text{R}}\mathcal{A}$. We can explicitly described the fixed data associated to the left and right companion algebras. In doing so, we will use the superscript ${}^{\text{C}}$ to indicate when we're considering an object in a generic companion algebra and the superscripts ${}^{\text{L}}$ and ${}^{\text{R}}$, respectively, to denote the corresponding notions in the left and right companion algebras. The data associated to the left and right companion algebras is summarized below.

	Left	Right
$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$
${}^{C}d_i$	${}^{\mathrm{L}}d_i = d_i r_i$	${}^{\mathrm{R}}d_i = \frac{d_i}{r_i}$
${}^{C}e_i$	${}^{\mathrm{L}}e_i = e_i$	${}^{\mathrm{R}}e_i = r_i e_i$
${}^{C}f_i$	${}^{\mathrm{L}}f_i = \frac{1}{{}^{\mathrm{L}}d_i}{}^{\mathrm{L}}e_i^* = \frac{1}{d_i r_i}e_i^* = \frac{1}{r_i}f_i$	${}^{\mathrm{R}}f_i = \frac{1}{{}^{\mathrm{R}}d_i}{}^{\mathrm{R}}e_i^* = \frac{r_i}{d_i}\frac{1}{r_i}e_i^* = f_i$
${}^{C}N$	${}^{\mathrm{L}}N = N$	${}^{\mathrm{R}}N = \text{span}\,\{r_i e_i\}$
${}^{C}N^\circ$	${}^{\mathrm{L}}N^\circ = \text{span}\,\{r_i(d_i e_i)\}$	${}^{\mathrm{R}}N^\circ = \text{span}\,\{d_i e_i\}$
${}^{C}M$	${}^{\mathrm{L}}M = \text{span}\,\{e_i^*\} = M$	${}^{\mathrm{R}}M = \text{span}\,\{\frac{1}{r_i}e_i^*\}$
${}^{C}M^\circ$	${}^{\mathrm{L}}M^\circ = \text{span}\,\{{}^{\mathrm{L}}f_i\}$	${}^{\mathrm{R}}M^\circ = \text{span}\,\{{}^{\mathrm{R}}f_i\}$
${}^{C}x_i$	${}^{\mathrm{L}}x_i = z^{{}^{\mathrm{L}}f_i} = x_i^{1/r_i}$	${}^{\mathrm{R}}x_i = z^{{}^{\mathrm{R}}f_i} = x_i$
${}^{C}y_i$	${}^{\mathrm{L}}y_i = z^{{}^{\mathrm{L}}e_i} = z^{e_i} = y_i$	${}^{\mathrm{R}}y = z^{{}^{\mathrm{R}}e_i} = z^{r_i e_i} = y_i^{r_i}$
${}^{C}\hat{y}$	${}^{\mathrm{L}}\hat{y} = z^{(v_i, e_i)} = \hat{y}$	${}^{\mathrm{R}}\hat{y} = z^{(r_i v_i, r_i e_i)} = \hat{y}^{r_i}$
${}^{C}v_i$	${}^{\mathrm{L}}v_i = \{{}^{\mathrm{L}}e_i, \cdot\} = \{e_i, \cdot\} = v_i$	${}^{\mathrm{R}}v_i = \{{}^{\mathrm{R}}e_i, \cdot\} = \{r_i e_i, \cdot\} = r_i v_i$
${}^{C}\langle d_i e_i, f_i\rangle$	$\langle {}^{\mathrm{L}}d_i {}^{\mathrm{L}}e_i, {}^{\mathrm{L}}f_i\rangle = \langle d_i r_i e_i, \frac{1}{r_i}f_i\rangle = 1$	$\langle {}^{\mathrm{R}}d_i {}^{\mathrm{R}}e_i, {}^{\mathrm{R}}f_i\rangle = \langle d_i r_i^{-1}(r_i e_i), f_i\rangle = 1$

Table 4.2: Fixed data for the companion algebras of an arbitrary generalized cluster algebra.

Note that it is possible for a right companion algebra to have fixed data such that some ${}^{\mathrm{R}}d_i$ are non-integral. In fact, ${}^{\mathrm{R}}d_i$ will be non-integral whenever $r_i \neq 1$. To ensure that all ${}^{\mathrm{R}}d_i$ are integral, one could scale ${}^{\mathrm{R}}d = ({}^{\mathrm{R}}d_i)$ by a factor of $\text{lcm}(r_i)$. Any computations done using a cluster scattering diagram generated by the scaled fixed data would then require that we account for this scaling.

In fact, the unscaled fixed data ${}^{\mathrm{R}}\Gamma$ can still be used to define a sensible cluster scattering diagram that allows for easy computation of cluster variables, etc. To understand the impact of non-integral values of ${}^{\mathrm{R}}d_i$ on the construction of a cluster scattering diagram, we can consider the impact on the associated lattices and initial cluster scattering diagram. It's clear that ${}^{\mathrm{R}}N$ will always be an integral lattice, since ${}^{\mathrm{R}}N = \text{span}\{r_i e_i\}$. The scaling of ${}^{\mathrm{R}}e_i$ guarantees that ${}^{\mathrm{R}}N^\circ$ is always an integral lattice, since

$${}^{\mathrm{R}}N^\circ = \text{span}\left\{{}^{\mathrm{R}}d_i {}^{\mathrm{R}}e_i\right\} = \text{span}\left\{\frac{d_i}{r_i} r_i e_i\right\} = \text{span}\left\{d_i e_i\right\}.$$

Note that this scaling also means that

$${}^{\mathrm{R}}M^\circ = \text{span}\{{}^{\mathrm{R}}f_i\} = \text{span}\left\{\frac{1}{{}^{\mathrm{R}}d_i}\left({}^{\mathrm{R}}e_i\right)^*\right\} = \text{span}\left\{\frac{r_i}{d_i}\frac{1}{r_i}e_i^*\right\} = \text{span}\left\{\frac{1}{d_i}e_i^*\right\} = \text{span}\{f_i\}.$$

By definition, ${}^{\mathrm{R}}\mathcal{A}$ has initial scattering diagram

$$ {}^{\mathrm{R}}\mathfrak{D}_{\mathrm{in}} = \left\{ (({}^{\mathrm{R}}e_i)^{\perp}, 1 + z^{{}^{\mathrm{R}}v_i}) \right\}_{i \in I_{\mathrm{uf}}} = \left\{ ((r_i e_i)^{\perp}, 1 + z^{r_i v_i}) \right\}_{i \in I_{\mathrm{uf}}}, $$

where all the wall-crossing automorphisms have integral exponents. As such, the algorithm for producing a consistent scattering diagram that was introduced by Kontsevich and Soibelman in two-dimensions and then extended to higher dimensions by Gross and Siebert can be applied to ${}^{\mathrm{R}}\mathfrak{D}_{\mathrm{in}}$. Likewise, other major results of Gross, Hacking, Keel, and Kontsevich [41] which rely on the wall-crossing automorphisms having integer exponents still hold for the resulting consistent scattering diagram.

Example 4.12.3. *Consider the generalized cluster algebra from Example 4.3.3,*

$$ \mathcal{A}\left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, ((1, a, a, 1), (1, 1))\right). $$

This generalized cluster algebra has companion algebras:

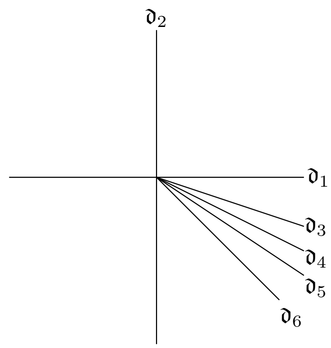
$$ {}^{L}\mathcal{A} = \left((x_1^{1/3}, x_2), (y_1, y_2), \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \right) $$
$$ {}^{R}\mathcal{A} = \left((x_1, x_2), (y_1^{3}, y_2), \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix} \right) $$

Recall from Example 4.1.4 that $\mathcal{A}$ has fixed data $d = (1,1)$, $r = (3,1)$, $I = I_{uf} = \{1,2\}$, $N = N^{\circ} = \langle e_1, e_2 \rangle$, $M = M^{\circ} = \langle e_1^*, e_2^* \rangle$, and skew-symmetric form $\{\cdot,\cdot\} : N^{\circ} \times N^{\circ} \to \mathbb{Z}$ specified by the exchange matrix. The fixed and seed data of its associated companion algebras are summarized in the following table.

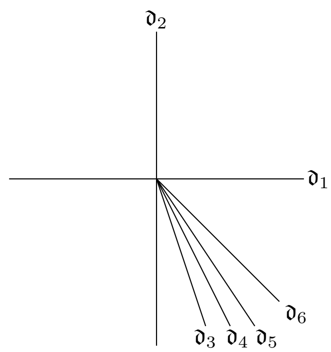
	${}^{L}\mathcal{A}$	${}^{R}\mathcal{A}$
${}^{C}d$	$(3,1)$	$(\frac{1}{3},1)$
${}^{C}e_i$	e_1, e_2	$3e_1, e_2$
${}^{C}f_i$	$\frac{1}{3}f_1, f_2$	f_1, f_2
${}^{C}N$	*span* $\{e_1, e_2\}$	*span* $\{3e_1, e_2\}$
${}^{C}N^{\circ}$	*span* $\{3e_1, e_2\}$	*span* $\{e_1, e_2\}$
${}^{C}M$	*span* $\{e_1^*, e_2^*\}$	*span* $\left\{\frac{1}{3}e_1^*, e_2^*\right\}$
${}^{C}M^{\circ}$	*span* $\left\{\frac{1}{3}f_1, f_2\right\}$	*span* $\{f_1, f_2\}$
${}^{C}v_i$	v_1, v_2	$r_1 v_1, r_2 v_2$

Table 4.3: Fixed data for the companion algebras of a generalized cluster algebra with $r = (3,1)$.

The cluster scattering diagrams for ${}^{L}\mathcal{A}$ and ${}^{R}\mathcal{A}$ are shown below (on the top and bottom, respectively).



$$\begin{aligned} f_{\mathfrak{d}_1} &= 1 + z^{(-3,0)} \\ f_{\mathfrak{d}_2} &= 1 + z^{(0,1)} \\ f_{\mathfrak{d}_3} &= 1 + z^{(-3,3)} \\ f_{\mathfrak{d}_4} &= 1 + z^{(-3,2)} \\ f_{\mathfrak{d}_5} &= 1 + z^{(-6,3)} \\ f_{\mathfrak{d}_6} &= 1 + z^{(-3,1)} \end{aligned}$$



$$\begin{aligned} f_{\mathfrak{d}_1} &= 1 + z^{(-1,0)} \\ f_{\mathfrak{d}_2} &= 1 + z^{(0,3)} \\ f_{\mathfrak{d}_3} &= 1 + z^{(-1,3)} \\ f_{\mathfrak{d}_4} &= 1 + z^{(-3,6)} \\ f_{\mathfrak{d}_5} &= 1 + z^{(-2,3)} \\ f_{\mathfrak{d}_6} &= 1 + z^{(-3,2)} \end{aligned}$$

Note that the left and right companion algebras are, up to isomorphism, Langlands dual according to the definition given in Section 4.12.1.

Proposition 4.12.4. *Let $\mathcal{A}$ be an arbitrary reciprocal generalized cluster algebra with companion algebras ${}^{\mathrm{L}}\mathcal{A}$ and ${}^{\mathrm{R}}\mathcal{A}$. Let Γ denote the fixed data of $\mathcal{A}$ and fix some choice of generalized torus seed $\mathbf{s} = \{(e_i, (a_{i,j})\}_{i \in I, j \in [r_i - 1]}$. Let ${}^{\mathrm{L}}\Gamma$ and ${}^{\mathrm{R}}\Gamma$ denote the fixed data of ${}^{\mathrm{L}}\mathcal{A}$ and ${}^{\mathrm{R}}\mathcal{A}$, respectively. Then,*

$$\left({}^{\mathrm{L}}\Gamma\right)^{\vee} \simeq {}^{\mathrm{R}}\Gamma \qquad \textit{and} \qquad \left({}^{\mathrm{R}}\Gamma\right)^{\vee} \simeq {}^{\mathrm{L}}\Gamma.$$

Proof. Let the fixed data Γ of $\mathcal{A}$ consist of the lattices N, M, M°, N°; index sets I and I_{uf}; collections of scalars $\{d_i\}$ and $\{r_i\}$; and the collection of formal variables $\{a_{i,j}\}_{i \in I, j \in [r_i - 1]}$. Fix a choice of associated generalized torus seed $\mathbf{s} = \{(e_i, (a_{i,j}))_{i \in I, j \in [r_i - 1]}\}$. Recall from previous definitions that the sets of fixed data ${}^{\mathrm{L}}\Gamma$, $({}^{\mathrm{L}}\Gamma)^\vee$, and ${}^{\mathrm{R}}\Gamma$ are:

	${}^{\mathrm{L}}\Gamma$	$({}^{\mathrm{L}}\Gamma)^\vee$	${}^{\mathrm{R}}\Gamma$
d_i	$d_i r_i$	$\frac{{}^{\mathrm{L}}D}{{}^{\mathrm{L}}d_i} = \frac{{}^{\mathrm{L}}D}{d_i r_i}$	$\frac{d_i}{r_i}$
e_i	e_i	$d_i e_i$	$r_i e_i$
N	span $\{e_i\}$	span $\{r_i d_i e_i\}$	span $\{r_i e_i\}$
N°	span $\{r_i d_i e_i\}$	span $\{e_i\}$	span $\{d_i e_i\}$
M	span $\{e_i^*\}$	span $\{\frac{1}{r_i d_i} e_i^*\}$	span $\{\frac{r_i}{d_i} e_i^*\}$
M°	span $\left\{\frac{1}{r_i d_i} e_i^*\right\}$	span $\{\frac{1}{r_i d_i D} e_i^*\}$	span $\{\frac{1}{d_i} e_i^*\}$

where ${}^{\mathrm{L}}D = \mathrm{lcm}({}^{\mathrm{L}}d_i)$. The isomorphism between $({}^{\mathrm{L}}\Gamma)^{\vee}$ and ${}^{\mathrm{R}}\Gamma$ is given by the maps

$$\begin{aligned}({}^{\mathrm{L}}d_i)^{\vee} &\mapsto \frac{d_i^2}{D}\cdot\left({}^{\mathrm{L}}d_i\right)^{\vee},\\ ({}^{\mathrm{L}}e_i)^{\vee} &\mapsto \frac{1}{r_i d_i}\cdot({}^{\mathrm{L}}e_i)^{\vee}.\end{aligned}$$

Observe that

$$({}^{\mathrm{L}}d_i)^{\vee} \mapsto \frac{d_i^2}{{}^{\mathrm{L}}D}\cdot\left({}^{\mathrm{L}}d_i\right)^{\vee} = \frac{d_i^2}{{}^{\mathrm{L}}D}\cdot\frac{{}^{\mathrm{L}}D}{d_i r_i} = \frac{d_i}{r_i} = {}^{\mathrm{R}}d_i$$

and

$$({}^{\mathrm{L}}e_i)^{\vee} \mapsto \frac{1}{r_i d_i}\cdot({}^{\mathrm{L}}e_i)^{\vee} = \frac{1}{r_i d_i}\cdot(r_i d_i e_i) = e_i = {}^{\mathrm{R}}e_i.$$

These maps induce the appropriate lattice isomorphisms. The argument that $\left({}^{\mathrm{R}}\Gamma\right)^{\vee} \simeq {}^{\mathrm{L}}\Gamma$ is analogous. □

Based on the explicit fixed data of the companion algebras, we can make several other useful observations. First, the c-vectors of the left companion algebras and the generalized algebras coincide because the torus seed basis vectors are the same. On the other hand, for the g-vectors we have

$$\begin{aligned}{}^{\mathrm{L}}\mathbf{g}_{\mathbf{s},j} &= \frac{1}{r_j}g_{\mathbf{s},j}\\ &= \frac{1}{r_j}\sum_i g_{ji}\mathbf{f}_{\mathrm{in},i}\\ &= \frac{1}{r_j}\sum_i \frac{r_i}{r_j}g_{ji}{}^{\mathrm{L}}\mathbf{f}_{\mathrm{in},i}.\end{aligned}$$

Hence, we have ${}^{\mathrm{L}}\mathbf{g}_{\mathbf{s},j} = [\frac{r_i}{r_j}g_{ji}]_{i\in I}$ as in Corollary 4.2 of [62]. We can similarly deduce that the g-vectors of the right companion algebras and the generalized algebra are the same, since $f_i = {}^{R}f_i$. The c-vectors are related as

$$\begin{aligned}{}^{\mathrm{R}}\mathbf{c}_{\mathbf{s},j} &= r_j\mathbf{c}_{\mathbf{s},j}\\ &= r_j\sum_i c_{ji}\mathbf{c}_{\mathrm{in},i}\\ &= \sum_i \frac{r_j}{r_i}c_{ji}{}^{\mathrm{R}}\mathbf{c}_{\mathrm{in},i},\end{aligned}$$

which also agrees with Corollary 4.1 of [62].

We can also explore the relationship between mutation of the generalized cluster algebra and mutation of its associated companion algebras. Consider a reciprocal generalized cluster algebra $\mathcal{A}$ with fixed data Γ. Again, let us consider a fixed generalized torus seed $\mathbf{s}$ and the associated scattering diagram with principal coefficients, $\mathfrak{D}_{\mathbf{s},\mathrm{prin}}$. The fixed data for the left and right companion algebras of $\mathcal{A}$ are as specified earlier in this section. By definition, the initial scattering diagram for ${}^{L}\mathcal{A}$ is of the form

$${}^{\mathrm{L}}\mathfrak{D}^{\mathrm{in}}_{\mathbf{s},\mathrm{prin}} = \left\{\left({}^{\mathrm{L}}\mathfrak{d}_k = (r_k d_k e_k, 0)^{\perp}, {}^{\mathrm{L}}f_{\mathfrak{d}_k} = 1 + z^{(v_k, e_k)}\right),\ \text{for } k\in I_{\mathrm{uf}}\right\}.$$

Note that the dual lattice of ${}^{\mathrm{L}}M^{\circ}$ is ${}^{\mathrm{L}}N^{\circ} = \operatorname{span}\{r_i d_i e_i\}_{i\in I}$. Hence, the primitive vectors normal to the wall $\mathfrak{d}_k$ in ${}^{\mathrm{L}}\mathfrak{D}^{\mathrm{in}}_{\mathbf{s},\mathrm{prin}}$ have the form $(\pm r_k d_k e_k, 0)$. By convention, we will choose to use the primitive normal vector $(r_k d_k e_k, 0)$ when calculating path-ordered products. Now, consider the cluster variables ${}^{\mathrm{L}}\hat{y}_i = z^{(v_i,e_i)}$ and let

$$v_i = \sum_k {}^{\mathrm{L}}\nu_{ik}{}^{\mathrm{L}}f_k = \sum_k {}^{\mathrm{L}}\nu_{ik}(r_k f_k).$$

Mutating ${}^{\mathrm{L}}\hat{y}_i$ in direction k is equivalent to applying the wall-crossing automorphism associated to ${}^{\mathrm{L}}\mathfrak{d}_k$ to the monomial $z^{(v_i,e_i)}$. Hence, we can calculate $\mu_k({}^{\mathrm{L}}\hat{y}_i)$ by computing

$$\begin{aligned}
{}^{\mathrm{L}}\hat{y}_i = z^{({}^{\mathrm{L}}v_i,e_i)} &\xmapsto{{}^{\mathrm{L}}\mathfrak{d}_k} z^{(v_i,e_i)}\left({}^{\mathrm{L}}f_{\mathfrak{d}_k}\right)^{\langle (r_k d_k e_k,0),(v_i,e_i)\rangle} \\
&= z^{({}^{\mathrm{L}}v_i,e_i)}\left({}^{\mathrm{L}}f_{\mathfrak{d}_k}\right)^{\langle r_k d_k e_k, \sum_{j\in I}{}^{\mathrm{L}}v_{ij}{}^{\mathrm{L}}f_j\rangle} \\
&= z^{({}^{\mathrm{L}}v_i,e_i)}\left({}^{\mathrm{L}}f_{\mathfrak{d}_k}\right)^{\sum_{j\in I}\langle r_k d_k e_k, {}^{\mathrm{L}}v_{ij}{}^{\mathrm{L}}f_j\rangle}
\end{aligned}$$

Observe that

$$\langle r_k d_k e_k, {}^{\mathrm{L}}v_{ij}{}^{\mathrm{L}}f_j\rangle = r_k d_k {}^{\mathrm{L}}v_{ij}\langle e_k, {}^{\mathrm{L}}f_j\rangle = r_k d_k {}^{\mathrm{L}}v_{ij}\langle e_k, \frac{1}{r_j d_j}e_j^*\rangle = \begin{cases} {}^{\mathrm{L}}v_{ij} & j = k \\ 0 & j \neq k \end{cases}$$

Hence, we have

$${}^{\mathrm{L}}\hat{y}_i = z^{({}^{\mathrm{L}}v_i,e_i)} \xmapsto{{}^{\mathrm{L}}\mathfrak{d}_k} z^{({}^{\mathrm{L}}v_i,e_i)}\left({}^{\mathrm{L}}f_{\mathfrak{d}_k}\right)^{{}^{\mathrm{L}}v_{ij}}$$

Recall that the variables of the generalized cluster algebra and the left companion algebra are related by ${}^{\mathrm{L}}x_i = x_i^{1/r_i}$ and ${}^{\mathrm{L}}y_i = y_i$. Recall also that ${}^{\mathrm{L}}v_i = v_i$ and therefore ${}^{\mathrm{L}}\hat{y}_i = z^{({}^{\mathrm{L}}v_i,e_i)} = z^{(v_i,e_i)} = \hat{y}_i$. Hence, we can rewrite the above map as

$$\hat{y}_i \xmapsto{{}^{\mathrm{L}}\mathfrak{d}_k} z^{(v_i,e_i)}\left({}^{\mathrm{L}}f_{\mathfrak{d}_k}\right)^{{}^{\mathrm{L}}v_{ik}r_k},$$

which agrees with the F-polynomial transformation given in Proposition 4.3 of [62]. The analogous computation for mutation in the right companion algebra also agrees with Proposition 4.6 of [62].

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