Conley Index Theory for Multivalued Dynamical Systems and Piecewise-Continuous Differential Equations

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Dedication

To my mother, Dr. Rita Thieme. Thanks for everything.

Abstract

Modern dynamical systems–particularly heuristic models–often take the form of piecewise-continuous differential equations. In order to better understand the behavior of these systems we generalize aspects of Conley index theory to this setting. Because nonsmooth models do not generally have unique solutions, this process involves organizing the solution set of the piecewise-continuous equation into a set-valued object called a *multiflow*. We prove several properties of this object, providing us with a foundation for extending Conley's techniques. This framework allows us to define isolating neighborhoods and demonstrate that they are stable under perturbation. We also provide an attractor-repeller pair decomposition of compact invariant sets for multiflows which helps us to understand the limiting behavior of solutions in such sets. This decomposition is shown to continue under perturbation. Because we assume very little structure in proving these results we are able to connect them to many different existing formulations of the Conley index for multivalued dynamical systems. Therefore we are able to identify isolating neighborhoods in a large class of differential inclusions, decompose the associated isolated invariant sets into an attractor-repeller pair, and provide the index of of the original isolated invariant set, the attractor, and the repeller; all of this information is stable under small perturbations. This process is carried out on a piecewise-continuous model from oceanography as an example.

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Chapter 1

Introduction

1.1 Broad Goals

In the mathematics community we often hear the phrase "all models are wrong, but some are useful." This aphorism tries to capture the idea that while a simple model could never capture the full truth of the universe, we can still hope that it tells us some useful information about reality. But for a model to identify physically relevant information we need to search for behavior that is not too dependent on the specifics of the equations that we write down; we need information that will hold for models that are, in some sense, similar.

In dynamical systems, our models are often differential equations of the form

$$\dot{x} = f(x)$$

where $f : U \mapsto \mathbb{R}^n$ is a smooth function and $U \subset \mathbb{R}^n$ is an open set. A solution of this equation is any differentiable function $x(\cdot)$ such that

$$\frac{d}{dt}(x(t)) = f(x(t))$$

for all t in some open interval I. In applications, the function f describes the rate of change of an abstract object that we are studying and the function x describes the position of the object. For example, the rate f might describe the population growth of some species; then x would be the population of the species at any given time. We solve for–or approximate–the solution x using mathematical techniques, but the function f is chosen by some application of domain expertise. Therefore we must accept that f represents only an approximation of the true rate of change, and so x represents only an approximation of the true position.

To account for the rough nature of these models we consider *perturbations* of the function f. We extend the domain of f to $U \times \Lambda$, where $\Lambda \subset \mathbb{R}^n$ is some interval around zero. This extension gives us a family of differential equations

$$\dot{x} = f(x, \lambda)$$

where our original model is now represented by $\dot{x} = f(x, 0)$. Our hope, then, is that the behavior of the system $\dot{x} = f(x, 0)$ is similar to the behavior of the systems $\dot{x} = f(x, \lambda)$ for $|\lambda|$ small.

It is well known in the field, however, that not all behavior described by the system $\dot{x} = f(x, 0)$ will necessarily persist for the system $\dot{x} = f(x, \lambda)$, no matter how small $|\lambda|$ is chosen. Such sudden changes are called *bifurcations*, and understanding these bifurcations is one of the central goals of the study of dynamical systems.

We may hope, however, that it is sometimes possible to avoid the difficulties of bifurcation theory. That is, we hope that some types of information that we find in our differential equations will always persist to nearby models. And indeed, we are able to find such behavior. In the 1960's and 1970's, Charles Conley and others were able to show that some qualitative, topological information about dynamical systems does persist under perturbation. Since this information can be shown to be truthful even though the model is inherently an approximation means that the information gleaned is generally somewhat vague. In Conley's words, "if such rough equations are to be of use it is necessary to study them in rough terms" [7].

The methods that came out of this line of thinking are now called Conley Index Theory in his honor. Many extensions of this theory have been made in the last half-century and this thesis aims to continue that trend by generalizing aspects of Conley index theory to the setting of piecewise-continuous differential equations and set-valued dynamical systems.

When differential equations of the form $\dot{x} = f(x)$ are studied we must put some smoothness hypotheses on the map f. Typically it is assumed to be at least Lipschitz continuous because this hypothesis guarantees that solutions are unique up to the choice of initial condition. However, differential equations where f is not Lipschitz continuous have received much less study. The reasons for this omission are essentially twofold. Firstly, from a mathematical point of view, the uniqueness of solutions which the Lipschitz hypothesis gives us is extremely valuable. Second, this mathematical simplification was historically justified by the applications that scientists studied. Most vector fields of interest were Lipschitz continuous, and hence the more complicated study of equations lacking this smoothness seemed unnecessary.

However, this second point is becoming less true in the modern world. There are now many models where the underlying differential equations are not Lipschitz continuous, or even continuous. This set includes models of friction, where an object can reach a rest point in finite time, and models involving mechanical switching, where a solution evolves according to one vector field till it reaches a certain point and then switches to another [8]. Low dimensional climate models also frequently exhibit non-smooth behavior; one particular model, Welander's ocean box model, will be examined in some detail throughout this essay in order to motivate the study of these systems. [39]. In general, many heuristic models feature discontinuities, where the sudden change represents a rapid–and poorly understood–shift between regions with different behavior.

Thus it has become important to study differential equations with discontinuous right-hand sides. This new area of study presents many unique challenges. Most notably, in order to define a solution of a piecewise-continuous differential equation in the region of discontinuity we must extend the map on the right hand side to a set-valued map. This consideration leads to the formulation of *differential inclusions*, which take the form

$$\dot{x} \in F(x)$$

where the function F is now multivalued. A solution of this inclusion is an absolutely continuous function $x(\cdot)$ satisfying

$$\dot{x}(t) \in F(x(t))$$

almost everywhere in the domain of F. As one might expect given the set-valued nature of F, there may be multiple solutions of such an inclusion with the same initial condition.

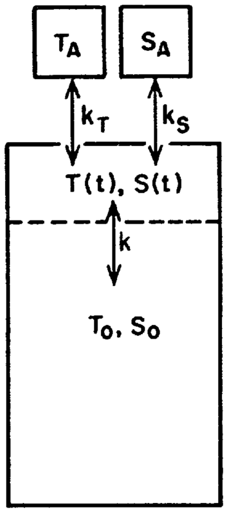
The overall aim of this thesis is to generalize aspects of Conley index theory to the multivalued setting of differential inclusions. Piecewise-continuous differential equations motivated this study, but the results are stated and proven in more general terms so that they apply to an extremely general class of differential inclusions. It is worth mentioning that control systems are often studied using differential inclusions, and the hypotheses we will assume in this manuscript are less stringent than those frequently assumed in that setting.

In the following section we will introduce the Welander model in slightly more detail in order to motivate the study of piecewise-continuous systems as well as indicate the sorts of results which we hope to obtain about these systems using Conley index theory. Section 1.3 provides a brief introduction to Conley index theory, including statements of the main results which are generalized in this thesis. The following section discusses generalizing this theory to a multivalued setting in somewhat broad terms and describes the goals and main results of the thesis. The closing section of the introduction discusses the structure of the rest of the text.

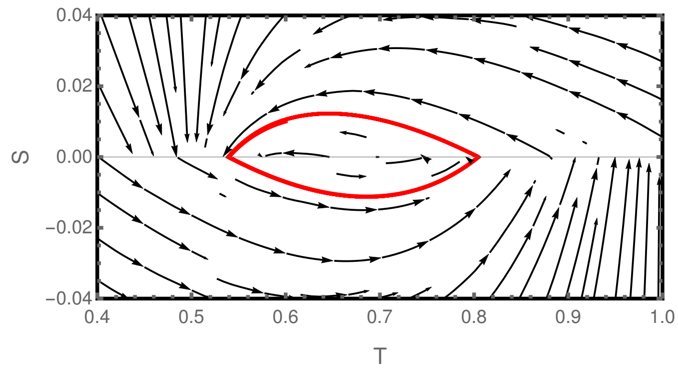
1.2 Piecewise-Continuous Differential Equations and Welander's Ocean Box Model

Piecewise-continuous differential equations often fit naturally into scientific schema, and conceptual climate models in particular often take this form [39, 31, 37, 35, 38, 36, 14, 13]. The Welander ocean box model [39] illustrates why this is the case. In this model the ocean is split into a deep ocean region and a shallow ocean region; this simplification is common in oceanography because many of the physical processes at play transition extremely rapidly at a certain depth. This split indirectly induces a discontinuity boundary in the underlying equations, which are written in terms of salinity and temperature. Welander studied this model in order to conceptually demonstrate that ocean circulation rates could oscillate without any outside forcing. He drew this conclusion by numerically finding a periodic orbit in the discontinuous model.

Actually proving the existence of this periodic orbit, however, is much more difficult



(a) Schematic of Welander's box model [39]



(b) Periodic orbit in a discontinuous model [22]

Figure 1.1: A box model split into two adjacent regions and the piecewise-continuous vector field that it induces.

because the mathematics behind nonsmooth systems is so complicated. Leifeld later provided this proof [22], but the methods used do not generalize to other models.

Moreover, researchers are often motivated to understand families of smooth systems which limit to the piecewise-continuous one. These discontinuous models are often heuristically understandable, allowing researchers to see why systems behave as they do, but usually they are considered simplifications of more realistic behavior. In the ocean box model, for instance, while there is not literally a plane dividing the ocean into two regions, we still hope this model reflects real dynamics.

However, understanding the relationship between a discontinuous model and nearby smooth ones is difficult. Given that two families of smooth vector fields, $\{f_\lambda\}$ and $\{g_\lambda\}$, both limit to the discontinuous vector field F, it is possible for the dynamics described by the equations $\{\dot{x} = f_\lambda(x)\}$ and $\{\dot{x} = g_\lambda(x)\}$ to be qualitatively different [16]. Therefore it is difficult to say what information studying F can provide us about either of these limiting smooth systems. However, a piecewise-continuous vector field is used in the Welander model both for heuristic reasons and because the transition between deep and shallow ocean regions is rapid and difficult to understand, so choosing any specific smooth model is somewhat undesirable. Therefore we would like to know what we actually can learn by studying F directly, and that is where Conley Index Theory comes in.

Conley Index Theory is a tool that describes the behavior of a dynamical system that is completely robust under perturbation [7]. The theory gives us a qualitative understanding of isolated invariant sets and the structure of their attractors. It sidesteps the problems associated with bifurcation theory by focusing on rough, topological features of the system. For many models, particularly models which aim to describe extremely complicated systems like the climate of our planet, we cannot hope that the equations that we write down perfectly describe the exact behavior of a given system. However, we hope that these models still tell us information about these systems, and that is precisely the aim of Conley theory.

This philosophy fits naturally with piecewise-continuous models. As in the Welander case, for many of these models the instantaneous transition is considered to be an approximation of a very abrupt and poorly understood, but smooth, transition. Since the information studied with Conley Index Theory is stable under small perturbations, we aim to use it in order to study the nonsmooth–but conceptually understandable–system, and carry that information over to limiting families of smooth systems.

1.3 Introduction to Conley Index Theory

For sufficiently smooth vector fields we can describe the solution set of the differential equation $\dot{x} = f(x)$ by a *flow*. A flow on a locally compact metric space Y is a continuous map $\varphi : \mathbb{R} \times Y \to Y$ satisfying the following group properties:

1. $\varphi(0, x) = x$ for all $x \in Y$.

2. $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for all $t, s \in \mathbb{R}$ and $x \in Y$.

The flow is the solution set for the differential equation in the sense that for each $x_0 \in Y$,

$$\frac{d}{dt}\varphi(t, x_0) = f(\varphi(t, x_0))$$

and so $\varphi(\cdot, x_0)$ is exactly the solution of the differential equation with initial condition $\varphi(0, x_0) = x_0$.

Having introduced this object, we can now rephrase our goal of understanding the behavior of the differential equation $\dot{x} = f(x)$ as an attempt to understand the behavior of the flow $\varphi : \mathbb{R} \times Y \to Y$. The main purpose of Conley's theory is to provide a robust qualitative description of the compact invariant sets of such a flow. We note here that this theory also works for somewhat less smooth differential equations–such as those which experience finite-time blowup and whose solution sets are instead local flows–but we work with flows here for the simplicity of the introduction.

A set $S \subset U$ is said to be *invariant* if

$$\varphi(\mathbb{R}, S) := \cup_{t \in \mathbb{R}} \varphi(t, S) = S.$$

In general, the dynamics of such an invariant set can be extremely complicated, and their structure can radically change under even small perturbations. These complications fall under the umbrella of bifurcation theory, and to obtain a detailed view of these dynamics requires a great deal of analysis. However, the Conley theory allows us to give a rough picture of compact invariant sets in a much more computationally simple manner.

The first object of interest in Conley index theory is the *isolating neighborhood*, which is a compact set whose maximal invariant set is contained in its interior. That is, a compact set $N \subset \mathbb{R}^n$ is an isolating neighborhood for the flow φ if

$$\operatorname{Inv}(N, \varphi) := \{x \in N | \varphi(t, x) \in N \, \forall t \in \mathbb{R}\} \subset \operatorname{int}(N).$$

These neighborhoods allow us to study *isolated invariant sets*, which are sets $S \subset \mathbb{R}^n$ satisfying $S = \operatorname{Inv}(N, \varphi)$ for some isolating neighborhood N.

The most important property of isolating neighborhoods is that they are stable under perturbation. More specifically, we associate a continuous family of flows

$$\varphi_\lambda : \mathbb{R} \times Y \mapsto Y, \qquad \lambda \in \Lambda$$

to the family of differential equations $\dot{x} = f(x, \lambda)$. If N is an isolating neighborhood for φ_0, then for $|\lambda|$ sufficiently small N is also an isolating neighborhood for φ_λ. It therefore seems reasonable that we might be able to study this stable neighborhood and glean limited but robust information about the invariant set in its interior; this goal is essentially the core of Conley index theory.

One key piece of information that we obtain through this theory is a description of the attractors of the invariant set S. We call a set $A \subset S$ an *attractor in* S if A is the ω-limit set of a neighborhood of itself in S. Associated to this attractor is a dual-repeller $R := \{x \in S | \omega(x) \not\subset A\}$, and for all other points $x \in S$, $\alpha(x) \subset R$ and $\omega(x) \subset A$. Moreover, this decomposition is stable in a topological sense; this attractor-repeller decomposition continues to nearby flows φ_λ for small $|\lambda|$.

The above results are the most important parts of Conley index theory for this thesis as they are replicated here in the multivalued setting. However, the theory actually provides us with far more information. For one, if (A, R) is an attractor-repeller pair decomposition of the set S, then the region $S \setminus (A \cup R)$ is gradient-like in the sense that there is a Lyapunov function defined on S which is strictly decreasing on $S \setminus (A \cup R)$. And looking beyond the basic attractor-repeller decomposition of S, the Conley theory defines a Morse decomposition of S which is similar in nature to the attractor-repeller decomposition but involves a larger partial ordering than the two sets used in that description. It also gives the so-called fundamental theorem of dynamical systems, showing that every flow can be decomposed into its chain-recurrent set and a gradient-like region.

All of this behavior is described algebraically using the Conley Index. This index is an index of an isolating neighborhood N, but it may also be described as an index of the maximal isolated invariant set S that this neighborhood contains. The introductory paper [28] cites three essential properties of this index:

1. *(Well-defined)* If N and N' are both isolating neighborhoods for φ and $\text{Inv}(N, \varphi) = \text{Inv}(N', \varphi)$, then the Conley index of N is the same as the Conley index of N'. This property allows us to view the Conley index as a well-defined index of isolated invariant sets.

2. *(Ważewski Property)* If the Conley index of N is not trivial then $\text{Inv}(N)$ is non-empty.

3. *(Continuation)* If $\{\varphi_\lambda | \lambda \in \Lambda\}$ is a continuous parametrized family of flows and N is an isolating neighborhood for each flow φ_λ, then the Conley index of N is the same for each flow φ_λ.

These three properties are the basis of how Conley index theory is used. We identify an isolating neighborhood and assign it an algebraic index. The first property tells us that all isolating neighborhoods containing the same maximal invariant set have the same index, and so we know that we can pass information that the index tells us about the isolating neighborhood to the associated isolated invariant set. One such piece of information is the existence of a nontrivial invariant set, as indicated by the Ważewski property. Finally, the third property gives the robustness of this information; since we already know that isolating neighborhoods are stable under small perturbations, we see from this property that the Conley Index, and any information we gain from it, is also stable under small perturbations.

This index ties directly to the attractor-repeller decomposition because attractors and repellers in an isolated invariant set are themselves isolated invariant sets. Therefore we can also use the Conley index to understand the topology of these sets and know that this information persists under small perturbations.

Importantly, it is worth highlighting that we do not actually need to know how the flow is perturbed in order to obtain these results. For example, if we identify an isolating neighborhood for the flow associated to the differential equation

$$\dot{x} = f(x)$$

then the Conley index of that neighborhood is the same for the flows generated by

$$\dot{x} = f(x) + \lambda g(x) \qquad \text{or} \qquad \dot{x} = f(x) + \lambda h(x)$$

for small $|\lambda|$ given any choice of smooth g and h. It is also stable under any other continuous perturbation of the ODE. The Conley index therefore allows us to address our original goal of understanding real-world behavior from an inherently simplified model.

For basic information on Conley theory, the reader is referred to the brief survey article [28]. For those interested in more detailed information on this topic, see Conley's manuscript [7] or the more complete survey [29].

1.4 Using Conley Index Theory in a Multivalued Setting

As stated previously, the main purpose of this thesis is to generalize aspects of Conley index theory to a multivalued setting. In particular, we hope to use this theory in order to study piecewise-continuous differential equations, and differential inclusions more generally. Some efforts have already been made along these lines. Versions of the index for certain differential inclusions and piecewise-continuous systems have been developed in [6, 9, 20, 32], each with their own strengths and drawbacks. Of these, the development of a cohomological version of the Conley index in [32] is closest to our goals. The ideas of the attractor-repeller pair decomposition for differential inclusions have been discussed in [5, 23]. Each of these papers and their results will be discussed in somewhat more detail later in the paper, but in this section we hope to indicate what distinguishes the results of this thesis from these earlier works.

As has been said a few times in this chapter, Conley index theory aims to study isolated invariant sets and the structure of their attractors in a way that is stable under small perturbations. This goal has, in some sense, two halves. On the one hand we have the structural component giving the Morse decomposition of the isolated invariant sets; the papers [5, 23] aim to generalize these ideas. The other half is the stable index itself, which is the subject of [32, 6, 9, 20]. The fact that attractors are themselves isolated invariant sets, and hence may be studied using the index, binds these two components together. Because of this property, it can be shown that the attractor-repeller pair decomposition continues in the same sense that isolated invariant sets do.

However, the various multivalued generalizations of the different aspects of Conley index theory mentioned above do not fit together; the results which build the index itself have different hypotheses than the ones that describe the structure of attractors, leaving the full power of Conley index theory out of reach. In particular, none of these papers addresses the notion of attractor-repeller pair continuation in a way that allows perturbation from a piecewise-continuous vector field to a limiting family of smooth vector fields. This situation is undesirable, and so the main goal of this thesis is to provide a unified framework for Conley Index theory which is general enough to study all piecewise-continuous differential equations.

A primary obstacle in achieving this goal is that the multivalued nature of these systems leads to competing notions of invariance. Roughly speaking, some authors consider a set to be invariant if every point in it has *an* orbit which remains in the set for all time; others demand that *every* orbit remain in the set for all time; still others demand *every* orbit remain in the set for all *forwards* time. Existing results on the stability of the isolating neighborhoods assume the less stringent first hypothesis while the results on attractors and repellers assume the more stringent second and third ones. Both of these assumptions are made for good reasons–which we will discuss in the text–but the fact that these different aspects of the theory do not work together leads to a fundamental barrier in its application. Therefore, in order to provide a framework for Conley index theory in this setting, this thesis will prove all of the relevant results using the first–and weaker–hypothesis of invariance.

In order to achieve this goal we introduce the idea of *multiflows*, a set-valued generalization of a flow introduced by Richard McGehee [25]. This definition is less restrictive than similar objects used in other works. In particular, we show in Theorem 5.1 that the solution set of any differential inclusion in Euclidean space that satisfies some basic set-valued conditions may be described by this object. In particular, any piecewise-continuous differential equation–even ones which are unbounded and experience finite-time blowup–will satisfy these conditions. However, this theorem uses an unusual conception of the solution set. Rather than considering the union of maximal solutions, we restrict our view to a compact set, and take the union of solutions up until any point where they leave through the boundary of the set. By considering the solution set in this way, we sidestep the complications associated to finite-time blowup and local flows. This has a valuable benefit in that it allows us to remove any global bounding conditions on our differential inclusions and so we may study a larger class of these objects. More fundamentally, however, it is this perspective which allows us to overcome the technical complications that arise due to competing notions of invariance.

This unusual view–where we consider solutions only in a compact set–actually arises quite naturally in Conley index theory. The Conley index of an isolating neighborhood is computed using only information about dynamics on the boundary of this neighborhood, and ignores the behavior of solutions once they leave this compact set. As we demonstrate throughout this thesis, this view is sufficient to generalize many important results from Conley index theory. Theorem 7.1 extends the stability of isolating neighborhoods to the setting of multiflows. A similar theorem is proven in [32] but assumes that the righthand side of the differential inclusion is bounded, an assumption that we avoid here. Theorem 8.1 extends the attractor-repeller decomposition; similar theorems are given in [5, 23], but they apply to a more limited set of differential inclusions than is considered here and utilize a different notion of invariance.

The continuation of the attractor-repeller decomposition is proven in theorem 8.2. As far as we can see, no comparable theorem has been proven previously, even for differential inclusions which satisfy global bounding hypotheses and are studied in the other works mentioned here. In particular, this theorem allows us to identify attractors in a piecewise-continuous system and continue them to certain nearby smooth systems.

In order to prove these results, several definitions and lemmas related to multiflows are necessary. One key definition is that of an *orbit* of a multiflow, which conveys the notion of a single possible path of a point among the potentially infinite possibilities of the multiflow. Theorem 5.3 demonstrates that in the case that the multiflow is generated by a differential inclusion these orbits are exactly the solutions of the differential inclusion. Any collection of these orbits is uniformly equicontinuous, as shown in Lemma 5.2, and Lemma 5.3 shows that the limit of any uniformly convergent sequence of orbits is also an orbit. These lemmas combine with the Arzèla-Ascoli theorem to give Theorem 5.4: given any family of orbits there is some orbit which is the uniform limit of a subsequence of the family on any compact subset of its domain. This result is used repeatedly in proving the Conley generalizations. Additionally, Theorem 6.1 and Lemma 4.2 help us understand perturbation in this setting.

In [32] it is demonstrated that differential inclusions which satisfy some basic hypotheses and are also bounded give rise to an object called a multivalued admissible flow, a strong generalization of the flow. We point out in Theorem 9.5 that recent results can replace a theorem cited in the proof given in [32], allowing us to relax the bounded hypothesis and require only linear growth. This change greatly increases the applicability of the results in [32]. For our purposes, it will allow us to combine the main results of this thesis with the results given in [32] and analyze the Welander model.

1.5 Description of Chapters and Sections

In Chapter 2 we introduce the basics of multivalued analysis and define upper-semicontinuity for set-valued maps. This chapter is analogous to introducing continuous maps in the single-valued setting and several classic results about continuous maps are generalized here. In particular, the closed graph theorem is generalized to this setting, a result which plays an important role in understanding multiflows. No original results are proven in this section.

The largest chapter, Chapter 3, introduces basic differential inclusions. This chapter explains how piecewise-continuous differential equations may be reframed as Filippov systems, a special case of basic differential inclusions. The basic behavior of Filippov systems is outlined. Here somewhat detailed information about the Welander model–including some bifurcation analysis–is discussed. Additionally, because control systems are also often described as basic differential inclusions they are also mentioned in this chapter. The final section in this chapter gives proofs of some fundamental results about basic differential inclusions, including Filippov's existence proof. Again, this section contains no original results, and is included as background material.

Perturbation plays a central role in Conley index theory, and so the idea of perturbation must be extended to the multivalued setting in order to understand this thesis. To do so, Chapter 4 discusses the perturbation of differential inclusions. The notion of perturbation used here is very general, and applies to perturbation in a control sense. However, since we are motivated by piecewise-continuous differential equations, how our notion of perturbation relates to these objects is the main point of discussion. In particular we highlight how Filippov systems may be perturbed both to other Filippov systems and to nearby smooth systems. While the results of this discussion are relatively straightforwards, they seem largely unknown in the Filippov systems community.

Finally, in Chapter 5, we introduce multiflows. This chapter includes Theorem 5.1, which demonstrates that the solution set of differential inclusions is a multiflow. It also introduces several definitions, such as *orbits* of a multiflow, which allow us to study multiflows as objects in their own rights. In order to demonstrate that the results which we state about multiflows apply to basic differential inclusions, Theorem 5.3 shows that these orbits in a multiflow generated by a differential inclusion are equivalent to the solutions of that differential inclusion.

In Chapter 6 we introduce the notion of a *well-parametrized family of multiflows*, which generalizes the notion of a continuous family of flows to this multivalued setting. The main result of this chapter is Theorem 6.1, which demonstrates that perturbed differential inclusions give rise to a well-parametrized family of multiflows, allowing us to conclude that the theorems proven about multiflows apply to the differential inclusions that we are interested in understanding.

We begin the extension of Conley index theory itself in Chapter 7. This includes introducing the idea of invariance, isolating neighborhoods, and isolated invariant sets. The stability of isolating neighborhoods is extended to this well-parametrized family of multiflows in Theorem 7.1; a similar theorem is proven for bounded differential inclusions in [32].

Chapter 8 introduces limit sets and attractors for multiflows, and extends the attractor-repeller decomposition in Theorem 8.1. Importantly, the continuation of the attractor-repeller pair decomposition is done in Theorem 8.2; this is probably the most original result of the thesis as we are unaware of similar statements even for more limited classes of differential inclusions.

Finally, in Chapter 9 we discuss the Conley index itself. Thus far the Conley index has not been extended to multiflows, but we make some observations and discuss other multivalued generalizations of the index. Importantly, because of Theorem 9.5, we may use a variant of the index developed in [32] in order to study differential inclusions which satisfy a linear growth bound. The Welander model is one such inclusion, and so we are able to give the stable index of an isolated invariant set and an attractor-repeller pair in that set.

Each of chapters 3-9 includes a section on the Welander model. These sections are included as a running example, showing how the ideas developed in each chapter may be applied to a specific scientific model. All of this analysis cumulates in the result described in the preceding paragraph. Each of these sections references the previous ones, and the notation that is adopted builds throughout each section. In case the reader would like to reference earlier sections for a reminder of the notation and analysis, a list of hyperlinks is included at the end of each section in order to simplify navigating between through this long example.

The final chapter–the conclusion–is quite brief, giving a summary of what has been discussed in the thesis and highlighting important open questions.

Separately, there is an appendix included at the end of the thesis. This appendix contains two sections. The first of these sections discusses the Arzèla-Ascoli Theorem because it plays a central role in proving many of the theorems in this thesis. The other section discusses convex sets and the Carathèodory Theorem which are used in reframing piecewise-continuous differential equations as differential inclusions.

Chapter 2

Set-Valued Maps

2.1 Introduction to Set-Valued Maps

Because this manuscript is devoted to the study of set-valued dynamical systems, we begin with an introduction to set-valued (or multivalued) maps for readers who are not used to working in this setting.

Definition 2.1. *For sets W and Y, a* ***set-valued map*** *(or* ***multivalued map****) $F : W \to \mathcal{P}(Y)$ is a mapping which associates a subset $F(w) \subset Y$ to each element $w \in W$.*

We will also use the word *function* interchangeably with map or mapping.

Of course, a set-valued map on its own has essentially no structure, and we will need to make certain assumptions on its behavior in order to describe interesting results. Usually we will assume (or prove) that the multivalued map in question is *compact valued* and *upper-semicontinuous*. In the following section we will define these terms and show that these objects behave like continuous single-valued maps in many ways. We will also highlight some of the ways in which these multivalued maps are not as well-behaved as typical continuous maps.

Because we will often be discussing both set-valued and single-valued maps in this paper, we will use capitalization in order to distinguish between the two concepts. Any maps which are assumed to be single-valued will be lower-case, like f or φ, and maps which are allowed to be set-valued will be capitalized, like F or Φ. We note here that single-valued maps are a special case of multivalued maps.

The results described in this chapter are commonly used throughout the literature of this niche subject and no ownership is claimed here.

2.2 Compact-Valued Upper-Semicontinuous Set-Valued Maps

In this section we generalize two standard properties of continuous maps to the multivalued setting. The first is that the composition of continuous maps yields another continuous map. The second is that continuous maps defined on compact domains are bounded. This second property is particularly useful in the study of differential equations because it bounds the velocity of solutions; its generalization serves a similar role in our study of differential inclusions.

To begin, we specify that for a multivalued map $F : W \to \mathcal{P}(Y)$ and any subset $U \subset W$, we denote the image of U under F as

$$F(U) = \cup_{w \in U} F(w)$$

Definition 2.2. *Let W and Y be topological spaces. A set-valued function $F : W \to \mathcal{P}(Y)$ is said to be* **upper-semicontinuous at the point** *$w_0 \in W$ if for any open neighborhood $V \subset Y$ of $F(w_0)$, there exists some open neighborhood $U \subset W$ of w_0 such that $F(U) \subset V$.*

Then F is said to be ***upper-semicontinuous*** *if it is upper semicontinuous at each $w \in W$.*

If we examine this definition, we notice that if an upper-semicontinuous set-valued map F is in fact single-valued then it is continuous in the traditional sense. Therefore all of the results proven in this section–generalizations of typical results about continuous maps–give the corresponding single-valued theorems as corollaries.

Because of this property, the definition of upper-semicontinuity provided here for set-valued maps contradicts the better known definition of upper-semicontinuity for real-valued functions, which are not continuous in general. This contradictory terminology is somewhat unfortunate but standard, and so we will use it here.

As is always the case with mathematics, it is helpful here to consider a simple example when digesting this abstract definition.

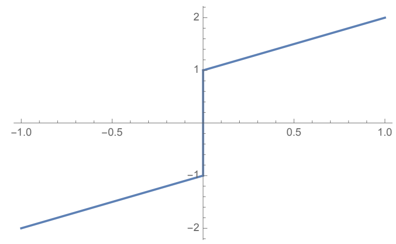


Figure 2.1: The upper-semicontinuous map of example 2.2.1.

Example 2.2.1. Consider $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by

$$F(x) = \begin{cases} x - 1 & x < 0 \\ [-1, 1] & x = 0 \\ x + 1 & x > 0 \end{cases}$$

We can directly verify that this map is upper-semicontinuous at each point in its domain. Away from the origin, F is single-valued and continuous, and hence upper-semicontinuous. We can check the upper-semicontinuity of F at the origin in a manner that is very similar to checking the continuity of single-valued maps. For any $\varepsilon > 0$ we have that $B_\varepsilon(F(0)) = (-1 - \varepsilon, 1 + \varepsilon)$; note that given any neighborhood V of $F(0)$, there is an ε sufficiently small that $B_\varepsilon(F(0)) \subset V$. Then choosing $0 < \delta < \varepsilon$, we get

$$F(B_\delta(0)) = F((-\delta, \delta)) = (-1 - \delta, 1 + \delta) \subset B_\varepsilon(F(0)) \subset V$$

and so F is upper-semicontinuous everywhere.

As in the single-valued case, we can give a well-defined composition of set-valued maps:

Definition 2.3. *Given multivalued maps $G : W \to \mathcal{P}(Y)$ and $F : Y \to \mathcal{P}(Z)$, we define the composition*

$$F \circ G : W \to \mathcal{P}(Z), \qquad F \circ G(w) := F(G(w))$$

Notice that this composition is somewhat unusual in that the range of G is $\mathcal{P}(Y)$ but the domain of F is just Y. However, the notation defined earlier–$F(G(w)) = \cup_{y \in G(w)} F(y)$–sidesteps this complication and allows $F \circ G$ to take input in W and output subsets of Z.

As is the case with single-valued maps and continuity, the composition of multivalued functions preserves upper-semicontinuity.

Property 2.1. *If maps $G : W \to \mathcal{P}(Y)$ and $F : Y \to \mathcal{P}(Z)$ are upper-semicontinuous then $F \circ G : W \to \mathcal{P}(Z)$ is also upper-semicontinuous.*

This proof is a straightforward generalization of the analogous result for single-valued continuous functions.

Proof. At an arbitrary $w \in W$, we must show that if $P \subset Z$ is an open neighborhood of $F \circ G(w)$ then there is some neighborhood $U \subset W$ of w such that $F \circ G(U) \subset P$.

By definition, if P is an open neighborhood of $F \circ G(W)$, then $P \supset F(y)$ for each $y \in G(w)$. By the upper-semicontinuity of F, for each $y \in G(w)$ there is some $V_y \supset y$ such that $F(V_y) \subset P$. Define $V := \cup_{y \in G(w)} V_y$ and notice that

$$G(w) \subset V, \qquad F(V) \subset P$$

By the upper-semicontinuity of G, there is some open neighborhood $U \ni w$ such that $G(U) \subset V$. For this choice of U we have that

$$F \circ G(U) = F(G(U)) \subset F(V) \subset P$$

and so $F \circ G$ is upper-semicontinuous at any arbitrary point in its domain. □

The given definition of an upper semicontinuous set-valued function applies to multivalued maps between any topological spaces. But in this paper we will work primarily with metric spaces, and so it is often more convenient for us to consider ε and δ neighborhoods rather than arbitrary neighborhoods. This convenience motivates the following definition:

Definition 2.4. *A set-valued function $F : W \to \mathcal{P}(Y)$ between metric spaces W and Y is* **ε-upper-semicontinuous at the point** *$w \in W$ if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that $F(B_\delta(w)) \subset B_\varepsilon(F(w))$.*

The set-valued map F is said to be ***ε-upper-semicontinuous*** *if it is ε-upper-semicontinuous at each $x \in G$.*

We can see immediately that upper-semicontinuity implies ε-upper-semicontinuity since $B_\varepsilon(S)$ is an open neighborhood of S for any set S and for any neighborhood U of a point $w \in W$ there is a sufficiently small δ such that $B_\delta(w) \subset U$. However, the converse is not necessarily true because given a neighborhood V of S, it is possible that $B_\varepsilon(S) \not\subset V$ for any $\varepsilon > 0$. For instance, consider the following example from [12]:

Example 2.2.2. Define the map $F : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ by $F(w) = \{w\} \times \mathbb{R}$. We can directly verify that this map is ε-upper-semicontinuous; for each $w \in \mathbb{R}$, $F(B_\delta(w)) = B_\delta(w) \times \mathbb{R} = B_\delta(F(w))$, and so given any $\varepsilon > 0$, if $0 < \delta < \varepsilon$ we have that $F(B_\delta(w)) \subset B_\varepsilon(F(w))$. We can also see, however, that this map is not upper-semicontinuous at any point; for simplicity we will show that it is not upper-semicontinuous at 0. Notice that the set $V := \{(w, y) \in \mathbb{R}^2 : |wy| < 1\}$ is an open neighborhood of $F(0)$. However, for any neighborhood U of 0, we see that $F(U) \not\subset V$.

Although the definitions of upper-semicontinuity and ε-upper-semicontinuity are not equivalent, we can still use the latter–and more convenient–characterization in most applications of interest. In addition to upper-semicontinuity, set-valued maps are frequently assumed to be *compact-valued*; that is, for each w in the domain of F, we demand that the set $F(w)$ be compact. We can immediately verify that this additional assumption allows us to pass freely between upper-semicontinuity and ε-upper-semicontinuity.

Property 2.2. *If W and Y are metric spaces and $F : W \to \mathcal{P}(Y)$ is compact-valued, then F is upper-semicontinuous if and only if it is ε-upper-semicontinuous.*

Proof. As was previously indicated, it is straightforward to see that upper-semicontinuity implies ε-upper-semicontinuity. Therefore we must only show that the converse holds whenever F is compact-valued.

Given an arbitrary $w \in W$, let V be a neighborhood of $F(w)$. For each $y \in F(w)$ there is some $\varepsilon_y > 0$ such that $B_{\varepsilon_y}(y) \subset V$. Since $F(w)$ is compact, there is a finite collection $\{y_i\}_{i=1}^n$ such that $\cup_{1\le i\le n} B_{\varepsilon_{y_i}}(y_i)$ covers $F(w)$. Defining $\varepsilon := \min_{1\le i\le n}(\varepsilon_{y_i})$, we have that $B_\varepsilon(F(w)) \subset V$.

□

Notice that as a direct consequence of this definition, the compact-valued map F is upper-semicontinuous at $w \in W$ if and only if given any sequence $w_i \to w$ and any choice of associated vectors $v_i \in F(w_i)$, for any $\varepsilon > 0$ there is some i sufficiently large that $v_i \in B_\varepsilon(F(w_i))$. This sequential characterization is frequently used to prove that maps are upper-semicontinuous in practical examples.

It is important to notice that the definition of upper-semicontinuity is inherently one-sided in the sense that for w, ε, δ as in the definition, $d(x, y) < \delta$ does not imply that $F(x) \subset B_\varepsilon(F(y))$. If F did have such a symmetric property it would be called *continuous*, but it turns out that that requirement is too stringent for many applications and so we will not assume it; notice that the set-valued map of example 2.2.1 is upper-semicontinuous but not continuous.

This one-sided nature of upper-semicontinuous functions implies, unfortunately, that there is no reasonable notion of *uniform* upper-semicontinuity. That is, even on a compact domain, for a fixed $\varepsilon > 0$ there is generally no single value of $\delta > 0$ which implies that $F(B_\delta(x)) \subset B_\varepsilon(F(x))$ for all $x \in X$. Example 2.2.1 demonstrates why this is the case; for any $x \neq 0$ and $\varepsilon < 2$, we see that $F(0) \not\subset N_\varepsilon(F(x))$.

However, there are many properties that we typically expect of continuous functions that also hold for compact-valued, upper-semicontinuous multivalued maps. Importantly, such set-valued maps are bounded on compact domains.

Lemma 2.1. *Let X, Y be metric spaces. If X is compact, then any upper-semicontinuous and compact-valued set-valued map $F : X \to \mathcal{P}(Y)$ is bounded.*

As was the case with the composition result, Lemma 2.1, the proof of this result is very similar to the proof that continuous functions with compact domains are bounded.

Proof. Since F is upper-semicontinuous, for each $\varepsilon > 0$ and $x \in X$ there exists some $\delta_x > 0$ such that $d_X(x, w) < \delta_x$ implies that $F(w) \subset B_\varepsilon(F(x))$. Balls of radius δ_x centered at each point $x \in X$ form an open cover of X, and so we may take a finite subcover of these balls centered at the points $\{x_n\}_{n=1}^k$. Then for each $x \in X$,

$$F(x) \subset \cup_{n=1}^k (B_\varepsilon(F(x_n)))$$

Since F is compact-valued, this union is a bounded set. □

Notice that the assumption of upper-semicontinuity is necessary for this result to hold just as continuity is essential to the analogous result in the single-valued case. The map $G : [-1, 1] \to \mathbb{R}$ given by

$$G(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not upper-semicontinuous and it is not bounded. The assumption that the set-valued map is compact-valued, on the other hand, may be relaxed; if the assumption that the image of each point is compact is replaced by the assumption that this image is bounded then the proof follows without any alterations. This weaker assumption is necessary, however, as the trivial set-valued map $H : [-1, 1] \to \mathbb{R}$, $x \mapsto \mathbb{R}$ demonstrates.

Actually, the preceding result can be strengthened; the image of a compact set is compact. As one might expect, this stronger result does require that F be compact valued (and not merely take bounded values).

Property 2.3. *Assume X and Y are metric spaces and that X is compact. If $F : X \to Y$ is compact-valued and upper-semicontinuous then $F(X)$ is compact.*

The following proof is taken from [2].

Proof. Let $\{V_\upsilon\}_{\upsilon \in \Upsilon}$ be an open cover of $F(X)$. Since $F(x)$ is assumed to be compact for each $x \in X$, we can cover $F(x)$ in a finite number $n(x)$ of these open sets. That is,

$$F(x) \subset \cup_{1 \leq i \leq n(x)} V_{\upsilon_i} =: V_x.$$

By the upper-semicontinuity of F, for each $x \in X$ there is some open neighborhood U_x of x such that $F(U_x) \subset V_x$. Notice that $\{U_x\}_{x \in X}$ is an open cover of X, and hence we may choose choose a finite subcover $\{U_{x_j}\}_{1 \leq j \leq p}$. Then

$$F(X) \subset \cup_{1 \leq j \leq p} F(U_{x_j}) \subset \cup_{1 \leq j \leq p} V_{x_j} = \cup_{1 \leq j \leq p} \cup_{1 \leq i \leq n(x_j)} V_{\upsilon_i}$$

and so we have found a finite open subcover of $F(X)$. □

2.3 The Closed Graph Theorem for Set-Valued Maps

There is an analogue of the closed graph theorem for upper-semicontinuous and compact-valued multivalued maps. This result is central to many of the ideas and theorems presented later in this thesis and so it is given its own section. To begin we formally define the graph of a multivalued map:

Definition 2.5. *If $F : W \to \mathcal{P}(Y)$ is a set-valued map, then the **graph of** F is the set*

$$\Gamma_F := \{(w, y) \in W \times Y | y \in F(w)\}$$

As a reminder, the typical closed graph theorem states that if W is a topological space and Y is a compact Hausdorff space, a map $f : Y \to Y$ is continuous if and only if its graph is a closed subset of $W \times Y$. If we do not assume that the range Y is compact then the graph of a continuous map is still closed but the converse no longer holds.

A direct analogue of the above statement does exist in the multivalued setting, but for our purposes it is simpler to assume that the spaces in question are metric spaces. This assumption does not restrict our applications at all and the proof becomes more reminiscent of the flavor of proof which will appear throughout this manuscript. We will state the theorem here and prove it in a sequence of lemmas. The proofs of these lemmas are sketched in [33], and these sketches were used in order to write down the proofs given here.

Theorem 2.1. *Let W, Y be metric spaces and Y be compact.*
The multivalued function $F : W \to \mathcal{P}(Y)$ is upper-semicontinuous and compact valued

$$\Updownarrow$$

the graph $\Gamma_F = \{(w, y) \in W \times Y | y \in F(w)\}$ is closed.

In order to prove Theorem 2.1, we begin with the following lemma which assumes upper-semicontinuity and compact-valuedness only at a single point w in the domain:

Lemma 2.2. *Let W and Y be metric spaces.*

A multivalued map $F : W \to \mathcal{P}(Y)$ is upper-semicontinuous at $w \in W$ and $F(w)$ is *compact*

$$\Updownarrow$$

for each sequence $\{w_n\}_{n=1}^{\infty}$ *such that* $w_n \to w$*, any sequence* $\{y_n\}_{n=1}^{\infty}$ *such that* $y_n \in F(w_n)$ *has a convergent subsequence with a limit in* $F(w)$*.*

Proof. We will first show that the sequence condition implies upper-semicontinuity by showing that if F is not upper-semicontinuous at w then we can find sequences $\{w_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ violating the stated conditions. If F is not upper-semicontinuous at w, then there is some open neighborhood V of $F(w)$ such that $F(U) \not\subset V$ for any neighborhood U of w. Let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\delta_n \to 0$ and choose an arbitrary $w_n \in B_{\delta_n}(w)$ for each n. Since F is not upper-semicontinuous at w, we can choose some $y_n \in F(w_n)$ such that $y_n \notin V$. Since $Y \setminus V$ is closed, the sequence $\{y_n\}_{n=1}^{\infty}$ does not have any limit points in V. More to the point, it does not have any limit points in $F(w)$.

We will directly show that the sequence condition implies that $F(w)$ is compact. Consider the stationary sequence $\{w_n\}_{n=1}^{\infty}$ where $w_n = w$ for all n. Then our condition tells us that any sequence $\{y_n\}_{n=1}^{\infty} \subset F(w)$ has a convergent subsequence. Notice that this property is exactly the sequential compactness of $F(w)$, which tells us that $F(w)$ is compact since Y is a metric space.

We will now give the other direction of the proof and assume that F is upper-semicontinuous and compact-valued at w. Let V be a bounded neighborhood of $F(w)$ and let U be a bounded neighborhood of w such that $F(U) \subset V$. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence such that $w_n \to w$ and choose $y_n \in F(w_n)$ for each n. For large enough n, $w_n \in U$, and so

$$(w_n, y_n) \in U \times V \subset \overline{U} \times \overline{V}$$

since $y_n \in F(w_n) \subset F(U) \subset V$. Because $\overline{U} \times \overline{V}$ is compact, this sequence has a convergent subsequence with some limit (w, y) (where w is the original point where we have assumed the upper-semicontinuity of F). We are done once we show that $y \in F(w)$.

For the sake of contradiction, assume that $y \notin F(w)$. Since $F(w)$ is compact, $b := d_Y(y, F(w))$ is greater than zero. Then $B_{b/2}(F(w))$ is a neighborhood of $F(w)$

such that for each basic neighborhood $B_r(w)$,

$$F(B_r(w)) \not\subset B_{b/2}(F(w))$$

because if the subsequence $\{y_{n_k}\}_{n=1}^{\infty}$ converges to $y \notin B_{b/2}(F(w))$, then $F(w_{n_k}) \not\subset B_{b/2}(F(w))$ even though $w_{n_k} \to w$. Thus we have a contradiction to our assumption that F was upper-semicontinuous at w. □

Lemma 2.2 does the bulk of the work in proving Theorem 2.1. The following lemmas (which have much shorter proofs) do the rest.

Lemma 2.3. *Assume W and Y are metric spaces and let $F : W \to \mathcal{P}(Y)$ be a multi-valued function. If for each sequence $\{w_n\}_{n=1}^{\infty}$ such that $w_n \to w$, any sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in F(w_n)$ has a convergent subsequence with a limit in $F(w)$, then the graph Γ_F of F is closed.*

Proof. Let (w, y) be a limit point of Γ_F. Then there is some sequence $\{(w_n, y_n)\}_{n=1}^{\infty} \subset \Gamma_F$ that limits to (w, y). Since $y_n \in F(w_n)$ by the definition of the graph, the sequence condition tells us that $y \in F(w)$. □

Lemma 2.4. *Assume that W and Y are metric spaces. Let Y be compact and let $F : W \to \mathcal{P}(Y)$ be a multivalued function. If the graph Γ_F of F is closed then for each sequence $\{w_n\}_{n=1}^{\infty}$ such that $w_n \to w$, any sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in F(w_n)$ has a convergent subsequence with a limit in $F(w)$.*

Proof. Let $w_n \to w$ and choose $y_n \in F(w_n)$ for each n. Since Y is compact, $\{y_n\}_{n=1}^{\infty}$ has some subsequence $\{y_{n_k}\}_{k=1}^{\infty}$which limits to some point y. Since $\{(w_{n_k}, y_{n_k})\}_{k=1}^{\infty} \subset \Gamma_F$ and Γ_F is closed, the limit point (w, y) belongs to Γ_F, and so $y \in F(w)$. □

Theorem 2.1 follows directly from these lemmas. Also, notice that, as in the single-valued case, the assumption of compactness on Y is necessary for only one direction; even if Y is not compact, a compact-valued and upper-semicontinuous set-valued map still has a closed graph.

We close this section with a few examples which demonstrate the necessity of each of the assumptions of Theorem 2.1.

Example 2.3.1. $F_1 : \mathbb{R} \to \mathcal{P}([-1,1])$ is not upper-semicontinuous but is compact-valued; its graph Γ_{F_1} is not closed:

$$F_1(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$F_2 : \mathbb{R} \to \mathcal{P}([-1,1])$ is not compact-valued but is upper semicontinuous; its graph Γ_{F_2} is not closed:

$$F_2(x) = \begin{cases} -1 & x < 0 \\ [-1,1) & x = 0 \\ 1 & x > 0 \end{cases}$$

The graph Γ_{F_3} is closed but $F_3 : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is not upper-semicontinuous (it is compact-valued):

$$F_3(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

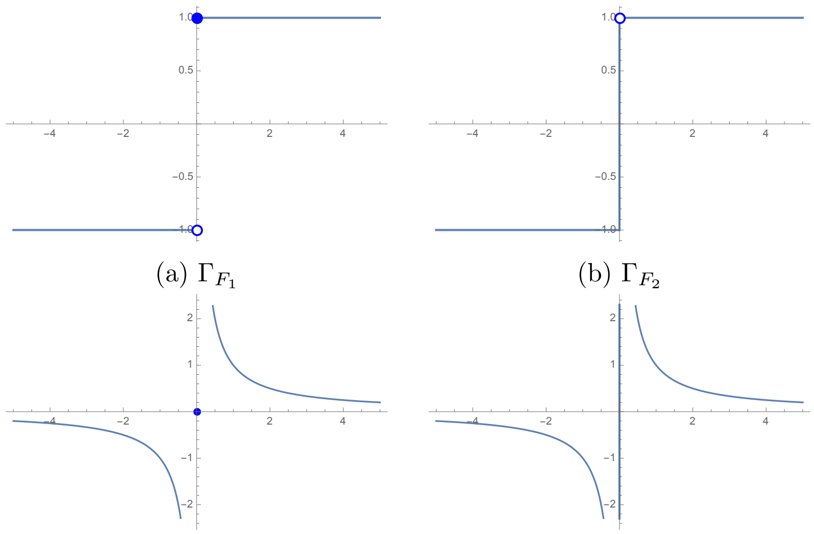
The graph Γ_{F_4} is closed but $F_2 : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is not compact-valued (it is upper-semicontinuous):

$$F(x) = \begin{cases} 1/x & x \neq 0 \\ \mathbb{R} & x = 0 \end{cases}$$

2.4 Multivalued Maps as Closed Relations

Another way of stating Theorem 2.1 is to say that upper-semicontinuous and compact-valued multivalued maps into compact metric spaces are equivalent to closed relations. In order to state this characterization formally, recall the following definition:

Definition 2.6. *Given sets W and Y, a* **relation over** *$W \times Y$ is any subset of $W \times Y$. If only one set W is specified, then a* ***relation on*** *W is any subset of $W \times W$.*



(a) Γ_{F_1} (b) Γ_{F_2}

(c) Γ_{F_3} (d) Γ_{F_4}

Figure 2.2: Graphs from example 2.3.1.

Clearly any set-valued map $F : W \to \mathcal{P}(Y)$ is associated with a unique relation on $W \times Y$, namely the graph Γ_F. But it is also true that any relation $\Gamma \subset W \times Y$ is associated with a unique multivalued map

$$F_\Gamma : W \to \mathcal{P}(Y), \qquad w \mapsto \{y \in Y | (w, y) \in \Gamma\}$$

Then the closed graph theorem tells us that for metric spaces W, Y with Y compact, a compact-valued and upper-semicontinuous multivalued map between W and Y is equivalent to a closed relation over $W \times Y$. We can even define the composition in these terms:

Definition 2.7. *The* ***composition of two relations*** $F \subset Y \times Z$ *and* $G \subset W \times Y$ *is the relation*

$$F \circ G = \{(w, z) \in W \times Z : \exists y \in Y \text{ such that } (w, y) \in G, (y, z) \in F\}$$

It follows from Property 2.1 that if the relevant spaces are metric spaces then the composition of closed relations is closed. In fact this result holds even if the spaces are only assumed to have a topological structure, but we will not work in that generality here.

The use of relations in the study of dynamical systems is quite powerful because of their generality. For readers unfamiliar with this perspective, this paper should contain sufficient information to follow the exposition, but for those interested in more detailed information the book [1] by Akin is recommended. For particular information about attractors and repellers in the setting of closed relations, see the article [25] by McGehee.

We will rely heavily on this result in this paper because we will often pass back and forth between these two characterizations. Many of the later results are stated more easily in one form, but their proofs are often more intuitive using the other interpretation.

Chapter 3

Differential Inclusions

3.1 Basic Differential Inclusions

We now turn to the objects which motivate this manuscript, differential inclusions. As mentioned in the introduction, the author's primary motivation for studying differential inclusions is that they are used in order to study piecewise-continuous differential equations, but these objects also have many other important applications, most notably in control theory [2]. Before beginning, we mention that most of the definitions and theorems found in this section are taken from A.F. Filippov's seminal work, [10].

We begin by explicitly defining the objects in question:

Definition 3.1. *A* ***differential inclusion*** *is a generalization of the concept of a differential equation. It takes the form*

$$\dot{x} \in F(x)$$

where F is a set-valued map.

A ***solution of the differential inclusion*** *is an absolutely continuous function $x(t)$ defined on some interval $I \in \mathbb{R}$ such that*

$$\frac{d}{dt}x(t) \in F(x(t))$$

almost everywhere in I.

Recall that on a compact interval $[a, b]$, an absolutely continuous function $x(t)$ may be written as

$$x(t) = x(a) + \int_a^t \dot{x}(s)ds$$

where the derivative $\dot{x}$ is Lebesgue integrable and exists almost everywhere. The motivation for considering these almost everywhere differentiable functions as the solutions to differential inclusions, rather than C^1 functions as we do for differential equations, is made clear in example 3.2.2.

Of course, in order to study these objects we must put some assumptions on the multivalued map F on the righthand side. In his seminal work [10], Filippov introduces what he calls the *basic conditions* for set-valued functions. These conditions are necessary in order to ensure that the differential inclusion satisfies certain essential properties; notably, the basic conditions guarantee the existence of solutions.

Definition 3.2. *Let the set-valued function $F : \mathbb{R}^n \supset G \to \mathcal{P}(R^n)$ be upper-semicontinuous. Also, for all $x_0 \in G$, assume that the set $F(x_0)$ is*

- *non-empty*
- *compact*
- *convex*

*Then F is said to satisfy the **Basic conditions**.*

For brevity, we will say that if F satisfies the basic conditions then the inclusion $\dot{x} \in F(x)$ is a *basic differential inclusion.* Such inclusions are very general, applying to a wide variety of dynamical systems. Note that any continuous single-valued function $f(x)$ trivially satisfies these conditions, and so any results about basic differential inclusions also apply to more typical differential equations of the form $\dot{x} = f(x)$.

Of course, since basic differential inclusions are more general than even differential equations with continuous righthand sides, we expect that their behavior may be very unusual. In particular, since it is well-known that differential equations where the righthand side is merely continuous–but not Lipschitz continuous–may have infinitely many solutions for a given initial condition (the prototypical example is $\dot{x} = \sqrt{x}$), we cannot expect any sort of uniqueness theorem for these systems. However, we will prove in section 3.5 that the solutions of these inclusions do satisfy the following properties:

Theorem 3.1 (Properties of Solutions to Basic Differential Inclusions)**.** *[10]*
Let F satisfy the basic conditions in an open domain $G \subset \mathbb{R}^n$ and consider the differential inclusion

$$\dot{x} \in F(x) \tag{3.1}$$

The following properties hold:

1. *For any $x_0 \in G$, there exists $\delta > 0$ and a solution $x : (-\delta, \delta) \to \mathbb{R}^n$ of* (3.1) *satisfying $x(0) = x_0$.*

2. *Solutions of* (3.1) *which lie in a compact domain are uniformly equicontinuous.*

3. *Solutions of* (3.1) *may be continued on both sides up to the boundary of any compact domain.*

4. *The limit of a uniformly convergent sequence of solutions to* (3.1) *is also a solution to that inclusion.*

In later sections we will see that these properties actually do imbue differential inclusions with a decent amount of structure; in particular, they are sufficient to generalize many aspects of Conley Index Theory.

We will now show that piecewise-continuous differential equations may be reframed as basic differential inclusions.

3.2 Filippov Systems: Piecewise-Continuous Differential Equations

3.2.1 Piecewise-Continuous Differential Equations as Basic Differential Inclusions

Although Filippov worked in greater generality, his name is primarily associated with the study of discontinuous differential equations. In this section we will see how piecewise-continuous differential equations may be reframed and studied as basic differential inclusions.

We begin by explicitly defining a piecewise-continuous differential equation on an open domain $G \subset \mathbb{R}^n$. Without loss of generality we may assume that G is connected because disconnected portions may simply be examined separately. The set G is divided into open, disjoint regions G_i, along with their boundary points. We will assume that the set of all boundary points of all G_i is measure zero in G, and denote it by Σ. We will refer to Σ as the *splitting boundary* or *discontinuity boundary*. For analytical reasons, we also impose the additional condition that any compact subset of G contains only finitely many G_i; this assumption is very useful mathematically, and does not impose a burden from a modeling standpoint. For the rest of this paper, we will call any connected, open domains partitioned in this way *Filippov domains*. This language is not standard in the literature, but it is useful to reference for our purposes.

We now consider a set of differential equations defined on the Filippov domain:

$$\dot{x} = p_i(x), \qquad x \in G_i \subset G$$

Each f_i is required only to be continuous, framing the system as a piecewise continuous one. We also assume that p_i are continuous up to the boundary of G_i so that $p_i(x)$ evaluates to a finite vector for each i and for all $x \in \Sigma$. In other words, each p_i is defined and continuous on the closure of G_i.

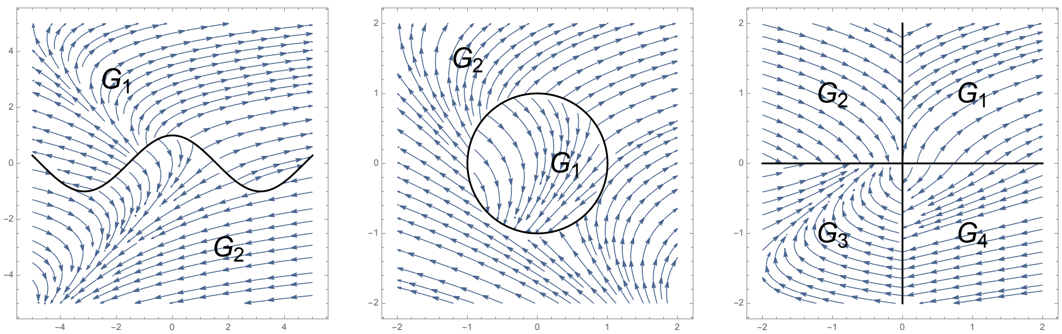
Of course, as written, this system is incomplete; there is no information about the vector field along Σ, and so there is no way to continue a solution which reaches the boundary of any G_i. This issue brings us to the concepts of differential inclusions and the Filippov convex combination method.

At each $x \in \Sigma$, there are multiple vector fields $p_i(x)$ that are defined. Because of that fact, it makes sense to introduce a differential inclusion $\dot{x} \in F(x)$. In the Filippov convex combination method we define our set-valued vector field $F(x)$ to be the single-valued functions $p_i(x)$ for all x in any of the open regions G_i. For $x \in \Sigma$, however, we take $F(x)$ to be the set-valued convex hull of all vectors $p_i(x)$ such that x is a boundary point of G_i. We will collect all of this information in the following definition.

Definition 3.3. *Let $G \subset \mathbb{R}^n$ be a Filippov domain such that each G_i is associated with a function p_i that is continuous on the closure of G_i.*

Define a set-valued function F as follows. For $x \in G_i$, let

$$F(x) = \{p_i(x)\}$$



(a) Σ is the curve $y = \cos(x)$.

(b) Σ is the circle $x^2 + y^2 = 1$.

(c) Σ is the union of the lines $x = 0$ and $y = 0$.

Figure 3.1: Some examples of Filippov systems in $\mathbb{R}^2$.

For $x \in \Sigma$, let $F(x)$ be the convex hull of all vectors $p_i(x)$ such that x is a boundary point of G_i.

Then the differential inclusion $\dot{x} \in F(x)$ is a ***Filippov System.***

Note that any differential equation $\dot{x} = f(x)$ where f is continuous will trivially fit into this framework. Classical systems, with Lipschitz continuous differential equations, may then be viewed as special cases of Filippov systems.

Now that we have defined a Filippov system $\dot{x} \in F(x)$, we should check that it does, in fact, meet the basic conditions.

Lemma 3.1. *If $\dot{x} \in F(x)$ is a Filippov system then F is upper-semicontinuous.*

Proof. It is clear that F is upper-semicontinuous at $x \in G_i$ for any i since $F(x)$ is defined by the single-valued continuous function $p_i(x)$ at such points. Thus it remains only to show that F is upper-semicontinuous at $x \in \Sigma$.

Fixing such an x, the set $F(x)$ is the convex hull of a finite number of vectors $\{p_i(x)\}_{i=1}^{p}$. Consider an arbitrary ε-neighborhood of $F(x)$, $B_\varepsilon(F(x))$. For each i, there is some δ_i such that $y \in \overline{G_i}$ and $|x-y| < \delta_i$ implies that $|p_i(x) - p_i(y)| < \varepsilon$. We can also write this condition as $p_i(y) \in B_\varepsilon(p_i(x))$. Let $\delta = \min_{1 \leq i \leq p}\{\delta_i\}$. Then for $y \in B_\delta(x)$, $F(y)$ is either a single valued function $p_i(y)$ (for $y \in G_i$) or the the convex hull of a finite set of vectors $\{p_i(y)\}_{i=1}^{q}$ (for $y \in \Sigma$) all satisfying the relationship $p_i(y) \in B_\varepsilon(p_i(x)) \subset B_\varepsilon(F(x))$.

This fact implies that for $y \in B_\delta(x)$, $F(y) \subset B_\varepsilon(F(x))$; to verify this statement, consider an arbitrary vector $f_y \in F(y)$. We can write this vector as $f_y = \sum_{i=1}^q \alpha_i p_i(y)$, where $1 \le q \le p$ and $\sum_{i=1}^q \alpha_i = 1$. Now consider the vector $f_x := \sum_{i=1}^q \alpha_i p_i(x) \in F(x)$.

$$\begin{aligned} |f_x - f_y| &= |\sum_{i=1}^q \alpha_i p_i(x) - \sum_{i=1}^q \alpha_i p_i(y)| \\ &= |\sum_{i=1}^q \alpha_i (p_i(x) - p_i(y))| \\ &< \sum_{i=1}^q \alpha_i |p_i(x) - p_i(y)| \\ &< \sum_{i=1}^q \alpha_i \varepsilon \\ &= \varepsilon \end{aligned}$$

Thus $F(B_\delta(x)) \subset B_\varepsilon(F(x))$, and so F is upper-semicontinuous at any x.

□

With this lemma, it is easy to see that Filippov systems satisfy the basic conditions of differential inclusions. At each point, the set-valued map F is well-defined and non-empty. Wherever the system is single valued, $F(x_0)$ is clearly compact and convex. For x_0 where the system is not single valued, $F(x_0)$ is still clearly closed and convex by definition, and it is bounded because we assume that p_i is defined on $\overline{G_i}$ for each i, and so $p_i(x_0)$ is always finite. We will summarize this information into a theorem:

Theorem 3.2. *If $\dot{x} \in F(x)$ is a Filippov system then F satisfies the basic conditions.*

We should note here that the Filippov convex combination convention is not the only possible convention that may be used to reframe piecewise-continuous differential equations as basic differential inclusions, though it is possibly the most natural and certainly the most common. In some sense the choice of set-valued vector field defined on Σ is related to families of smooth systems which limit to the piecewise-continuous vector field. This concept and other possible conventions to use on Σ will be discussed in Section 4.2.

3.2.2 Standard Behavior of Filippov Systems

In general it is possible to consider systems where Σ can be very complicated, and when theorems are quoted in this paper they will apply to these general systems defined above. However, in most models, Σ is simply a codimension 1 manifold. There are a few notable common examples where the dimensionality of Σ is not well defined everywhere–for instance, Σ could be two intersecting lines–and so it would not be a manifold. But for many applications, the set Σ is a manifold, and in such cases we will refer to it as the *splitting manifold*.

In fact, the majority of non-smooth models in the literature have only two distinct regions, G_1 and G_2, as in Figure 3.1 a. When only two regions border Σ, as in this case, the convex hull may be written compactly as the convex combination of the two vectors $p_1(x)$ and $p_2(x)$:

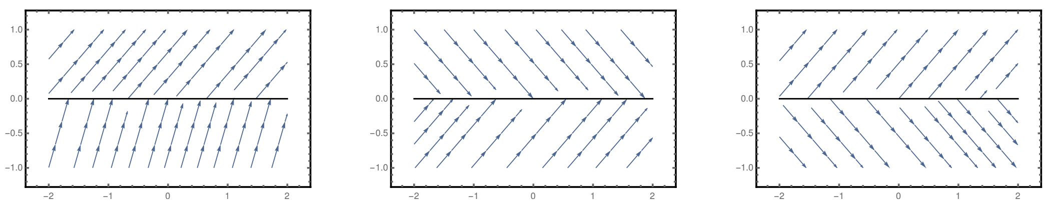
$$F(x) = \{\alpha p_1(x) + (1-\alpha)p_2(x) : \alpha \in [0,1]\}, \qquad x \in \Sigma$$

Thus the entire Filippov system may be written as

$$\dot{x} \in F(x) = \begin{cases} p_1(x), & x \in G_1 \\ p_2(x), & x \in G_2 \\ \{\alpha p_1(x) + (1-\alpha)p_2(x) : \alpha \in [0,1]\}, & x \in \Sigma \end{cases}$$

For a Filippov system where the dynamics are defined by two regions split by a codimension-1 manifold, the behavior of the solutions usually falls into one of three categories: crossing, attracting, or repelling. These names are meant to be descriptive, and do not give a full description of all possible behavior of Filippov systems. But typically, solutions which are near the splitting manifold are either drawn to cross over it, attracted to it, or repelled away from it. Explaining these categories rigorously can be somewhat technical, but the ideas can be easily understood graphically, as in Figure 3.2.

It is easiest to mathematically describe these categories in the simple case where there are only two open regions, G_1 and G_2, and the splitting manifold Σ has a well defined tangent at each point. Since most models meet those requirements, we will work



(a) Crossing Region (b) Attracting Region (c) Repelling Region

Figure 3.2: Examples of behavior of Filippov systems in $\mathbb{R}^2$

in this setting in order to convey the basic ideas as simply as possible, but we note that these categories can be abstracted to fit general Filippov systems. Again, this section draws extensively from the work of Filippov [10].

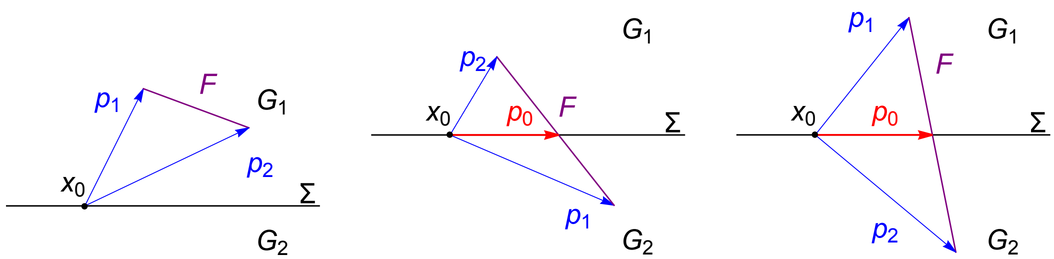
On the splitting manifold Σ, the set $F(x)$ is a line segment connecting the two vectors $p_1(x)$ and $p_2(x)$. The simplest situation is when the entirety of this line segment lies on one side or the other of the tangent to the splitting manifold at x; this arrangement is called a *crossing case*. Example 3.2.2 is a crossing case. Here the solution is uniquely determined and simply passes from G_1 to G_2 or vice-versa. In fact, the concept of a differential inclusion is unnecessary to explain these systems. Since the differential equation $\dot{x} = f(x)$ is known to be equivalent to the integral equation

$$x(t) = x(0) + \int_0^t f(x(s))ds$$

we may use the concept of the Lebesgue integral in order to find solutions. The only issue that arises is a loss of differentiability on a subset of measure zero, but since such a subset is irrelevant to the Lebesgue integral, we face no major difficulties.

The cases where the line segment $F(x)$ intersects the tangent plane, on the other hand, do require the concept of a differential inclusion for the full description. Let the normal to the tangent plane of Σ at x be directed into G_1, and let $p_1^N(x)$ and $p_2^N(x)$ denote the projections of $p_1(x)$ and $p_2(x)$ onto this normal vector. As long as the line segment $F(x)$ is not entirely contained in the tangent plane, there is a unique vector

$$p_0(x) = \alpha p_1(x) + (1-\alpha)p_2(x), \qquad \alpha = \frac{p_2^N(x)}{p_2^N(x) - p_1^N(x)}$$



(a) Crossing Region (b) Attracting Region (c) Repelling Region

Figure 3.3: Vectors at a point x_0 on a discontinuity boundary Σ. The set of vectors $F(x_0)$ is represented by the purple line segment. Its boundaries are determined by the blue vectors $p_1(x_0)$ and $p_2(x_0)$. The sliding vector $p_0(x_0) \in F(x_0)$ is given in red when it exists.

that allows the solution to remain on the splitting manifold. When a solution remains on the splitting manifold for some positive time it is said to be a *sliding solution* or to exhibit *sliding behavior*. Since p_1^N and p_2^N have opposite signs in non-crossing cases, α is between zero and one, and this value is found by solving the equation

$$0 = \alpha p_1^N(x) + (1-\alpha)p_2^N(x)$$

From here it is important to break into two further cases. The first case is that $p_1^N(x) < 0$ and $p_2^N(x) > 0$. This is called the *attracting case*. Here, if a solution attempted to leave the splitting manifold it would be immediately pushed back onto it, and so the only possible choice of vector in $F(x)$ is $p_0(x)$. If the splitting manifold is attracting at x then forward time solutions are uniquely determined locally.

If $p_1^N(x) > 0$ and $p_2^N(x) < 0$ then a solution may proceed into either domain G_1 or G_2 and obey the vector field there, or it may remain on the splitting manifold and obey $p_0(x)$. This is called the *repelling case*. In this case $F(x)$ is truly set valued and the behavior that results is non-deterministic; it is this loss of determinism that motivates many of the considerations of this paper.

There is actually another possible behavior that does not fit into any of these classifications. This remaining possibility is that the line segment $F(x)$ is entirely contained in the tangent plane. Then all solutions remain on the splitting manifold but the velocity is not uniquely determined, as in the following example; this case is very rare, and we

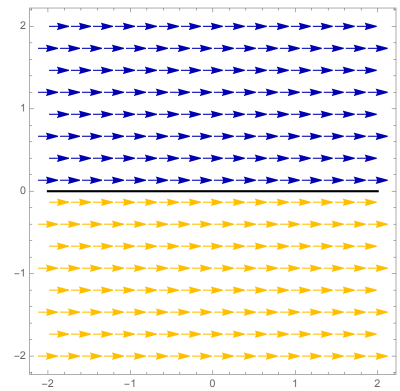


Figure 3.4: Graphs from example 3.2.1. The color of the vector indicates its magnitude.

have not found it in practical applications.

Example 3.2.1. For the Filippov system

$$\dot{(x,y)} \in F(x,y) = \begin{cases} \{(1,0)\}, & y > 0 \\ [1,2] \times \{0\}, & y = 0 \\ \{(2,0)\}, & y < 0 \end{cases}$$

with the initial condition $(x,y)(0) = (0,0)$, both

$$(x,y)_1(t) = (t,0) \qquad \& \qquad (x,y)_2(t) = (2t,0)$$

are solutions to the differential inclusion.

To be clear, in dimensions greater than one, a single Filippov system may exhibit any and all of the above cases. A splitting manifold may have crossing behavior in some regions, attracting behavior in others, and repelling behavior elsewhere. Additionally, we note that in backwards time, an attracting case becomes a repelling case and vice-versa, so except in the simple crossing case, we will never expect to see uniqueness in both forwards and backwards time.

Additionally, we note that while the discussion in this particular section has been limited to describing the case where the Filippov system is split into two regions locally, examples involving more regions can be examined using similar methods. For instance,

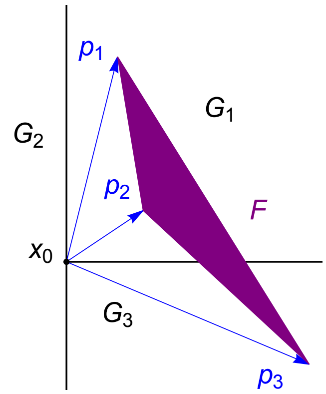


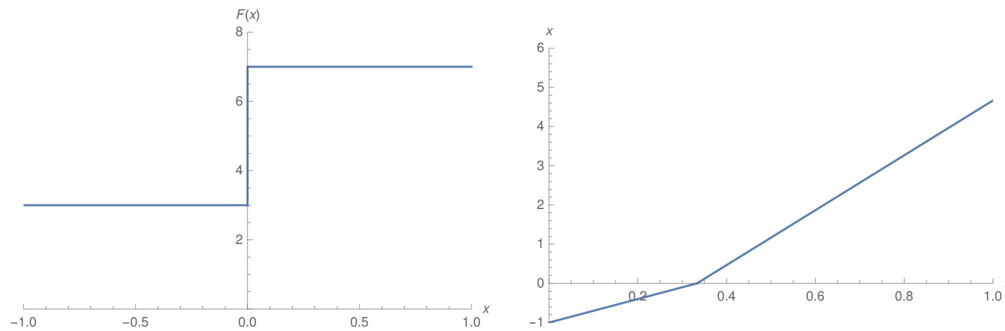
Figure 3.5: Convex combination of vectors at a point on a splitting boundary which borders three open regions.

if a point lies on the boundary of three regions then the convex combination which determines its potential velocities will be a triangle (unless the three vectors happen to be colinear) and so the velocity will take any value in that triangle. Whether a solution actually exists which travels in the direction of a given velocity vector in the triangle depends on the behavior of the vectors in a neighborhood of the point; this limitation is analogous to the attracting case, where the set of vectors under consideration was a whole line, but only a single one of those vectors could actually be used in a solution.

Before we move on to a scientific application of Filippov systems, it is worth returning to the definition of a solution of a differential inclusion as these Filippov systems show us why we want to demand only that solutions be differentiable almost everywhere. When a solution approaches the splitting boundary Σ, its derivative limits to a certain value. However, on the other side of that boundary, the solution may rapidly change velocity since the defining vector fields p_i and p_j need not have any relationship to one another. Hence we typically expect a loss of differentiability when solutions reach Σ. The following simple example demonstrates this behavior explicitly:

Example 3.2.2. Consider the one-dimensional basic differential inclusion

$$\dot{x} \in F(x) = \begin{cases} 3, & x < 0 \\ [3,7], & x = 0 \\ 7, & x > 0 \end{cases}$$



(a) Righthand side of $\dot{x} \in F(x)$, example 3.2.2.

(b) The solution $x(t)$ with $x(0) = -1$.

Figure 3.6: The multivalued map and solution to the basic differential inclusion of example 3.2.2.

Away from $\{x = 0\}$ we can solve this inclusion locally as either $x(t) = 3t + x_0$ or $x(t) = 7t + x_0$ using techniques from standard ODE theory. However, any solution with initial condition $x_0 < 0$ will eventually move to the right and reach the discontinuity boundary at time $t^* = \frac{-x_0}{3}$. Since all $v \in F(0)$ are greater than zero, the solution immediately leaves the splitting boundary and proceeds to the right. After time t^*, the solution proceeds with velocity $\dot{x} = 7$, and so the unique solution for any initial condition $x_0 < 0$ may be written as

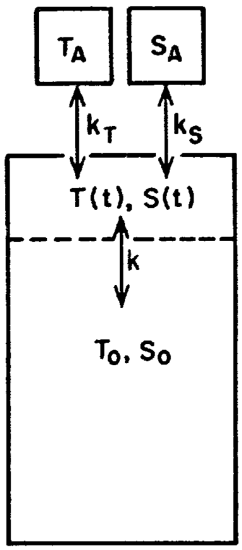
$$x(t) = \begin{cases} 3t + x_0, & t < \frac{-x_0}{3} \\ 7(t + \frac{x_0}{3}), & t \geq \frac{-x_0}{3} \end{cases}$$

This function is absolutely continuous, but we see that it is not differentiable at $t^* = \frac{-x_0}{3}$.

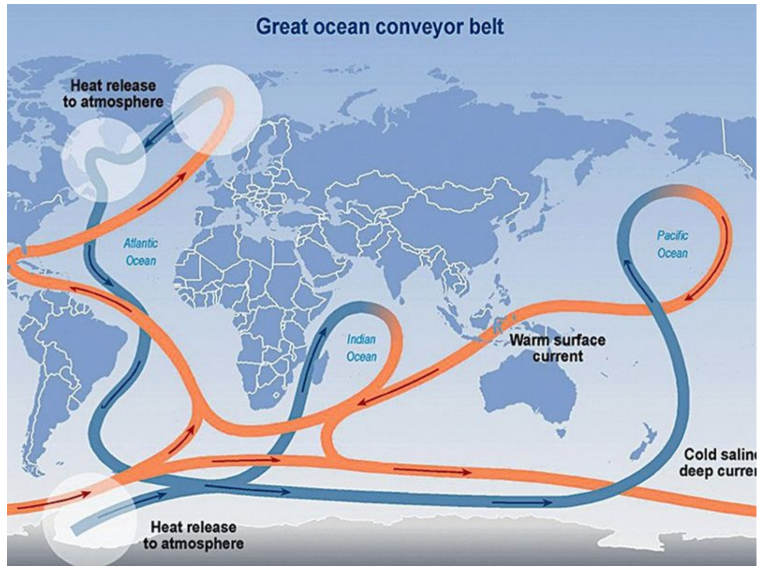
3.3 A Scientific Application of Filippov Systems: Welander's Ocean Box Model

3.3.1 Description of the Model

Before delving further into the theory of Filippov systems we will take a brief look at one of its applications. In 1982, Pierre Welander published a conceptual ocean circulation



(a) Schematic of Welander's box model [39]



(b) Atlantic Meridional Overturning Circulation [15]

Figure 3.7: Images related to ocean circulation box models.

box model [39]. This model splits the ocean into two boxes: deep ocean and shallow ocean. Ordinary differential equations dependent on two variables, temperature T and salinity S, describe the flow of water between these two boxes. Welander's scientific goal in this paper was to demonstrate that an ocean box model could oscillate between convective and non-convective modes without external influences, which would provide an explanation for behavior that had been observed in the Atlantic Ocean.

The model that he used has the following form:

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

The values k_T, k_S, T_A, S_A, α, and γ are all constant. The pressure of the system, given by ρ, is an equation of state, and $k(\rho)$ is a non-negative function. Welander performs analysis of this system using both a smooth and non-smooth version of k, analytically finding a periodic orbit in the smooth case and numerically finding one in the non-smooth one. This periodic orbit provided proof of concept for his convective oscillation.

In the smooth case, Welander chooses the function

$$k(\rho) = \frac{1}{\pi}\tan^{-1}(\frac{\rho - \varepsilon}{a}) + \frac{1}{2}$$

As the parameter a goes to infinity, this equation limits to the one that Welander uses in his non-smooth case:

$$k(\rho) = \begin{cases} k_1 & \rho > \varepsilon \\ 0, & \rho < \varepsilon \end{cases}$$

In our language, the splitting manifold Σ is the line $\rho = \varepsilon$. The periodic orbit that Welander finds is created because the linear equations which govern each region would have globally stable equilibria if either of them governed the whole system; however, the locations that these equilibria would be at are on opposite sides of Σ from the vector fields that determine them. Such would-be equilibria are called *virtual equilibria*. In Welander's model, solutions head towards one virtual equilibria, cross the manifold, and then begin to head towards the other, causing the periodic orbit that Welander finds numerically.

It seems that Welander chose to examine this non-smooth system because he thought it was simpler than the similar smooth version. Indeed, because it is linear away from the splitting manifold, this system is very simple in some sense. However, Welander does not elaborate on the behavior of solutions on the splitting manifold, so his analysis is not mathematically rigorous.

3.3.2 Filippov Analysis of the Model

While Welander only finds a periodic orbit in the non-smooth model numerically, Julie Leifeld proved its existence by performing more detailed, Filippov style analysis than Welander had performed [22]. In doing this analysis she discovered two different non-smooth bifurcations in the system. One of these bifurcations, the *fused focus bifurcation*, is analogous to a supercritical Hopf bifurcation and had been discussed in prior literature on Filippov systems. The other, however, was a special type of *border collision* *bifurcation* (meaning that under parameter changes an equilibrium collided with the splitting manifold) that was qualitatively different from any that had been discussed before.

The new bifurcation is important because previous prominent literature claimed to have classified all planar codimension one bifurcations [21]. The omission of this bifurcation from this prominent work indicates that there are foundational difficulties in the study of Filippov systems. Because bifurcations of Filippov systems are not well understood–even in the planar case–it is desirable to have a theory which sidesteps these complications, motivating the study of Conley index theory in this setting. But for now, let us focus on Leifeld's bifurcation analysis of this model. behavior

Before performing her analysis, Leifeld performs a coordinate transformation so that the system is written as

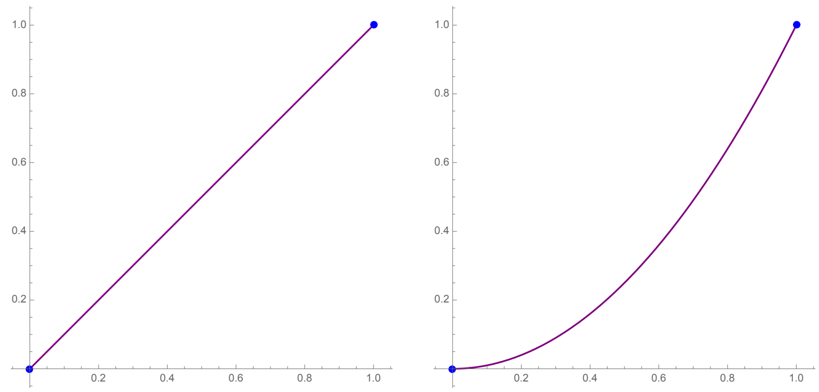
$$\dot{x} = 1 - x - k(y)x$$
$$\dot{y} = \beta - \beta\varepsilon - k(y)\varepsilon - \alpha - (\beta + k(y))y - (\alpha\beta - \alpha)x$$

Now, $k(y)$ is written as

$$k(y) = \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases}$$

This coordinate transformation takes the splitting manifold to the x-axis, a technique that is very popular in Filippov analysis. But we should note that this specific coordinate change is just a linear transformation composed with a translation. This fact is important because such a transformation preserves the Filippov convex combination and therefore the behavior of the system. If the original splitting manifold had been nonlinear, then a continuous transformation sending the splitting manifold to the x-axis could alter the underlying Filippov convention. That issue would make any analysis of the transformed system inapplicable to the original system; see Figure 3.8.

For completeness, we will write down the Filippov system version of Welander's



(a) A set K in blue $\mathrm{co}(K)$ in purple.

(b) The set $f(K)$ in blue and $f(\mathrm{co}(K))$ purple.

Figure 3.8: Letting $K = \{(0,0),(1,1)\}$ and $f(x,y) = (x,y^2)$, we see that even under smooth mappings the image of the convex hull of a set is not the same as the convex hull of the image of a set. This simple example indicates that if we apply nonlinear transformations to a Filippov system the result is not necessarily a Filippov system.

model here:

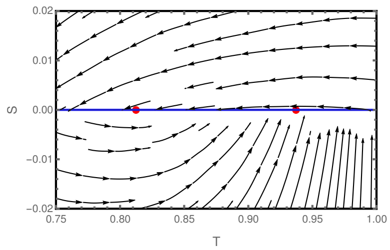
$$\dot{\begin{bmatrix} x \\ y \end{bmatrix}} \in W_0(\begin{bmatrix} x \\ y \end{bmatrix}) := \begin{cases} \begin{bmatrix} 1-2x \\ \beta - \beta\varepsilon - \varepsilon - \alpha - (\beta+1)y - (\alpha\beta - \alpha)x \end{bmatrix}, & y > 0 \\ \{ \begin{bmatrix} 1 - x - \tau x \\ \beta - \beta\varepsilon - \alpha - \beta y - \alpha\beta x + \alpha x - \tau y \end{bmatrix} | \tau \in [0,1]\}, & y = 0 \\ \begin{bmatrix} 1-x \\ \beta - \beta\varepsilon - \alpha - \beta y - \alpha\beta x + \alpha x \end{bmatrix}, & y < 0 \end{cases} \quad (3.2)$$

The choice of appending the subscript 0 to this function will be explained in section 4.7 when we examine perturbations of this model.

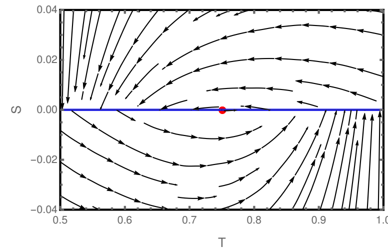
For the sake of simplicity, we will fix the parameters

$$\alpha = 4/5, \quad \beta = 1/2$$

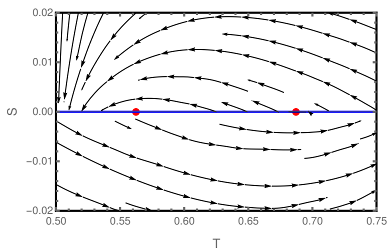
for the duration of our analysis of this model.



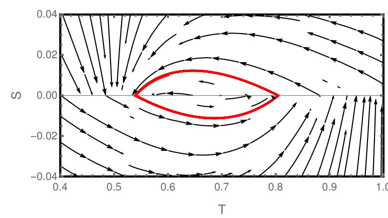
(a) $\varepsilon > 0$: one stable Filippov equilibrium and a stable sliding region.



(b) Bifurcation at $\varepsilon = 0$: the sliding region narrows to a point and this point is a stable equilibrium.



(c) $\varepsilon < 0$: the sliding region is unstable.

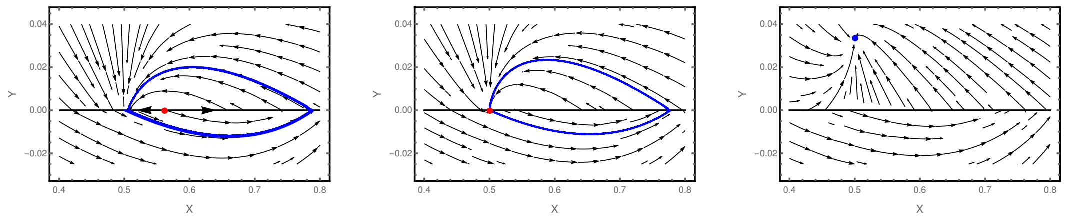


(d) $\varepsilon < 0$: periodic orbit

Figure 3.9: Fused Focus Bifurcation; figures from [22].

After making this coordinate transformation, the bifurcation analysis becomes simpler. The first bifurcation that Leifeld finds–the fused focus bifurcation–is analogous to a supercritical Hopf bifurcation in smooth systems. For $\varepsilon > 0$ there is a stable sliding region that contains a stable equilibrium. For $\varepsilon < 0$ there is an unstable sliding region containing an unstable equilibrium. At $\varepsilon = 0$ the sliding region is reduced to a single point, which is also a stable equilibrium.

Finally, when $\varepsilon < 0$ there is a periodic orbit which crosses the interval $(1/2, 3/4 + 15/4\varepsilon) \times \{0\}$; this orbit is the one that Welander was concerned with when he originally published his model. There is also a unique unstable node in the interval $(3/4 + 15/4\varepsilon, 3/4 + 5/4\varepsilon) \times \{0\}$, which corresponds to the sliding region in this model for this parameter range. Everywhere else on the x-axis is a crossing region. This birth of a periodic is analogous to a supercritical Hopf bifurcation, and has been found in other sources. Notably, [21] includes it in the classification of planar bifurcations in Filippov systems.



(a) $\varepsilon > \varepsilon_0$: one unstable Filippov equilibrium and a stable periodic orbit.

(b) Bifurcation at $\varepsilon = \varepsilon_0$: the periodic orbit becomes a homoclinic orbit.

(c) $\varepsilon < \varepsilon_0$ the periodic orbit does not exist and all solutions are attracted to a globally stable equilibrium.

Figure 3.10: Border Collision Bifurcation; figures from [22].

The second bifurcation that Leifeld finds, however, is not included in [21], which motivates a further and more rigorous study of this field. This bifurcation is a type of *border collision bifurcation*. The name comes from the fact that a virtual equilibrium collides with the splitting boundary as the parameter is varied, causing the bifurcation. Although border collision bifurcations had been discussed previously, this specific one displays qualitatively different behavior than any that had been previously described.

Here, for $\varepsilon > \varepsilon_0 := \frac{-1}{15}$, there is a stable periodic orbit that passes through the splitting manifold, and inside of this orbit and on the splitting manifold there is an unstable *Filippov equilibrium*. A Filippov equilibrium is just an equilibrium of the system that exists on the splitting manifold. At $\varepsilon = \varepsilon_0$, this Filippov equilibrium collides with the edge of the periodic orbit, transforming it into a homoclinic orbit. The stability of this equilibrium is very strange. All solutions in the region $y > 0$ converge to it while solutions in the region $y < 0$ are repelled from it. If we restrict our view to the splitting manifold it is a repelling equilibrium. We note that this behavior is qualitatively impossible in continuous systems. Solutions which begin on the line $y = 0$ are repelled from the equilibrium while solutions that begin in the region $y > 0$ are attracted to it, regardless of how close the initial condition is to the line $y = 0$. This behavior can be topologically ruled out in smooth systems. Additionally, solutions may leave the equilibrium in finite time and go into the region $y < 0$; this occurs because the Filippov convex combination at this point includes both 0 and terms with a negative y component, so a solution can stay at the equilibrium an arbitrary length of time before heading downwards.

For $\varepsilon < \varepsilon_0$ this Filippov equilibrium leaves the splitting manifold and becomes a globally stable equilibrium; the periodic orbit no longer remains. The change in stability is what the classification in [21] misses. While border collision bifurcations are recognized in that paper, the equilibrium is said to either be annihilated or to persist without a change in stability. This case–where the equilibrium persists but changes stability–is missing.

That classification of bifurcations misses this type of border collision bifurcation because of some of the assumptions that the paper makes regarding transformations. As discussed above, continuous but nonlinear transformations can alter the Filippov combination and make the behavior of the transformed system qualitatively different from the original system. Said another way, if we apply a nonlinear transformation h to a Fillipov vector field F, where $F(x)$ is defined by the convex combination at the point x, then $h(F(x))$ will not necessarily be the convex combination of the transformed system. The paper [21] ignores this issue, and that leads to an incomplete classification of qualitatively interesting behavior.

This bifurcation is also important because it has no analogue in smooth systems; the discontinuous stability of the equilibrium at ε_0 makes this clear. This fact indicates that the behavior of Filippov systems is unique and further motivates their study.

Navigating Sections on Welander's Model:

- Goals for Welander's Model, Section 1.2
- Introduction to Welander's Model and Bifurcation Analysis, Section 3.3
- Perturbation and Welander's Model, Section 4.7
- Welander's Model as a Multiflow, Section 5.5
- Welander's Model as a Well-Parametrized Family of Multiflows, Section 6.2
- Isolating Neighborhood in Welander's Model, Section 7.4
- Attractor-Repeller Pair Decomposition for Welander's Model, Section 8.5
- The Conley Index and Welander's Model, Section 9.5

3.4 Vector Fields with Bounded, Nonautonomous Control

Throughout this section we fix the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$. We are interested in perturbing the ordinary differential equation

$$\dot{x} = f(x)$$

by locally integrable and essentially bounded control functions. To state that more explicitly, consider the family of functions

$$\{g : \mathbb{R} \supset I \to \mathbb{R}^n\}$$

such that g are Lebesgue integrable functions bounded in the sup norm by some positive constant $r \geq 0$; that is,

$$||g||_\infty := \sup_{t \in I} |g(x)| \leq r$$

We now consider the perturbed the differential equations

$$\dot{x} = f(x) + g(t) \tag{3.3}$$

for any g in this family. As is the case with differential inclusions, a solution to (3.3) is an absolutely continuous function $x : \mathbb{R} \supset J \to \mathbb{R}^n$ such that

$$\frac{dx}{dt} = f(x) + g(t)$$

almost everywhere. This notion of a solution comes from Caratheòdory theory [10].

Now for this same f, define the set valued function

$$F_r(x) = \{y \in \mathbb{R}^n : |y - f(x)| \leq r\}$$

The differential inclusion

$$\dot{x} \in F_r(x) \tag{3.4}$$

seems very similar to the differential equation (3.3). In fact, they are identical:

Lemma 3.2. *[24] The function*

$$x : \mathbb{R} \supset J \to \mathbb{R}^n$$

is a solution to (3.3) *if and only if it is a solution to* (3.4)*.*

Proof. We first assume that $x : \mathbb{R} \supset J \to \mathbb{R}^n$ satisfies $\dot{x}(t) = f(x(t)) + g(t)$ almost everywhere for some Lebesgue integrable g that is bounded by r. Then

$$|\dot{x}(t) - f(x(t))| = |g(t)| \leq r$$

and so $\dot{x}(t) \in F_r(x(t))$.

Conversely, suppose that $x : \mathbb{R} \supset J \to \mathbb{R}^n$ is a solution of (3.4). Since x is absolutely continuous, $\dot{x}$ is a Lebesgue integrable function satisfying $|\dot{x}(t) - f(x(t))| \leq r$. Then if we define the function

$$g(t) := \dot{x}(t) - f(x(t))$$

we see that g is Lebesgue integrable (since both $\dot{x}$ and f are) and that $||g||_\infty \leq r$. Then $\dot{x}(t) = f(x(t)) + g(t)$ and so x solves (3.3). □

This lemma is useful because it allows us to immediately apply any results about basic differential inclusions to the solutions of the family of differential equations (3.3) since it is clear that F_r satisfies the basic conditions. It is interesting to note, however, that F_r actually has a good deal more structure than we typically assume in this manuscript. Given any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we can find $\delta > 0$ such that for $|x-y| < \delta$ implies that $F_r(y) \subset B_\varepsilon(F_r(x))$ and $F_r(x) \subset B_\varepsilon(F_r(y))$; that is, F_r is actually continuous and not merely upper-semicontinuous.

3.5 Solutions of Basic Differential Inclusions

In this section we give proofs of some standard results about differential inclusions; these results may all be found in [10].

The first few results build to the existence theorem for differential inclusions. The proof of that theorem is a modification of the classic Cauchy-Peano existence proof for differential equations. The Cauchy-Peano proof relies on the construction of approximate solutions to the differential equation $\dot{x} = f(x)$, and so it is therefore necessary to define an approximate solution to a differential inclusion $\dot{x} \in F(x)$.

Definition 3.4. *A* δ***-solution*** *of the differential inclusion* $\dot{x} \in F(x)$ *is an absolutely continuous function* $y(t)$ *that almost everywhere satisfies the differential inclusion*

$$\dot{y}(t) \in F_\delta(y(t))$$

where $F_\delta(y) := \overline{B_\delta(co(F(\overline{B_\delta(y)})))}$

One of the key ideas of the Cauchy-Peano existence proof is that a sequence of increasingly accurate approximate solutions to the differential equation (the Euler broken line approximations) converges to an exact solution. This step is very different in the case of differential inclusions, and so we will present the analogous result in the next two lemmas.

Lemma 3.3. *[10] Let* $\{x_k : [a,b] \to \mathbb{R}^n\}_{k=1}^{\infty}$ *be a sequence of absolutely continuous functions that limit to a function* $x(t)$*, and assume that* $\dot{x_k}(t) \in D$ *almost everywhere, where* $D \subset \mathbb{R}^n$ *is a compact, convex set. Then* $x(t)$ *is absolutely continuous and* $\dot{x}(t) \in D$ *wherever it is defined, namely, almost everywhere on* $[a,b]$*.*

Proof. Since D is bounded, by Lemma 2.1 there is some $m > 0$ such that $|\dot{x_k}(t)| \leq m$ for all k and $t \in [a,b]$. Then for $t_1, t_2 \in [a,b]$ we have:

$$\begin{aligned} |x(t_1) - x(t_2)| &= \lim_{k\to\infty} |x_k(t_1) - x_k(t_2)| \\ &= \lim_{k\to\infty} |\int_{t_2}^{t_1} \dot{x_k}(t)\, dt| \\ &\leq \lim_{k\to\infty} \int_{t_2}^{t_1} m\, dt \\ &= m|t_1 - t_2| \end{aligned}$$

Thus x is Lipschitz continuous, and hence absolutely continuous.

To see that $\dot{x}(t) \in D$ wherever it is defined, arbitrarily fix $t \in (a,b)$ and take h small enough that $[t-h, t+h] \subset (a,b)$. We claim that

$$q_k^h := \frac{x_k(t+h) - x_k(t)}{h} = \int_t^{t+h} \frac{\dot{x_k}(t)}{h}\, dt \in D$$

In order to prove this claim, we consider the Riemann definition of the above integral. Note that since the x_k are absolutely continuous functions on the real line, the Riemann and Lebesgue definitions of the integral are equivalent, and so we consider the Riemann sum for simplicity. The q_k^h are supremums (or infimums) of integral sums of the form

$$\sum \frac{\Delta_i \dot{x_k}(t_i)}{h}, \qquad \sum \frac{\Delta_i}{h} = 1$$

This presentation shows that because of the averaging $\frac{1}{h}$ factor, the integral sums are convex combinations of points $\dot{x_k}(t_i) \in D$, and hence belong to the convex set D. Since D is compact, the supremum (or infimum) over the set of integral sums also belongs to D, and so the claim is verified.

Since D is compact,

$$\lim_{k\to\infty} q_k^h = \frac{x(t+h) - x(t)}{h} \in D$$

Note that the above statement remains true for arbitrarily small h. Then again using the compactness of D, this statement implies that

$$\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h\to 0} q_k^h \in D$$

whenever that limit exists. Since $x(t)$ is absolutely continuous, the limit must exist almost everywhere on the interval (a,b).

□

In addition to its use in demonstrating solution existence for basic differential inclusions, the following lemma is fundamental in demonstrating that isolating neighborhoods are stable in this multivalued setting. As we will see in chapter 4, solutions to perturbed differential inclusions are approximate solutions of the original differential inclusion, and so this lemma allows us to study these perturbed inclusions.

Lemma 3.4. *[10] Let $F(x)$ satisfy the basic conditions in a domain G and let $\delta_k \to 0$ as $k \to \infty$. Then the limit $x(t)$ of a uniformly convergent sequence $\{x_k : [a,b] \to G\}$*

of δ_k-solutions to the differential inclusion $\dot{x} \in F(x)$ is a solution to that inclusion (as long as $x(t) \in G$).

Proof. Choose an arbitrary $t_0 \in (a, b)$ and $\varepsilon > 0$. We will show that in a neighborhood of t_0, $\dot{x_k}(t) \in \overline{B_{2\varepsilon}(F(x(t_0)))}$. By Lemma 3.3, this relationship implies that in that neighborhood, $x(t)$ is absolutely continuous and $\dot{x}(t) \in \overline{B_{2\varepsilon}(F(x(t_0)))}$ wherever that derivative exists (almost everywhere in the neighborhood). Since our choices of t_o, ε were arbitrary, we see a few things. First, we see that $x(t)$ is absolutely continuous on the whole of the interval $[a, b]$ since it is absolutely continuous in a neighborhood of each t_0 and $[a, b]$ is compact. Second, it shows that $\dot{x}(t_0) \in F(x(t_0))$ for all t_0 where x is differentiable (again, almost everywhere on the interval) since the choice of ε was arbitrary and t_0 clearly belongs to any neighborhood of itself. Thus, in order to prove this lemma, we only need to prove the claim.

During this proof, bear in mind that we only consider $t \in [a, b]$, and so for $t_0 = a$ or $t_0 = b$ the neighborhoods we will find are one-sided. Now, let $x_0 := x(t_0)$. By the upper-semicontinuity of F, there exists some $\eta > 0$ such that $|y - x_0| < 3\eta$ implies that $F(y) \subset \overline{B_\varepsilon(F(x_0))}$. Since $\delta_k \to 0$ and the $x_k(t)$ converge uniformly to $x(t)$, there is also some k_0 such that $k > k_0$ implies that $\delta_k < \min(\eta, \varepsilon)$ and $|x_k(t) - x(t)| < \eta$ for all $t \in [a, b]$. Additionally, by the continuity of $x(t)$ (clear since $x_k \to x$ uniformly), there is some $\gamma \in (0, \eta)$ such that $|t - t_0| < \gamma$ implies that $|x(t) - x(t_0)| < \eta$. For such t, k, η and γ, the following are true:

1. $\overline{B_{\delta_k}(t)} \subset \overline{B_{2\eta}(t_0)}$: This relationship is clear from the choice of t.

2. $\overline{B_{\delta_k}(x_k(t))} \subset \overline{B_{3\eta}(x_0)}$: This fact follows from our assumption that $\delta_k < \eta$ and the inequalities

$$|x_k(t) - x(t_0)| \leq |x_k(t) - x(t)| + |x(t) - x(t_0)| \leq \eta + \eta$$

3. $F(\overline{B_{\delta_k}(x_k(t))}) \subset \overline{B_\varepsilon(F(x_0))}$: This final relationship follows from the prior one and our choice of η.

From this third insight we see that

$$\begin{aligned}\dot{x_k}(t) &\in \overline{B_{\delta_k}(co(F(\overline{B_{\delta_k}(x_k(t))})))} \\ &\subset \overline{B_{\delta_k}(co(\overline{B_\varepsilon(F(x_0))}))} \\ &\subset \overline{B_{2\varepsilon}(F(x_0))}\end{aligned}$$

The final inclusion follows from the condition that $\delta_k < \varepsilon$ and the fact that $F(x_0)$, and hence $\overline{B_\varepsilon(F(x_0))}$, are already convex. By Lemma 3.3, it follows that $\dot{x}(t_0) \in \overline{B_{2\varepsilon}(F(x_0))}$, completing the proof.

□

The preceding lemma also gives us the following corollary, which is used in showing that the solution set of basic differential inclusions is a multiflow.

Corollary 3.4.1. *If $F(x)$ satisfies the basic conditions in G, then the limit of a uniformly convergent sequence of solutions to the differential inclusion $\dot{x} \in F(x)$ is also a solution.*

Now that we have that lemma, we are in a position to prove the main existence result. The proof of that theorem is very similar to the proof of the classic Cauchy-Peano existence theorem, using a sequence of Euler broken line approximations that limit to the desired solution. One small alteration that must be made is that when we iteratively define the Euler broken lines at a point x, we choose any arbitrary vector in $F(x)$ since we do not have a unique choice $f(x)$. The fact that these approximate solutions of the differential inclusion converge to an exact solution follows from Lemma 3.4.

Theorem 3.3. *[10] Let F satisfy the basic conditions in an open domain $G \subset \mathbb{R}^n$. Then for any $x_0 \in G$, there exists a solution of the differential inclusion*

$$\dot{x} \in F(x), \qquad x(0) = x_0$$

on some interval $[-c_-, c_+]$, where $c_-, c_+ > 0$.

Proof. Without loss of generality, we will demonstrate solution existence on a closed positive interval $[0, c]$. The existence proof in backwards time is symmetric.

Since G is open, we may choose r small enough that the closed ball $\overline{B_r(x_0)}$ is contained in G. Let us denote this ball Z. Next, let $m := \sup_Z |F(x)|$. By Lemma 2.1, $m < \infty$. The length of our interval is $c := \frac{r}{m}$. We are now ready to begin to define the sequence of Euler broken lines.

For $k = 1, 2, \cdots$, define a step size $h_k := \frac{c}{k}$; clearly, $h_k \to 0$ as $k \to \infty$. We partition the interval $[0, c]$ into k subintervals. Let $t_k^i := ih_k$ for $i = 0, 1, \cdots, k$. Note that the superscript here is an index rather than an exponential. We will define a family of continuous functions $x_k : [0, c] \to Z$ that are linear on the intervals $[t_k^i, t_k^{i+1}]$.

We initiate an iterative process by declaring that $x_k(0) = x_0$. In order to define $x_k(t)$ for $t \in (t_k^i, t_k^{i+1}]$, first choose any vector $v_k^i \in F(x_k(t_k^i))$. Again, the superscript here denotes an index. Then for $t \in (t_k^i, t_k^{i+1}]$,

$$x_k(t) := x_k(t_k^i) + (t - t_k^i)v_k^i$$

The functions $x_k(t)$ are absolutely continuous since they are continuous and piecewise linear. Additionally, if we define $\delta_k := h_k$, then

$$\dot{x_k}(t) = v_k^i \in F(x_k(t_k^i)) \subset F(x_k(\overline{N_{\delta_k}(t)})) \subset F_{\delta_k}(x_k(t))$$

and so the x_k are δ_k-solutions to the differential inclusion $\dot{x} \in F(x)$.

We also see that $x_k(t) \in Z$ for $t \in [0, c]$ because we make at most k steps of length $h_k = \frac{r}{mk}$ and the maximum velocity is m. More formally, for $t \in (t_k^l, t_k^{l+1}]$ $(0 \leq l < k)$, we have the following:

$$
\begin{aligned}
|x_k(t) - x_0| &= |\int_0^t \dot{x_k}(s)\, ds| \\
&= |\int_{t_k^l}^t \dot{x_k}(s)\, ds + \sum_{i=0}^{l-1} \int_{t_k^i}^{t_k^{i+1}} \dot{x_k}(s)\, ds| \\
&\leq \int_{t_k^l}^t |\dot{x_k}(s)|\, ds + \sum_{i=0}^{l-1} \int_{t_k^i}^{t_k^{i+1}} |\dot{x_k}(s)|\, ds \\
&= \int_{t_k^l}^t |v_k^l|\, ds + \sum_{i=0}^{l-1} \int_{t_k^i}^{t_k^{i+1}} |v_k^i|\, ds \\
&\leq \sum_{i=0}^{k} \int_{t_k^i}^{t_k^{i+1}} m\, ds \\
&= \sum_{i=0}^{k} (h_k m) \\
&= k(\frac{r}{mk})m \\
&= r
\end{aligned}
$$

Then since the family of functions $\{x_k\}_{k=1}^{\infty}$ is uniformly bounded (contained in Z) and equicontinuous ($|\dot{x_k}(t)| \leq m$), by the Arzela-Ascoli theorem we can choose a uniformly convergent subsequence with a limit $x(t)$. Since Z is compact, $x(t) \in Z$ for $t \in [0, c]$, and so by Lemma 3.4, the function $x : [0, c] \to G$ is a solution of the inclusion $\dot{x} \in F(x)$. □

The basic existence result is very important for the study of differential inclusions. Unfortunately, there is no general uniqueness result for basic differential inclusions; many of these systems do, in fact, have multiple solutions for a given initial condition. However, the solutions of these differential inclusions do behave in other ways that are reminiscent of solutions to standard differential equations. For starters, any family of approximate solutions on a common time interval is uniformly equicontinuous.

Lemma 3.5. *Assume F satisfies the basic conditions and fix $\delta > 0$. For any ε such that $0 < \varepsilon < \delta$, all ε-solutions to the differential inclusion $\dot{x} \in F(x)$ which lie in a* *compact domain X share a uniform Lipschitz bound (which does not depend on ε). Thus ε-solutions are uniformly equicontinuous irrespective of ε.*

Proof. Since upper-semicontinuous functions are bounded on compact domains, there is some $M > 0$ such that $|F(x)| < M$ for all $x \in X$. Note that $|F_\varepsilon(x)| \leq M + \delta$. Then for any ε-solution $\psi : I \to X$,

$$|\psi(t) - \psi(s)| = |\int_s^t \dot{\psi}(\tau)d\tau| \leq |\int_s^t |\dot{\psi}(\tau)|d\tau| \leq |\int_s^t (M+\delta)\, d\tau| = (M+\delta)|t-s|$$

□

It is worth emphasizing that the previous lemma gives us the following corollary:

Corollary 3.5.1. *[10] If $F(x)$ satisfies the basic conditions in a closed, bounded domain D, all solutions of the differential inclusion $\dot{x} \in F(x)$ in that domain are uniformly equicontinuous.*

Proof. This result may be seen by considering the definition of a solution. A solution $x(t)$ has a derivative $\dot{x}(t) \in F(x(t))$ almost everywhere, and it satisfies the Lebesgue integral equation

$$x(t) = x(0) + \int_0^t \dot{x}(s)ds$$

Using the bound of F in D from Lemma 2.1, we see that solutions are equicontinuous:

$$|x(s_1) - x(s_2)| = |\int_{s_2}^{s_1} \dot{x}(s)ds| \leq \int_{s_2}^{s_1} |\dot{x}(s)|ds \leq \int_{s_2}^{s_1} M ds = M|s_1 - s_2|$$

□

Using the preceding lemmas and theorems, we can also show that any solution can be continued until it reaches the boundary of a compact domain. The basic intuition of this claim is clear; if our solution terminates somewhere in the interior of a compact set, we can extend it using the existence theorem. Below, we state and prove this result more rigorously.

Theorem 3.4. *[10] Let the set-valued function $F(x)$ satisfy the basic conditions in a closed, bounded domain D. Then each solution of the differential inclusion $\dot{x} \in F(x)$ lying within D can be continued on both sides up to the boundary of the domain D.*

Proof. From the existence Theorem 3.3, we know that at any initial condition there is a solution $x(t)$ to the differential inclusion on some closed interval $[0, c_1]$. The idea of this proof is to now consider the point $x(c_1)$ and extend the solution from there by again using the existence theorem. This process is iterated indefinitely, and it either yields a solution defined for all time (meaning the solution remains in the interior of D for all forwards time) or the position of the solution at the endpoints limits to the boundary of D. The process in backwards time is symmetric.

Now, more formally, take an arbitrary $x_0 \in D$. There is some ε_1 such that $\overline{B_{\varepsilon_1}(x_0)}$ is contained in the interior of D. Following the method of the proof of the existence Theorem 3.3, there is a solution $x : [0, c_1] \to \overline{B_{\varepsilon_1}(x_0)}$ where $c_1 = \frac{\varepsilon_1}{m}$ and $m = \sup_D |F(x)|$ ($m < \infty$ by Lemma 2.1). Denoting the boundary of D by Γ, if $d(x(c_1), \Gamma) > \varepsilon_1$ then we can extend the solution to a further interval of length c_1. We either repeat this process indefinitely (giving us a solution which remains in D for all forwards time) or until $d(x(kc_1), \Gamma) \leq \varepsilon_1$ for some k. In the latter case, let $t_1 = kc_1$ and $x_1 = x(t_1)$.

Choosing $\varepsilon_2 < d(x_1, \Gamma) < \varepsilon_1$, we repeat this process, letting $t_2 = t_1 + jc_2$ and $x_2 = x(t_2)$ if we reach a point that $d(x(t_1 + jc_2), \Gamma) \leq \varepsilon_2$. In fact, we may iterate this process either until the algorithm yields a solution remaining in the interior of D for all time at some i^{th} step or we get sequences

$$t_1 < t_2 < \cdots, \qquad x_1, x_2, \cdots$$

If $t_i \to \infty$ as $i \to \infty$, then the solution x remains in D for all forwards time. Otherwise, there exists some T such that $t_i < T$ for all i. Thus $\{t_i\}$ is a bounded, monotonic sequence, and hence limits to some t^*. This bound implies that $\varepsilon_i \to 0$ since $c_i = \frac{\varepsilon_i}{m}$. We also see that the x_i converge to some x^* because by the equicontinuity of solutions (Lemma 3.5.1), $|x(t_i) - x(t_j)| \leq m|t_i - t_j|$. Letting $x(t^*) = x^*$, we obtain a solution $x : [0, t^*] \to D$ which reaches the boundary of D. □

Chapter 4

Perturbation of Differential Inclusions

4.1 The Notion of Perturbation

A differential equation of the form

$$\dot{x} = f(x)$$

where $f : \mathbb{R}^n \supset G \to \mathbb{R}^n$ may be used in order to model many different types of systems. However, we generally do not believe that this model is an exact representation of the reality, but rather an approximation to it. Therefore it is natural to consider perturbing this equation in some way. One very natural and general way to do so is to consider a parametrized differential equation

$$\dot{x} = f(x, \lambda)$$

where now $f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$. Here we assume that Λ is an interval containing zero, f varies continuously with λ, and we identify the original map with the value $\lambda = 0$: $f(x) = f(x, 0)$.

Our goal for this section is to determine an equivalent method of perturbing an arbitrary basic differential inclusion $\dot{x} \in F(x)$. To do so, we will consider a set-valued function

$$F : G \times \Lambda \to \mathcal{P}(\mathbb{R}^n)$$

for some open subset G of $\mathbb{R}^n$ and interval $\Lambda \subset \mathbb{R}$, which satisfies the basic conditions. That is, we assume that F is upper-semicontinuous in its domain and that the image of any point is compact-valued, convex-valued, and non-empty.

Definition 4.1. *Let $G \subset \mathbb{R}^n$ be open and $\Lambda \subset \mathbb{R}$ be an interval. If $F : G \times \Lambda \to \mathcal{P}(\mathbb{R}^n)$ satisfies the basic conditions then the family of inclusions*

$$\dot{x} \in F(x, \lambda), \qquad \lambda \in \Lambda$$

is said to be a basic parametrized differential inclusion.

We note that this definition of perturbation is also used in [32].

Notice that in the special case where F is single-valued, the assumption that F satisfies the basic conditions reduces to the assumption that F be continuous. Therefore the basic parametrized differential inclusion $\dot{x} \in F(x, \lambda)$ is a direct generalization of a continuous parametrized differential equation $\dot{x} = f(x, \lambda)$.

Because this multivalued setting is unusual and sometimes counterintuitive, it is worthwhile to consider whether this generalization of the notion of perturbation is reasonable. This concept is inherently subjective, but we will try and demonstrate that this definition has many features that we would expect it to have.

First, we notice that because the composition of upper-semicontinuous maps is upper-semicontinuous (Property 2.1), continuously perturbing the input of a basic multivalued map gives us an allowable perturbation:

Property 4.1. *Let $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ satisfy the basic conditions and let $g : \mathbb{R}^n \times J \to \mathbb{R}^n$ be a continuous map. The set-valued map*

$$H : \mathbb{R}^n \times J \to \mathcal{P}(\mathbb{R}^n) \qquad (x, \lambda) \mapsto F(g(x, \lambda))$$

satisfies the basic conditions.

Because our primary motivation is the study of Filippov systems, we devote the next two sections to understanding how this notion of perturbation applies to these objects.

4.2 Perturbing Filippov Systems to Other Filippov Systems

Given a Filippov system, any reasonable notion of perturbation should allow us to continuously deform the splitting boundary Σ or continuously perturb any of the defining vector fields f_i. We will show in this section that these modifications do fit into our definition.

For the sake of simplicity we will focus on Filippov systems with a single discontinuity boundary:

$$\dot{x} \in F(x) = \begin{cases} p_1(x), & h(x) > 0 \\ p_2(x), & h(x) < 0 \\ \{\alpha p_1(x) + (1-\alpha)p_2(x) : \alpha \in [0,1]\} & h(x) = 0 \end{cases}$$

We will show that continuously perturbing either h or the p_i yields an allowable perturbation in our sense.

Lemma 4.1. *Let $G \subset \mathbb{R}^n$ be open and assume $p_i : G \times [-1,1] \to \mathbb{R}^n$ and $h : G \times [-1,1] \to \mathbb{R}^n$ are continuous. Then the set-valued function*

$$F(x,\lambda) = \begin{cases} p_1(x,\lambda), & h(x,\lambda) > 0 \\ p_2(x,\lambda), & h(x,\lambda) < 0 \\ \{\alpha p_1(x,\lambda) + (1-\alpha)p_2(x,\lambda) : \alpha \in [0,1]\} & h(x,\lambda) = 0 \end{cases}$$

satisfies the Filippov conditions.

Proof. For a fixed λ_0, the fact that $F(\cdot,\lambda_0)$ is compact, convex, and non-empty valued is obvious. Therefore we only need to check that F is upper-semicontinuous at any (x_0,λ_0) in the domain.

Fix (x_0,λ_0). If $h(x_0,\lambda_0) \neq 0$ then by the continuity of h we can choose (x,λ) close enough to (x_0,λ_0) that $h(x,\lambda) \neq 0$, and so the upper-semicontinuity of F at (x_0,λ_0) follows trivially from the continuity of p_1 and p_2. Therefore we will assume that $h(x_0,\lambda_0) = 0$.

Fix $\varepsilon > 0$. By the continuity of the p_i, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|(x_0, \lambda_0) - (x, \lambda)| < \delta_i$$

implies that

$$|p_i(x_0, \lambda_0) - p_i(x, \lambda)| < \varepsilon$$

Define $\delta = \min(\delta_1, \delta_2)$ and arbitrarily choose (x, λ) such that $|(x_0, \lambda_0) - (x, \lambda)| < \delta$. In this case it is possible for $h(x, \lambda)$ to be positive, negative, or zero.

We will first address the case that $h(x, \lambda) \neq 0$, and assume without loss of generality that $h(x, \lambda) > 0$. Note that $F(x, \lambda) = \{p_1(x, \lambda)\}$ and that $p_1(x_0, \lambda_0) \in F(x_0, \lambda_0)$. By our choice of δ we have that $|p_1(x_0, \lambda_0) - p_1(x, \lambda)| < \varepsilon$, and so $F(x, \lambda) \subset B_\varepsilon(F(x_0, \lambda_0))$.

The remaining possibility is that $h(x, \lambda) = 0$, so

$$F(x, \lambda) = \{\alpha p_1(x, \lambda) + (1 - \alpha)p_2(x, \lambda) : \alpha \in [0, 1]\}$$

Choose any arbitrary vector $v \in F(x, \lambda)$. Then $v = \alpha p_1(x, \lambda) + (1 - \alpha)p_2(x, \lambda)$ for some fixed $\alpha \in [0, 1]$. Note that for this fixed α, the vector

$$v_0 := \alpha p_1(x_0, \lambda_0) + (1 - \alpha)p_2(x_0, \lambda_0)$$

lies in $F(x_0, \lambda_0)$. Then by our choice of δ,

$$\begin{aligned}
|v_0 - v| &= |(\alpha p_1(x_0, \lambda_0) + (1 - \alpha)p_2(x_0, \lambda_0)) - (\alpha p_1(x, \lambda) + (1 - \alpha)p_2(x, \lambda))| \\
&= |\alpha(p_1(x_0, \lambda_0) - p_1(x, \lambda)) + (1 - \alpha)(p_2(x_0, \lambda_0) - p_2(x, \lambda))| \\
&\leq \alpha|p_1(x_0, \lambda_0) - p_1(x, \lambda)| + (1 - \alpha)|p_2(x_0, \lambda_0) - p_2(x, \lambda)| \\
&\leq \alpha\varepsilon + (1 - \alpha)\varepsilon \\
&= \varepsilon
\end{aligned}$$

Thus $F(x, \lambda) \subset B_\varepsilon(F(x_0, \lambda_0))$ and so F is upper-semicontinuous. □

4.3 Perturbing Filippov Systems to Nearby Smooth Systems

4.3.1 Limiting Smooth Systems which are not Perturbations of the Filippov System

As stated in the introduction, our primary motivation for generalizing Conley index theory to the setting of differential inclusions was to be able to study piecewise-continuous differential equations and carry the information that we gather there to nearby smooth systems. However, it is well-known that two different families of differential equations $\{\dot{x} = f(x, \lambda)\}$ and $\{\dot{x} = g(x, \lambda)\}$ may have qualitatively different behavior even if the parametrized vector fields p and g both limit to the same piecewise-continuous vector field away from the discontinuity boundary as $\lambda \to 0$ [16]. In example 5.3.1 it is shown that this distinction can even impact the Conley index. Therefore we do not expect our notion of perturbation to allow us to pass from a Filippov system to *any* limiting family of smooth systems. Instead, what our notion of perturbation does is allow us to easily distinguish whether a limiting smooth system is an appropriate perturbation of a given Filippov system, or whether a different convention should be used to define the differential inclusion from the piecewise-continuous differential equation.

Let us begin by considering a typical piecewise-continuous differential equation with a discontinuity boundary given by the zeros of a continuous scalar-valued function $h : \mathbb{R}^n \to \mathbb{R}$:

$$\dot{x} \in f(x) = \begin{cases} p_1(x), & h(x) > 0 \\ p_2(x), & h(x) < 0 \end{cases}$$

Although this setup is more narrow than Filippov systems in general (because we have restricted ourselves to two regions of continuity and assumed that the discontinuity boundary Σ is given by the set $\{x | h(x) = 0\}$) this formulation is by far the most common in applications, and the ideas discussed here can be easily generalized to allow more regions once understood.

Given such a vector field $f : (\mathbb{R}^n \setminus \Sigma) \to \mathbb{R}^n$, the Filippov convex combination convention defines a unique basic differential inclusion

$$\dot{x} \in F(x) = \begin{cases} p_1(x), & h(x) > 0 \\ p_2(x), & h(x) < 0 \\ \{\alpha p_1(x) + (1-\alpha)p_2(x) : \alpha \in [0,1]\} & h(x) = 0 \end{cases}$$

which we call a Filippov system. However, a continuous vector field $g(x, \lambda)$ may limit to f on its domain and yet not be a perturbation of F.

To understand this subtle issue it is helpful to consider a concrete example. Define the piecewise-continuous function

$$\hat{f}(x) = \begin{cases} 1, & x < 0 \\ 3, & x > 0 \end{cases}$$

This function can be extended to a basic differential inclusion by using the Filippov convex combination method at the splitting manifold $\hat{\Sigma} = \{0\}$. However, we will see that for certain limiting families of smooth systems that the resulting Filippov system does not have the same qualitative behavior as the smooth family.

Define the functions

$$f_\lambda(x) = \tanh(\frac{x}{\lambda}) + 2 \qquad g_\lambda(x) = \tanh(\frac{x}{\lambda}) + 2 - 2 * \mathrm{e} * \mu(\frac{x}{\lambda})$$

where μ is the smooth mollifier

$$\mu(x) = \begin{cases} 0, & |x| > 1 \\ \exp(\frac{-1}{1-x^2}), & |x| \leq 1 \end{cases}$$

As $\lambda \to 0$, both f_λ and g_λ limit to $\hat{f}$ pointwise on its domain $\mathbb{R} \setminus \{0\}$. However, the dynamics of the differential equations

$$\dot{x} = f_\lambda(x) \qquad \dot{x} = g_\lambda(x)$$

are qualitatively different. For all λ, $\dot{x} = f_\lambda(x)$ has no equilibria, but for any λ there is

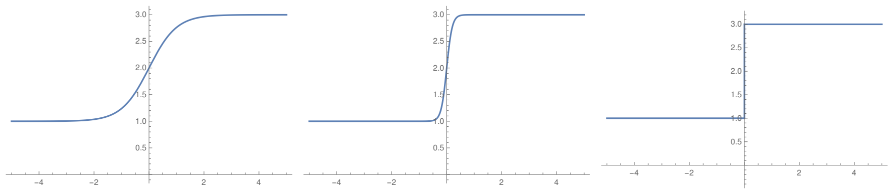


Figure 4.1: The functions f_1, $f_{1/5}$, and F_0 respectively.

an equilibrium of the ODE $\dot{x} = g_\lambda(x)$.

If we use the convex combination method on $\hat{f}$ we get the set-valued map

$$F_0(x) = \begin{cases} 1, & x < 0 \\ 3, & x > 0 \\ [1,3], & x = 0 \end{cases}$$

Qualitatively, the Filippov system $\dot{x} \in F_0(x)$ behaves like the differential equations $\dot{x} = f_\lambda(x)$, and so it seems natural to consider f_λ to be a nearby smooth system for F_0. Indeed, if we define the function $F : \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(x,\lambda) = \begin{cases} F_0(x), & \lambda = 0 \\ f_\lambda(x), & \lambda > 0 \end{cases}$$

we can easily verify that F satisfies the basic conditions. Therefore f_λ can be said to be a perturbation of F_0 in our sense.

However, g_λ is not an allowable perturbation of F_0; the function

$$H(x,\lambda) = \begin{cases} F_0(x), & \lambda = 0 \\ g_\lambda(x), & \lambda > 0 \end{cases}$$

is not upper-semicontinuous.

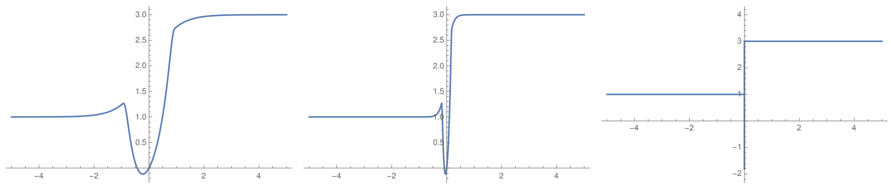


Figure 4.2: The functions g_1, $g_{1/5}$, and G_0 respectively.

The more appropriate set-valued limit of g_λ would be the function

$$G_0(x) = \begin{cases} 1, & x < 0 \\ 3, & x > 0 \\ [\tau, 3], & x = 0 \end{cases}$$

where $\tau = \min_{x\in\mathbb{R}} g_\lambda(x) < 0$ (this minimum is independent of λ). It is straightforward to verify that G_0 satisfies the basic conditions, and so we can reasonably consider the differential inclusion $\dot{x} \in G_0(x)$. Moreover, note that G_0 preserves the qualitative features of g_λ; the function $\psi \equiv 0$ solves the differential inclusion $\dot{x} \in G_0(x)$, showing that this inclusion has an equilibrium.

Moreover, if we define the function $G : \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$G(x, \lambda) = \begin{cases} G_0(x), & \lambda = 0 \\ g_\lambda(x), & \lambda > 0 \end{cases}$$

we can again verify that G satisfies the basic conditions, and so g_λ is an appropriate perturbation of G_0.

This example illustrates a general principle about these perturbations: piecewise-continuous systems may be the pointwise limit of a family of continuous systems, but that doesn't mean that family is an appropriate perturbation of the non-smooth system.

Readers familiar with the Conley index may notice that in the prior example, if we take a neighborhood of $\Sigma = \{0\}$ and compute its index as we would in the smooth case (we note here that no proofs yet exist to claim that this index is meaningful in the case of differential inclusions, though we hope to change that) then the index is trivial for both F_0 and G_0. That is, the Conley index does not distinguish between these systems even though our notion of perturbation does. However, that coincidence is simply due to the fact that the boundary of any neighborhood of $\{0\}$ does not intersect the discontinuity boundary $\Sigma = \{0\}$. In higher dimensions, when we are interested in isolating neighborhoods whose boundaries intersect the discontinuity, we will see that the Filippov convention plays an important role (example 5.3.1).

What this definition of perturbation does provide is a simple way to check whether a limiting smooth system is close to a given differential inclusion. Since single-valued functions are automatically compact, convex, and non-empty valued (at each point) checking the Filippov conditions simply requires checking upper-semicontinuity, a task which is about as easy as checking continuity in the classical case. We will discuss this idea more in the following subsection.

4.3.2 Smooth Approximations of the Filippov System

As we saw in the previous subsection, when we consider a typical piecewise-continuous differential equation

$$\dot{x} = f(x) = \begin{cases} p_1(x), & h(x) > 0 \\ p_2(x), & h(x) < 0 \end{cases}$$

with a discontinuity boundary given by the zeros of a continuous scalar-valued function $h : \mathbb{R}^n \to \mathbb{R}$, the choice of set-valued vector field on the discontinuity boundary $\Sigma = \{x|h(x) = 0\}$ plays a large role; crucially, the Filippov convention is only appropriate for certain smooth approximations to f. In this section we will give a large class of smooth approximations which *are* perturbations for the Filippov system

$$\dot{x} \in F(x) = \begin{cases} p_1(x), & h(x) > 0 \\ p_2(x), & h(x) < 0 \\ \{\alpha p_1(x) + (1-\alpha)p_2(x) : \alpha \in [0,1]\} & h(x) = 0 \end{cases}$$

that extends the general piecewise-continuous differential equation we were originally considering.

It is instructive to view f in another form. Consider the step function

$$\gamma(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

and notice that this function is the classic Heaviside step function except that the domain excludes $\{0\}$. With this function we can rewrite our discontinuous vector field as

$$f(x) = \gamma(h(x))p_1(x) + (1 - \gamma(h(x)))p_2(x).$$

We now consider smooth approximations to γ and f.

If the smooth family of functions $\{\gamma_\lambda : \mathbb{R} \to \mathbb{R}\}_{\lambda \in (0,1]}$ satisfies the condition that $\gamma_\lambda(t) \to \gamma(t)$ as $\lambda \to 0$ for $t \in \mathbb{R} \setminus \{0\}$, then the function

$$g(x, \lambda) = \gamma_\lambda(h(x))p_1(x) + (1 - \gamma_\lambda(h(x)))p_2(x)$$

limits to f pointwise on its domain. Some possible choices of functions γ_λ meeting these conditions are

$$\hat{\gamma}_\lambda(t) = \frac{1 + \tanh(t/\lambda)}{2}$$

$$\overline{\gamma}_\lambda(t) = \frac{1}{\pi} \arctan(t/\lambda) + \frac{1}{2}$$

$$\tilde{\gamma}_\lambda(x) = \frac{1 + \tanh(t/\lambda)}{2} - 2 * \mathrm{e} * \mu(\frac{t}{\lambda})$$

where μ is the smooth mollifier

$$\mu(t) = \begin{cases} 0, & |t| > 1 \\ \exp(\frac{-1}{1-t^2}), & |t| \leq 1 \end{cases}$$

Notice that in the example of the prior subsection, $\hat{\gamma}_\lambda$ is used in order to give a perturbation of the Filippov system, while $\tilde{\gamma}_\lambda$ gives a perturbation of the other differential inclusion. The difference is that $\hat{\gamma}_\lambda(0) \to 1/2$ as $\lambda \to 0$, whereas $\tilde{\gamma}_\lambda(0)$ limits to a value outside of $[0, 1]$. In fact, this behavior extends beyond that example.

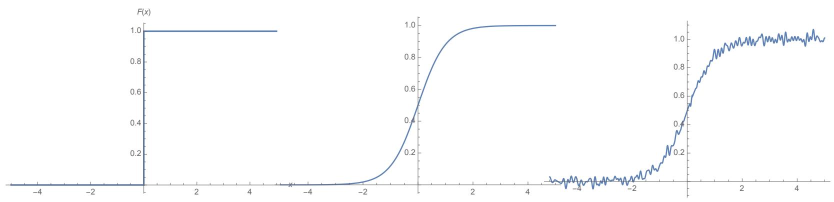


Figure 4.3: The Heaviside function and potential approximations.

The set-valued Heaviside function

$$\Gamma(t) = \begin{cases} 0, & t < 0 \\ [0,1], & t = 0 \\ 1, & t > 0 \end{cases} \tag{4.1}$$

and we consider a uniformly bounded continuous family of single-valued functions

$$\{\gamma_\lambda : \mathbb{R} \to \mathbb{R}\}_{\lambda \in (0,1]}$$

which limits to the Heaviside function as $(\lambda, t) \to (0, t_0)$. That is, we assume that $\lim_{(\lambda,t)\to(0,t_0)} \gamma_\lambda(t) = 0$ for $t_0 < 0$, $\lim_{(\lambda,t)\to(0,t_0)} \gamma_\lambda(t) = 1$ for $t_0 > 0$, and for any sequence $(\lambda_n, t_n) \to (0, 0)$ the set $\{\gamma_{\lambda_n}(t_n)\}$ contains no limit points outside of the interval $[0, 1]$. Notice that we do not demand that $\lim_{(\lambda,t)\to(0,0)} \gamma_\lambda(t)$ exists.

If we define the continuous vector fields

$$f_\lambda(x) = \gamma_\lambda(h(x))p_1(x) + (1 - \gamma_\lambda(h(x)))p_2(x) \tag{4.2}$$

then it is straightforward to verify that f_λ is a basic perturbation of F_0. Stated more directly, the set-valued map

$$F(x, \lambda) = \begin{cases} f_\lambda(x), & \lambda > 0 \\ F_0(x), & \lambda = 0 \end{cases} \tag{4.3}$$

satisfies the basic conditions.

This result is important for our applications because we will ultimately use these basic perturbations in order to make Conley type statements about families of differential inclusions. As in the classical case, the results obtained in this way are all stable under perturbation, and so these formulas gives us a large class of smooth approximations to the Filippov system which have the same qualitative features we identify using isolating neighborhoods.

It is worth mentioning that here when we say smooth approximation, what we mean is continuous approximation. For the sake of generality we choose to impose the minimal structure that we are able to work with, and that is continuity. Because differential equations with continuous right-hand side may have multiple solutions to a given initial condition, that means that it is possible that there are non-unique solutions to the equations $\dot{x} \in f_\lambda(x)$. Assuming that the functions which define the model, p_1 and p_2, are themselves sufficiently smooth, avoiding this behavior is easy–simply demand that γ_λ be locally Lipschitz continuous for all $\lambda \in (0, 1]$ in addition to the demands that we have placed above.

Finally, we remark that for many heuristic models (which is often the motivation for considering Filippov systems) the assumptions that we have made here are not a burden. In these heuristic models, the discontinuity is often an approximation of a rapid and difficult to understand transition. Then in many cases, the assumption that a smooth approximation to the Heaviside switching function does not contain limits outside of the interval $[0, 1]$ is very reasonable. However, we should remark explicitly that this statement of what might be a reasonable assumption for a heuristic model is not a mathematical one but rather depends on the context of the application, and therefore must be evaluated by the modeler.

As a final remark, we note that this section does not prove that the *only* smooth perturbations of a Filippov system are built using approximations to the Heaviside function, but only that such approximations will work. These approximations are highlighted because they are used commonly in applications.

4.4 Solutions of Perturbed Inclusions and Approximate Solutions

So far this chapter has been devoted to justifying the given notion of perturbation of differential inclusions. However, we also need to be able to make interesting statements about these perturbed inclusions. For each $\lambda \in \Lambda$, the basic differential inclusion $\dot{x} \in F(x, \lambda)$ may be examined using Filippov's theorems. However, we need some way to relate solutions for different values in Λ to each other. In particular, for our later Conley applications we will need that a uniformly convergent sequence of solutions to $\dot{x} \in F(x, \lambda)$ with $\lambda \to \lambda_0$ is a solution of the inclusion $\dot{x} \in F(x, \lambda_0)$. This desired property will follow from Filippov's Lemma 3.4 after the following lemmas demonstrate that solutions of a perturbed inclusion are approximate solutions of the original inclusion.

Lemma 4.2. *Assume $X \subset \mathbb{R}^n$ is compact. If $F : X \times [-1, 1] \to \mathbb{R}^n$ satisfies the basic conditions, then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\lambda| < \delta$ implies that*

$$F(x, \lambda) \subset F_\varepsilon(x, 0) := \overline{B_\varepsilon(co(F(\overline{B_\varepsilon(x)}, 0)))}$$

Proof. Choose $\varepsilon > 0$. Since F is upper-semicontinuous in both x and λ, for each $(x, 0)$ in the subspace $X \times \{0\}$ there is some δ'_x such that

$$|(x, 0) - (y, \lambda)| < \delta'_x \implies F(y, \lambda) \subset B_\varepsilon(F(x, 0))$$

Let $\delta_x := \min(\frac{\delta'_x}{2}, \varepsilon)$. The sets $\{B_{\delta_x}(x)\}_{x \in X}$ cover X, and since the space X is assumed to be compact we can find some finite subcover $\{B_{\delta_{x_k}}(x_k)\}_{k=1}^m$. Let

$$\delta := \min_{1 \le k \le m} \delta_{x_k}$$

The basic open sets

$$\{B_{\delta_{x_k}}(x_k) \times (-\delta, \delta)\}_{k=1}^m$$

form an open cover of $X \times (-\delta, \delta)$, and so for any $(x, \lambda) \in X \times (-\delta, \delta)$, there is some k such that

$$|(x_k, 0) - (x, \lambda)| \le |(x_k, 0) - (x, 0)| + |(x, 0) - (x, \lambda)| < \delta_{x_k} + \delta \le \delta_{x_k} + \delta_{x_k} \le \delta'_{x_k}$$

Thus $F(x,\lambda) \subset B_\varepsilon(F(x_k,0))$.

Since we have assumed that $\delta_{x_k} \leq \varepsilon$, we have that $|x - x_k| < \varepsilon$. Therefore

$$F(x,\lambda) \subset B_\varepsilon(F(x_k,0)) \subset \overline{B_\varepsilon(co(F(x_k,0)))} \subset \overline{B_\varepsilon(co(F(\overline{B_\varepsilon(x)},0)))} = F_\varepsilon(x,0)$$

□

Our purpose in proving that lemma is to get the following corollary:

Corollary 4.2.1. *If $F(x,\lambda)$ satisfies the Filippov conditions, then for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $|\lambda| < \delta$ implies that solutions to the perturbed differential inclusion $\dot{x} \in F(x,\lambda)$ are ε-solutions of the differential inclusion $\dot{x} \in F(x,0)$.*

Proof. A solution to the perturbed differential inclusion $\dot{x} \in F(x,\lambda)$ is an absolutely continuous function ψ mapping from an interval of the real line into the state space which satisfies

$$\dot{\psi}(t) \in F(\psi(t),\lambda)$$

almost everywhere.

By Lemma 4.2, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|\lambda| < \delta$ implies that $F(x,\lambda) \subset F_\varepsilon(x,0)$ for all x. Then for such λ,

$$\dot{\psi}(t) \in F(\psi(t),\lambda) \subset F_\varepsilon(\psi(t),0)$$

wherever the derivative exists (almost everywhere).

□

4.5 Welander's Model and Perturbations

We conclude this chapter on the perturbation of differential inclusions by examining what this notion of perturbation means for the Welander model. Recall that Welander used both a smooth and nonsmooth version of his model. After making a coordinate transformation, the nonsmooth model is given by the Filippov system of inclusion 3.2.

The transformed smooth version that Welander uses is given by the equations

$$\dot{\begin{bmatrix} x \\ y \end{bmatrix}} = \hat{w}_\lambda(x,y) := \begin{bmatrix} 1 - x - \hat{\gamma}_\lambda(y)x \\ \beta - \beta\varepsilon - \hat{\gamma}_\lambda(y)\varepsilon - \alpha - (\beta + \hat{\gamma}_\lambda(y))y - (\alpha\beta - \alpha)x \end{bmatrix} \tag{4.4}$$

where

$$\hat{\gamma}_\lambda(t) = \frac{1 + \tanh(t/\lambda)}{2}$$

As shown in Section 4.3.2, this family of smooth functions is a basic perturbation of the Filippov system given by inclusion 3.2. Therefore, letting $\hat{w}_\lambda$ and W_0 be defined as in 4.6 and 3.2 respectively, we see that

$$\hat{W}_\lambda(x,y) = \begin{cases} \hat{w}_\lambda(x,y), & \lambda > 0 \\ W_0(x,y), & \lambda = 0 \end{cases} \tag{4.5}$$

satisfies the basic conditions, and $\dot{(x,y)} \in \hat{W}_\lambda(x,y)$ is a parametrized basic differential inclusion.

More generally, however, we can consider any family of smooth functions

$$\dot{\begin{bmatrix} x \\ y \end{bmatrix}} = w_\lambda(x,y) := \begin{bmatrix} 1 - x - \gamma_\lambda(y)x \\ \beta - \beta\varepsilon - \gamma_\lambda(y)\varepsilon - \alpha - (\beta + \gamma_\lambda(y))y - (\alpha\beta - \alpha)x \end{bmatrix} \tag{4.6}$$

where γ_λ limits to the set-valued Heaviside function as $\lambda \to 0$ and it follows from Section 4.3.2 that

$$W_\lambda(x,y) = \begin{cases} w_\lambda(x,y), & \lambda > 0 \\ W_0(x,y), & \lambda = 0 \end{cases} \tag{4.7}$$

is a basic parametrized differential inclusion.

Navigating Sections on Welander's Model:

- Goals for Welander's Model, Section 1.2
- Introduction to Welander's Model and Bifurcation Analysis, Section 3.3
- Perturbation and Welander's Model, Section 4.7
- Welander's Model as a Multiflow, Section 5.5

- Welander's Model as a Well-Parametrized Family of Multiflows, Section 6.2
- Isolating Neighborhood in Welander's Model, Section 7.4
- Attractor-Repeller Pair Decomposition for Welander's Model, Section 8.5
- The Conley Index and Welander's Model, Section 9.5

Chapter 5

Multiflows

5.1 Defining Multiflows

Recall that for sufficiently smooth vector fields (for instance, vector fields with a global Lipschitz bound) we can describe the solution set of the differential equation $\dot{x} = f(x)$ by a *flow.* A flow on a locally compact metric space Y is a continuous map $\varphi : \mathbb{R} \times Y \to Y$ satisfying the following group properties:

1. $\varphi(0, x) = x$ for all $x \in Y$.

2. $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for all $t, s \in \mathbb{R}$ and $x \in Y$.

The flow is the solution set for the differential equation in the sense that for each $x_0 \in Y$,

$$\frac{d}{dt}\varphi(t, x_0) = f(\varphi(t, x_0))$$

and so $\varphi(\cdot, x_0)$ is exactly the solution of the differential equation with initial condition $\varphi(0, x_0) = x_0$.

Because basic differential inclusions may have multiple solutions for a single initial condition, we cannot write their solution set as a flow as we would for typical differential equations. In order to overcome that problem we work with a multivalued object defined by Richard McGehee, the *multiflow* [26].

Definition 5.1. *A **multiflow** on a locally compact metric space Y is a set-valued map*

$$\Phi : \mathbb{R}^+ \times Y \to \mathcal{P}(Y)$$

which is upper-semicontinuous and compact-valued, and which satisfies the monoid properties:

1. $\Phi(0, y) = \{y\}$
2. $\Phi(t, \Phi(s, y)) = \Phi(t + s, y)$

Here we also introduce the notation that for $A \subset Y$ and $I \subset \mathbb{R}^+$, we may write

$$\Phi(I, A) = \cup_{t \in I} \cup_{x \in A} \Phi(t, y)$$

When the space in question is compact, it is frequently more convenient to characterize a multiflow in terms of its graph. In order to do so, we introduce some notation. Let $\Phi \subset [0, \infty) \times X \times X$; we write

$$\Phi^t = \{(x, y) : (t, x, y) \in \Phi\}$$

That is, for each $t \geq 0$, Φ^t defines a relation on X. With this notation in mind, the closed graph theorem for set-valued functions (Theorem 2.1) gives the following property immediately:

Definition 5.2. *Let X be a compact metric space. A **multiflow** on X is a closed subset of $[0, \infty) \times X \times X$ satisfying the two monoid properties:*

1. $\Phi^0 = \{(x, x) \in X \times X\}$
2. $\Phi^{t+s} = \Phi^t \circ \Phi^s$ *for all* $t, s \geq 0$.

Multiflows are clearly similar to flows, with the most obvious difference being that they are set-valued. But in addition to being multivalued, a multiflow only considers forward time, making it more closely akin to a semiflow than a complete flow. Indeed, we see that a multiflow with the additional assumption that $\Phi(t, x)$ is a single point for each (t, x) is a semiflow. The possibility of intersecting trajectories necessitates working

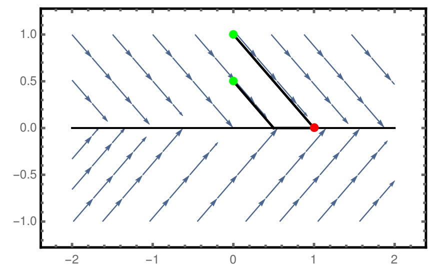


Figure 5.1: In this simple Filippov system, two different initial conditions (green) reach the same point (red) at time $t = 1$. This collision means that any analogue of a flow for Filippov systems cannot have a group action; $\varphi^1 \circ \varphi^{-1}$ would not be the identity.

only with forward time. With differential inclusions, it is possible for solutions that begin at distinct initial conditions to reach the same point in finite time. Therefore a solution could move forwards for time t to one location, then backwards for time $-t$ to a location other than the initial condition. This makes retaining the group action of $\mathbb{R}$ impossible because the identity requirement would not hold in general. Therefore we only examine solutions in forward time and settle for a monoid action.

It is, however, quite straightforward to examine backwards time behavior as a separate system. To do so we introduce the *dual multiflow*:

Definition 5.3. *Given a multiflow* $\Phi : \mathbb{R}^+ \times Y \times Y$*, the* **dual multiflow to** Φ *is the set-valued map* $\Phi^* : \mathbb{R}^+ \times Y \to \mathcal{P}(Y)$ *defined pointwise by*

$$\Phi^*(t, b) := \{a \in Y \,|\, b \in \Phi(t, a)\}$$

This naming convention is justified by two easily-verified properties: Φ^* is itself a multiflow over X, and the dual is well-defined ($(\Phi^*)^* = \Phi$).

Of course, one might also wonder why we have gone out of our way to give a second characterization of multiflows over compact spaces X. We have done so primarily because this setting will be the primary one in which we will work. Adding the assumption of compactness gives an extremely helpful simplification.

But we should note here that this assumption of compactness is not a serious barrier to our goal of extending Conley Index Theory to a multivalued setting. The information derived from Conley index theory is determined using only the behavior of the dynamical system on the boundary of certain compact neighborhoods, and its goal is to understand the dynamics of objects contained in these sets. Moreover, we will see in Theorem 5.1 that the solution set of basic differential inclusions which are defined over an unbounded space can still be described using a multiflow defined only on a compact subset of that space. Therefore multiflows, as defined here over compact spaces, are sufficient for extending elements of Conley Index Theory to basic differential inclusions which are defined over any open subset of $\mathbb{R}^n$.

5.2 Basic Differential Inclusions as Multiflows

In this section we tie together the concepts of differential inclusions and multiflows.

Let $\dot{x} \in F(x)$ be any basic differential inclusion defined on an open domain $G \subset \mathbb{R}^n$, and let X be any compact subset of G. We are interested in considering all solutions of the differential inclusion which are contained entirely in X. If a solution begins in X, but then leaves, we only monitor the solution up until the time it leaves. We want to show that the union of the graphs of all of these solutions forms a multiflow.

More explicitly, define the object Φ as follows:

Definition 5.4. *Assume that $G \subset \mathbb{R}^n$ is open and that the set-valued map $F : G \to \mathcal{P}(\mathbb{R}^n)$ satisfies the basic conditions. Let $X \subset G$ be compact and define the set*

$$\Phi \subset \mathbb{R}^+ \times X \times X$$

to be the union of all points $(T, a, b) \in \mathbb{R}^+ \times X \times X$ such that there exists a solution $x(t) : [0, T] \to X$ to the basic differential inclusion $\dot{x} \in F(x)$ with $x(0) = a$ and $x(T) = b$.

We call Φ the **multiflow over X associated to the differential inclusion $\dot{x} \in F(x)$**.

Of course, the name used in that definition is not yet justified; it is unclear that Φ is actually a multiflow. The following theorem remedies that issue.

Theorem 5.1. *Let $F : G \to \mathcal{P}(\mathbb{R}^n)$ satisfy the basic conditions on the open domain $G \subset \mathbb{R}^n$. If $X \subset G$ is compact then the multiflow Φ over X associated to the differential inclusion $\dot{x} \in F(x)$ is actually a multiflow.*

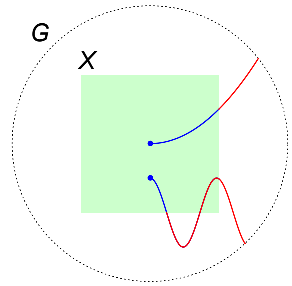


Figure 5.2: Explanation of the solution set in Theorem 5.1. The blue and red curves represent solutions of the differential inclusion. Only points on the blue portion of the curves are included in Φ.

Before beginning this proof, we should note that Richard McGehee proved an analogous result for differential equations $\dot{x} = f(x)$ where f is assumed only to be continuous (but not necessarily Lipschitz continuous). This proof owes a deep debt to that result.

Proof. The monoid properties are relatively trivial to see, although it does take a bit of space to write their proof. We see that $\Phi^0 = \{(a, a) \in X \times X\}$ because by Theorem 3.3, for each $a \in X$, there is at least one solution, and obviously a solution cannot begin at a and go to any other point in zero time.

Next, note that the second monoid property, $\Phi^{t+s} = \Phi^t \circ \Phi|_S$ for all $t, s \geq 0$, is equivalent to the following statement: $(t + s, a, c) \in \Phi$ if and only if there exists $b \in X$ such that $(t, a, b) \in \Phi$ and $(s, b, c) \in \Phi$. In our case, points in Φ may be written as $(t, x(0), x(t))$. Then we must show two things. First, if there is a solution z such that $z(0) = a$ and $z(t + s) = c$, then there must be some point b and solutions x and y such that $x(0) = a$, $x(t) = b = y(0)$, and $y(s) = c$. Conversely, if there is some point b and solutions x and y such that $x(0) = a$, $x(t) = b = y(0)$, and $y(s) = c$, then there must be a solution z such that $z(0) = a$ and $z(t + s) = c$.

Let us first assume that there is some point b and solutions x and y such that $x(0) = a$, $x(t) = b = y(0)$, and $y(s) = c$. In brief, pasting these solutions together yields the desired solution. More rigorously, define the function $z : [0, t + s] \to X$ by the equation

$$z(r) = \begin{cases} x(r) & r \leq t \\ y(r - t) & r \geq t \end{cases}$$

It is clear that z is absolutely continuous since both x and y are, and by its definition it is obvious that $z(0) = a$ and $z(t+s) = c$. It also satisfies the differential equation $\dot{z} \in F(z)$ almost everywhere because for almost all $r \in [0,t]$,

$$\frac{d}{dr}z(r) = \frac{d}{dr}x(r) \in F(x(r)) = F(z(r))$$

and for almost all $r \in [t, t+z]$

$$\frac{d}{dr}z(r) = \frac{d}{dr}y(r-t) \in F(y(r-t)) = F(z(r))$$

Thus z is a solution to the differential inclusion.

Now assume that there is a solution z such that $z(0) = a$ and $z(t+s) = c$. In brief, splitting this function into two functions at time t yields the desired solutions. More rigorously, define the functions $x : [0,t] \to X$ and $y : [0,s] \to X$ by the equations

$$x(r) = z(r) \qquad y(r) = z(r+t)$$

Again, it is clear that x and y are absolutely continuous functions which evaluate to the desired points at the appropriate times. Additionally,

$$\frac{d}{dr}x(r) = \frac{d}{dr}z(r) \in F(z(r)) = F(x(r))$$

almost everywhere and

$$\frac{d}{dr}y(r) = \frac{d}{dr}z(r+t) \in F(z(r+t)) = F(y(r))$$

almost everywhere. Therefore x and y are the desired solutions to the differential inclusion, and Φ satisfies the monoid properties.

The difficultly of this proof comes from showing that Φ is closed. Luckily, Filippov's theorems do much of the hard work for us. Let (T, a, b) be a limit point of Φ; we will show that $(T, a, b) \in \Phi$.

Since (T, a, b) is a limit point of Φ, there is some sequence of points in Φ

$$(T_n, x_n(0), x_n(T_n)) \to (T, a, b)$$

where each $x_n(t) \in F(x_n(t))$ for almost all t in the interval $[0, T_n]$.

The basic idea of the proof is to find a subsequence of $\{x_n\}$ which exist on (or can be extended to) the common interval $[0, T]$. Then we apply the Arzela-Ascoli theorem to this family of solutions in order to get a uniformly convergent subsequence. By Theorem 3.4, this subsequence converges to a solution $x^*(t)$; the proof will then be complete once we show that $x^*(0) = a$, $x^*(T) = b$, and $x^*(t) \in X$ for each $t \in [0, T]$.

We begin by taking a compact neighborhood K of X; that is, K is a compact set satisfying

$$X \subset \text{int}(K) \subset K \subset G$$

By Lemma 2.1 there is some constant M such that $|F(x)| \leq M$ on K. Combining that result with Corollary 3.5.1, any family of solutions $\{x : [0, T] \to K\}$ is equicontinuous and

$$|x(s_1) - x(s_2)| \leq M|s_1 - s_2|$$

At this point, however, the sequence of solutions we are considering are not necessarily defined on the common interval of time $[0, T]$, and in order to get equicontinuity and apply Arzela-Ascoli, we need them to be. If $T_n > T$ then this presents no obstacle, as we simply consider $x_n|_{[0,T]}$. However, we must show that for sufficiently large n we can extend x_n to the interval $[0, T]$ even if $T_n < T$.

By Theorem 3.4, we know that any solution can be extended at least until it reaches the boundary of K. Since $X \subset \text{int}(K)$, we can extend any x_n to be defined on some interval $[0, T'_n]$, where $T'_n > T_n$. Let $\delta := d(K, \overline{K}) > 0$, and choose an n_0 such that $n \geq n_0$ implies that $|T_n - T| < \frac{\delta}{M}$. Then for such n, if $T_n < T'_n < T$ we get that

$$|x_n(T_n) - x_n(T'_n)| < M|T_n - T'_n| < \delta$$

and so $x_n(T'_n) \in N^0$, and hence may be continued further. Thus we may assume that for $n \geq n_0$, the solution x_n may be extended to the interval $[0, T]$.

We are now in a position to apply Arzela-Ascoli; the solutions are clearly uniformly bounded (they are all contained in the compact set K) and we know that they are equicontinuous. Thus, there is a convergent subsequence of $\{x_n : [0, T] \to K\}$. By Theorem 3.4, this subsequence converges to a solution of the differential inclusion; let us call this solution $x^*(t)$. Since $x_n(0) \to a$ by definition, it is clear that $x^*(0) = a$. Then to show that $(T, a, b) \in \Phi$, we just need to show that $x^*(T) = b$ and that $x^*(t) \in X$ $\forall t \in [0, T]$.

To show that $x^*(T) = b$, we will show that for any ε, $|x_n(T) - b| < \varepsilon$ for sufficiently large n.

$$\begin{aligned}|x_n(T) - b| &= |x_n(T) - x_n(T_n) + x_n(T_n) - b| \\ &\leq |x_n(T) - x_n(T_n)| + |x_n(T_n) - b| \\ &\leq M|T - T_n| + |x_n(T_n) - b|\end{aligned}$$

Since $T_n \to T$ and $x_n(T_n) \to b$ by assumption, we can guarantee that $|T - T_n| < \frac{\varepsilon}{2M}$ and $|x_n(T_n) - b| < \varepsilon/2$ for sufficiently large n, and so $x^*(T) = b$.

Now, for sake of contradiction, assume that $x^*(t) \in K \setminus X$ for some $t \in [0, T]$. Let $\tau := T - t$. For sufficiently large n, $|T - T_n| < \tau$. Then for these large n, $x_n(t) \in X$, since we have chosen n large enough to guarantee that $t \in [0, T_n]$ and $x_n : [0, T_n] \to X$ by definition. Then $x_n(t)$ must limit to a point in X as $n \to \infty$ since X is compact. Thus we have a contradiction, and we see that $x^* : [0, T] \to X$.

Thus $(T, x^*(0), x^*(T)) = (T, a, b) \in \Phi$, and so Φ is a multiflow. □

Although multiflows are defined only for positive time, we may use the dual multiflow to examine the negative time solution set of a basic differential inclusion as well. The following lemma shows that this characterization is justified:

Lemma 5.1. *Let $\Phi \subset \mathbb{R}^+ \times X \times X$ be the multiflow associated to the basic differential inclusion $\dot{x} \in F(x)$ as in Theorem 5.1. Then*

$$(T, b, a) \in \Phi^* \subset \mathbb{R}^+ \times X \times X$$

if and only if there exists a solution $x : [-T, 0] \to X$ of $\dot{x} \in F(x)$ such that $x(-T) = a$ and $x(0) = b$.

Proof. Assume that there exists a solution $x : [-T, 0] \to X$ of $\dot{x} \in F(x)$ such that $x(-T) = a$ and $x(0) = b$. Consider the function $y : [0, T] \to X$ defined by $y(t) := x(t - T)$. Clearly y is absolutely continuous, $y(0) = x(0 - T) = a$, and $y(T) = x(T - T) = b$.

Also, the chain rule gives us that

$$\dot{y}(t) = \dot{x}(t - T) \in F(x(t - T)) = F(y(t))$$

and so y is also a solution. Then $(T, a, b) \in \Phi$, and so $(T, b, a) \in \Phi^*$.

Conversely, assume that $(T, b, a) \in \Phi^*$. Then by definition, $(T, a, b) \in \Phi$, and so there is some solution $x(t) : [0, T] \to X$ of the differential inclusion $\dot{x} \in F(x)$ with $x(0) = a$ and $x(T) = b$. Defining $y : [-T, 0] \to X$ by $y(t) := x(t + T)$, we see that y is absolutely continuous, $y(-T) = x(0) = a$, and $y(0) = x(T) = b$. Also, the chain rule gives us that

$$\dot{y}(t) = \dot{x}(t + T) \in F(x(t + T)) = F(y(t))$$

and so y is also a solution.

□

5.3 Other Multivalued Generalizations of Flows

The concept of a solution set used to relate multiflows and differential inclusions–solutions monitored only until they reach the boundary of a compact set–is somewhat unusual. More typically, given a multivalued vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, we would consider the solution set of the differential inclusion $\dot{x} \in F(x)$ to be the multivalued function

$$\pi : \mathbb{R} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$$

where $b \in \pi(T, a)$ if and only if there is a solution $x : [0, T] \to \mathbb{R}^n$ (or $x : [T, 0] \to \mathbb{R}^n$ in the case where $T < 0$) such that $x(0) = a$ and $x(T) = b$. In other words, typical conceptions of the solution set would consider the entire domain of the multivalued vector field.

It is known in the field [2, 4] that if, in addition to the basic conditions, we assume that F satisfies the bounding assumption

$$\sup_{v \in F(x)} |v| \leq c(1 + |x|) \tag{5.1}$$

for all $x \in \mathbb{R}^n$ and some fixed $c > 0$, then the set-valued map π defined above is a *multivalued flow*:

Definition 5.5. *Let Y be a locally compact metric space. An upper-semicontinuous, compact-valued, multivalued map*

$$\pi : \mathbb{R} \times Y \to \mathcal{P}(Y)$$

which satisfies the monoid properties

1. $\pi(0, y) = \{y\}$ *for all* $y \in Y$
2. $\pi(t + s, y) = \pi(t, \pi(s, y))$ *for all* $ts \geq 0$
3. $y \in \pi(t, x) \iff x \in \pi(-t, y)$
4. $\pi(t, x) \neq \emptyset$ *for all* $(t, x) \in \mathbb{R}$.

is called a **multivalued flow** *over* Y.

This definition comes from [32] but similar generalizations are found in many other places [4, 33, 3, 34]. We have chosen to highlight this object because [32] also deals with Conley index theory, but the other definitions are only superficially different.

Notice that Property 2 indicates that although π is defined on all of $\mathbb{R}$, the action is still only a monoid action. Property 3 is essentially equivalent to introducing the dual multiflow Φ^* for a multiflow Φ.

Indeed, we see that a multivalued flow is nearly identical to the multiflow. However, notice that Property 4 demands that the image of any point $(t, y) \in \mathbb{R} \times Y$ is not the empty set. This requirement is what forces us to place the additional Bounding Assumption (5.1) on any differential inclusions $\dot{x} \in F(x)$ which we wish to consider using a multivalued flow.

If we drop Property 4, we get the following object [32]:

Definition 5.6. *Let Y be a locally compact metric space. An upper-semicontinuous, compact-valued, multivalued map*

$$\pi : \mathbb{R} \times Y \to \mathcal{P}(Y)$$

which satisfies the monoid properties

1. *$\pi(0, y) = \{y\}$ for all $y \in Y$*

2. *$\pi(t+s, y) = \pi(t, \pi(s, y))$ for all $ts \geq 0$*

3. *$y \in \pi(t, x) \iff x \in \pi(-t, y)$*

is called a **partial multivalued flow***.*

This object is essentially identical to a multiflow. More directly, if Φ is a multiflow over a locally compact metric space Y, then

$$\pi : \mathbb{R} \times Y \to \mathcal{P}(Y), \qquad (t, y) \mapsto \begin{cases} \Phi(t, y), & t \geq 0 \\ \Phi^*(t, y), & t < 0 \end{cases}$$

is a partial multivalued flow. Conversely, if $\pi : \mathbb{R} \times Y \to \mathcal{P}(Y)$ is assumed to be a partial multivalued flow on a locally compact space Y, then

$$\Phi : \mathbb{R}^+ \times Y \to \mathcal{P}(Y), \qquad (t, x) \mapsto \pi(t, x)$$

is a multiflow over Y.

While this second object is defined in [32], it is not really used there. Importantly for this thesis, none of the Conley theorems (such as the stability of isolating neighborhoods or the attractor-repeller decomposition) are proven for this object. Probably this omission is due to the fact that Property 4 is actually quite useful; a key difficulty in most of the proofs in this thesis is the fact that the image of a point may be empty.

But it is also important to mention that without the bounding assumption (hypothesis 5.1) the full solution set of a basic differential inclusion $\dot{x} \in F(x)$ does *not* form a partial multivalued flow on $\mathbb{R}^n$. That is, without our unusual view of restricting to a compact set, we can find no object which deals with the problem of finite-time blowup as a local flow does in the single-valued case. The issue, as the following example demonstrates, is that the set-valued map defined using the typical conception of the solution set may not be closed-valued (and hence not compact-valued).

Example 5.3.1. We define a Filippov system in $\mathbb{R}^2$ using the multivalued map

$$F(x,y) = \begin{cases} (-x^2, 0) & y < 0 \\ \{(-\alpha x^2, 1-\alpha) | \alpha \in [0,1]\} & y = 0 \\ (0,-1) & y \in (0,1) \\ \{(\alpha x^2, 1-\alpha) | \alpha \in [0,1]\} & y = 1 \\ (x^2, 0) & y > 1 \end{cases}$$

Notice, in particular, that $(0,0) \in F(x,y)$ if and only if $x = 0$, and so all equilibria lie on that line.

Define

$$\Phi(t,x,y) = \{(w,z) \in \mathbb{R}^2 | \exists\, \psi : I \to \mathbb{R}^2,\, \dot{\psi}(t) \in F(\psi(t))\text{a.e.},\, \psi(0) = (x,y),\, \psi(t) = (w,z)\}$$

We will show that this map is not compact valued at some points in the domain. In particular, it is not compact-valued at $(3,1,1)$.

A point (w,z) is in the set $\Phi(3,1,1)$ if and only if there is some solution $\psi : [0,3] \to \mathbb{R}^2$ with $\psi(0) = (1,1)$ and $\psi(3) = (w,z)$; although a priori any interval containing $[0,3]$ may be the domain of a solution we may consider the restriction of such maps to $[0,3]$. There are infinitely many such maps ψ, but their behavior follows a simple pattern. Since $(x^2,0) \in F(x,1)$, solutions may slide along the line $\{y = 1\}$ for any amount of time $t \in [0,1)$ (notice that $t = 1$ is when the finite-time blowup occurs for the IVP $\dot{x} = x^2, x(0) = 1$). Once they leave this line, however, their behavior is uniquely determined; they follow a vertical line path down to reach the line $y = 0$, and then head left towards the equilibrium at the origin. In particular, notice that by the symmetry of the system, if the solution left the line $\{y = 1\}$ at time τ, then it reaches the point $(1,0)$ at time $2\tau+1$, and then continues left of this point for the remainder of the time interval.

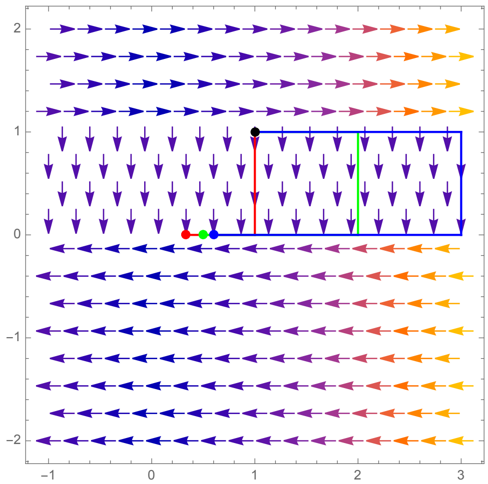


Figure 5.3: Trajectories plotted for $\tau = 0, 1/2, 2/3$ (red, green, blue). Notice that the trajectories intersect and cover each other in the figure.

More specifically, for each $\tau \in [0, 1)$, there is a solution

$$\psi_\tau(t) = \begin{cases} (\frac{1}{1-t}, 1) & t \in [0, \tau] \\ (\frac{1}{1-\tau}, 1 - (t - \tau)) & t \in [\tau, \tau + 1] \\ (\frac{1}{t-2\tau}, 0) & t \in [\tau + 1, 3] \end{cases}$$

and $\Phi(3, 1, 1) = \{\psi_\tau(3) | \tau \in [0, 1)\}$.

In particular, notice that $\psi_\tau(3) = (\frac{1}{3-2\tau}, 0)$ limits to $(1, 0) \in \mathbb{R}^2$ as $\tau \to 1$, and so $(1, 0)$ is a limit point of the set $\Phi(3, 1, 1)$. However, there is no solution to the differential inclusion which begins at $(1, 1)$ and terminates at $(1, 0)$, and so $\Phi(3, 1, 1) = [1/3, 1) \times \{0\}$ is not compact.

In the previous example, the multivalued map Υ was not compact-valued because $\Upsilon(3, 1, 1)$ was not closed. The following example will demonstrate that it is also possible for similar maps to have images which are not bounded.

Example 5.3.2. Consider the Filippov system $\dot{x} \in G(x)$ where

$$G(x) = \begin{cases} 0 & x < 0 \\ [0,1] & x = 0 \\ (1+x)^2 & x > 0 \end{cases}$$

Define the set-valued map $\Pi : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by saying that $y \in \Pi(t,x)$ if and only if there is some $\psi : [0,t] \to \mathbb{R}$ satisfying $\dot{\psi}(t) \in G(\psi(t))$ a.e. and $\psi(0) = x$, $\psi(t) = y$. Since $0 \in G(0)$, solutions where $\psi(0) = 0$ may remain at the origin for any amount of time, including for all time (so $\hat{\psi} \equiv 0$ is a solution). The unique solution to the initial value problem ($\dot{x} = (1+x)^2$, $x(0) = 0$) is $x(t) = \frac{-t}{t-1}$, and so if the solution leaves the origin at time $t = \tau \in \mathbb{R}^+$, we may write it as

$$\psi_\tau(t) = \begin{cases} 0, & t < \tau \\ \frac{-(t-\tau)}{(t-\tau)-1}, & t \geq \tau \end{cases}$$

(note that $\psi_\tau : [0, \tau+1) \to \mathbb{R}$). Therefore $\Pi(1,0) = \cup_{\tau>0}\{\psi_\tau(1)\} \cup \{0\} = [0,\infty)$.

This loss of compactness presents difficulties because set-valued maps which are not compact-valued are not very well-behaved. Among other complications, even upper-semicontinuous maps with compact domains may be unbounded if they are not compact-valued. For the applications at hand, the primary obstacle obstacle comes from the fact that the proofs of many of the results in this paper rely on the convergence of certain sequences, and that convergence depends on the compact-valued nature of Φ.

We also emphasize that example 5.3.1 demonstrates that the typical way of handling finite-time blowup in the single-valued setting–restricting the domain to an open subset of $\mathbb{R} \times \mathbb{R}^n$ and considering a local flow–does not translate directly to this multivalued setting. This difficulty is one of the reasons that we have introduced our unusual conception of the solution set (solutions monitored only until they reach the boundary of a finite set). Although this perspective shift introduces some complications, it allows us to overcome the problem of finite-time blowup in a multivalued setting. We will also see in Chapter 8 that this perspective allows us to overcome certain technical complications regarding the attractor-repeller pair decomposition of compact invariant sets in the multivalued setting.

5.4 Orbits of Multiflows

In this section we introduce the notion of an *orbit* on a multiflow and demonstrate various properties of these orbits. The main goal is to begin considering multiflows as objects in their own right, rather than just as the solution sets to basic differential inclusions.

Definition 5.7. *Let Φ be a multiflow over a locally compact metric space Y. An orbit on Φ is a continuous function $\psi : I \to Y$, where $I \subset \mathbb{R}$ is an interval, such that $\psi(t) \in \Phi(t-s, \psi(s))$ whenever $s, t \in I$ and $t > s$.*

This definition is intended to invoke the idea of a path in a multiflow. Examining this definition we notice that if Φ is the multiflow over a compact space X associated with the differential inclusion (3.1) as in Theorem 5.1, then solutions of (3.1) are orbits of Φ. The converse is not obvious but it is true; see Theorem 5.3. This abstract notion of orbits, then, allows us to analyze similar objects without assuming that our multiflow comes from a differential inclusion.

Of course, for this definition to be useful in the study of multiflows we need to know that an arbitrary multiflow is guaranteed to have orbits. Kate Meyer has elegantly demonstrated this property.

Theorem 5.2. *[27] If $\Phi \subset \mathbb{R}^+ \times X \times X$ is a multiflow on the compact metric space X and $(T, a, b) \in \Phi$, then there is an orbit $\psi : [0, T] \to X$ on Φ satisfying $\psi(0) = a$ and $\psi(T) = b$.*

Proof. Meyer's proof consists of three main steps. In the first step we define a mapping $\hat{\psi} : D \to X$, where D is a dense subset of $[0, s]$, which satisfies

$$(t-s, \hat{\psi}(s), \hat{\psi}(t)) \in \Phi$$

for $t, s \in D$ and $t > s$. This step is the most difficult; the key idea here is that given $(t, x, y) \in \Phi$, the semigroup property of the multiflow Φ guarantees some $z \in X$ such that $(t/2, x, z) \in \Phi$ and $(t/2, z, y) \in \Phi$. We demonstrate the Cauchy continuity of this function in the second step. Then for the final step we demonstrate that the unique extension of this function to $[0, s]$ is an orbit of Φ.

For the first step, consider the set of dyadic rationals scaled by T:

$$D := \{rT \mid r = \frac{p}{2^q},\, p, q \in \mathbb{N},\, p \leq 2^q\}$$

For each $j \in \mathbb{N}$, let D_j be the finite subset $D_j := \{\frac{i}{2^j} s\}_{i=0}^{2^j} \subset D$. Notice that

$$D_0 \subset D_1 \subset D_2 \subset \cdots, \qquad \cup_{j \in \mathbb{N}} D_j = D$$

We will inductively define functions $\psi_j : D_j \to X$ such that $\psi_{j+1}|_{D_j} \equiv \psi_j$ and

$$(t - s, \psi_j(s), \psi_j(t)) \in \Phi \tag{5.2}$$

for all $t \in D_j$. To begin this process we define $\psi_0 : D_0 \to X$ by $\psi_0(0) = a$ and $\psi_0(T) = b$ (note that $D_0 = \{0, T\}$). By assumption, $(T, \psi_0(0), \psi_0(T)) \in \Phi$, and by the identity property of Φ, $(0, \psi_0(0), \psi_0(0)) \in \Phi$, and so ψ_0 satisfies (5.2). We will then inductively assume that we can define ψ_j satisfying (5.2) and use it to define ψ_{j+1}.

Notice that if $t \in D_{j+1} \setminus D_j$ then

$$t_- := t - \frac{T}{2^{j+1}} \in D_j, \qquad t_+ := t + \frac{T}{2^{j+1}} \in D_j$$

and so $(\frac{T}{2^j}, \psi_j(t_-), \psi_j(t_+) \in \Phi$ by our inductive hypothesis. Since $\frac{T}{2^j} = \frac{T}{2^{j+1}} + \frac{T}{2^{j+1}}$, the semigroup property of Φ implies that for each $t \in D_{j+1} \setminus D_j$ there is some $\zeta_t \in X$ such that $(\frac{T}{2^{j+1}}, \psi_j(t_-), \zeta_t) \in \Phi$ and $(\frac{T}{2^{j+1}}, \zeta_t, \psi_j(t_+)) \in \Phi$. We choose such a ζ_t for each $t \in D_{j+1} \setminus D_j$ and define

$$\psi_{j+1} : D_{j+1} \to X, \qquad t \mapsto \begin{cases} \psi_j(t), & t \in D_j \\ \zeta_t, & t \in D_{j+1} \setminus D_j \end{cases}$$

By the construction of ψ_{j+1}, we see that

$$(\frac{T}{2^{j+1}}, \psi_{j+1}(t), \psi_{j+1}(t + \frac{T}{2^{j+1}})) \in \Phi$$

for all $t \in D_{j+1} \setminus \{T\}$. If $t, s \in D_{j+1}$ and $t > s$ then $t - s = \frac{kT}{2^{j+1}}$ for some positive integer $k \leq 2^{j+1}$, and so the semigroup property of Φ gives us that

$$(t - s, \psi_{j+1}(s), \psi_{j+1}(t)) = (\frac{kT}{2^{j+1}}, \psi_{j+1}(s), \psi_{j+1}(s + \frac{kT}{2^{j+1}})) \in \Phi$$

and so (5.2) holds for $\psi_{j+1} : D_{j+1} \to X$, completing the induction. The first step is then completed by letting $\hat{\psi}(t) := \psi_j(t)$, where here j is chosen to be the least integer such that $t \in D_j$.

For the second step let $\{t_i\}_{i=1}^{\infty} \subset D$ be a Cauchy sequence. For the sake of contradiction assume that $\{\hat{\psi}(t_i)\}_{i=1}^{\infty}$ is not a Cauchy sequence. Then for some $\varepsilon > 0$ and each $M \in \mathbb{N}$ there are some $m_M, n_M > M$ such that $d_X(\hat{\psi}(t_{m_M}), \hat{\psi}(t_{n_M})) \geq \varepsilon$. Without loss of generality we assume that $m_M > n_M$.

Since $X \times X$ is compact and $\{t_i\}_{i=1}^{\infty}$ is Cauchy, there is a limit point of the set

$$\{(t_{m_M} - t_{n_M}, \hat{\psi}(t_{m_M}), \hat{\psi}(t_{n_M}))\}_{M \in \mathbb{N}} \subset \Phi$$

which we shall denote by $(0, x, y)$. It follows that $d_X(x, y) \geq \varepsilon$. Since Φ is closed, $(0, x, y) \in \Phi$, contradicting the identity property of Φ; this concludes the second step.

Finally, we define the continuous function $\psi : [0, T] \to X$ as the natural extension of $\hat{\psi}$:

$$\psi(t) = \begin{cases} \hat{\psi(t)}, & t \in D \\ \lim_{t_k \to t} \hat{\psi}(t_k), & t \in [0, T] \setminus D \end{cases}$$

Here $\{t_k\}_{k=1}^{\infty}$ is any sequence which limits to t; $\psi(t)$ is unique and well-defined since $\hat{\psi}$ is Cauchy continuous and X is complete (since it is compact).

By construction $\psi(0) = a$ and $\psi(T) = b$, and so it remains only to check that $(t - s, \psi(s), \psi(t)) \in \Phi$ for all $t, s \in [0, T]$ with $t > s$. Choose sequences $\{t_i\}_{i=1}^{\infty} \subset D$ and $\{s_i\}_{i=1}^{\infty}$ with $t_i \to t$ and $s_i \to t$. Then

$$(t - s, \psi(s), \psi(t)) = \lim_{i \to \infty} (t_i - s_i, \psi(s_i), \psi(t_i)) = \lim_{i \to \infty} (t_i - s_i, \hat{\psi}(s_i), \hat{\psi}(t_i)) \in \Phi$$

since Φ is closed and $(t_i - s_i, \hat{\psi}(s_i), \hat{\psi}(t_i)) \in \Phi$ was demonstrated in step one. $\square$

It is clear from definition 5.7 that if Φ is the solution set of a basic differential inclusion $\dot{x} \in F(x)$ then the solutions to that differential inclusion are orbits. The converse is less clear. That is, it seems possible that some orbit of Φ may not be a solution of the differential inclusion. The following theorem, however, rules out that behavior.

Theorem 5.3. *If X is a compact metric space and $\Phi \subset \mathbb{R}^+ \times X \times X$ is the multiflow over X associated to the basic differential inclusion $\dot{x} \in F(x)$, then all orbits in Φ are solutions to the differential inclusion.*

Before beginning this proof, we should mention that Kate Meyer proved an analogous result for the special case that $F(x) = B_r(f(x))$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a globally Lipschitz continuous function. The methods of this more general proof, however, differ substantially.

Proof. Let $\psi : [a, b] \to X$ be an orbit on Φ. We will construct a solution $x^* : [0, b-a] \to X$ to the differential inclusion $\dot{x} \in F(x)$ such that $x^*(h) = \psi(a+h)$ for all $h \in [0, b-a]$. Then we will have that $\dot{\psi}$ exists almost everywhere and that

$$\dot{\psi}(t) = \dot{x}(t-a) \in F(x(t-a)) = F(\psi(t))$$

which directly demonstrates that ψ is a solution to the differential inclusion. In other words, ψ is simply a reparametrization of the constructed solution $x^*(\cdot)$.

Let $\{h_i\}_{i=1}^{\infty}$ be an enumeration of $(\mathbb{Q} \cup \{b-a\}) \cap [0, b-a]$ such that $h_1 = b - a$. Since

$$(b-a, \psi(a), \psi(b)) \in \Phi$$

there is some solution $x_1 : [0, h_1]$ such that $x_1(0) = \psi(a)$ and $x_1(h_1) = \psi(b)$ by the definition of Φ. Similarly, since

$$(h_2 - a, \psi(a), \psi(a+h_2)) \in \Phi \qquad (h_1 - h_2, \psi(a+h_2), \psi(a+h_1)) \in \Phi$$

there are solutions $y : [0, h_2] \to X$ and $z : [0, h_1 - h_2] \to X$ such that $y(0) = \psi(a)$, $y(h_2) = z(0) = \psi(a+h_2)$, and $z(h_1 - h_2) = \psi(b)$. We can reparametrize z by the map $t \mapsto t + h_2$ and concatenate y and z into the solution $x_2 : [0, h_1] \to X$ satisfying $x_2(0) = \psi(a)$, $x_2(h_2) = \psi(a+h_2)$, and $x_2(h_1) = \psi(b)$.

We continue this process inductively as follows. Reorder the finite set $\{h_i\}_{i=1}^{j}$ as $\{\delta_i\}_{i=1}^{j}$ where $\delta_i > \delta_{i+1}$. Then by the definition of ψ we have that

$$\{(\delta_i - \delta_{i+1}, \psi(a + \delta_{i+1}), \psi(a + \delta_i))\}_{i=1}^{j-1} \subset \Phi$$

and

$$(\delta_j - a, \psi(a), \psi(a + \delta_j)) \in \Phi$$

By the definition of Φ, there are solutions $\{y_i : [0, \delta_i - \delta_{i+1}] \to X\}_{i=1}^{j-1}$ and $y_j : [0, \delta_j - a] \to X$ that agree with ψ on their endpoints (in other words, $y_i(\delta_i - \delta_{i+1}) = \psi(a + \delta_i)$). Reparametrizing and concatenating these solutions as before, we get a solution $x_j : [0, h_1] \to X$ that satisfies $x_j(0) = \psi(a)$ and $x_j(h_i) = \psi(a + h_i)$ for $i \leq j$.

The family of solutions

$$\{x_i : [0, h_1] \to X\}_{i=1}^{\infty}$$

is uniformly bounded (their range is the compact set X) and equicontinuous (Corollary 3.5.1). Then by the Arzaela-Ascoli lemma, there is a uniformly convergent subsequence of this family. Its limit, which we denote $x^*(\cdot)$, is a solution by Lemma 3.4. It is also clear that $x^*(0) = \psi(a)$ and $x^*(h_i) = \psi(a+h_i)$ since $x_j(0) = \psi(a)$ and $x_j(h_i) = \psi(a+h_i)$ for $j \geq i$.

It follows that $x^*(h) = \psi(a + h)$ for all $h \in [0, h_1]$ because $x^*(\cdot)$ and $\psi(\cdot)$ are both continuous and the rational numbers are dense in $\mathbb{R}$. From this construction it follows that ψ is a solution of the differential inclusion $\dot{x} \in F(x)$. □

As the preceding proof demonstrates, the uniform equicontinuity of solutions of basic differential inclusions is quite useful. The following lemma indicates that orbits of general multiflows have this same property. Since such orbits are also uniformly bounded (the range is compact), we see that any family of orbits of a multiflow satisfies the hypotheses of the Arzelà-Ascoli theorem, a property which we will utilize repeatedly in the remainder of this thesis.

Lemma 5.2. *Let Φ be a multiflow over a compact metric space X. Assume that $I \subset \mathbb{R}$ is an interval and let $\Upsilon := \{\psi : I \to X | \psi$ orbits$\}$ be a family of orbits on Φ. Then Υ is uniformly equicontinuous.*

Proof. Since the result is trivial if Υ is finite, we will assume that it is infinite. For the sake of contradiction assume that there is some $\varepsilon > 0$ such that for each $\delta > 0$ there exists $t_\delta, s_\delta \in I$ and $\psi_\delta \in \Upsilon$ such that $|t_\delta - s_\delta| < \delta$ and $|\psi_\delta(t_\delta) - \psi_\delta(s_\delta)| \geq \varepsilon$.

Without loss of generality, we may choose a subsequence $\delta_k \to 0$ such that $t_{\delta_k} > s_{\delta_k}$ (if this were not the case, we would simply have to switch the order of a tuple later in the proof). For convenience, we will drop the δ part of the subscript, leaving us with sequences $(t_k - s_k) \to 0$ and $\{\psi_k(\cdot)\}_{k=1}^{\infty} \subset \Upsilon$.

Since X is compact,

$$\{(t_k - s_k, \psi_k(t_0), \psi_k(t_k))\}_{k=1}^{\infty} \subset [0, \max_{1 \leq k < \infty}(t_k - s_k)] \times X \times X$$

has a convergent subsequence $\{(t_{k_i} - s_{k_i}, \psi_{k_i}(t_0), \psi_{k_i}(t_{k_i}))\}_{i=1}^{\infty}$. Its limit, which we will denote $(0, x, y)$, satisfies the property that $|x - y| \geq \varepsilon$ since $|\psi_\delta(t_\delta) - \psi_\delta(s_\delta)| \geq \varepsilon$ for all δ.

By our definition of an orbit, $\{(t_k - s_k, \psi_k(t_0), \psi_k(t_k))\}_{k=1}^{\infty} \subset \Phi$ (it is here that the tuple order would be switched if the condition $t_k > s_k$ could not be met). Since Φ is closed we have that $(0, x, y) \in \Phi$. Since $x \neq y$, this result contradicts the definition of a multiflow. Therefore all families of orbits in a multiflow are uniformly equicontinuous. □

Having shown that families of orbits of multiflows satisfy the hypotheses of the Arzèla-Ascoli theorem and therefore contain uniformly convergent subsequences, we now demonstrate that the limit of such subsequences is also an orbit.

Lemma 5.3. *Let Φ be a multiflow over the compact space X and let $I \subset \mathbb{R}$ be any interval. If $\psi : I \to X$ is the limit of a uniformly convergent sequence of orbits $\{\psi_n : I \to X\}_{n=1}^{\infty}$ on Φ, then ψ is also an orbit on Φ.*

Proof. Since the convergence is uniform, ψ is continuous. For each $t, s \in I$ with $t > s$ we have that

$$(t - s, \psi(s), \psi(t)) = \lim_{n \to \infty}(t - s, \psi_n(s), \psi_n(t)) \in \Phi$$

since $(t - s, \psi_n(s), \psi_n(t)) \in \Phi$ by assumption and Φ is closed. □

Because our applications are concerned with invariant sets, the common domain of families of orbits that we consider will often be all of $\mathbb{R}$. However, it is not true that any uniformly bounded and equicontinuous family of functions will contain a uniformly convergent subsequence; that is, the hypothesis that the domain is compact is necessary for the Arzèla-Ascoli theorem to hold. However, it is a straightforward consequence of the theorem (Lemma A.3) that for any such family there is a continuous function which is the uniform limit of a subsequence of the family when restricted to any compact interval contained in $\mathbb{R}$. The following lemma demonstrates that for a family of orbits on a multiflow, this function is also an orbit.

Lemma 5.4. *Let $\Phi \subset \mathbb{R}^+ \times X \times X$ be a multiflow over a compact metric space X. Given any sequence $\{\psi_k : \mathbb{R} \to X\}_{k=1}^{\infty}$ of orbits of Φ, there is an orbit*

$$\psi : \mathbb{R} \to X$$

on Φ such that on any compact interval $[a, b] \subset \mathbb{R}$, there is a subsequence of the restricted family

$$\{\psi_k|_{[a,b]} : [a, b] \to X\}_{k=1}^{\infty}$$

which converges uniformly to $\psi|_{[a,b]}$.

Proof. Since orbits are continuous functions, by Lemma A.3 we know that there exists a continuous function ψ such that on any compact interval $[a, b] \subset \mathbb{R}$, there is a subsequence of the restricted family $\{\psi_k|_{[a,b]} : [a, b] \to X\}_{k=1}^{\infty}$ which converges uniformly to $\psi|_{[a,b]}$. By Lemma 5.3, $\psi|_{[a,b]}$ is an orbit on Φ for any $[a, b]$. We can see directly that $\psi : \mathbb{R} \to X$ is an orbit on the entire real line. Since its restriction to any compact interval is continuous, Ψ is continuous on $\mathbb{R}$. Now choose any arbitrary $t > s$. Take m large enough that $t, s \in [-m, m]$. We see that

$$(t - s, \psi(s), \psi(t)) = (t - s, \psi_{[-m,m]}(s), \psi_{[-m,m]}(t)) \in \Phi$$

since $\psi|_{[-m,m]}$ is an orbit of Φ and so we are finished. $\square$

5.5 Welander's Model and Multiflows

As we remarked in earlier chapters, Inclusion 3.2 is a basic differential inclusion. Therefore, given any compact subset $X \subset \mathbb{R}^2$, we have a multiflow Ω_0 over X associated to $\dot{(x, y)} \in W_0(x, y)$. For our purposes it is sufficient to fix $X = B_{10}(0, 0)$. In later chapters we will extend certain aspects of Conley index theory to multiflows, and so this formulation also allows us to use these techniques to study the Welander model.

Additionally, it is worth noting that the set-valued map $W_0 : \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$ given in 3.2 is the Filippov convex combination of two linear maps. Therefore it satisfies Bounding Assumption 5.1, and hence its full solution set over $\mathbb{R}^2$ is a multivalued flow.

Navigating Sections on Welander's Model:

- Goals for Welander's Model, Section 1.2
- Introduction to Welander's Model and Bifurcation Analysis, Section 3.3
- Perturbation and Welander's Model, Section 4.7
- Welander's Model as a Multiflow, Section 5.5
- Welander's Model as a Well-Parametrized Family of Multiflows, Section 6.2
- Isolating Neighborhood in Welander's Model, Section 7.4
- Attractor-Repeller Pair Decomposition for Welander's Model, Section 8.5
- The Conley Index and Welander's Model, Section 9.5

Chapter 6

Families of Multiflows and Continuation

The primary motivation for Conley index theory is to gain information about dynamical systems which is robust under perturbation. In Chapter 4 we discussed the perturbation of differential inclusions, and here we define a notion of perturbation for multiflows.

As always, we assume in this chapter that X is a compact metric space.

6.1 Families of Multiflows

Our goal for this section is to define a set-valued analogue of a continuous family of flows

$$\{\varphi_\lambda\}_{\lambda\in\Lambda}$$

where Λ is some interval of the real line.

In order to do so, we begin by considering a parametrized families of multiflows

$$\{\Phi_\lambda\}_{\lambda\in\Lambda}$$

That is, for each λ in the interval Λ, Φ_λ is a multiflow over X.

Of course, in order to study such an object, we need each of these multiflows to have some sort of relation to each other which is an appropriate generalization of the continuity of the family of flows. As might be expected by this point in the exposition, that generalization is upper-semicontinuity.

Definition 6.1. *We consider a family of multiflows over* X

$$\{\Phi_\lambda\}_{\lambda\in\Lambda}$$

where $\Lambda \subset \mathbb{R}$ *is an interval. This family is said to be* **well-parametrized** *if the set-valued map*

$$\mathbb{R}^+ \times X \times \Lambda \supset (t, x, \lambda) \mapsto \Phi_\lambda(t, x) \subset X$$

is upper-semicontinuous.

Notice that by the closed graph theorem for set-valued maps, this definition is equivalent to the following one if the parametrizing interval Λ is compact.

Equivalent Definition 6.1.1. *Let* $X \subset \mathbb{R}^n$ *be compact and* $\Lambda \subset \mathbb{R}$ *be a compact interval. A* **well-parametrized family of multiflows** *over* X *and* Λ *is a closed subset*

$$\Phi_\Lambda \subset \mathbb{R}^+ \times X \times X \times \Lambda$$

such that Φ_λ *is a multiflow over* X *for each* $\lambda \in \Lambda$*.*

Of course, even though we have shown throughout this paper that upper-semicontinuity and compact-valuedness are a natural generalization of continuity, we still must justify that this generalization of a continuous family of flows is sufficient for our applications. This justification comes from Theorems 7.1 and 8.2, which generalize the stability of isolating neighborhoods and the continuation of attractor-repeller pairs, as well as Theorem 6.1, which demonstrates that perturbed differential inclusions may be studied using well-parametrized families of multiflows.

Throughout the duration of this thesis we will often be concerned with collections of functions which are each orbits on one multiflow in a parametrized family. Because of the importance of these families we give them a name in the following definition.

Definition 6.2. *Assume that* $I \subset \mathbb{R}$ *is an interval and* $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ *is a family of multiflows. The collection* $\Upsilon := \{\psi : I \to X\}$ *is said to be a* **family of orbits on** $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ *if each* $\psi \in \Upsilon$ *is an orbit on the multiflow* Φ_λ *for some* λ*.*

One crucial property of well-parametrized families of multiflows is that if the parametrizing interval Λ is compact then any family of orbits on these multiflows is uniformly equicontinuous. The proof is nearly the same as the proof of the same property for a single multiflow.

Lemma 6.1. *Assume $J \subset \mathbb{R}$ is a compact interval and let $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ be a well-parametrized family of multiflows over a compact metric space X. Assume that $I \subset \mathbb{R}$ is an interval and let $\Upsilon := \{\psi : I \to X\}$ be a family of orbits on $\{\Phi_\lambda\}$. Then Υ is uniformly equicontinuous.*

Proof. Since the result is trivial if Υ is finite, we will assume that it is infinite. For the sake of contradiction assume that there is some $\varepsilon > 0$ such that for each $\delta > 0$ there exists $t_\delta, s_\delta \in I$, $\lambda_\delta \in [-1, 1]$, and $\psi_\delta \in \Upsilon$ such that ψ_δ is an orbit on Φ_δ, $|t_\delta - s_\delta| < \delta$, and $|\psi_\delta(t_\delta) - \psi_\delta(s_\delta)| \geq \varepsilon$.

Without loss of generality, we may choose a subsequence $\delta_k \to 0$ such that $t_{\delta_k} > s_{\delta_k}$ (if this were not the case, we would simply have to switch the order of a tuple later in the proof). For convenience, we will drop the δ part of the subscript, leaving us with sequences $(t_k - s_k) \to 0$, $\{\lambda_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty \subset \Upsilon$.

Since X is compact,

$$\{(t_k - s_k, \psi_k(s_k), \psi_k(t_k), \lambda_k)\}_{k=1}^\infty \subset [0, \max_{1\leq k<\infty}(t_k - s_k)] \times X \times X \times \Lambda$$

has a convergent subsequence $\{(t_{k_i} - s_{k_i}, \psi_{k_i}(s_{k_i}), \psi_{k_i}(t_{k_i}), \lambda_{k_i})\}_{i=1}^\infty$. Its limit, which we will denote $(0, x, y, \lambda_0)$, satisfies the property that $|x - y| \geq \varepsilon$ since $|\psi_\delta(t_\delta) - \psi_\delta(s_\delta)| \geq \varepsilon$ for all δ.

By the definition of an orbit, $\{(t_k - s_k, \psi_k(s_k), \psi_k(t_k), \lambda_k)\}_{k=1}^\infty \subset \Phi_\Lambda$ (it is here that the tuple order would be switched if the condition $t_k > s_k$ could not be met). Since $\Phi \subset \mathbb{R}^+ \times X \times X \times \Lambda$ is closed we have that $(0, x, y, \lambda_0) \in \Phi_\Lambda$. Since $x \neq y$, this result demonstrates that Φ_{λ_0} is not a multiflow, contradicting our original hypothesis. Therefore all families of orbits in a well-parametrized family of multiflows are uniformly equicontinuous. □

In order to demonstrate that isolating neighborhoods are stable under perturbation we will need to show that uniformly convergent families of orbits of a well-parametrized family of multiflows limit to an orbit. Once again, the proof of this result is almost identical to the proof of the same result for a single multiflow.

Lemma 6.2. *Assume $J \subset \mathbb{R}$ is a compact interval and let $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ be a well-parametrized family of multiflows over a compact metric space X and let $I \subset \mathbb{R}$ be any interval. If $\psi : I \to X$ is the limit of a uniformly convergent sequence of orbits $\{\psi_n : I \to X\}_{n=1}^{\infty}$ of $\{\Phi_\lambda\}_{\lambda\in\Lambda}$, then ψ is an orbit on Φ_λ for some λ.*

Proof. Since the convergence is uniform, ψ is continuous.

Note that for each n there is some λ_n such that ψ_n is an orbit on Φ_{λ_n} by definition, meaning that for each $t, s \in I$ with $t > s$ we have that $(t - s, \psi_n(s), \psi_n(t), \lambda_n) \in \Phi_J$. Since Λ is compact there is some subsequence $\{\lambda_{n_i}\}_{i=1}^{\infty}$of $\{\lambda_n\}_{n=1}^{\infty}$ which converges; we will call its limit λ_0.

Then for each $t, s \in I$ with $t > s$ we have that

$$(t - s, \psi(s), \psi(t), \lambda_0) = \lim_{i\to\infty} (t - s, \psi_{n_i}(s), \psi_{n_i}(t), \lambda_{n_i}) \in \Phi_\Lambda$$

since Φ_Λ is closed. Therefore ψ is an orbit on Φ_{λ_0}. □

Putting the previous lemmas together with the Arzela-Ascoli theorem gives us the following result:

Lemma 6.3. *Let $\Phi_\Lambda \subset \mathbb{R}^+ \times X \times X \times J$ be a well-parametrized family of multiflows over a compact metric space X and compact interval Λ. Given any sequence $\{\psi_k : \mathbb{R} \to X\}_{k=1}^{\infty}$ of orbits of Φ_J, there is an orbit*

$$\psi : \mathbb{R} \to X$$

on Φ_λ for some $\lambda \in \Lambda$ such that on any compact interval $[a, b] \subset \mathbb{R}$, there is a subsequence of the restricted family

$$\{\psi_k|_{[a,b]} : [a, b] \to X\}_{k=1}^{\infty}$$

which converges uniformly to $\psi|_{[a,b]}$.

Proof. Since orbits are continuous functions, by Lemma A.3 we know that there exists a continuous function ψ such that on any compact interval $[a, b] \subset \mathbb{R}$, there is a subsequence of the restricted family $\{\psi_k|_{[a,b]} : [a, b] \to X\}_{k=1}^{\infty}$ which converges uniformly to $\psi|_{[a,b]}$. By Lemma 6.2, there is some $\lambda \in \Lambda$ such that $\psi|_{[a,b]}$ is an orbit on Φ_λ for any $[a, b]$. Notice that the same λ is fixed for all $[a, b]$ because the subsequence chosen for any interval is a subsequence of the one chosen for any smaller interval which it contains. We can see directly that $\psi : \mathbb{R} \to X$ is an orbit on the entire real line. Since its restriction to any compact interval is continuous, Ψ is continuous on $\mathbb{R}$. Now choose any arbitrary $t > s$. Take m large enough that $t, s \in [-m, m]$. We see that

$$(t - s, \psi(s), \psi(t)) = (t - s, \psi_{[-m,m]}(s), \psi_{[-m,m]}(t)) \in \Phi_\lambda$$

since $\psi|_{[-m,m]}$ is an orbit of Φ_λ and so we are finished. □

6.2 Perturbed Differential Inclusions and Families of Multiflows

The following straightforward theorem demonstrates that well-parametrized families of multiflows may be used in order to study perturbed differential inclusions. Essentially, this theorem allows us to prove results about well-parametrized families of multiflows and use those results in order to make statements about perturbed differential inclusions in the same manner that we use families of flows in order to study perturbed differential equations.

Theorem 6.1. *For an interval $\Lambda \subset \mathbb{R}$ and an open subset G of R^n, assume that $F : G \times [-1, 1] \to \mathbb{R}^n$ meets the basic conditions. Fix any compact $X \subset G$. For each $\lambda \in \Lambda$ let Φ_λ be the multiflow over X associated to the basic parametrized differential inclusion $\dot{x} \in F(x, \lambda)$. Then $\{\Phi_\lambda : \mathbb{R}^+ \times X \to \mathcal{P}(X)\}_{\lambda \in \Lambda}$ is a well-parametrized family of multiflows.*

Proof. We must only show that the set $\Phi_\Lambda \subset \mathbb{R}^+ \times X \times X \times \Lambda$ is closed. Let (T, a, b, λ) be a limit point of Φ_Λ and consider any sequence $\{(T_n, a_n, b_n, \lambda_n)\}_{n=1}^{\infty}$ which limits to it.

By definition there exists $\{\psi_n : [0, T_n] \to X\}_{n=1}^{\infty}$, $\psi_n(0) = a_n$, $\psi_n(T_n) = b_n$, and $\dot{\psi}_n(t) \in F(\psi_n(t), \lambda_n)$ for almost all t in the interval. As in the proof of Theorem 5.1, for sufficiently large n we can ensure that each function in this family is actually defined on the interval $[0,T]$. That is, if $T_n > T$ we simply consider the restriction $\psi_n|_{[0,T]}$ and if $T_n < T$ but n sufficiently large then we can extend ψ_n by Lemmas 2.1 and 3.4.

Let $\delta_k \to 0$. By Corollary 4.2.1, there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that for each k, $\psi_{n_k} : [0,T] \to X$ is a δ_k-solution to the differential inclusion $\dot{x} \in F(x,\lambda)$. By Lemma 3.4, a subsequence of $\{\psi_{n_k}(\cdot)\}_{k=1}^{\infty}$ converges to a solution $\psi : [0,T] \to X$ of $\dot{x} \in F(x,\lambda)$. It is clear that $\psi(0) = a$, and again through an argument identical to one made in the proof of Theorem 5.1 we see that $\psi(T) = b$. Therefore $(T,a,b,\lambda) \in \Phi_\Lambda$, and so Φ_Λ is a well-parametrized family of multiflows.

□

6.3 Well-Perturbed Families of Multiflows and Welander's Model

We begin by recalling the basic parametrized differential inclusion $\dot{(x,y)} \in W_\lambda(x,y)$ given by inclusion 4.7. As a reminder, $W_\lambda(x,y)$ is a single-valued function for λ positive, but we do not define a particular form of this equation; rather, W_λ may be any continuous function which limits to W_0 as discussed in Section 4.3.2. The particular smooth function that Welander uses in his original paper is a special case of these possibilities.

Let X be a large closed disk centered at the origin; for our purposes in later chapters, a disk of radius 10 will suffice. For each $\lambda \in [0,1]$, let $\Omega_0 : \mathbb{R}^+ \times X \to X$ be the multiflow over X associated to $\dot{(x,y)} \in W_\lambda(x,y)$. Then the collection

$$\{\Omega_\lambda\}_{\lambda \in [0,1]}$$

is a well-perturbed family of multiflows.

We remark that Ω_λ may be multivalued for $\lambda > 0$ because we have imposed minimal structure on W_λ and even single-valued differential equations may have multiple solutions for a given initial condition. If we further demand that $\{\gamma_\lambda\}_{\lambda \in (0,1]}$ satisfy a local Lipschitz bound then we would avoid this behavior.

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Chapter 7

Isolating Neighborhoods and Isolated Invariant Sets

7.1 Invariance for Multiflows

As mentioned in the introduction, there are several possible notions of invariance in a multivalued setting. In this section we will choose a definition of invariance and give some reasons why this notion is more natural than competing ones for Conley index theory.

Definition 7.1. *A set $S \subset X$ is called **invariant** under the multiflow Φ if for each $x \in S$ there exists an orbit*

$$\psi : \mathbb{R} \to S$$

on Φ with $\psi(0) = x$.

We will often also consider the maximal invariant subset of a given set U under the multiflow Φ, which is

$$\mathrm{Inv}(U, \Phi) := \{x \in U | \exists \,\mathrm{orbit}\, \psi : \mathbb{R} \to U \,\mathrm{on}\, \Phi,\ \psi(0) = x\}$$

If the choice of multiflow Φ is clear, we will sometimes shorten this notation to $\mathrm{Inv}(U)$.

Note that because we are working with set-valued systems, it is possible for points in an invariant set to have an orbit which leaves the invariant set. That is, since a given point may have many different orbits, some of these orbits can leave the invariant set. All that is required is that each point in the invariant set has *at least one* orbit which stays in the set for all time.

This definition of invariance–which is sometimes called *weak invariance*–is not the only possible notion of invariance for multiflows. The following definition–which we will call *strong invariance*–may at first glance appear to be the more natural generalization of the classical notion of invariance.

Definition 7.2. *A set S is* **strongly invariant under** Φ *if* $\Phi(t, S) = S$ *for all* $t \in \mathbb{R}^+$.

We note that an attractor-repeller pair decomposition of strongly invariant sets is given in [23]; we perform this decomposition of weakly invariant sets in Chapter 8. It is also worth noting that it is shown in [23] that strong invariance implies weak invariance for compact sets.

One reason for studying choosing our notion of invariance, rather than strong invariance, is that with this perspective the closure of an invariant set is invariant.

Property 7.1. *The closure of an invariant set is invariant.*

Proof. Assume the set K is invariant. Let x be a limit point of K, so that there is some sequence $\{x_k\}_{k=1}^{\infty} \subset K$ converging to x. Since K is invariant, we have an associated sequence of orbits $\{\psi_k : \mathbb{R} \to K\}_{k=1}^{\infty}$ with $\psi_k(0) = x_k$. By Lemma **??** there is an orbit $\psi : \mathbb{R} \to \overline{K}$ where $\{\psi_k : \mathbb{R} \to K\}_{k=1}^{\infty}$ converges uniformly to ψ on any compact interval. Clearly $\psi(0) = x$. Since the choice of $x \in \overline{K}$ was arbitrary, we see that the closure of K is invariant. $\square$

The following simple Filippov system shows that the closure of a strongly invariant set is not necessarily invariant.

Example 7.1.1. Let

$$F(x) = \begin{cases} 0, & x < 0 \\ [0,1], & x = 0 \\ 1, & x > 0 \end{cases}$$

and let Φ be the multiflow over $[-1, 1]$ associated to $\dot{x} \in F(x)$. The set $[-1, 0)$ is strongly invariant, but its closure, $[-1, 0]$, is not strongly invariant. Note that $[-1, 0]$ is invariant, however.

7.2 Isolating Neighborhoods and Isolated Invariant Sets for Multiflows

In this section we define isolating neighborhoods–the basic building blocks of Conley index theory–and demonstrate the crucial property that these neighborhoods are stable under perturbation. Following this fundamental theorem we examine alternative notions of invariance and demonstrate that similar stability properties do not hold under these alternative definitions.

Definition 7.3. *A compact set $N \subset X$ is called an **isolating neighborhood** if its maximal invariant set lies in its interior; that is,*

$$\mathrm{Inv}(N) \cap \partial N = \emptyset$$

The most important property of isolating neighborhoods is that they are stable under perturbation. This property is generalized here for multiflows.

Theorem 7.1. *Let $X \subset \mathbb{R}^n$ be compact and $\Lambda \subset \mathbb{R}$ be a compact interval. Let $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ be a well-parametrized family of multiflows. If N is an isolating neighborhood for the multiflow Φ_{λ_0} then there exists some $\varepsilon > 0$ such that $|\lambda - \lambda_0| < \varepsilon$ implies that N is an isolating neighborhood for Φ_λ.*

Proof. Without loss of generality, assume that $0 \in \Lambda$ and $\lambda_0 = 0$. We proceed by contradiction. Suppose that for each $\varepsilon > 0$ there was some λ with $|\lambda| < \varepsilon$ such that N was not an isolating neighborhood for Φ_λ. Then we can make sequences

$$\{\varepsilon_k\}_{k=1}^{\infty}, \qquad \{\lambda_k\}_{k=1}^{\infty}$$

such that $\varepsilon_k \to 0$ as $k \to \infty$, $|\lambda_k| < \varepsilon_k$, and N is not an isolating neighborhood for Φ_{λ_k}. Since N is not an isolating neighborhood for Φ_{λ_k}, for each k there is some orbit

$$\psi_k : \mathbb{R} \to N$$

of Φ_{λ_k} such that $\psi_k(0) \in \partial N$.

The sequence $\{\psi_k\}_{k=1}^{\infty}$ is uniformly bounded (since the range of each orbit is the compact space X) and it is uniformly equicontinuous by Lemma 6.1. Since $\lambda_k \to 0$, by Lemma 6.3 there is an orbit

$$\psi : \mathbb{R} \to N$$

of Φ_0 such that on any compact interval I, there is a subsequence of $\{\psi_k|_I\}_{k=1}^{\infty}$ that converges uniformly to ψ. Since ∂N is compact and $\psi_k(0) \in \partial N$ for each k, $\psi(0) \in \partial N$. This contradicts our assumption that N is an isolating neighborhood for Φ_0.

□

Here we should mention that a key motivation in considering invariance as we have defined it, rather than strong invariance, is that analogously defined isolating neighborhoods for strongly invariant sets are actually not stable under perturbation.

Example 7.2.1. Let

$$F(x) = \begin{cases} -1, & x < -1 \\ [-1, 0], & x = -1 \\ 0, & -1 < x < 0 \\ [0, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

and let Φ_λ be the multiflow over $[-2, 1]$ associated to $\dot{x} \in F(x + \lambda)$. The set $[-1, 0]$ is compact and under Φ_0 its maximal strongly invariant set $(-1, 0)$ is contained in its interior. However, for any $\lambda < 0$, $\Phi_\lambda(t, 0) = 0 = \Phi_\lambda^*(t, 0)$ for all $t \in \mathbb{R}^+$. That is, the maximal strongly invariant set contained in $[-1, 0]$ under Φ_λ includes the boundary point $\{0\}$ for any $\lambda < 0$.

We notice that in example 7.2.1 the maximal strongly invariant set was an open interval. This fact again demonstrates that the closure of a strongly invariant set need not be strongly invariant.

Still, we may wonder whether the definitions of isolated invariant sets and isolating neighborhoods could be further massaged so that Conley index theory could be applied to the study of strongly invariant sets as we are doing with invariant sets. Perhaps an alternative definition of an isolating neighborhood could be a compact set whose maximal strongly invariant set is a compact set contained in its interior. However, as

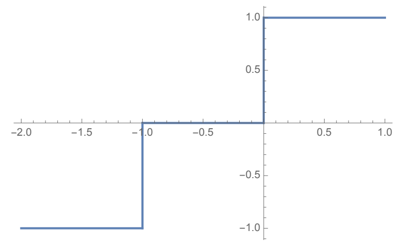

Figure 7.1: The set-valued map F from example 7.2.1

the following example shows, the key issue is that a boundary point which contains a *psuedo-equilibirium*–where solutions may stay for all time or leave in finite time–may perturb to a standard rest point in the nearby smooth system. Avoiding this issue is the primary motivation for defining invariant as we have.

Example 7.2.2. Consider the Filippov vector field

$$F_0(x) = \begin{cases} -x, & x \leq 1 \\ -1, & 1 < x < 2 \\ [-1, 1], & x = 2 \\ 1, & x > 2 \end{cases}$$

as well as the continuous vector fields

$$f_\lambda(x) = \begin{cases} -x, & x \leq 1 \\ -1, & 1 < x < 2 - \lambda \\ \frac{x-2}{\lambda}, & 2 - \lambda < x < 2 + \lambda \\ 1, & x > 2 + \lambda \end{cases}$$

for $\lambda \in (0, 1]$. Let Φ_λ be the well-parametrized family of multiflows over the interval $X := [-1, 3]$ associated to the basic parametrized family of differential inclusions

$$\dot{x} \in F(x, \lambda) = \begin{cases} F_0(x), & x = 0 \\ f_\lambda, & x \in (0, 1] \end{cases}$$

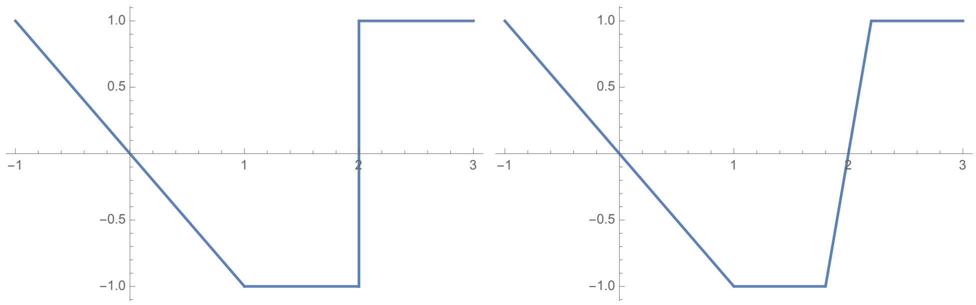

Figure 7.2: The functions F_0 and $f_{1/2}$ from example **??**.

For $N := [-1/2, 2]$, the maximal strongly invariant compact set contained in N is $S_0 := \{0\}$ for Φ_0. However, for Φ_λ, the maximal strongly invariant compact set contained in N is $S_\lambda = [0, 2]$, which does not lie in the interior of N.

The main idea that the preceding example is intended to convey is that at a particular parameter valued, the maximal *strongly* invariant set of a compact set N may be a compact set contained in the interior of N. However, under any small parameter change, this maximal set grows discontinuously. As we will discuss in the following section, by defining invariance as we do–as opposed to this notion of strong invariance–we avoid this major problem.

7.3 Isolated Invariant Sets and Continuation

Our main interest in defining isolating neighborhoods is to use them in order to study the associated isolated invariant sets, which we define here for the case of multiflows.

Definition 7.4. *A set $S \subset \mathbb{R}^n$ is called an* ***isolated invariant set*** *if it is the maximal invariant set in some isolating neighborhood. That is, S is an isolated invariant set if there is an isolating neighborhood N such that*

$$S = \mathrm{Inv}(N)$$

One key property of isolated invariant sets is that they are compact; therefore we may apply the results of Chapter 8 to these objects.

Property 7.2. *Isolated invariant sets are compact.*

Proof. Let S be an isolated invariant set. Since S is contained in a compact neighborhood N by assumption, the boundedness condition is immediate. Also, $\overline{S}$ is invariant by Property 7.1. Then since $\overline{S} \subset N$ is invariant, and $\text{Inv}(N)$ is contained in the interior of N by assumption, we see that $\overline{S} = S$.

□

Isolated invariant sets are associated to isolating neighborhoods, and the key property of isolating neighborhoods is that they are stable under perturbation. Therefore we need some way of associating this stability to the isolated invariant sets. The following definition gives us the language to do so.

Definition 7.5. *Let $N \subset X$ be a compact neighborhood, and denote $S_\lambda := \text{Inv}(N, \Phi_\lambda)$. Two isolated invariant set S_{λ_0} and S_{λ_1} are* ***related by continuation*** *or S_{λ_0}* ***continues to*** *S_{λ_1} if N is an isolating neighborhood for all Φ_λ, $\lambda \in [-\lambda_0, \lambda_1] \subset [-1, 1]$.*

Note that this definition is exactly the same as the definition given in classical Conley Index theory, once the notion of invariance and perturbation has been understood. Then, as in the classical case, it is worth mentioning here that continuation says nothing explicitly about the invariant sets S_λ, and is only a statement about isolating neighborhoods. Indeed, the structure of the invariant sets is allowed to change somewhat drastically while remaining related by continuation. For instance, a degenerate fixed point is often continued to the empty set. For a simple example, consider the family of differential equations $\dot{x} = x^2 + \lambda$. Then the interval $[-1, 1]$ is an isolating neighborhood for all $\lambda \in [0, 1]$, and therefore $S_0 = \{0\}$ continues to $S_1 = \emptyset$.

This property of continuation is actually a feature of Conley theory and not a bug, allowing us to avoid the complications of bifurcation theory. By using the Conley index, we can use knowledge of the behavior on the boundary of the isolating neighborhoods to obtain topological information about the associated isolated invariant sets. At this point in time, the Conley index itself has not been generalized to multiflows, but the results of this paper lead us to believe that this generalization is possible.

One interesting remark about the continuation of isolated invariant sets is that the invariant sets only change semicontinuously. That is, isolated invariant sets which are related by continuation may suddenly shrink, as the example involving the degenerate fixed point and the empty set shows, but they can only grow in a continuous way. We see this by noticing that if S is an isolated invariant set, then for arbitrarily small δ, the set $\overline{B_\delta(S)}$ is an isolating neighborhood for S. Since this isolating neighborhood is stable under perturbation, the continuation of S is a subset of $\overline{B_\delta(S)}$ for sufficiently small perturbations of the multiflow. Then because δ can be made arbitrarily small, it is clear that S cannot grow discontinuously. Since this result is used in the proof of one of our main theorems, we will state it formally as the following lemma.

Lemma 7.1. *Let S_0 continue to S_λ for $\lambda \in I$, where I is a closed interval around 0. Then if $\lambda_n \to 0$ and $x_n \in S_{\lambda_n}$, then any convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ must limit to a point in S_0.*

7.4 Isolating Neighborhoods and The Welander Model

We return once again to Welander's model. In this section we will identify an isolating neighborhood homeomorphic to a disk in the system. Proceeding from the setup in Section 6.3, we have that $\{\Omega_\lambda\}_{\lambda\in[0,1]}$ is a well-parametrized family of multiflows and that Ω_0 corresponds to the nonsmooth version of the model while Ω_λ comes from the nearby continuous systems. The associated differential inclusion is $(\dot{x}, y) \in W_\lambda(x, y)$, and is given by inclusion 4.7. Recall that W_0 takes the form

$$W_0(x,y) = \begin{cases} p_1(x,y), & y > 0 \\ \{\tau p_1(x,y) + (1-\tau)p_2(x,y) : \tau \in [0,1]\}, & y = 0 \\ p_2(x,y), & y < 0 \end{cases}$$

where both p_1 and p_2 are linear functions. This linearity means that we can compute the associated trajectories explicitly, and we will exploit this fact throughout this analysis.

We begin with a few pieces of information from [22]. When we analyzed Filippov bifurcations of this model in section 3.3.2, we allowed the parameter ε to vary; for the duration of this manuscript we will fix $\varepsilon := -1/20$ and consider perturbations of this fixed Filippov system to nearby smooth systems. This parameter choice means that the sliding region is $(9/16, 11/16) \times \{0\} =: (\upsilon, \eta) \times \{0\}$; recall that all other parts of the

x-axis are crossing regions, which simplifies our analysis. Further, it is given in [22] that there is a periodic which intersects the interval $(1/2, \upsilon) \times \{0\}$. Examining inclusion 3.2, we see that $p_1(x, y)$ contains a vertical trajectory at $x = 1/2$. This asymptote will help us to define our isolating neighborhood.

Our isolating neighborhood is homeomorphic to a disk, and we will define it by giving the curve on its boundary. In describing this curve, it is useful to let $\varphi_i(t, x, y)$ be the flow associated to the differential equation $(\dot{x, y}) = p_i(x, y)$; again, the equations for these flows may be explicitly computed because of the linearity of the functions p_i.

Consider the point $(1/2, 0)$. This point lies in a crossing region, and so $\Omega_0(t, 1/2, 0)$ is determined completely by $\varphi_2(t, 1/2, 0)$ until this trajectory intersects the x-axis. We denote by t_a the first positive time such that $\varphi_2(t_a, 1/2, 0)$ does return to the x-axis, and further denote $(x_a, 0) := \varphi_2(t_a, 1/2, 0)$; note that $x_a > 3/4$. Conveniently, $(x_a, 0)$ also lies in a crossing region, and so $\Phi_0(t_a + t, 1/2, 0)$ is uniquely determined by $\varphi_1(t, x_a, 0)$ until this trajectory intersects the x-axis. We now denote by t_b the first positive time such that $\varphi_1(t_b, x_a, 0)$ returns to the x-axis and let $(x_b, 0) := \varphi_1(t_b, x_a, 0)$. Since the x-component of the vector $p_1(x, 0)$ is positive for $x > 3/4 + 15/4 * \varepsilon = \upsilon$, the trajectory of φ_1 cannot return to the x-axis on this interval, and so $x_b \in (1/2, \upsilon)$; note that the lower bound of this interval is determined by the vertical trajectory of p_1 at $x = 1/2$. We now define the closed curve C as the union of the curves $\varphi_2([0, t_a], 1/2, 0)$, $\varphi_1((0, t_b], x_a, 0)$, and $(1/2, x_b) \times \{0\}$. This allows us to define the compact set N to be the region bounded by this curve (including C itself).

Notice that N is an isolating neighborhood. For all (x, y) on the curve $\varphi_2([0, t_a], 1/2, 0) \cup \varphi_1(t_b, x_a, 0)$, $\Phi_0^*((t_a + t_b, \infty), x, y) \subset \{1/2\} \times (0, \infty)$; we see this behavior because these trajectories are uniquely determined by following the flows discussed in the preceding paragraph backwards until the points enter this vertical trajectory. For $(x, y) \in (1/2, x_b) \times \{0\}$, $\Phi_0^*(t, x, y) \notin P$ for arbitrarily small $t > 0$; we see this because $(1/2, x_b) \times \{0\}$ is part of a crossing region.

Morevoer, we are able to conclude from Theorem 7.1 that N is an isolating neighborhood for Ω_λ if λ is sufficiently small.

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- Goals for Welander's Model, Section 1.2

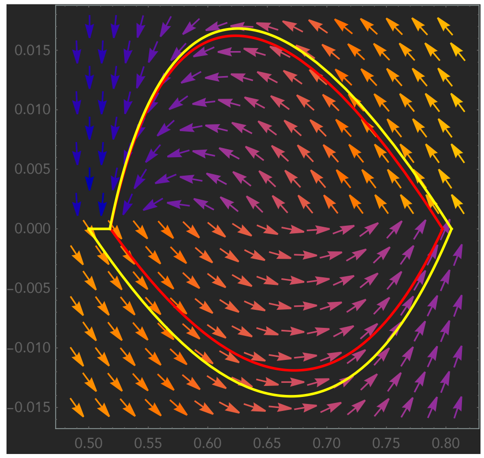

Figure 7.3: The boundary of an isolating neighborhood (yellow) and a periodic orbit (red) in Welander's model.

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Chapter 8

The Attractor-Repeller Pair Decomposition for Multiflows

As indicated by the title, the goal of this chapter is to extend Conley's attractor-repeller decomposition to the setting of multiflows. For the remainder of this chapter, we will assume that $\Phi \subset \mathbb{R}^+ \times X \times X$ is a multiflow over a compact metric space X.

8.1 Limit Sets for Multiflows

In order to discuss attractors and repellers, we need to define the concepts of limit sets for multiflows.

Definition 8.1. *The ω-limit set of a set U is defined by*

$$\omega(U) = \cap_{t \geq 0} \overline{\Phi([t, \infty), U)}$$

This definition of an ω-limit set for multiflows is a direct generalization of the classical definition for flows. Note that $\omega(U)$ is the set of all points $x \in X$ such that

$$x = \lim_{n \to \infty} \psi_n(t_n)$$

where ψ_n is an orbit on Φ, $\psi_n(0) \in U$ and $t_n \to \infty$.

As is the case with flows, we can also consider the α-limit set, which is essentially the ω-limit set in backwards time.

Definition 8.2. *The α-limit set of a set U is defined by*

$$\alpha(U) = \cap_{t\geq 0}\overline{\Phi^*([t,\infty),U)}$$

Unfortunately, since some solutions can leave invariant sets, the ω-limit set is not extremely well behaved. For instance, if we have a multiflow over some space X which is not itself invariant, it is possible that the ω-limit set is also not invariant.

Example 8.1.1. Consider the differential inclusion

$$\dot{x} \in F(x) = \begin{cases} [0,1] & x = 0 \\ 1 & x \neq 0 \end{cases}$$

Let Φ_F be the multiflow over the compact interval $[0,1]$ associated to this differential inclusion. Then for Φ_F, $\omega(0) = [0,1]$, but $[0,1]$ is not invariant. Moreover, we note that $S = \{0\}$ is an invariant set for this multiflow, but $\omega(S) \not\subset S$.

The fact that it is possible that $\omega(S) \not\subset S$ for an invariant set S means that when we are trying to describe the dynamics on S, we need to take an extra step to restrict our view to S. Therefore we will be concerned with the following object:

Definition 8.3. *Let $\Phi : \mathbb{R}^+ \times X \to \mathcal{P}(X)$ be a multiflow and let $S \subset X$. Then $\Phi|_S : \mathbb{R}^+ \times S \to S$ is **the multiflow** Φ **restricted to** S and is defined pointwise by saying that $b \in \Phi|_S(T,a)$ if and only if there exists an orbit $\psi : [0,T] \to S$ on Φ such that $\psi(0) = a$ and $\psi(T) = b$.*

Note in particular that the orbits here are followed only until they leave S, even if they return. For instance, it is possible that there is some point $b \in S$ such that $b \in \Phi(T,a)$ for some $(T,a) \in \mathbb{R}^+ \times S$, but $b \notin \Phi|_S(T,a)$. Because of this fact, this terminology contradicts the better known terminology for the typical restriction of a map.

We can verify that $\Phi|_S : \mathbb{R}^+ \times S \to S$ is itself a multiflow over the compact invariant set S using the same methods as Theorem 5.1. Therefore the view that we take in the remainder of this section is essentially to study the behavior of a multiflow with the additional property that

$$\Phi|_S(T, a) \neq \emptyset.$$

The value of first considering multiflows which do not have this property is that we may consider a wider class of differential inclusions. That is, we can begin with a differential inclusion defined over an unbounded space–like $\mathbb{R}^n$, for instance–and use this restricted object in order to understand the structure of the attractors of its invariant sets.

Definition 8.4. *Let S be a closed invariant set for the multiflow Φ and let $U \subset S$. Then the ω_S and α_S limit sets of U are the sets*

$$\omega_S(U) = \cap_{t \geq 0} \overline{\Phi|_S([t, \infty), U)}, \qquad \alpha_S(U) = \cap_{t \geq 0} \overline{(\Phi|_S)^*([t, \infty), U)}$$

This formulation trivially gives us the convenient property that $\omega_S(S) \subset S$ for any invariant set S of the multiflow Φ.

Similar to the case of the general ω-limit set, we notice that $\omega_S(U)$ is the set of all points $x \in S$ such that

$$x = \lim_{n \to \infty} \psi_n(t_n)$$

where ψ_n is an orbit on $\Phi|_S$, $\psi_n(0) \in U$ and $t_n \to \infty$. Since S is invariant, however, we also may assume that ψ_n is defined for all time. Although this domain does not follow a priori from the definition–the definition of invariance only requires that each point have an orbit which remains in S for all time, and not that all orbits remain in S for all time–we notice that we can always extend any orbit on $\Phi|_S$ to a maximal orbit which exists for all time since S is invariant. Said another way, although the definition only directly implies that $\psi_n : I_n \to S$, where I_n is an interval containing $[0, t_n]$, we can extend ψ_n beyond this interval because ψ_n evaluated at the endpoints of I_n (or the limiting value, in the case where I_n is not closed) is a point in S, and therefore there is some orbit which exists for all time at these points which we may append to ψ_n. Therefore, when we consider the ω_S-limit set, we only need to think about maximal orbits which are defined on all of the real line.

Notice that in the special case where Φ is a flow, $\omega_S(U) = \omega(U)$ for all $U \subset S$, and so this object is simply a generalization of the classic ω-limit set. And as the following lemma begins to demonstrate, this object is much more well-behaved than the general ω-limit set of a multiflow.

Lemma 8.1. *Let S be a compact invariant set for Φ. For nonempty $U \subset S$, both $\omega_S(U)$ and $\alpha_S(U)$ are nonempty and invariant.*

Proof. We will prove the result for the ω_S-limit set; the same result for the α_S-limit set follows by symmetry.

Let $\{\psi_n\}_{n=1}^{\infty}$ be any sequence of orbits such that $\psi_n(\mathbb{R}) \subset S$ and $\psi_n(0) \in U$. We know that such a sequence must exist because S is invariant (and we do not demand that the ψ_n are unique). Let $t_n \to \infty$ and consider the sequence of points $\{\psi_n(t_n)\}_{n=1}^{\infty} \subset S$. Since S is compact, this sequence must contain a convergent subsequence with some limit x. By definition, $x \in \omega_S(U)$, and therefore $\omega_S(U)$ is non-empty.

Now let $x \in \omega_S(U)$, so $x = \lim_{n\to\infty} \psi_n(t_n)$. For each $s \in \mathbb{R}$, let $\gamma_n(s) = \psi_n(t_n + s)$. By Lemma 5.4, there is some orbit $\gamma : \mathbb{R} \to S$ such that on any compact interval $[a, b]$, the family $\{\gamma_n\}_{n=1}^{\infty}$ has some subsequence which converges uniformly to γ. We see that $x = \gamma(0)$ and that $\gamma(s) = \lim_{n_k\to\infty} \gamma_{n_k}(s) = \lim_{n_k\to\infty} \psi_{n_k}(t_{n_k} + s) \in \omega_S(U)$ (for any given s we can take $[a, b]$ to be large enough that we get the subsequence in that previous equality). □

The α_S- and ω_S-limit sets also satisfy the following properties which are used in proving Theorem 8.1, the attractor-repeller decomposition.

Lemma 8.2. *Let S be a compact invariant set for Φ. For any $U \subset S$, if $\omega_S(U) \subset U$ then* $\mathrm{Inv}(U) = \omega_S(U)$.
Symmetrically, if $\alpha_S(U) \subset U$ then $Inv(U) = \alpha_S(U)$.

Proof. The inclusion $\omega_S(U) \subset \mathrm{Inv}(U)$ follows from Lemma 8.1. Therefore we must only show that $\mathrm{Inv}(U) \subset \omega_S(U)$.
Let $x \in \mathrm{Inv}(U)$. Then by definition there is some orbit $\psi : \mathbb{R} \to U$ such that $\psi(0) = x$. Given a sequence $\{t_n\} \to \infty$, $\psi(-t_n) \in U$, so

$$x \in \Phi|_S(t_n, \psi(-t_n)) \subset \Phi|_S([t_n, \infty), U)$$

for each $n \in \mathbb{N}$, and so $x \in \omega_S(U)$. □

8.2 Attractor-Repeller Decomposition of Compact Invariant Sets

For the remainder of this section, assume that $S \subset X$ is a compact invariant set for Φ.

With the α_S and ω_S limit sets defined, we are now ready to define attractors and repellers for multiflows. As in the traditional setting, an attractor is a set which is the ω_S-limit set of a neighborhood of itself, and a repeller is a set which is the α_S-limit set of some neighborhood of itself.

Definition 8.5. *A set $A \subset S$ is said to be an **Attractor in S** if there is a neighborhood U of A in S such that $\omega_S(U) = A$.*

*A set $R \subset S$ is said to be an **repeller in S** if there is a neighborhood U^* of R in S such that $\alpha_S(U^*) = R$.*

It follows directly from Lemma 8.1 that these attractors and repellers are invariant. However, it is worth mentioning that if A is an attractor, $\Phi^*(t, A) \neq A$ in general; in particular, orbits may leave an attractor in backwards time. The following example demonstrates this behavior, and similar examples demonstrate that we have symmetric issues with repellers.

Example 8.2.1. Let

$$F(x) = \begin{cases} x+2, & x < -1 \\ [0,1], & x = -1 \\ 0, & x \in (-1,1) \\ [-1,0], & x = 1 \\ x-2, & x > 1 \end{cases}$$

and consider the multiflow Φ over $S := [-2, 2]$ associated to the Filippov system $\dot{x} \in F(x)$. We see that S is a compact invariant set and that $[-1, 1]$ is an attractor in S. However, the orbit $\psi : \mathbb{R} \to S$ given by

$$\psi(t) = \begin{cases} 2 - \exp(t), & t \leq 0 \\ 1, & t > 0 \end{cases}$$

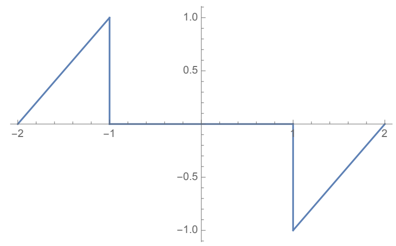

Figure 8.1: The set-valued map F from example 8.2.1. Since orbits may enter the attractor $[-1,1]$ from outside of it, we see that orbits may leave attractors in backwards time.

has initial condition $\psi(0) = 1 \in [-1,1]$ but $\psi((-\infty,0)) \not\subset [-1,1]$, and so $\Phi^*(t,[-1,1]) \not\subset [-1,1]$ for any $t < 0$.

A crucial aspect of the attractor-repeller decomposition is that for a given attractor we can associate a specific dual repeller. Symmetrically, if we begin with a repeller, we can associate a specific dual attractor.

Definition 8.6. *If $A \subset S$ is an attractor in S, then the **dual repeller of A in S** is the set*

$$R = \{x \in S | \omega_S(x) \not\subset A\}$$

*If $R \subset S$ is a repeller in S, then the **dual attractor of R in S** is the set*

$$A = \{x \in S | \alpha_S(x) \not\subset R\}$$

At this point, it is unclear that the dual repeller is actually a repeller, or that the dual attractor is actually an attractor. Moreover, it is unclear that the term *dual* is justified–that the dual of the dual object is the original object. However, we will ultimately see that this terminology is justified in Theorem 8.1. This symmetry also distinguishes our work from earlier work extending the attractor-repeller decomposition to differential inclusions in [23]. In that paper, the repeller is not defined as an object in its own right, and only a definition of dual repeller is given. By considering the notion of a repeller as its own object we see that more of the structure of the attractor-repeller decomposition carries over to this setting than was previously shown. Before diving more into this structure, however, we should make a few remarks about our definition.

In the classical theory of Conley index for flows, the dual repeller is defined as the set $\{x \in S | \omega(x) \cap A = \emptyset\}$. A simple lemma then shows that $\omega(x) \cap A \neq \emptyset$ if and only if $\omega(x) \subset A$, and so the definition that we have provided here for multiflows does indeed generalize the traditional definition. However, in the case of differential inclusions it is possible for the ω_S-limit set of a point to intersect an attractor without being a subset of that attractor, motivating our definition. To see this phenomenon consider the following example:

Example 8.2.2. Let $F : [-1, 1] \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0 & x \in [-1, 0) \\ [0, 1] & x = 0 \\ 1 - x & x \in (0, 1] \end{cases}$$

Note that F satisfies the basic conditions, and let Φ be the associated multiflow. Notice that $S = [-1, 1]$ is invariant and $A = \{1\}$ is an attractor in S. Still we see that $\omega_S(0) = [0, 1]$, and so $\omega_S(0) \cap A \neq \emptyset$ and $\omega_S(0) \not\subset A$.

Also, notice that the set

$$\{x \in [-1, 1] | \omega_S(x) \cap A = \emptyset\} = [-1, 0)$$

is not a repeller, but the set

$$\{x \in [-1, 1] | \omega_S(x) \not\subset A\} = [-1, 0]$$

is a repeller (it is the dual-repeller to A).

We also note that if A is an attractor and R is the associated dual-repeller, then $A \cap R = \emptyset$. Therefore we can complete the decomposition of S by simply taking the set of all remaining points, which we will call the connecting region.

Definition 8.7. *Given an attractor A in S and its dual repeller R, define the* ***connecting region between** A **and** R* *as*

$$C(R, A) := S \setminus (A \cup R)$$

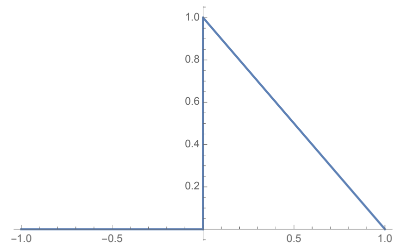

Figure 8.2: The set-valued map F from example 8.2.2

Given these definitions, we see that

$$S = A \cup R \cup C(R, A).$$

We will call the pair (A, R) an **attractor-repeller pair decomposition of the invariant set** S, and we will list its properties in Theorem 8.1.

Theorem 8.1. *Let $\Phi : \mathbb{R}^+ \times X \to \mathcal{P}(X)$ be a multiflow over a compact space $X \subset \mathbb{R}^n$ and assume that $S \subset X$ is compact and invariant under Φ. Let A be an attractor in S, R its dual repeller, and $C(R, A)$ the connecting region between them.*

1. *$S = A \cup R \cup C(R, A)$ and the sets A, R and $C(R, A)$ are all disjoint.*

2. *R is a repeller in S.*

3. *$C(R, A) = \{x \in S | \, \omega_S(x) \subset A, \, \alpha_S(x) \subset R\}$.*

4. *A is the dual attractor to R.*

Proof. Item 1 follows directly from the definitions of the relevant sets and is included only for emphasis. Item 4 follows directly after proving items 2 and 3.

Proof of 2:

Let U be a neighborhood of A in S such that $\omega_S(U) = A$. Then there is some time $t^* > 0$ such that $\overline{\Phi|_S([t^*, \infty), U)} \subset U$; if that were not the case, we could find a sequence of image points of U whose limit was not in A. Define $U^* := S \setminus \overline{\Phi|_S([t^*, \infty), U)}$; note that $S = U \cup U^*$. We will show that $R = \alpha_S(U^*)$.

We can see that $(\Phi|_S)^*([t^*,\infty),U^*) \subset S \backslash U \subset U^*$. If not, there would be some points $y \in U^*$ and $x \in U$ and some time $\tau > t^*$ such that $x \in (\Phi|_S)^*(\tau, y)$. But then we would have that $y \in \Phi|_S(\tau, x)$, contradicting our assumption on t^*. From this inclusion it follows that U^* is a neighborhood of $\alpha_S(U^*)$, and so by Lemma 8.2, $\text{Inv}(U^*) = \alpha_S(U^*)$. Therefore we can show that $R \subset \alpha_S(U^*)$ by showing that $R \subset \text{Inv}(U^*)$.

We want to show that if $x \in R$ then there is some orbit with initial condition x that remains in U^* for all time. Since S is invariant, there is some orbit $\psi : \mathbb{R} \to S$ with $\psi(0) = x$. Note that if *every* orbit originating at x had to enter U in positive time then it follows that $\omega_S(x) \subset \omega_S(U)$ and so $x \notin R$. Therefore, without loss of generality, we can assume that $\psi(\mathbb{R}^+) \cap U = \emptyset$. Now, if $\psi(-t) \in U$ for any $t > 0$ then $\omega_S(x) \subset \omega_S(\psi(-t)) \subset A$, contradicting the assumption that $x \in R$. Thus $\psi(\mathbb{R}^-) \subset U^*$. Therefore $x \in \text{Inv}(U^*) = \alpha_S(U^*)$ and so $R \subset \alpha_S(U^*)$.

To see that $\alpha_S(U^*) \subset R$, we start by noting that if $x \in \alpha_S(U^*)$ then $\omega_S(x) \cap \alpha_S(U^*) \neq \emptyset$ by the invariance of the α_S-limit set. Since $\alpha_S(U^*) \subset U^*$ and $U^* \cap A = \emptyset$, we conclude $\omega_S(x) \not\subset A$ and so $x \in R$.

Proof of 3:

It follows directly from the definition that $\omega_S(x) \subset A$ for $x \in S \setminus R$, so we must only show that $\alpha_S(x) \subset R$ for $x \in S \setminus A$.

Let $x \in S \setminus A$, and call $\delta := \text{dist}(x, A)$. Let U' be a neighborhood of A in S such that $\omega_S(U') = A$. Then $U := U' \cap B_{\delta/2}(A)$ also satisfies $\omega_S(U) = A$. As shown in the proof of part 2, there is some time $t^* > 0$ such that $\overline{\Phi|_S([t^*,\infty),U)} \subset U$ and $U^* := S \setminus \overline{\Phi|_S([t^*,\infty),U)}$ satisfies $\alpha_S(U^*) = R$. Then since $x \in U^*$, $\alpha_S(x) \subset R$. □

We will close out this section with a lemma that gives a way to more easily identify attractors. If the multiflow moves the closure of a set into its interior at some positive time then the set's ω_S-limit set is an attractor. This condition is very helpful in identifying attractors because it relies only on checking a single positive time.

Lemma 8.3. *Suppose $U \subset S$ and $\Phi|_S(t, \overline{U}) \subset \text{int}(U)$ for some $t > 0$. Then $\omega_S(U)$ is an attractor contained in the interior of U.*

Proof. Since $\Phi|_S(t, \overline{U}) \subset \text{int}(U)$, there is an open set V such that $\Phi|_S(t, \overline{U}) \subset V \subset$

$\overline{V} \subset \text{int}(U)$. Then there is some $\varepsilon > 0$ such that $\Phi|_S((t-\varepsilon, t+\varepsilon), \overline{U}) \subset V$. If that were not the case then there would be a sequence of times $t_n \to t$ and associated points $x_n \in \overline{U}$ and orbits ψ_n with $\psi_n(0) = x_n \in \overline{U}$ and $\psi_n(t_n) \not\subset V$. By Lemma 5.3, there is an orbit ψ such that on any compact interval $I \subset \mathbb{R}$ there is some sequence $\{n_k\}_{k=1}^{\infty}$ where $\psi_{n_k}|_I \to \psi|_I$. But $\psi(0) = \lim_{k\to\infty} \psi_{n_k}(0) \in \overline{U}$ and $\psi(t) \lim_{k\to\infty} \psi_{n_k}(t) \in V^c$, contradicting our assumption that $\Phi|_S(t, \overline{U}) \subset V$.

Then if $t' > t^2/\varepsilon$, we can write $t' = s_1 + \cdots + s_m$ where $s_i \in (t-\varepsilon, t+\varepsilon)$. Then

$$\Phi|_S(t', \overline{U}) = \Phi|_S(s_1 + \cdots + s_m, \overline{U}) = \Phi|_S(s_m, \Phi|_S(s_{m-1}(\cdots(\Phi|_S(s_2, \Phi|_S(s_1, \overline{U})))\cdots))) \subset V$$

We can see this by noticing that $\Phi|_S(s_i, \overline{U}) \subset \Phi|_S((t-\varepsilon, t+\varepsilon), \overline{U}) \subset V \subset \text{int}(U)$, and hence $\Phi|_S(s_j, \Phi|_S(s_i, \overline{U})) \subset \Phi|_S((t-\varepsilon, t+\varepsilon), \overline{U})$. Then for $t' > t^2/\varepsilon$, $\Phi([t', \infty), \overline{U}) \subset \overline{V} \subset \text{int}(U)$. Therefore $\omega_S(U) \subset \text{int}(U)$ and so $\omega_S(U)$ is an attractor. □

8.3 Attractor-Repeller Pair Continuation

The ultimate goal of the Conley Index theory is to obtain results which are stable under perturbation. Therefore we would like to show that, in some sense, our attractor-repeller pairs are stable up to perturbation of the differential inclusion.

Recall that isolated invariant sets are compact (Property 7.2), and hence we may give an attractor-repeller pair decomposition of any isolated invariant set. Importantly, these attractors and repellers are also isolated invariant sets.

Lemma 8.4. *Let S be an isolated invariant set for the multiflow Φ. If $A \subset S$ is an attractor in S, then A is an isolated invariant set for the multiflow Φ. Symmetrically, a repeller in S is an isolated invariant set.*

Proof. Let N be an isolating neighborhood for S and let $U' \subset S$ be a neighborhood of A in S such that $\omega_S(U') = A$. Let $d_H(W, Z)$ denote the Hausdorff distance of the compact sets W and Z, and define

$$\delta := \min(d_H(A, \partial N), d_H(A, \partial U'))$$

Now let $U := U' \cup B_{\delta/2}(A)$. We will show that $\overline{U} \subset \text{int}(N)$ is an isolating neighborhood for A in Φ.

We need to show that $\text{Inv}(\overline{U}) \subset \text{int}(U)$, so let $x \in \partial U$. If $x \notin S$, then we know that $x \notin \text{Inv}(\overline{U})$ because $S = \text{Inv}(N)$ and $\overline{U} \subset \text{int}(N)$. If $x \in S$, then $\alpha_S(x) \subset R$, where R is the dual-repeller of A in S. Since $R \cap U' = \emptyset$, there cannot be an orbit with initial condition x that remains in $\overline{U}$ for all time, and so $x \notin \text{Inv}(\overline{U})$.

□

With this lemma stated, we are ready to prove one of the key results of this paper, showing that the attractor-repeller decomposition described in the prior section is stable under perturbation.

Theorem 8.2. *Attractor-repeller pair decompositions continue.*

To our knowledge, no comparable theorem exists in the differential inclusions literature.

In the proof of this result, we will need to discuss the interiors and closures of sets relative to other sets. To do so, we will adopt the convention that $\text{int}(W; Z)$ and $\text{cl}(W; Z)$ respectively denote the interior and closure of the set W relative to Z. Additionally, the notation $W \setminus Z$ does not imply here that $Z \subset W$, but merely is intended to convey the notion $W \setminus (W \cap Z)$.

Additionally, before beginning this proof, we should acknowledge the role Richard Moeckel played in its development. He offered some extremely valuable insights into the nature of continuation that come into play here.

Proof. Assume that S_0 is an isolated invariant set for the multiflow Φ_0 with isolating neighborhood N, and let (A_0, R_0) be an attractor-repeller pair decomposition of S_0. Then since A_0 and R_0 are themselves isolated invariant sets for Φ_0 by Lemma 8.4, they have isolating neighborhoods $N_A \subset N$ and $N_R \subset N$. Since isolating neighborhoods are stable under perturbation by Theorem 7.1, there is some $\lambda_S > 0$ such that $|\lambda| \leq \lambda_S$ implies that N is an isolating neighborhood for Φ_λ. Similarly, there exist $\lambda_A > 0$ and $\lambda_R > 0$ such that N_A and N_R remain isolating neighborhoods for $\lambda \in [-\lambda_A, \lambda_A]$ and $\lambda \in [-\lambda_R, \lambda_R]$ respectively.

Let $\lambda_0 := \min(\lambda_S, \lambda_A, \lambda_R)$. Then for $\lambda \in [-\lambda_0, \lambda_0]$, the isolated invariant sets $A_\lambda := \text{Inv}(N_A, \Phi_\lambda)$ are related by continuation, the $R_\lambda := \text{Inv}(N_R, \Phi_\lambda)$ are related by continuation, and the $S_\lambda := \mathrm{Inv}(N, \Phi_\lambda)$ are related by continuation. Thus all that remains to check is that (A_λ, R_λ) is an attractor-repeller pair decomposition for S_λ for for sufficiently small $|\lambda|$.

We will start by showing that A_λ is an attractor in S_λ for small enough $|\lambda|$. We know that is some time t^* such that

$$\Phi_0^{S_0}(t^*, N_A \cap S_0) \subset \mathrm{int}(N_A \cap S_0; S_0)$$

since A_0 is assumed to be an attractor in S_0. For a sufficiently small $|\lambda|$, we will show that

$$\Phi_\lambda^{S_\lambda}(t^*, N_A \cap S_\lambda) \subset \mathrm{int}(N_A \cap S_\lambda; S_\lambda)$$

which implies that $\omega_{S_\lambda}(N_A \cap S_\lambda)$ is an attractor in S_λ by Lemma 8.3.

If this were not the case, we would have a sequence $\lambda_n \to 0$ and associated points $x_n \in N_A \cap S_{\lambda_n}$ and orbits ψ_n on Φ_{λ_n} satisfying

$$\psi_n(0) = x_n \in N_A \cap S_{\lambda_n}, \qquad \psi_n(t) \notin \mathrm{int}(N_A \cap S_{\lambda_n}; S_{\lambda_n})$$

By Lemma 6.3, on a compact interval I containing 0 and t, we can take some subsequence $\{n_k\}_{k=1}^\infty$ of these orbits which converge uniformly to an orbit ψ on Φ_0. Notice that

$$x_{n_k} = \psi_{n_k}(0) \to \psi(0) \in N_A \cap S_0$$

by Lemma 7.1. But Lemma 7.1 also shows that $\psi(t) \in \mathrm{cl}(S_0 \setminus N_A; S_0)$, contradicting our assumption that $\Phi_0^{S_0}(t, N_A \cap S_0) \subset \mathrm{int}(N_A \cap S_0)$. Therefore $\omega_{S_\lambda}(N_A \cap S_\lambda)$ is an attractor in S_λ for small enough $|\lambda|$. Since $\omega_{S_\lambda}(N_A \cap S_\lambda) = A_\lambda$ by Lemma 8.2, we see that A_λ is an attractor as desired.

We can follow a symmetric argument to see that R_λ is a repeller in S_λ for small enough $|\lambda|$, and so it only remains to show that R_λ is the dual-repeller to A_λ in S_λ. That is, we must show that $\omega_{S_\lambda}(x) \subset A_\lambda$ for all $x \in S_\lambda \setminus R_\lambda$ for small enough $|\lambda|$.

In fact, we actually only need to show this property for for $x \in S_\lambda \setminus N_R$. This restriction is possible because $x \in (S_\lambda \cap N_R) \setminus R_\lambda$ implies that for *any* orbit ψ on $\Phi_\lambda^{S_\lambda}$ such that $\psi(0) = x$, there is some time t such that $\psi(t) \in S_\lambda \setminus N_R$ because N_R is an isolating neighborhood and S_λ is an invariant set. Then if $\omega_{S_\lambda}(x) \not\subset A_\lambda$, then also

$\omega_{S_\lambda}(\psi(t)) \not\subset A_\lambda$ for some such orbit.

For the sake of contradiction, assume that this is not the case and $\omega_{S_\lambda}(x) \not\subset A_\lambda$ for all $x \in S_\lambda \setminus N_R$ for small enough $|\lambda|$. Then there is some sequence $\lambda_n \to 0$ and associated points $x_n \in S_{\lambda_n} \setminus N_R$ and $y_n \notin A_{\lambda_n}$ such that $y_n \in \omega_{S_{\lambda_n}}(x_n)$. By the definition of the ω-limit set, that means that for each n there is a sequence of orbits $\{\psi_n^k\}_{k=1}^\infty$ and a sequence of times $t_n^k \to \infty$ such that

$$\psi_n^k(0) = x_n, \qquad \psi_n^k(t_n^k) \to y_n, \qquad k \to \infty$$

Without loss of generality, we may assume that $t_n^k < k$ for all n.

As we just saw, however, for $x \in N_A \cap S_\lambda$ and $|\lambda|$ sufficiently small, we have that $\omega_{S_\lambda}(x) \subset A_\lambda$. Since $y_n \in \omega_{\lambda_n}(\psi_n^k(t_n^k))$ for any k or n, we therefore must have that

$$\psi_n^k(t_n^k) \in S_{\lambda_n} \setminus N_A$$

for all n and k.

By Lemma 6.3, for each k there is an orbit ψ^k on Φ_0 such that on any compact interval, $\{\psi_n^k\}_{n=1}^\infty$ has some subsequence which converges uniformly to ψ^k. Taking further subsequences if necessary, we also find limit points t^k of $\{t_n^k\}_{n=1}^\infty \subset [0, k]$, x of $\{x_n\}_{n=1}^\infty \subset N$, and y^k of $\{\psi_n^k(t_n^k)\}_{n=1}^\infty \subset N$. Notice that $x = \psi^k(0)$ and that $y^k = \psi^k(t^k)$. Additionally, note that Lemma 7.1 implies that $x \in \mathrm{cl}(S_0 \setminus N_R; S_0)$ and $y^k \in \mathrm{cl}(S_0 \setminus N_A; S_0)$.

Since $y^k \in \mathrm{cl}(S_0 \setminus N_A; S_0)$, there is some convergent subsequence

$$y^{k_m} \to y \in \mathrm{cl}(S_0 \setminus N_A; S_0)$$

as $k_m \to \infty$. That is,

$$\psi^{k_m}(t^{k_m}) \to y \in \mathrm{cl}(S_0 \setminus N_A; S_0)$$

This implies that $\omega_{S_0}(x) \not\subset A_0$, even though $x \in S_0 \setminus R_0$, contradicting our assumption that (A_0, R_0) is an attractor-repeller decomposition of S_0. Therefore $\omega_{S_\lambda}(x) \subset A_\lambda$ for all $x \in S_\lambda \setminus R_\lambda$ for small enough $|\lambda|$, and attractor-repeller decompositions continue. $\square$

8.4 Basins of Attraction and Attractor Blocks

This brief section of the chapter does not prove any substantial results; instead we introduce some definitions that are used in the classical Conley index theory and address a few questions that had been posed by Richard McGehee. Throughout this section assume that Φ is a multiflow over a compact space X, $S \subset X$ is a compact invariant set for Φ, and $A \subset X$ is an attractor in S.

Definition 8.8. *The* **basin of attraction for** A *is the set*

$$\mathcal{B}(A) = \{x \in S | \omega(x) \subset A\}.$$

Notice that Theorem 8.1 immediately gives two properties of these basins. Namely, basins of attraction are open and $B(A)^c$ is a repeller.

Definition 8.9. *An* **attractor block** *is a compact subset $B \subset S$ such that*

$$\Phi|_S(t, B) \subset \text{int}(B), \qquad t > 0$$

It follows from Lemma 8.3 that $\omega_S(B)$ is an attractor. However, one crucial question that McGehee raised about this object remains: given an attractor A and a neighborhood U of A, are we guaranteed the existence of an attractor block $B \subset U$ such that $\omega_S(U) = A$?

8.5 Attractor Repeller Pair Decomposition in Welander's Model

We proceed from the work done in Section 7.4, where for the multiflow Ω_0 we identified an isolating neighborhood N and associated isolated invariant set S_0. Although the index theory is not sufficiently developed to allow us to conclude that S_0 is non-empty, ad-hoc methods do allow us to make this claim because [22] identified a periodic orbit in the interior of N, and S_0 must contain the disk bounded by this periodic orbit. Our goal for this section is to establish an attractor-repeller pair decomposition (A_0, R_0) of S_0.

We will identify a set $U \subset S_0$ such that $\omega_{S_0}(U) = A_0$; the techniques used will be nearly identical to those used in defining N. Consider the point $(\upsilon, 0)$, which lies on the boundary of the crossing region. Because of this crossing behavior, $\Omega_0(t, \upsilon, 0)$ is determined by $\varphi_2(t, \upsilon, 0)$ for t sufficiently small. We denote by t_c the first positive time such that $\varphi_2(t_c, \upsilon, 0)$ returns to the x-axis, and further denote $(x_c, 0) := \varphi_2(t_c, \upsilon, 0)$; note that $x_c > 3/4$. Conveniently, $(x_c, 0)$ also lies in a crossing region, and so $\Omega_0(t_c + t, x_f, 0)$ is uniquely determined by $\varphi_1(t, x_f, 0)$ until this trajectory intersects the x-axis. We now denote by t_d the first positive time such that $\varphi_1(t_d, x_f, 0)$ returns to the x-axis and let $(x_d, 0) := \varphi_1(t_d, x_c, 0)$. Since the x-component of the vector $p_1(x, 0)$ is positive for $x > \upsilon$, the trajectory of φ_1 cannot return to the x-axis on this interval, and so $x_d \in (x_b, \upsilon)$. Here the lower bound is determined by the fact that $(x_d, 0) \in S_0 \subset N$. Let D be the union of the curves $\varphi_2([0, t_c], \upsilon, 0)$, $\varphi_1((0, t_d), x_c, 0)$, and $(x_d, \upsilon) \times \{0\}$, and notice that $D \subset \operatorname{int}(S_0)$ because D is in the interior of the periodic orbit identified by [22]. We then define U to be the exterior of the curve D intersected with the set S_0, as well as the curve D itself. Notice that for $t > t_c + t_d$, we have that $\Omega|_{S_0}(t, U) \subset \operatorname{int}(U, S_0)$. Therefore by Lemma 8.3, $\omega_{S_0}(U) =: A_0$ is an attractor contained in the interior of U.

We know that A_0 is itself an isolated invariant set by Lemma 8.4. If we define N_A as the closed annulus bounded by the curves C and D, we can see that N_A is an isolating neighborhood for A_0. We have already demonstrated that points on the curve C leave N_A in backwards time (since C is the curve defining the boundary of N). We also notice that for $(x, y) \in D$ have the same property because for for $t > t_c + t_d$, we have that $\Omega_0^*(t, x, y)$ lies in the interior of the curve D, i.e. outside of the set N_A. Therefore N_A is in fact an isolating neighborhood. Further notice that $\operatorname{Inv}(N_A, \Omega_0) = \operatorname{Inv}(U, \Omega_0|_{S_0})$ because $N_A \subset N$ and $\operatorname{Inv}(N_A, \Omega_0) = S_0$. Finally, by Lemma 8.2, $\omega_{S_0}(U) = \operatorname{Inv}(U, \Omega_0|_{S_0})$, and so $A_0 = \operatorname{Inv}(N_A, \Omega_0)$.

Letting N_R be the interior of the curve D as well as the curve itself, symmetric arguments imply that $\alpha_{S_0}(N_R)$ is the dual-repeller R_0 to A_0 in S_0. Moreover, N_R is an isolating neighborhood for R_0. Also notice that both Theorem 8.1 and Lemma 8.3 imply on their own that R_0 is non-empty.

Finally, denoting $A_\lambda := \operatorname{Inv}(N_A, \Omega_\lambda)$, $R_\lambda := \operatorname{Inv}(N_R, \Omega_\lambda)$, and $S_\lambda := \operatorname{Inv}(N, \Omega_\lambda)$, by Theorem 8.2, (A_λ, R_λ) is an attractor-repeller pair decomposition of the isolated invariant set S_λ for λ sufficiently small.

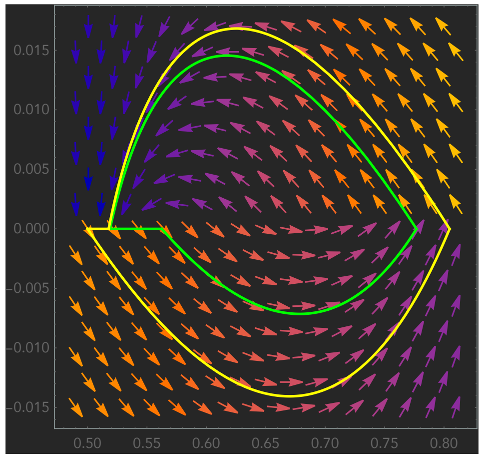

Figure 8.3: The curves C (yellow) and D (green) which define the annulus N_A in Welander's model.

Navigating Sections on Welander's Model:

- Goals for Welander's Model, Section 1.2
- Introduction to Welander's Model and Bifurcation Analysis, Section 3.3
- Perturbation and Welander's Model, Section 4.7
- Welander's Model as a Multiflow, Section 5.5
- Welander's Model as a Well-Parametrized Family of Multiflows, Section 6.2
- Isolating Neighborhood in Welander's Model, Section 7.4
- Attractor-Repeller Pair Decomposition for Welander's Model, Section 8.5
- The Conley Index and Welander's Model, Section 9.5

Chapter 9

The Conley Index for Multiflows

9.1 The Conley Index for Flows

Before discussing a multivalued generalization of the Conley Index we should review its basic properties in the classical setting. This review section borrows heavily from [29]. Throughout this section we assume that φ is a flow. As was stated in the introduction, the Conley Index of an isolating neighborhood N for φ satisfies three crucial properties:

1. *(Well-defined)* If N' is another isolating neighborhood for φ and $\text{Inv}(N, \varphi) = \text{Inv}(N', \varphi)$, then the Conley index of N is the same as the Conley index of N'. This property allows us to view the Conley index as a well-defined index of isolated invariant sets.

2. *(Ważewski Property)* If the Conley index of N is not trivial then $\text{Inv}(N)$ is non-empty.

3. *(Stable)* If $\{\varphi_\lambda\}_\lambda \in [0, 1]$ is a continuous parametrized family of flows and N is an isolating neighborhood for each flow φ_λ, then the Conley index of N is the same for each flow φ_λ.

Because isolating neighborhoods are robust under small perturbations, the hypotheses of the continuation property are always guaranteed to be satisfied for some small range of λ. That means that if we can use the Ważewski property of the Conley index to identify a non-empty invariant set in an isolating neighborhood N, that same neighborhood will contain a non-empty invariant set for nearby flows.

In Section 9.2 we will see that we need to adopt additional hypotheses on multiflows in order to retain the Ważewski property. Without this property, it is difficult to see what value the index provides.

Before seeing why these additional assumptions must be made, however, we should make a few more remarks about the Conley index in the setting of flows. To do so we must first introduce a few other terms.

Definition 9.1. *Let $L \subset N$. The set L is said to be* **positively invariant relative to** N *under the flow* φ *if given* $x \in L$ *and* $\varphi([0,t],x) \subset N$*, we have that* $\varphi([0,t],x) \subset L$*.*

Intuitively, the preceding definition says that points in L remain in L under the flow φ for as long as they remain in its superset N.

Definition 9.2. *Let $L \subset N$. The set L is said to be* **an exit set for** N *under the flow* φ *if for each* $x \in N$*,* $\varphi(t_1,x) \notin N$ *and* $t_1 > 0$ *implies that there is some* $t_0 \in [0,t_1)$ *such that* $\varphi([0,t_0],x) \subset N$ *and* $\varphi(t_0,x) \in L$*.*

The name *exit set* captures the notion it defines very well; a point may only leave the set N in forward time by passing through the exit set.

These definitions allow us to define an *index pair* for an isolated invariant set; this pair is what is actually used to compute the Conley index.

Definition 9.3. *A pair of compact sets* (N,L) *with* $L \subset N$ *is called an* **index pair** *for the isolated invariant set* S *if:*

1. $S = \mathrm{Inv}(\overline{N \setminus L})$ *and* $N \setminus L$ *is a neighborhood of* S*.*

2. L *is positively invariant in* N*.*

3. L *is an exit set for* N*.*

There is always an index pair associated to any isolated invariant set [29].

In order to tie this concept of an index pair to the attractor-repeller pair decomposition of an isolated invariant set S, we introduce the notion of an *index triple.*

Definition 9.4. *Let S be an isolated invariant set and let* (A,R) *be an attractor-repeller pair decomposition of* S*. An* **index triple** *for* (A,R) *is a collection of compact sets* (N_2,N_1,N_0) *such that* $N_0 \subset N_1 \subset N_2$ *and*

1. (N_2, N_0) *is an index pair for* S*;*

2. (N_1, N_0) *is an index pair for* A*;*

3. (N_2, N_1) *is an index pair for* R*.*

There exists an index triple for any attractor-repeller pair decomposition of any isolated invariant set [29]. Although we will not discuss this object in great detail, we note that it is useful in understanding connecting orbits between attractors and repellers (or more generally between Morse sets). We mention these triples here because we will see that there are problems with this object for the index developed in [32]; we will discuss this index and these problems in Section 9.3.

The notion of an index pair for an isolated invariant set allows us to finally define the Conley index. This index has both a homotopy and homology variant; since homotopic spaces have the same homology as well, this twin notion is well-defined.

Definition 9.5. *Given any index pair* (N, L) *for the isolated invariant set* S*, the* **homotopy Conley index of** S *is the homotopy type of the pointed topological space obtained by identifying the exit set to a point:*

$$h(S) = h(S, \varphi) \sim (N/L, [L])$$

The **homological Conley index** *is given by*

$$CH_*(S) := H_*((N/L, [L]))$$

Keeping these definitions in mind we now proceed to the next section and demonstrate that additional assumptions must be made on multiflows in order to meaningfully define the Conley index in the multivalued setting.

9.2 Necessity of Additional Conditions on Multiflows

The results of the previous chapters indicate that multiflows have sufficient structure for generalizing Conley Index Theory to the multivalued setting. However, the following example shows that this is not the case, and that we must make additional assumptions on these objects in order to develop a meaningful Conley index theory.

Consider the set-valued map

$$F(x) = \begin{cases} -1, & x < 0 \\ \{-1, 1\}, & x = 0 \\ 1, & x > 0 \end{cases}$$

Notice that $\dot{x} \in F(x)$ is *not* a basic differential inclusion because $F(0)$ is not a convex set. For such a simple differential inclusion, however, we may explicitly compute all solutions and therefore do not require any general existence theorem. For initial conditions $x_0 \geq 0$, we have

$$x : [0, \infty) \to \mathbb{R}, \qquad x_0 \to x_0 + t$$

and for initial conditions $x_0 \leq 0$ we have

$$x : [0, \infty) \to \mathbb{R}, \qquad x_0 \to x_0 - t$$

Notice, in particular, that there are exactly two solutions with initial condition $x_0 = 0$.

We can also directly verify that the set

$$\Phi = \{(T, a, b) \in \mathbb{R}^+ \times X \times X | \exists \text{ solution } \psi : [0, T] \to X \text{ of } \dot{x} \in F(x),\ \psi(0) = a, \psi(T) = b\}$$

is a multiflow for any compact interval $X \subset \mathbb{R}$. Let's take $X = [-2, 2]$.

The compact interval $N = [-1, 1]$ is an isolating neighborhood for Φ. Moreover, because the behavior of the multiflow at the endpoints $\{-1, 1\}$ is actually single-valued, under any possible generalization of the notion of an exit set, the set $L := \{-1, 1\}$ must be an exit set for N. We also see that L is positively invariant relative to N, and so (N, L) is an index pair for $S := \text{Inv}(\overline{N \setminus L})$.

The homotopy type of $(N/L, [L])$ is that of a pointed circle; in particular, $h(S)$ is nontrivial. However, $S = \emptyset$, and so the Ważewski property does not hold in this setting.

The problem here is that $\Phi(t, \{0\})$ is a two-point set for $t \in (0, 2]$ (the right endpoint of this interval is the time when the image of this point leaves the compact set $X = [-2, 2]$). This change to the topological structure of the point as it is mapped forwarded in time contradicts the basic structure of Conley index theory. This issue is, of course, a serious problem.

We also remark that the reason that the Ważewski property fails for this example is that one of the other critical properties–that the index is well-defined–also fails. While (N, L) is an index pair for the empty set, so is $(\emptyset, \emptyset)$, and the homotopy types of the associated pointed spaces are different.

However, as we mentioned earlier, F does not satisfy the basic conditions. If we use the Filippov convex combination method in order to extend F to the multivalued map

$$F'(x) = \begin{cases} -1, & x < 0 \\ [-1, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

this problem goes away. Letting Φ' be the multiflow associated to $\dot{x} \in F(x)$ over the compact set $X = [-2, 2]$, we see that again $(N, L) = ([-1, 1], \{-1, 1\})$ is an index pair and $(N/L, [L])$ is homotopy equivalent to a pointed circle. Now, however, $\mathrm{Inv}(\overline{N \setminus L}) = \{0\}$, and so we do not have a contradiction to the Ważewski property. This example indicates that we may still be able to generalize Conley index theory to differential inclusions by placing additional assumptions on multiflows which are compatible with the differential inclusions that we have discussed. As we will see in the following section, a version of the index actually has been developed for a class of differential inclusions that is slightly less general than the basic differential inclusions that we have considered thus far.

9.3 The Conley Index for Multivalued Dynamical Systems

Our primary purpose for this section is to discuss some of the results of "A Cohomological Index of Conley Type for Multi-valued Admissible Flows" [32], which defines an index for these objects in a way that preserves the three crucial properties of the index. We will see in this section that an admissible multivalued flow is a multiflow with some additional assumptions. Moreover, all definitions given for multiflows in earlier parts of this text align with the definitions given for the more restrictive objects used in [32], and therefore by combining this index with the attractor-repeller decomposition and continuation given earlier we are able to analyze certain differential inclusions. In fact, as we will see in the following section, any basic differential inclusion satisfying a linear growth bound may be studied with these techniques. That allows us to identify the Conley index of the isolated invariant set and its attractor-repeller pair decomposition identified in the Welander model in Section 8.5; this analysis is carried out in Section 9.5.

Before proceeding we should make a few remarks. First, this section is far less detailed than earlier sections; in particular, important theorems here will be cited and not proven. Second, to the extent possible, terms introduced in [32] have been translated into the language that we have introduced throughout this thesis, and so may differ in appearance (but not substance) from the definitions given in [32].

We begin with some notation. As indicated in the title, [32] uses cohomology rather than homology in defining the index. Specifically, the Alexander-Spanier cohomology is used. In order to define this theory we need a bit of notation. First, let $\mathbf{Top}_2$ denote the category of topological pairs; that is, pairs (P, Q) of topological spaces such that $Q \subset P$. Further let **GMod** denote the category of graded $\mathbb{Z}$-modules and linear maps of degree zero. Then the Alexander-Spanier cohomology is here considered as a functor $H^* : \mathbf{Top}_2 \to \mathbf{GMod}$. We can consider the cohomology of a single space Y by identifying it with the pair $(Y, \emptyset)$. Note that the notation H^* is intended to indicate an infinite sequence of functors; that is, for each $i \in \mathbb{N}$, we have a functor H^i. We now proceed with a series of necessary definitions from topology.

Definition 9.6. *A continuous map $f : W \to Y$ between topological spaces is said to be* **proper** *if the preimage of any compact set is compact. That is, given $K \subset Y$ compact, $f^{-1}(K) \subset W$ is also compact.*

Definition 9.7. *A non-empty compact space Y is said to be* **acyclic** *if it has the same cohomology as a one point space:*

$$H^*(Y) = \{\mathbb{Z}, 0, 0, 0, \cdots\}$$

Definition 9.8. *A continuous map $f : W \to Y$ between topological spaces is said to be* **Vietoris** *if it is proper and the preimage $f^{-1}(y)$ of any point $y \in Y$ is acyclic.*

We also need the following definition for set-valued maps:

Definition 9.9. *A set-valued map* $F : W \to \mathcal{P}(Y)$ *between topological spaces is said to be* **strongly admissible** *if there exists a topological space* Υ*, a Vietoris map* $p : \Upsilon \to [0,T] \times Y$*, and a continuous map* $q : \Upsilon \to Y$ *such that*

$$q(p^{-1}(y)) = \Phi(y)$$

for all $y \in Y$*.*

Note that any single-valued map f is strongly admissible since $f \equiv f \circ \mathrm{id}^{-1}$ and id is Vietoris.

In order to define the cohomological index of Conley type, [32] introduces the concept of an *admissible multivalued flow.* This object is a multiflow with two additional properties.

Definition 9.10. *Let* Y *be a locally compact metric space. A multiflow*

$$\Phi : \mathbb{R}^+ \times Y \to \mathcal{P}(Y)$$

which satisfies the additional criteria

1. *(nonempty)* $\Phi(t,y) \neq \emptyset$*,* $\Phi^*(t,y) \neq \emptyset$ *for all* $(t,y) \in \mathbb{R}^+$
2. *(admissibility) there exists a time* $T > 0$ *such that the restriction of* Φ *to the time interval* $[0,T]$ *is strongly admissible*

is called an **admissible multivalued flow***.*

We note that the definition of the admissibility criterion given in [32]–where it is defined in terms of certain morphisms–highlights the fact that the choice of p, q, and W is generally not unique, but rather a whole equivalence class. We do not need the more technical information for our purposes, however.

There are two important things to notice about this definition. The first is that the restriction of a flow $\varphi : \mathbb{R} \times Y \to Y$ to forwards time is automatically an admissible multivalued flow because it is single-valued. Therefore this definition is truly a generalization of the idea of a flow to a set-valued setting.

The second crucial fact is that the admissibility criterion rules out problematic behavior like we saw in section 9.2. Acyclic sets are path-connected, and the continuous image of a path-connected set is again path-connected. Therefore the admissibility criterion rules out the possibility that $\Phi(t, x)$ is two disjoint points, like we saw in that example.

Continuing now to the use of this object in Conley index theory, we note that invariance, isolated invariant sets, and isolating neighborhoods are all defined in [32] as we have defined them here; no additional phrasing is needed to introduce them here because admissible multivalued flows are multiflows with additional assumptions. This alignment is valuable because it allows us to use the attractor-repeller structure developed in this thesis with the index defined in [32]. However, we do need to introduce the following notation for maximal forwards and backwards invariant sets:

$$\mathrm{Inv}^{+}(U, \Phi) := \{x \in U | \exists \, \mathrm{orbit} \, \psi : \mathbb{R}^{+} \to U \, \mathrm{on} \, \pi, \, \psi(0) = x\}$$

$$\mathrm{Inv}^{-}(U, \Phi) := \{x \in U | \exists \, \mathrm{orbit} \, \psi : \mathbb{R}^{-} \to U \, \mathrm{on} \, \pi, \, \psi(0) = x\}$$

In order to define an index pair–and hence the Conley index itself–[32] introduces an object identical to the restricted multiflow. Note that given a compact set N and an admissible multivalued flow Φ, it is generally not true that $\Phi|_N$ is an admissible multivalued flow; however, it is still a multiflow (since admissible multivalued flows are a special case of multiflows).

In stating this definition, we will also need the terminology that a set $A \subset N$ is **strongly positively invariant with respect to** $\Phi|_N$ if $\Phi|_N([0, \infty), A) \subset A$.

Definition 9.11. *The pair of subsets (P, Q) of an isolating neighborhood N is said to be an* **index pair in N with respect to Φ** *if the following conditions are satisfied:*

1. *P and Q are compact and strongly positively invariant with respect to $\Phi|_N$.*
2. $\mathrm{Inv}^{-}(N) \subset \mathrm{int}(P, N), \qquad \mathrm{Inv}^{+}(N) \subset N \setminus Q$
3. $\overline{P \setminus Q} \subset \mathrm{int}(N)$

Notice that it is not assumed that an index pair is a topological pair; that is, we do not demand that $Q \subset P$. As in the classical setting, for any isolated invariant set S there is always an index pair [32].

In [32] it is remarked that this notion of an index pair does not reduce to Conley's notion of an index pair in the single-valued case [7], and is in fact more restrictive. The final condition–that $\overline{P \setminus Q} \subset \text{int}(N)$–induces some technical difficulties in finding index pairs. More importantly, as we will see at the end of the chapter, it is not generally true that index triples exist with this definition. Nevertheless, these index pairs may still be used to construct a cohomological index.

Definition 9.12. *If S is an isolated invariant set with isolating neighborhood N, then the* **cohomological index of Conley type** *of S is*

$$C(S) := H^*(P, P \cap Q)$$

where (P, Q) is an index pair in N with respect to Φ.

A priori this index is not well-defined; given a different isolating neighborhood of S, or a different choice of index pair, the index of S could be different. However, the following theorem indicates that this is not the case:

Theorem 9.1. *[32] The cohomological index of Conley type of an isolated invariant set S depends only on S.*

We remark again that the example given in the previous section demonstrates that this property does not hold for multiflows without the admissibility criterion. However, since the index is well-defined with the additional assumption of the admissibility criterion, it follows that this index also has the Ważewski property. Notice that for any admissible multivalued flow, the empty set is an isolating neighborhood and its associated isolated invariant set is also the empty set. Additionally, $(\emptyset, \emptyset)$ is an index pair for the empty set. Thus, we see that

$$C(\emptyset) = \{0, 0, 0, 0, \cdots\}.$$

The converse of this result gives us the Ważewski property; although this property is a direct corollary of Theorem 9.1, we label it as a theorem because of its importance.

Theorem 9.2. *If the cohomological index of Conley type of an isolating neighborhood is not trivial, then the associated isolated invariant set is non-empty.*

Finally, [32] also proves the final crucial property of the Conley index: it is stable under perturbation. To give this statement, we need to formally define a **well-parametrized family of multivalued admissible flows** $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ as a well-parametrized family of multiflows with the additional assumption that each multiflow Φ_λ be an admissible multivalued flow.

Theorem 9.3. *[32] Let $\Lambda \subset \mathbb{R}$ be a compact interval. If*

$$\{\Phi_\lambda\}_{\lambda\in\Lambda}$$

is a well-parametrized family of admissible multivalued flows and N is an isolating neighborhood for Φ_{λ_0}, then

$$C(\mathrm{Inv}(N, \Phi_{\lambda_0})) = C(\mathrm{Inv}(N, \Phi_{\lambda_1}))$$

for λ_1 sufficiently close to λ_0.

The preceding theorems demonstrate that the index developed in [32] retains the fundamental properties of the classic Conley index. However, there are still some open questions about this theory which we would like to answer in the future.

First, we would like to understand what class of differential inclusions may be studied using this theory. Without demanding linear growth, it is not true that the full solution set of basic differential inclusions yields a multivalued admissible flow. Of course this behavior–finite time blowup–is well-known even for flows, but we remark again that the problem is more substantial in this setting. As demonstrated in Example 5.3.1, we see that the resulting set-valued map may not have closed values, and its graph may not be closed. However, by restricting to a compact set–as we have done in the bulk of this work–we avoid these issues. An interesting question, then, is how–or even whether–we can rephrase the admissibility criterion for multiflows over compact spaces. It seems reasonable to expect that the Conley index can be defined for such objects since they retain all of the features necessary for computing the index. Understanding these issues is an important goal for future research.

Moving on from this question we get to a detail more intimately related to the cohomological index itself. If we define an index triple as in Section 9.1, then we get the following unfortunate property:

Property 9.1. *Index triples do not generally exist for the cohomological index of Conley type for multivalued admissible flows.*

Proof. We consider a multivalued admissible flow Φ with non-empty isolated invariant set S. Assume that (A, R) is an attractor-repeller pair decomposition of S and further assume R is a proper subset of S; that is, $S \not\subset R$.

Let N be an isolating neighborhood of S (so $S = \text{Inv}(N)$) and assume that (N_2, N_0) is an index pair in N with respect to φ. Since $\text{Inv}^-(N) \subset \text{int}(N_2, N)$, we see that $S \subset N_2$.

Now let N' be an isolating neighborhood of R and assume that (N_2, N_1) is an index pair in N' with respect to φ. Then $N_2 \subset N'$, and hence $S \subset N'$. Since S is invariant, $S \subset \text{Inv}(N') = R$, giving our contradiction.

□

As was noted earlier, the definition of index pair given in [32] is more restrictive that the classical definition. Here we see that this restriction impedes our ability to find index triples. Also notice that this issue is not directly related to the multivalued nature of the object at hand, but rather is due to the more restrictive definition; if the multivalued admissible flow Φ used in the proof is assumed to be a single-valued flow we run into the same problem. This property is unfortunate because these triples are used in the construction of connection matrices, which allow us to understanding the structure of connecting orbits via linear algebra. Therefore we would like to understand whether or not it is possible to give a less restrictive definition of index pairs and still retain the fundamental properties of the Conley index in this multivalued setting.

With this question in mind, we note that [32] is not the only work in the literature to consider the Conley index in the setting of differential inclusions. In [9], the Conley index is developed for special multivalued flows in a Hilbert space. One advantage of this theory over the one developed in [32] is that the definition of index pairs is the same as in the classical setting, and so we avoid the problem of index triples that we have identified with [32]. However, the additional hypotheses assumed on the differential inclusions in [9] exclude Filippov systems. Essentially, [9] defines a Conley index with all of the desired properties in a multivalued setting, but in a more restricted multivalued setting that does not include the objects we are interested in. However, this adaptation of the index does seem very well suited to the applications in 3.4. Future research should examine whether or not the techniques given can be adapted to our setting.

A version of the Conley index for certain discontinuous vector fields is developed in [6]. This work is much closer to the setting that we desire, but we remark that many additional conditions are placed on the orientation of the vector fields and the nearby smooth systems considered are more limited (the transition functions are assumed to be monotonic, among other things). Moreover, the Conley index theory developed is not quite as robust as the one given in [32]. The isolating neighborhoods considered are less general, and the stability of the index is assumed as a hypothesis, rather than demonstrated. Therefore we are not able to use this index for our goals, but as with [9], further research should examine these techniques and see if they can be strengthened.

Earlier, [20] proves the existence of a well-defined Conley index for classes of differential inclusions which is far closer to those we would like to consider. The index there is defined by passing first to single-valued approximations and using the classical index defined there. However, this approach presents computational difficulties, as the modeller is then forced to demonstrate that the index computed in a single-valued approximation is stable over the whole parameter range of single-valued approximations before passing to the nonsmooth model. Our interest, instead, is in computing the index in the nonsmooth case–where one frequently benefits from the model being composed of relatively simple pieces–and passing from that single computation to any nearby smooth system. The machinery developed in [32] seems closest to that goal, which is the reason that we have focused on that work here.

Before moving on, we emphasize that in each of these papers more structure is assumed than we have done for multiflows. Therefore the attractor-repeller pair decomposition and continuation which we have proven in this thesis may be used in conjunction with any of the indices developed in [32, 20, 9, 6].

9.4 Basic Differential Inclusions with Linear Growth as Admissible Multivalued Flows

It is demonstrate in [32] that the full solution set of a basic differential inclusion $\dot{x} \in F(x)$ is a multivalued admissible flow if we also assume that F is bounded. Our goal in this section is to relax that bounded hypothesis in order to allow linear growth; the result is Theorem 9.5. This relaxation is possible because of work done in the years since [32] was published. Essentially, the proof given in [32] references a theorem analogous to Theorem 9.4 which required that F be bounded. With the updated version of Theorem 9.4 presented here we are able to weaken this constraint. This update is important because it greatly increases the class of differential inclusions which may be studied using the cohomological index of Conley type. In particular, it allows us to identify the Conley index of isolated invariant sets in the Welander model.

We introduce a few definitions and results from [11].

Definition 9.13. *An upper-semicontinuous map $F : W \to \mathcal{P}(Y)$ between topological spaces is said to be* **acyclic** *if $F(w)$ is an acylic set for each $w \in W$.*

Denote the graph of F by $\Gamma_F = \{(w, y) \in W \times Y | y \in F(w)\}$ and let $p_F : \Gamma_F \to W$ and $q_F : \Gamma_F \to W$ be the associated projections. We have the following property:

Property 9.2. *[11, 32.3] If $F : W \to \mathcal{P}(Y)$ is an acyclic map then the projection $p_F : \Gamma_F \to W$ is Vietoris.*

Notice that $F(w) = q_F(p_F^{-1}(w))$. Therefore we get the following corollary:

Corollary 9.2.1. *An acyclic map $F : W \to \mathcal{P}(Y)$ is strongly admissible.*

We now formally state the linear growth bound on F that we will need in order to show that $\dot{x} \in F(x)$ gives rise to an admissible multivalued flow.

Definition 9.14. *A multivalued map $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to have* **linear growth** *if it satisfies Bounding Assumption 5.1; that is, if there is some constant $c > 0$ such that*

$$\sup_{v \in F(x)} |v| \leq c(1 + |x|).$$

For our purposes, it is worth mentioning that if a Filippov system is formed from finitely single-valued maps that all satisfy comparable linear growth bounds, then the multivalued map on the righthand side of that inclusion has linear growth; we simply take the constant c to be the maximum of all such constants from the single-valued pieces. Because this situation is so common in the Filippov systems community, it is worth recording this trivial observation as a formal property.

Property 9.3. *Let $\dot{x} \in F(x)$ be a Filippov system in $\mathbb{R}^n$ that is defined by finitely many single-valued vector fields $\{p_i\}_{i=1}^k$. If there are constants $\{c_i\}_{i=1}^k$ such that*

$$|p_i(x)| \leq c_i(1 + |x|)$$

then F has linear growth.

For the duration of this section, let us fix that $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ satisfies the basic conditions and has linear growth.

For an interval $I \subset \mathbb{R}$, let $C(I, \mathbb{R}^n)$ be the space of continuous functions from I to $\mathbb{R}^n$ with the norm $||f|| = \sup_{t \in I} |f(t)|$. Consider the following sets:

$$S_T(x) = \{\psi \in C([0,T], \mathbb{R}^n) \mid \psi(0) = x, \, \dot{\psi}(t) \in F(\psi(t)) \, a.e.\}$$

Theorem 9.4. *[11, 70.6]*

The set $S_T(x)$ is acyclic and compact for each $x \in \mathbb{R}^n$.

It is worth mentioning that [11, 70.6] actually assumes slightly weaker hypotheses and gives a slightly stronger result than what we have stated here. However, introducing these statements would require introducing a good deal of terminology, and the above result is sufficient for our purposes.

In addition to this theorem we will require the following simple lemma:

Lemma 9.1. *For each $T \in \mathbb{R}$, the multivalued map $S_T : \mathbb{R}^n \to C([0,T], \mathbb{R}^n)$ given by $x \mapsto S_T(x)$ is upper-semicontinuous.*

Proof. Fix an arbitrary $x \in \mathbb{R}^n$. If this result were not true then there would be some $\varepsilon > 0$ such that for each $\delta > 0$ there would be some $\psi_\delta \in S_T(B_\delta(x))$ with $\psi_\delta \notin B_\varepsilon(S_T(x))$. This statement contradicts Lemma 3.4. □

Notice that the direct implication of Theorem 9.4 and Lemma 9.1 is that the map S_T is an acyclic–and hence admissible–mapping.

Lemma 9.2. *[11, 40.6] If $F : W \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ are strongly admissible then the composition $G \circ F$ is also strongly admissible.*

We are now ready to prove the main result of this section. To state this theorem, define the set valued map $\Phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ by

$$(t, x) \mapsto \{\psi(t) \,|\, \psi \in S_T(x)\}.$$

Notice that Φ is the multiflow associated to $\dot{x} \in F(x)$ over $\mathbb{R}^n$.

Theorem 9.5. *Φ is a multivalued admissible flow.*

Proof. Note that the fact that Φ is a multivalued flow–that is, a multiflow with the additional property that $\Phi(t, x) \neq \emptyset$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$–is well known [4, 33, 3, 34]. It remains to be demonstrated that there exists a time $T > 0$ such that the restriction of Φ to the time interval $[0, T]$ is strongly admissible.

Choose any $T > 0$. Define the map $E_T : S_T(\mathbb{R}^n) \times [0, T] \to \mathbb{R}^n$ by

$$(\psi, t) \to \psi(t).$$

We will show that this map is continuous (note that any single-valued continuous map is automatically strongly admissible); when we have done so the proof will be complete since the restriction of Φ to the interval $[0, T]$ equals $E_T \circ S_T$ and the composition of strongly admissible maps is strongly admissible (Lemma 9.2).

Choose an arbitrary $(\psi_0, t_0) \in S_T(\mathbb{R}^n) \times [0, T]$ and a compact neighborhood $K \subset S_T(\mathbb{R}^n)$ of ψ_0. Fix $\varepsilon > 0$. By Lemma 3.5, all maps in K are equicontinuous, and so in particular we will find $\delta > 0$ such that

$$|t - t_0| < \delta \implies |\psi(t) - \psi(t_0)| < \varepsilon/2$$

for all $\psi \in K$.

Further choose V to be a neighborhood of ψ_0 such that

$$\psi \in V \implies ||\psi - \psi_0|| < \varepsilon/2.$$

We can now verify the continuity of E_T. For any $(\psi, t) \in (V \cap K) \times B_\delta(t_0)$, we have the following inequalities:

$$\begin{aligned}|E_T(\psi,t)-E_T(\psi_0,t_0)| &= |\psi(t)-\psi_0(t_0)| \\ &\le |\psi(t)-\psi(t_0)|+|\psi(t_0)-\psi_0(t_0)| \\ &\le \varepsilon/2+||\psi-\psi_0|| \\ &< \varepsilon.\end{aligned}$$

□

This improved result greatly increases the class of differential inclusions which may be analyzed using the techniques in [32]. For Filippov systems in particular, it is generally not the case in applications that the system is actually bounded; however, many common models do have linear growth. We will continue our analysis of one such model–the Welander model–in the next section.

9.5 The Conley Index and Welander's Model

We return, for the final time, to Welander's model. As in each of these sections, we will continue the notation built in the previous Welander's model sections. Recall that the model is given by the well-parametrized differential inclusion $(\dot{x,y}) \in W_\lambda(x,y)$, where W_0 is the piecewise-continuous vector field and W_λ is a continuous single-valued vector field for $\lambda > 0$. For each $\lambda \in [0,1]$ define the map $\Upsilon_\lambda : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$ by

$$\Upsilon_\lambda(t,x,y) = \{(w,z) \in \mathbb{R}^n \,|\, \exists \psi : [0,t] \to \mathbb{R}^n,\, \psi(0) = (x,y),\, \psi(t) = (w,z),\, \dot{\psi(t)} \in W_\lambda(\psi(t)) a.e.\}.$$

For each $\lambda \in [0,1]$, the map W_λ has linear growth. For $\lambda = 0$ this fact follows from the linearity of the vector fields p_i which define the Filippov system and Property 9.3. For $\lambda > 0$ this fact follows again from the linearity of the vector fields p_i and the boundedness of the family of transition functions $\{\gamma_\lambda\}_{\lambda\in(0,1]}$. Therefore, by Theorem 9.5, $\{\Upsilon_\lambda\}_{\lambda\in[0,1]}$ is an a well-parametrized family of multivalued admissible flows. Notice that $\Upsilon_\lambda|_X \equiv \Omega_\lambda$.

In Section 8.5 we identified an isolated invariant set S_λ with attractor-repeller pair decomposition (A_λ, R_λ) for sufficiently small λ. Here we will strengthen that result by giving the index of each of these isolated invariant sets (possibly restricting to smaller λ). We phrase this result as the following theorem:

Theorem 9.6. *For λ sufficiently small, Υ_λ contains an isolated invariant set S_λ with*

$$C(S_\lambda) = \{\mathbb{Z}, 0, 0, \cdots\}.$$

Moreover, there is an attractor-repeller pair decomposition (A_λ, R_λ) of S_λ such that

$$C(A_\lambda) = \{\mathbb{Z}, \mathbb{Z}, 0, \cdots\}$$

and

$$C(R_\lambda) = \{0, 0, \mathbb{Z}, 0, 0, \cdots\}.$$

Notice that if we were able to use the classic definition of an index pair (Definition 9.3) then we would be able to use $(N, N_A, \emptyset)$ as an index triple in order to get this result. Unfortunately, the more stringent definition of an index pair required for the cohomological index of Conley type prevents us from using these sets directly–indeed, as the proof of Property 9.1 demonstrates, index triples do not exist for this attractor-repeller pair decomposition–but it is still possible to find index pairs giving these results.

The construction of these index pairs is very similar to the construction of N and N_A; again we will define certain closed curves by taking advantage of the single-valued crossing region, and these curves will be the boundaries of relevant sets. Before beginning this process, we note that the indexing of points which we identify in these constructions continues from Sections 8.5 and 7.4. Moreover, the fact that we have to identify points each time a curve crosses the discontinuity boundary, combined with the large number of curves which we will have to construct, makes the proofs of these lemmas somewhat long and tedious, and perhaps makes the proofs appear more difficult than they actually are. Ultimately, the idea behind each of these constructions is the same: we follow trajectories and use them to define the boundary curves of isolating neighborhoods.

We prove Theorem 9.6 as the following sequence of lemmas.

Lemma 9.3. *The cohomological index of Conley type of the isolated invariant set S_0 of Υ_0 is $C(S_0) = \{\mathbb{Z}, 0, 0, \cdots\}$.*

Proof. We will ultimately see that $(N, \emptyset)$ is an index pair for S_0, but we first must specify an additional isolating neighborhood M such that $N \subset \text{int}(M)$. We must take this extra step–which we would not typically need to do in single-valued systems–because of the additional restriction placed on index pairs in [32] that the closure of the first element of the pair with the second one removed must be contained in the interior of an isolating neighborhood for the invariant set.

Choose $\mu < 1/2$; we will use the point $(\mu, 0)$ in order to define the boundary of M in a manner that is very similar to how $(1/2, 0)$ was used in order to define the boundary of N. We denote by t_e the first positive time such that $\varphi_2(t_e, \mu, 0)$ returns to the x-axis, and further denote $(x_e, 0) := \varphi_2(t_e, \mu, 0)$; note that $x_e > 3/4$. Conveniently, $(x_e, 0)$ also lies in a crossing region, and so $\Upsilon_0(t_e + t, x_e, 0)$ is uniquely determined by $\varphi_1(t, x_e, 0)$ until this trajectory intersects the x-axis. We now denote by t_f the first positive time such that $\varphi_1(t_f, x_e, 0)$ returns to the x-axis and let $(x_f, 0) := \varphi_1(t_f, x_e, 0)$. Since the x-component of the vector $p_1(x, 0)$ is positive for $x > 3/4 + 15/4 * \varepsilon = \upsilon$, the trajectory of φ_1 cannot return to the x-axis on this interval, and so $x_f \in (1/2, x_b)$; note that the lower bound of this interval is determined by the vertical trajectory of p_1 at $x = 1/2$ and the upper bound comes from the fact that trajectories of φ_1 cannot intersect.

At this point, our method deviates from the one used to define N. We cannot use a straight line to close the curves defined in the previous paragraph as we did in that case because we need $N \subset \text{int}(M)$. Instead, we choose $0 < t_{ff} < t_f$ to be small enough that $\varphi_1(-t, [\mu, x_f] \times \{0\})$ lies in the region $y > 0$ for all $t \in (0, t_{ff}]$; notice that this is clearly possible because of the linearity of p_1. Further note that $\varphi_1(-t_{ff}, [\mu, x_f] \times \{0\})$ is a curve homeomorphic to a closed interval. We define the closed curve C' as the union of the curves $\varphi_1((-t_{ff}, 0), \mu, 0)$, $\varphi_2([0, t_e), \mu, 0)$, $\varphi_2([0, t_f - t_{ff}), x_e, 0)$, and $\varphi_1(-t_{ff}, [\mu, x_f] \times \{0\})$. We now define M to be the curve C' and its interior. We can verify that M is also an isolating neighborhood in a manner that is almost identical to proving that N was an isolating neighborhood.

It is clear that N and $\emptyset$ are strongly positively invariant with respect to $\Upsilon_0|_M$, that $\text{Inv}^+(M) \subset M \setminus \emptyset = M$, and $\overline{N \setminus \emptyset} = N \subset \text{int}(M)$. We can further verify that $\text{Inv}^-(M) \subset \text{int}(N, M)$ by examining the backwards time trajectories of points in $M \setminus \text{int}(N)$. Therefore $(N, \emptyset)$ is an index pair for M, and M contains an isolated invariant set S such that

$$C(S) = H^*(N, \emptyset) = \{\mathbb{Z}, 0, 0, \cdots\}.$$

Moreover, since $\mathrm{Inv}(M) \subset \mathrm{Inv}^-(M) \subset N$, we have that $S = S_0$. □

Lemma 9.4. *The attractor A_0 in S_0 of Υ_0 satisfies $C(A_0) = \{\mathbb{Z}, \mathbb{Z}, 0, 0, 0, \cdots\}$.*

Proof. We will not use $(N_A, \emptyset)$ as an index pair for A_0–even though $H^*(N_A, \emptyset) = \{\mathbb{Z}, \mathbb{Z}, 0, 0, 0, \cdots\}$–because the restrictive definition of an index pair would further require finding another isolating neighborhood of A_0 that contained N_A in its interior. Because of the location of the sliding region in Welander's model, it is simpler to construct new sets. However, we are still able to recycle much of the construction of the curve D–one of the closed curves which bounds the annulus N_A–in constructing a new isolating neighborhood M'.

Choose a time $0 < t_{dd} < t_d$ so that $\varphi_1(-t, [x_d, \upsilon] \times \{0\})$ is contained entirely in the upper-half plane for all $t \in (0, t_{dd}]$; it is clearly possible to do so because p_1 is linear. Define the closed curve D' to be the union of the curves $\varphi_1((-t_{dd}, 0), \upsilon, 0)$, $\varphi_2([0, t_c), \upsilon, 0)$, $\varphi_1([0, t_d - t_{dd}), x_c, 0)$, and $\varphi_1(-t_{dd}, [x_d, \upsilon] \times \{0\})$. Let M' be the closed annulus that lies between the curves C' and D', including the curves. It is straightforwards to verify that M' is an isolating neighborhood as we have done with other neighborhoods.

Choose $\rho \in (x_d, \upsilon)$. Follow $\varphi_2(t, \rho, 0)$ until it reaches the x-axis for the first time; label the time t_g and the point $(x_g, 0)$. Then follow $\varphi_1(t, x_g, 0)$ until it reaches the x-axis; label the time t_h and the point $(x_h, 0)$. Define the curve D'' as the union of $\varphi_2([0, t_g), \rho, 0)$, $\varphi_1([0, t_h), x_g, 0)$, and $[x_h, \rho] \times \{0\}$. Further define N'_A to be the annulus bounded by the curves D'' and C, including these curves. Then it is routine to verify that $(N'_A, \emptyset)$ is an index pair in M' because the boundaries of these sets lie completely in crossing regions or away from the discontinuity boundary. Therefore there is an isolated invariant set $A := \mathrm{Inv}(M') \subset \mathrm{int}(N'_A)$ and

$$C(A) = \{\mathbb{Z}, \mathbb{Z}, 0, 0, \cdots\}.$$

Moreover, since $N_A \subset M'$, we have that $\mathrm{Inv}(N_A) \subset \mathrm{Inv}(M') \subset \mathrm{int}(N'_A)$, and therefore $A = A_0$.

□

Lemma 9.5. *The dual repeller R_0 of A_0 in S_0 for Υ_0 satisfies $C(R_0) = \{0, 0, \mathbb{Z}, 0, 0, ...\}$.*

Proof. Let N_1 be the disk bounded by the curve D'' as well as that curve. Let N_2 be the annulus bounded by the curves D'' and D' as well as those curves. It is routine to check that (N_1, N_2) is an index pair in N_1, and so $R := \mathrm{Inv}(N_1)$ satisfies

$$C(R) = \{0, 0, \mathbb{Z}, 0, 0, ...\}.$$

We also notice that $R \subset N_1 \setminus N_2 \subset \mathrm{int}(N_R)$. Then since $N_R \subset N_1$ we have that $R = R_0$. □

Finally, combining these lemmas with Theorems 8.2 and 9.3, we obtain Theorem 9.6.

Navigating Sections on Welander's Model:

- Goals for Welander's Model, Section 1.2
- Introduction to Welander's Model and Bifurcation Analysis, Section 3.3
- Perturbation and Welander's Model, Section 4.7
- Welander's Model as a Multiflow, Section 5.5
- Welander's Model as a Well-Parametrized Family of Multiflows, Section 6.2
- Isolating Neighborhood in Welander's Model, Section 7.4
- Attractor-Repeller Pair Decomposition for Welander's Model, Section 8.5
- The Conley Index and Welander's Model, Section 9.5

Chapter 10

Conclusion and Discussion

In this thesis we have demonstrated that Conley's qualitative theory of isolated invariant sets can–to some extent–be extended to the setting of piecewise-continuous differential equations.

Before extending Conley's theory we first needed to organize these equations in a manner analogous to the introduction of a flow for classical dynamical systems. We borrowed from Filippov's study of piecewise-continuous differential equations and saw that they may be rephrased as differential inclusions, objects which are better known for their use in control theory. The introduction of the multiflow object allows us to organize the solution set of basic differential inclusions, even when the set-valued vector field under consideration does not satisfy any bounding conditions. Passing then to the study of multiflows as objects in their own right, we have seen that the orbits of multiflows are relatively well-behaved, and that we can consider well-parametrized families of multiflows in order to study perturbed differential inclusions.

With this structure developed we are able to demonstrate that multiflows have some of the same qualitative structure identified for flows in [7]. We are able to define isolating neighborhoods for multiflows and demonstrate that they are stable under perturbation. The associated isolated invariant sets may then be decomposed into attractor-repeller pairs and so we can understand the limiting behavior of trajectories in these sets. Moreover, this decomposition continues under small perturbations.

In order to actually define the Conley index itself we saw that more structure is needed than multiflows assume. However, others have developed the Conley index for various classes of differential inclusions [9, 20, 32]. Importantly, multiflows assume less structure than any of these papers demand, and therefore the attractor-repeller pair decomposition and continuation developed here may be applied to any of the inclusions studied by [9, 20, 32]. To this end, we have examined [32] in particular because the development in that manuscript is closest to our goal of studying piecewise-continuous differential equations. Through a simple observation we extended the class of inclusions which may be considered by [32] to all basic differential inclusions with linear growth, and therefore we are able to understand the indices associated to any attractor-repeller pair decomposition in such systems, and continue this result to nearby systems. This process was carried out for Welander's model, hopefully demonstrating the utility of this theory in the setting of conceptual climate models.

Still, many open questions remain about the structure of multivalued dynamical systems. We would like to demonstrate that one aspect of the attractor-repeller pair decomposition not studied in this thesis–the existence of a Lyapunov function–will still hold in our general setting. Similar results have been established for strongly invariant sets in [23, 5], and so these papers should be helpful roadmaps in proving the related result for our (weakly) invariant sets. We would also like to extend the attractor-repeller pair decomposition to the more general Morse decomposition, as is done in [23] for strongly invariant sets.

We introduced the notion of an attractor block in Section 8.4, and mention that an important question surrounding this object remains for multiflows: given an attractor A in S and a neighborhood $U \subset S$ of A, are we guaranteed the existence of an attractor block $B \subset U$ such that $\omega_S(B) = A$?

Turning more to the subject of the Conley index itself, we recall that [32] develops this index for an object called and admissible multivalued flow, which is a multiflow with the addition of an nonempty criterion and an admissibility criterion. Because the full solution set of basic differential inclusions does not generally yield a compact-valued map, and we avoid this issue by restricting to a compact set and removing the nonempty criterion, we would like to know if it is possible to rephrase the admissibility criterion while allowing empty images. If so, we should be able to develop a consistent Conley index for basic differential inclusions without assuming any bounds on the growth.

In Section 9.3 we noted that the index pairs defined in [32] are more restrictive than those given in the classic theory. This strengthened hypothesis prevents the existence of index triples, providing a barrier to the development of connection matrices [29, 30]. Therefore an important avenue of future research would be to understand whether or not these restrictions may be removed in the multivalued setting, or if there is some alternative method of constructing the connection matrices. The index defined in [9] does not have this same restriction on the index pairs and so it is worth pursuing the existence of index triples and connection matrices for this setting (which we note is a more restricted class of inclusions than assumed by [32], and may not be applied to Filippov systems).

On a somewhat different note, we remark that recent work has reframed the structure of attractors in the algebraic language of lattices and posets [17, 18, 19]. This perspective has the advantages of being both extremely general and highly computable. At this time, it is not known that piecewise-continuous differential equations, or differential inclusions more generally, can fit into this abstract framework, but determining that result certainly merits future efforts.

Finally, we remark that we would like to actually use the abstract machinery developed in this thesis in applications. We have applied the theorems given here to one example–Welander's ocean box model–but many other conceptual climate models are also described as Filippov systems [39, 31, 37, 35, 38, 36, 14, 13]. In the future, we would like to examine what Conley index theory can tell us about these–and many other–nonsmooth models.

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Appendix A

Appendix

A.1 Uniform Convergence and the Arzelà-Ascoli Theorem

The basic purpose in including this appendix is to state and prove the Arzelà-Ascoli theorem that is used repeatedly throughout the manuscript. This theorem–as stated and proven here, and used throughout this manuscript–gives sufficient conditions (uniform boundedness and equicontinuity) for a family of functions to contain a uniformly convergent subsequence. This theorem may be substantially generalized, removing hypotheses on the relevant spaces and also giving necessary conditions, but these generalizations are not included here because they are not necessary for our results.

To begin, we recall the definition of uniform convergence:

Definition A.1. *For a topological space S and a metric space M, a sequence of functions $\{f_n : S \to M\}$* **converges uniformly** *to $f : S \to M$ if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$d_M(f_n(x), f(x)) < \varepsilon, \quad n > n_0$$

for all $x \in S$.

Without explicitly referencing the limit f, we sometimes say that such a sequence is **uniformly convergent**.

The primary reason for considering uniform convergence–as opposed to pointwise convergence–is that this convergence preserves continuity.

Property A.1. *The limit of a uniformly convergent sequence of continuous functions is continuous.*

We also give a related definition that is used in the proof of the Arzelà-Ascoli theorem.

Definition A.2. *For a set S and a metric space M, a sequence of functions $\{f_n : S \to M\}$ is considered to be* **uniformly Cauchy** *if for each $\varepsilon > 0$ there is some $N > 0$ such that $d_M(f_n(x), f_m(x)) < \varepsilon$ for all $x \in S$ and $m, n > N$.*

As the following lemma demonstrates, given reasonable hypotheses on S and M, these definitions are actually equivalent. The proof is standard and omitted.

Lemma A.1. *For a topological space S and a complete metric space M, any uniformly Cauchy sequence of functions $\{f_n : S \to M\}$ converges uniformly.*

The following lemma is also used in the proof of the Arzelà-Ascoli theorem; again, this result is standard in analysis courses and the proof is not given here.

Lemma A.2. *Any compact metric space is complete.*

The basic hypotheses of the Arzelà-Ascoli Theorem are *uniform boundedness* and *equicontinuity.* We include these definitions here.

Definition A.3. *Given a set S and a metric space M, a family of functions $\{f : S \to M\}_{f\in\mathcal{F}}$ is said to be* **uniformly bounded** *if there is some $m \in M$ and $0 < K < \infty$ such that*

$$d_M(f(s), m) < K$$

for all $s \in S$.

Definition A.4. *Let P, Q be metric spaces. A family Υ of continuous functions from P to Q is said to be* ***equicontinuous at a point*** *$p_0 \in P$ if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_P(p_0, p) < \delta$ implies that $d_Q(\upsilon(p_0), \upsilon(p)) < \varepsilon$ for each $\upsilon \in \Upsilon$.*

The family Υ is said to be ***equicontinuous*** *if it is equicontinuous at each point.*

The family Υ is said to be ***uniformly equicontinuous*** *if the choice of δ does not depend on the point p_0.*

Theorem A.1 (Arzelà-Ascoli Theorem)**.** *Let Υ be a family of continuous functions from a compact interval $I \subset \mathbb{R}$ to a compact metric space X. If Υ is equicontinuous, then there is a sequence $\{\psi_n : I \to X\}_{n=1}^{\infty} \subset \Upsilon$ which converges uniformly on I.*

Notice that the assumption that X is compact replaces the usual assumption that Υ be uniformly bounded, and also guarantees that X is complete.

Proof. Let $\{t_i\}_{i=1}^{\infty}$ be an enumeration of $I \cap \mathbb{Q}$, the rationals in I. The set of points $\{\psi(t_1)\}_{\psi \in \Upsilon} \subset X$ contains a convergent sequence $\{\psi_{n_1}(t_1)\}_{n_1=1}^{\infty}$ since X is compact. We use this sequence of points in order to identify the sequence of functions $\{\psi_{n_1}\}_{n_1=1}^{\infty} \subset \Upsilon$.

We now consider the sequence of points $\{\psi_{n_1}(t_2)\}_{n_1=1}^{\infty}$; again by the compactness of X, this sequence contains a convergent subsequence $\{\psi_{n_2}(t_2)\}_{n_2=1}^{\infty}$ which we use in order to identify the subsequence of functions $\{\psi_{n_2}\}_{n_2=1}^{\infty} \subset \{\psi_{n_1}\}_{n_1=1}^{\infty} \subset \Upsilon$. Notice that

$$\lim_{n_2 \to \infty} \psi_{n_2}(t_1) = \lim_{n_1 \to \infty} \psi_{n_1}(t_1)$$

Continuing in this way, for each $i \in \mathbb{N}$ we inductively identify a sequence $\{\psi_{n_i}\}_{n_i=1}^{\infty} \subset \Upsilon$ such that

$$\{\psi_{n_1}\}_{n_1=1}^{\infty} \supset \{\psi_{n_2}\}_{n_2=1}^{\infty} \supset \{\psi_{n_3}\}_{n_3=1}^{\infty} \supset \{\psi_{n_4}\}_{n_4=1}^{\infty} \supset \cdots$$

and $\{\psi_{n_k}(t_i)\}_{n_k=1}^{\infty}$ converges for $1 \leq i \leq k$.

We now take the diagonal sequence $\{\psi_n\}_{n=1}^{\infty}$ defined by $\psi_n = \psi_{n_n}$. Notice that $\{\psi_n(t_i)\}_{n=1}^{\infty}$ converges for each $i \in \mathbb{N}$ by construction.

Then for each $\varepsilon > 0$ and $t_k \in I \cap \mathbb{Q}$, there is some $M(\varepsilon, t_k)$ such that

$$d_X(\psi_n(t_k), \psi_m(t_k)) \leq \varepsilon/3$$

for all $n, m > M(\varepsilon, t_k)$.

By the equicontinuity of Υ, for each $t \in I$ there exists some interval $U_t \subset I$ containing t such that

$$d_X(\psi(r), \psi(s)) \leq \varepsilon/3$$

for each $\psi \in \Upsilon$ and $r, s \in U_t$. Because $\{U_t\}_{t \in I}$ forms an open cover of the compact interval I, we may choose a finite subcover which we relabel $\{U_p\}_{p=1}^{P}$.

Choose a rational $t_{k(p)} \in U_p$ for each p and let $K > \max_{1 \leq p \leq P} k(p)$. Then because each $t \in I$ lies in U_p for some $1 \leq p \leq P$, we see that for any $t \in I$

$$\begin{aligned} d_X(\psi_n(t), \psi_m(t)) &\leq d_X(\psi_n(t), \psi_n(t_{k(p)})) + d_X(\psi_n(t_{k(p)}), \psi_m(t_{k(p)})) + d_X(\psi_m(t_{k(p)}), \psi_m(t)) \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \end{aligned}$$

for each $n, m > \max_{1 \leq k(p) \leq K}(M(\varepsilon, t_{k(p)}))$. Therefore the sequence $\{\psi_n\}_{n=1}^{\infty}$ is Cauchy continuous and converges uniformly.

□

The following example shows that the assumption in the Arzèla-Ascoli theorem that the family of functions in question has a compact domain is necessary.

Example A.1.1. Define the function $g : \mathbb{R} \to [0, 1]$ by

$$g(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

Now let $f_n(x) := g(x - n)$. Then the family of functions $\{f_n : \mathbb{R} \to [0, 1]\}_{n=0}^{\infty}$ is uniformly bounded and equicontinuous. Additionally, it converges pointwise to the continuous function $f \equiv 0$. However, this convergence is not uniform; for each n, we have that $|f_n(x) - f(x)| = 1$ for $x > n + 2$.

This property is somewhat problematic for our applications; because we are interested in invariant sets, the domain of the functions that we are concerned with is generally $\mathbb{R}$. However, this issue is not catastrophic. In the example above, we notice that there is still a continuous function which, when restricted to any compact interval, is the uniform limit of the sequence of functions restricted to that compact interval. The following lemma–basically a corollary of the Arzèla-Ascoli theorem, and in some texts actually called the Arzèla-Ascoli theorem–shows that this property is generic in a sense made specific here.

Lemma A.3. *If X is a compact metric space, then for any equicontinuous sequence of* *functions* $\{\psi_k : \mathbb{R} \to X\}_{k=1}^{\infty}$ *there is a continuous function*

$$\psi : \mathbb{R} \to X$$

such that on any compact interval $[a, b] \subset \mathbb{R}$*, there is a subsequence of the restricted sequence*

$$\{\psi_k|_{[a,b]} : [a, b] \to X\}_{k=1}^{\infty}$$

which converges uniformly to $\psi|_{[a,b]}$*.*

Alternatively, this lemma may be phrased as saying that any family of functions which meets the assumed hypotheses is compact under the topology of uniform convergence on compacta.

Proof. The idea of this proof is relatively straightforward but may be obfuscated by indexing, and so we will first sketch the basic intuition. We consider a nested sequence of compact intervals that grow to encompass all of $\mathbb{R}$. On each interval the Arzela-Ascoli theorem gives us a uniformly convergent subsequence that limits to a continuous function defined on that compact interval. In successive steps we take subsequences of the prior subsequences so that the limit function defined on the larger interval agrees with the limit function on any smaller interval. Then we use these limit functions in order to pointwise define the function $\psi(\cdot)$ on all of $\mathbb{R}$ that satisfies the desired requirements.

In more detail, the proof goes as follows:

For all k, the function ψ_k is defined on the interval $[-1, 1]$, and so we can consider the restricted family $\{\psi_k|_{[-1,1]} : [-1, 1] \to X\}_{k=1}^{\infty}$. Since this family of functions is uniformly bounded (the range is X) and equicontinuous, by the Arzela-Ascoli theorem there is some subsequence $\{\psi_{k_i}|_{[-1,1]} : [-1, 1] \to X\}_{i=1}^{\infty}$ which converges uniformly to a continuous function $\psi^1 : [-1, 1] \to X$. To avoid any possible confusion, we note that the superscript used here differentiates functions, and does not denote an exponent.

In the next step, we note that ψ_{k_i} is defined on the interval $[-2, 2]$ for all i, and so we may consider the uniformly bounded and equicontinuous family $\{\psi_{k_i}|_{[-2,2]} : [-2, 2] \to X\}_{i=1}^{\infty}$. As in the last step, this has some subsequence which converges to a continuous function $\psi^2 : [-2, 2] \to X$. Here we note that since $\psi_{k_i}(t) \to x^1(t)$ for $t \in [-1, 1]$, we have that $\psi^2|_{[-1,1]} \equiv \psi^1$.

We iterate this process inductively. That is, at step m we use the subsequence defined in step $(m-1)$ and then pass to another subsequence that converges on the larger interval $[-m, m]$. In this way we get subsequences of $\{\psi_k|_{[-m,m]} : [-m, m] \to X\}_{k=1}^{\infty}$ which converge uniformly to continuous functions $\psi^m : [-m, m] \to X$ and satisfy

$$\psi^m|_{[-q,q]} \equiv x^q$$

for any $q \leq m$.

Finally, we pointwise define the function $\psi(t) := \psi^m(t)$, where here m is taken to be the least integer greater than or equal to t. Clearly ψ is continuous. To see that for any compact interval $[a, b]$ there is a subsequence of the restricted family $\{\psi_k|_{[a,b]} : [a, b] \to X\}_{k=1}^{\infty}$ which converges uniformly to $\psi|_{[a,b]}$, take m large enough that $[a, b] \subset [-m, m]$ and use the subsequence from step m of the induction.

□

A.2 Convex sets and the Caratheòdory Theorem

Filippov's method for studying piecewise-continuous differential equations as differential inclusions involves taking the convex combination of a finite number of vectors. Here we examine a few basic facts about convex sets which we require for this technique. The definitions, results, and proofs in this section are simplified from ones found in [10].

Definition A.5. *A set $C \subset \mathbb{R}^n$ is said to be* **convex** *if for every pair of points, $x, y \in C$, the line segment connecting these points, $\{\alpha x + (1-\alpha)y | \alpha \in [0, 1]\}$, is a subset of C.*

Definition A.6. *A* **convex combination** *of a finite number of points $\{x_i\}_{i=1}^{k} \subset \mathbb{R}^n$ is any point of the form*

$$y = \sum_{i=1}^{k} \alpha_i x_i$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$.

Property A.2. *A set is convex if and only if it contains all possible convex combinations of its points.*

Proof. If a set contains all possible convex combinations of its points then it clearly contains the line segment connecting any two of its points.

For the other direction we require induction. Let K be convex. If $x_1, x_2 \in K$, α_1 and α_2 are both non-negative, and $\alpha_1 + \alpha_2 = 1$, then by definition $\alpha_1 x_1 + \alpha_2 x_2 \in K$. Using this as the basis for our induction, we assume that if $\{x_i\}_{i=1}^k \subset K$, $\alpha_i \geq 0$ and $\sum_{i=1}^k \alpha_i = 1$, then $\sum_{i=1}^k \alpha_i x_i \in K$.

Now consider the convex combination $\sum_{i=1}^{k+1} \lambda_i y_i$ of points in K. Define $\Lambda = \sum_{i=1}^k \lambda_i$; notice that trivially $\sum_{i=1}^k \frac{\lambda_i}{\Lambda} = 1$ and hence $\sum_{i=1}^k \frac{\lambda_i}{\Lambda} y_i \in K$ by our inductive hypothesis. Therefore

$$\sum_{i=1}^{k+1} \lambda_i y_i = (\sum_{i=1}^{k} \lambda_i y_i) + \lambda_{k+1} y_{k+1} = \Lambda(\sum_{i=1}^{k} \frac{\lambda_i}{\Lambda} y_i) + (1-\Lambda) y_{k+1} \in K$$

since $0 \leq \Lambda \leq 1$ and K is convex, proving our result. □

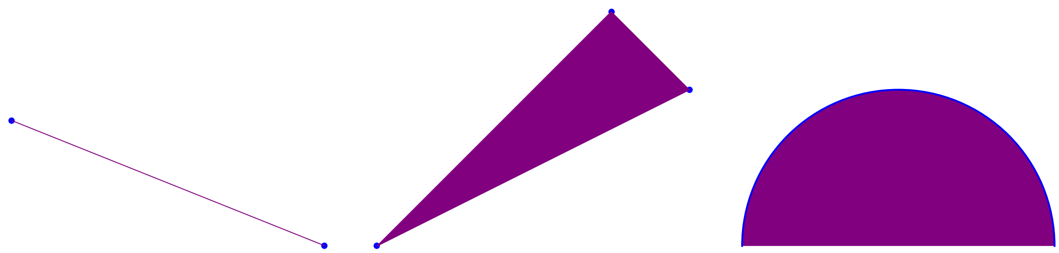
Property A.3. *The intersection of any collection of convex sets is convex.*

Proof. If the intersection is empty or contains only a single point, then the result follows trivially, and so we examine the case where this intersection contains at least two distinct points. Since these two points are contained in each set in the collection, the line between them is also contained in each set in the collection. Therefore that line is in the intersection, and the intersection is convex.

□

Given a set $K \subset \mathbb{R}^n$, we are often concerned with the minimal convex subset containing K. By minimal we mean that this set is contained in any other convex subset containing K. A priori it is unclear that such a set exists or that it is unique, but we can see that both of these statements are true by considering the intersection of *all* convex subsets which contain K. This intersection is well-defined and unique, and is convex by Property A.3. Additionally, it is contained in any other convex subset which contains K by definition. Therefore this intersection is precisely the minimal convex subset containing K, and we give it a name in the following definition:

Definition A.7. *The* **convex hull** *of a set $K \subset \mathbb{R}^n$ is the unique minimal convex set in $\mathbb{R}^n$ which contains K. We denote the convex hull of K by $co(K)$.*



(a) K is two points. (b) K is three points. (c) K is the upper half unit circle.

Figure A.1: The convex hull co(K) (blue and purple) of various sets K (blue).

It is immediate that the convex hull of a bounded set is bounded, and we will later show that the convex hull of a compact set is compact. However, it is not true that the convex hull of a closed set is closed, as the following example demonstrates:

Example A.2.1. Let K be the set

$$\{(x, y) \in \mathbb{R}^2 | y \geq \frac{1}{1 + x^2}\}$$

Then $\text{co}(K) = \{(x, y) \in \mathbb{R}^2 | y > 0\}$.

Property A.4. *The convex hull of a set K is the set of all possible convex combinations of points in K.*

Proof. Since co(K) is convex, by Property A.2 it contains all possible convex combinations of points in K. Conversely, the set of all possible convex combinations of the set K is a convex set; if $\sum_{i=1}^{k} \alpha_i x_i$ and $\sum_{j=1}^{p} \lambda_j y_j$ are convex combinations of points in K, then any point on the line between them may be written as

$$\beta(\sum_{i=1}^{k} \alpha_i x_i) + (1 - \beta)(\sum_{j=1}^{p} \lambda_j y_j) = (\sum_{i=1}^{k} \beta\alpha_i x_i) + (\sum_{j=1}^{p} (1 - \beta)\lambda_j y_j)$$

which is also a convex combination of points in K since $\sum_{i=1}^{k} \beta\alpha_i + \sum_{j=1}^{p} (1 - \beta)\lambda_j = 1$. Therefore the set of all convex combinations of points in K contains the intersection of all convex subsets containing K, which is co(K). □

With these properties of convex sets we are now in a position to state and prove one of the fundamental results of convex analysis, the Carathèodory theorem.

Theorem A.2 (Carathèodory Theorem)**.** *If K is a subset of $\mathbb{R}^n$, then any point in the convex hull of K may be written as the convex combination of at most $n+1$ points in K. That is, if $x \in co(K)$, then there are $\{x_i\}_{i=1}^{n+1} \subset K$, $\alpha_i \geq 0$, and $\sum_{i=1}^{n+1} \alpha_i = 1$ such that*

$$x = \sum_{i=1}^{n+1} \alpha_i x_i$$

Proof. Pick an arbitrary $x \in \text{co}(K)$. By Property A.4 we may write x as a convex combination of points in K:

$$x = \sum_{i=1}^{k} \alpha_i x_i$$

Without loss of generality, assume that $\alpha_i > 0$. If $k \leq n+1$ then there is nothing to prove so assume that $k > n+1$. Therefore the $k-1$ vectors

$$x_2 - x_1, x_3 - x_1, \cdots, x_k - x_1$$

must be linearly dependent and so we can choose $k-1$ real numbers $\mu_2, \cdots, \mu_k$, not all of which are zero, such that

$$\sum_{i=2}^{k} \mu_i (x_i - x_1) = 0$$

Define $\mu_1 := \sum_{i=2}^{k} \mu_i$ and notice that

$$\sum_{i=1}^{k} \mu_i x_i = \mu_1 x_1 + \sum_{i=2}^{k} \mu_i x_i = \sum_{i=2}^{k} (-\mu_i x_1) + \sum_{i=2}^{k} \mu_i x_i = \sum_{i=2}^{k} \mu_i (x_i - x_1) = 0$$

Also, since $\sum_{i=1}^{k} \mu_i = 0$ and not all μ_i are zero, we must have that $\mu_i > 0$ for some i. Using that fact, we know that there is some j such that

$$\frac{\alpha_j}{\mu_j} = \min_{1 \leq i \leq k} \{ (\frac{\alpha_i}{\mu_i}) | \mu_i > 0 \}$$

is a positive number.

We can therefore write

$$x = \sum_{i=1}^{k} \alpha_i x_i - \frac{\alpha_j}{\mu_j} \sum_{i=1}^{k} \mu_i x_i = \sum_{i=1}^{k} (\alpha_i - \frac{\alpha_j}{\mu_j} \mu_i) x_i = \sum_{\substack{i=1 \\ i \neq j}}^{k} (\alpha_i - \frac{\alpha_j}{\mu_j} \mu_i) x_i$$

Notice that $(\alpha_i - \frac{\alpha_j}{\mu_j}\mu_i) \geq 0$ for all $i \in \{1, \cdots, k\}$ and that

$$\sum_{\substack{i=1 \\ i \neq j}}^{k} (\alpha_i - \frac{\alpha_j}{\mu_j} \mu_i) = \sum_{i=1}^{k} (\alpha_i - \frac{\alpha_j}{\mu_j} \mu_i) = \sum_{i=1}^{k} \alpha_i - \frac{\alpha_j}{\mu_j} \sum_{i=1}^{k} \mu_i = 1 - 0$$

Therefore we have written x as the convex combination of $k-1$ points in K. Since our only assumption on k was that $k > n+1$, we can iterate this process until we write x as the convex combination of at most $n+1$ points in K.

$\square$

Our interest in the Caratheòdory theorem is in showing that the convex hull of a compact set is compact. To see this corollary, we rephrase the Caratheòdory theorem in order to give an explicit formula for co(K). First, define the map

$$\zeta : \mathbb{R}^{n(n+1)} \times [0,1]^{n+1} \to \mathbb{R}^n, \qquad (x_1, x_2, \cdots, x_{n+1}, \alpha_1, \cdots, \alpha_{n+1}) \mapsto \sum_{i=1}^{n+1} \alpha_i x_i$$

Clearly ζ is continuous.

Notice that we can identify any combination of $n+1$ points in the set K with a single point in $K \times K \times \cdots \times K = K^{n+1} \subset \mathbb{R}^{n(n+1)}$. Letting

$$\Delta_n := \{(\alpha_1, \cdots, \alpha_{n+1}) | \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\} \subset [0,1]^{n+1}$$

the Caratheòdory theorem tells us that

$$\text{co}(K) = \zeta(K^{n+1} \times \Delta_n)$$

This formula is not particularly useful for computational purposes. What it does, rather, is allow us to view co(K) as the image of a continuous map. In particular, since

Δ_n is a compact set, we see immediately that if K is compact then $\text{co}(K)$ is compact. This corollary is used in reframing piecewise-continuous differential equations as basic differential inclusions and so we label it here:

Property A.5. *The convex hull of a compact set is compact.*