Cyclic Actions in Combinatorial Invariant Theory

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Dedication

In loving memory of Corrine Townsend.

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Chapter 1

Introduction

Invariant theory is the subfield of representation theory that considers groups acting on polynomial rings and analogous structures. The discipline is so named because of the particular emphasis placed on the **invariant polynomials**, those which are fixed under the action of every group element.

The roots of invariant theory can be traced back to Newton in the 1600s. He showed that every symmetric polynomial, that is, those which are unchanged by permuting variables, is a polynomial in the "elementary symmetric polynomials"

$$e_k(x_1, \ldots, x_r) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_r}.$$

However, invariant theory truly began to blossom in the mid 19$^{\text{th}}$ century, alongside the development of solid foundations for representation theory and Coxeter's investigations of reflection groups. Of particular note was the celebrated result due first to Shephard and Todd [ST54], and then uniformly to Chevalley [Che55], which identified the distinguished role that complex reflection groups play in the theory.

Cyclic actions arguably entered the field via the work of Kraśkiewicz and Weyman (published considerably later [KW01]), as well as Springer's seminal paper [Spr74] on regular elements in reflection groups W. These authors showed that a natural cyclic action on the "coinvariant algebra", which for these W is a canonical graded construction of the regular representation, arises in a straightforward manner in the regular representation as well.

The turn of the 21st century has seen renewed interest in cyclic actions in invariant theory, spurred by a paper of Reiner, Stanton, and White's [RSW04] on the cyclic sieving phenomenon, as well as Armstrong, Reiner, and Rhoades's paper [ARR15] on parking spaces. Both of these papers are well-motivated by historical developments. The latter describes, among other things, an "algebraic parking space" with a root-of-unity action directly analogous to the one that Springer identifies on the coinvariant ring. The former may be understood as the foundations for a systematic theory of such actions, that also generalizes Stembridge's "$q = -1$ phenomenon" [Ste94a] to cyclic actions of order $m > 2$.

The main results of this dissertation, in turn, draw inspiration from both of these investigations and use some of the machinery they develop. Indeed, these two different sources largely separate this document into two distinct parts, and so we give a brief outline in hopes of clarifying the path forward. First, Chapter 2 consists of well-established material on group representations and reflection groups. The practicing representation theorist will find much of it familar, except perhaps the rather more specialized final section, which is nevertheless essential background for Chapter 4. The next three chapters are the main material, each of which centers around one novel contribution:

- Theorem 3.3.1 and Proposition 3.3.3, which together show that a natural q-analogue of the rational Schröder polynomial is (separately) unimodal in both its even and odd coefficient sequences.

- Theorem 4.1.2, which, for certain parameters, defines an elementary $(W \times C)$-action on the classical parking space for a Weyl group. When this action is defined, it agrees with the more technical algebraic construction of Armstrong, Reiner, and Rhoades.

- Theorem 5.1.3, a general cyclic sieving result which in particular recovers the $q = -1$ phenomenon for Catalan necklaces, as well as higher-order sieving for a more general family of necklaces.

The first two of these, which involve considerations arising directly from the rational Cherednik algebra, are rather distinct from the more enumerative and group-theoretic part in Chapter 5. Nevertheless, these two parts find common ground in Conjecture 6.1.1, which we consider to be of particular importance. More explicitly, this conjecture states that the parity-unimodality results of Chapter 3 also apply to the generalized Schröder-like polynomials arising from necklaces considered in Chapter 5.

We conclude in Chapter 6 by describing some more speculative observations, which includes the construction of a poset that explains parity-unimodality for the $a = 3$ family of rational Catalan polynomials. Finally, in the Appendix, we include the scripts used in the exceptional-type computations of Chapter 4.

Chapter 2

Background

This chapter serves as a rather terse review of the background in reflection groups necessary for our later constructions. All of this material is known in the literature; with the exception of Section 2.4 it can be found in much greater detail in most texts on representation theory or invariant theory (e.g. [DF03], [Kan01])

Throughout this chapter, we let $\mathbb{F}$ be a field of characteristic zero. Primarily we will focus our attention on $\mathbb{F} = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}$, and very little will be lost by reading with these in mind. Nevertheless, it will be important at times (e.g. in Section 2.3) that we do not restrict exclusively to $\mathbb{F} = \mathbb{C}$.

2.1 Representations and Gradings

Let G be a finite group and $\mathbb{F}$ be a field of characteristic zero. Recall that a **representation** V **(of** G **over** $\mathbb{F}$**)** is an $\mathbb{F}$-vector space together with a group homomorphism $\varphi : G \to \mathrm{GL}(V)$; in other words, an action of G on V by linear automorphisms. Said in yet another way, representations of G are precisely the (left) modules of the group algebra $\mathbb{F}[G]$.

The **character** of a finite-dimensional representation V is the function $\chi^V : G \to \mathbb{F}$

that records the trace of the group elements on V, that is, $\chi^V(g) = \mathrm{tr}(\varphi(g))$. These derive their importance from the following fact: when $\mathbb{F}$ is algebraically closed (in particular, when $\mathbb{F} = \mathbb{C}$), we have $V_1 \cong V_2$ (as $\mathbb{F}[G]$-modules) if and only if $\chi^{V_1} = \chi^{V_2}$; in this case we say that $V_1 \cong V_2$ **as G-representations**. This gives a computationally straightforward algorithm for determining whether two representations are isomorphic. A bit of additional work can be saved in the computations by observing that χ^V is a **class function**, that is $\chi^V(g) = \chi^V(h^{-1}gh)$ for any $g,h \in G$. Thus, two representations are isomorphic if and only if their characters agree on an element from each conjugacy class of G.

We say that V is an **irreducible** representation of G if there is no subspace of V that is stabilized by G. The set of characters χ^λ for irreducible representations λ of G forms a basis for the set of class functions.

We recall several ways to construct representations. Perhaps the first large class of examples are those which only use the $\mathbb{F}$-linear structure incidentally. If X is a (finite) set on which G acts, we may construct a representation on $\mathbb{F}X := \mathrm{span}_{\mathbb{F}}(X)$, the formal $\mathbb{F}$-linear combinations of elements in X, simply by extending the action on X linearly. We will call such representations **permutation representations**, and one easily checks that the characters of permutation representations simply count the number of fixpoints: $\chi^{\mathbb{F}X}(g) = |\{x \in X : g \cdot x = x\}|$.

For any two representations V_1 and V_2 of G, their direct sum $V_1 \oplus V_2$ is a representation of G in the natural way: $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$. Additionally, for a representation V its dual space $V^* = \{f : V \to \mathbb{F} \text{ linear}\}$ is also a representation, with $g \cdot f$ being the functional $v \mapsto g^{-1}v$.

A very useful, if more complicated, construction is as follows: given a ring R, a right R-module M and a left R-module N, the **tensor product** $M \otimes_R N$ is the abelian group generated by the symbols $\{m \otimes n : m \in M, n \in N\}$ with relations

$$(m+m') \otimes n = m \otimes n + m' \otimes n,$$

$$m \otimes (n+n') = m \otimes n + m \otimes n',$$

$$mr \otimes n = m \otimes rn,$$

for all $m, m' \in M$ and $n, n' \in N$, and $r \in R$. In particular, if $R = \mathbb{F}$ is a field then $M \otimes_{\mathbb{F}} N$ is an $\mathbb{F}$-vector space, with a basis given by $\{m_i \otimes n_j\}$, where $\{m_i\}$ and $\{n_j\}$ are bases for M and N, respectively.

Given any two representations V_1 and V_2 of G over $\mathbb{F}$, the tensor product $V_1 \otimes_{\mathbb{F}} V_2$ is also a representation, where G acts diagonally: $g \cdot (v_1 \otimes v_2) = gv_1 \otimes gv_2$. In particular, $V \otimes_{\mathbb{F}} V$ and more generally $T^i V := V^{\otimes i}$ is a representation of G for nonnegative i (where we interpret $V^{\otimes 0}$ as $\mathbb{F}$). Therefore, so too is is the **tensor algebra** $TV := \bigoplus_{i=0}^{\infty} V^{\otimes i}$. These have two natural quotients:

- the **symmetric algebra** $SV := \bigoplus_{i=0}^{\infty} S^i V$, where $S^i V$ is the quotient of $T^i V$ by all relations of the form

$$v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_i = v_1 \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes v_i.$$

  Typically we remove the $\otimes$ symbols in elements of the symmetric algebra and $S^i V$.

- the **exterior algebra** $\wedge V := \bigoplus_{i=0}^{\infty} \wedge^i V$, where $\wedge^i V$ is the quotient of $T^n V$ by all relations of the form

$$v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_i = -(v_1 \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes v_i)$$

  As usual we have assumed that $\mathbb{F}$ has characteristic zero; in particular, more care is needed in characteristic 2. Typically we replace the $\otimes$ symbols with $\wedge$, in elements of the exterior algebra and $\wedge^i V$.

Since the set of relations in both cases are stabilized by G, these quotients are also representations of G. Note that if V is finite-dimensional, then $\wedge V$ is finite-dimensional, since $\wedge^i V = 0$ if $i > \dim_{\mathbb{F}}(V)$.

The tensor, symmetric, and exterior algebras are classical examples of graded algebras. We say that an $\mathbb{F}$-algebra A is **graded** if A has an $\mathbb{F}$-vector space decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ which respects the multiplication: $A_i \cdot A_j \subseteq A_{i+j}$. We say A_i is the i^{th} **graded component**, that nonzero elements f in some A_i are **homogeneous**, and that f has **degree** i, writing $\deg(f) = i$. In this document we will typically, but not always, consider **positively-graded** algebras, which means $A_i = 0$ for all $i < 0$. Finally, we say that a (left) A-module V is **graded** if A is a graded $\mathbb{F}$-algebra and V is an A-module with a vector space decomposition $V = \bigoplus_{i\in\mathbb{Z}} V_i$ such that $A_i \cdot V_j \subseteq V_{i+j}$

Example 2.1.1. The prototypical example of a graded algebra is the **polynomial algebra** $A = \mathbb{F}[\mathbf{x}] := \mathbb{F}[x_1,\ldots,x_r]$ with the usual (coarse) grading: $\deg(x_i) = 1$. In particular this means that all monomials $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ are homogeneous, and are defined to have degree $|\mathbf{a}| := a_1 + \cdots + a_r$. This is closely related to the symmetric algebra: if V is a representation of G and $(x_1,\ldots,x_r)$ is a basis for V^*, then the diagonal representation $\mathbb{F}[\mathbf{x}]$ is naturally isomorphic to SV^*.

Example 2.1.2. An even simpler example is furnished by taking $A = \mathbb{F}$, letting all elements be homogeneous of degree 0. In this case, graded A-modules are simply **graded $\mathbb{F}$-vector spaces**, that is, vector spaces with a distinguished ($\mathbb{Z}$-indexed) decomposition.

We may transport all the basic definitions into the graded setting. A graded A-module $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is a **graded representation** of G if V_i is a representation of G for all $i \in \mathbb{Z}$. Two graded representations $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and $V' = \bigoplus_{i\in\mathbb{Z}} V'_i$ are **isomorphic (as graded**

representations) if V_i is isomorphic to V_i' for each $i \in \mathbb{Z}$. Finite-dimensional graded representations are isomorphic if and only if their **graded characters** $\chi^V(g;t) := \sum_{i\in\mathbb{Z}} \chi^{V_i}(g)t^i$ are equal as Laurent polynomials. Related is the **Hilbert series** of a graded vector space V, defined as $\mathrm{Hilb}(V;t) := \sum_{i\in\mathbb{Z}} \dim_{\mathbb{F}}(V_i)t^i$ when every graded component V_i is finite-dimensional.

2.2 Reflection Groups

For the rest of this chapter, we restrict to the situation where $\mathbb{F}$ is a subfield of $\mathbb{C}$.

Let V be a finite-dimensional $\mathbb{F}$-vector space. Recall that a **reflection** for V is a linear transformation s with finite order whose subspace of fixpoints has codimension 1. Such s will be diagonalizable over $\overline{\mathbb{F}}$ (the algebraic closure of $\mathbb{F}$). Note that v is a fixpoint of s if and only if it is an eigenvector with eigenvalue 1, and so s has exactly one eigenvalue $\lambda_s \neq 1$, which must be contained in $\mathbb{F}$. The order of s is then equal to the order of $\lambda_s \in \mathbb{F}^\times$. In particular if $\mathbb{F}$ is a subfield of $\mathbb{R}$ then λ_s must be -1 and all reflections have order 2, but for other fields, reflections may have larger orders.

Remark. Note that in positive characteristic, reflections need not be diagonalizable over $\overline{\mathbb{F}}$. For instance $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is clearly not diagonalizable, but it fixes a one-dimensional space and has order p (hence, is a reflection) over any field with characteristic p.

Any subgroup $W \leq \mathrm{GL}(V)$ has a set (possibly empty) of reflections, which we denote by R. We say that W is a **reflection group** if it is finite and is generated as a group by this subset R. Moreover, W is a **real** or **complex** reflection group if the underlying field is $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, respectively, and is a **Weyl group** if $\mathbb{F} = \mathbb{Q}$.

Clearly V is a representation of W, which might be called the reflection representation. However, since $\mathbb{F}$ is a subfield of $\mathbb{C}$, we will prefer the technical convenience of saying that the **reflection representation** of W is instead $V \otimes_{\mathbb{F}} \mathbb{C}$. The group W is called **irreducible** if its reflection representation is irreducible.

Example 2.2.1. The group of $a \times a$ permutation matrices forms a subgroup of $\mathrm{GL}_a(\mathbb{Q})$ and thus is a Weyl group because it is generated by its reflections, the transposition matrices. It is not irreducible because $V = \{(x_1, \ldots, x_a) \in \mathbb{C}^a : \sum x_i = 0\}$ is a stable subspace. However, this V is an irreducible representation of S_a and thus the same matrices, considered instead as operators in $\mathrm{GL}(V)$, form an irreducible Weyl group.

Example 2.2.2. Recall that a **monomial matrix** is a matrix with exactly one nonzero entry in each row and column. The set of $a \times a$ monomial matrices, such that all the nonzero entries are n^{th} roots of unity for a fixed n, forms a finite subgroup of $\mathrm{GL}_a(\mathbb{C})$. This turns out to be a complex reflection group; if $n > 1$ it is irreducible, and we have analyzed the $n = 1$ case in the previous example.

These examples also have the property that the standard inner product $\langle u, v \rangle = \sum u_i \overline{v_i}$ is invariant with respect to the action of all the group elements. Not all representations on $\mathbb{C}^a$ have this property, but they do if we are willing to consider other inner products. More precisely, given a representation V of G equipped with an inner product $\langle \cdot, \cdot \rangle$, we say that V is **unitary** if the inner product is **G-invariant**, that is, if $\langle g \cdot u, g \cdot v \rangle = \langle u, v \rangle$ for all $g \in G$. Of course, not every representation of G is unitary, but in fact every representation of G is isomorphic to a unitary representation. Equivalently, a representation V with *any* inner product, and in particular any finite-dimensional representation, admits a G-invariant inner product.

While there are evidently geometric reasons to study reflection groups, there is also a purely algebraic characterization in the language of invariant theory. Given a representation V of any group G, the **ring of invariants** SV^G is the subring of elements in SV which are fixed by all elements of G:

$$SV^G := \{f \in SV : g \cdot f = f \text{ for all } g \in G\}.$$

We may also consider the **coinvariant ring** $SV_G := SV/I_G^+$, where I_G^+ is the ideal generated by all $f \in SV^G$ in which all terms of f have strictly positive degree.

Example 2.2.3. When $G = S_a$ is the symmetric group and V is the (reducible) permutation representation $\mathbb{C}^a$, the invariants are the polynomials which do not change if the variables are permuted. These are in turn generated by the **elementary symmetric polynomials** $e_k(\mathbf{x}) := \sum_{1 \le i_1 < \cdots < i_k \le a} x_{i_1} x_{i_2} \ldots x_{i_k}$ for $1 \le k \le a$, known to be algebraically independent. Hence the ring of invariants is a polynomial ring with a nonstandard grading; indeed, the generator $e_k(\mathbf{x})$ has degree k.

In general, both SV^G and SV_G are graded representations of G. Of course, the graded representation structure on SV^G is only as interesting as the graded vector space structure, since by definition G acts trivially. Nevertheless it is very striking that the *ring* structure of SV^G is so simple for the symmetric group. Amazingly, this simple structure completely characterizes complex reflection groups:

Theorem 2.2.4. *Let V be an r-dimensional $\mathbb{C}$-vector space, and W be any finite subgroup of $\mathrm{GL}(V)$. Then the following are equivalent:*

(a) W is a reflection group,

(b) SV^W is generated by r algebraically independent elements,

(c) $SV_W \cong \mathbb{C}[W]$ as W-representations.

Remark. The theorem as stated here is due to Chevalley [Che55], although Shephard and Todd [ST54] were the first to prove that (a) is equivalent to (b). Additionally, the fact that

(a) implies (b) for real reflection groups W had earlier been conjectured by Todd and proven by Racah (via Coxeter [Cox51]).

Although there is not a unique choice for the r algebraically independent elements of part (b), their degrees $d_1, \ldots, d_r$ depend only on the complex reflection group W. One way to see this is to note that any such choice makes SV^W into a polynomial ring with nonstandard grading. Therefore, the Hilbert series of SV^W is

$$\mathrm{Hilb}(SV^W;t) = \frac{1}{1-t^{d_1}} \cdots \frac{1}{1-t^{d_r}}.$$

Hence the d_i cannot depend on the choice of elements, and to emphasize this invariance, they are often called the **degrees of** W.

Aside from the Hilbert series of SV^W, a number of important numerical invariants can be read off the degrees of W. For instance, the order of W is equal to the product of the degrees. Another invariant which we will need is the *largest* degree; when W is a real reflection group it is called the **Coxeter number** of W and is usually denoted h.

Remark. The above definition of *Coxeter number* is grammatical for all complex reflection groups, but there is a sense in which this definition is only “correct” when W is “well-generated,” so we restrict here to the real case. A lengthy discussion of the well-generated case may be found in Reiner–Ripoll–Stump [RRS17], along with a few comments about the situation for general complex reflection groups (e.g. Remarks 1.11 and 3.5).

2.3 Classification of Crystallographic Root Systems

Shephard and Todd [ST54] gave a simple classification of irreducible complex reflection groups. We will not have need of their complete classification, but it will be convenient to recall here some theory that leads to the classification in the rational case.

Definition 2.3.1. Let V be a real inner product space. A **(crystallographic) root system** **Φ (of V)** is a finite collection of vectors in V such that

(a) $\operatorname{span}_{\mathbb{R}}(\Phi) = V$,

(b) if $\alpha \in \Phi$ then $k\alpha \in \Phi$ if and only if $k = \pm 1$,

(c) $s_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$, where s_α is the reflection about the hyperplane orthogonal to α, that is, $s_\alpha : v \mapsto v - 2\frac{\langle v,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$, and

(d) for any $\alpha, \beta \in \Phi$, the real number $\frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}$ is an integer.

If the dimension of V is r, we say that Φ has **rank** r.

Remark. Often a "root system" is defined to only satisfy the first three conditions, and root systems that also satisfy the integrality condition (d) are called **crystallographic**. For us, however, root systems will always be assumed crystallographic. As we will see below, crystallographic root systems are used to classify Weyl groups; general root systems similarly classify real reflection groups. The Shephard–Todd classification in the complex case has a rather different flavor, although recently a theory of "complex root systems" has emerged as well [BCM18].

There are two routine ways to construct root systems from others. First, a root system $\Phi \subseteq V$ is **isomorphic** to another root system $\Phi' \subseteq V'$ if there is an invertible and orthogonal linear transformation $\psi : V \to V'$ and a scalar $\delta \in \mathbb{R}$ such that $\Phi' = \delta \cdot \psi(\Phi)$. Second, a root system is **irreducible** if there is no nontrivial orthogonal decomposition $V = V_1 \oplus V_2$ and disjoint partition $\Phi = \Phi_1 \cup \Phi_2$, such that Φ_1 is a root system of V_1 and Φ_2 is a root system of V_2. (Here, a trivial decomposition means that V_1 or V_2 is the zero subspace).

Given a root system $\Phi \subseteq V$, the group generated by s_α is finite, and thus is a real reflection group $W(\Phi)$; in fact, the integrality condition ensures that it is a Weyl group. Note that every element of $W(\Phi)$ preserves the inner product on V and if Φ is irreducible then so is $W(\Phi)$. Conversely, every Weyl group W arises in this manner from a root system, and if W is irreducible, that root system may be taken to be irreducible as well.

For each $\alpha \in \Phi$, let H_α be the hyperplane orthogonal to α; equivalently, the fixed space of s_α. Choose a vector v which is not contained in any H_α. The elements of the set $\Phi^+ := \{\alpha \in \Phi : \langle v, \alpha \rangle > 0\}$ are called **positive roots**. There is a unique subset of positive roots called the **simple roots** such that every positive root is a positive integer combination of the simple roots. In particular, the simple roots $\alpha_1, \ldots, \alpha_r$ form a basis of V, which we call the **root basis (with respect to v)**.

Remark. Note that although the simple roots depend on the initial choice of v, any two choices of will yield root bases that differ only by (reindexing and) the action of some $w \in W(\Phi)$.

Theorem-Definition 2.3.2. *Let $e_1, \ldots, e_N$ be the standard basis for $\mathbb{R}^N$. Every irreducible root system is isomorphic to exactly one of the systems described in Figure 2.1, called the **Cartan types**. The first four (families of) types are called **classical**, and the others are called **exceptional**.*

Remark. For all the Cartan types we take V to be the span of the given root basis contained in $\mathbb{R}^N$ for sufficiently large N; in all types except A and E this gives $V = \mathbb{R}^r$.

The following observation then completes the classification of real reflection groups. There are no isomorphisms among the Cartan types (that is, at the level of root systems), and all of their $W(\Phi)$ are distinct except for $W(\mathrm{B}_r) = W(\mathrm{C}_r)$, for $r \geq 3$. For future use, we also record an explicit description of the reflection groups for classical types:

- $W(\mathrm{A}_r)$ is the group of $(r+1) \times (r+1)$ permutation matrices, when considered as operators on $V = \{(x_1, \ldots, x_{r+1}) : \sum x_i = 0\}$ and is isomorphic (as abstract groups) to the symmetric group S_{r+1}. (See Example 2.2.1.)

Cartan type	Rank	Root basis (simple roots)	Coxeter number
A_r	$r \geq 1$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r$	$r+1$
B_r	$r \geq 2$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$, and $\alpha_r = e_r$	$2r$
C_r	$r \geq 3$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$, and $\alpha_r = 2e_r$	$2r$
D_r	$r \geq 4$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$, and $\alpha_r = e_{r-1} + e_r$	$2r-2$
E_r	$r = 6, 7, 8$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$, and $\alpha_r = -\frac{1}{2}(\sum_{i=1}^{3} e_i - \sum_{j=4}^{8} e_j)$	$12, 18, 30$
F_4		$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3$, and $\alpha_4 = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$	12
G_2		$\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_1 - 2e_2 + e_3$	6

Figure 2.1: All irreducible (crystallographic) root systems.

- $W(\mathrm{B}_r)$ is the group of $r \times r$ **signed permutation matrices**, which are the monomial matrices with only 1 and -1 permitted in the nonzero entries. (See Example 2.2.2.) In particular, $W(\mathrm{B}_r)$ is defined over $\mathbb{Q}$, and hence is indeed a Weyl group.

- As noted above, $W(\mathrm{C}_r) = W(\mathrm{B}_r)$.

- $W(\mathrm{D}_r)$ is the subgroup of $r \times r$ signed permutation matrices with an even number of -1s. It is a subgroup of $W(\mathrm{B}_r)$, and so also a Weyl group.

Finally, because they are defined over the rationals, Weyl groups are distinguished among the real reflection groups as those which stabilize a lattice. In particular, for a (crystallographic) root system $\Phi \subseteq V$, we define the **root lattice** $Q(\Phi)$, or Q if there is no risk of confusion, to be the $\mathbb{Z}$-span of Φ; that is,

$$Q := \left\{ \sum_{\alpha \in \Phi} z_\alpha \alpha \in V : z_\alpha \in \mathbb{Z} \text{ for all } \alpha \in \Phi \right\}.$$

Since $W(\Phi)$ by definition stabilizes Φ, it also stabilizes Q. The root basis for V also turns out to be a $\mathbb{Z}$-basis for Q, so we may also write $Q = \operatorname{span}_{\mathbb{Z}}\{\alpha_1, \ldots, \alpha_r\}$.

2.4 Parking Spaces and Very Good Integers

Classically, a **parking function** is a sequence $(a_1, \ldots, a_n)$ of integers $1 \leq a_i \leq n$ such that if $(a'_1, \ldots, a'_n)$ is the same sequence sorted in increasing order, then $a'_i \leq i$. They were first considered and enumerated by Pyke [Pyk59] and independently by Konheim and Weiss [KW66].

Shortly after their introduction, Pollak (via Riordan [Rio69]) gave a different but particularly elegant proof of their enumeration. This proof shows that the natural inclusion of parking functions $(a_1, \ldots, a_n)$ into $(\mathbb{Z}/(n+1)\mathbb{Z})^n$ becomes an S_n-equivariant bijection after passing to the quotient by the all-ones vector, that is, $(\mathbb{Z}/(n+1)\mathbb{Z})^n / \langle (1, \ldots, 1) \rangle$. The quotient is isomorphic to $Q/(n+1)Q$ for (Cartan) type A_{n-1}, and thus begins the study of parking functions in combinatorial representation theory.

Indeed, with Pollak's argument in mind, it is sensible to consider the "finite torus" Q/bQ for any root system Φ and integer b. Because the Weyl group W preserves the root lattice Q, we know that $\mathbb{C}[Q/bQ]$ is a permutation representation of W. Sommers [Som97, Proposition 3.9] showed that for all b coprime to the Coxeter number h (and in fact a slightly larger family than this, described below) the character of this representation has a simple formula: $\chi^{\mathbb{C}[Q/bQ]}(w) = b^{\dim_{\mathbb{C}} \ker(1-w)}$.

For $W = S_a$ (i.e. in type A), Haiman [Hai94, Proposition 2.5.3] proved the existence of a graded representation with graded character

$$\chi_b(w;t) = \frac{\det(1-t^b w)}{\det(1-tw)}$$

if and only if b is coprime to a (which is the Coxeter number of S_a). Such a representation is a graded version of the finite torus in the sense that $\chi_b(w;1) = b^{\dim\ker(1-w)}$; this can be seen writing λ_j for the eigenvalues of w and calculating

$$\lim_{t\to 1}\frac{\det(1-t^b w)}{\det(1-tw)} = \prod_{j=1}^{r}\left(\lim_{t\to 1}\frac{1-t^b\lambda_j}{1-t\lambda_j}\right) = \left(\lim_{t\to 1}\frac{1-t^b}{1-t}\right)^{\left|\{j:\lambda_j=1\}\right|} = b^{\dim_{\mathbb{C}}\ker(1-w)}.$$

The middle equality holds by directly evaluating the limit of the corresponding factor to be 1 whenever $\lambda_j \neq 1$, and $\left|\{j : \lambda_j = 1\}\right| = \dim_{\mathbb{C}}\ker(1-w)$ in the last equality because w is diagonalizable.

In a yet-unpublished paper [IO15], Ito and Okada prove a remarkable classification theorem that unifies these observations, as well as more recent developments in the theory of rational Cherednik algebras (which we will discuss in the following chapter).

Definition 2.4.1. Let W be a complex reflection group. A **graded parking space** for W with **parameter** b is a graded representation with graded character

$$\chi_b(w;t) = \frac{\det(1-t^b w)}{\det(1-tw)}.$$

Similarly, the **parking space** for W with parameter b is a representation with character $\chi_b(w;1)$; that is, $w \mapsto b^{\dim_{\mathbb{C}}\ker(1-w)}$.

A (graded) parking space is unique for each W and b when it exists, in the sense that (graded) representations are determined by their (graded) characters, up to isomorphism.

However, as Haiman's result implies, that existence is nontrivial; the χ_b in the definition of a graded parking space is always a class function, but not always the character of an actual representation. The content of the aforementioned paper by Ito and Okada is to give a complete classification of those b which are parameters for (graded) parking spaces, for all complex reflection groups. Their description is type-dependent, and for brevity we record only the Weyl group cases, which we will use in Chapter 4:

Theorem 2.4.2 (Ito–Okada [IO15])**.** *If W is an irreducible Weyl group, then a (graded) parking space with parameter b exists if and only if b is an integer that is*

- *coprime to the Coxeter number h, if W has type A, E, F, or G; or*
- *odd, if W has type B or D.*

Remark. Curiously, for all complex reflection groups except for the dihedral groups, the parameters b which admit ungraded parking spaces all also admit graded parking spaces. Also, the Ito–Okada conditions on b in the Weyl group case are exactly the (sufficient) type-independent conditions predicted by Sommers.

Remark. In case b satisfies the Ito–Okada conditions for Weyl group W, we will call $\mathbb{C}[Q/bQ]$ the **classical parking space**.

Unfortunately, Ito and Okada's proof does not provide any hints as to a natural construction: they proceed by explicitly expanding χ_b as a weighted sum of characters of irreducible representations, and check whether the coefficients the sum are positive integers. However, there does exist a framework for finding parking spaces "in the wild," essentially proposed by Haiman [Hai94]:

Definition 2.4.3. Let W be an irreducible complex reflection group, and let $\mathbf{x} = (x_1, \ldots, x_r)$ be a basis for V^*. We say that an integer b is **very good (for W)** if there exist elements $(\theta_1, \ldots, \theta_r) \in S^b V^*$ (that is, polynomials in $\mathbb{C}[\mathbf{x}]$ of degree b) such that

- $(\theta_1, \ldots, \theta_r)$ is a **homogeneous sequence of parameters** for SV^*; that is to say, $SV^*/\langle \theta_1, \ldots, \theta_r \rangle$ is finite-dimensional, and

- the linear map $\theta_\bullet : V^* \to S^b V^*$ defined by $x_i \mapsto \theta_i$, is W**-equivariant**; that is to say, $w \cdot \theta_i = \theta_\bullet(w \cdot x_i)$.

In this case, we say that $(\theta_1, \ldots, \theta_r)$ is an **exhibiting hsop** for (the very goodness of) b.

Remark. This terminology "very good" is due to Sommers [Som97].

This is a somewhat elaborate definition, and so it is worth noting that in some cases we have very simple choices of exhibiting hsops.

Example 2.4.4. Recall that the Weyl group of type B acts on $\mathbb{C}^r$ (and $(\mathbb{C}^r)^*$) by signed permutation matrices, and hence it acts on $\mathbb{C}[\mathbf{x}]$ by accordingly permuting coordinates and swapping signs. Now let b be an odd integer, and write $\theta_i = x_i^b$ for all $1 \leq i \leq r$. On one hand $\mathbb{C}[\mathbf{x}]/\langle \theta_1, \ldots, \theta_r \rangle$ is evidently an r^b-dimensional vector space. On the other, if w sends x_i to $\pm x_j$ then

$$w \cdot (\theta_i) = w(x_i^b) = (\pm x_j)^b = (\pm 1)^b x_j^b = \pm x_j^b,$$

which is indeed $\theta_\bullet(w \cdot x_i)$. Notice the last equality requires b to be odd.

This example shows that odd integers are very good for the Weyl groups of type B, and also type D, since $W(\mathrm{D}_r) \leq W(\mathrm{B}_r)$. In addition, Haiman [Hai94, Proposition 2.5.3] gives a construction, only slightly more difficult, for b coprime to r in type A_{r-1}.

In any case, the definition of a very good integer explicitly requires the existence of a particular finite-dimensional quotient of SV^*. That quotient is a graded representation of W, and in fact it is a graded parking space:

Proposition 2.4.5. *Let b be very good for an irreducible complex reflection group W, and $\theta_1, \ldots, \theta_r$ be an exhibiting hsop for b. Write $\Theta = \mathrm{span}_{\mathbb{C}}(\theta_1, \ldots, \theta_r)$; then $SV^*/\langle \Theta \rangle$ is a* *graded parking space. Moreover, we have the following resolution of the quotient $SV^*/\langle\Theta\rangle$ as a graded $\mathbb{C}[S_a]$-module:*

$$0 \leftarrow SV^*/\langle\Theta\rangle \leftarrow SV^* \leftarrow \left(SV^* \otimes_{\mathbb{C}} \wedge^1\Theta\right) \leftarrow \cdots \leftarrow \left(SV^* \otimes_{\mathbb{C}} \wedge^r\Theta\right) \leftarrow 0. \tag{2.1}$$

Proof (sketch). To avoid an extended digression on the requisite commutative algebraic definitions, we do not give a full proof here. These arguments are already in Haiman [Hai94, Proposition 2.5.2] in type A, and in Chmutova–Etingof [CE03, Theorem 2.3]; the latter assumes an additional hypothesis to achieve a more refined module structure.

The key idea is that the Koszul complex $K(\theta_1,\dots,\theta_r)$ is the desired resolution (2.1). The finite-dimensionality condition on the exhibiting hsop, together with the fact that SV^* is Cohen-Macaulay, implies that this complex is exact and hence a resolution. It is a general principle about resolutions (by careful bookkeeping with the rank-nullity theorem) that the alternating sum of the graded characters is zero, and so

$$\chi^{SV^*/\langle\Theta\rangle}(w;t) = \sum_{i=0}^{r}(-1)^i\chi^{SV^*\otimes\wedge^i\Theta}(w;t).$$

A careful eigenvalue computation then shows that the graded character on the right-hand side is in fact $\frac{\det(1-t^bw)}{\det(1-tw)}$, as desired. (In particular, it is at this step that we use the hypothesis that $\Theta \cong V^*$ as W-representations.) □

Hence, we have constructions of every graded parking space in the classical types. In fact, for all irreducible Weyl groups, every integer satisfying the Ito–Okada conditions is very good, but we do not know a simple proof of this fact in the exceptional types. We will thus defer this question, resolving it with a (much) more elaborate construction after discussing the machinery of the rational Cherednik algebra.

Chapter 3

The Rational Cherednik Algebra

The main goal of this chapter is to present a unimodality property enjoyed by the rational Schröder polynomials, on account of the existence of a certain $\mathfrak{sl}_2(\mathbb{C})$-module. We achieve this goal in Section 3.3. To describe this module and its relationship to the rational Schröder polynomials, we first must cut a path through several foundational results in the representation theory of the rational Cherednik algebra.

The literature on this subject is quite large, interconnected, and highly technical. We hope that this (rather opinionated) account will be helpful for newer, combinatorially-minded readers to engage with it. In particular, we have performed explicit calculations in a number of places which most primary sources and expository works have left as exercises.

3.1 Elementary Properties

For any complex reflection group $W \leq GL(V)$, we use the following notation. Let r be the rank of W, let R be the set of all reflections in W, and let $\langle \cdot, \cdot \rangle$ be a W-invariant innner product on V. For each $s \in R$, let H_s be its fixed hyperplane, $\alpha_s \in V^*$ be a functional with

$\ker(\alpha_s) = H_s$ (unique up to scalars), and $\alpha_s^\vee \in V$ be the unique element of $H_s^\perp$ such that $\alpha_s(\alpha_s^\vee) = 2$. Finally, let λ_s be the nontrivial eigenvalue of s, so that we have an explicit formula for s acting on $y \in V$, namely $s(y) = y + \frac{\lambda_s - 1}{2}\alpha_s(y)\alpha_s^\vee$; an analogous formula holds for $s(x) \in V^*$. Finally, in this chapter, all tensor products without a subscript are to be taken over $\mathbb{C}$, and we omit the $\otimes$ symbol in elements even of the tensor algebra.

For a general finite group G and a $\mathbb{C}[G]$-algebra A, the **semidirect product** (or **smash product**) $A \rtimes G$ is the algebra which as a vector space is $A \otimes \mathbb{C}[G]$, and whose product structure given by $(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh$.

Definition 3.1.1. Suppose $c : R \to \mathbb{C}$ is a function invariant on W-conjugacy classes. The **rational Cherednik algebra** $\mathscr{H}_c$ is $(T(V \oplus V^*) \rtimes W)/I_c$, where I_c is the ideal generated by the following relations for all $x, x' \in V^*$ and $y, y' \in V$:

$$\begin{aligned} xx' &= x'x, \\ yy' &= y'y, \\ yx - xy &= x(y) - \sum_{s \in R} c(s) \cdot \alpha_s(y) \cdot x(\alpha_s^\vee) \cdot s. \end{aligned}$$

Remark. The reader familar with this algebra from other sources may be aware that it is typically defined with two parameters: c as well as $t \in \mathbb{R}$. However, these algebras are isomorphic for all $t \neq 0$. The algebra for $t = 0$ has a distinctly different flavor to its representation theory, and will not be treated here; the interested reader may wish to consult the discussion preceeding Theorem 1.7 of Etingof–Ginzburg [EG02].

Notice that in the right-hand side of the last relation, all products are taken in $\mathbb{C}[S_a]$, but for the left-hand side and the first two relations the products are in $T(V \oplus V^*)$.

We may recognize a (non-positive) grading on $T(V \oplus V^*) \rtimes W$ via $\deg(x) = 1$ for all $x \in V^*$ and $\deg(y) = -1$ for all $y \in V$, and $\deg(w) = 0$ for all $w \in W$. This grading makes I_c homogeneous and thus descends to $\mathscr{H}_c$. This gives an explicit, if messy, characterization of all the homogeneous elements in $\mathscr{H}_c$. However, the following PBW-type theorem cleans up this description considerably:

Theorem 3.1.2 (c.f. [EG02, Theorem 1.3])**.** *Let* $\mathbf{y} = (y_1, \ldots, y_r)$ *be a basis for* V *and* $\mathbf{x} = (x_1, \ldots, x_r)$ *be the dual basis for* V^**. Then* $\{\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}w : \mathbf{a}, \mathbf{b} \in \mathbb{Z}^r_{\geq 0}$ *and* $w \in W\}$ *is a basis for* $\mathscr{H}_c$*.*

The result cited by Etingof and Ginzburg is in fact much stronger than this. They prove that the existence of a PBW-type theorem characterizes the rational Cherednik algebras among the class of *symplectic reflection algebras*, which are those whose generators satisfy similar "quadratic" relations. The theorem as quoted above can be achieved much more quickly by using the Dunkl embedding; an exposition of this approach may be found in [Eti07].

An interesting feature of $\mathscr{H}_c$ is that it contains the following "grading" element:

$$H = \sum_{i=1}^{r} x_i y_i + \frac{r}{2} - \sum_{s \in R} \frac{2c(s)}{1-\lambda_s} s. \tag{3.1}$$

The motivation for this name will have to wait for Proposition 3.2.3, which (after some auxiliary constructions have been appropriately defined) is a direct corollary of the following calculation.

Proposition 3.1.3. *We have* $Hz - zH = \deg(z)z$ *for any homogeneous element* $z \in \mathscr{H}_c$ *(with the grading as described above).*

Proof. We verify the claim for the "pure monomials"; that is, $w \in W$, $\mathbf{x}^{\mathbf{a}}$, and $\mathbf{y}^{\mathbf{b}}$. This will comprise the majority of the work; a brief inductive argument will deal with the other basis elements.

Beginning with $w \in W$, we use the definition for H and the relations in $\mathscr{H}_c$:

$$\begin{aligned} wH &= w\sum_{i=1}^{r} x_i y_i + w\frac{r}{2} - \sum_{s\in R}\frac{2c(s)}{1-\lambda_s} ws \\ &= w\left(\sum_{i=1}^{r} x_i y_i\right) w + \frac{r}{2}w - \sum_{s\in R}\frac{2c(s)}{1-\lambda_s}(wsw^{-1})w. \end{aligned}$$

In this way, each term has a w factor on the right, and so we may compare terms to Hw. The middle term is clearly the middle term of Hw. Since R is closed under conjugation and $\frac{2c(s)}{1-\lambda_s}$ is invariant under conjugation, the last term is the last term of Hw. The first terms would also agree if $\sum_{i=1}^{r} x_i y_i$ were W-invariant. This is indeed the case: because $\mathbf{x}$ and $\mathbf{y}$ are dual bases, the natural isomorphism $V \otimes V^* \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ sends $\sum_{i=1}^{r} x_i \otimes y_i \mapsto \mathrm{id}_V$. This isomorphism is in particular W-invariant, and so w commutes with $\sum_{i=1}^{r} x_i \otimes y_i \in V \otimes V^*$; thus it must also commute with its projection $\sum_{i=1}^{r} x_i y_i \in \mathscr{H}_c$. We conclude that $wH = Hw$, and so $Hw - wH = 0 = \deg(w)w$.

We now prove the claim for $\mathbf{y^b}$, proceeding by induction on $|\mathbf{b}|$; a nearly identical argument proves the claim for $\mathbf{x^a}$ as well. Beginning with the base case, we will prove slightly more than needed: let y be an arbitrary element of V (rather than just a basis element y_j). Using the explicit formula for $s(y)$, we have:

$$\begin{aligned} Hy - yH &= \left(\sum_{i=1}^{r} x_i y_i y - y x_i y_i\right) - \sum_{s\in R}\frac{2c(s)}{1-\lambda_s}(sy - ys) \\ &= \sum_{i=1}^{r}(x_i y - y x_i) y_i - \sum_{s\in R}\frac{2c(s)}{1-\lambda_s}(s(y) - y)\cdot s \\ &= \sum_{i=1}^{r} -\left(x_i(y) - \sum_{s\in R} c(s)\alpha_s(y) x_i(\alpha_s^\vee)\cdot s\right) y_i - \sum_{s\in R}\frac{2c(s)}{1-\lambda_s}\cdot\frac{\lambda_s - 1}{2}\alpha_s(y)\alpha_s^\vee \cdot s \\ &= \left(\sum_{i=1}^{r} -x_i(y) y_i + \sum_{s\in R} c(s)\alpha_s(y) x_i(\alpha_s^\vee)\cdot s y_i\right) + \sum_{s\in R} c(s)\alpha_s(y)\alpha_s^\vee \cdot s. \end{aligned}$$

We now recall that $\mathbf{x}$ and $\mathbf{y}$ are dual bases, so that $\sum x_i(v) y_i = v$. Thus the first term is $-y$, and the remaining terms cancel:

$$\begin{aligned} Hy - yH &= -y + \sum_{s\in R} c(s)\alpha_s(y)\cdot s\left(\sum_{i=1}^{r} x_i(\alpha_s^\vee)y_i\right)s + c(s)\alpha_s(y)\alpha_s^\vee s \\ &= -y + \sum_{s\in R} c(s)\alpha_s(y)\cdot\big((-\alpha_s^\vee)s + \alpha_s^\vee s\big), \end{aligned}$$

as desired. The inductive step is more straightforward. Given the pure monomial $\mathbf{y}^{\mathbf{b}}$, choose an index i such that $b_i > 0$, and write $\mathbf{b}' = (b_1, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots b_r)$. Then since all the y_i commute, we apply the inductive hypothesis and then the base case:

$$H\mathbf{y}^{\mathbf{b}} = H\mathbf{y}^{\mathbf{b}'}y_i = \left(-|\mathbf{b}'|\mathbf{y}^{\mathbf{b}'} + \mathbf{y}^{\mathbf{b}'}h\right)y_i = -|\mathbf{b}'|\mathbf{y}^{\mathbf{b}} + \mathbf{y}^{\mathbf{b}'}(y_i + y_iH) = -\big(|\mathbf{b}'|+1\big)\,\mathbf{y}^{\mathbf{b}} + \mathbf{y}^{\mathbf{b}}H.$$

Therefore, $H\mathbf{y}^{\mathbf{b}} - \mathbf{y}^{\mathbf{b}}H = -|\mathbf{b}|\mathbf{y}^{\mathbf{b}}$, completing the inductive step.

Finally, the PBW theorem implies that any homogeneous element of degree j is a sum of elements $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}w$ such that $|\mathbf{a}| - |\mathbf{b}| = j$, and by the above results we compute

$$\begin{aligned} H(\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}w) - (\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}w)H &= (|\mathbf{a}|\mathbf{x}^{\mathbf{a}} + \mathbf{x}^{\mathbf{a}}H)\mathbf{y}^{\mathbf{b}}w - \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}(Hw) \\ &= \Big((|\mathbf{a}|\mathbf{x}^{\mathbf{a}} + \mathbf{x}^{\mathbf{a}}H)\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}H\Big)\,w \\ &= \Big(|\mathbf{a}|\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} + \mathbf{x}^{\mathbf{a}}(H\mathbf{y}^{\mathbf{b}} - \mathbf{y}^{\mathbf{b}}H)\Big)\,w \\ &= \Big(|\mathbf{a}|\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{a}}|\mathbf{b}|\mathbf{y}^{\mathbf{b}}\Big)\,w \\ &= j\cdot\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}w, \end{aligned}$$

as desired. □

Recall that $\mathfrak{sl}_2(\mathbb{C})$ is a three-dimensional semisimple Lie algebra; a certain choice of basis is traditionally denoted by E, F, and H. In any algebra, we say that three elements (E, F, H) form a $\mathfrak{sl}_2(\mathbb{C})$**-triple** if they obey the same commutation relations of $\mathfrak{sl}_2(\mathbb{C})$;

namely, if

$$HE - EH = 2E, \qquad EF - FE = H, \qquad HF - FH = -2F.$$

In particular, note that if a $\mathbb{C}$-algbera contains an $\mathfrak{sl}_2(\mathbb{C})$-triple, this makes it (and any of its modules) into a $\mathfrak{sl}_2(\mathbb{C})$-module.

Proposition 3.1.4. *Suppose that $W \leq \mathrm{GL}(V)$ is a real reflection group, $\mathbf{x} = (x_1, \dots, x_r)$ is an orthonormal basis for V^*, and $\mathbf{y} = (y_1, \dots, y_r)$ is its dual basis. Then (E, F, H) an $\mathfrak{sl}_2(\mathbb{C})$-triple, where H is as defined in (3.1), and*

$$E = -\frac{1}{2}\sum_{i=1}^{a-1} x_i^2, \qquad \textit{and} \qquad F = \frac{1}{2}\sum_{i=1}^{a-1} y_i^2.$$

Moreover, the action of this triple commutes with that of W.

Proof. That H commutes with the action of S_a follows immediately from Proposition 3.1.3, since $w \in W$ is homogeneous of degree 0. For E and F, this follows from the orthormality of the bases $\mathbf{x}$ and $\mathbf{y}$. For instance, since W is real we have that $E(v) = -\frac{1}{2}\langle v, v\rangle$, and thus $\sigma(E) = E$ because the inner product is W-invariant. Notice that this means that E commutes with all group elements in $\mathscr{H}_c$, since $w \cdot E = w(E) \cdot w = E \cdot w$. A completely analogous argument can be made for F.

It remains to be shown that (E, F, H) is an $\mathfrak{sl}_2(\mathbb{C})$-triple. The first and last relations follow immediately from Proposition 3.1.3, since E and F are homogeneous elements of degree 2 and -2, respectively.

For ease of reading, let us define $\sigma_{ij} = \sum_{s \in R} c(s)\alpha_s(y_j)x_i(\alpha_s^\vee)s$. Applying the relations for $\mathscr{H}_c$ one step at a time yields the following expression for $EF - FE$:

$$
\begin{aligned}
y_j x_i &= x_i y_j + \delta_{ij} - \sigma_i, \\
y_j x_i^2 &= x_i^2 y_j + 2\delta_{ij} x_i - (\sigma_{ij} x_i + x_i \sigma_{ij}), \\
y_j^2 x_i^2 &= x_i^2 y_j^2 + 2\delta_{ij}(y_j x_i + x_i y_j) - (y_j \sigma_{ij} x_i + y_j x_i \sigma_{ij} + \sigma_{ij} x_i y_j + x_i \sigma_{ij} y_i), \\
EF - FE &= \frac{1}{2}\sum_{i=0}^{r}(y_i x_i + x_i y_i) - \frac{1}{4}\sum_{i,j=0}^{r}(y_j x_i \sigma_{ij} + y_j \sigma_{ij} x_i + \sigma_{ij} x_i y_j + x_i \sigma_{ij} y_i).
\end{aligned}
$$

In the last expression, we claim that the first summation is H, and the second is 0. The former claim is simple, just apply the relations one more time:

$$
\begin{aligned}
\frac{1}{2}\sum_{i=0}^{r}(y_i x_i + x_i y_i) &= \frac{1}{2}\sum_{i=0}^{r}(2x_i y_i + 1 - \sigma_{ii}) \\
&= \left(\sum_{i=0}^{r} x_i y_i\right) + \frac{r}{2} - \frac{1}{2}\sum_{s\in R} c(s)\left[\sum_{i=1}^{r}\alpha_s(y_i)x_i(\alpha_s^\vee)\right] s.
\end{aligned}
$$

The term in square brackets is indeed 2 because **x** and **y** are dual bases and the definition of $\alpha_s^\vee$. More explicitly, we can use linearity to view this term as α_s applied to $\sum_{i=1}^{r} x_i(\alpha_s^\vee)y_i$, which is $\alpha_s(\alpha_s^\vee) = 2$.

The second summation is somewhat more tedious. Note that by commuting scalars appropriately we have

$$
\sum_{i,j=0}^{r}\sigma_{ij}x_i y_j = \sum_{s\in R} c(s)s\cdot\left(\sum_{i=1}^{r} x_i(\alpha_s^\vee)x_i\right)\left(\sum_{j=1}^{r}\alpha_s(y_j)y_j\right).
$$

An analogous formula holds for the other three terms, where the three factors of the outer summation— the $c(s)s$, the (inner) i-summation, and the j-summation— appear in the appropriate ordering. The i-summation is a scalar multiple of α_s because for any v in the hyperplane H_s, the functional $\sum_{i=1}^{r} x_i(\alpha_s^\vee)x_i$ sends v to $\langle\alpha_s^\vee, v\rangle$, which is zero by definition of $\alpha_s^\vee$. Similarly, the j-summation is a scalar multiple of $\alpha_s^\vee$.

In particular, this means that when we evaluate s on the i-summation or the j-summation we get the same back but with the sign flipped. This finally yields the desired cancellation:

$$\sum_{i,j=0}^{r} \sigma_{ij} x_i y_j = \sum_{i,j=0}^{r} \sigma_{ij}(x_i)\sigma_{ij} y_j = -\sum_{i,j=0}^{r} x_i \sigma_{ij} y_j,$$
$$\sum_{i,j=0}^{r} y_j \sigma_{ij} x_i = \sum_{i,j=0}^{r} y_j \sigma_{ij}(x_i)\sigma_{ij} = -\sum_{i,j=0}^{r} y_j x_i \sigma_{ij}.$$

□

3.2 Constructions of $\mathscr{H}_c$-modules

In this section we describe a construction for some $\mathscr{H}_c$-modules, and recall some more concrete constructions which are equivalent as W-representations. Most of the tools will be analogues of classical tools from Lie theory. Indeed, the analogy turns out to be very robust; for this reason it is traditional in the literature to write $\mathfrak{h}$ where we have written V. We will only be following this analogy in this section, though, and so have opted for our more neutral notation throughout.

Notice that $\mathscr{H}_c$ contains $SV \rtimes \mathbb{C}[W]$ as a subalgebra, and hence is a right $(SV \rtimes \mathbb{C}[W])$-module. Additionally, given any representation λ of W, we may extend it to an left $(SV \rtimes \mathbb{C}[W])$-module by letting each $y \in V$ act by zero. This makes sense of the following definition:

Definition 3.2.1. The **standard module** with respect to λ is $M_c(\lambda) := \mathscr{H}_c \otimes_{SV \rtimes \mathbb{C}[W]} \lambda$, which has a natural left $\mathscr{H}_c$-module structure by (left-)multiplication on the left tensor factor.

Remark. Notice that although $y \in V$ acts on λ by zero, it does not act on $M_c(\lambda)$ by zero, since y must first be commuted past any x's in in the left tensor factor, which introduces additional terms. In fact, elements of V act via "Dunkl operators," i.e. their images in the Dunkl embedding, although we will not need this here.

As a $\mathbb{C}$-vector space, $M_c(\lambda) \cong SV^* \otimes \lambda$; certainly PBW states that is spanned by $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} w \otimes_{\mathbb{C}[W] \rtimes SV} v$ for $v \in \lambda$, but we can ignore the $w \in W$ since it is invertible and so we may simply absorb it into v, and then we may assume $\mathbf{b} = 0$ since otherwise the non-trivial $\mathbf{y}^{\mathbf{b}} \in SV$ factor kills the entire element. In particular, $M_c(\lambda)$ is a graded $\mathscr{H}_c$-module by importing the grading from the polynomial ring: $(M_c(\lambda))_i \cong S^i V^* \otimes \lambda$.

This module $M_c(\lambda)$ happens to be irreducible for all but a countable set of c. When it is not, there is a particular quotient of interest:

Theorem-Definition 3.2.2 (Dunkl–Opdam [DO03] Propositions 2.20 and 2.34)**.** *The $\mathscr{H}_c$-module $M_c(\lambda)$ has a unique maximal submodule $J_c(\tau)$, which is also graded. Let the (graded, irreducible) quotient $M_c(\lambda)/J_c(\tau)$ be denoted by $L_c(\lambda)$.*

We will now restrict our discussion to real reflection groups W, since we will be using the $\mathfrak{sl}_2(\mathbb{C})$-triple defined in Proposition 3.1.4. In this case, $M_c(\lambda)$ and $L_c(\lambda)$ are clearly graded $\mathfrak{sl}_2(\mathbb{C})$-modules since we have the identification $\mathfrak{sl}_2(\mathbb{C}) \subseteq \mathscr{H}_c$.

In general, for a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V, the **weight space** $V_{(j)}$ (of weight j) is the j-eigenspace of H, that is, $V_{(j)} = \{m \in V : H \cdot m = jm\}$. In our situation, the weight spaces turn out to be the same as the graded components up to a shift in degree. Here is a more precise statement. Because c is a class function, the element $\frac{r}{2} - \sum_{s \in R} c(s)s$ is in the center of the group algebra $\mathbb{C}[W]$. Thus, by Schur's lemma it acts as a scalar on any irreducible representation; for a given irreducible representation λ, we denote that scalar by $h_c(\lambda)$.

Proposition 3.2.3. *Let W be a real reflection group and (E,F,H) be the $\mathfrak{sl}_2(\mathbb{C})$-triple defined in Proposition 3.1.4. If λ is an irreducible representation of $W \leq GL(V)$, the weight spaces of $M_c(\lambda)$ (and $L_c(\lambda)$) are also its graded components. More precisely, the*

j^{th} graded component $S^jV \otimes \lambda$ is the weight space of weight $j + h_c(\lambda)$. Moreover, the action of $\mathfrak{sl}_2(\mathbb{C})$ on $M_c(\lambda)$ (and $L_c(\lambda)$) commutes with that of W.

Proof. The commutativity statement here follows directly from the one in Proposition 3.1.4. It remains to show that the weight spaces and graded components coincide; since the homogeneous elements form a spanning set of $M_c(\lambda)$, it suffices to show that they have the claimed weights on $M_c(\lambda)$. In particular, this proves the same statement for $L_c(\lambda)$.

As suggested by the proposition statement, we first identify $M_c(\lambda) \cong S^iV^* \otimes \lambda$, so that the j^{th} graded component is spanned by $\mathbf{x}^{\mathbf{a}} \otimes v$, where $v \in \lambda$ and $|\mathbf{a}| = j$. By Proposition 3.1.3, we have that

$$\begin{aligned} H \cdot \mathbf{x}^{\mathbf{a}} \otimes v &= j\mathbf{x}^{\mathbf{a}} \otimes v + \mathbf{x}^{\mathbf{a}} h \otimes v \\ &= j\mathbf{x}^{\mathbf{a}} \otimes v + \mathbf{x}^{\mathbf{a}} \left(\sum_{i=1}^{r} x_i y_i + \frac{r}{2} - \sum_{s \in R} c(s)s. \right) \otimes v \\ &= j\mathbf{x}^{\mathbf{a}} \otimes v + \left(\sum_{i=1}^{r} x_i y_i \right) \otimes v + \mathbf{x}^{\mathbf{a}} \otimes h_c(\lambda)v \end{aligned}$$

In the definition of H, note that $\frac{2}{1-\lambda_s} = 1$ because W is real: the nontrivial eigenvalue of a real reflection must be -1. Finally, when we pass each y_i across the tensor symbol, it acts as zero on λ and hence kills the term. Thus the summation term vanishes and we conclude that $H \cdot (\mathbf{x}^{\mathbf{a}} \otimes v) = (j + h_c(\lambda))(\mathbf{x}^{\mathbf{a}} \otimes v)$, as desired. □

Finally, we return to our discussion of graded parking spaces. Recall that we discussed a resolution (2.1) which holds for very good parameters. Evidently $b = 1$ is a very good parameter, since we can simply take $\theta_i = x_i$; in this case the resolution is

$$0 \leftarrow \mathbb{C} \leftarrow SV^* \leftarrow SV^* \otimes \wedge^1 V \leftarrow \cdots \leftarrow SV^* \otimes \wedge^r V.$$

Berest, Etingof, and Ginzburg consider at length the modules $L_c(\lambda)$ when $\lambda = \mathbb{C}$ is the trivial representation. Among many other results, they explicitly compute [BEG03, Proposition 2.1] that when c is the constant function $c(s) = \frac{1}{h}$, then $L_{1/h}(\mathbb{C}) = \mathbb{C}$ (recall here, and throughout, that h is the Coxeter number of W). As we saw above, $M_c(\wedge^i\mathbb{C}) = SV^* \otimes \wedge^i V$, which means that we can rewrite (2.1) as a **BGG (Bernstein–Gelfand–Gelfand) resolution** for $L_{1/h}(\mathbb{C})$; that is:

$$0 \leftarrow L_c(\mathbb{C}) \leftarrow M_c(\mathbb{C}) \leftarrow M_c(\wedge^1 V) \leftarrow \cdots \leftarrow M_c(\wedge^r V) \tag{3.2}$$

when $c = \frac{1}{h}$. This rewriting is at first somewhat uncomfortable because we have used the notation of $\mathscr{H}_c$-modules, but we only know that the homomorphisms are graded $\mathbb{C}[W]$-module maps. Fortunately, there is also a fairly quick argument [BEG03, Proposition 2.3] to show that they are indeed $\mathscr{H}_{1/h}$-module maps as well.

They then go on to describe an equivalence of categories between parameters c and $c+1$ for the "Fuss parameters" $c = m + \frac{1}{h}$. In so doing, they show that there are BGG resolutions of $L_{(mh+1)/h}(\mathbb{C})$ for all integers m. Rouquier later used the "Knizhnik–Zamolodchikov functor" to achieve a much more subtle equivalence [Rou08, Corollary 5.13]; the following is a corollary of that work:

Theorem 3.2.4. *Let W be a Weyl group, and b be an integer coprime to the Coxeter number h. The module $L_{b/h}(\mathbb{C})$ admits a BGG resolution:*

$$0 \leftarrow L_{b/h}(\mathbb{C}) \leftarrow M_{b/h}(\mathbb{C}) \leftarrow M_{b/h}(\wedge^1 V) \leftarrow \cdots \leftarrow M_{b/h}(\wedge^r V)$$

In particular, b is very good for W: an appropriate basis for the b^{th} graded component of $J_{b/h}(\mathbb{C})$ serves as an exhibiting hsop.

Remark. Although it uses only a tiny amount of the power of Rouquier's result, it appears likely that the result as stated above (in particular, for all Weyl groups) was not implicit in earlier work. In particular, the methods for proving that $L_c(\mathbb{C})$ is a parking space for various parameters c have generally either used explicit constructions in the classical types (e.g. [CE03]) or produced functors on module categories in a manner similar to Rouquier.

Recall that in section 2.4 we gave constructions in the classical types. We also noted that Ito–Okada requires the very good interegers for exceptional types to be coprime to h. Hence, this equivalence of Rouquier at last completes our characterization of very good integers for Weyl groups.

3.3 The Rational Schröder Polynomials

We will now shift attention away from $L_c(\mathbb{C})$ iteself, and toward its intertwiners with certain other representations. For any nonnegative integer k, any integer b which is coprime to the Coxeter number h, and any representation M of W, we define the **rational generalized W-Kirkman polynomial** $C_b^k(W,M;q)$ to be the following Hilbert series:

$$C_b^k(W,M;t) := t^{-\binom{k}{2}} \operatorname{Hilb}\Big(\operatorname{Hom}_{\mathbb{C}[W]}\big(\wedge^k M, L_{b/h}(\mathbb{C})\big); t\Big).$$

Remark. We give this name to these polynomials because they generalize the "ordinary" rational W-Kirkman polynomial, which is the case $M = V^*$. Setting $k = 0$ then yields the usual **rational W-Catalan polynomials**, which specialize further to the traditional W-Catalan polynomials at $b = h+1$. The additional generality is useful in type A_r and $k > 0$, where in many situations it is more common to consider the **rational Schröder polynomial** which corresponds to the *reducible* reflection representation $M = \mathbb{C}^{r+1} \cong V^* \oplus \mathbb{C}$.

These polynomials have an unusual unimodality property. Recall that a finite or infinite sequence $(\ldots, a_{-1}, a_0, a_1, a_2, \ldots)$ is **unimodal** if there is an integer c such that $a_{i-1} \leq a_i$ for all $i \leq c$, and $a_i \geq a_{i+1}$ for all $i \geq c$. Let us say that a (Laurent) polynomial $f(t) = \sum_i a_i t^i$

with nonnegative coefficients is **parity-unimodal** if both subsequences $(\dots,a_0,a_2,a_4,\dots)$ and $(\dots,a_1,a_3,a_5,\dots)$ are unimodal. Moreover, f is called **symmetric** if $f(t^{-1}) = q^{c'}f(t)$ for some integer c' (when f is a polynomial, we may take $c' = -\deg f$).

Such polynomials naturally arise from $\mathfrak{sl}_2(\mathbb{C})$-representations. Recall that H plays a distinguished role in the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. The irreducible (and finite-dimensional) $\mathfrak{sl}_2(\mathbb{C})$-modules may be characterized by their weight spaces: the weights with nontrivial weight spaces are $\{-\ell,-\ell+2,\dots,\ell\}$, and each such weight space is one-dimensional. In light of this, for an $\mathfrak{sl}_2(\mathbb{C})$-module A we consider the generating function of its the weight space dimensions, $\operatorname{ch}(A) := \sum \dim(A_{(j)})t^j$, called the **formal character**. The semisimplicity of $\mathfrak{sl}_2(\mathbb{C})$ means precisely that every A is the direct sum of irreducible modules, thus $\operatorname{ch}(A)$ is a Laurent polynomial that is symmetric and parity-unimodal about t^0.

With this theory in hand, we easily observe the following.

Theorem 3.3.1. *Let W be a Weyl group, k be any nonnegative integer, b be an integer coprime to h, and M be any $\mathbb{C}$-representation of W. Then the corresponding generalized W-Kirkman polynomial $C_b^k(W,M;t)$ is symmetric and parity-unimodal.*

Proof. This is essentially a corollary of Proposition 3.2.3; the only reason for the additional hypothesis $c = b/h$ is to ensure that $C_b^k(W,M;t)$ is well-defined. In more detail, the actions of W and $\mathfrak{sl}_2(\mathbb{C})$ commute on $L_c(\mathbb{C})$, and so $A := \operatorname{Hom}_{\mathbb{C}[W]}\left(\wedge^k M, L_c(\mathbb{C})\right)$ is a graded $\mathfrak{sl}_2(\mathbb{C})$-module. We conclude that

$$C_m^k(W,M;t) \;=\; t^{-\binom{k}{2}}\operatorname{Hilb}(A;t) \;=\; t^{-\binom{k}{2}+h_{b/a}(\mathbb{C})}\operatorname{ch}(A),$$

and hence is symmetric and parity-unimodal, as desired. □

The interested reader may find a growing body of related research on parity-unimodality. We mention three works in particular: first, Xin and Zhong [XZ20, Conjectures 3 and 4] prove Theorem 3.3.1 when $k=0$, $M=V^*\oplus\mathbb{C}$, and $W=S_a$ for small values of a. Although their work is not as general, it has the advantage of being considerably more elementary than ours. Second, a conjectural generalization of the Catalan case was given by Billey, Konvalinka, and Swanson [BKS20, Conjecture 4.3] that is related to the major index for standard Young tableaux. Finally, Galashin and Lam [GL20] discuss a more refined notion called "(q,t)-unimodality" which they verify for the rational (q,t)-Catalan polynomial.

We conclude with an explicit description of the rational Schröder polynomials, which in particular verifies that they are a "standard" t-analogue of the rational Schröder numbers. Throughout, we write $C^k_{a,b}(t)$ as an abbreviation for $C^k_b(S_a, V\oplus\mathbb{C};t)$.

Definition 3.3.2. Let a be a nonnegative integer and $\alpha=(\alpha_1,\ldots,\alpha_j)$ be a **composition** of n, that is, a sequence of nummbers that sums to n. We denote the **standard t-analogues** of the multinomial coefficients, factorials, and integers as follows:

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_t = \frac{[a]!_t}{[\alpha_1]!_t\cdots[\alpha_j]!_t},$$
$$[n]!_t = [n]_t[n-1]_t\cdots[2]_t[1]_t,$$
$$[n]_t = 1+t+t^2+\cdots+t^{n-1}.$$

Remark. Traditionally the variable q is used, and this is called a "q-analogue," to emphasize certain analogies with geometry and linear algebra over finite fields. However, we will be working with finite fields in a more explicit (and incompatible) manner in Chapter 4. We thus write the variable as t throughout to avoid confusion, even though, admittedly, this change of notation does at times detract from the salience of the terminology.

Proposition 3.3.3. *Let a, b, and k be nonnegative integers which satisfy* $\gcd(a,b)=1$ *and* $0\le k\le a<b$. *Then the rational Schröder polynomial* $C^k_{a,b}(t)$ *has the following product*

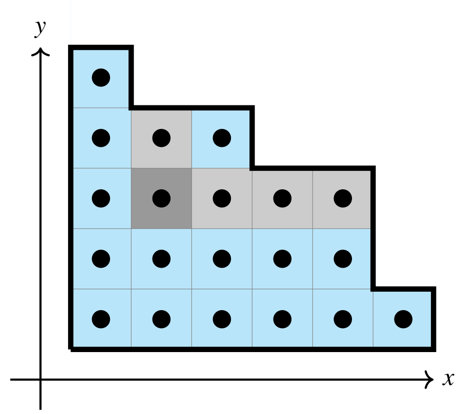
formula:

$$C^k_{a,b}(t) = \frac{1}{[a+b-k]_t}\begin{bmatrix} a+b-k \\ k, a-k, b-k \end{bmatrix}_t.$$

The proof is a computation using the BGG resolution in type A together with [KP90, Theorem 1], corrected and simplified by Molchanov in [Mol92], which we review before performing the computation.

Recall that a list of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ that sums to a is said to be a **partition** of a, written $\lambda \vdash a$, if it is also non-increasing: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A foundational result in the representation theory of S_a is that partitions of a without zero entries are in explicit bijective correspondence with irreducible representations of S_a. We write S^λ for the representation corresponding to λ, and χ^λ for its character. Of particular interest for us are the (dual) irreducible reflection representation $V^* \cong S^{(a-1,1)}$, as well as the permutation representation $\mathbb{C}^a \cong S^{(a-1,1)} \oplus S^{(a)}$, and its higher exterior powers $\wedge^k\mathbb{C}^a$, isomorphic to $S^{(a-k,1^k)} \oplus S^{(a-k+1,1^{k-1})}$.

To state the Molchanov result, the following notation will be useful. Given a partition $\lambda = (\lambda_1, \dots \lambda_n)$, write $(i, j) \in \lambda$ to mean that i and j are positive integers with $j \leq n$ and $i \leq \lambda_j$. This notation is justified by thinking of λ as its Ferrers diagram in French notation:



When $(i, j) \in \lambda$, the **hook-length** $h(i, j)$ is defined as $(\lambda_i - j) + (\lambda'_j - i) + 1$, where λ' is the conjugate partition given by $(i, j) \in \lambda \Leftrightarrow (j, i) \in \lambda'$. The diagram above illustrates this definition, in particular that for $\lambda = (6,5,5,3,1)$ we have $h(2,3) = 5$.

Theorem 3.3.4 (Molchanov [Mol92]). *Let V be the irreducible reflection representation of S_a and λ a partition of a. Then*

$$\sum_{i,j\geq 0} \mathrm{Hom}_{\mathbb{C}[S_a]}\left(S^\lambda, S^iV^* \otimes \wedge^j V^*\right) \cdot t^i z^j \;=\; \frac{1-t}{1+z} \prod_{(i,j)\in\lambda} \frac{t^{i-1}+zt^{j-1}}{1-t^{h(i,j)}}.$$

Proof (of Proposition 3.3.3). From Theorem 3.2.4, we obtain the following identity on graded $\mathbb{C}[S_a]$-characters:

$$\chi_{L_{b/a}(\mathbb{C})} = \sum_{j=0}^{a-1} (-1)^j \chi_{M_{b/a}(\wedge^j\mathbb{C})},$$

From this we conclude that a similar identity holds for each λ-isotypic component:

$$\mathrm{Hilb}\left(\mathrm{Hom}_{\mathbb{C}[S_a]}\left(S^\lambda, L_{b/a}(\mathbb{C})\right);\, t\right) \;=\; \sum_{j=0}^{a-1} (-1)^j\, \mathrm{Hilb}\left(\mathrm{Hom}_{\mathbb{C}[S_a]}\left(S^\lambda, M_{b/a}(\wedge^j(\mathbb{C})\right);\, t\right).$$

As described in Section 3.2, we have $M_{b/a}(\wedge^j\mathbb{C}) \cong SV^* \otimes \wedge^j V^*$ as a graded vector space. For the sake of clarity, we write the right tensor factor as $\wedge^j\Theta$, and so $\Theta \cong V^*$ as an ungraded representation. Moreover, we claim that elements $\theta \in \Theta$ have degree b.

The first nonzero maps of the resolution (3.2) are $SV^*/J_{b/a}(\mathbb{C}) \leftarrow SV^* \leftarrow SV^* \otimes V^*$. Since the kernel of the left map is the ideal $J_{b/a}(\mathbb{C})$, so is the image of the right map. Thus, $J_{b/a}(\mathbb{C}) \subseteq SV^*$ is generated as an $\mathscr{H}_{b/a}$-module by the images of $1 \otimes \theta \in SV^* \otimes \Theta$ under the right map. Since $J_{b/a}(\mathbb{C})$ is also generated by elements of degree b (again from Theorem 3.2.4), this implies that the elements $1 \otimes \theta$ are all of degree b (at least one must have degree b, and by the irreducibility of Θ and the definition of the graded components in $SV^* \otimes \Theta$, the others must have the same degree).

This proves the claim, and so we conclude that

$$\mathrm{Hilb}\left(\mathrm{Hom}_{\mathbb{C}[S_a]}\left(S^\lambda, L_{b/a}(\mathbb{C})\right);\, t\right) \;=\; \sum_{j=0}^{a-1} (-1)^j\, \mathrm{Hilb}\left(\mathrm{Hom}_{\mathbb{C}[S_a]}\left(S^\lambda, SV^* \otimes \wedge^j\Theta\right);\, t\right),$$

which is almost the same form as the left-hand side of Theorem 3.3.4. The differences are that we have lost z^j, picked up a factor of $(-1)^j$, and in the exterior powers, the degree-1 elements of V are now the degree-b elements of Θ. These are not so severe; they simply amount to evaluating Theorem 3.3.4 at $z = -t^b$:

$$\operatorname{Hilb}\Big(\operatorname{Hom}_{\mathbb{C}[S_a]}\left(S^\lambda, L_{b/a}(\mathbb{C})\right);\, t\Big) \;=\; \frac{1-t}{1-t^b}\prod_{(i,j)\in\lambda}\frac{t^{i-1}-t^{j-1+b}}{1-t^{h(i,j)}}\,.$$

To complete the calculation of the rational Schröder polynomials, we apply the above formula for each lambda in the irreducible decomposition of $\wedge^k\mathbb{C}^a$. Namely, these are the hook shapes $\lambda = (a-k, 1^k)$ and, when k is positive, $\lambda = (a-k+1, 1^{k-1})$, each with multiplicity 1. We then add these together and obtain the desired product formula after routine simplification. □

Remark. The skeptical reader may be concerned about how we used our control on the generators of $J_{b/a}$ when arguing that Θ is concentrated in degree b, since this seems to not extend easily to show $SV^* \otimes \wedge^j V^* \cong SV^* \wedge^j \Theta$ as (graded) $\mathscr{H}_{b/a}$-modules for $j > 1$. It is possible, if tedious, to circumvent this difficulty by showing inductively that the images of $1 \otimes x_{i_1} \wedge \cdots \wedge x_{i_j}$ have degree jb by directly calculating $h_c(\wedge^j\Theta)$ and invoking Proposition 3.2.3. In particular, we do not need to assume that the BGG resolution comes from a Koszul complex to complete this calculation. This is important if we are using Rouquier's result, since we need not have such fine control under the equivalence of categories (but see Theorem 2.3 and Section 4 of Chmutova–Etingof [CE03] for an alternative approach).

3.4 Notes

There appears to be some grey area regarding the novelty of some results in Section 3.3. For instance, Theorem 3.3.1 is a direct consequence of the graded $\mathfrak{sl}_2(\mathbb{C})$-module structure on rational parking spaces, although it appears to not have been observed explicitly as such. Perhaps more surprisingly, we were unable to find a reference for the product formula for (even as the definition of) the rational Schröder polynomials, although it is almost certainly folklore.

Despite these questions, Proposition 3.3.3 appears to be new, and justifies the name "Schröder polynomial" for the representation-theoretic definition. Indeed, the fact that the parity-unimodality statement of Theorem 3.3.1 applies to the *product formula* is what we consider to be our major original contribution (c.f. Conjecture 6.1.1).

Chapter 4

Parking Spaces

In this chapter we will finally make good on the first half of this dissertation's title by introducing certain cyclic actions on our two constructions of parking spaces, and then showing that they are equivalent when they are both defined.

We recall that a graded parking space for a reflection group W with parameter b is a graded W-representation with the following class function as its graded character

$$\chi_b(w;t) = \frac{\det(1-t^b w)}{\det(1-tw)}.$$

Similarly, a parking space with parameter b is a W-representation with character $\chi_b(w;1)$, that is, $w \mapsto b^{\dim_{\mathbb{C}} \ker(1-w)}$. Recall also the explicit description of very good integers for irreducible Weyl groups:

- b is very good for an irreducible Weyl group of type A, E, F, or G if and only if it is coprime to the Coxeter number h.
- b is very good for an irreducible Weyl group of type B or D if and only if it is odd.

Finally, all tensor products in this chapter are assumed to be over $\mathbb{Z}$. Notice that this is a different convention than in the previous chapter.

4.1 Cyclic Actions on Parking Spaces

Let W be a rank-r Weyl group, b be an integer that is very good for W, and fix an exhibiting hsop $\theta_1, \ldots, \theta_r \in \mathbb{C}[\mathbf{x}]$. Write $\mathrm{Park}_b(W)$ to be the corresponding graded parking space $\mathbb{C}[\mathbf{x}]/\langle \theta_1, \ldots, \theta_r \rangle$.

We prescribe an action of the cyclic group C (with generator c) of order $b-1$ on $\mathrm{Park}_b(W)$ in a manner following Armstrong, Reiner, and Rhoades [ARR15]. Let ζ be a primitive root $(b-1)^{\text{th}}$ root of unity, and let C act by $c^d : \mathbf{x}^{\mathbf{a}} \mapsto \zeta^{|\mathbf{a}|d} \mathbf{x}^{\mathbf{a}}$. In particular, this is a scalar multiplication on each graded component, and so clearly it commutes with the action of W.

It is perhaps not clear why we should demand that the order of C to be $b-1$. One motivation is to ensure that the map $x_i \mapsto \theta_i$ defined by the exhibiting hsop is not only W-equivariant, but in fact $(W \times C)$-equivariant.

Armstrong, Reiner, and Rhoades have an even more substantial need for this demand, since rather than taking the quotient by $\langle \theta_1, \ldots, \theta_r \rangle$, they are instead taking the quotient by $\langle \theta_1 - x_1, \ldots, \theta_r - x_r \rangle$. Thus the C must have order (dividing) $b-1$ even to stabilize the ideal. (This quotient is also a parking space, ungraded but with better geometric properties than $\mathrm{Park}_b(W)$.) They moreover show that a different permutation representation, the "non-crossing parking space," has a natural combinatorial C-action, and that it is equivalent as $(W \times C)$-representations to $\mathrm{Park}_b(W)$ for all irreducible W aside from the exceptional types E_7 and E_8, which remain conjectural.

However, they left open the problem of finding an appropriate combinatorial action on a different parking space that they call the non-*nesting* parking space (see, for instance, Armstrong–Reiner–Rhoades' Problem 11.4 [ARR15]). This non-nesting parking space is canonically isomorphic to $\mathbb{C}[Q/bQ]$ in the Weyl group case. Etingof [Eti12] conjectured a partial solution: when $b = p$ is prime, the finite torus Q/bQ is in fact a vector space over

$\mathbb{F}_p$, and so we may consider the action of $C \cong \mathbb{F}_p^\times$ via scalar multiplication in the vector space: $c^d : \alpha \mapsto c^d \alpha$.

We may be tempted to extend this to powers of primes, but unfortunately if $q = p^e$ then Q/qQ is not a vector space over $\mathbb{F}_q$. One naïve fix is to modify the classical parking space to enforce this vector space structure.

Definition 4.1.1. Let W be an irreducible Weyl group and q be a prime power $q = p^e$ which is very good for W. Then we write $\text{Park}_q^\sim := Q \otimes \mathbb{F}_q$.

When $q = p$, this definition agrees with the classical parking space because we have $\mathbb{C}[Q/qQ] \cong \mathbb{C}[Q \otimes (\mathbb{Z}/q\mathbb{Z})]$. For general q it has the same number of elements, but by swapping the abelian group, we now have an action of the cyclic group C of order $q-1$: match a generator of C to one of $\mathbb{F}_q^\times$ and then perform scalar multiplication on the left tensor factor. After making this adjustment, we find that there is indeed an isomorphism of $(W \times C)$-representations.

Theorem 4.1.2. *Let W be an irreducible Weyl group, q be a prime power $q = p^e$ which is very good for W, and C be the cyclic group of order $q-1$. Then as $(W \times C)$-representations,*

$$\text{Park}_q(W) \cong \text{Park}_q^\sim(W).$$

This theorem is already rather remarkable when the C-action is ignored, since it implies that $\text{Park}_q^\sim(W) \cong \text{Park}_q'(W)$ as W-representations! Since $\mathbb{F}_{p^e} \cong (\mathbb{Z}/p\mathbb{Z})^e$ as abelian groups, we may think of these two parking spaces as living on "opposite ends" of a family of representations given by $\mathbb{C}[Q \otimes A]$, where A ranges over the abelian groups of order p^e. With this picture in mind, it is natural to suspect that every member of the family is a parking space. This is true, and in fact, the analogous statement holds for all very good b.

Theorem 4.1.3. *Let Q be a root lattice for an irreducible Weyl group W, b be an integer very good for W, and A be any finite abelian group of order b. Then $\mathbb{C}[Q \otimes A]$ is a parking space.*

In light of this theorem, we will call $\mathbb{C}[Q \otimes A]$, for any abelian group A of order b, a **semi-classical parking space** with parameter b. (This is perhaps a bit premature, since we have not proven the theorem yet, but it will occasionally be useful to refer to them by a name.)

We prove both of these theorems by character computation. Thus, we begin by writing down simpler formulas for the characters, with the purpose of reducing the statements to their linear-algebraic cores.

Because the generator $c \in C$ acts on the degree-i polynomials as multiplication by ζ^i, the ungraded $(W \times C_{q-1})$-character evaluations of wc^d simply substitute $t = \zeta^d$ in $\chi(w;t)$. Thus, if w has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ then

$$\chi(wc^d) = \lim_{t \to \zeta^d} \frac{\det(1 - t^b w)}{\det(1 - tw)} = \prod_{j=1}^{r} \left(\lim_{t \to \zeta^d} \frac{1 - t^b \lambda_j}{1 - t\lambda_j} \right).$$

For any $\lambda_j \neq \zeta^{-d}$ we may evaluate the limit for the corresponding factor by direct evaluation, and it contributes a factor of 1 because ζ is a $(b-1)^{\text{th}}$ root of unity. Thus we need only consider the eigenvalues which are equal to ζ^{-d}. In other words, if ζ^{-d} is an eigenvalue of w with multiplicity κ, we have

$$\chi(wc^d) = \left(\left[\frac{1 - t^b \zeta^{-d}}{1 - t\zeta^{-d}} \right]_{t=\zeta^d} \right)^{\kappa} = \left[\frac{1 - (t\zeta^{-d})^b}{1 - t\zeta^{-d}} \right]_{t=\zeta^d}^{\kappa} = b^{\kappa}.$$

Thus, we have that the character evaluation is a power of b. Since w is diagonalizable over $\mathbb{C}$, that power must be $\kappa = \dim_{\mathbb{C}} \ker(\zeta^{-d} - w)$.

Switching to the semi-classical parking space, we unify the settings of the theorems somewhat. (Indeed, the reader will notice a remarkable similarity between the proofs throughout; it seems to us that there should be some common generalization.) Let A be a finite abelian group of order b and suppose that $Z = \langle z \rangle$ is a cyclic group acting $\mathbb{Z}$-linearly on A. In particular, we may always take Z to be the trivial group if we do not wish to consider this cyclic action. Observe that the actions of W and Z commute on $Q \otimes A$, because they they act on the left and right tensor factors, respectively. Thus $\mathbb{C}[Q \otimes A]$ is a permutation $(W \times Z)$-representation; we denote its character by $\tilde{\chi}_A$.

Since it is a permutation representation, the character simply counts fixpoints, which we write as

$$\tilde{\chi}(wz^d) = |\{\alpha \in Q \otimes A : w\alpha = z^{-d}\alpha\}|.$$

These fixpoints form a subgroup of $Q \otimes A$, written explicitly as $\ker_{Q\otimes A}(z^{-d} - w)$. We would like these subgroups to have their orders be "appropriate" powers of b. More precisely if we may embed $Z \leq C$ with $z = c^k$ for some k (that is, if $k|Z| \equiv 0 \bmod b-1$), then we may consider both $\mathrm{Park}_b(W)$ and $\mathbb{C}[Q \otimes A]$ as $(W \times Z)$-representations, and we see that they are isomorphic if and only if

$$|\ker_{Q\otimes A}(z^d - w)| = b^{\dim_{\mathbb{C}} \ker(\zeta^d - w)}$$

for all integers d and all $w \in W$. By this line of reasoning, we reduce the main theorems to the following lemmata.

On one hand, let us restrict to $A = \mathbb{F}_q$, where $q = p^e$ is a prime power that is very good for W. (In particular, this means that p is also very good for W.) In this case there is a natural choice for Z, namely $\mathbb{F}_q^\times$. Since $|Z| = b-1$ in this case, we write $Z = C$. Moreover, $Q \otimes A$ is a vector space over $\mathbb{F}_q$ and $\ker_{Q\otimes A}(c - w)$ is a subspace. Thus it suffices only to consider the dimensions, and so Theorem 4.1.2 is equivalent to this lemma:

Lemma 4.1.4. *Let Q be a root lattice for an irreducible Weyl group W, and suppose that*

$q = p^e$ is a prime power which is very good for W. Then for any $w \in W$ and any integer d, we have

$$\dim_{\mathbb{C}} \ker(\zeta^d - w) = \dim_{\mathbb{F}_q} \ker_{Q\otimes\mathbb{F}_q}(c^d - w)$$

where ζ is a primitive $(q-1)^{th}$ root of unity, and c is a generator of $\mathbb{F}_q^{\times}$.

On the other hand, for any A we may take Z to be the trivial group. The following is then a slight algebraic strengthening of Theorem 4.1.3:

Lemma 4.1.5. *Let Q be a root lattice for an irreducible Weyl group W, and b be very good for W. Then for any abelian group A of order b and any $w \in W$,*

$$\ker_{Q\otimes A}(1-w) \cong A^{\dim_{\mathbb{C}} \ker(1-w)}$$

as abelian groups, where we have written 1 *as shorthand for* $\mathrm{id}\otimes\mathrm{id}$*, and w for* $w\otimes\mathrm{id}$.

Roughly speaking, these lemmata state that the dimensions of the eigenspaces for w over $\mathbb{C}$ agree with the "dimensions" for corresponding "eigenspaces" over $Q\otimes A$. However, even in the finite field case this cannot is not precisely true, since in general w will not be diagonalizable over $\mathbb{F}_q$ (whereas it is over $\mathbb{C}$). Lemma 4.1.4 succeeds despite this fact because it does not detect all the complex eigenspaces, only those for the $(q-1)^{\text{th}}$ roots of unity.

4.2 An Inequality

Our proofs proceed in a largely case-by-case fashion, computing the characters of the representations, but it is worth noting that the equality in Lemma 4.1.4 can be replaced by an inequality in a rather general setting.

Proposition 4.2.1. *Let $q = p^e$ be a prime power. For any $r \times r$ matrix $w \in \mathrm{Mat}_{r\times r}(\mathbb{Z})$, any $\alpha \in \mathbb{C}^\times$ with finite order ℓ dividing $q-1$, and any $\beta \in \mathbb{F}_q^\times$ also with order ℓ:*

$$\dim_{\mathbb{C}} \ker(\alpha - w) \leq \dim_{\mathbb{F}_q} \ker(\beta - \pi(w)),$$

where π is the unique ring map $\mathbb{Z} \to \mathbb{F}_q$, namely reduction mod p.

There are some technicalities in this proof, but heuristically the statement is sensible because $\mathbb{F}_q$ has positive characteristic and so "more elements are zero." This should make it easier, or at least no harder, to find eigenvectors. To make this precise, we first briefly recall the machinery of Smith normal forms, and for use later in the chapter, we work at the generality of arbitrary commutative rings.

Definition 4.2.2. For any commutative ring R and any matrices $X, Y \in \mathrm{Mat}_{r\times s}(R)$, let us say that X and Y are **RC-equivalent (over R)** if there exist matrices $U \in \mathrm{GL}_r(R)$ and $V \in \mathrm{GL}_s(R)$ such that $Y = UXV$.

It is easy to check that this is, indeed, an equivalence relation. We will only work with square matrices, so we set $r = s$ to simplify the statements somewhat, although the theory holds generally. We record in the following proposition some of its elementary properties that we will use later:

Proposition 4.2.3. *Let R be a commutative ring and $X, Y \in \mathrm{Mat}_{r\times r}(R)$.*

(a) If there is a sequence of invertible row and column operations that transform X into Y, then X and Y are RC-equivalent over R.

(b) If X and Y are similar matrices (equivalently, if they represent the same linear transformation on R^r in different bases), then $tI - X$ and $tI - Y$ are RC-equivalent over $R[t]$.

(c) For an R-module M, let $\mathrm{Mat}_{r\times r}(R)$ *act on* M^r *by matrix (left-)multiplicition (i.e. make the identification* $M^r \cong R^r \otimes_R M$*). Then if X and Y are RC-equivalent matrices over R, we have* $\ker_{M^r}(X) \cong \ker_{M^r}(Y)$ *as R-modules.*

Remark. Proposition 4.2.3(a) is the reason for our terminology "RC-equivalent;" RC is short for "row/column."

Recall that when R is a PID, there is a (mostly) canonical choice of representative for RC-equivalence classes:

Theorem 4.2.4. *Let* $X \in \mathrm{Mat}_{r\times r}(R)$ *be a matrix with entries in a principal ideal domain R. Then X is RC-equivalent to a diagonal matrix* $D = \mathrm{diag}(d_1, \ldots, d_r)$ *such that* $d_i | d_{i+1}$ *for each* $1 \leq i < r$*. Moreover, the matrix D is unique up to multiplication of each* d_i *by a unit of R.*

We may consider such forms over any commutative ring R. That is, say that a matrix $X \in \mathrm{Mat}_{r\times r}(R)$ **has a Smith normal form (over** R**)** if it is RC-equivalent to a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_r)$ such that $d_i | d_{i+1}$. In this way, the classical Theorem 4.2.4 says that every matrix over a PID has a Smith normal form. Moreover we say that the matrix equation $X = UDV$, with D as before and $U, V \in \mathrm{GL}_n(R)$, is a **Smith factorization of** X **(over** R**)**. Note that Smith factorizations are not unique, even up to units; the uniqueness statement of the theorem only applies to the diagonal part D.

It is a small class of matrices $M \in \mathrm{Mat}_{r\times r}(R)$ such that $tI - M$ has a Smith normal form in $R[t]$ (see e.g. [MR09, Proposition 8.9]). In general, Weyl group elements need not be among them. For instance, if W is the Weyl group of type B_2, and $w \in W(B_2)$ is the signed permutation with matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, it is easy to check that $tI - w$ has no Smith normal form over $\mathbb{Z}[t]$ by computing cokernels at $t = 1$.

We are now ready to prove Proposition 4.2.1. The plan is as follows: since ζ is a primitive $(q-1)^{\mathrm{th}}$ root of unity, α is a power of ζ. So we would like to compute a Smith factorization of $\alpha - w$ over $\mathbb{Z}[\zeta]$ and then apply π to see that the diagonal elements divisible by p vanish.

This approach fails on numerous levels, which necessitates a number of technicalities at the beginning of the proof. The most fundamental problem is that $\mathbb{Z}[\zeta]$ need not be a PID, so it is not clear that $\alpha - w$ should even have a Smith normal form. However, going all the way to $\mathbb{Q}(\zeta)$, we lose the ability to apply π. We can thread this needle by localizing $\mathbb{Z}$ away from p, but there is one last technical hiccup, which is resolved by completing to the p-adic integers $\mathbb{Z}_p$.

Proof (of Proposition 4.2.1). As observed above, $\alpha - w$ is defined over $\mathbb{Q}(\zeta)$, and so has the same rank after extending to either $\mathbb{C}$ or $\mathbb{Q}_p(\zeta)$. In this proof we use the latter for computations.

Because $\mathbb{Z}_p$ is a complete DVR whose (unique) maximal ideal is generated by p, then $\mathbb{Z}_p[\zeta]$ is also a DVR with maximal ideal $\mathfrak{p}$ generated by p (e.g. [Ser79, Proposition II.2.3]). Notice that the map π sends all elements not in the ideal $p\mathbb{Z}$ to units, and hence we apply universal properties (of localization and completion) to lift it first to $\mathbb{Z}_{(p)} \to \mathbb{F}_q$ and then to $\mathbb{Z}_p \to \mathbb{F}_q$. Finally, by sending ζ to a generator $c \in \mathbb{F}_q^\times$, this extends to a (nonzero) map $\mathbb{Z}_p[\zeta] \to \mathbb{F}_q$, which we also denote by π, whose kernel contains p and hence is $\mathfrak{p}$.

Recall that α is a power of ζ, and hence $\pi(\alpha)$ also has order ℓ. For any $\alpha' \in \mathbb{Q}_p(\zeta)$ with order ℓ, there is an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)$ such that $\sigma(\alpha) = \alpha'$. Moreover, any automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)$ yields a bijection $\ker(\alpha - w) \to \ker(\sigma(\alpha) - w)$, since w is defined over $\mathbb{Q}$. Therefore, any two eigenvalues of w that have the same multiplicative order will also have eigenspaces of the same dimension.

From all of this, we may assume without loss of generality that $\beta = \pi(\alpha)$. Hence it suffices to prove that

$$\dim_{\mathbb{Q}_p(\zeta)} \ker(\alpha - w) \leq \dim_{\mathbb{F}_q} \ker(\pi(\alpha - w)).$$

Let $\alpha - w = UDV$ be the Smith normal form of A over $\mathbb{Z}_p[\zeta]$. In particular, this means U and V are invertible matrices in $\mathrm{GL}_r(\mathbb{Z}_p[\zeta])$. Moreover, the ideal structure of $\mathbb{Z}_p[\zeta]$ means that $D = \mathrm{diag}(1,\ldots,1,p^{e_1},\ldots,p^{e_j},0,\ldots 0)$ for some positive integers $e_1,\ldots,e_j$. Writing κ for the number of zeros on the diagonal of D, we claim that $\dim_{\mathbb{Q}_p(\zeta)} \ker(\alpha - w) = \kappa$ and $\dim_{\mathbb{F}_q} \ker(\beta - \pi(w)) = j + \kappa$.

For the former claim, simply view $\alpha - w = UDV$ as a matrix equation over $\mathbb{Q}_p(\zeta)$. Evidently U and V are invertible in $\mathbb{Q}_p(\zeta)$, and so $\alpha - w$ is RC-equivalent to D. Powers of p are now invertible as well, so $\dim_{\mathbb{Q}_p(\zeta)} \ker(D) = \kappa$, and the claim follows from Theorem 4.2.3(c).

For the latter claim, apply π to the equation $\alpha - w = UDV$. The invertiblity of U and V in $\mathrm{GL}_n(\mathbb{Z}_p[\zeta])$ means that $\det(U)$ and $\det(V)$ are units; that is, not divisible by p and hence not in $\ker\pi$. Since 0 is the only non-invertible element in the field $\mathbb{Z}_p[\zeta]/\mathfrak{p}$, this means that $\pi(U)$ and $\pi(V)$ are still invertible, and so $\pi(\alpha - w)$ is RC-equivalent to $\pi(D) = \mathrm{diag}(1,\ldots,1,0,\ldots,0,0,\ldots,0)$. Again applying Theorem 4.2.3(c) we see that

$$\dim_{\mathbb{Q}_p(\zeta)} \ker(\alpha - w) = \kappa \ \leq \ j + \kappa = \dim_{\mathbb{F}_q} \ker(\beta - \pi(w)),$$

which completes the proof of the lemma. □

A concluding comment: Proposition 4.2.1 together with the classification of Ito and Okada shows that Lemma 4.1.4 in some sense characterizes the primes (or prime powers) which are very good.

Corollary 4.2.5. *Let W be a Weyl group, p be any prime number, and suppose that $q = p^e$ is any prime power. Then for any $w \in W$ and any integer d, we have*

$$\dim_{\mathbb{C}} \ker(\zeta^d - w) \leq \dim_{\mathbb{F}_q} \ker(c^d - w)$$

where ζ is a primitive $(q-1)^{th}$ root of unity, and c is a generator of $\mathbb{F}_q^\times$. Moreover, equality holds for all $w \in W$ and all integers d if and only if q (equivalently, p) is very good for W.

Proof (conditional on Lemma 4.1.4). The inequality comes directly from Proposition 4.2.1. If q is very good for W, equality is precisely the statement of Lemma 4.1.4. Conversely, if q is not very good for W, then it does not satisfy condition (iii) of Theorem 1.4 in [IO15], and thus $w \mapsto \left[\frac{\det(1-t^q w)}{\det(1-tw)}\right]_{t=1}$ cannot be the character of a permutation representation of W. As argued in Section 4.1, we have $\chi(w) = q^{\dim_{\mathbb{C}} \ker(1-w)}$. But $\mathbb{C}[Q \otimes \mathbb{F}_q]$ is manifestly a permutation representation of W for any prime power q. Therefore, there must be some $w \in W$ such that

$$q^{\dim_{\mathbb{C}} \ker(1-w)} = \chi(w) \neq \chi'(w) = q^{\dim_{\mathbb{F}_q} \ker(1-w)},$$

which shows the inequality is strict, as desired. □

4.3 Classical Types

The remainder of this section is devoted to a case-by-case proof of Lemmas 4.1.4 and 4.1.5, and hence of Theorems 4.1.2 and 4.1.3 for the classical Weyl groups. A central component of these proofs are certain partial Smith form calculations; these calculations are considerably more involved in type A, and so we defer it to the last subsection.

4.3.1 Type B

Let us recall the basic data. A root system of type B is isomorphic to one with the following "normalized" simple roots:

$$\mathrm{B}_r: \ (\alpha_1^{\mathrm{B}}, \ldots, \alpha_{r-1}^{\mathrm{B}}, \alpha_r^{\mathrm{B}}) = (e_1 - e_2, \ldots, e_{r-1} - e_r, e_r).$$

Any root lattice for a root system of type B is isomorphic to $Q = \mathbb{Z}^r$. Moreover, by viewing its action in the standard basis, any Weyl group element w acts on Q by a signed permutation matrix. Ignoring the signs, the underlying permutation has a cycle type $\lambda = (\lambda_1, \ldots, \lambda_k)$, which we call the cycle type of w. In particular, any w is conjugate to some w^* whose matrix is block diagonal matrix $\mathrm{diag}(w_1, \ldots, w_k)$, where each w_j is a $\lambda_j \times \lambda_j$ matrix of the form

$$\begin{bmatrix} 0 & & & & \varepsilon_j \\ 1 & 0 & & & \\ & 1 & & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix}.$$

where ε_j is either 1 or -1. We call the tuple $(\varepsilon_1 \lambda_1, \ldots, \varepsilon_k \lambda_k)$ the **signed cycle type** of w. Finally, an integer b is very good for any Weyl group of type B if and only if it is odd.

We are now in position to state the following partial Smith normal form calculation for $tI - w$.

Proposition 4.3.1. *Let V be the irreducible reflection representation of B_r, and $w \in W(B_r)$ be a signed permutation matrix with signed cycle type $\lambda = (\varepsilon_1 \lambda_1, \ldots, \varepsilon_k \lambda_k)$. Then $tI - w$ is RC-equivalent over $\mathbb{Z}[t]$ to the diagonal matrix* $\mathrm{diag}(I_{r-k}, t^{\lambda_1} - \varepsilon_1, \ldots, t^{\lambda_k} - \varepsilon_k)$.

Proof. By Proposition 4.2.3(b), we may assume that w is of the form w^* for the rest of the argument. Each nonzero block of the matrix is of the form

$$\begin{bmatrix} t & & & & -\varepsilon_j \\ -1 & t & & & \\ & -1 & & & \\ & & \ddots & t & \\ & & & -1 & t \end{bmatrix}.$$

Beginning at the second-to-last row $\mathbf{R}_{\lambda_j - 1}$ and successively performing row operations $\mathbf{R}_i \mapsto \mathbf{R}_i + t\mathbf{R}_{i+1}$ up to the top, we obtain

$$\begin{bmatrix} & & & & t^j - \varepsilon_j \\ -1 & & & & t^{j-1} \\ & -1 & & & \\ & & \ddots & & t^2 \\ & & & -1 & t \end{bmatrix}.$$

Cycling the first row down to the bottom and cleaning up the others using column operations, we obtain $\operatorname{diag}(1, \dots, 1, t^{\lambda_j} - \varepsilon_j)$. Doing this for each cycle and then permuting the diagonal entries appropriately completes the proof. □

Corollary 4.3.2. *Lemma 4.1.4 and Lemma 4.1.5 both hold for $W = W(B_r)$.*

Let $W = W(\mathrm{B}_r)$, and Q be the normalized root lattice of type B. Each $w \in W$ acts as a signed permutation matrix on Q with the standard basis, so we may apply Proposition 4.3.1 for each element $w \in W$. Below, we will identify w with this matrix without further comment.

For Lemma 4.1.4, we first tensor with any field $\mathbb{F}$, and thus conclude that $tI - w$ is RC-equivalent to $\operatorname{diag}(1, \dots, 1, t^{\lambda_1} - \varepsilon_1, \dots, t^{\lambda_k} - \varepsilon_k)$ over $\mathbb{F}[t]$. Then evaluating t at any nonzero element of the field, we see that the $\mathbb{F}$-nullity of this matrix is precisely the number of λ_j such that $t^{\lambda_j} = \varepsilon_j$. Thus what remains to be shown is that for any odd prime power q and any integer d, the number of j such that $c^{d\lambda_j} = \varepsilon_j$ in $\mathbb{F}_q$ is equal to the number of j such that $\zeta^{d\lambda_j} = \varepsilon_j$ in $\mathbb{C}$.

Let us say that a cycle of w is **negative** if it has an odd number of -1s, that is, if the corresponding $\varepsilon_j = -1$, and **positive** otherwise. For positive cycles $\varepsilon_j = 1$, the condition in either case is equivalent to $d\lambda_j$ being divisible by $q-1$, because the order of ζ and c are each $q-1$ in $\mathbb{C}^\times$ and $\mathbb{F}_q^\times$ respectively.

For negative cycles, the condition in either case is equivalent to $\frac{d\lambda_j}{q-1}$ being a half-integer. This makes use of the fact that q is odd: $-1 = \zeta^{(q-1)/2} \in \mathbb{C}^\times$, and for any odd q we have $-1 = c^{(q-1)/2} \in \mathbb{F}_q^\times$. Moreover, $(q-1)/2$ is the smallest exponent d' such that $-1 = \zeta^{d'}$ or $-1 = c^{d'}$, so the only d' that satisfy these equations must differ from $(q-1)/2$ by a multiple of $q-1$. Since the condition on j is the same for both $\mathbb{F} = \mathbb{F}_q$ and $\mathbb{F} = \mathbb{C}$, for both positive and negative ε_j, this completes the proof of Lemma 4.1.4.

For Lemma 4.1.5 we evaluate at $t = 1$. We see that $I - w$ is RC-equivalent over $\mathbb{Z}$ to $\operatorname{diag}(I_{r-n-m}, 2I_n, 0_m)$ where n is the number of negative cycles of w and m is the number of positive cycles.

Let A be an abelian group whose order is very good for W. That is, the order of A is odd, so multiplication by 2 is an invertible operator on A. Therefore, we conclude that

$$\ker_{Q\otimes A}(1-w) \cong \ker_{A^r}(I-w) \cong \ker_{A^r} \operatorname{diag}(I_{r-n-m}, 2I_n, 0_m) \cong A^m$$

as abelian groups, with the middle isomorphism due to Proposition 4.2.3(c). Finally, by tensoring instead with $\mathbb{C}$, we find that $m = \dim_{\mathbb{C}} \ker(1-w)$, which concludes the proof of Lemma 4.1.5 for type B.

4.3.2 Types C and D

As before, we recall the basic data. We have the following relationships among Weyl groups of root systems: $W(\mathrm{D}_r) \leq W(\mathrm{C}_r) = W(\mathrm{B}_r)$. An integer b is very good for a Weyl group of type C or D if and only if it is odd. Any root lattice for a root system of these types is isomorphic to $Q = \{(z_1, \ldots, z_r) \in \mathbb{Z}^r : \sum z_i \in 2\mathbb{Z}\}$, and any root system of type C_r is isomorphic to a "normalized" root system compatible with the the one for B_r in the previous section:

$$\begin{aligned} B_r: \quad & (\alpha_1^{\mathrm{B}}, \ldots, \alpha_{r-1}^{\mathrm{B}}, \alpha_r^{\mathrm{B}}) = (e_1 - e_2, \ldots, e_{r-1} - e_r, e_r), \\ C_r: \quad & (\alpha_1^{\mathrm{C}}, \ldots, \alpha_{r-1}^{\mathrm{C}}, \alpha_r^{\mathrm{C}}) = (e_1 - e_2, \ldots, e_{r-1} - e_r, 2e_r). \end{aligned}$$

In particular, after extending scalars to $\mathbb{Q}$, we find the change of basis matrix from type B to type C to be $T := \operatorname{diag}(I_{r-1}, 2)$.

Corollary 4.3.3. *Lemma 4.1.4 and Lemma 4.1.5 both hold for Weyl groups $W = W(\mathrm{C}_r)$ and $W = W(\mathrm{D}_r)$.*

Proof. Each element $w \in W(\mathrm{C}_r)$ acts on Q in the type C root basis, and we identify w with this matrix. In particular, this means that TwT^{-1} is the matrix representing the action of w on $\mathbb{Z}^r$ in the type B root basis. Hence Lemma 4.1.4 follows trivially for type C from statement in type B:

$$\begin{aligned} \dim_{\mathbb{C}} \ker(\zeta^d - w) &= \dim_{\mathbb{C}} \ker(T(\zeta^d - w)T^{-1}) \\ &= \dim_{\mathbb{C}} \ker(\zeta^d - TwT^{-1}) \\ &= \dim_{\mathbb{F}_q} \ker(c^d - TwT^{-1}) \\ &= \dim_{\mathbb{F}_q} \ker(c^d - w), \end{aligned}$$

the first and last equalities holding since multiplication by invertible matrices does not change the dimension of the kernel.

For Lemma 4.1.5, let A be any abelian group with odd order. As in the previous argument, Corollary 4.3.2 states that $\ker_{\mathbb{Z}^r \otimes A}(1 - TwT^{-1}) \cong A^b$. Moreover, observe that multiplication by 2 is an invertible map on A, and so $z \otimes a \mapsto 2z \otimes \frac{1}{2}a$ is a well-defined map $\mathbb{Z}^r \otimes A \to Q \otimes A$. Since this map is clearly invertible, we have $\mathbb{Z}^r \otimes A \cong Q \otimes A$. In addition, T (and T^{-1}) is invertible as a map on $Q \otimes A$, and we hence conclude that $\ker_{Q \otimes A}(1 - w) \cong A^b$

Finally, every root lattice of type D is also isomorphic to Q. Hence the statements follow immediately in type D because $W(\mathrm{D}_r) \leq W(\mathrm{C}_r)$. $\square$

4.3.3 Type A

We normalize the root system of type A_r by explicitly choosing the simple roots to be the following vectors in $\mathbb{R}^{r+1}$:

$$\mathrm{A}_r: \ (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r) = (e_2 - e_1, \ldots, e_r - e_{r-1}, e_{r+1} - e_r).$$

The Weyl groups are $W(\mathrm{A}_r) = S_{r+1}$, and so the conjugacy classes of this Weyl group are determined by the corresponding cycle types. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k > 0$ is the cycle type of w (a partition of $r+1$), then w is conjugate to

$$w_\lambda = (1\ 2\ \cdots\ \Lambda_1)\ (\Lambda_1{+}1\ \ \Lambda_1{+}2\ \cdots\ \Lambda_2)\ \cdots\ (\Lambda_{k-1}{+}1\ \ \Lambda_{k-1}{+}2\ \cdots\ \Lambda_k),$$

where $\Lambda_j = \sum_{i=1}^{j} \lambda_i$. We wish to describe the matrix action of w_λ; for this it will be helpful to have some additional notation. Let $f = \lambda_2'$ be the number of cycles of λ which do not have length 1, and write $\alpha_{(j-1,j)}$ as an abbreviation for the sum of simple roots $\displaystyle\sum_{i=\Lambda_{j-1}+1}^{\Lambda_j - 1} \alpha_i$. Then, observe that w_λ acts on the above basis of $Q(\mathrm{A}_r)$ by

$$w_\lambda \cdot \alpha_i = \begin{cases} -\alpha_{(j-1,j)} & \text{if } i = \Lambda_j - 1 \text{ for some } j \leq f, \\ \alpha_{(j-1,j)} + \alpha_i + \alpha_{i+1} & \text{if } i = \Lambda_j \text{ for some } j < f, \\ \alpha_{(f-1,f)} + \alpha_i & \text{if } i = \Lambda_f, \\ \alpha_i & \text{if } i > \Lambda_f, \\ \alpha_{i+1}. & \text{otherwise} \end{cases}$$

Finally, we recall that an integer b is very good for $W(\mathrm{A}_r) = S_{r+1}$ if and only if it is coprime to $h = r+1$. We again state a partial Smith form computation, whose proof is unfortunately is somewhat more complicated:

Proposition 4.3.4. *Let V be the irreducible reflection representation of S_{r+1}, and let the element $w \in S_{r+1}$ have cycle type $\lambda = (\lambda_1, \dots, \lambda_k)$. Then $tI - w$ is RC-equivalent over $\mathbb{Z}[t]$ to the block-diagonal matrix* $\mathrm{diag}(I_{r-1-k}, \Lambda(t))$ *where $\Lambda(t)$ is the $k \times k$ matrix*

$$\Lambda(t) = \begin{bmatrix} [\lambda_1]_t & [\lambda_2]_t & [\lambda_3]_t & \cdots & [\lambda_k]_t \\ 0 & t^{\lambda_2}-1 & 0 & & 0 \\ 0 & 0 & t^{\lambda_3}-1 & & 0 \\ & & & & \\ 0 & 0 & 0 & & t^{\lambda_k}-1 \end{bmatrix}.$$

where $[m]_t = 1 + t + \cdots + t^{m-1}$ (as in Definition 3.3.2).

Proof. By Proposition 4.2.3(b), we may assume that w is of the form w_λ for the rest of the argument. The matrix $tI - w_\lambda$, which has size $r \times r$, has the form described in Figure 4.1, where:

- the black rows consist entirely of 0s except where they intersect a region of another color; the last row that is not black is $\Lambda_f - 1$.
- the i^{th} pink (light-shaded) block on the diagonal is a square submatrix of size $\lambda_i - 1$ which has the form

$$\begin{bmatrix} t & & & & 1 \\ -1 & t & & & 1 \\ & -1 & \ddots & & \vdots \\ & & & t & 1 \\ & & & -1 & 1+t \end{bmatrix}.$$

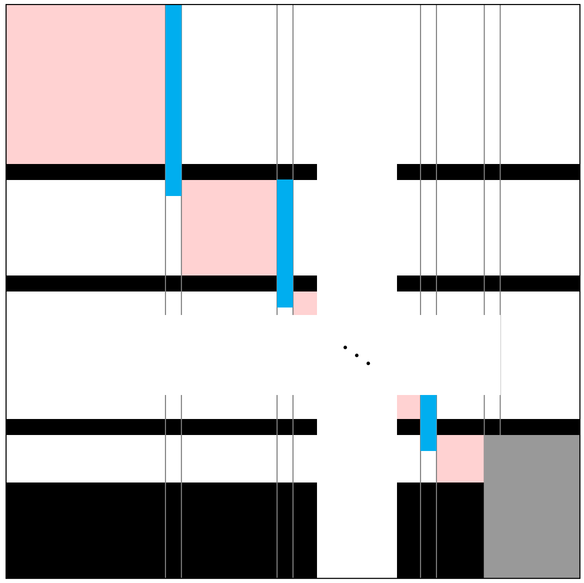


Figure 4.1: Schematic form of the matrix $tI - w$

- the blue (dark-shaded) regions are columns that have height $\lambda_i + 1$ consist of -1s except in the second-to-last entry (which is on the diagonal) which contains a $t-1$.
- the white regions are also 0s.
- the gray region in the bottom-right is an $(r - \Lambda_{f-1}) \times (k - f)$ matrix having the form

$$\begin{bmatrix} 1 & & & \\ \vdots & & & \\ 1 & & & \\ t-1 & & & \\ & t-1 & & \\ & & \ddots & \\ & & & t-1 \end{bmatrix}.$$

Notice that if λ has no parts of size 1 (that is, if $\Lambda_f = n$) then there are no black rows on the bottom of the matrix, and the gray region is absent.

Beginning with the bottom row and going upward, for each $i \neq \Lambda_j, \Lambda_j + 1$, perform the row operation $\mathbf{R}_i \mapsto \mathbf{R}_i + t\mathbf{R}_{i+1}$.

In so doing, for all $i \neq \Lambda_j, \Lambda_j + 1$, the -1 below the diagonal is the only nonzero entry in column $i-1$, and so we can perform column operations to clear the other entries in row $\mathbf{R}_i$. After doing this, the matrix has the same form, but the meaning of the colored submatrices has changed:

- the i^{th} pink (light-shaded) block on the diagonal now has the form

$$\begin{bmatrix} 0 & & & & [\lambda_i]_t \\ -1 & & & & 0 \\ & -1 & & & 0 \\ & & \ddots & & \vdots \\ & & & -1 & 0 \end{bmatrix}.$$

- the i^{th} blue (dark-shaded) region now has the form

$$\begin{bmatrix} -[\lambda_i - 1]_t \\ 0 \\ \vdots \\ 0 \\ t-1 \\ -1 \end{bmatrix}.$$

- the gray region in the bottom-right now has the form

$$\begin{bmatrix} -[\lambda_f - 1]_t & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \\ t-1 & & & \\ & t-1 & & \\ & & \ddots & \\ & & & t-1 \end{bmatrix}.$$

Notice the off-by-one distinction between the pink and blue entries in each row, so if we perform the column operations $\mathbf{C}_{\Lambda_k} \mapsto \mathbf{C}_{\Lambda_j} + \mathbf{C}_{\Lambda_j - 1}$ for $1 \le j \le f$, this changes the top entry of each blue region to t^{λ_j} (and of the gray region to t^{λ_f}).

We may permute rows and columns to put the matrix in the form $\mathrm{diag}(A(t), I, (t-1)I)$, where for $k > f$ the identity matrices are of size $r - k - f$ and $k - f - 1$ respectively, and the submatrix of interest $A(t)$ is

$$A(t) = \begin{bmatrix} [\lambda_1]_t & t^{\lambda_1} & & & & & & \\ & t-1 & & & & & & \\ & -1 & [\lambda_2]_t & t^{\lambda_2} & & & & \\ & & & t-1 & & & & \\ & & & & \ddots & & & \\ & & & & & t^{\lambda_{f-1}} & & \\ & & & & & t-1 & & \\ & & & & & -1 & [\lambda_f]_t & t^{\lambda_f} \\ & & & & & & & t-1 \end{bmatrix}.$$

If $k = f$, that is if λ has no 1-cycles, then omit the last (row and) column of $A(t)$ as well as the $(t-1)I$ block.

Beginning with $i = f-1$ and decreasing to $i = 1$ we perform the following operations:

$$\begin{aligned} \mathbf{R}_{2i} &\mapsto \mathbf{R}_{2i} + (t-1)\mathbf{R}_{2i+1}, & \mathbf{C}_{2i+1} &\mapsto \mathbf{C}_{2i+1} + [\lambda_f]_t \mathbf{C}_{2i}, \\ \mathbf{R}_{2i-1} &\mapsto \mathbf{R}_{2i-1} + t^{\lambda_i}\mathbf{R}_{2i+1}, & \mathbf{C}_{2i+3} &\mapsto \mathbf{C}_{2i+3} + [\lambda_f]_t \mathbf{C}_{2i} \quad (i < f-1). \end{aligned}$$

(The order in which these operations are performed does not matter, as long as all four are completed before continuing to the preceding i.)

The effect of these operations is to clear the even-index columns $\mathbf{C}_{2i}$ and odd-index rows $\mathbf{R}_{2i+1}$ of their entries with positive degree (excepting $\mathbf{R}_1$, as the notation suggests). Ultimately this yields a matrix of the form:

$$\begin{bmatrix} [\lambda_1]_t & 0 & t^{\lambda_1}[\lambda_2]_t & & t^{\lambda_1+\lambda_2}[\lambda_3]_t & & & t^{\Lambda_{f-1}}[\lambda_f]_t & t^{\Lambda_f} \\ & 0 & t^{\lambda_2}-1 & & t^{\lambda_2}(t^{\lambda_3}-1) & & & t^{\bullet}(t^{\lambda_f}-1) & t^{\bullet}(t-1) \\ & -1 & 0 & 0 & 0 & & & 0 & 0 \\ & & & 0 & t^{\lambda_3}-1 & & & t^{\bullet}(t^{\lambda_f}-1) & t^{\bullet}(t-1) \\ & & & -1 & 0 & & & 0 & 0 \\ & & & & & \ddots & & & \\ & & & & & & 0 & t^{\lambda_f}-1 & t^{\lambda_f}(t-1) \\ & & & & & & -1 & 0 & 0 \\ & & & & & & & & t-1 \end{bmatrix}.$$

The bullets in column $2i-1$ (and $2f$) above are explicit but immaterial expressions in λ, and we have omitted them for ease of reading. The important observation is that, first row notwithstanding, all other entries in column $2i-1$ (and $2f$) are divisible by their bottom-most nonzero entry $t^{\lambda_i}-1$ (or $t-1$), which allows us to clear most of the off-diagonal entries:

$$\begin{bmatrix}
[\lambda_1]_t & 0 & t^{\lambda_1}[\lambda_2]_t & & t^{\lambda_1+\lambda_2}[\lambda_3]_t & & & t^{\Lambda_{f-1}}[\lambda_f]_t & t^{\Lambda_f}[1]_t \\
 & 0 & t^{\lambda_2}-1 & & 0 & & & 0 & 0 \\
 & -1 & 0 & 0 & 0 & & & 0 & 0 \\
 & & & 0 & t^{\lambda_3}-1 & & & 0 & 0 \\
 & & & -1 & 0 & & & 0 & 0 \\
 & & & & & \ddots & & & \\
 & & & & & & 0 & t^{\lambda_f}-1 & 0 \\
 & & & & & & -1 & 0 & 0 \\
 & & & & & & & & t-1
\end{bmatrix}.$$

The "extra" factors of t in the first row can be cleared by appropriate row operations, namely $\mathbf{R}_1 \mapsto \mathbf{R}_1 - [\Lambda_j]_t \mathbf{R}_{2j}$ for $1 \leq j \leq f$. Finally, focusing away from the submatrix A and back to the entire matrix, an extra $[1]_t = 1$ may be inserted into the last $k-f-1$ entries of the first row by an appropriate column and then row operation, e.g.:

$$\begin{bmatrix} 1 & 0 \\ t-1 & 0 \\ 0 & t-1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ t-1 & t-1 \\ 0 & t-1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ t-1 & 0 \\ 0 & t-1 \end{bmatrix}.$$

The resulting matrix is RC-equivalent (over $\mathbb{Z}[t]$) to the one in the theorem statement by permuting rows and columns and flipping signs. □

Corollary 4.3.5. *Lemma 4.1.4 and Lemma 4.1.5 both hold for $W = W(\mathrm{A}_r)$.*

Let $W = W(\mathrm{A}_r)$ and Q be the normalized root lattice of type A. Each element $w \in W$ acts on Q as discussed above, so we may apply Proposition 4.3.4 for each element $w \in W$. Below, we will identify w with this matrix without further comment.

For Lemma 4.1.4, we first tensor with any field $\mathbb{F}$. We now evaluate t at any non-unity element of the field $t_0 \neq 1$ (which is dealt with below), so that $t_0 - 1$ is invertible. Thus by performing the row operations

$$\mathbf{R}_{r-k+1} \mapsto (t_0 - 1)\mathbf{R}_{r-k+1},$$
$$\mathbf{R}_{r-k+1} \mapsto \mathbf{R}_{r-k+1} - \mathbf{R}_{r-k+i} \quad (2 \leq i \leq k),$$

we conclude that $t_0 I - w$ is RC-equivalent to $\mathrm{diag}(t_0^{\lambda_1} - 1, \ldots, t_0^{\lambda_k} - 1)$ over $\mathbb{F}$.

We thus see that the $\mathbb{F}$-nullity of this matrix is precisely the number of λ_i such that $t_0^{\lambda_i} = 1$. Thus what remains to be shown is that for any very good prime power q and any integer d, the number of i such that $c^{d\lambda_i} \equiv 1$ in $\mathbb{F}_q$ is equal to the number of i such that $\zeta^{d\lambda_i} = 1$. As in type B, the condition in either case is equivalent to $d\lambda_i$ being divisible by $q-1$, and so the eigenspaces have equal dimension, as desired.

For Lemma 4.1.5 we evaluate at $t = 1$. All entries of $\Lambda(t)$ except those in the first row are zero, and by iterating the Euclidean algorithm we find that $I - w$ is RC-equivalent to $\mathrm{diag}(I_{r-k}, \gcd(\lambda), 0, \ldots, 0)$, where we write $\gcd(\lambda)$ as shorthand for $\gcd(\lambda_1, \ldots, \lambda_k)$. Clearly, $\gcd(\lambda)$ must divide $h = r + 1 = \sum \lambda_i$, and thus any number coprime to h is also coprime to $\gcd(\lambda)$.

Therefore, multiplication by $\gcd(\lambda)$ is an invertible operator on A, since the order of A is very good for W (that is, coprime to h). We conclude that

$$\ker_{Q \otimes A}(1 - w) \cong \ker_{A^r}(I - w) \cong \ker_{A^r} \mathrm{diag}(I_{r-k}, \gcd(\lambda), 0_{k-1}) \cong A^{k-1}$$

as abelian groups, with the middle isomorphism due to Proposition 4.2.3(c). Finally, Lemma 4.1.5 follows by tensoring instead with $\mathbb{C}$ to find $\dim_{\mathbb{C}} \ker(1 - w) = k - 1$ as well.

4.4 Exceptional Types

We now turn to the exceptional types. Broadly, we reduce the proofs to a finite calculation for each Weyl group W, and then check the five exceptional types (E_6, E_7, E_8, F_4, and G_2) with a computer script.

For Lemma 4.1.5 we need to compute fixpoints, and so we evaluate at $t = 1$. After making this substitution, $I - w$ is a matrix over $\mathbb{Z}$, and it is a simple matter to convince the computer algebra package `gap3` to compute a Smith factorization $I - w = UDV$. The diagonal entries of D are generally not 1, but this is not necessary for the kernel of $I - w$ to be a sum of copies of A. As in the proofs for the classical types, they need only be integers whose action on A (by multiplication) is an invertible operator.

Hence, we check whether, for all nonzero diagonal entries $d_i \in D$, the primes in the prime factorization of d_i are also all in the prime factorization of h (which is the relevant condition for the exceptional types). In case they are, we can divide the rows through by the corresponding entries to get a diagonal matrix D' of 0s and 1s, and so

$$\ker_{A^r}(I - w) \cong A^\kappa,$$

where κ is the number of 0s on the diagonal of D'. But changing D to D' simply amounts to making a change of the left or right factors $I - w = U'D'V'$; the entries of U' and V' are not integers but they certainly are complex numbers, and clearly $\ker_{\mathbb{C}^r}(I - w) \cong \mathbb{C}^\kappa$. Thus, if the condition on the d_i holds, it would imply the statement of Lemma 4.1.4.

Finally, we have again included our `gap3` script in the Appendix. Checking and running the code verifies that every d_i does satisfy the condition for the Weyl groups of exceptional type, and hence completes the proof of Lemma 4.1.5, and hence Theorem 4.1.3. $\square$

Lemma 4.1.4, is somewhat more involved. An essential difficulty in the proofs has been that w is not diagonalizable over $\mathbb{F}_q$, and so we have had to work with the eigenspaces of Weyl group elements directly, not merely their characteristic polynomials. However, if we fix W, then this problem disappears for most q.

Lemma 4.4.1. *Let $\mathbb{F}$ be a field and π the unqiue ring map $\mathbb{Z} \to \mathbb{F}$. Let $w \in \mathrm{GL}_r(\mathbb{Z})$ be any element of finite order m such that $\pi(m) \in \mathbb{F}^\times$. Then any $\beta \in \mathbb{F}$ has $\dim_{\mathbb{F}}(\beta - w)$ vanishing unless β has finite order ℓ dividing m, in which case $\dim_{\mathbb{F}}(\beta - w)$ is the multiplicity of the irreducible cyclotomic polynomial $\Phi_\ell(t)$ as a factor of $\det(t-w) \in \mathbb{Z}[t]$.*

This implies that for large p, Weyl group elements are "as diagonalizable as possible" over $\mathbb{F}_q$; that is, their eigenspaces coincide with their generalized eigenspaces. In particular, if $p > m$ (or, more generally, if p does not divide m), and $\alpha \in \mathbb{C}^\times$ and $\beta \in \mathbb{F}_q^\times$ both have order m, it implies that $\dim_{\mathbb{C}} \ker(\alpha - w) = \dim_{\mathbb{F}_q} \ker(\beta - w)$. Conversely, it is worth noting that the $p > m$ is a sharp bound in the sense that Lemma 4.4.1 can fail for any q such that the corresponding prime p divides m, even if it is very good for W:

Example 4.4.2. Let w be the simple transposition $(1\,2) \in S_3$; then w acts on the type A root lattice $Q = \mathrm{span}_{\mathbb{Z}}(e_1 - e_2, e_2 - e_3) \subseteq \mathbb{Z}^3$ by the matrix

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Over $\mathbb{F}_{2^e}$, for any e, this matrix is a Jordan block and is not diagonalizable. (Note that this is not a contradiction since the actual eigenspace for $1 = c^0$ over $\mathbb{F}_{2^e}$ is one-dimensional, same as for $1 = \zeta^0$ over $\mathbb{C}$.)

Assuming Lemma 4.4.1 for now, we can prove Lemma 4.1.4 for each W by checking it for only finitely many primes p. More precisely, for each $w \in W$, we need only check the list of primes that divide its order m. This is still an infinite calculation because there are many finite fields of characteristic p, but here again Lemma 4.4.1 is useful. Since we need only check the β whose order divides m, and there are only m of these in $\overline{\mathbb{F}}_p$, it suffices to perform the check at the minimal power $q = p^e$ in which all of them appear; that is, for which m divides $q-1$.

In this way, for each W we have reduced Lemma 4.1.4 to a finite calculation. After making the easy optimization of testing only one w for each conjugacy class of W, it turns out to be computationally cheap for all of the exceptional types. We have included our `gap3` script in the Appendix, so that the skeptical reader may verify that the code does indeed execute the plan outlined above; after it is verified and run, the proof of Lemma 4.1.4, and hence Theorem 4.1.2, is complete. □

It remains only to prove Lemma 4.4.1. For this, we recall some elementary facts about cyclotomic polynomials:

Proposition 4.4.3. *Let $\mathbb{F}$ be a field, $\overline{\mathbb{F}}$ be its algebraic closure, and π be the unique ring map $\mathbb{Z} \to \mathbb{F}$. For any integer d with $\pi(\ell) \in \mathbb{F}^\times$, we have that $t^\ell - 1$ and $\Phi_\ell(t)$ factor with distinct roots in $\overline{\mathbb{F}}[t]$ as*

(a) $t^\ell - 1 = \prod_{\beta \in \overline{\mathbb{F}}:\ \beta^\ell = 1} (t-\beta)$,

(b) $\Phi_\ell(t) = \prod_{\beta \in \overline{\mathbb{F}}:\ ord(\beta) = \ell} (t-\beta)$,

where $ord(\beta)$ is the multiplicative order of $\beta \in \mathbb{F}$.

Proof. For part (a), note that $f(t)$ has distinct roots in $\overline{\mathbb{F}}[t]$ because

$$\gcd(f(t), f'(t)) = \gcd(t^\ell - 1, \ell t^{\ell-1}) = \gcd(t^\ell - 1, t^{\ell-1}) = 1.$$

The middle equality is where we use the assumption that $\pi(\ell) \in \mathbb{F}^\times$. Part (a) then follows by the observation that $\overline{\mathbb{F}}$ contains ℓ such roots, and part (b) follows from part (a) by Möbius inversion. □

Proof (of Lemma 4.4.1). Since $w^m = 1$, the minimal polynomial for w divides $t^m - 1$, and so by Proposition 4.4.3(a) it has distinct roots in $\overline{\mathbb{F}}$. Therefore w is diagonalizable and so $\dim_{\mathbb{F}}(\beta - w)$ is the same as the algebraic multiplicity of β as a root of the characterstic polynomial $\det(t - w)$ of w.

If $\dim_{\mathbb{F}}(\beta - w) > 0$ then necessarily $\beta^m = 1$, therefore ℓ divides m and in particular $\pi(\ell) \in \mathbb{F}^{\times}$. Then by Proposition 4.4.3(b), we conclude that $\dim_{\mathbb{F}}(\beta - w)$ is the same as the multiplicity of $\Phi_\ell(t)$ as a factor of $\det(t - w)$, as desired. $\square$

Chapter 5

Cyclic Sieving for Necklaces

In this chapter we switch gears, restricting to type A and moving away from the rational Cherednik algebra to consider a cyclic action in a different setting entirely. The material in this chapter is borrowed from [Stu21].

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, the multinomial coefficient

$$\binom{n}{\alpha} = \binom{n}{\alpha_1, \ldots, \alpha_k} := \frac{n!}{\alpha_1! \cdots \alpha_k!}$$

is a positive integer, counting the number of words having exactly α_i occurrences of the letter i for each $i = 1, 2, \ldots, k$. The symmetric group S_n acts on the set of such words by permuting positions, and when one restricts this action to the cyclic subgroup $C = \langle c \rangle$ generated by the n-cycle $c = (1, 2, \ldots, n)$, the orbits are called **necklaces** with α_i **beads** of color i; we refer to these as α-necklaces. It is easily seen that the C-action on α-necklaces will be free if and only if $\gcd(\alpha) = \gcd(\alpha_1, \ldots, \alpha_k) = 1$, and thus the number of α-necklaces in this case is given by $C(\alpha) = \frac{1}{n}\binom{n}{\alpha}$.

When $\alpha = (a, a+1)$, this is the well-known **Catalan number**:

$$C(a,a+1) = \frac{1}{2a+1}\binom{2a+1}{a} = \frac{1}{a+1}\binom{2a}{a}.$$

For example, when $\alpha = (3,4)$, there are $C(3,4) = \frac{1}{7}\binom{7}{3} = \frac{1}{4}\binom{6}{3} = 5$ such necklaces with 3 black beads and 4 white beads, shown here:

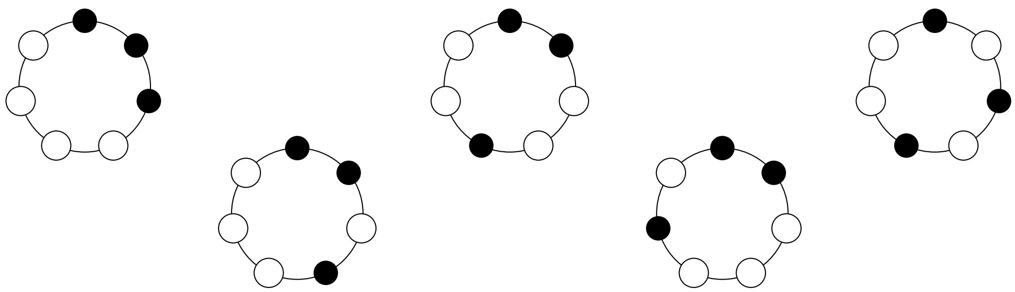


Figure 5.1: The five $(3,4)$-necklaces.

Recall from Reiner, Stanton, and White [RSW04] that for a set X carrying the action of a cyclic group $\langle \tau \rangle$ of order m, and a polynomial $X(t) \in \mathbb{Z}[t]$, one says that $(X, X(t), \langle \tau \rangle)$ exhibits the **cyclic sieving phenomenon** if for every integer d one has that

$$\left| \{x \in X : \tau^d(x) = x\} \right| = [X(t)]_{t=\zeta^d},$$

where ζ is a primitive m^{th} root of unity.

The case in which $m = 2$, so that τ is an involution; that is,

$$X(1) = |X|,$$
$$X(-1) = |\{x \in X : \tau(x) = x\}|.$$

was considered earlier by Stembridge [Ste94a], and so it is traditional to say that $(X, X(t), \tau)$ exhibits **Stembridge's** $t = -1$ **phenomenon**.

When Y is the set of necklaces, there is of course a natural cyclic action τ_0 of order 2 on Y: simply reflect a necklace over a line; orbits for this τ_0-action are called **bracelets**. We say that a bracelet is **asymmetric** if, as a τ_0-orbit of necklaces, it has size two. The enumeration of bracelets (and asymmetric bracelets) reduces to a fairly straightforward exercise, e.g. using Burnside's lemma.

We provide a different perspective on this enumeration. Define the standard t-analogue of $C(\alpha)$ to be

$$C(\alpha;t) = \frac{1}{[n]_t} \begin{bmatrix} n \\ \alpha \end{bmatrix}_t .$$

It turns out (and we will argue below) that $C(\alpha,t)$ is a polynomial whenever $\gcd(\alpha) = 1$, and indeed it helps exhibit sieving with respect to τ_0:

Theorem 5.0.1. *When* $\gcd(\alpha) = 1$*, the set Y of α-necklaces along with the polynomial $Y(t) := C(\alpha;t)$ and its τ_0-action by reflection exhibits Stembridge's $t = -1$ phenomenon. That is, if $C(\alpha;t) = \sum_i a_i t^i$, then*

$$\begin{aligned} \frac{1}{2}\left(C(\alpha;1) + C(\alpha;-1)\right) &= a_0 + a_2 + a_4 + \cdots \quad \textit{and} \\ \frac{1}{2}\left(C(\alpha;1) - C(\alpha;-1)\right) &= a_1 + a_3 + a_5 + \cdots , \end{aligned}$$

respectively, count the total number of bracelets and the number of asymmetric bracelets.

In the example of $\alpha = (3,4)$, one has

$$C(\alpha;t) = \frac{1}{[7]_t} \begin{bmatrix} 7 \\ 3 \end{bmatrix}_t = 1 + t^2 + t^3 + t^4 + t^6,$$

with $\frac{1}{2}(C(\alpha;1) + C(\alpha;-1)) = 4$ and $\frac{1}{2}(C(\alpha;1) + C(\alpha;-1)) = 1$. This agrees with the fact that the five necklaces shown above give rise to four bracelets, only one of which is asymmetric.

Theorem 5.0.1 will be deduced in Section 5.1 from a much more general statement. Notice that the reflection τ_0, considered as an element of S_n, is contained in the normalizer

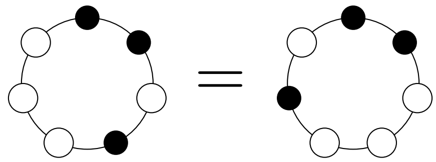


Figure 5.2: The unique asymmetric $(3,4)$-bracelet.

of C. In particular, we provide a sufficient condition for other $\tau \in N_{S_n}(C)$ acting on α-necklaces to exhibit a cyclic sieving phenomenon as well.

5.1 Group-Theoretic Formulation

We begin by reviewing a cyclic sieving phenomenon that specializes a result from [RSW04]. To avoid uninteresting trivialities in the more technical results, we will assume from here on that $n \geq 3$.

Given any subgroup H of S_n, let X be the coset space $X = S_n/H$, and $C = \langle c \rangle$ be the cyclic subgroup of S_n generated by the n-cycle $c = (1\,2\,\ldots\,n)$. Recall that S_n, and hence H, acts on the graded ring of n-variable polynomials $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ by permuting indices. Denote the fixed space of this S_n-action by let $\mathbb{C}[\mathbf{x}]^{S_n}$, and similarly for $\mathbb{C}[\mathbf{x}]^H$. Then [RSW04, Theorem 8.2] implies that the triple $(X, X(t), C)$ exhibits the cyclic sieving phenomenon, where

$$X(t) = \frac{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H, t)}{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n}, t)}. \tag{5.1}$$

Remark. The $X(t)$ used in [RSW04] is defined as $\mathrm{Hilb}((\mathbb{C}[\mathbf{x}]_{S_n})^H, t)$; recall that $\mathbb{C}[\mathbf{x}]_{S_n}$ is the coinvariant ring for S_n. This is just a mild difference in notation. The fact that $\mathrm{Hilb}((\mathbb{C}[\mathbf{x}]_{S_n})^H, t)$ is the same polynomial as $\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H; t)/\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n}; t)$ is a standard fact from invariant theory; see for instance [BRSW11, Corollary 1.2.2].

We write $\mathrm{Conj}_G(\gamma)$ to denote the set of elements in a group G conjugate to γ. Recall that elements are conjugate in S_n if and only if they have the same cycle type. When $\mu = \mathrm{cyc}(\gamma)$, that is, when μ is the cycle type of γ, we may abuse notation and write $\mathrm{Conj}_{S_n}(\mu)$ instead of $\mathrm{Conj}_{S_n}(\gamma)$. Generally, we say that $H \le S_n$ **avoids** μ if $H \cap \mathrm{Conj}_{S_n}(\mu) = \varnothing$.

We will be interested in $X = S_n/H$ as a set on which C acts by left-multiplication, particularly for those H such that the C-action is free. Note that the freeness of this action is equivalent to the condition that no nontrivial power of c is conjugate in S_n to an element of H, and hence to the statement that H avoids $(\ell^{\frac{n}{\ell}})$ for any divisor $\ell > 1$ of n. In this case, we aim to set up a further cyclic sieving triple. We begin with the polynomial:

Proposition 5.1.1. *Let C be a cyclic group acting freely on a set X, and $X(t) \in \mathbb{Z}[t]$. Then the triple $(X, X(t), C)$ exhibits the cyclic sieving phenomenon if and only if $Y(t) = \frac{1}{[n]_t} X(t)$ is a polynomial in $\mathbb{Z}[t]$.*

Proof. Let ζ be a primitive n^{th} root of unity. Both conditions are equivalent to the fact that $X(\zeta^d) = 0$ for any $1 \le d \le n-1$, because $[n]_t = \frac{1-t^n}{1-t} = \prod_{d=1}^{n-1}(t - \zeta^d)$. □

Moreover, notice that elements τ of the normalizer $N_{S_n}(C)$ can act on $Y = C\backslash S_n/H$, the collection of double-cosets CgH, via the rule

$$\tau \cdot CgH = \tau CgH = C\tau gH. \tag{5.2}$$

Example 5.1.2. For instance, let τ_0 be the permutation that fixes n and otherwise sends i to $n-i$, for any $1 \le i \le n$. Note that τ_0 also fixes exactly one other vertex, namely $\frac{n}{2}+1$, when n is even. In the case that $H = S_{\alpha_1} \times \cdots \times S_{\alpha_k}$ this has a natural geometric interpretation on words, which descends to (the unique) reflection on necklaces, c.f. Figure 5.3.

We now write our main theorem for this Chapter, a generalization of Theorem 5.0.1 that allows some flexibility with both τ and H:

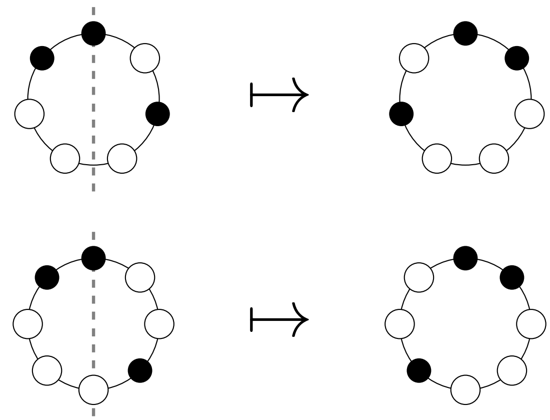


Figure 5.3: Because necklaces are fixed by rotation, all reflections are equal to τ_0.

Theorem 5.1.3. *Fix an element $\tau \in N_{S_n}(C)$ with cycle type either $(m^{\frac{n-1}{m}}, 1)$ or $(m^{\frac{n-2}{m}}, 1, 1)$, for some integer m. Suppose that $H \leq S_n$ is a subgroup that avoids the following cycle types:*

- $(\ell^{\frac{n}{\ell}})$ *for any divisor $\ell > 1$ of n,*
- $(4, 2^{\frac{n-4}{2}})$ *if m is even,*
- $\left(\ell^{\frac{n-2}{\ell}}, 2\right)$ *for any even divisor $\ell > 1$ of m, and*
- $\left((2\ell)^{\frac{n-2}{2\ell}}, 2\right)$ *for any odd divisor $\ell > 1$ of m.*

Finally, let $\langle\tau\rangle$ act on $Y = C\backslash S_n/H$ via the rule (5.2) and $Y(t) = \frac{1}{[n]_t}X(t)$, where $X(t)$ is defined by (5.1). Then $(Y, Y(t), \langle\tau\rangle)$ exhibits the cyclic sieving phenomenon.

The technicalities here are unfortunate, but the restrictions on H, at least, capture real difficulties. For instance, the desired sieving fails for $H = \langle(1234)(5678)(90)\rangle \leq S_{10}$ and $\tau = (1)(2408)(3795)(6)$, even though $\tau \in N_{S_{10}}(C)$ has cycle type $(4^{\frac{10-2}{2}}, 1, 1)$.

On the other hand, it may be possible to allow a broader class of τ if we appropriately restrict H, but the restrictions on τ given here are needed for our argument. The precise role they play is explicated at the end of Section 5.2, where in particular it is clear that these are the only cycle types that can reasonably be expected to yield such a cyclic sieving result whenever $n \equiv 1,2 \mod m$. However, when $n \not\equiv 1,2 \mod m$ the situation appears much more delicate, and we do not have a general conjecture.

Nevertheless, this theorem is already permissive enough to resolve Theorem 5.0.1. We recall the statement here:

Theorem 5.0.1. *When* $\gcd(\alpha) = 1$*, the set* Y *of* α*-necklaces along with the polynomial* $Y(t) := C(\alpha;t)$ *and its* τ_0*-action by reflection exhibits Stembridge's* $t = -1$ *phenomenon. That is, if* $C(\alpha;t) = \sum_i a_i t^i$*, then*

$$\frac{1}{2}\left(C(\alpha;1) + C(\alpha;-1)\right) = a_0 + a_2 + a_4 + \cdots \quad \textit{and}$$
$$\frac{1}{2}\left(C(\alpha;1) - C(\alpha;-1)\right) = a_1 + a_3 + a_5 + \cdots,$$

respectively, count the total number of bracelets and the number of asymmetric bracelets.

Proof (of Theorem 5.0.1). Let $H \leq S_n$ be the subgroup $S_{\alpha_1} \times \cdots \times S_{\alpha_k}$. Since S_n and H are both complex reflection groups, the invariant rings $\mathbb{C}[\mathbf{x}]^{S_n}$ and $\mathbb{C}[\mathbf{x}]^H$ are generated by algebraically independent elements. Indeed, as noted in Section 2.2, these may be taken to be elementary symmetric polynomials (in appropriate subsets of the variables), so that

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n}, t) = \prod_{i=1}^{n} \frac{1}{1-t^i} \qquad \text{and} \qquad \mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{H}, t) = \prod_{j=1}^{k} \prod_{i=1}^{\alpha_j} \frac{1}{1-t^i}.$$

From this we deduce that

$$X(t) = \frac{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{H}, t)}{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n}, t)} = \begin{bmatrix} n \\ \alpha \end{bmatrix}_t.$$

Write $X := S_n/H$, so that X is equivalent to the set of words having exactly α_i occurences of the letter i, and so C acts freely on X if and only if $\gcd(\alpha) = 1$. In this case, the associated $Y(t)$ is $C(\alpha;t)$. In Example 5.1.2, we saw that τ_0 acts on Y by reflection, and that its cycle type is $(2^{\frac{n-1}{2}},1)$ for odd n and $(2^{\frac{n-2}{2}},1,1)$ for even n. Moreover, we observe the following facts about cycle types of elements in H:

- As discussed above, the fact that C acts freely on X is equivalent to H avoiding $(\ell^{\frac{n}{\ell}})$ for any divisor $\ell > 1$ of n.
- H cannot contain elements with cycle type $(2^{\frac{n-4}{2}},4)$, because otherwise every α_i would have to be even, but we have $\gcd(\alpha) = 1$.
- The only divisor of 2 aside from 1 is $\ell = 2$ itself, for which $\left(\ell^{\frac{n-2}{\ell}},2\right) = (2^{\frac{n}{2}})$. Again H avoids this cycle type by the freeness of C on X.

Therefore, τ_0 and H satisfy the conditions of Theorem 5.1.3, and thus we conclude that the triple $(Y, Y(q), \tau_0)$ exhibits Stembridge's $q = -1$ phenomenon, as desired. □

Before beginning the proof of Theorem 5.1.3, we wish to make two more remarks.

First, it is clear that the latter three cycle conditions apply only when $n \equiv 2 \bmod m$. It is tempting to think that the only problem with extending to $n \equiv 3 \bmod m$ is an unwieldy proliferation of cycle type restrictions. This may indeed be the case, but our argument will not yield this; see the remarks following the proof of Theorem 5.4.1.

Second, in Section 5.3 we recall facts from elementary number theory that provide some insight into which $\tau \in S_n$ have the cycle types required by Theorem 5.1.3. In particular, this reveals a fairly general setting in which all of the technicalities simplify. When n is an odd prime, it happens that every $\tau \in N_{S_n}(C)$ is either in C itself, or has cycle type $(m^{\frac{n-1}{m}},1)$ for some m. Moreover, as described above, we only need the first cycle type restriction. Therefore, we obtain the following pleasing corollary:

Corollary 5.1.4. *Fix an odd prime p, an element $\tau \in N_{S_p}(C) \setminus C$, and a subgroup $H \leq S_p$ for which C acts freely on S_p/H. Additionally, let $\langle\tau\rangle$ act on $Y = C\backslash S_p/H$ via the rule (5.2)* *and let $Y(t) = \frac{1}{[p]_t} X(t)$, where $X(t)$ is defined by (5.1). Then the triple $(Y, Y(t), \langle\tau\rangle)$ exhibits the cyclic sieving phenomenon.*

The remainder of this chapter is devoted to the proof of Theorem 5.1.3. To that end, we fix some notation. For any group G acting on some set S, and any $g \in G$, write $\mathrm{Fix}_S(g)$ to denote the set of g-fixpoints: $\{s \in S : g \cdot s = s\}$. Any two G-conjugate elements have the same number of fixpoints in S. So, in particular, for a partition μ of n, we abuse notation and write $|\mathrm{Fix}_S(\mu)|$ to mean the number of points in S that are fixed by any permutation with cycle type μ.

In the next two sections we will complete the bulk of a single root-of-unity calculation, and then we will bundle them together with some concluding details. For the intermediate results, the following definition is useful:

Definition 5.1.5. Suppose that $\tau' \in N_{S_n}(C)$ has cycle type either $\left(m^{\frac{n-1}{m}}, 1\right)$ or $\left(m^{\frac{n-2}{m}}, 1, 1\right)$, for some integer m. Write $k = \left\lfloor \frac{n-1}{m} \right\rfloor$ for the number of m-cycles that τ' has. Moreover, suppose that $H \leq S_n$ is a subgroup such that H avoids the following cycle types:

- $(\ell^{\frac{n}{\ell}})$ for any divisor $\ell > 1$ of n,
- $(m^k, 2)$ if m is even, and
- $((2m)^{\frac{k}{2}}, 2)$ if m is odd.

In this case, we say that the pair (τ', H) is ***C*-admissible**.

Part of the bundling process is the observation that the conditions of Theorem 5.1.3 on τ and H are equivalent to the statement that (τ^d, H) is C-admissible for every integer $1 \leq d \leq m-1$, and also H avoids $(4, 2^{\frac{n-4}{2}})$. We will see that the latter cycle type restriction arises from a different consideration than C-admissibility does.

5.2 Evaluating $Y(\zeta)$

This proposition gives an explicit formula relating $Y(\zeta)$ and various fixpoints in X. Notice that we only use the freeness condition on H, and not the other cycle type restrictions.

Proposition 5.2.1. *Fix a subgroup $H \leq S_n$ such that C acts freely on $X = S_n/H$. Fix an integer $m \geq 2$, and then define ζ to be a primitive m^{th} root of unity, and $k = \lfloor \frac{n-1}{m} \rfloor$. Moreover, for any partition λ of $n-km$, write c_i to denote the number of parts in λ with size i. Then, defining*

$$X(t) = \frac{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H;t)}{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n};t)}$$

and $Y(t) = \frac{1}{[n]_t}X(t)$, we have the following:

(a) If $n \not\equiv 0$ mod m, then

$$Y(\zeta) = (1-\zeta)\prod_{i=1}^{n-1-km}(1-\zeta^i)\left[\sum_{\lambda \vdash n-km} \frac{\left|\mathrm{Fix}_X(m^k,\lambda)\right|}{\prod_{i\geq 1}(i(1-\zeta^i))^{c_i}c_i!}\right].$$

(b) If $n \equiv 0$ mod m, then

$$Y(\zeta) = (1-\zeta)\left[\frac{mk}{4}\left|\mathrm{Fix}_X(2m,m^{k-1})\right| + \sum_{\substack{\lambda \vdash m \\ \lambda_1 \neq m}} \frac{\left|\mathrm{Fix}_X(m^k,\lambda)\right|}{\prod_{i\geq 1}(i(1-\zeta^i))^{c_i}c_i!}\right].$$

Proof. Observe that

$$\begin{aligned} X(t) &= \frac{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H;t)}{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n};t)} \\ &= \mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H;t)\prod_{i=1}^{n}(1-t^i) \\ &= \mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H;t)\cdot(1-t)^n[n]!_t. \end{aligned}$$

Thus, $Y(t) = (1-t)^n [n-1]!_t \cdot \mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H;t)$. We can explicitly calculate the Hilbert series of the H-invariants using Molien's formula [Mol97]:

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H;t) = \frac{1}{|H|}\sum_{h\in H}\frac{1}{1-\det(1-th)} = \frac{1}{|H|}\sum_{h\in H}\prod_{\text{cycles } z \text{ of } h}\frac{1}{1-t^{|z|}},$$

where the determinant of $1-th$ is taken with respect to its action on $\mathbb{C}[\mathbf{x}]$, and $|z|$ is the length of the cycle z.

Putting the Hilbert series aside momentarily, notice that

$$\begin{aligned}(1-t)^n[n-1]!_t &= (1-t)\prod_{i=1}^{n-1}(1-t^i)\\ &= (1-t)\prod_{j=1}^{k}\left((1-t^{jm})\prod_{d=1}^{m-1}(1-t^{jm+d})\right)\prod_{i=km+1}^{n-1}(1-t^i).\end{aligned}$$

For each $h\in H$, define the auxiliary quantity

$$F_h(\zeta) := \lim_{t\to\zeta}\prod_{j=1}^{k}(1-t^{jm})\cdot\prod_{\text{cycles } z \text{ of } h}\frac{1}{1-t^{|z|}},$$

so that

$$Y(\zeta) = \frac{1}{|H|}(1-\zeta)\cdot\left(\prod_{d=1}^{m-1}(1-\zeta^d)\right)^k\cdot\prod_{i=km+1}^{n-1}(1-\zeta^i)\cdot\sum_{h\in H}F_h(\zeta).$$

The second of the four factors in the above expression has a simple evaluation. Because $x=\zeta^d$ is a root of $[m]_x$ for all integers $1\le d\le m-1$, we conclude that $\prod_{d=1}^{m}(x-\zeta^d)=[m]_x$, and hence

$$Y(\zeta) = \frac{1}{|H|}(1-\zeta)\cdot m^k\cdot\prod_{i=km+1}^{n-1}(1-\zeta^i)\cdot\sum_{h\in H}F_h(\zeta).$$

It remains to compute $F_h(\zeta)$. The first factor of $F_h(\zeta)$ has a zero of multiplicity k at $t=\zeta$, and so $F_h(\zeta)=0$ unless the second factor has a pole of order at least k at $t=\zeta$. We can see that this occurs precisely when h has at least k cycles whose lengths divide m. By definition of k, the element h can never have more than $k+1$ cycles whose lengths divide m, because $(k+1)m \leq n$. In fact, h has at most k such cycles: equality would occur if and only if $m|n$, but then $\text{cyc}(h) = (m^{k+1})$, which contradicts that C acts freely on S_n/H. This, in turn, means that either

- m does not divide n, in which case $\text{cyc}(h) = (m^k, \lambda)$ for some $\lambda \vdash n - km$,

- m divides n and $\text{cyc}(h) = (m^k, \lambda)$ for some $\lambda \vdash m$ (since $n - km = m$), or

- m divides n and $\text{cyc}(h) = (2m, m^{k-1})$; this is the only way some cycle of h can have length greater than m, while still having at least k cycles that divide m.

For λ a partition with ℓ parts, define the further auxiliary quantity

$$F_\lambda(\zeta) := \prod_{j=1}^{\ell} \frac{1}{1-\zeta^{\lambda_j}} = \prod_{i \geq 1} \frac{1}{(1-\zeta^i)^{c_i}},$$

so that

$$F_h(\zeta) = \lim_{t \to \zeta} \prod_{j=1}^{k}(1-t^{jm}) \cdot \prod_{j=2}^{k} \frac{1}{1-t^m} \cdot \begin{cases} \dfrac{1}{1-t^{2m}} & \text{if } \text{cyc}(h) = (2m, m^{k-1}), \\[2em] \dfrac{F_\lambda(\zeta)}{1-t^m} & \text{otherwise.} \end{cases}$$

In both cases, the k factors of $1-t^m$ in the denominator may be pulled into the first product, yielding the t-factorial $[k]!_t$ evaluated at $t = \zeta^m = 1$. Hence, $F_h(\zeta)$ is either $\frac{k!}{1+\zeta^m} = \frac{1}{2}k!$ for the exceptional cycle type, or otherwise is $k!\,F_\lambda(\zeta)$.

Putting this all together,

$$Y(\zeta) = \frac{(1-\zeta)m^k}{|H|} \prod_{i=km+1}^{n-1} (1-\zeta^i) \left(\frac{k!}{2} \left| \{h \in H : \mathrm{cyc}(h) = (2m, m^{k-1})\} \right| \right.$$
$$\left. + k! \sum_{\lambda} \left| \{h \in H : \mathrm{cyc}(h) = (m^k, \lambda)\} \right| F_\lambda(\zeta) \right).$$

For the moment, let us assume that m does not divide n, so that the first parenthesized term vanishes. Plugging in the definition of $F_\lambda(\zeta)$, we observe that

$$\begin{aligned} Y(\zeta) &= \frac{1-\zeta}{|H|} \prod_{i=km+1}^{n-1} (1-\zeta^i) \cdot \left(\sum_{\lambda} \left| \mathrm{Conj}_{S_n}(m^k, \lambda) \cap H \right| m^k k! \prod_{i \geq 1} \frac{1}{(1-\zeta^i)^{c_i}} \right) \\ &= \frac{1-\zeta}{|H|} \prod_{i=km+1}^{n-1} (1-\zeta^i) \cdot \left(\sum_{\lambda} \left| \mathrm{Conj}_{S_n}(m^k, \lambda) \cap H \right| \left| Z_{S_n}(m^k, \lambda) \right| \prod_{i \geq 1} \frac{1}{((1-\zeta^i)i)^{c_i} c_i!} \right). \end{aligned}$$

In the last line, we have written $\left| Z_{S_n}(m^k, \lambda) \right|$ to denote the size of the centralizer of any element in S_n with cycle type $\mu = (m^k, \lambda)$.

In the case when m divides n, note that the initial product simplifies, since in that case it includes all roots of unity except 1 itself, and thus as argued before evaluates to m. Then, performing similar calculations to the above yields:

$$Y(\zeta) = \frac{(1-\zeta)}{|H|} \left(\frac{mk}{4} \left| \mathrm{Conj}_{S_n}(2m, m^{k-1}) \cap H \right| \left| Z_{S_n}(2m, m^{k-1}) \right| \right.$$
$$\left. + \sum_{\lambda} \left| \mathrm{Conj}_{S_n}(m^k, \lambda) \cap H \right| \left| Z_{S_n}(m^k, \lambda) \right| \prod_{i \geq 1} \frac{1}{((1-\zeta^i)i)^{c_i} c_i!} \right).$$

Comparing this to the desired formula, it would suffice to show that

$$|\mathrm{Fix}_X(\gamma)| = \frac{1}{|H|} \cdot \left| \mathrm{Conj}_{S_n}(\gamma) \cap H \right| |Z_{S_n}(\gamma)|$$

for any $\gamma \in S_n$. In fact, the analogous statement is true for any group, not just S_n: see

Lemma 5.2.2 below. ☐

In analogy to the symmetric group notation, for any group G and any $g \in G$, write $Z_G(\gamma)$ to denote the centralizer of γ in G.

Lemma 5.2.2. *For any finite group G, any subgroup $H \leq G$, and any $\gamma \in G$:*

$$\left|\mathrm{Fix}_{G/H}(\gamma)\right| = \frac{1}{|H|} \cdot |Z_G(\gamma)|\,|\mathrm{Conj}_G(\gamma) \cap H|.$$

Proof. Note that any $g \in G$ satisfies $\gamma g H = gH$ if and only if $g^{-1}\gamma g \in H$, so the left-hand side is zero if and only if the right side is zero. Suppose that the right-hand side is not zero; in particular, that there exists an element $\eta \in H \cap \mathrm{Conj}_G(\gamma)$. Notice that $|Z_G(\eta)| = |Z_G(\gamma)|$ and $|\mathrm{Fix}_{G/H}(\eta)| = |\mathrm{Fix}_{G/H}(\gamma)|$, so we may assume without loss of generality that $\gamma \in H$.

We want to show that $|H| \cdot |\mathrm{Fix}_{G/H}(\gamma)| = |Z_G(\gamma)| \cdot |H \cap \mathrm{Conj}_G(\gamma)|$, or, since all cosets have the same size $|H|$, we may write the left-hand side as $|\{g \in G : g^{-1}\gamma g \in H\}|$. To show this equality, we observe that the map $\phi : \{g \in G : g^{-1}\gamma g \in H\} \to H \cap \mathrm{Conj}_G(\gamma)$ given by $\phi(g) = g^{-1}\gamma g$ is surjective, and then it suffices to show that every $\phi^{-1}(h)$ has size $|Z_G(\gamma)|$. In fact, $\phi^{-1}(h) = gZ_G(\gamma)$ where g is any element in $\phi^{-1}(h)$, because

$$(xg^{-1})\gamma(xg^{-1})^{-1} = \gamma \qquad \Longleftrightarrow \qquad g^{-1}\gamma g = x^{-1}\gamma x.$$

The left equality states that $x \in gZ_G(\gamma)$; the right equality states that $\phi(x) = \phi(g) = h$. ☐

5.3 Evaluating $|\mathrm{Fix}_Y(\tau)|$

We have now written $Y(\zeta)$ in terms of X-fixpoints, but this is only useful for cyclic sieving if there is some relationship between X-fixpoints and Y-fixpoints. Fortunately, such a relationship exists when C acts freely. Before describing it, we recall some facts from elementary number theory:

Proposition 5.3.1. *Let S_n be the symmetric group on $Z_n = \{1, \ldots, n\}$, and suppose that $\tau \in N_{S_n}(C)$.*

(a) There exists unique $s \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $r \in \mathbb{Z}/n\mathbb{Z}$ such that $\tau(x) \equiv sx + r \bmod n$. In particular, $s = 1$ if and only if $\tau \in C$.

(b) τ is S_n-conjugate to $\tilde{\tau} : x \mapsto sx + r'$ where r' is the smallest nonnegative integer such that $r' \equiv r \bmod \gcd(n, s-1)$.

(c) If $\gcd(n, s-1)$ divides r then $|\mathrm{Fix}_{Z_n}(\tau)| = \gcd(n, s-1)$, and otherwise $\mathrm{Fix}_{Z_n}(\tau) = \varnothing$.

If moreover $\mathrm{cyc}(\tau) = (m^k, 1, 1)$ for some integers m and k, then n is even and:

(d) using the notation of (b) above, $r' = 0$ and the fixpoints of $\tilde{\tau}$ are n and $\frac{n}{2}$;

(e) if $\frac{n}{4} \in \mathbb{Z}$, then $m = 2$.

Proof. The proofs are routine, and we leave them to the end of the chapter. □

For the arguments that follow, by far the most important fact here is part (c). It implies that if there is any $\tau \in N_{S_n}(C)$ with cycle type $(m^k, 1, 1)$, then n must necessarily be even. In particular, this means that n must in fact be even whenever (τ, H) is C-admissible and $m \equiv 2 \bmod n$ (where m is the order of τ, as usual).

We also follow up on the remark in Section 5.1 preceeding Corollary 5.1.4. Clearly τ is determined by its output mod n for each input, and so part (a) states that the scaling factor s of elements with the desired cycle type has order m as an element of $(\mathbb{Z}/n\mathbb{Z})^\times$, and the translation r must be even if $\mathrm{cyc}(\tau) = (m^k, 1, 1)$. This necessary condition can be elevated to a sufficient condition if $s^b x \not\equiv x$ for any x except 0 and perhaps $\frac{n}{2}$, for any $1 \leq b < m$. Often this condition is quite restrictive; for instance if n is divisible by 3, then taking $x = \frac{n}{3}$

shows that $s^b \not\equiv 1 \bmod 3$, and this forces $m = 2$. However, if n is prime, then we can divide through by any nonzero x, and hence every nonzero x is in a cycle of the same length m.

The idea of the computation of $|\mathrm{Fix}_Y(\tau)|$ is summarized in the following proposition.

Proposition 5.3.2. *Let (τ, H) be C-admissible. Then writing m for the order of τ, as well as $X = S_n/H$ and $Y = C\backslash S_n/H$, the canonical quotient map $\pi : X \to Y$ restricts to a surjective map $\pi_F : \mathrm{Fix}_X(\tau) \to \mathrm{Fix}_Y(\tau)$. Indeed, for any $y \in Y$,*

$$|\pi_F^{-1}(y)| = \begin{cases} 1 & \textit{if } n \equiv 1 \bmod m, \\ 2 & \textit{if } n \equiv 2 \bmod m. \end{cases}$$

Proof. First, observe that π_F is well-defined, that is, $\pi_F(\mathrm{Fix}_X(\tau)) \subseteq \mathrm{Fix}_Y(\tau)$, since for any $\gamma H \in \mathrm{Fix}_X(\tau)$, we have $\tau \cdot C\gamma H = C\tau\gamma H = C\gamma H$.

Now suppose that $CgH \in \mathrm{Fix}_Y(\tau)$, that is, $g \in S_n$ is an element such that $\tau \cdot CgH = CgH$. In particular, this means

$$\tau g H \subseteq \coprod_{d=1}^{n} c^d gH,$$

where the cosets on the right are all distinct because C acts freely on X. Thus, there must be some d_0 such that $\tau gH = c^{d_0} gH$.

Note that all elements of $\pi^{-1}(CgH)$ have the form $c^z gH$ for some integer $0 \le z < n$. Since C acts freely on X, these are distinct for distinct z. Letting s be the element of $(\mathbb{Z}/n\mathbb{Z})^\times$ guaranteed by Proposition 5.3.1(a), we may write:

$$\tau(c^z gH) = c^{sz} \tau gH = c^{sz+d_0} gH.$$

Thus $\tau(c^z gH) = c^z gH$ if and only if $sz + d_0 \equiv z \bmod n$. In other words, for any $CgH \in Y$,

$$\left|\pi_F^{-1}(CgH)\right| = |\{z \in \mathbb{Z}/n\mathbb{Z} : (s-1)z \equiv -d_0 \bmod n\}|.$$

We remark that the right-hand side is not *a priori* independent of the element CgH, because d_0 generally does depend on gH.

In this way, the $n \equiv 1 \bmod m$ case resolves immediately. Since (τ, H) is C-admissible, we know τ has a unique fixpoint in $\mathbb{Z}/n\mathbb{Z}$ and thus $\gcd(n, s-1) = 1$ by Proposition 5.3.1(c). In other words, $s-1$ is invertible mod n, and so for any value of d_0 we have the unique solution $z \equiv -d_0(s-1)^{-1} \bmod n$.

The argument for $n \equiv 2 \bmod m$ is similar, but we require a technical prerequisite. By definition, $\tau g \in c^{d_0} gH$, or equivalently, $g^{-1}c^{-d_0}\tau g \in H$. By Lemma 5.3.3 below, the fact that (τ, H) is C-admissible implies that $-d_0$ must not be odd. Therefore, we may divide both sides of the congruence by 2 and solve. Namely,

$$(s-1)z \equiv -d_0 \bmod n \qquad \Longleftrightarrow \qquad z \equiv -\tfrac{d_0}{2}\left(\tfrac{s-1}{2}\right)^{-1} \bmod \tfrac{n}{2},$$

where the right congruence is well-defined by applying Proposition 5.3.1(c): because $\gcd(n, s-1) = 2$, we know $\frac{s-1}{2}$ is invertible mod $\frac{n}{2}$. This solution z is unique mod $\frac{n}{2}$, and hence there are precisely two solutions mod n, as desired. □

Finally, we prove the required technical lemma to complete the proof of Proposition 5.3.2.

Lemma 5.3.3. *Fix an element $g \in S_n$ and let d be an integer. For any $\tau \in N_{S_n}(C)$ such that* $\operatorname{cyc}(\tau) = (m^k, 1, 1)$ *for some integers m and k, then (n is even and):*

$$\operatorname{cyc}(c^d\tau) = \begin{cases} (m^k, 1, 1) & \text{if } d \text{ is even,} \\ (\ell^{\frac{n}{\ell}}) & \text{for some } \ell | n, \text{ if } d \text{ is odd and } m = 2, \\ (m^k, 2) & \text{if } d \text{ is odd and } m > 2 \text{ is even,} \\ ((2m)^{\frac{k}{2}}, 2) & \text{if } d \text{ is odd and } m \text{ is odd.} \end{cases}$$

In particular, if we additionally choose $H \leq S_n$ such that (τ, H) is C-admissible, then no element conjugate to $c^d\tau$ is contained in H for any odd j.

Proof. The cycle type of τ together with Proposition 5.3.1(c) imply that $\gcd(n, s-1) = 2$, and so by Proposition 5.3.1(b) we have that the cycle type of $c^d\tau$ depends only on the parity of d. In particular, all $c^d\tau$ are conjugate to either τ or $c\tau$, and so it suffices to compute the cycle type of the latter.

Moreover, we may replace τ with $\tilde{\tau}$, and together with Proposition 5.3.1(d), we may say that $\tau(x) = sx$ without loss of generality. Because $c\tau(x) \equiv sx + 1 \bmod n$,

$$(c\tau)^d(x) = x \qquad \Longleftrightarrow \qquad \left(s^{d-1} + \cdots + s + 1\right)\left(1 + (s-1)x\right) \equiv 0 \bmod n.$$

Note that m is the order of τ in S_n, and thus the order of s in $(\mathbb{Z}/n\mathbb{Z})^\times$. Moreover, both n and $s-1$ are even by Proposition 5.3.1(c), and hence the congruence has no solution unless d is even: the left-hand side would be the product two odd numbers and hence nonzero.

This is enough to resolve case $m = 2$, in which $s^2 \equiv 1 \bmod n$. Using this fact, we have the following congruence mod n:

$$\begin{aligned}\left(s^{d-1} + \cdots + s + 1\right)\left(1 + (s-1)x\right) &\equiv \tfrac{d}{2}(s+1)\left(1 + (s-1)x\right) \\ &\equiv \tfrac{d}{2}\left((s+1) + (s^2-1)x\right) \\ &\equiv \tfrac{d}{2}(s+1).\end{aligned}$$

This expression is independent of x. Thus, letting ℓ denote the smallest positive integer d such that $\frac{d}{2}(s+1) \equiv 0 \bmod n$, we have shown that every cycle of $c\tau$ has length ℓ.

We now assume that $m > 2$. It thus suffices to show that $c\tau$ has exactly one two-cycle, and every other x is contained in a cycle of length m if m is even, or $2m$ if m is odd. It will be convenient to write the prime factorization $n = 2^{e_0} p_1^{e_1} \cdots p_a^{e_a}$, where the p_i are distinct odd primes and each e_i is a positive integer. By Sun Tzu's Theorem (i.e. the Chinese Remainder Theorem), we may take $p_0 = 2$ and then restate the equivalence above as

$$(c\tau)^d(x) = x \quad \Longleftrightarrow \quad \left(s^{d-1} + \cdots + s + 1\right)\left(1 + (s-1)x\right) \equiv 0 \bmod p_i^{e_i} \text{ for all } 0 \le i \le a.$$

Recall that $(n, s-1) = 2$, via Proposition 5.3.1(c), and thus none of the odd p_i divide $s-1$. In particular, $s-1$ is invertible mod $p_i^{e_i}$ and thus we again apply Sun Tzu's Theorem

$$\begin{aligned} s^m &\equiv 1 \bmod n, \\ (s-1)\left(s^{m-1} + \cdots + s + 1\right) &\equiv 0 \bmod p_i^{e_i} \qquad \text{for all } 1 \le i \le a, \\ s^{m-1} + \cdots + s + 1 &\equiv 0 \bmod p_i^{e_i} \qquad \text{for all } 1 \le i \le a. \end{aligned}$$

Thus the desired equivalence $\left(s^{m-1} + \cdots + s + 1\right)\left(1 + (s-1)x\right) \equiv 0 \bmod p_i^{e_i}$ holds for the odd primes. For the prime $p_0 = 2$ we use our assumption that $m > 2$. By Proposition 5.3.1(e), this means that $\frac{n}{4} \notin \mathbb{Z}$ and thus $2^{e_0} = 2$. Therefore,

$$s^{m-1} + \cdots + s + 1 \equiv m \bmod 2^{e_0}.$$

So, if m is even then every cycle has length dividing m, but if m is odd then every cycle has length dividing $2m$ and (in particular) *no* cycle has length m.

We conclude by showing that for any $x \not\equiv -(s-1)^{-1} \bmod \frac{n}{2}$, that $(c\tau)^d(x) \neq x$ for any $d < m$. Before doing so, we make two observations. First, the inverse is well-defined because, as above, $\frac{n}{2}$ is odd, so $\gcd(s-1, \frac{n}{2}) = 1$ by Proposition 5.3.1(c). Second, in so doing we will complete the proof: it will guarantee that each such x lies in a $c\tau$-cycle of size at least m. Taken together with the previous paragraph, this means that all but two x lie in $c\tau$-cycles of the desired size. Moreover, the other two must form a 2-cycle, since by Proposition 5.3.1(c) neither is a fixpoint, which yields the desired cycle type for $c^d\tau$.

Proposition 5.3.1(d) already gives an analogous statement for τ: for every $1 \le d < m$, all but two $x \in \mathbb{Z}/n\mathbb{Z}$ are contained in an m-cycle, and thus $s^d x \not\equiv x$ for any $x \not\equiv 0 \bmod \frac{n}{2}$. In particular, choosing $x_i = \frac{n}{p_i}$ for each $1 \le i \le a$ shows that:

$$\frac{n}{p_i}(s-1)\left(s^{d-1}+\cdots+s+1\right) \not\equiv 0 \bmod n \qquad \text{for all } 1 \le i \le a$$
$$(s-1)\left(s^{d-1}+\cdots+s+1\right) \not\equiv 0 \bmod p_i \qquad \text{for all } 1 \le i \le a$$

Hence, $s^{d-1}+\cdots+s+1$ is not divisible by any p_i and so is invertible mod $\frac{n}{2}$. Therefore, we have the following necessary condition:

$$(c\tau)^d(x) = x \qquad \Longrightarrow \qquad 1+(s-1)x \equiv 0 \bmod \tfrac{n}{2}.$$

As discussed above, $s-1$ is invertible mod $\frac{n}{2}$. Therefore, for any $x \not\equiv -(s-1)^{-1} \bmod \frac{n}{2}$ we conclude that $(c\tau)^d(x) \neq x$, as desired. $\square$

5.4 Completing the Proof of Theorem 5.1.3

We combine the previous two subsections into the following result:

Theorem 5.4.1. *Suppose that (τ, H) is C-admissible, and write m for the order of τ. If H additionally avoids the cycle type $(4, 2^{\frac{n-4}{2}})$, and we write ζ for a primitive m^{th} root of unity, $Y := C\backslash S_n/H$, and*

$$Y(t) := \frac{1}{[n]_t} \cdot \frac{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^H, t)}{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^{S_n}, t)},$$

then $Y(\zeta) = |\mathrm{Fix}_Y(\tau)|$.

Proof. We begin with the case $\mathrm{cyc}(\tau) = (m^{\frac{n-1}{m}}, 1)$. The calculation from Proposition 5.2.1 simplifies considerably since there is only one λ that partitions $n - km$, namely $\lambda = (1)$.

Therefore, if ζ is a primitive m^{th} root of unity, then

$$Y(\zeta) = (1-\zeta)\frac{\left|\mathrm{Fix}_X(m^k,1)\right|}{1-\zeta} = |\mathrm{Fix}_X(\tau)|\,.$$

Thus, by Proposition 5.3.2 we have $Y(\zeta) = |\mathrm{Fix}_Y(\tau)|$, as desired.

If instead $\mathrm{cyc}(\tau) = (m^{\frac{n-1}{m}},1,1)$, Proposition 5.2.1 simplifies similarly: there are now two λ that partition $n-km$, namely $\lambda = (1,1)$ and $\lambda = (2)$. Therefore, if ζ is a primitive m^{th} root of unity, then

$$Y(\zeta) = \begin{cases} \frac{1}{2}\left|\mathrm{Fix}_X(2^k,1,1)\right| + k\left|\mathrm{Fix}_X(4,2^{k-1})\right| & \text{if } m=2, \\ \frac{1}{2}\left|\mathrm{Fix}_X(m^k,1,1)\right| + \left(\frac{1-\zeta}{1+\zeta}\right)\left|\mathrm{Fix}_X(m^k,2)\right| & \text{if } m>2. \end{cases}$$

In either case, Lemma 5.2.2 and the conditions on H imply that the second term vanishes and $Y(\zeta) = \frac{1}{2}|\mathrm{Fix}_X(\tau)|$. Thus by Proposition 5.3.2 we have $Y(\zeta) = |\mathrm{Fix}_Y(\tau)|$, as desired.

□

Remark. As mentioned before, trying to extend this argument to $n \equiv 3 \bmod m$ is more troublesome. For simplicity let us suppose that $m > 3$, then we may see the difficulty by using Proposition 5.2.1 again:

$$Y(\zeta) = \tfrac{(1-\zeta)(1-\zeta^2)}{3(1+\zeta+\zeta^2)}\left|\mathrm{Fix}_X(m^k,3)\right| + \tfrac{1-\zeta}{2}\left|\mathrm{Fix}_X(m^k,2,1)\right| + \tfrac{1+\zeta}{6}\left|\mathrm{Fix}_X(m^k,1,1,1)\right|.$$

None of these coefficients are rational, and so if $Y(\zeta)$ is to evaluate to a positive integer, it must have contributions from multiple terms. Unlike for the $n \equiv 2 \bmod m$ case, we cannot simply exclude the cycle type giving complex contribution and focus on the only τ remaining.

This is nearly all of Theorem 5.1.3; to complete the proof, we must compute $Y(\zeta)$ at non-primitive roots of unity. Recall that the conditions on τ and H of Theorem 5.1.3

are equivalent to the fact that (τ^d, H) is C-admissible for all $1 \leq d \leq m-1$ and also H avoids $(4, 2^{\frac{n-4}{2}})$. Thus, we may apply Theorem 5.4.1 to all powers τ^d of τ, in which case the corresponding order will be $\frac{m}{\gcd(m,d)}$ and hence the corresponding root of unity may be chosen to be ζ^d.

Hence we have shown that $\left|\mathrm{Fix}_Y(\tau^d)\right| = Y(\zeta^d)$ for all $1 \leq d \leq m-1$. Since the $d = 0$ case is straightforward, this completes the proof of cyclic sieving.

□

Finally, for the sake of completeness, we return to prove the number-theoretic facts from Proposition 5.3.1:

Proof (of Proposition 5.3.1). Throughout the proof, let $\delta = \gcd(n, s-1)$.

Part (a): Because $\tau C = C\tau$, there is some unique $0 \leq s < n$ such that $\tau c = c^s \tau$. Note that also there is an e such that $\tau c^{s'} = c\tau$ for some s', and thus $ss' \equiv 1 \bmod n$, so s is invertible in $\mathbb{Z}/n\mathbb{Z}$. Additionally,

$$\tau(x+1) = (\tau c)(x) = (c^s \tau)(x) \equiv \tau(x) + s \bmod n. \tag{5.3}$$

Hence, the fact that $\tau(x) = sx + r$ for some r follows by induction. Finally, plugging in $x = n$, the unique $r \in \mathbb{Z}/n\mathbb{Z}$ that satisfies the equation is $r = \tau(n)$, or $r = 0$ if $\tau(n) = n$.

Remark. Equation (5.3) is the only point in the proof of Theorem 5.1.3 where we genuinely use the assumption that $c = (1\,2\,\ldots\,n)$ instead of taking it to be a general n-cycle. Clearly, we could introduce a "twist" to eliminate this assumption, but this would complicate the statement of Proposition 5.3.1, and hence the (many) parts of the proof that rely on it, while providing minimal further generality.

Part (b): Suppose that a and b are integers such that $an + b(s-1) = \delta$; these exist by

Bézout's lemma. Moreover, let q be the integer such that $r = r' + q\delta$. Then, modulo n:

$$\begin{aligned}(c^{qb}\tau c^{-qb})(x) &\equiv \tau(x-qb)+qb \\ &\equiv s(x-qb)+r+qb \\ &\equiv sx+r-qb(s-1) \\ &\equiv sx+[r-q\delta],\end{aligned}$$

and so τ is conjugate to $\tilde{\tau}$, as desired.

Part (c): The number of Z_n-fixpoints is the number of 1s in the cycle type, and so we may replace τ with $\tilde{\tau}$. Then x is a Z_n-fixpoint precisely when $sx+r \equiv x \bmod n$, that is, when $(s-1)x \equiv -r' \bmod n$. The left-hand side is divisible by δ, and thus there are no solutions to this equation unless r' is divisible by δ; by definition of r', this happens only if $r' = 0$. In this case we may divide through by δ. That is, the following are equivalent:

$$\begin{aligned}(s-1)x &\equiv 0 \bmod n \\ \tfrac{s-1}{\delta}\cdot x &\equiv 0 \bmod \tfrac{n}{\delta} \\ x &\equiv 0 \bmod \tfrac{n}{\delta},\end{aligned}$$

where the last statement holds because $\frac{s-1}{\delta}$ is invertible mod $\frac{n}{\delta}$. Thus the solutions to this congruence mod n, and hence the Z_n-fixpoints of τ, are $\{1\cdot\frac{n}{\delta}, 2\cdot\frac{n}{\delta}, \ldots, \delta\cdot\frac{n}{\delta}\}$.

For parts (d) and (e) we now have $\delta = 2$. In particular, note that n must be even.

Part (d): Repeating the proof of part (c) we see that $\delta | r$ and hence $r' = 0$, and moreover the fixpoints are $\{1\cdot\frac{n}{2}, 2\cdot\frac{n}{2}\}$, as desired.

Part (e): Because $s \in (\mathbb{Z}/n\mathbb{Z})^\times$, we know s must be odd, so write $s = 2s'+1$. If $\frac{n}{4} \in \mathbb{Z}$, we compute

$$\tau^2(\tfrac{n}{4}) \equiv s^2\cdot\tfrac{n}{4} \bmod n \equiv 4(s'^2+s')\tfrac{n}{4}+\tfrac{n}{4} \bmod n.$$

Thus, $\frac{n}{4}$ is a fixpoint of τ^2; that is, it is contained in a cycle of τ whose length divides 2. From part (d) we know that it is not a fixpoint of τ, and thus it must be in a 2-cycle. But τ only has cycles of length 1 and m; hence $m = 2$. □

Chapter 6

Further Questions

6.1 Parity-Unimodality Revisited

Encouraged by our success with the rational Schröder polynomials in Chapter 3, we considered what results might translate to other necklace polynomials $C(\alpha;q)$. A simple script in `SAGE` shows that the parity-unimodality property holds for all compositions of n for $n \leq 30$. We thus conjecture that this property holds in general.

Conjecture 6.1.1. *When* $\gcd(\alpha) = 1$*, the polynomial* $C(\alpha;t)$ *is parity-unimodal.*

This conjecture appears to be difficult and we do not have a concrete proposal for a proof. Still, we wish to make a few remarks to contextualize it. As mentioned in Chapter 3, such parity-unimodality conjectures have been gaining traction, with many generalizations of the Catalan case $\alpha = (a, a+1)$ either known or conjectural.

The recent results of Galashin and Lam [GL20] are particularly striking. These authors define a two-variable polynomial $F(s,t) = \sum_{i,j\geq 0} f_{i,j} s^i t^j$ to be (s,t)**-unimodal** if for each i, the sequences $f_{i,0}, f_{i-1,1}, \ldots, f_{0,i}$ are unimodal. Galashin and Lam prove that the so-called *rational* (s,t)*-Catalan polynomial* $C_{a,b}(s,t)$ has this property, and this is a refinement of the rational Catalan case of our theorems, because $C_{a,b}(t^{-1},t) = t^{-\frac{(a-1)(b-1)}{2}} C^0_{a,b}(t)$. More explicitly, our “skip by 2 steps” is refined by their “ diagonal steps”: decreasing the degree of $s = t^{-1}$ while also increasing the degree of t.

Unfortunately, constructions of the rational (s,t)-Catalan polynomial are quite tricky, even with the machinery of the rational Cherednik algebra available. As one might expect, we may define the Hilbert series of a (positively) “bigraded” algebra or module— that is, having graded components indexed by a pair of integers— which will have two variables. However, there is no known bigrading on $L_{b/a}(\mathbb{C})$ which produces these polynomials! As a replacement for the second grading, one may find an appropriate filtration and then consider the associated graded algebra of $L_{b/a}(\mathbb{C})$ with respect to this filtration; in the literature there are multiple distinct candidates for such a filtration (see, for instance, the summary provided in the introduction of Gorsky–Oblomkov–Rasmussen–Shende [GORS14]).

In particular, Galashin and Lam construct this polynomial in a completely different manner: it is the mixed Hodge polynomial of a certain positroid variety. It is interesting that their proof does not arise from an $\mathfrak{sl}_2(\mathbb{C})$-module, but instead the “curious Lefschetz property” of Lam and Speyer [LS16] (following the framework originally described by Hausel and Rodriguez-Villegas [HRV08]). While Hodge structures and $\mathfrak{sl}_2(\mathbb{C})$-modules are often related, we do not see a straightforward relationship here.

It seems extremely likely that an analogue of Theorem 3.3.1 should hold for (s,t)-Schröder polynomials as well. It is less clear what an appropriate analogue of Proposition 3.3.3 would be. In particular, we would like to be able to conjecture that $C(\alpha;s,t)$ is (s,t)-unimodal, but it is not clear what the (s,t)-analogue of $C(\alpha,t)$ should be outside the Schröder case.

We now pivot to a different appearance of $\mathfrak{sl}_2(\mathbb{C})$ that one might consider relevant in light of the combinatorial interpretation afforded to $C^k_{a,b}(t)$ afforded by necklaces. Let P be a finite ranked poset with maximum rank ρ, and P_i be the set of elements with rank i. The **rank generating function** of P is the polynomial $\sum_{i\geq 0} |P_i| t^i$. We say that P is

rank-symmetric if $|P_i| = |P_{\rho-i}|$ for all i, and that P is **rank-unimodal** if the sequence $(|P_0|, |P_1|, \ldots, |P_\rho|)$ is unimodal. Finally, P is **strongly Sperner** if for each $j \geq 1$, there are no j antichains whose union has more elements than the k largest P_i.

Theorem 6.1.2 (Proctor [Pro82]). *A ranked poset P with top rank ρ is rank-symmetric, rank-unimodal, and strongly Sperner if and only if it "carries a representation of $\mathfrak{sl}_2(\mathbb{C})$" in the following sense: letting P_i be the set of elements with rank i, there exist linear operators e and f acting on $\bigoplus_{i=0}^{\rho} \mathbb{C}P_i$ such that*

$$e(p) = \sum_{p \gtrdot q} c_{p,q}\, q \qquad \text{and} \qquad f(p) = \sum_{\mathrm{rank}(q)=\mathrm{rank}(p)+1} c'_{p,q}\, q$$

for some coefficients $c_{p,q}, c'_{p,q} \in \mathbb{C}$, and for which each restriction $(ef - fe)|_{\mathbb{C}P_i}$ acts by scalar multiplication $v \mapsto (2i - \rho)v$.

If a poset satisfies either of the equivalent conditions in this theorem, it is said to be **Peck**. Because one of these conditions is the existence of an $\mathfrak{sl}_2(\mathbb{C})$ action, it is reasonable to ask if there is a Peck poset which explains the parity-unimodality of the rational Schröder polynomials. In particular, if the poset and its corresponding representation are reasonably straightforward, this could provide a significantly more elementary demonstration of parity-unimodality than the proof from Chapter 3.

Definition 6.1.3. Let $C^k_{a,b}(t) = \sum_{i \geq 0} c_i t^i$ be the coefficient expansion of the rational Schröder polynomials. The **even** and **odd rational Schröder polynomials**, respectively, are defined to be

$$EC^k_{a,b}(t) = c_0 + c_2 t + c_4 t^2 + \cdots,$$
$$OC^k_{a,b}(t) = c_1 + c_3 t + c_5 t^2 + \cdots.$$

Additionally, recall that a **symmetric chain decomposition** of a ranked poset P with finite maximum rank ρ is a partition (of its ground set) $P = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$ into pairwise disjoint saturated chains Γ_i, such that $\mathrm{rank}(\min \Gamma_i) + \mathrm{rank}(\max \Gamma_i) = \rho$ for all i. Having a symmetric chain decomposition is a much stronger condition on P than being Peck, but it is somewhat more elementary, and is satisfied by many combinatorially significant posets.

Question 6.1.4. *If a, b, and k be positive integers satisfying $\gcd(a,b) = 1$ and $0 \le k \le a < b$, do there exist "natural" ranked posets $\beta_{a,b}^k$ and $\tilde{\beta}_{a,b}^k$ with the following properties (1)–(4)?*

(1) The ground sets of these posets are respectively the $(k, a-k, b-k)$-bracelets and asymmetric $(k, a-k, b-k)$-bracelets.

(2) These posets each admit symmetric chain decompositions.

(3) The rank generating functions of these posets are respectively $EC_{a,b}^k(t)$ and $OC_{a,b}^k(t)$.

(4) The identity map is an order-preserving injection $\tilde{\beta}_{a,b}^k \to \beta_{a,b}^k$.

We cannot elevate this question to the status of a conjecture because an appropriate notion of "naturalness" is both unclear and also required. Without it, we could make the following trivial construction: arbitrarily assign ranks to bracelets to achieve the correct rank sizes for both $\beta_{a,b}^k$ and $\tilde{\beta}_{a,b}^k$, and then take $A \le B$ if and only if $\mathrm{rank}(A) \le \mathrm{rank}(B)$.

If Conjecture 6.1.1 holds, it may be worthwhile to ask this question for other α-bracelets. We cannot expect it to hold for all α, however. At minimum, the analogously defined $EC(\alpha;t)$ and $OC(\alpha;t)$ must both be symmetric and unimodal; i.e. $C(\alpha;t)$ must have even degree, and some symmetric α-bracelet must exist.

We can answer in the affirmative in the rational Catalan case for $a = 3$. Let us say that a poset structure on the set of α-bracelets is **generated by local moves** if for each covering relation $A \lessdot B$, we can obtain B from A by swapping two adjacent beads. This is one property which we might consider "natural", and we wish to make some extended remarks about it.

Proposition 6.1.5. *If b is not divisible by 3, there exist posets $\beta^0_{3,b}$ and $\tilde{\beta}^0_{3,b}$ which satisfy the four properties of Question 6.1.4 and are also generated by local moves.*

Proof (sketch). Let us say that the 3 beads of color 2 are **white**, and the b beads of color 3 are **black**. (There are $k = 0$ beads of color 1.)

Recall that the **dominance order** $\mathscr{D}(b)$ on partitions of b is defined in this way: pad all paritions of b with infinitely many zeros, and then $(\lambda_1, \lambda_2, \dots) \leq (\mu_1, \mu_2, \dots)$ whenever the partitions satisfy the partial sum inequalities $\lambda_1 + \cdots + \lambda_n \leq \mu_1 + \cdots + \mu_n$ for all $n \geq 1$. It turns out that there is an interval $[\bot, \top]$ in the dominance order that is a satisfactory model for $\beta^0_{3,b}$. The top element is the partition $\top = (b)$, and the bottom is the partition $\bot = (\lceil b/3 \rceil, [b/3], \lfloor b/3 \rfloor)$, where $[x]$ is the nearest integer to x.

For any bracelet, each of the three adjacent pairs of white beads has a number of black beads between them. These numbers can be viewed as the parts of a partition, because the white beads can be arbitrarily permuted by rotations and reflections to put the numbers in decreasing order. In particular, $\top$ and $\bot$ correspond respectively to the bracelets where all three white beads are next to each other, and where they are as evenly distributed as possible.

We omit a careful calculation that the ground set of $[\bot, \top]$ is in bijection with the set of $(0, 3, b)$-necklaces and also asymmetric $(0, 3, b+3)$-necklaces. The fact that it is generated by local moves is then straightforward, since a local move corresponds precisely to increasing one part of the partition by 1, and decreasing another by 1; all cover relations in $\mathscr{D}(b)$ are of this form.

Finally, one may check the following in a straightfoward manner. First, that the map

$$(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1 + 4, \lambda_2 + 1, \lambda_3 + 1)$$

is an injection $\beta^0_{3,b-6} \to \beta^0_{3,b}$. Then, that after making these identifications inductively we have $[\perp,\top] = \Gamma_b \cup \Gamma_{b-6} \cup \Gamma_{b-12} \cup \cdots \cup \Gamma_{b-6\lfloor \frac{b}{6} \rfloor}$ is a symmetric chain decomposition of $[\perp,\top]$, where $\Gamma_{b'} = \{\lambda \in \beta^0_{3,b'} : \lambda_3 = 0\} \cup \{\lambda \in \beta^0_{3,b'} : \lambda_1 - \lambda_2 \leq 2\}$. Last, that an explicit computation of $EC^0_{3,b}(t)$ and $OC^0_{3,b}(t)$ gives the required rank generating functions. □

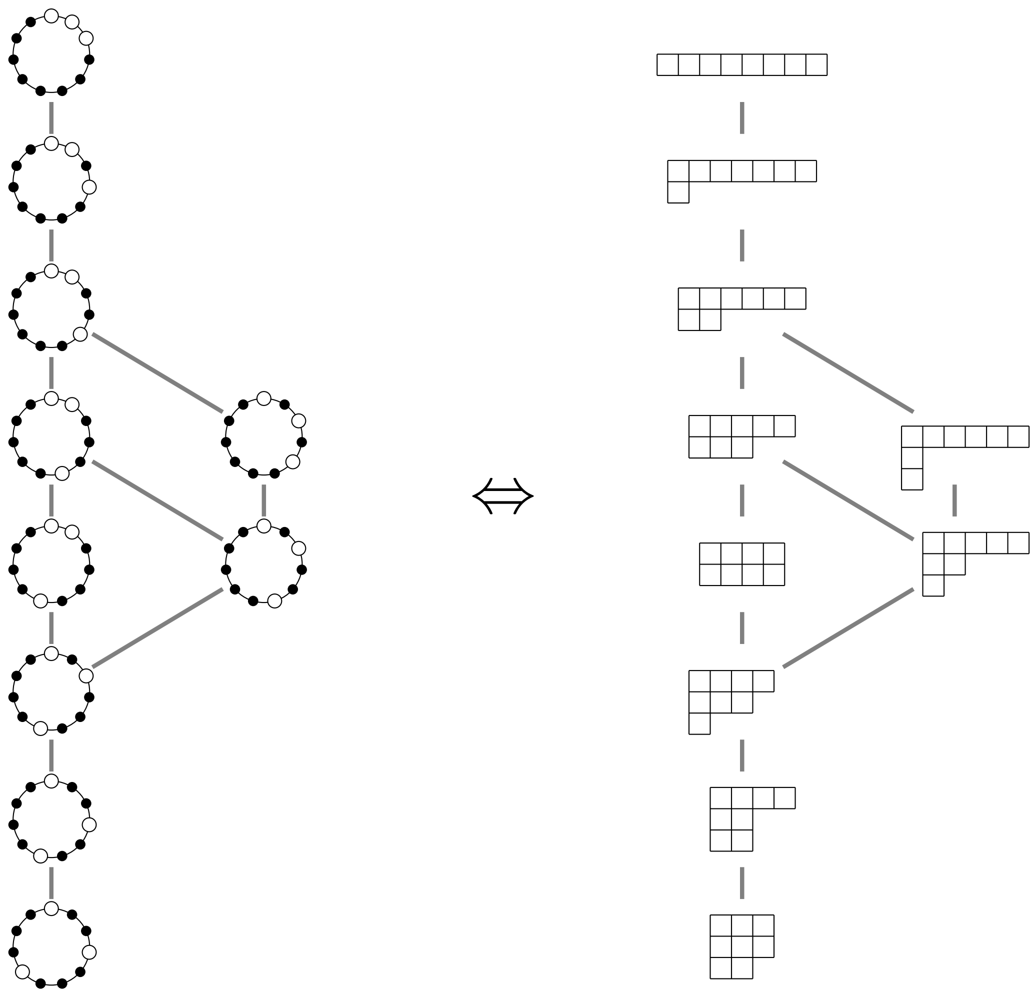


Figure 6.1: The poset $\beta^0_{3,8}$, arranged into two columns corresponding to the symmetric chain decomposition $\Gamma_8 \cup \Gamma_2$. Note $EC_{3,8}(t) = 1+t+t^2+2t^3+2t^4+t^5+t^6+t^7$ is its rank generating function.

The argument above suggests two candidates for making the question into a precise conjecture. Unfortunately, both are false. First, being generated by local moves is not a viable condition for higher values of a. For instance, a case-by-case analysis reveals that there are no bounded ranked posets generated by local moves on $(4,5)$-bracelets that have rank generating function $EC^0_{4,5}(t)$.

As a second attempt, there is nothing in the definition of dominance which forces us to restrict our attention to partitions. So, on one hand, while not preserving the bracelet by arbitrarily permuting the gaps between white beads for $a > 3$, we may still consider the dominance order on representatives of the bracelets. Unfortunately, choosing appropriate representatives still appears difficult. For instance, calculations with the $(4,5)$-bracelets already show that any refinement or coarsening of the lexicographic or revlex orders fail to have the correct rank sizes (when they are ranked at all).

6.2 Semi-Classical Parking Spaces

Let W be a Weyl group, Q be a root lattice for W, and ζ be a $(b-1)^{\text{th}}$ root of unity, where b is an integer very good for W. Recall that we called $\mathbb{C}[Q/bQ]$ the classical parking space, and in light of Theorem 4.1.3 we had called $\mathbb{C}[Q \otimes_{\mathbb{Z}} A]$ "semi-classical parking spaces," for abelian groups A with order b. The content of the aforementioned theorem is that these are indeed parking spaces for W with parameter b.

In our personal communication with Arreche [Arr19], the original formulation of the prime power case of Theorem 4.1.2 had a more arithmetic flavor. Rather than "forcing" the $\mathbb{F}_q$-vector space structure, it may be found perhaps more naturally in the following way. Let $\widetilde{Q}$ be the $\mathbb{Z}[\zeta]$-span of the root basis rather than just the $\mathbb{Z}$-span; that is, $\widetilde{Q} = Q \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$. Now the extension $\mathbb{Z} \subseteq \mathbb{Z}[\zeta]$ is unramified at the prime p; that is, $\mathfrak{p} = p\mathbb{Z}[\zeta]$ is a prime ideal. Because of this, $\mathbb{Z}[\zeta]/\mathfrak{p} \cong \mathbb{F}_q$ and therefore we may construct $Q \otimes \mathbb{F}_q$ as $\widetilde{Q}/\mathfrak{p}\widetilde{Q}$.

Notice in particular that $\widetilde{Q}$ is not a lattice. However, it is a "$\mathbb{Z}[\zeta]$-lattice." Generally, given a number field $\mathbb{F}$ with ring of integers $\mathscr{O}$, an **$\mathscr{O}$-lattice** is a finitely generated $\mathscr{O}$-submodule of the vector space $\mathbb{F}^r$ for some r, such that $L \otimes_{\mathbb{Z}} \mathbb{F}$ is all of $\mathbb{F}^r$. This is an intriguing reformulation because every complex reflection group can be defined over a number field (easy to see since there are only finitely many matrix entries in W). Since the computational heart of Lemma 4.1.4 is a verification that we gain no new eigenvectors when thinking of $w \in W$ as an $\mathbb{F}_q$-matrix rather than a $\mathbb{C}$-matrix, it is natural to wonder if we similarly gain no new eigenvectors when thinking of $w \in \mathrm{Mat}_{r\times r}(\mathscr{O}/\mathfrak{p})$.

Question 6.2.1. *Let $\mathbb{F}$ be a number field, $\mathscr{O}$ be its ring of integers, and W be an $\mathbb{F}$-reflection group. For which $\mathscr{O}$-lattices L and which prime ideals $\mathfrak{p}$ do we have, for all integers d,*

$$\dim_{\mathbb{C}} \ker(\zeta^d - w) = \dim_{\mathscr{O}/\mathfrak{p}} \ker_{L/\mathfrak{p}L}(c^d - w)?$$

(As usual, ζ is a primitive $(q-1)^{th}$ root of unity, and c is a generator of $(\mathscr{O}/\mathfrak{p})^\times \cong \mathbb{F}_q^\times$.)

We now return to the case of Weyl groups to make a more concrete conjecture.

As we mentioned in Section 4.2, it need not be the case that group elements $w \in W$ give rise to $t - w$ having a Smith normal form over $\mathbb{Z}[t]$. Of course, every $t - w$ has a Smith normal form over $\mathbb{Q}[t]$, and so we may blame this failure on "necessary denominators" appearing in every Smith factorization. Because the partial calculations ended up being extremely useful, we may wonder about which demoninators would actually be necessary to complete them. Computer calculations with $r \leq 10$ suggest that there are not so many:

Conjecture 6.2.2. *Let Q be a root lattice for an irreducible Weyl group with Coxeter number h. Define*

$$h' = \begin{cases} 1 & \text{if } W \text{ is of type } \mathrm{A} \\ 2 & \text{if } W \text{ is of type } \mathrm{B}, \mathrm{C}, \text{ or } \mathrm{D} \\ h & \text{if } W \text{ is of exceptional type } (\mathrm{E}, \mathrm{F}, \text{ or } \mathrm{G}). \end{cases}$$

Then $t-w$ has a Smith normal form over $\mathbb{Z}[\frac{1}{h'},t]$ for each $w\in W$, in the sense that this is true when w is identified with an integer matrix describing its action on Q in some basis (e.g. the root basis).

This conjecture is perhaps somewhat surprising because for most of the types, h' has a natural interpretation: the integers coprime to h' are those which are very good for W. Mysteriously, though, in type A it seems that we do not have any necessary denominators at all. It is not clear whether this numerical invariant has a more conceptual, type-independent definition.

In particular, the $r\leq 10$ calculations included all the exceptional types. Briefly, we describe the procedure, which is somewhat more elaborate than the other computer computations in Chapter 4. Since `gap3` has no native functionality to compute the Smith normal form over polynomial rings, we instead wrote code in `Maple` to compute Smith factorizations $tI-w=UDV$ over $\mathbb{Q}[t]$, and then check the prime factorizations for the denominators of all entries in U and V, verifying their invertibility over $\mathbb{Z}[\frac{1}{h'},t]$.

Unfortunately, `Maple` does not natively support computations with Weyl groups as matrix groups, so we used a `gap3` script from the other conjectures to generate a list of conjugacy class representatives. Loading these into `Maple` and running the code yields the perhaps-alarming result that one matrix from E_7 has entries of U and V with denominator 5 (coprime to $h=12$), and ten matrices from E_8 also have "bad" denominators. (These examples do not disprove the conjecture because Smith factorizations are not unique— all we have learned is that `Maple`'s algorithm yields U and V with bad denominators.)

We identified these 11 group elements which gave bad denominators and returned to `gap3` to compute N random elements in the corresponding conjugacy classes, and then ran the `Maple` code again on each of the resulting lists. When we took $N\geq 10$ we consistently found at least one representative that confirmed the existence of an appropriate Smith factorization, as desired.

6.3 Improving Theorem 5.1.3

Finally, we return to our cyclic sieving phenomenon from Theorem 5.1.3. While it does encompass a large number of triples $(C\backslash S_n/H, X(q), \langle\tau\rangle)$, it is not particularly elegant. We have already discussed that the technical conditions on H are essentially required for the class of τ, but also that other τ are possible and we have no intuition for any conditions on H in those cases, much less a conjectural organizing scheme for them.

The result of Reiner, Stanton and White that we quoted in Chapter 5 [RSW04, Theorem 8.2] is in fact stated for arbitrary reflection groups. We might at first be discouraged from generalizing to this setting because of the characterization of our H which is so specialized to type A.

However, it is worth observing that taking $H = S_{\alpha_1} \times \cdots \times S_{\alpha_k}$, greatly simplifies the conditions of Theorem 5.1.3. Whenever τ has the correct cycle types (that is, $(m^k, 1, 1)$ or $(m^k, 1)$ for some m) then the triple $(S_n/H, \begin{bmatrix} n \\ \alpha \end{bmatrix}_t, \langle\tau\rangle)$ exhibits the cyclic sieving phenomenon when $\gcd(\alpha) = 1$. Checking the cycle conditions in Theorem 5.1.3, we find: if H does not avoid $(\ell^{\frac{n}{\ell}})$ for some divisor $\ell > 1$ of n then $\ell | \gcd(\alpha)$, and if H does not avoid any of $(4^{\frac{n-2}{4}}, 2), \left((2m)^{\frac{n-2}{2m}}, 2\right)$, or $\left((2\ell)^{\frac{n-2}{2\ell}}, 2\right)$ for some divisor $\ell > 1$ of m, then $2 | \gcd(\alpha)$. In all cases we contradict $\gcd(\alpha) = 1$.

Such H (without the $\gcd(\alpha) = 1$ condition) are called the **standard parabolic subgroups** of S_n. A general **parabolic subgroup** of S_n is any conjugate of such an H, and so in particular it also satisfies the cycle avoidance conditions. Thus, we have an elementary condition for which parabolic subgroups admit a triple, for our choices of τ. For arbitrary complex reflection groups W, one typically defines a **parabolic subgroup** of $W \leq \mathrm{GL}(V)$ to be any subgroup of elements which act trivially on a subspace of V. Moreover, a typical generalization of the long cycle $(1\,2\,\ldots\,n)$ to complex reflection groups W is given by the so-called **regular elements**: $w \in W$ which have an eigenvector that is not fixed by any non-identity element $w' \in W$. This leads to the following question:

Question 6.3.1. *For a complex reflection group $W \leq \mathrm{GL}(V)$, let C be the cyclic group generated by a regular element, and $\tau \in N_{W(C)}$. Is there a simple characterization of which parabolic subgroups P admit a triple $(C\backslash W/P, X(t), \langle\tau\rangle)$, where $X(t) = \mathrm{Hilb}\left((SV_W^*)^P; t\right)$? Perhaps if the order of c is small modulo the order of τ?*

This question is quite broad; even for $W = S_n$ this is not fully resolved by Theorem 5.1.3, since both n- and $(n-1)$-cycles are regular elements in S_n. A complete resolution to the question even in this case would be quite interesting.

There is another, more subtle, deficiency. The manner in which Theorem 5.1.3 is proved is, at the highest level, simply evaluating both sides of $Y(\zeta^d) = |\mathrm{Fix}_Y(\tau^d)|$. The "best" proofs of cyclic sieving are those fitting into a certain **representation theory paradigm** with roots in the work of Stembridge [Ste94a] [Ste94b], and clarified for $m > 2$ in the original paper of Reiner, Stanton, and White [RSW04, Lemma 2.4]. We refer the interested reader to their papers for a more complete description, but we recall here the equivalence that makes it work:

Proposition 6.3.2. *Let C be a cyclic group of order m generated by c, and ζ be a primitive m^{th} root of unity. The triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon if and only if $\mathbb{C}(X)$ is isomorphic to the following graded C-representation $A_X = \bigoplus_{i\geq 0} A_{X,i}$. The Hilbert series of A_X (which fully determines the graded vector space structure) is $X(q)$, and on each graded component $A_{X,i}$, the generator $c \in C$ acts by scalar multiplication: $c \cdot a = \zeta^i a$.*

Proofs that fit into this paradigm are known for some triples, including the one for complex reflection groups referenced above. While our proof makes use of representation theory via Molien's formula, it clearly does not follow this paradigm, and we expect such a proof to be rather difficult. Indeed, even some of the triples known already in Reiner–Stanton–White to exhibit cyclic sieving have not yet found representation-thereoretic proofs. Given the opportunities for improvement identified above, it seems fair to say that even the statement of the theorem needs a bit more clarification before attempting such a proof, but we mention it as a longer-term goal.

6.4 Secondary Cyclic Sieving

Another way that we might more abstractly "improve" Theorem 5.1.3 is to place it in the context of some general framework. We briefly outline such a framework here, which would also encompass the complex reflection group setting in Question 6.3.1.

Definition 6.4.1. Suppose that a group G acts on a set X, and $(X, X(t), C)$ exhibits the cyclic sieving phenomenon for some polynomial $X(t)$ and some cyclic subgroup $C \leq G$ which acts freely on X. Define $Y = C\backslash X$, and the action of $N_G(C)$ on Y via $\tau \cdot Cx = C(\tau \cdot x)$, and the polynomial $Y(q) = \frac{1}{[n]_t} X(t)$. Then for any τ such that $(Y, Y(t), \langle\tau\rangle)$ exhibits the cyclic sieving phenomenon, we say that the triple $(X, X(t), C)$ **exhibits a secondary cylic sieving phenomenon** with respect to τ.

Remark. As before, $Y(t)$ is a polynomial with integer coefficients because of Proposition 5.1.1.

In particular, Theorem 5.0.1 can be reformutated as stating that if $\gcd(\alpha) = 1$ and $W(\alpha)$ is the collection of words having exactly α_i occurrences of the letter i, then $(W(\alpha), \begin{bmatrix} n \\ \alpha \end{bmatrix}_q, C)$ exhibits a secondary cyclic sieving phenomenon with respect to the action of reflection. Similarly, Theorem 5.1.3 can be understood as describing some sufficient conditions for the triple $(S_n/H, X(t), C)$ to exhibit a secondary cyclic sieving phenomenon with respect to an element $\tau \in S_n$, for the polynomial $X(t)$ as given in (5.1).

The language here is quite general, but one must admit that the examples are lacking. The sensible next step would be to turn to the now-extensive cyclic sieving literature (for instance, [BR16], [Gor19], [RS18], [SW20], [Thi17]) with the following question in mind:

Question 6.4.2. *To what extent do previously known cyclic sieving results admit secondary cyclic sieving phenomena?*

We know of at least one other example. Consider the set X_k of noncrossing partitions $\{1,\ldots,n\}$ with k blocks. Recall that a set partition of $\{1,\ldots,n\}$ is called **noncrossing** if for any four numbers $w<x<y<z$ such that w and y are in the same block, and x and z are in the same block, then in fact all four are in the same block. Noncrossing partitions admit a geometric action by the subgroup $D_n \cong \langle c, \tau_0 \rangle$ of S_n, where c and τ_0 are the same elements that act on $W(\alpha)$ as described in the introduction of Chapter 5.

The elements of X_k are enumerated by the **Narayana numbers** $N(n,k)$, which have a product formula that suggests a t-analogue:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}; \qquad X_k(t) = \frac{1}{[n]_t}\begin{bmatrix} n \\ k \end{bmatrix}_t \begin{bmatrix} n \\ k-1 \end{bmatrix}_t.$$

It is shown in [RSW04, §7] that $(X_k, X_k(t), \langle c \rangle)$ exhibits the cyclic sieving phenomenon. Moreover, we can check that $\langle c \rangle$ acts freely whenever $\gcd(n,k) = \gcd(n,k-1) = 1$. In particular, this implies that n must be odd, and thus $[n]_t$ evaluates to 1 at $t=-1$. Thus:

$$Y_k(-1) = \frac{1}{[n]_t^2}\begin{bmatrix} n \\ k \end{bmatrix}_t \begin{bmatrix} n \\ k-1 \end{bmatrix}_t \Bigg|_{t=-1} = \frac{1}{[n]_t}\begin{bmatrix} n \\ k \end{bmatrix}_t \begin{bmatrix} n \\ k-1 \end{bmatrix}_t \Bigg|_{t=-1} = X_k(-1).$$

In [Din16, §3.2] Ding computes the number of τ_0-fixed elements of X_k to be $X_k(-1)$, and hence $Y_k(-1)$. Also, in [CS05, §4], Callan and Smiley show the surprising fact that the number of τ_0-fixed elements of Y_k is the same as the number of τ_0-fixed elements of X_k. We thus conclude that $Y(-1) = \#\{y \in Y_k : \tau_0(y) = y\}$. Hence, $(Y_k, Y_k(t), \langle \tau_0 \rangle)$ exhibits the cyclic sieving phenomenon, or in other words, the triple $(X_k, X_k(t), \langle c \rangle)$ exhibits a secondary cyclic sieving phenomenon with respect to the reflection τ_0.

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Appendix

Code for Lemma 4.1.4

The following is `gap3` code used to prove Lemma 4.1.4 for the exceptional types.

```
#------------------#
# HELPER FUNCTIONS #
#------------------#
# Finds the degree of the minimal extension
# of F_p which contains (d)th roots of unity
#
DegOfMinExtWithRootOfUnity := function(p,d)
    local exp;
    exp := 0;

    repeat
        exp := exp+1;
     until RemInt(p^exp-1,d)=0;
    return exp;
 end;


#---------------------#
# WORKHORSE FUNCTIONS #
#---------------------#
# Checks whether the eigenspace dimensions
#  match up for a specific matrix wMat and
#  pair of eigenvalues gi and zi

CheckEigens := function(wMat, gi, zi)
    local rk, wpdiAgree,
```

```
        ParkAEig, ParkAMat, ParkACharExp,
        ParkQEig, ParkQMat, ParkQCharExp;

  rk := DimensionsMat(wMat)[1];

  # eigenspace dimension for C
  ParkAEig := zi*IdentityMat(rk,Field(zi));
  ParkAMat := wMat - ParkAEig;
  ParkACharExp := Length(NullspaceMat(ParkAMat));

  # eigenspace dimension for Fpe
  ParkQEig := gi*IdentityMat(rk,Field(gi));
  ParkQMat := wMat - ParkQEig;
  ParkQCharExp := Length(NullspaceMat(ParkQMat));

  wpdiAgree := (ParkACharExp=ParkQCharExp);
  if not wpdiAgree then
      # This code runs if the dimensions don't match
      # So, should never happen for the main loop
      Print(wMat); Print("\n\n");
      Print(gi); Print("\n");
      Print(ParkACharExp); Print(" ");
      Print(ParkQCharExp);
      Print("\n------\n\n");
   else
      Print(ParkQCharExp); Print("\n");
   fi;
  return wpdiAgree;
end;

# Checks whether the eignspace dimensions
# match up for a specific matrix and for all
# field elements of C and F_q of order d
#
CheckFieldEltsOfOrder := function(d, p, wMat)
  local e, k, g, z, i,
        fpeElt, clxElt,
        wpdAgree, wpdiAgree;

  # name primitive roots in fields F_p^e and C
  e := DegOfMinExtWithRootOfUnity(p,d);
  k := (p^e-1)/d;
  g := Z(p^e)^k;
```

```
    z := E(p^e-1)^k;  #equals E(d)
    Print(" p="); Print(p);
    Print(" d="); Print(d);
    Print(" e="); Print(e);

    wpdAgree := true;
    # [[to be falsified in the loop below]]

    for i in [1..d] do
        if GcdInt(i,d)=1 then
            Print(i); Print(": ");
            wpdiAgree := true;
            fpeElt := g^i;
            clxElt := z^i;
            wpdiAgree := CheckEigens(wMat,fpeElt,clxElt);
            wpdAgree := wpdAgree and wpdiAgree;
         fi;
     od;
    Print("\n");
    return wpdAgree;
 end;


#------------------#
# HELPER FUNCTIONS #
#------------------#
# Finds a prime bigger than order if one exists,
#  and checks if the eignspace dimensions match
#
CheckPrime := function(p, wMat, order)
    local d, wpAgree, wpdAgree;

    wpAgree := true;
    for d in [1..order] do
        if RemInt(order,d) = 0 and GcdInt(p,d)=1 then
            wpdAgree := CheckFieldEltsOfOrder(d,p,wMat);
            wpAgree := wpAgree and wpdAgree;
         fi;
     od;
    return wpAgree;
 end;
```

```
#---------------#
# MAIN FUNCTION #
#---------------#
# Checks the lemma for an exceptional Weyl
# group (Also works for type A but for types
# B/C/D, does not check the lemma for the
# odd primes that divide h.)
#
CheckLemmaFor := function(W)
    local rk, h, r,
          class, w, wMat, order,
          charsAgree, wAgree, wpAgree;

    rk:= W.rank;
    h:= Length(W.roots)/rk;

    charsAgree := true;
    # [[to be falsified in the loop below]]

    for class in ConjugacyClasses(W) do
        w := class.representative;
        wMat := MatYPerm(W, w);

        order := Order(W, w);
        Print("Order "); Print(order);
        Print(" element\n");

        wAgree := true;
        # [[to be falsified in the loop below]]

        for p in [1..order] do
            if IsPrimeInt(p) and
             GcdInt(p,h)=1 and RemInt(order,p)=0 then
                wpAgree := CheckPrime(p,wMat,order);
                wAgree := wAgree and wpAgree;
                # [[if any wp is false, then w is also]]
             fi;
         od;

        charsAgree := charsAgree and wAgree;
     od;
    return charsAgree;
end;
```

```
#-----------#
# MAIN LOOP #
#-----------#
# Checks the lemma for all exceptional types
#
for W in [
 CoxeterGroup("G",2,Rationals),
 CoxeterGroup("F",4,Rationals),
 CoxeterGroup("E",6,Rationals),
 CoxeterGroup("E",7,Rationals),
 CoxeterGroup("E",8,Rationals)
         ] do
    Print("For the group "); Print(W); Print(",\n");

    if CheckLemmaFor(W) then
        Print("the parking spaces are isomorphic");
        Print(" for any very good p!\n\n");

     else
        Print("...the lemma is false :(\n\n");
     fi;
 od;
```

Code for Lemma 4.1.5

The following is `gap3` code used to prove Lemma 4.1.5 for the exceptional types.

```
#-----------------#
# HELPER FUNCTION #
#-----------------#
# Given a nonzero number n, and a number h,
# returns true if n has a prime factor that h
# does not have
#
ContainsBadPrimes := function(n,h)
    local g;

    g := GcdInt(n,h);
    if g = 1 then
        if n=1 then
            return false;
         else
            return true;
         fi;
     else
        return ContainsBadPrimes(n/g,h);
     fi;
 end;


#-----------------#
# HELPER FUNCTION #
#-----------------#
# Given a matrix in Hermite normal form M,
# and a number h, returns true if any pivots
# have a prime factor that h does not have
#
ContainsBadPrimesInPivots := function(M,h)
    local ans, row, elt,
          ansRow, seenPivot;

    ans := false;
    for row in M do
        ansRow := false;
        seenPivot := false;
        for elt in row do
```

```
            if not seenPivot and not elt = 0 then
                ansRow := ContainsBadPrimes(elt,h);
                seenPivot := true;
             fi;
         od;
        ans := ans or ansRow;
     od;
    return ans;
 end;


#------------#
# OUTER LOOP #
#------------#
# Checks the lemma for all exceptional types
#
for W in [
CoxeterGroup("G",2),
CoxeterGroup("F",4),
CoxeterGroup("E",6),
CoxeterGroup("E",7),
CoxeterGroup("E",8)
         ] do
    Print("For the group "); Print(W); Print(",\n");

    r := W.rank;
    h:= Order(W,Product(W.reflections));
    for class in ConjugacyClasses(W) do
        w := class.representative;
        wMat := MatYPerm(W, w);
        wEig := IdentityMat(r) - wMat;
        wEigNorm := SmithNormalFormIntegerMat(wEig);
        if ContainsBadPrimesInPivots(wEigNorm, h) then
            # This code should never run
            Print("\n"); PrintArray(wEigNorm);
            Print("\n");
        fi;

     od;
    Print("If no matrices have printed, then");
    Print(" the parking spaces are isomorphic");
    Print(" for any A of very good order!\n\n");
 od;
```