Advanced Optimal Control Strategies for Nonlinear Systems with Application to Wind Energy

A THESIS

SUBMITTED TO THE FACULTY OF

THE UNIVERSITY OF MINNESOTA

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

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June, 2021

Acknowledgement

I would like to express my utmost gratitude to my advisor, Dr. Desineni Subbaram Naidu, who introduced me to the concept of nonlinear optimal control which became my motivation to conduct this research. For the last two and a half years at the University of Minnesota Duluth (UMD), he has always been very supportive and inspiring mentor to me. I truly thank him for his exceptional and unconditional support and guidance.

Secondly, I'm expressing my appreciation and gratitude to Dr. Jing Bai and Dr. Arshia Khan for their time and effort to serve my thesis committee. Their kind advice and corrections have made this work more rigorous.

I would also like to thank Ms. Shey Peterson at the Electrical Engineering Department who was always by my side whenever I had any trouble. She made my life easier at UMD. I am also grateful to all the stuff and faculty members at the Electrical Engineering department for helping me conducting my course works, research and TA sessions.

Lastly, I would like to thank my parents, younger sister, and friends for their continuous support.

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Abstract

The research in this Masters thesis presents the theory and application of the existing and simplified version of the Finite-Horizon State-Dependent Riccati Equation (SDRE) nonlinear optimal control techniques. SDRE technique for closed-loop optimal control of nonlinear systems has been an active research area during the last decade. Although SDRE provides great advantages to the control systems designers by providing design flexibility on the state matrices, the existing technique for finite-horizon control is approximate and involves several steps which makes it computationally complex. The SDRE technique for finite-horizon optimal problem involves first representing any given dynamical system in the state-dependent coefficient (SDC) form and then solving the SDRE at each small time interval during the given finite-horizon period. The process then is to assume that during the small intervals the Riccati coefficient and vector coefficient are constant and hence use the algebraic Riccati equation and algebraic vector equation. This assumption makes the solution suboptimal. In this research, without the assumption of SDRE coefficients being constant during each small interval, a simplified SDRE technique is presented by employing the analytic solution for the matrix differential Riccati equation and vector differential equation, hence avoiding the suboptimality and eliminating the several steps associated with the existing SDRE technique. The validity of the proposed simplified SDRE method is illustrated and compared with the existing SDRE for both regulation and tracking problems by implementing them in a nonlinear, sixth-order model of a permanent magnet synchronous generator-based wind energy system. The research conducted in this thesis resulted in the publication of three international conferences and one journal article (under preparation).

Chapter 1

Introduction

1.1 Background

The modern control theory dealing with multiple inputs and multiple outputs (MIMO) is based on state variable representation in terms of a set of first-order differential equations. Here, the system is characterized by state variables in linear time-invariant form as [31]

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{1.1.1}$$

where $\mathbf{A}(t)$ is nxn state matrix, $\mathbf{B}(t)$ is nxr control matrix, $\mathbf{C}(t)$ is mxn output matrix, $\mathbf{D}(t)$ is mxr transfer matrices, $\mathbf{x}(t)$ is the system state, and the control signal $\mathbf{u}(t)$ is unconstrained.

The modern control theory prescribes that all or most of the the state variables should be fed back after suitable weighting. Fig 1.1 demonstrates a modern control

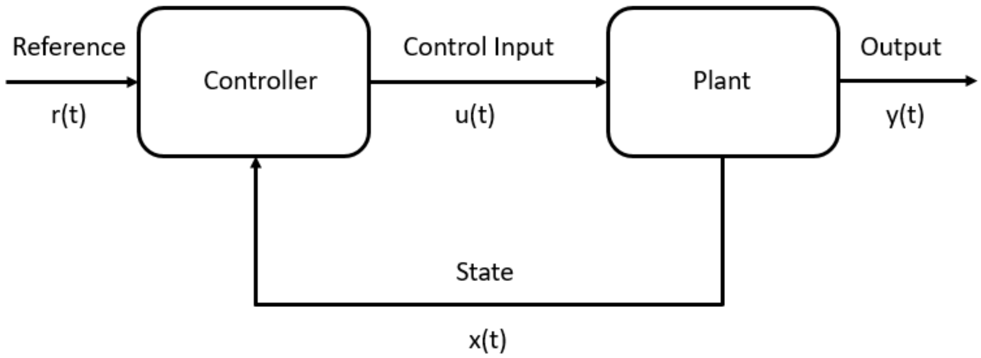

Figure 1.1: State feedback control configuration

configuration where the input $\mathbf{u}(t)$ to the plant is determined by the controller driven by reference signal $\mathbf{r}(t)$ and system states $\mathbf{x}(t)$. Here, all or most of the state variables are available for feedback control, and it depends on well-established matrix theory.

Optimization is a very desirable feature in modern control application. Optimal control theory provides a design tool for manipulating a real-world system in a way where a trade-off between control effort and control objective is maintained [30]. Linear optimal control is a well-established area of research and has its application in various fields such as engineering, economics, aerospace and finance [38]. However, the performance of this linear optimal controllers become limited, when the system is highly nonlinear in nature. The availability of the powerful low-cost microprocessor, the advancement in the simulation and modeling techniques, and the modern mathematical software greatly motivated the control community to come up with a number of effective nonlinear control technique such as feedback linearization, model predictive control, adaptive control, sliding mode control etc [33]. However, most of them are limited in application and depends on choosing between various factors such as cost, performance, optimality, computation time, and memory requirements [9, 22].

Recently, State-Dependent Riccati Equation (SDRE) technique for designing nonlinear controller gained a great attention among the control community through the last decade [10, 11, 17, 23]. Some great features of SDRE are that it provides great design flexibility to the system designer through state-dependent weighting matrices, allows nonlinearities in the system states in a linear-like structure, and has applicability to a large class of nonlinear systems [10]. Substantial research work has been done to implement SDRE in the areas of robotics, generators, RC batteries, aircrafts, missiles, satellite systems, wind energy conversion systems, biomedical systems etc [3, 13, 40, 42, 43]. Although SDRE still didn't receive a wide range of implementation in real-world physical systems, it is showing promising results in simulations and laboratory experiments.

SDRE has been studied for both infinite- and finite horizon case. For infinite horizon SDRE, control signal is obtained by the solution of an algebraic Riccati equation (ARE) with SDC matrices. On the other hand, for finite-horizon case, the solution of the control law is time dependent and a differential Riccati equation requires to be solved [22].

Finite-horizon optimal control via SDRE technique were solved for both regulation and tracking problems [21, 22]. In finite horizon optimal regulation, the control law is found in terms of the solution of the state-dependent differential Riccati equation (DRE) at each small time step of a finite-horizon period. In case of finite-horizon tracking, a vector differential equation (VDE) along with DRE is solved simultaneously at each time step. Unfortunately, the solution of DRE and VDE not only involves several steps that makes the control solution computationally complex, the assumption of DRE and VDE being constant at each time step make the solution suboptimal.

1.2 Literature Review

The State Dependent Riccati Equation (SDRE) method, due to its simple algorithms and design flexibility has emerged as a nonlinear control system design technique in a variety of practical and meaningful applications such as robotics, aircrafts, process control, biomedical systems, power systems, automotive systems, satellites and spacecrafts, missiles, unmanned aerial vehicles, ships, and underwater vehicles [22].

SDRE technique was first proposed by Pearson (1962) and later expanded by Wernli Cook (1975), Mracek Cloutier (1998), and Friedland (1996). In [19], it is shown that the infinite-horizon SDRE feedback scheme for nonlinear optimal control problem- in the multivariable case, is locally asymptotically stable and locally asymptotically optimal, and in the scalar case, is optimal. It is also shown in the general multivariable case that the Pontryagin necessary conditions for optimality are satisfied asymptotically by the algorithm. This research actually initiated a growing use of SDRE methods in a wide variety of nonlinear control applications [9].

From the literature, the available methods to find the solution of SDRE can be classified to classical and intelligent methods. One way in the classical methods is to determine the open-loop solution using some numerical methods and then use predictive techniques for closing the control loop [18,20]. However, the dependency of the solution to a pre-specified initial condition is major drawback in this technique. On the other hand, the intelligent methods uses offline training of weights and then online usage of neural networks for determining control signals. Unfortunately, if the resulted trajectory lies outside of the neural network trained domain this methods becomes useless [8,14,16]. A method based on polynomial fitting for the solution of SDRE was proposed in [47] and was applied in the design of missile guidance. Although this method has high computational efficiency and accuracy, it can only be used to those systems which satisfy the special conditions.

Research was also performed to formulate the SDRE controller for non-affine nonlinear systems. In [4], power series approximation (PSA) and online control update (OCU) methods were used to solve the SDRE control problem for non-affine nonlinear systems. The PSA method solves the control problem by transforming the state-dependent Riccati equation into Riccati equation and multiple Lyapunove equations that are solved offline. However, the inability to implement the control method in real-time makes it less effective for real-world application. Suboptimality is also an issue for this control method which depends on the number of terms that can be used for approximation of the state matrix $A(x, u)$ and input gain matrix $B(x, u)$. A method for obtaining the closed-form state-dependent coefficient (SDC) matrices for synthesis of SDRE controller for non-affine nonlinear systems were also presented in [44].

Even though a significant amount of research were conducted in infinite-horizon SDRE, finite-horizon SDRE is still a challenging problem among the control community due to the complexity of time-dependency of the Hamilton Jacobi Bellman (HJB) differential equation [17]. It is well known that the solution of the finite-horizon SDRE cannot be found analytically, except for some limited simple non-linear systems. Though Taylor series and interpolation methods can be used to approximate the offline solution, it is hard to find the solution of SDRE with these methods for highly nonlinear systems [22].

In [45], a dynamic solution method is proposed by the authors to solve the HJB equation. This paper proposed a method to construct dynamically an exact solution of a HJB equation, i.e. considering the immersion of the system to be solved into an augmented system, without actually solving a partial differential equation. Although this method avoids the explicit solution of the HJB equation, a proper selection of the initial condition of the dynamic controller is required to find the control law. Reference [34] proposed a patchy technique [2] to solve the HJB equations and applied it to first and second order systems. Unfortunately, this method doesn't mentioned about the boundary condition of the outer patches. Research were also done to find the solution of the HJB equation using power series method. In [27], the authors proposed a two stage way to solve the HJB. First, the HJB is solved in a neighborhod of the origin using the power series method. Then, the ordinary differential equations are formulated for the higher order partial derivatives of the solution along the extremal. Authors in [1], focused on the class of systems whose right-hand sides are analytic functions, and they have utilized the quadratization technique to lift the HJB equation into an operator equation that resembles the Riccati equation.

Being motivated by the SDRE method for infinite-horizon nonlinear optimal controls which gives the control law through an online solution to the state-dependent algebraic Riccati equation, a state-dependent differential Riccati equation method is introduced in [15] for finite-horizon regulation problem. This research introduced an approximate analytical approach that converts the DRE to the linear differential Lyapunov equation (DLE) [35], then solves the DLE in real-time. Authors in [22], also followed the similar approach but extended the method to solve the tracking problem where in addition to solving the DRE, a vector differential equation (VDE) is solved simultaneously to find the control law. VDE is solved in a similar approximate way as DRE. Although this approximate methods provide effective ways to find the control law, the algorithm not only involves suboptimality, but also involves several steps which makes it computationally complex. In [24], the authors utilized the same approach mentioned above, to solve the finite-horizon optimal regulation and tracking of stochastic systems. Here, Kalman filter has been integrated with the finite-horizon SDRE method where the Kalman filter estimates the states which is corrupted with noise.

1.3 Research Goal and Contribution

In this research, the SDRE based finite-horizon nonlinear optimal control problem for both regulation and tracking is addressed. The main goal is to eliminate the approximate nature of the solution and reduce the computational complexities involved in solving DRE and VDE in the existing finite-horizon SDRE regulation and tracking problems. In the existing SDRE regulation, the state-dependent Riccati coefficient $\mathbf{P}(\mathbf{x},t)$ is considered to be constant during each small time step of a finite-horizon period. Similarly, in case of finite-horizon tracking, state-dependent vector coefficient, $\mathbf{V}(\mathbf{x},t)$, along with $\mathbf{P}(\mathbf{x},t)$ is assumed to be constant at each intervals. These assumptions makes the control solution suboptimal in nature. Apart from that, the steps involved in solving DRE and VDE makes the algorithm computationally complex.

To solve the above mentioned problems, this research proposes a simplified SDRE for both the regulation and tracking problems. In the proposed simplified SDRE method, the analytic solution of matrix DRE and VDE and the associated MATLAB program *lqrnss* (non-steady state linear quadratic regulator) and *lqtnss* (non-steady state linear quadratic tracking) were used at each time step of the finite-horizon period. The simplified SDRE method is illustrated with the finite-horizon optimal regulation and tracking of a complex sixth-order permanent magnet synchronous generator (PMSG) based wind energy conversion system (WECS). As a highly nonlinear electro-mechanical system, PMSG based WECS would be a perfect application for the validation of the proposed method.

1.4 Chapters Review

This dissertation is composed of **five** chapters covering the following topics:

1. **Chapter 1** provides an introduction, literature review, research goals and contributions of the work.

2. **Chapter 2** presents an overview of finite-horizon linear quadratic regulation and finite-horizon linear quadratic tracking problems and implements these control strategies on a simple DC motor.

3. **Chapter 3** illustrates the finite-horizon regulation via existing SDRE and simplified SDRE and implements them in a complex sixth-order nonlinear model of a wind energy conversion system (WECS).

4. **Chapter 4** presents the theory and application of finite-horizon tracking via existing SDRE and simplified SDRE for complex nonlinear systems. Validation of simplified SDRE and a comparison with existing SDRE by implementing them in wind energy conversion system is also illustrated in this chapter.

5. **Chapter 5** presents the advantages and disadvantages of the proposed simplified SDRE over the existing method and discusses some future research scope in these areas.

Chapter 2

Finite-Horizon LQR and LQT

Optimal control of linear systems is now a well-established area of research and already have its application in various fields of science and engineering [5, 26, 41]. Since most of the physical systems in nature are nonlinear, the control community are working with various nonlinear control theory by utilizing the advanced modeling techniques and the modern mathematical software. Although researchers have already come up with numerous nonlinear control theory, the linear controls not only dominates in the real-world application, but also they are the building block of most of the modern nonlinear controls [7]. In this chapter, the mathematical background and implementation of linear quadratic regulator (LQR) and linear quadratic tracking (LQT) [31] optimal control theories are discussed. Although, both LQR and LQT are widely applied in the real-world systems, the mathematical background behind these theories will help to understand the state-dependent Riccati equation (SDRE) method, which is the research focus of this thesis.

2.1 Finite-Horizon Linear Quadratic Regulator (LQR): Overview

Consider a linear time-varying system as [21]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t)\end{aligned} \tag{2.1.1}$$

The cost function is defined as

$$\begin{aligned}J &= \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) \\ &+\frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt\end{aligned} \tag{2.1.2}$$

where, $\mathbf{A}(t)$ is a nxn state matrix, $\mathbf{B}(t)$ is a nxr control matrix, $\mathbf{C}(t)$ is a mxn output matrix, the control signal $\mathbf{u}(t)$ is unconstrained, the final time t_f is specified, $\mathbf{x}(t_f)$ is unknown, the terminal cost weighted matrix $\mathbf{F}(t_f)$ and the state weighted matrix $\mathbf{Q}(t)$ are nxn symmetric positive semidefinite matrices, and the control weighted matrix $\mathbf{R}(t)$ is a rxr symmetric, positive definite.

A feedback control law for minimizing the cost function is defined as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}(t) = -\mathbf{K}(t)\mathbf{x}(t) \tag{2.1.3}$$

where, the Kalman gain $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$ and the Riccati coefficient matrix, $\mathbf{P}(t)$, the nxn symmetric *positive definite* matrix (for all $t \in [t_0,\, t_f]$), are the solution of the matrix differential Riccati equation (DRE)

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) \tag{2.1.4}$$

with the final condition

$$\mathbf{P}(t = t_f) = \mathbf{F}(t_f) \tag{2.1.5}$$

Table 2.1: Summary of Finite-Time Linear Quadratic Regulator System: Time-Varying Case

The Problem Statement	
Given the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, the terminal cost function $J = \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt$, with the boundary conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, t_f is specified, and $\mathbf{x}(t_f)$ is unknown, find the optimal control input, state and performance index.	
The Problem Solution	
Step 1	Solve the matrix differential Riccati equation $\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$ *backward* in time starting with the *final* condition $\mathbf{P}(t = t_f) = \mathbf{F}(t_f)$.
Step 2	Solve the optimal state $\mathbf{x}(t)$ from $\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}(t)$ *forward* in time starting with the *initial* condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain the optimal control $\mathbf{u}(t)$ as $\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t)$, where $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$.
Step 4	Obtain the optimal performance index as $J = \frac{1}{2}\mathbf{x}'(t)\mathbf{P}(t)\mathbf{x}(t)$.

Then, the optimal state can be found as

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}(t) \tag{2.1.6}$$

and the optimal cost is

$$J = \frac{1}{2}\mathbf{x}'(t)\mathbf{P}(t)\mathbf{x}(t) \tag{2.1.7}$$

The optimal control $\mathbf{u}(t)$, given by (2.1.3), is linear in the optimal state $\mathbf{x}(t)$. The finite-time linear optimal control regulator procedure is summarized in Table 2.1.

2.2 Analytical Solution of MDRE

This section presents the analytical solution of the matrix differential Riccati equation, $\mathbf{P}(t)$ [46]. The Hamiltonian system of the state and costate equations for the time-invariant case as can be written as:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}^{'} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \lambda(t) \end{bmatrix} \tag{2.2.1}$$

where, $\mathbf{E} = \mathbf{B}\ \mathbf{R}^{-1}\mathbf{B}'.Let$

$$\triangle = \begin{bmatrix} \mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}^{'} \end{bmatrix}$$

Let us also recall the solution of the matrix differential Riccati equation as

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) \tag{2.2.2}$$

with the final condition

$$\mathbf{P}(t = t_f) = \mathbf{F}(t_f) \tag{2.2.3}$$

Now, the solution $\mathbf{P}(t)$ can be obtained analytically in terms of the eigenvalues and eigenvectors of the Hamiltonian matrix. But before that, it is necessary to show that if is an eigenvalue of the Hamiltonian matrix $\triangle$, then it implies that $-\mu$ is also the eigenvalue of $\triangle$. Let's define

$$\Gamma = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

so that $\Gamma^{-1} = -\Gamma$. Now, after pre- and post-multiplication with Γ, it is found that

$$\triangle = \Gamma\triangle^{'}\Gamma = -\Gamma\triangle^{'}\Gamma^{-1} \tag{2.2.4}$$

Again, if μ is an eigenvalue of $\triangle$ with eigenvector $\mathbf{v}$,

$$\triangle\mathbf{v} = \mu\mathbf{v} \tag{2.2.5}$$

then,

$$\Gamma\triangle^{'}\Gamma\mathbf{v} = \mu\mathbf{v}, \triangle^{'}\mathbf{\Gamma v} = -\mu\mathbf{\Gamma v} \tag{2.2.6}$$

Next, rearranging the eigenvalues of $\triangle$ as

$$\mathbf{D} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}$$

where, $\mathbf{M}(-\mathbf{M})$ is a diagonal matrix with right half plane (left half plane) eigenvalues. Now, let's define $\mathbf{W}$ as the modal matrix of eigenvectors corresponding to $\mathbf{D}$, where,

$$\mathbf{W} = \begin{bmatrix} \mathbf{W_{11}} & \mathbf{W_{12}} \\ \mathbf{W_{21}} & \mathbf{W_{22}} \end{bmatrix}$$

Here, $[\mathbf{W_{11}}\ \mathbf{W_{21}}]'$ are n eigenvectors of left half plane eigenvalues of $\triangle$. Also,

$$\mathbf{W^{-1}}\triangle\mathbf{W} = \mathbf{D} \tag{2.2.7}$$

Let's define a state transformation as

$$\begin{bmatrix} \mathbf{x}(t) \\ \lambda(t) \end{bmatrix} = \mathbf{W} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{W_{11}} & \mathbf{W_{12}} \\ \mathbf{W_{21}} & \mathbf{W_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix} \tag{2.2.8}$$

Then, combining (2.2.7) and (2.2.8), the Hamiltonian system (2.2.1) becomes

$$\begin{bmatrix}\dot{\mathbf{w}}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix} = \mathbf{W^{-1}}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\lambda}(t)\end{bmatrix} = \mathbf{W^{-1}}\triangle\begin{bmatrix}\mathbf{x}(t)\\ \lambda(t)\end{bmatrix} = \mathbf{W^{-1}}\triangle\mathbf{W}\begin{bmatrix}\mathbf{w}(t)\\ \mathbf{z}(t)\end{bmatrix} = \mathbf{D}\begin{bmatrix}\mathbf{w}(t)\\ \mathbf{z}(t)\end{bmatrix} \tag{2.2.9}$$

Now, solving (2.2.9) in terms of known final condition as

$$\begin{bmatrix}\mathbf{w}(t)\\ \mathbf{z}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{e}^{-\mathbf{M}(t-t_f)} & \mathbf{0}\\ \mathbf{0} & \mathbf{e}^{\mathbf{M}(t-t_f)}\end{bmatrix}\begin{bmatrix}\mathbf{w}(t_f)\\ \mathbf{z}(t_f)\end{bmatrix} \tag{2.2.10}$$

Rewriting (2.2.10), we get,

$$\begin{bmatrix}\mathbf{w}(t_f)\\ \mathbf{z}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{e}^{\mathbf{M}(t-t_f)} & \mathbf{0}\\ \mathbf{0} & \mathbf{e}^{\mathbf{M}(t-t_f)}\end{bmatrix}\begin{bmatrix}\mathbf{w}(t)\\ \mathbf{z}(t_f)\end{bmatrix} \tag{2.2.11}$$

Next, from (2.2.8) and using the final condition (2.2.3)

$$\begin{aligned}\lambda(t_f) &= \mathbf{W_{21}}\mathbf{w}(t_f) + \mathbf{W_{22}}\mathbf{z}(t_f)\\ &= \mathbf{F}\mathbf{x}(t_f)\\ &= \mathbf{F}[\mathbf{W_{11}}\mathbf{w}(t_f) + \mathbf{W_{12}}\mathbf{z}(t_f)]\end{aligned} \tag{2.2.12}$$

Now, solving this for $\mathbf{z}(t_f)$ in terms of $\mathbf{w}(t_f)$

$$\begin{aligned}\mathbf{z}(t_f) &= \mathbf{T}(t_f)\mathbf{w}(t_f), where\\ \mathbf{T}(t_f) &= -[\mathbf{W_{22}} - \mathbf{F}\mathbf{W_{12}}]^{-1}[\mathbf{W_{21}} - \mathbf{F}\mathbf{W_{11}}]\end{aligned} \tag{2.2.13}$$

Again, from (2.2.11)

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{e}^{-\mathbf{M}(t_f-t)}\mathbf{z}(t_f)\\ &= \mathbf{e}^{-\mathbf{M}(t_f-t)}\mathbf{T}(t_f)\mathbf{w}(t_f)\\ &= \mathbf{e}^{-\mathbf{M}(t_f-t)}\mathbf{T}(t_f)\mathbf{e}^{-\mathbf{M}(t_f-t)}\mathbf{w}(t)\end{aligned} \tag{2.2.14}$$

Rewriting the equation as

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{T}(t)\mathbf{w}(t), where, \\ \mathbf{T}(t) &= \mathbf{e}^{-\mathbf{M}(t_f - t)}\mathbf{T}(t_f)\mathbf{e}^{-\mathbf{M}(t_f - t)} \end{aligned} \tag{2.2.15}$$

Now, from (2.2.8), we can write,

$$\begin{aligned}\lambda(t) &= \mathbf{W_{21}}\mathbf{w}(t) + \mathbf{W_{22}}\mathbf{z}(t) \\ &= \mathbf{P}(t)\mathbf{x}(t) \\ &= \mathbf{P}(t)[\mathbf{W_{11}}\mathbf{w}(t) + \mathbf{W_{12}}\mathbf{z}(t)] \end{aligned} \tag{2.2.16}$$

Combining (2.2.15) and (2.2.16), it can be written as

$$[\mathbf{W_{21}} + \mathbf{W_{22}}\mathbf{T}(t)]\mathbf{w}(t) = \mathbf{P}(t)[\mathbf{W_{11}} + \mathbf{W_{12}}\mathbf{T}(t)]\mathbf{w}(t) \tag{2.2.17}$$

Since the previous relation should hold good for all $x(t_0)$ and hence for all states $\mathbf{w}(t)$, it implies that the analytical expression to the solution of $\mathbf{P}(t)$ is given by

$$\mathbf{P}(t) = [\mathbf{W_{21}} + \mathbf{W_{22}}\mathbf{T}(t)][\mathbf{W_{11}} + \mathbf{W_{12}}\mathbf{T}(t)]^{-1} \tag{2.2.18}$$

2.3 Finite-Horizon LQR: Application and Simulation

This section presents the simulation of finite-horizon linear quadratic regulator control of a DC motor. The electric equivalent circuit of the armature and the free-body diagram of the rotor are shown in Fig. 2.1. Here, the input of the system is the voltage source, V_m which is applied to the motor's armature, and the output is the rotational speed of the shaft, ω_m. The rotor and shaft are assumed to be rigid. The physical parameters of the system are given in Table 2.2.

The state-space model of the DC motor according to (2.1.1) can be written as

$$\begin{bmatrix}\dot{\omega_m}(t) \\ \dot{i_m}(t)\end{bmatrix} = \begin{bmatrix}-\frac{b}{J_m} & \frac{K_m}{J_m} \\ -\frac{K_m}{L_m} & -\frac{R_m}{L_m}\end{bmatrix}\begin{bmatrix}\omega_m(t) \\ i_m(t)\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{L_m}\end{bmatrix}\begin{bmatrix}V_m(t)\end{bmatrix} \tag{2.3.1}$$

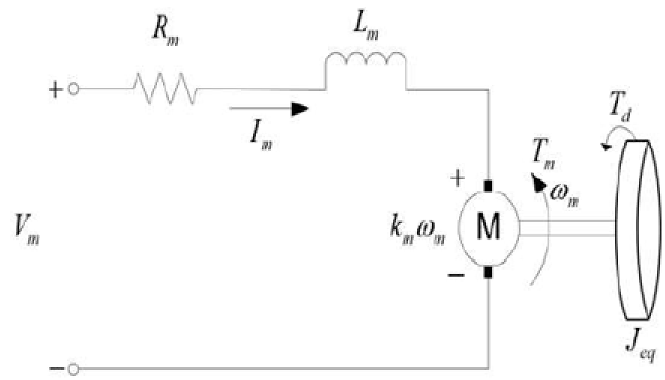

Figure 2.1: Electrical equivalent circuit of DC motor

where, the system states $\mathbf{x}(t) = \begin{bmatrix} \omega_m(t) \\ i_m(t) \end{bmatrix}$, system input, $\mathbf{u}(t) = V_m(t)$, system matrix, $\mathbf{A}(t) = \begin{bmatrix} -\frac{b}{J_m} & \frac{K_m}{J_m} \\ -\frac{K_m}{L_m} & -\frac{R_m}{L_m} \end{bmatrix}$, and input matrix, $\mathbf{B}(t) = \begin{bmatrix} 0 \\ \frac{1}{L_m} \end{bmatrix}$. Now, let us select the weighted matrices as

$$\mathrm{Q} = \mathrm{diag}[1,1],\ \mathrm{R} = 0.25$$

Table 2.2: DC Motor Parameters

Notions	**Descriptions**	**Values**
J_m	Motor moment of inertia	0.01 $kg.m^2$
b	Motor viscous friction constant	0.1 $N.m.s$
R_m	Stator resistance	1 ohm
L_m	Stator inductance	0.5 H
K_m	Motor torque constant	0.01 $N.m/Amp$

The simulations are performed for 5 seconds and the initial conditions are taken as, $x_0 = [5, -3]'$. The resulting state trajectories are presented in Fig. 2.2 and the optimal control is presented in Fig. 2.3. In Fig. 2.2, the solid line denotes the motor speed ω_m, and the doted line denotes stator current, i_m. Fig. 2.2 clearly illustrates the finite-horizon linear quadratic regulator in DC motor.

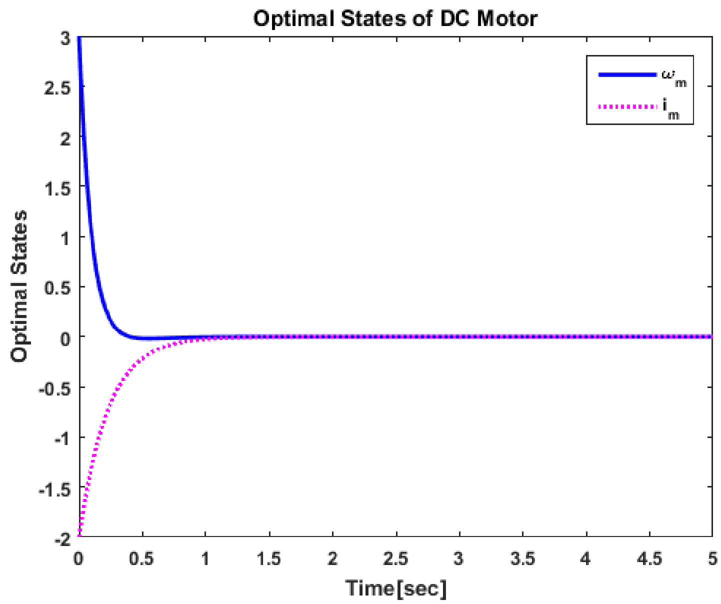

Figure 2.2: Optimal states of DC motor by finite-horizon LQR control

2.4 Finite-Horizon Linear Quadratic Tracking (LQT): Overview

Consider a linear time-varying system as [31]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t)\end{aligned} \tag{2.4.1}$$

and a quadratic performance index

$$J = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \tag{2.4.2}$$

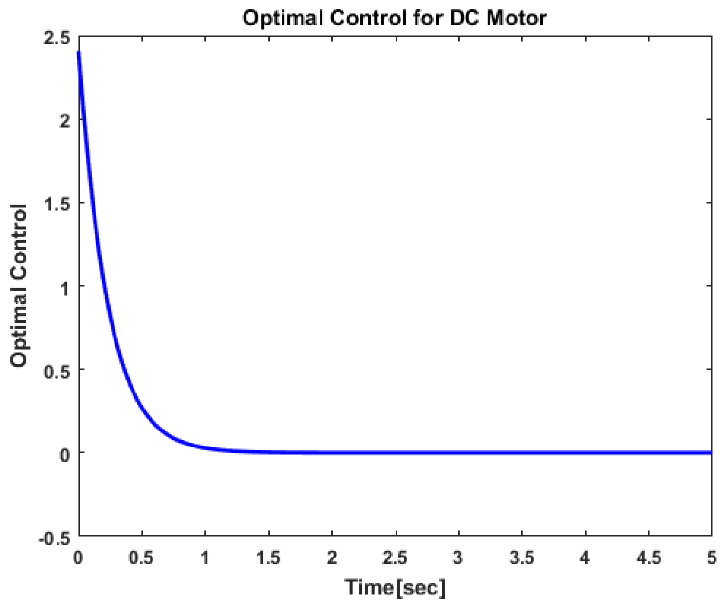

Figure 2.3: Finite-horizon LQR based optimal control signal of DC motor

where, $\mathbf{A}(t)$ is a nxn state matrix, $\mathbf{B}(t)$ is a nxr control matrix, $\mathbf{C}(t)$ is a mxn output matrix, the control signal $\mathbf{u}(t)$ is unconstrained, the final time t_f is specified, $\mathbf{x}(t_f)$ is unknown, the terminal cost weighted matrix $\mathbf{F}(t_f)$ and the state weighted matrix $\mathbf{Q}(t)$ are nxn symmetric positive semidefinite matrices, and the control weighted matrix $\mathbf{R}(t)$ is a rxr symmetric, positive definite, $\mathbf{z}(t)$ is the desired output, and the error is $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t)$.

The optimal control $\mathbf{u}(t)$ is defined as

$$\begin{aligned} \mathbf{u}(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\left[\mathbf{P}(t)\mathbf{x}(t) - \mathbf{g}(t)\right] \\ &= -\mathbf{K}(t)\mathbf{x}(t) + \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t) \end{aligned} \tag{2.4.3}$$

where, the Riccati coefficient matrix $\mathbf{P}(t)$, is the solution of the matrix differential Riccati equation (MDRE)

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t) \tag{2.4.4}$$

with the final condition

$$\mathbf{P}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f) \tag{2.4.5}$$

and $\mathbf{g}(t)$, is the solution of the linear nonhomogeneous vector differential equation

$$\dot{\mathbf{g}}(t) = -\left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]' \mathbf{g}(t) - \mathbf{W}(t)\mathbf{z}(t)] \tag{2.4.6}$$

with the final condition

$$\mathbf{g}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f) \tag{2.4.7}$$

Here, the matrices $\mathbf{E}(t)$, $\mathbf{V}(t)$ and $\mathbf{W}(t)$ are defined as

$$\begin{aligned} \mathbf{E}(t) &= \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t) \\ \mathbf{V}(t) &= \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t) \\ \mathbf{W}(t) &= \mathbf{C}'(t)\mathbf{Q}(t) \end{aligned} \tag{2.4.8}$$

Then, the optimal state for the trajectory tracking can be obtained from

$$\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]\mathbf{x}^*(t) + \mathbf{E}(t)\mathbf{g}(t)] \tag{2.4.9}$$

and the optimal performance index J is defined as

$$J(t_0) = \frac{1}{2}\mathbf{x}'(t_0)\mathbf{P}(t_0)\mathbf{x}(t_0) - \mathbf{x}'(t_0)\mathbf{g}(t_0) + h(t_0) \tag{2.4.10}$$

The overall procedure of the linear quadratic tracking (LQT) control is summarized in Table 2.3 [31]

2.5 Finite-Horizon LQT : Application and Simulation

To illustrate the finite-horizon tracking control, let us consider the same system described is section (2.3). The task is to control the motor speed at a desired

Table 2.3: Summary of Linear Quadratic Tracking System

The Problem Statement	
Given the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \quad \mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t),$ the terminal cost function $J = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right]dt,$ with the boundary conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, t_f is specified and $\mathbf{x}(t_f)$ is unknown, find the optimal control input, state and performance index.	
The Problem Solution	
Step 1	Solve the matrix differential Riccati equation (DRE) $\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t),$ with the final condition $\mathbf{P}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f)$, and the non-homogeneous vector differential equation (DVE) $\dot{\mathbf{g}}(t) = -\left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]'\mathbf{g}(t) - \mathbf{W}(t)\mathbf{z}(t),$ with final condition $\mathbf{g}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f)$ where $\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t), \quad \mathbf{V}(t) = \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t),$ $\mathbf{W}(t) = \mathbf{C}'(t)\mathbf{Q}(t)$.
Step 2	Solve the optimal state $\mathbf{x}^*(t)$ from $\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]\mathbf{x}^*(t) + \mathbf{E}(t)\mathbf{g}(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain optimal control $\mathbf{u}^*(t)$ from $\mathbf{u}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t),$ where $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$.
Step 4	The optimal cost $J^*(t_0)$ is $J^*(t_0) = \frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0) - \mathbf{x}^*(t_0)\mathbf{g}(t_0) + h(t_0)$ where $h(t)$ is the solution of $\dot{h}(t) = -\frac{1}{2}\mathbf{g}'(t)\mathbf{E}(t)\mathbf{g}(t) - \frac{1}{2}\mathbf{z}'(t)\mathbf{Q}(t)\mathbf{z}(t)$ with final condition $h(t_f) = -\mathbf{z}'(t_f)\mathbf{P}(t_f)\mathbf{z}(t_f)$.

trajectory.

Let the selected weighted matrices be

$$Q = \text{diag}[50{,}50],\ R = [10]$$

The simulations are performed for 6 seconds and the resulting speed trajectories are shown in Fig. 3.4, where the dashed line (blue) denotes the reference angle trajectory, and the solid line (red) denotes the actual angle. The optimal control signal is shown in Fig. 2.5. The error between the actual and the estimated speed trajectories is shown in Fig. 2.6.

Comparing these trajectories in Fig. 2.4, its clear that the finite-horizon LQT controller provides a very good results as the estimated speed is making a very good tracking. As shown in Fig. 2.6, the error between the actual and reference speed is very small, the average error for this example is 0.4%.

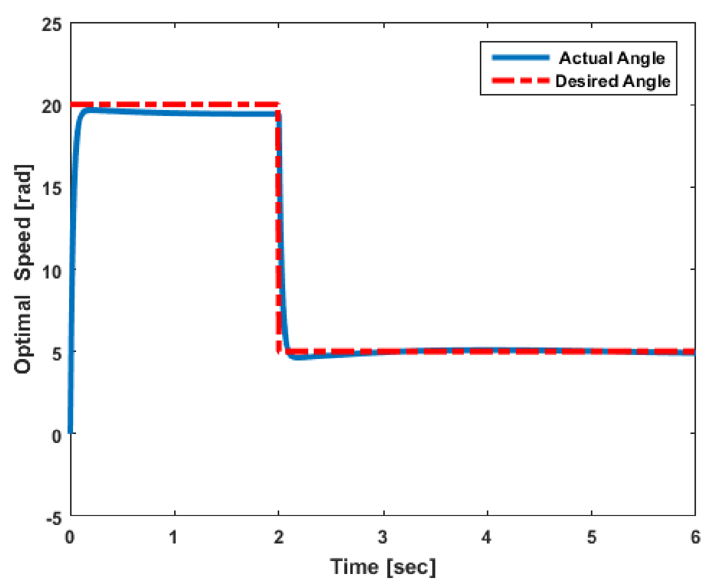

Figure 2.4: Speed trajectories for DC motor by finite-horizon LQT

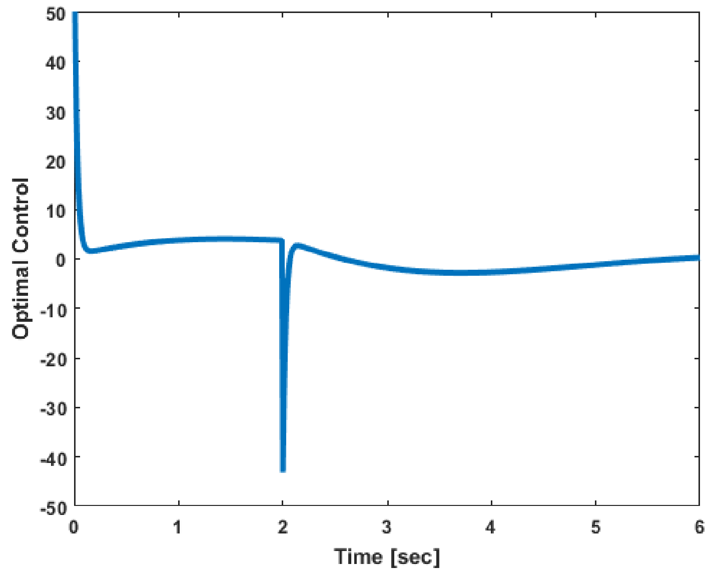

Figure 2.5: Finite-horizon LQT based control signal of DC motor

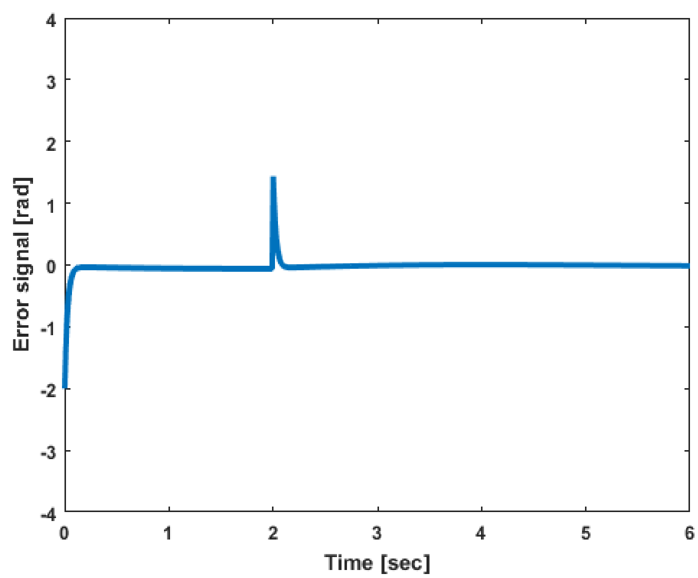

Figure 2.6: Optimal error of trajectory tracking for DC motor

2.6 Conclusion

This chapter illustrated the mathematical background and application of the finite-horizon linear quadratic regulator (LQR) and linear quadratic tracking (LQT) optimal control theory. Although both of the theories are well-established and widely used in the real-world physical systems, their discussion will assist in understanding the finite-horizon optimal regulation and tracking of nonlinear systems via state-dependent Riccati equation (SDRE) which are presented in the upcoming chapters.

Chapter 3

Finite-Horizon Regulation using Simplified SDRE

The complexity of the time-dependency of the Hamilton Jacobi-Bellman (HJB) partial differential equation makes the finite-horizon optimal regulation of nonlinear systems a challenging problem [32]. To handle this complexity, the potential of the SDRE technique in infinite-horizon control is utilized to find the solution of the finite-horizon case. Finite horizon SDRE understands the limitation on time and generates the control signal such that the state error eliminates or becomes very small at final time. Some crucial problems in control engineering such as path planning, guidance can be considered under the category of finite-horizon control [15]. In this section, the solution of the finite-horizon regulation via existing SDRE and simplified SDRE is discussed.

3.1 Statement of Problem

Consider the time-varying linear system [39]

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}) \end{aligned} \tag{3.1.1}$$

The nonlinear system can be transformed to a linear-like structure with state dependent coefficient (SDC) form as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{x})\mathbf{x}(t)\end{aligned} \tag{3.1.2}$$

The goal is to obtain a state feedback optimal control of the form $\mathbf{u}(\mathbf{x},t) = -\mathbf{K}(\mathbf{x},t)\mathbf{x}(t)$, which minimizes a cost function [31]

$$\begin{aligned}J &= \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt\end{aligned} \tag{3.1.3}$$

where, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{F}$ are symmetric *positive semi-definite* matrices, and $\mathbf{R}(\mathbf{x})$ is a symmetric *positive definite* matrix. $\mathbf{x}'(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t)$ is a measure of state accuracy and $\mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})$ is a measure of control effort.

3.2 General Solution of the Problem

A state feedback control law to minimize the cost function (5) can be defined as

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= -\mathbf{K}(\mathbf{x},t)\mathbf{x}(t) \\ &= -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x},t)\mathbf{x}(t)\end{aligned} \tag{3.2.1}$$

Here, $\mathbf{P}(\mathbf{x}, t)$ is the symmetric, positive definite solution of the SDRE or strictly speaking State Dependent Differential Riccati Equation (SD-DRE) of the form

$$\begin{aligned}-\dot{\mathbf{P}}(\mathbf{x}, t) = \mathbf{P}(\mathbf{x},t)\mathbf{A}(\mathbf{x}) + \mathbf{A}'(\mathbf{x})\mathbf{P}(\mathbf{x},t) \\ - \mathbf{P}(\mathbf{x},t)\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x},t) + \mathbf{Q}(\mathbf{x}),\end{aligned} \tag{3.2.2}$$

with the final condition

$$\mathbf{P}(t = t_f) = \mathbf{F}(t_f) \tag{3.2.3}$$

Then, the SD-DRE controlled trajectory becomes the solution of the state-dependent closed loop dynamics

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}(t) \tag{3.2.4}$$

3.3 Existing SDRE Method

In the existing SDRE, the nonlinear system is first factorized into the product of a matrix valued function- the SDC matrix, which depends on the state itself, and a state vector. So, the SDRE is presented in the linear-like structure with SDC form and contains the nonlinearities of the system. Here, the SDC matrices should be formed in such a way that the $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ matrices are controllable or at least stablizable. Then, the SDRE is solved at small intervals of time during the given period of initial time t_0 to final time t_f. During each of these small time step, the Riccati coefficient $\mathbf{P}(\mathbf{x},t)$ is considered to be constant and hence algebraic Riccati equation (ARE) is used to obtain the steady-state Riccati coefficient $\mathbf{P_{ss}}(\mathbf{x})$. This assumption leads to a sub-optimal control theory. Then, by using change of variable technique, a new variable $\mathbf{K}(\mathbf{x},t)$ is introduced in terms of unknown $\mathbf{P}(\mathbf{x},t)$ and known $\mathbf{P_{ss}}(\mathbf{x})$ as $\mathbf{K}(\mathbf{x},t) = [\mathbf{P}(\mathbf{x},t) - \mathbf{P_{ss}}(\mathbf{x})]^{\mathbf{-1}}$; transforming into a linear differential Lyapunov equation (DLE) which can be solved online at each time step [37]. Basically, $\mathbf{P}(\mathbf{x},t)$ is obtained in terms of $\mathbf{K}(\mathbf{x},t)$, the analytic solution of DLE, which itself requires the solution of ARE and algebraic Lyapunov equation [12]. Finally, $\mathbf{P}(\mathbf{x},t)$ is calculated using $\mathbf{K}(\mathbf{x},t)$ from the previous transformation as $\mathbf{P}(\mathbf{x},t) = \mathbf{K}^{-1}(\mathbf{x},t) + \mathbf{P_{ss}}(\mathbf{x})$. Once $\mathbf{P}(\mathbf{x},t)$ is found, optimal control can be determined using (3.2.1). In the existing SDRE technique, the following steps can be followed to solve the DRE at each time step [22, 33]:

1. Solve ARE to find the steady-state Riccati coefficient $\mathbf{P_{ss}}(\mathbf{x})$.

$$\mathbf{P_{ss}}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) - \mathbf{P_{ss}}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = 0 \tag{3.3.1}$$

2. Use change of variable technique and assume

$$\mathbf{K}(\mathbf{x}, \mathbf{t}) = [\mathbf{P}(\mathbf{x}, \mathbf{t}) - \mathbf{P_{ss}}(\mathbf{x})]^{-1} \tag{3.3.2}$$

3. Calculate the value for $\mathbf{A_{cl}}(\mathbf{x})$ as

$$\mathbf{A_{cl}}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R^{-1}}\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) \tag{3.3.3}$$

4. Solve algebraic Lyapunov equation to find the value $\mathbf{D}$

$$\mathbf{A_{cl}}\mathbf{D} + \mathbf{D}\mathbf{A}'_{\mathbf{cl}} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}' = 0 \tag{3.3.4}$$

5. Solve differential Lyapunov equation

$$\dot{\mathbf{K}}(\mathbf{x}, t) = \mathbf{K}(\mathbf{x}, t)\mathbf{A}'_{\mathbf{cl}}(\mathbf{x}) + \mathbf{A_{cl}}(\mathbf{x})\mathbf{K}(\mathbf{x}, t) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}'(\mathbf{x}) \tag{3.3.5}$$

The solution of (13) is found as

$$\mathbf{K}(\mathbf{x}, t) = \mathbf{e}^{\mathbf{A_{cl}}(\mathbf{t}-\mathbf{t_f})}(\mathbf{K}(\mathbf{x}, t_f) - \mathbf{D})\mathbf{e}^{\mathbf{A_{cl}}'(\mathbf{t}-\mathbf{t_f})} + \mathbf{D} \tag{3.3.6}$$

6. Obtain $\mathbf{P}(\mathbf{x}, t)$ from the following

$$\mathbf{P}(\mathbf{x}, t) = \mathbf{K}^{-1}(\mathbf{x}, t) + \mathbf{P_{ss}}(t) \tag{3.3.7}$$

7. Finally, obtain the optimal control $\mathbf{u}(\mathbf{x}, t)$ as

$$\mathbf{u}(\mathbf{x}, t) = -\mathbf{R}^{-1}\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x}, t)\mathbf{x}(t) \tag{3.3.8}$$

The overall algorithmic procedures to find the control law using the existing SDRE method is also presented in Fig. 3.1.

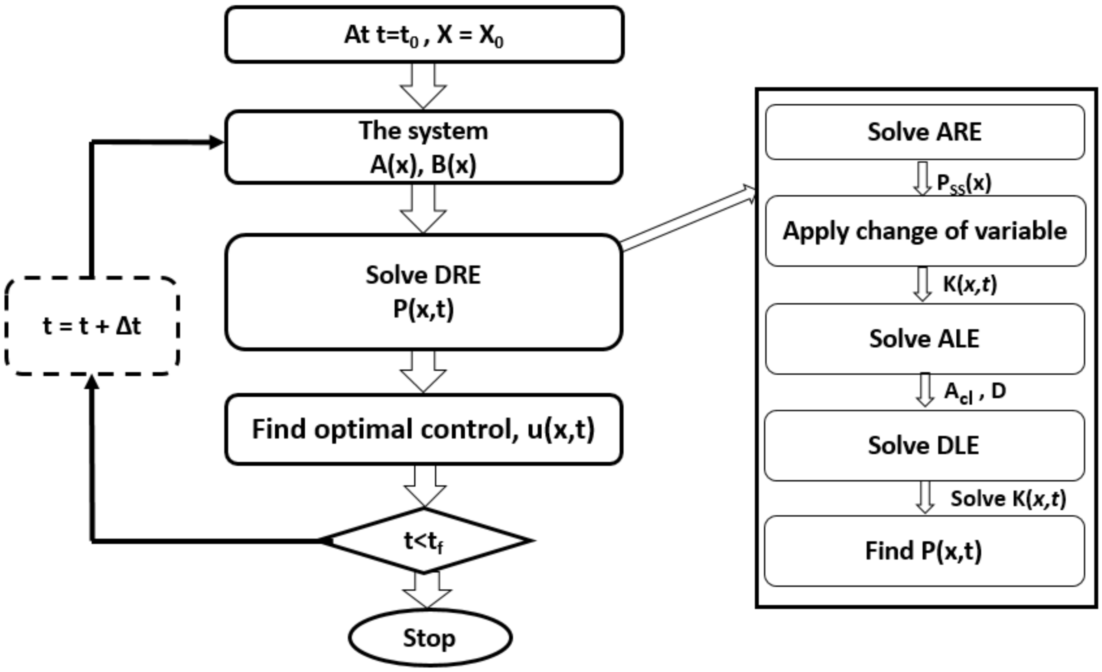

Figure 3.1: Flow chart of finite-horizon optimal regulation via existing SDRE

3.4 Simplified SDRE Method

In this section, a simplified SDRE technique is proposed which reduces the computational complexities involved in finding the Riccati coefficient $\mathbf{P}(\mathbf{x},t)$ described in section 3.3. In the existing SDRE method, it is assumed that during the small intervals between the initial time and final time, the state-dependent Riccati coefficient $\mathbf{P}(\mathbf{x},t)$ is constant and hence, the steady-state Riccati coefficient $\mathbf{P_{ss}}(\mathbf{x})$ is used. This assumption seems to be due to the lack of the analytic solution for the finite-time differential Riccati equation (DRE) and the corresponding MATLAB program for real-time implementation. On the other hand, there exist a MATLAB command *lqr* (linear quadratic regulator) for solving the ARE. However, a MATLAB-based program *lqrnss* for solving finite-time regulator in optimal control was developed in [31, 41] which indicates that the program is *lqr* for non-steady state conditions.

The solution of the finite time (non-steady-state) LQR problem requires a backward integration of the matrix differential Riccati equation. The common method

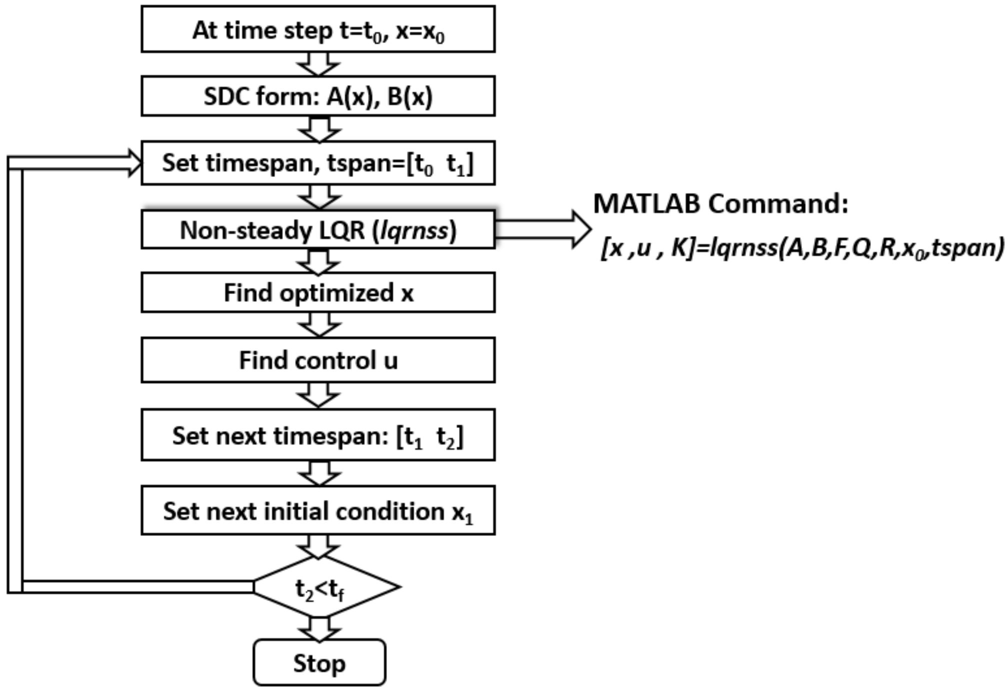

Figure 3.2: Flow chart of finite-horizon optimal regulation via simplified SDRE

for solving this problem requires rewriting the matrix equation into a set of scalar equations, then performing the required numerical backward integration. This method is time consuming, as the complexity and computation time grow as the order of the system grows. Further, this method does not lend itself to the development of a generic LQR controller for online implementation to a variety of systems. A better and easier method is to utilize the analytical solution that is provided in various texts [28, 31]. A MATLAB implementation of the analytical solution for finite-time LQR was developed by the authors in [31,41], where in addition, a MATLAB program for finite-time linear quadratic tracking (LQT) was also developed [33].

In the proposed simplified SDRE method, the DRE in finite time regulation problem is solved at each of the small steps using MATLAB command *lqrnss* for finite-time matrix differential Riccati equation (MDRE) as opposed to using the steady-state algebraic Riccati equation (ARE) in the existing SDRE technique. Thus, simplified SDRE eliminates the suboptimality present in the existing SDRE method. Not only that, in this proposed technique, there is no need for several steps involved in the existing SDRE method presented in the section (3.3). Hence, it reduces the computational complexities. The algorithmic steps for solving a nonlinear, finite-horizon regulator via simplified SDRE technique is represented in a flowchart shown in Fig. 3.2.

3.5 Finite-Horizon Regulation via Existing and Simplified SDRE

To illustrate the validation of the simplified SDRE, we implemented both the existing and simplified SDRE method to a finite-time optimal regulation of variable speed, variable pitch wind energy con-version system (VSVP-WECS) of sixth order. A comparison of the system states and the control signals resulted from these two methods were also presented in this section.

3.5.1 System Modeling

A wind energy conversion system (WECS) is basically a converter that transforms the kinetic energy of the wind to the electric power. It can be divided into two groups in terms of physical nature (electrical and mechanical) and four groups in terms of dynamics (aerodynamics, drive train dynamics, generator dynamics and structural dynamics) as shown in Fig. 3.3 [39]. The aerodynamic block converts the wind's kinetic energy into rotational (mechanical) energy. The drive train block increases this slower rotational speed and transmits the speed to the generator block. Generator block converts this rotational speed into electrical power.

For the control purposes, only aerodynamics, drive train dynamics and generator dynamics are taken into consideration in this research. The most complex dynamics among these three- Aerodynamics [36], takes wind speed V and turbine rotor speed ω_r as the input and provides the output in terms of aerodynamics torque. The

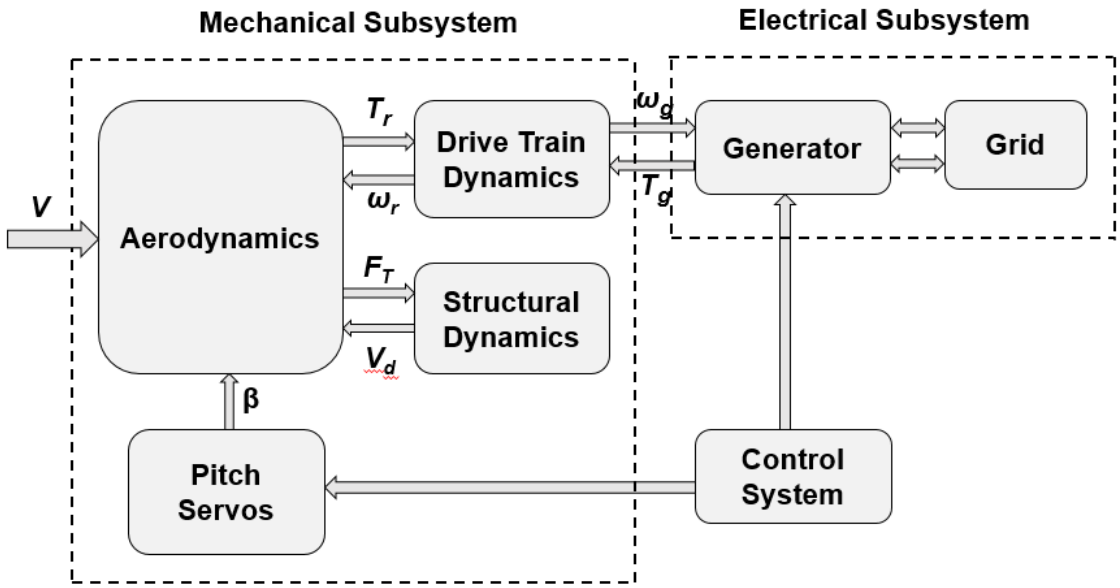

Figure 3.3: WECS block diagram

dynamical equation can be written as

$$T_r = \frac{1}{2}\pi\rho R^3 C_Q(\lambda, \beta)V^2 \tag{3.5.1}$$

where, ρ is the air density, R is the radius of the wind rotor plane, C_Q is the torque coefficient as a function of pitch angle β and tip speed ratio λ. Again, tip speed ratio can be mathematically expressed as

$$\lambda = \frac{\omega_r R}{V} \tag{3.5.2}$$

where, ω_r is the wind rotor angular speed. The torque coefficient C_Q can be approximated by a polynomial function of tip speed λ and pitch angle β as [6]

$$C_Q(\lambda, \beta) = c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\lambda + c_5\lambda\beta + c_6\lambda\beta^2 + c_7\lambda\beta^3 \tag{3.5.3}$$

The main part of the drive train block is the gearbox which connects a low-speed shaft and a high-speed shaft. In this research, a flexible drive train is considered and the dynamic model is given as follows:

$$\begin{aligned}
\dot{\omega}_r &= -\frac{i}{\eta J_r}T_H + \frac{1}{J_r}T_r, && (3.5.4)\\
\dot{\omega}_g &= \frac{1}{J_g}T_H - \frac{1}{J_g}T_g, && (3.5.5)\\
\dot{T}_H &= iK_g\omega_r - K_g\omega_g - B_g(\frac{1}{J_g} + \frac{i^2}{\eta J_r})T_H \quad + \frac{iB_g}{J_r}T_r + \frac{B_g}{J_g}T_g, && (3.5.6)
\end{aligned}$$

where, T_H is the internal torque, ω_g is the generator speed, T_g is the generator electromagnetic torque, J_r is the wind rotor inertia, J_g is the generator inertia, K_g is the stiffness coefficient of the high-speed shaft, B_g is the damping coefficient of the high-speed shaft, i is the gearbox ratio and η is the gearbox efficiency.

The pitch actuator servo is a nonlinear servo that takes input from the control system which decides blade angle (desired pitch angle), β_d and gives output as the final pitch angle of the blade [6]. The dynamic model of the pitch actuator is expressed as

$$\dot{\beta} = -\frac{1}{\tau}\beta + \frac{1}{\tau}\beta_d \qquad (3.5.7)$$

where, τ is the time constant of the pitch actuator system.

Among the three popular types of generators used in WECS- Squirrel Cage Induction Generator(SCIG), Doubly Fed Induction Generator (DFIG), and Permanent Magnet Synchronous Generator (PMSG) [29]- this research uses the PMSG which has the model in (d, q) axes as

$$\begin{aligned}
\dot{i}_d &= -\frac{R_s}{L_d}i_d + \frac{pL_q}{L_d}i_q\omega_g - \frac{1}{L_d}u_d, && (3.5.8)\\
\dot{i}_q &= -\frac{R_s}{L_q}i_q - \frac{p}{L_q}(L_d i_d - \phi_m)\omega_g - \frac{1}{L_q}u_q, && (3.5.9)\\
T_g &= p\phi_m i_q, && (3.5.10)
\end{aligned}$$

where, i_d and i_q are the d and q component current of the stator; L_d and L_q are the d and q component inductance of the stator; u_d and u_q are the d and q component voltage of the stator; p is the number of pole pairs, and R_s is the stator resistance.

Now, By combining (3.5.1) to (3.5.10), a complete sixth order nonlinear model of the PMSG based VSVP-WECS is obtained. By transforming this nonlinear model to SDC form using (3.1.1) and (3.1.2), the $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ matrix becomes,

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} A_{11} & 0 & -\frac{i}{\eta J_r} & A_{14} & 0 & 0 \\ 0 & 0 & \frac{1}{J_g} & 0 & 0 & -\frac{1}{J_g}p\phi_m \\ A_{31} & -K_g & A_{33} & A_{34} & 0 & \frac{B_g}{J_g}p\phi_m \\ 0 & 0 & 0 & -\dfrac{1}{\tau} & 0 & 0 \\ 0 & \frac{pL_q}{L_d}i_q & 0 & 0 & -\frac{R_s}{L_d} & 0 \\ 0 & A_{62} & 0 & 0 & 0 & -\frac{R_s}{L_q} \end{bmatrix}$$

and,

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} \frac{1}{2J_r}\rho\pi R^3(a_0V + a_4\omega_r R) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{iB_g}{2J_r}\rho\pi R^3(a_0V + a_4\omega_r R) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau} \\ 0 & -\frac{1}{L_d} & 0 & 0 \\ 0 & 0 & -\frac{1}{L_q} & 0 \end{bmatrix}$$

where,

$$\begin{aligned} A_{11} &= \frac{1}{2J_r}\rho\pi R^3(c_5R\beta V + c_6R\beta^2V + c_7R\beta^3V), \\ A_{14} &= \frac{1}{2J_r}\rho\pi R^3(c_1V^2 + c_2\beta V^2 + c_3\beta^2V^2), \\ A_{31} &= iK_g + \frac{iB_g}{2J_r}\rho\pi R^3(c_5R\beta V + c_6R\beta^2V + c_7R\beta^3V), \\ A_{33} &= -B_g(\frac{1}{J_g} + \frac{i^2}{\eta J_r}), \\ A_{34} &= \frac{iB_g}{2J_r}\rho\pi R^3(c_1V^2 + c_2\beta V^2 + c_3\beta^2V^2), \\ A_{62} &= \frac{p}{L_q}(L_di_d - \phi_m). \end{aligned}$$

Table 3.1: PMSG-based VSVP Wind Turbine Parameters

Notions	Descriptions	Values
ρ	Air density	1.25 kg/m^3
R	Wind rotor plane radius	2.5 m
i	Gearbox ratio	6
η	Gearbox efficiency	1
J_r	Wind rotor inertia	2.88 $kg.m^2$
J_g	Generator inertia	0.22 $kg.m^2$
K_g	High-speed shaft stiffness coefficient	75 Nm/rad
B_g	High-speed shaft damping coefficient	0.3 $Kg.m^2/s$
p	Number of pole pairs	3
R_s	PMSG stator resistance	3.3 Ω
ϕ_m	PMSG flux linkage	0.4382 Wb
L_d	PMSG stator $d-axis$ inductance	41.56 mH
Lq	PMSG stator $q-axis$ inductance	41.56 mH
$c_0 - c_7$	$\mathrm{c}_0 = 0.0068; c_1 = -0.002; c_2 = 0.006; c_3 = -9.15\text{x}10^{-4}$ $\mathrm{c}_4 = -6.54\text{x}10^{-5}; c_5 = 1.30\text{x}10^{-5}; c_6 = -4.54\text{x}10^{-7}$ $\mathrm{c}_7 = 6.5\text{x}10^{-7}$	

3.5.2 Simulation Results

In this section, nonlinear finite horizon-regulation is simulated for both existing SDRE and simplified SDRE. PMSG based VSVP-WECS parameters are shown in Table 3.1 [36]. The polynomial coefficients ($c_0 - c_7$) are obtained by curve fitting using (3.5.3). The state vector and control vector of the described sixth order model are $\mathbf{x} = [\omega_r\ \omega_g\ T_H\ \beta\ i_d\ i_q]^T$ and $\mathbf{u} = [V\ u_d\ u_q\ \beta_d]^T$ respectively. The initial conditions are taken as $\mathbf{x}(0) = [2, 1, 3, 4, 4, 3]^T$. Existing and simplified SDRE controllers for VSVP-WECS system are obtained using MATLAB and SIMULINK. Simulations are performed by considering the final time (t_f) as 4 seconds and incremental time step as 0.001s. Weighting matrices Q, R and F are chosen as trial and error basis. The simulation results for optimal states are presented in Fig. 3.4 - Fig. 3.9 and optimal controls are shown in Fig. 3.10 and Fig. 3.11.

It is found that the state responses of all the six states of simplified SDRE controller are close to that of existing SDRE controller. In both of the cases, the

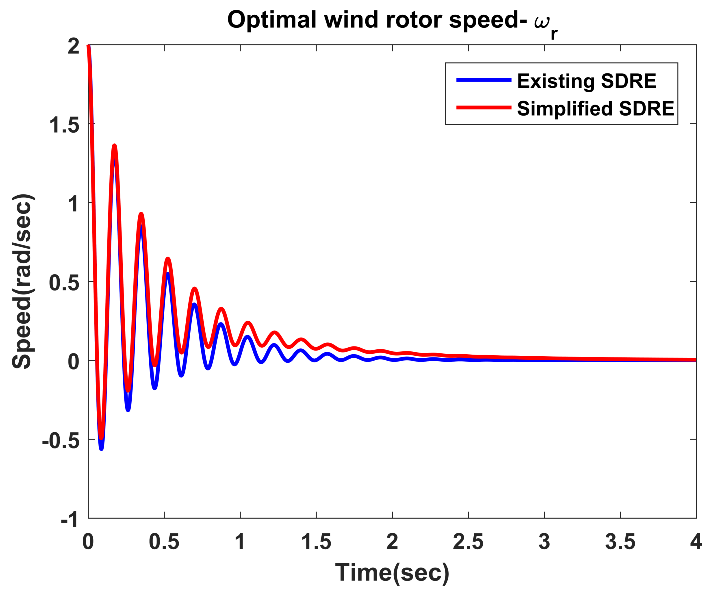

Figure 3.4: State ω_r response with existing SDRE and simplified SDRE

system is able to finish the task of regulation within 4 seconds while the control inputs are within acceptable limits.

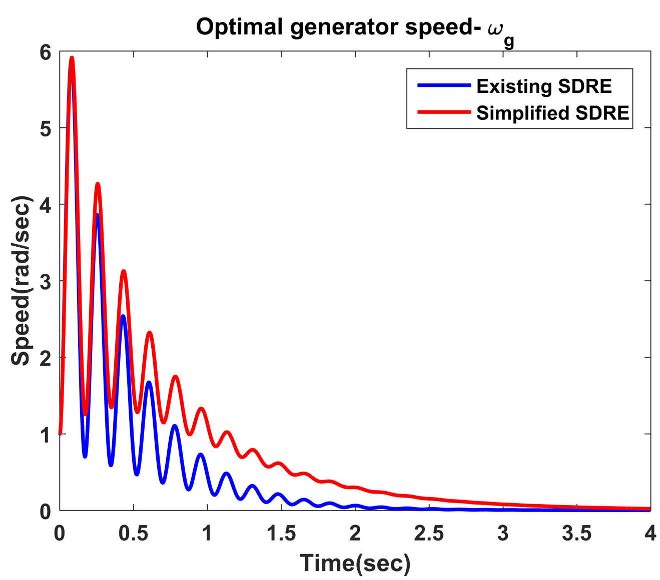

Figure 3.5: State ω_g response with existing SDRE and simplified SDRE

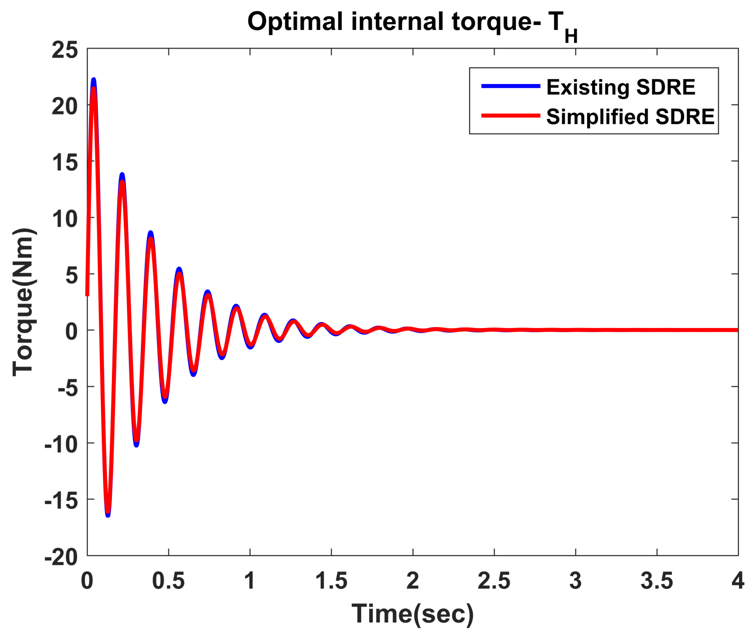

Figure 3.6: State T_H response with existing SDRE and simplified SDRE

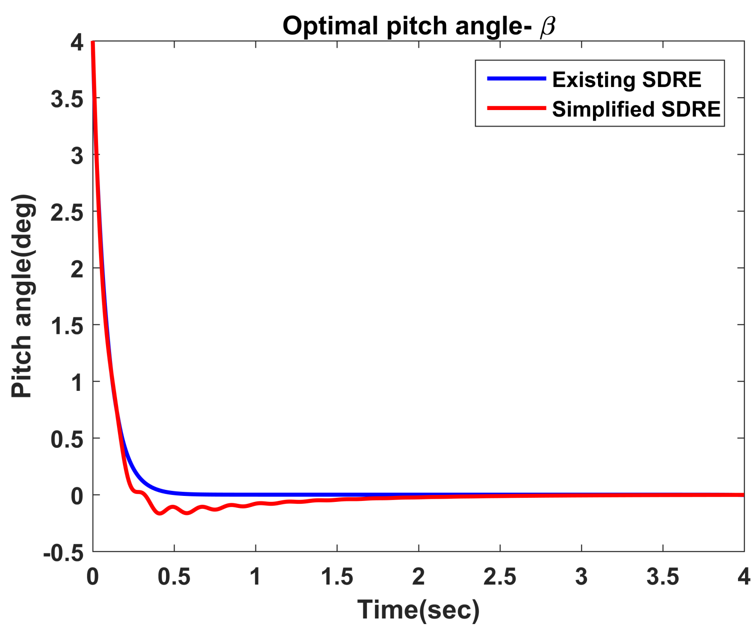

Figure 3.7: State β response with existing SDRE and simplified SDRE

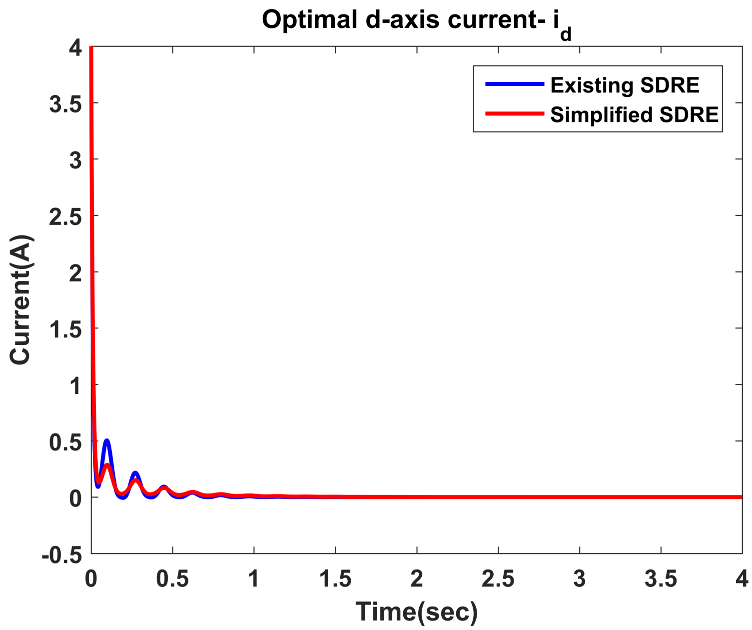

Figure 3.8: State i_d response with existing SDRE and simplified SDRE

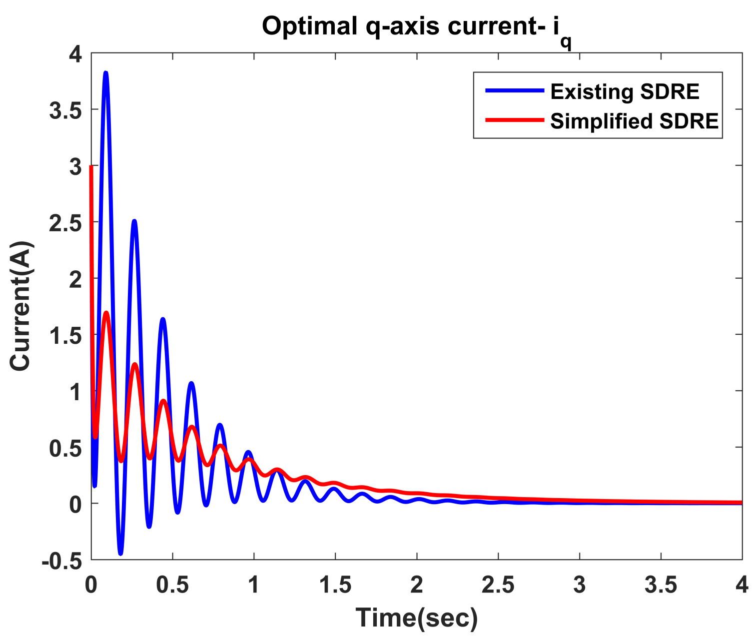

Figure 3.9: State i_q response with existing SDRE and simplified SDRE

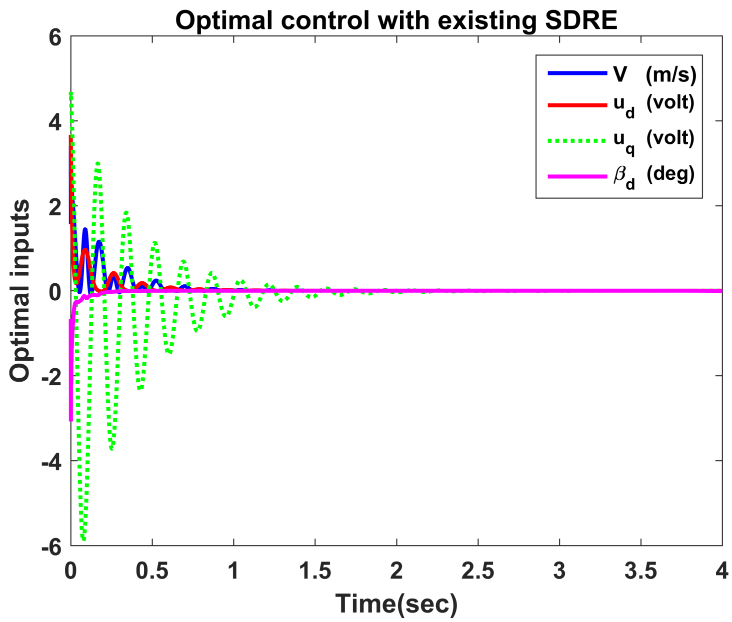

Figure 3.10: Nonlinear optimal controller with existing SDRE

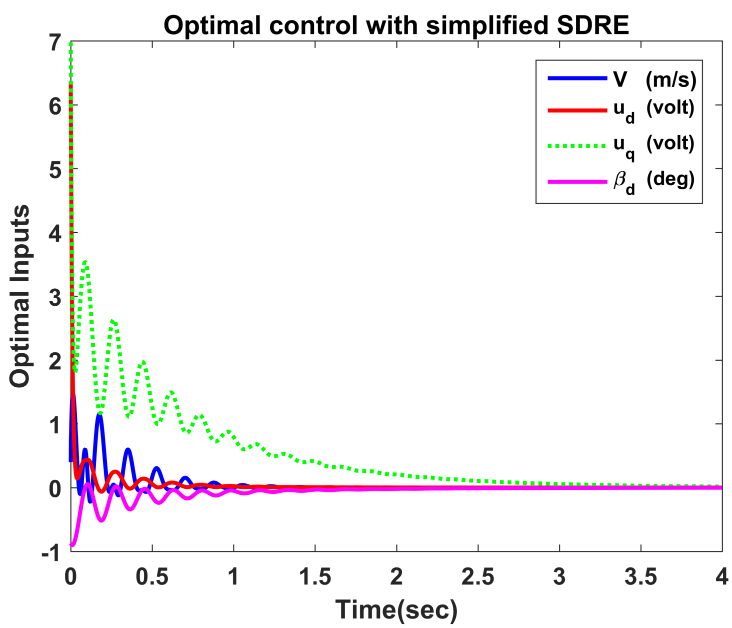

Figure 3.11: Nonlinear optimal controller with simplified SDRE

3.6 Conclusion

This chapter presents the theoretical background and implementation of finite-horizon regulation via existing SDRE and simplified SDRE. The main idea of the proposed simplified SDRE method is to use the analytic solution of the matrix differential riccati equation and the associated MATLAB program at each time step instead of using the steady-state Riccati coefficient. Simplified SDRE not only eliminates the approximate nature of the solution presents in the existing SDRE but also reduce the computational complexities. The simulation results and the comparison between these two approaches clearly shows the validity of the proposed simplified SDRE method.

Chapter 4

Finite-Horizon Tracking using Simplified SDRE

Finite-horizon optimal tracking of nonlinear systems is a challenging problem in the control field due to the time-dependency of the Hamilton Jacobi-Bellman (HJB) partial differential equation [25]. Finite-horizon regulation via SDRE is discussed in the previous chapter where differential Riccati equation (DRE) needs to be solved at each time step of a finite-horizon period. For finite-horizon tracking, in addition to the solution of the nonlinear DRE, a vector differential equation (VDE) is also required to be solved. In this chapter, the solution of the finite-horizon tracking via both existing SDRE and simplified is discussed.

4.1 Statement of Problem: SDRE

Consider the nonlinear system in the form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t), \tag{4.1.1}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}). \tag{4.1.2}$$

The nonlinear system can be converted to the state dependent coefficient (SDC) form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \tag{4.1.3}$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{x})\mathbf{x}(t), \tag{4.1.4}$$

where, $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}(t)$, $\mathbf{B}(\mathbf{x})= \mathbf{g}(\mathbf{x})$, and $\mathbf{h}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\mathbf{x}(t)$. The goal is to evaluate a state feedback control law which minimizes the cost function [31]

$$\begin{aligned}\mathbf{J}(\mathbf{x},\mathbf{u}) = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}\mathbf{e}(t_f) + \\ \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{e}'(t)\mathbf{Q}(\mathbf{x})\mathbf{e}(t) + \mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})\right] dt,\end{aligned} \tag{4.1.5}$$

Here, $e(t)$ is the closed loop error as $e(t) = z(t) - y(t)$ where $z(t)$ is the desired output or trajectory. $\mathbf{Q}(\mathbf{x})$ and $\mathbf{F}$ are symmetric *positive semi-definite* matrices, and $\mathbf{R}(\mathbf{x})$ is a symmetric *positive definite* matrix.

4.2 General Solution of the Problem

A feedback control law for minimizing the cost function (4.1.5) is defined as

$$\mathbf{u}(\mathbf{x},t) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})[\mathbf{P}(\mathbf{x},t)\mathbf{x}(t) - \mathbf{g}(\mathbf{x},t)], \tag{4.2.1}$$

where, $\mathbf{P}(\mathbf{x},t)$ is the symmetric, positive definite solution of state dependent differential Riccati equation (DRE) given as

$$\begin{aligned}-\dot{\mathbf{P}}(\mathbf{x},t) = \mathbf{P}(\mathbf{x},t)\mathbf{A}(\mathbf{x}) + \mathbf{A}'(\mathbf{x})\mathbf{P}(\mathbf{x},t) \\ - \mathbf{P}(\mathbf{x},t)\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x},t) + \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{C}(\mathbf{x}),\end{aligned} \tag{4.2.2}$$

with the final condition

$$\mathbf{P}(\mathbf{x},t_f) = \mathbf{C}'(t_f)\mathbf{F}\mathbf{C}(t_f), \tag{4.2.3}$$

and $\mathbf{g}(\mathbf{x},t)$ is the solution of the nonhomogeneous state dependent vector differential equation (VDE) which has the form

$$\dot{\mathbf{g}}(\mathbf{x},t) = -[\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-\mathbf{1}}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x},t)]'\mathbf{g}(\mathbf{x},t) \\ - \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(\mathbf{x}), \quad (4.2.4)$$

having the final condition

$$\mathbf{g}(\mathbf{x},t_f) = \mathbf{C}'(t_f)\mathbf{F}\mathbf{z}(t_f). \quad (4.2.5)$$

Then, the SDRE controlled trajectory becomes the solution of the state dependent closed loop dynamics

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x},t)]\mathbf{x}(t) \\ + \mathbf{B}(\mathbf{x})\mathbf{R}^{-\mathbf{1}}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{g}(\mathbf{x},t). \quad (4.2.6)$$

4.3 Existing SDRE Method

In the existing SDRE method, the nonlinear dynamics of the system in the state-space form is first transformed into the state-dependent coefficient (SDC) form. Then, the existing SDRE method assumes that during each of the small time steps of the finite-horizon period, the Riccati coefficient $\mathbf{P}(\mathbf{x},t)$ is constant and an algebraic Riccati equation (ARE) is considered to evaluate the steady-state Riccati coefficient $\mathbf{P_{ss}}(\mathbf{x})$. Then, a new variable $\mathbf{K}(\mathbf{x},t)$ is introduced in terms of unknown coefficient $\mathbf{P}(\mathbf{x},t)$ and known coefficient $\mathbf{P_{ss}}(\mathbf{x})$ as $\mathbf{K}(\mathbf{x},t)=[\mathbf{P}(\mathbf{x},t)-\mathbf{P_{ss}}(\mathbf{x})]^{-\mathbf{1}}$ using the change of variable technique. Now, substituting the expression for $\mathbf{K}(\mathbf{x},t)$, the nonlinear DRE is transformed into linear differential Lyapunov equation (DLE) which is solved in real-time during each time interval [37]. Once the Lyapunov coefficient $\mathbf{K}(\mathbf{x},t)$ is known, the Riccati coefficient $\mathbf{P}(\mathbf{x},t)$ is recovered from $\mathbf{P}(\mathbf{x},t)=\mathbf{K}^{-1}(\mathbf{x},t)+\mathbf{P_{ss}}(\mathbf{x})$.

Similarly, for obtaining the solution of vector differential equation $\mathbf{g}(\mathbf{x},t)$, the existing SDRE technique first assumes that during each small time step, $\mathbf{g}(\mathbf{x},t)$ is constant and use steady-state vector coefficient $\mathbf{g_{ss}}(\mathbf{x})$ by solving vector algebraic equation (VAE). A change of variable technique is then applied by introducing a new variable $\mathbf{K_g}(\mathbf{x},t)$ as $\mathbf{K_g}(\mathbf{x},t)$=[$\mathbf{g}(\mathbf{x},t)-\mathbf{g_{ss}}(\mathbf{x})$] which after substituting in VDE leads to a differential equation in terms of $\mathbf{K_g}(\mathbf{x},t)$ [22]. Finally, $\mathbf{g}(\mathbf{x},t)$ is recovered from $\mathbf{g}(\mathbf{x},t)$=$\mathbf{K_g}(\mathbf{x},t)$ + $\mathbf{g_{ss}}(\mathbf{x})$. After evaluating both $\mathbf{P}(\mathbf{x},t)$ and $\mathbf{g}(\mathbf{x},t)$, finite-horizon optimal *tracking* control is obtained using (4.2.1). The following steps for evaluating $\mathbf{P}(\mathbf{x},t)$ and $\mathbf{g}(\mathbf{x},t)$ can be followed at each time step for the solution of finite-horizon *tracking* via existing SDRE method [21, 22, 32]:

1. Calculate steady-state Riccati coefficient $\mathbf{P_{ss}}(\mathbf{x})$ by solving ARE.

$$\begin{aligned}\mathbf{P_{ss}}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) \\ - \mathbf{P_{ss}}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) = 0.\end{aligned} \quad (4.3.1)$$

2. Use change of variable technique and assume

$$\mathbf{K}(\mathbf{x},t) = [\mathbf{P}(\mathbf{x},t) - \mathbf{P_{ss}}(\mathbf{x})]^{-\mathbf{1}}. \quad (4.3.2)$$

3. Evaluate the value of $\mathbf{A_{cl}}(\mathbf{x})$ as

$$\mathbf{A_{cl}}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-\mathbf{1}}\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}). \quad (4.3.3)$$

4. Solve algebraic Lyapunov equation to evaluate the value of $\mathbf{D}$ [12]

$$\mathbf{A_{cl}}\mathbf{D} + \mathbf{D}\mathbf{A}'_{\mathbf{cl}} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}' = \mathbf{0}. \quad (4.3.4)$$

5. Solve the differential Lyapunov equation

$$\begin{aligned}\dot{\mathbf{K}}(\mathbf{x},t) = \mathbf{K}(\mathbf{x},t)\mathbf{A}'_{\mathbf{cl}}(\mathbf{x}) + \mathbf{A_{cl}}(\mathbf{x})\mathbf{K}(\mathbf{x},t) \\ - \mathbf{B}(\mathbf{x})\mathbf{R}^{-\mathbf{1}}(\mathbf{x})\mathbf{B}'(\mathbf{x}).\end{aligned} \quad (4.3.5)$$

   The solution of (16) is found as

$$\mathbf{K}(\mathbf{x},t) = \mathbf{e}^{\mathbf{A_{cl}}(\mathbf{t}-\mathbf{t_f})}(\mathbf{K}(\mathbf{x},t_f) - \mathbf{D})\mathbf{e}^{\mathbf{A}'_{\mathbf{cl}}(\mathbf{t}-\mathbf{t_f})} + \mathbf{D}. \quad (4.3.6)$$

6. Find $\mathbf{P}(\mathbf{x},t)$ from the equation

$$\mathbf{P}(\mathbf{x},t) = \mathbf{K}^{-1}(\mathbf{x},t) + \mathbf{P_{ss}}(\mathbf{x}). \tag{4.3.7}$$

7. Find steady-state vector coefficient $\mathbf{g_{ss}}(\mathbf{x})$ as

$$\mathbf{g_{ss}}(\mathbf{x}) = [\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x})]'^{-1} \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(\mathbf{x}). \tag{4.3.8}$$

8. Use change of variable technique and assume

$$\mathbf{K_g}(\mathbf{x},t) = [\mathbf{g}(\mathbf{x},t) - \mathbf{g_{ss}}(\mathbf{x})]. \tag{4.3.9}$$

9. Solve the differential equation

$$\dot{\mathbf{K}}_{\mathbf{g}}(\mathbf{x},t) = -[\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x},t)]' \mathbf{K_g}(\mathbf{x},t) \tag{4.3.10}$$

   The solution of (21) is found as

$$\mathbf{K_g}(\mathbf{x},t) = \mathbf{e}^{-(\mathbf{A}-\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P})'(\mathbf{t}-\mathbf{t_f})}[\mathbf{g}(\mathbf{x},t_f) - \mathbf{g_{ss}}(\mathbf{x})]. \tag{4.3.11}$$

10. Evaluate $\mathbf{g}(\mathbf{x},t)$ from the following

$$\mathbf{g}(\mathbf{x},t) = \mathbf{K_g}(\mathbf{x},t) + \mathbf{g_{ss}}(\mathbf{x}). \tag{4.3.12}$$

11. Finally, using $\mathbf{P}(\mathbf{x},t)$ and $\mathbf{g}(\mathbf{x},t)$ calculate the value of optimal control $\mathbf{u}(\mathbf{x},t)$ as

$$\mathbf{u}(\mathbf{x},t) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})[\mathbf{P}(\mathbf{x},t)\mathbf{x}(t) - \mathbf{g}(\mathbf{x},t)] \tag{4.3.13}$$

The algorithmic steps for finding the solution of finite-horizon *tracking* via existing SDRE is presented in Fig. 4.1.

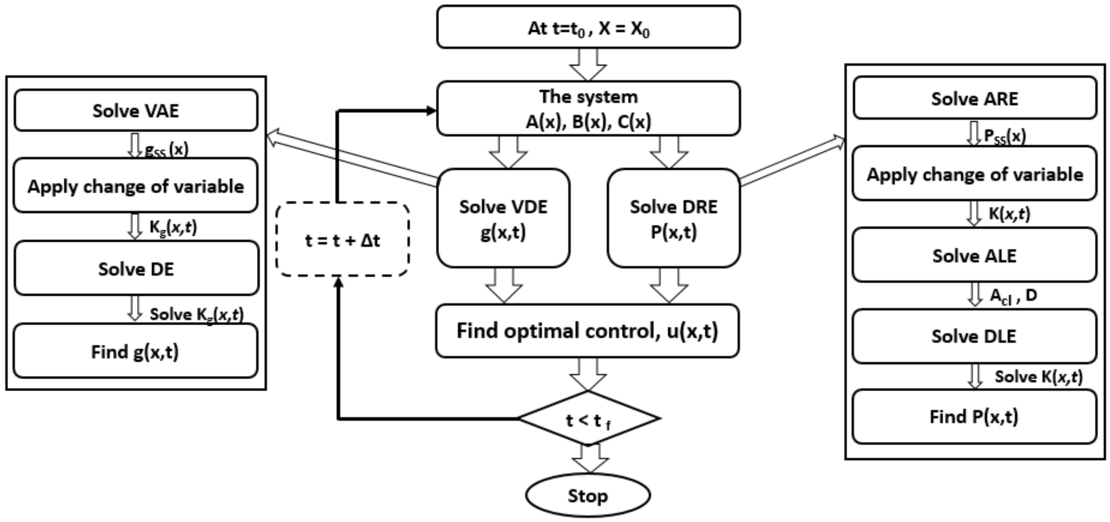

Figure 4.1: Flow chart of finite-horizon optimal tracking via existing SDRE

4.4 Simplified SDRE Method

In this section, a simplified SDRE technique for nonlinear finite-horizon *tracking* is proposed. In the existing SDRE, it is assumed that the Riccati coefficient, $\mathbf{P}(\mathbf{x},t)$ and the vector coefficient, $\mathbf{g}(\mathbf{x},t)$ is constant during each small time steps between initial and final time of a finite-horizon period. This assumption definitely leads to a sub-optimal control theory. This approximation is due to the lack of analytic solution of the nonlinear DRE and the associated real-time implementable MATLAB program. However, a MATLAB based program *lqtnss* was developed in [41] for solving the finite time linear quadratic *tracking* for optimal control systems where *lqtnss* stands for linear quadratic tracking (*lqt*) for non-steady state condition.

Recently, a simplified SDRE *regulation* is proposed by employing a MATLAB based program *lqrnss* (linear quadratic regulator for non-steady state condition) developed by the authors in [31, 41] at each time step of a finite-horizon period [33]. The program *lqrnss* contains the analytic solution of matrix differential Riccati equation (DRE) [31, 41]. However, for finite-horizon *tracking*, in addition to the solution of matrix DRE, a vector differential equation (VDE) needs to be solved simultaneously at each time step. The MATLAB based program *lqtnss* solves both matrix DRE and VDE analytically [41].

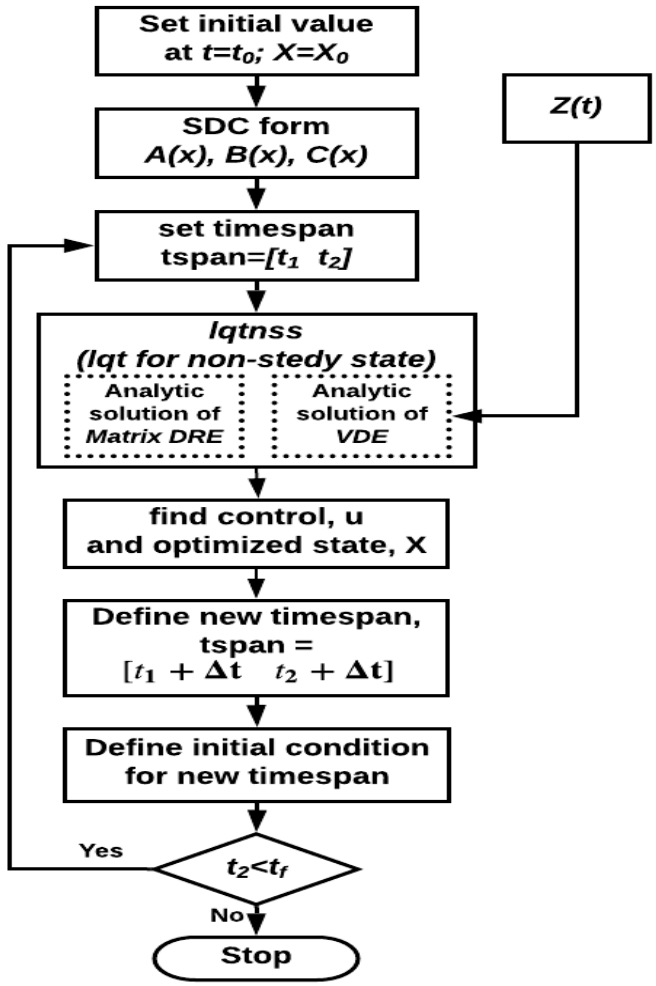

Figure 4.2: Flow chart of finite-horizon optimal tracking via simplified SDRE

In the proposed simplified SDRE *tracking* technique, the SD-DRE and the SD-VDE arising in finite-horizon *tracking* problem is solved by using the MATLAB command *lqtnss* which employs the analytic solution of matix DRE and VDE at each small time steps. The implementation of *lqtnss* eliminates the approximate nature of the solution by avoiding the use of ARE and VAE at each time step, which ultimately avoids sub-optimality. In addition, the simplified SDRE method reduces the computational complexities involved in finding the solution of SD-DRE and SD-VDE in existing SDRE method as explained in section 4.3. The algorithmic steps for solving a nonlinear, finite-horizon *tracking* via simplified SDRE technique is represented in a flowchart shown in Fig. 4.2.

4.5 Finite-Horizon Tracking via Existing and Simplified SDRE

To validate the effectiveness of the simplified SDRE tacking, we implemented both the existing and simplified SDRE method to a finite-time optimal tracking of a wind energy con-version system (WECS) of fifth order. A comparison of the system states and the control signals resulted from these two methods is also presented in this section. Here, the goal of the control technique is to track the optimal reference rotor speed in the partial load region so that the optimal energy conversion is achieved.

4.5.1 System Modeling

The nonlinear model of WECS used in this section is almost same as the model used in section 2.5.1 except the pitch actuator part. For simplification, the pitch actuator dynamics is neglected in this section. In the nonlinear modeling, aerodynamics, drive-train dynamics and generator dynamics are considered. Since most of the part of this section are very similar to section 2.5.1, instead of going through the details of each blocks, only the nonlinear dynamical equations are presented here.

$$\dot{\omega}_r = -\frac{i}{\eta J_r}T_H + \frac{1}{J_r}T_r, \tag{4.5.1}$$

$$\dot{\omega}_g = \frac{1}{J_g}T_H - \frac{1}{J_g}T_g, \tag{4.5.2}$$

$$\dot{T}_H = iK_g\omega_r - K_g\omega_g - B_g(\frac{1}{J_g} + \frac{i^2}{\eta J_r})T_H \tag{4.5.3}$$

$$+ \frac{iB_g}{J_r}T_r + \frac{B_g}{J_g}T_g,$$

$$\dot{i}_d = -\frac{R_s}{L_d}i_d + \frac{pL_q}{L_d}i_q\omega_g - \frac{1}{L_d}u_d, \tag{4.5.4}$$

$$\dot{i}_q = -\frac{R_s}{L_q}i_q - \frac{p}{L_q}(L_d i_d - \phi_m)\omega_g - \frac{1}{L_q}u_q, \tag{4.5.5}$$

$$T_g = p\phi_m i_q, \tag{4.5.6}$$

where, T_r is the aerodynamic torque, T_H is the internal torque, ω_g is the generator speed, T_g is the generator electromagnetic torque, J_r is the wind rotor inertia, J_g is the generator inertia, K_g is the stiffness coefficient of the high-speed shaft, B_g is the damping coefficient of the high-speed shaft, i is the gearbox ratio and η is the gearbox efficiency, i_d and i_q are the d and q component current of the stator; u_d and u_q are the d and q component voltage of the stator; L_d and L_q are the d and q component inductance of the stator; p is the number of pole pairs, ϕ_m is the flux and R_s is the stator resistance.

4.5.2 Simulation Results

In this section, the nonlinear finite-horizon *tracking* is simulated with both existing SDRE and simplified SDRE for PMSG based WECS. This nonlinear model presented in (4.5.1) to (4.5.6), after transforming to the SDC form using (4.1.1)-(4.1.4) and considering the state vector and the control vector as $\mathbf{x} = [\omega_r\ \omega_g\ T_H\ i_d\ i_q]^T$ and $\mathbf{u} = [u_d\ u_q]^T$ respectively, $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ matrices becomes,

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} A_{11} & 0 & -\frac{i}{\eta J_r} & 0 & 0 \\ 0 & 0 & \frac{1}{J_g} & 0 & -\frac{1}{J_g}p\phi_m \\ A_{31} & -K_g & A_{33} & 0 & \frac{B_g}{J_g}p\phi_m \\ 0 & \frac{pL_q}{L_d}i_q & 0 & -\frac{R_s}{L_d} & 0 \\ 0 & A_{52} & 0 & 0 & -\frac{R_s}{L_q} \end{bmatrix}$$

and,

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{L_d} & 0 \\ 0 & -\frac{1}{L_q} \end{bmatrix}$$

where,

$$\begin{aligned} A_{11} &= \frac{\rho\pi R^2 C_P(\lambda,\beta)V^3}{2J_r\omega_r^2}, \\ A_{31} &= iK_g + \frac{iB_g\pi\rho R^2 C_P(\lambda,\beta)V^3}{2J_r\omega_r^2}, \\ A_{33} &= -B_g(\frac{1}{J_g} + \frac{i^2}{\eta J_r}), A_{52} = -\frac{p}{L_q}(L_d i_d - \phi_m). \end{aligned}$$

Here, the initial conditions of the states are taken as $\mathbf{x}(0) = [20, 120, 3, 5, 3]^T$. A staircase wind profile presented in Fig. 4.3 is considered as the reference wind speed in this work. For a given wind speed V, the turbine needs to track a reference speed which is found as follows [48]

$$\omega_{r,ref} = \frac{\lambda_{opt}V}{R}. \tag{4.5.7}$$

The simulations of the *tracking* controllers are performed in MATLAB/Simulink environment by considering the final time (t_f) as 20 seconds and the incremental time step as 0.005 second. Weighting matrices Q, R and F are chosen based on trial and error. The purpose of the control mechanism is to track an optimal reference

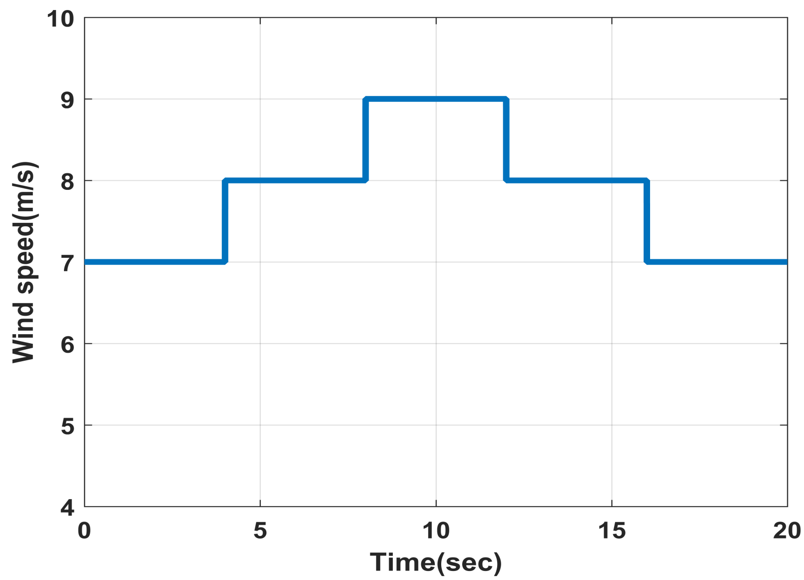

Figure 4.3: Wind speed profile

speed given by (4.5.7). The simulation results are presented in Fig. 4.4 - Fig. 4.6. Fig. 4.4 illustrates the finite horizon *tracking* performance, where the dotted green line presents the reference speed, solid blue line presents the tracking performance via existing SDRE and the solid red line presents the tracking performance via simplified SDRE.

It is found that the proposed simplified SDRE controller is making an excellent tracking of the reference rotor speed and maintaining a very close tracking performance with the existing SDRE controller. Both the existing and simplified SDRE controllers are maintaining a negligible tracking error as shown in Fig. 4.5. The optimal control signals for both the controllers are presented in Fig. 4.6 which are within the acceptable limits.

4.6 Conclusion

This chapter illustrates the theoretical background and implementation of finite-horizon tracking via existing SDRE and simplified SDRE. The main idea of the proposed simplified SDRE method is to use the analytic solution of the matrix

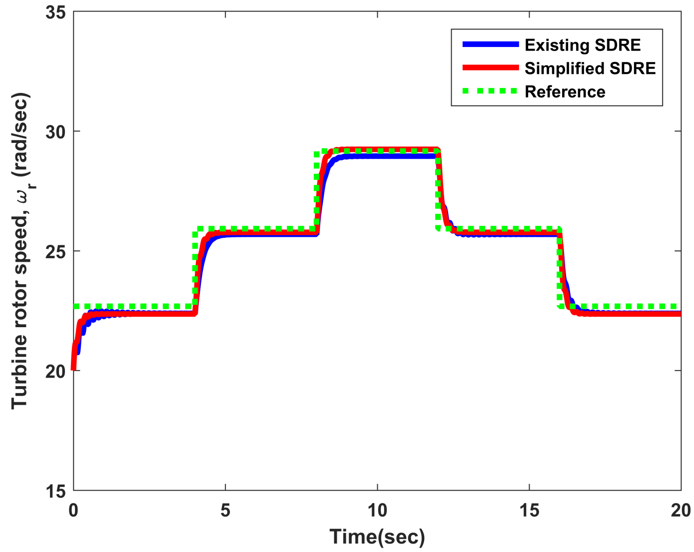

Figure 4.4: Tracking performance via existing SDRE and simplified SDRE

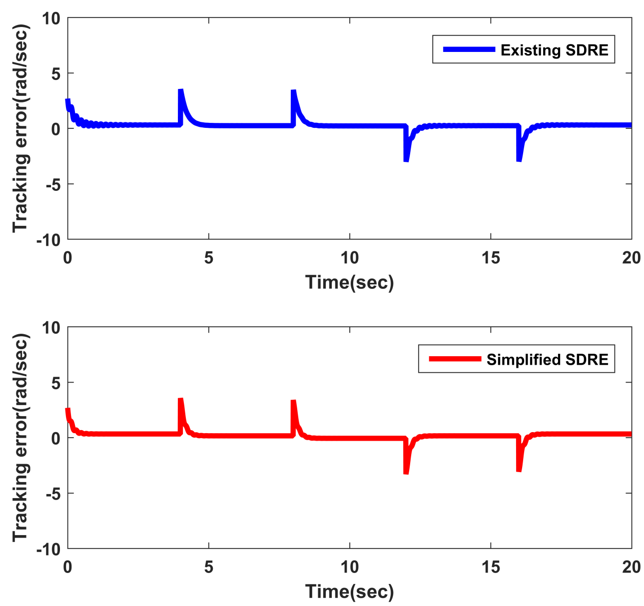

Figure 4.5: Tracking error via existing SDRE and simplified SDRE

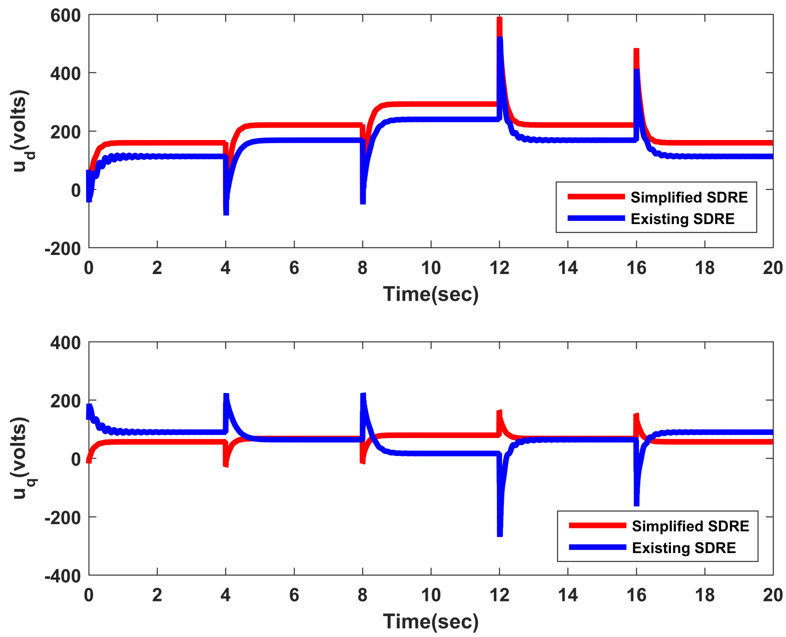

Figure 4.6: Nonlinear finite-horizon tracking controller via existing SDRE and simplified SDRE

differential riccati equation (MDRE) and vector differential equation (VDE) and their associated MATLAB program (*lqtnss*) at each time step instead of using the steady-state Riccati coefficient and steady-state vector coefficient. Simplified SDRE tracking not only eliminates the approximate nature of the solution presents in the existing SDRE but also reduce the computational complexities. The simulation results and the comparison between these two methods clearly shows the validity of the simplified SDRE for finite-horizon tracking.

Chapter 5

Conclusion and Future Work

5.1 Discussion and Conclusion

In this research, a detailed investigation of advanced finite-horizon regulation and tracking of nonlinear systems via state-dependent Riccati equation (SDRE) has been presented. Although SDRE provides an effective algorithm for optimal control of nonlinear systems, the existing SDRE method suffers from suboptimality and computational complexities. These issues were encountered in this research and a simplified SDRE method for both regulation and tracking is proposed.

In case of finite-horizon regulation with existing SDRE method, the Riccati coefficient is assumed to be constant at each small time step of a given time period, which ultimately leads to a suboptimality in the solution. Apart from that, the solution of the differential Riccati equation (DRE) using existing method requires several steps which makes the algorithm computationally complex. In this research, instead of assuming the Riccati coefficient being constant at each time step, the analytical solution of the matrix differential Riccati equation (MDRE) and its associated MATLAB program *lqrnss* is used at each small time step of the finite-horizon period. This approach not only eliminates the suboptimal nature of the solution presents in the existing SDRE, but also eliminates the computational complexity involved in solving the DRE.

A similar approach is followed in solving the finite-horizon tracking problem via SDRE method. In finite-horizon tracking, in addition to solving the DRE, a vector differential equation (VDE) needs to be solved simultaneously at each time step. In the exiting SDRE, the consideration of DRE and VDE being constant at each time step makes the method suboptimal. It also makes the algorithm computationally complex. In the simplified SDRE tracking, instead of considering the DRE and VDE to be contant at each small time step, the analytical solution of MDRE and VDE, and their associated MATLAB program *lqtnss* is used. *lqtnss* eliminates the suboptimality of the solution, as well as reduces the computational complexity of the algorithm.

The simplified SDRE method is validated by implementing it in a complex nonlinear model of a permanent magnet synchronous generator based wind energy conversion system (PMSG-WECS) and compared with the existing SDRE. The simulation results and the comparison between these two methods clearly shows the effectiveness of the simplified SDRE.

5.2 Publications from Thesis

Portions of the research contributions of this thesis have so far been published in three international peer-reviewed conference papers as follows:

1. D. S. Naidu, **S. Paul**, A. Khamis and C. R. Rieger, "A Simplified SDRE Technique for Finite Horizon Tracking Problem in Optimal Control Systems," 2019 Sixth Indian Control Conference (ICC), 2019, pp. 170-175 doi: 10.1109/ICC47138.2019.9123230 [32].

2. **S. Paul** and D. S. Naidu, "Nonlinear Optimal Tracking Control of Wind Energy Conversion System in Partial Load Region," 2019 North American Power Symposium (NAPS), 2019, pp. 1-6, doi: 10.1109/NAPS46351.2019.9000319 [39].

3. D. S. Naidu, **S. Paul** and C. R. Rieger, "A Simplified SDRE Technique for Regulation in Optimal Control Systems," 2019 IEEE International Conference on Electro Information Technology (EIT), 2019, pp. 327-332, doi: 10.1109/EIT.2019.8834201 [33].

4. **S. Paul**, D. S. Naidu, C. R. Rieger, "A Simplified Technique for Finite-Horizon Optimal Regulation of Stochastic System" [Under Preparation].

5.3 Future Work

The successful implementation of the simplified SDRE in complex nonlinear systems gives us hope to extend its research scope in the future and accomplish the following goals:

1. Theoretical development and implementation of the finite-horizon *regulation* via simplified SDRE for stochastic systems.

2. Theoretical development and application of the finite-horizon *tracking* via simplified SDRE for stochastic systems.

3. Development of finite-horizon optimal *regulation* and *tracking* via simplified SDRE for nonlinear, discrete-time systems.

4. Finally, hardware implementation of simplified SDRE to simple and complex nonlinear systems.

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