Exterior Cube L-function for GL_6

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Shengmei An

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Professor Dihua Jiang

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Contents

Chapter 1

Introduction

In this paper, we study the analytic properties of exterior cube L-function for the group GL_6 introduced by Ginzburg and Rallis in [1]. More precisely, let $\pi = \otimes_v \pi_v$ be a cusp form of $GL_6(A_F)$, we can associate to each unramified representation π_v a semisimple conjugacy class t_v in the L-group, then we can define the local L-function as

$$L_v(\pi_v \otimes \chi_v, \Lambda^3, s) = \det(I - \Lambda^3(t_v)\chi_v(p)q^{-s})^{-1}$$

where I is the 20×20 identity matrix. Let S be a finite set of places in F including the archimedean ones such that π_v and χ_v are unramified outside S, so we can define the partial exterior L-function as

$$L^S(\pi \otimes \chi, \Lambda^3, s) = \prod_{v \notin S} L_v(\pi_v \otimes \chi_v, \Lambda^3, s)$$

Our goal is to show that the partial L-function is entire for all characters χ and for all generic cup forms π on $GL_6(A)$ except when $\omega_\pi^2 \chi^4 = 1$ and $\omega_\pi \chi^2 \neq 1$. We also generalized the analytic properties to the general cases.

Our method is to use the Rankin-Selberg method. In section 2, we will introduce the Langlands L-function Conjecture, and describe the typical steps in contructing Rankin-selberg integrals and their properties. For more details can be found in [2]. In section 3, we will talk about the basic case of Rankin-Selberg integral: Tate's integral following [3], which will be used to show the convergence and meromorphic continuation of our integral in section 7. In section 4, we will discuss the Fourier-Whittaker Expansion of cusp forms for GL_n following [4], and we will use a similar approach when contructing the global integral.

We will follow through the steps in [1] in the rest of the section with detailed proofs. In section 5, we will construct the exterior cube integral for GL_6 with Siegel-Eisenstein series. In section 6, we will calculate the integral with unramified data. In section 7, we will give a new proof about the convergence and meromorphic continuation of our integral using Gauge theory [5]. We will also prove that we can choose the data nicely so that the integral is nonzero at $s = s_0$. In section 8, we prove the properties of the partial exterior cube L-function that the partial L-function is entire for all characters χ and for all generic cusp forms π on $GL_6(A)$ except when $\omega_\pi^2\chi^4 = 1$ and $\omega_\pi\chi^2 \neq 1$.

Chapter 2

Langlands Conjecture and Rankin-Selberg Convolutions

2.1 Satake correspondence

In Langlands' program, Satake correspondence gives a correspondence between unramified representation of a reductive group G over a local field and conjugacy classes in the Langlands dual group ${}^LG$.

Let F be a local field with ring of integers $\mathcal{O}$ and a uniformizer ϖ. $\mathbf{G}$ is a split connected reductive group, K be a compact open subgroup of G, T be the torus group of G, $T_0 = T \cap K$. $H(G, K)$ denotes the spherical function algebra.

Theorem 2.1.1 (Satake Isomorphism)**.** *There is an algebra isomorphism $\mathcal{H}(F, K) \to \mathcal{H}(T, T_0)$ given by*

$$f \mapsto (\mathcal{S}f)(t) = \delta(t)^{1/2} \int_N f(tn)dn$$

where $\delta(t) = |\det(Ad(t)|_{\mathfrak{n}})|^{-1}$.

Corollary 2.1.1.1. *(Unramified Local Langlands) Fix an embedding $\hat{T} \subset GL_n(\mathbb{C})$, there is a bijection between isomorphism classes of irreducible unramified representations of G and semisimple unramified representations $\phi : W_F \to GL_n(\mathbb{C})$, where W_F is the Weyl group.*

2.2 L-function data

Let G be a connected reductive linear algebraic group over F, $\pi = \otimes_v \pi_v$ be an irreducible unitary automorphic cuspidal representation of $G(A)$, r be a finite dimensional representation of the L-group ${}^LG$ of G. These data are refered to as the L-function data (G, π, r). The data is called unramified at a finitely place v if G is unramified over F_v and π_v is unramified.

In this case, π_v corresponds to a semisimple conjugacy class t_v in the local Langlands dual group ${}^L G_v$ by the Satake homomorphism, so we can define the local Langlands L-factor as

$$L(s, \pi_v, r_v) = [\det(1 - r_v(t_v)q_v^{-s})]^{-1}$$

Let S be a finite set of places of F such that for $v \notin S$, the L-function data are unramified. Then Langlands defined the global L-function as

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v)$$

In [6], he proved the convergence of the Euler product $L_S(s, \pi, r)$ when $\Re s > s_0$ for some s_0 for irreducible unirary representation π. Later in [7], he extended the $L_S(s, \pi, r)$ to a meromorphic function in $\mathbb{C}$ for automorphic cuspidal π.

Later he conjectured[8]:

Conjecture 1. *Given any G, π, r, S, $L(s, \pi, r)$ (in our case it means L_S) is a meromorphic function in the entire complex plane $\mathbb{C}$ with only a finite number of poles and statisfies the functional equation*

$$L(s, \pi, r) = \epsilon(s, \pi, r)L(1 - s, \pi, \widetilde{r})$$

and

$$\epsilon(s, \pi, r) = \prod_{v} \epsilon(s, \pi_p, r_p, \psi).$$

For the generalization of this conjecture, there are two approaches proven successful . One is by Langlands' original approach using the theory of Eisenstein series. The other one is to undertand the analytic properties of Zeta integrals, which could give an explicit construction of the L-functions. We will introduce the second approach in a very simple case $G = GL_2$ to show the basic ideas and standard steps.

2.3 GL_2 Jacquet-Langlands method

Let π be an automorphic cuspidal representation of GL_2 over $A_{\mathbb{Q}}$, which is the generalization of a cusp form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

Define the "complete" L-function attached to f as

$$L(s, f) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \int_0^{\infty} f(iy)y^{s-1}dy$$

f corresponds to automorphic cuspidal representation $\pi_f = \otimes\pi_p$, where unramified π_p of $GL_2(\mathbb{Q}_p)$ is of the form $Ind_{B_p}^{GL_2(\mathbb{Q}_p)} \mu_1 \otimes \mu_2$ (μ_i are unramified characters in the form of $|x|_P^{s_i}$ i=1,2).

The L-group of GL_2 is ${}^L G = GL_2(\mathbb{C})$, and the conjugacy class t_p is $\begin{pmatrix} \mu_1(p) & \\ & \mu_2(p) \end{pmatrix}$. Let r be the standard representation of ${}^L G$. Then

$$L(s, \pi_p, r) = [\det(I - r(t_p)p^{-s})]^{-1} = (1 - a_p p^{-s'} + p^{k-1-2s'})^{-1}, \quad s' = \frac{k-1}{2} + s$$

then

$$\prod_{p<\infty} L(s, \pi_p, r) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s'}}.$$

Define the global Zeta-integral

$$Z(\varphi, s) = \int_{\mathbb{Q}^* \backslash \mathbb{A}^*} \varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|_{\mathbb{J}}^{s-\frac{1}{2}} d^* a$$

where φ is a function in the subspace of $L_0^2(\chi)$ realizing π, then φ is rapidly decreasing at infinity, i.e. for any compact set C in $G_{\mathbb{A}}$, and $t > 0$, there exists C_N such that

$$\varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) \leqslant C_N |a|^{-N}, \quad g \in C, |a| > t$$

Now we want to relate $Z(\varphi, s)$ to the automorphic L-function $L(s, \pi)$. By using the Fourier expansion, we can write a cusp form as

$$\varphi(g) = \sum_{\xi \in \mathbb{Q}^*} W_\varphi\left(\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} g\right)$$

where ψ is a nontrivial character on $\mathbb{Q}\backslash\mathbb{A}$, and

$$W_\varphi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \overline{\varphi}(x) dx$$

Then $W_\varphi(g)$ is rapidly decreasing at infinity, and for $\Re s$ large enough,

$$Z(\varphi, s) = \int_{\mathbb{A}^*} W_\varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^* a$$

Since the Whittaker model $W(\pi, \psi)$ equals the tensor product of local Whittaker models $W(\pi_p, \psi_p)$, we have

$$W_\varphi(g) = \prod_p W_p(g)$$

where $W_p \in W(\pi_p, \psi_p)$ depends on φ. The local Zeta-integral

$$Z(W_p, s) = \int_{\mathbb{Q}_p^*} W_p\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^* a$$

It converges absolutely for $\Re s > s_0$, so we obtain the Euler product expansion for $\Re s$ large enough

$$Z(\varphi, s) = \prod_{p \leqslant \infty} Z(W_p, s)$$

The following general results in local theory are expected for Zeta/Rankin-Selberg integerals:

(a) $Z(W, s)$ converges for $\Re s > s_0$.

(b) $Z(W, s)$ is a rational functions in p^{-s}, hence meromorphic in $\mathbb{C}$.

(c) $\{Z(W, s)\}$ admits a common denominator as a polynomial P such that $P(p^{-s})Z(W, s) \subset \mathbb{C}[p^{-s}, p^s]$, for all $W \in W(\pi_p)$.

(d) There exists W_0 such that $Z(W_0, s) \equiv 1$.

(e) There exists a meromorphic function $\gamma(\pi_p, \psi_p, s)$, such that

$$\widetilde{Z}(W^w, 1 - s) = \gamma(\pi_p, \psi_p, s) Z(W, s)$$

where $W^w(g) = W(gw)$, $\widetilde{Z}(W, s) = \int_{\mathbb{Q}_p^*} \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) |a|^{s-\frac{1}{2}} \chi^{-1}(a) d^* a$, where χ_p is the central character of π_p.

(f) If π_p is unramified, then for K-invariant W^0,

$$Z(W^0, s) = \det[I - r(t_\pi) p^{-s}]^{-1} = L(s, \pi_p, r_p).$$

From the theory of asymptotic expansions[**?**], we can derive the first four results.

Now we go to the global theory. Since we haven't define $L(s, \pi_p)$ for ramified places, so we first restrict them to unramified places, i.e. partial L-function

$$L_S(s, \pi, r) = \prod_{r \text{ unramified}} L(s, \pi_p, r_p)$$

For ramified p, pick W_p such that $Z(W_p, s) \equiv 1$. Then for $W = \prod_p W_p$, where $W_p = W_p^0$ (the unique right K-invariant function such that $W^0(k) \equiv 1$ for all $k \in K_p$) for unramified p, we have

$$Z(\varphi, s) = e^{g(s)} L_S(s, \pi)$$

where $g(s)$ is some holomorphic function, thus we can deduce the holomorphy of $L_S(s, \pi)$ from the holomorphy of the Zeta integral $Z(\varphi, s)$.

As for the functional equation of $L_S(s,\pi)$, we first recall the functional equation of Zeta integral $Z(\varphi, s)$, then

$$\begin{aligned} Z(\varphi,s) &= \prod_{p\leqslant\infty} Z(W_p,s) \\ &= (\prod_{p\in S} Z(W_p,s))L_S(s,\pi) \\ &= (\prod_{p\in S}\gamma^{-1}(\pi_p,\psi_p,s))\prod_{p\in S}\widetilde{Z}(W_p^w,1-s)L_S(s,\pi) \end{aligned}$$

By the global functional equation:

$$Z(\varphi,s) = \widetilde{Z}(\varphi^w,1-s) = \prod \widetilde{Z}(W_p^w,1-s) = \prod_{p\in S}\widetilde{Z}(W_p^w,1-s)L_S(s,\pi)$$

For $W_p = W_p^0$,

$$\widetilde{Z}(W_p^w,s) = [(1-\mu_1^{-1}(p)p^{-s})(1-\mu_2^{-1}(p)p^{-s})]^{-1} = \det[I - r(\widetilde{t_p})p^{-s}]^{-1}$$

where $\widetilde{t_p} = \begin{pmatrix} \mu_1^{-1}(p) & \\ & \mu_2^{-1}(p) \end{pmatrix}$ is the conjugacy class in $GL_2(\mathbb{C})$ associated to the contragredient representation $\widetilde{\pi} = ind\mu_1^{-1}\otimes\mu_2^{-1}$. Choose W_p so that $\widetilde{Z}(W_p^w, 1-s)$ is nonvanishing for $p\in S$, so

$$L_S(s,\pi,r) = (\prod_{p\in S}\gamma(\pi_p,\psi_p,s))L_S(1-s,\pi,\widetilde{r})$$

Next we want to explain that $L(s,\pi_p)$ can be defined as the greatest common divisor of the family of Zeta integrals $Z(W_p,s)$.

We know from earlier that there exists a polynomial P such that $P(p^{-s})Z(W,s)\in \mathbb{C}[p^{-s},p^s]$ for all $W\in W(\pi_p)$. Let A_p denote the $\mathbb{C}[p^s,p^{-s}]$ module generated by all the Zeta integrals $Z(W_p,s)$ for $W_p\in W(\pi_p,\psi_p)$. Since A_p is a subspace of $\mathbb{C}(p^{-s})$ and there is a common denominator P such that $PA_p\subset \mathbb{C}[p^s,p^{-s}]$, so A_p is a fractional ideal of $\mathbb{C}[p^s,p^{-s}]$. Denote $L(s,\pi_p)$ as any of these generators, so it can be written as $\frac{Q}{R}$, with $Q,R\in\mathbb{C}[p^{-s}]$ relatively prime, and Q has constant term 1. We know that $Z(W,s)=1$ for at least one W_p, so the numerator cannot have any zeroes, hence $L(s,\pi_p)$ can be uniquely defined by $\frac{1}{Q(p^{-s})}$, with $Q(0)=1$. Then $A_p = L(s,\pi)\mathbb{C}[p^s,p^{-s}]$, so for unramified π_p,

$$L(s,\pi_p) = Z(s,W_p^0) = \det[I - r(t_p)p^{-s}]^{-1}$$

And $L(s,\pi_p)$ is indeed the greatest common divisor of the family of the Zeta integrals $Z(W_p,s)$. Simillarly, define $L(s,\widetilde{\pi}_p) = \frac{1}{\widetilde{Q}(p^{-s})}$ as the greatest common divisor of the contragredient Zeta integrals $\widetilde{Z}(W_p,s)$.

For the functional equation $L(s,\pi) = \prod_p L(s,\pi_p)$, choose W_p so that $Z(W_p, s) = L(s,\pi_p)$. We have

$$\begin{aligned}
L(s,\pi) = \prod Z(W_p, s) = Z(\varphi, s) =& \widetilde{Z}(\varphi^w, 1-s) \\
=& (\prod_{p\in S} \widetilde{Z}(W^w, 1-s)) \prod_{p\notin} L(1-s, \widetilde{\pi}_p) \\
=& (\prod_{p\in S} \frac{\widetilde{Z}(W^w, 1-s))}{L(1-s,\widetilde{\pi})_p} L(1-s,\widetilde{\pi}) \\
=& (\prod_{p\in S} \frac{\gamma(\pi_p, \psi_p, s) Z(W, s)}{L(1-s,\widetilde{\pi})_p} L(1-s,\widetilde{\pi}) \\
=& (\prod_{p\in S} \epsilon(\pi_p, \psi_p, s)) L(1-s,\widetilde{\pi})
\end{aligned}$$

where $\epsilon(\pi_p, \psi_p, s) = \dfrac{\gamma(\pi_p, \psi_p, s) Z(W, s)}{L(1-s,\widetilde{\pi})_p}$, and by checking that $\epsilon(s,\pi)$ has no zeros or poles, we have

Theorem 2.3.1. *The automorphic L-function $L(s,\pi) = \prod L(s,\pi_p)$ can be extended to an entire function of s in $\mathbb{C}$, and satisfies the functional equation*

$$L(s,\pi) = \epsilon(s,\pi) L(1-s,\widetilde{\pi})$$

where $\epsilon(s,\pi) = \prod_{p\in S} \epsilon(s,\pi_p,\psi_p)$ is a monomial function of s.

Chapter 3

Tate thesis-local zeta function (non-Archimedean case)

Let F be a non-Archimedean local field, and $|\cdot|_F$ be the normalized absolute value. Let $\mathfrak{o}_F$ be the ring of integers in F, $\mathfrak{p} \subset \mathfrak{o}_F$ be the maximal ideal, π_F be the uniformizer of $\mathfrak{o}_F$.

Definition 3.0.1. *For $f \in \mathscr{S}(F)$ and quasi-character $\chi : F^\times \to \mathbb{C}^\times$, define the associated local zeta function as*

$$Z(f, \chi) = \int_{F^\times} f(x)\chi(x)d^*x$$

Theorem 3.0.1. *For $f \in \mathscr{S}(F)$, $\chi = \tilde{\chi}|\cdot|^s$ where $\tilde{\chi}$ is the unitary part of χ. Let $\sigma = \Re(s)$, then*

(a) $Z(f, \chi)$ is holomorphic and absolutely convergent if $\sigma > 0$.

(b) If $0 < \sigma < 1$, then there is a functional equation

$$Z(\hat{f}, \hat{\chi}) = \gamma(\chi, \psi, dx)Z(f, \chi)$$

for some $\gamma(\chi, \psi, dx)$, which is independent of f and meromorphic as a function of s. So $Z(f, \chi)$ admits a meromorphic continuation to the whole s plane.

(c) There exists a $\epsilon-$factor $\epsilon(\chi, \psi, dx) \in \mathbb{C}^\times$ satisfying:

$$\gamma(\chi, \psi, dx) = \epsilon(\chi, \psi, dx)\frac{L(\hat{\chi})}{L(\chi)}$$

Therefore,

$$L(\chi)Z(\hat{f}, \hat{\chi}) = \epsilon(\chi, \psi, dx)L(\hat{\chi})Z(f, \chi)$$

Hence, $L(\chi) = Z(f_0, \chi)$ for some f_0.

Proof. (a) Since F is non-Archimedean, let q be the order of the residue field $k_F = \mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$. Since $f \in \mathscr{S}(F)$, then f factors through the finite quotient group $\mathfrak{p}^{-m}/\mathfrak{p}^n$, $m, n \in \mathbb{Z}$, $-m \leqslant n$. So it is enought to prove for $f = \chi_{\mathfrak{p}^n}$. Let π_F be a uniformizer of $\mathfrak{p}$. Since

$$\pi_F^n \mathfrak{o}_F - \{0\} = \bigcup_n^{\infty} \pi_F^k \mathfrak{o}_F^{\times}$$

let $d^*x = cdx$, we have

$$\begin{aligned}
|Z(f,\chi)| =& c\int_{F-\{0\}} |f(x)||x|_F^{\sigma-1}dx \\
=& c\int_{F-\{0\}} \chi(\pi_F^n)|x|_F^{\sigma-1}dx \\
=& c\sum_{k=n}^{\infty}\int_{\pi_F^k\mathfrak{o}_F^{\times}} |x|_F^{\sigma}d^*x \\
=& \sum_{k=n}^{\infty}\int_{\mathfrak{o}_F^{\times}} |\pi_F^k x|_F^{\sigma}d^*x \\
=& \sum_{k=n}^{\infty} q^{-k\sigma}\int_{\mathfrak{o}_F^{\times}} d^*x \\
=& \frac{q^{-n\sigma}}{1-q^{-\sigma}} Vol(\mathfrak{o}_F^{\times}, d^*x)
\end{aligned}$$

which is finite for $\sigma > 0$, therefore $Z(f,\chi) = Z(f,\tilde{\chi},x)$ is holomorphic and absolutely convergent for $\sigma > 0$.

(b)

Lemma 3.0.2. *For all $\chi \in X(F^{\times})$ with $0 < \sigma < 1$, we have*

$$Z(f,\chi)Z(\hat{g},\hat{\chi}) = Z(\hat{f},\hat{\chi})Z(g,\chi)$$

In fact,

$$\begin{aligned}
Z(f,\chi)Z(\hat{g},\hat{\chi}) &= \int_{F^{\times}}(\int_{F^{\times}} f(x)\hat{g}(xy)|x|d^{\times}x)\chi(y^{-1})|y|d^{\times}y \\
&= \int_{F^{\times}}(\int_F\int_F f(x)\hat{g}(z)\phi(xyz)dzdx)\chi(y^{-1})|y|d^{\times}y
\end{aligned}$$

The above integral is symmetric about f and g, thus we finished the proof of lemma 3.2. □

Fix a Schwartz function $f_0 \in \mathscr{S}(F)$, let

$$\gamma(\chi,\psi,dx) = \frac{Z(\hat{f}_0,\hat{\chi})}{Z(f_0,\chi)}.$$

By the lemma above, γ is independent of the choice of f_0, and hence we have

$$Z(\hat{f}, \hat{\chi}) = \gamma(\chi, \psi, dx) Z(f, \chi)$$

Since $\gamma(\chi, \psi, dx)$ is independent of d^*x, and Haar measure is unique up to a constant, so

$$\begin{aligned}
\gamma(\chi, \psi, dx) =& \frac{Z(\hat{f}, \hat{\chi})}{Z(f, \chi)} \\
=& \frac{\int_{F^\times} \hat{f}(x)\hat{\chi}(x) d^*x}{\int_{F^\times} f(x)\chi(x) d^*x} \\
=& \frac{\int_{F^\times} \hat{f}(x)\hat{\chi}(x) d^\times x}{\int_{F^\times} f(x)\chi(x) d^\times x}
\end{aligned}$$

We will show in part (c) that $\gamma(\chi, \psi, dx)$ is meromorphic as a function of s. Since $Z(f, \chi)$ is holomorphic for $\sigma > 0$ and $Z(\hat{f}, \hat{\chi})$ is holomorphic for $\sigma < 1$, so there is a memorphic continuation of $Z(f, \chi)$ to the entire complex plane.

(c) For test function $f \in \mathscr{S}(F)$, define an entire nonzero function h such that

$$Z(f, \chi) = h(f, \chi, \psi, dx) L(\chi)$$

$$Z(\hat{f}, \chi) = h(\hat{f}, \hat{\chi}, \psi, dx) L(\chi)$$

We fix again $d^*x = dx/|x|_F$, so

$$\gamma(\chi, \psi, dx) = \frac{Z(\hat{f}, \hat{\chi})}{Z(f, \chi)} = \frac{h(\hat{f}, \hat{\chi}, \psi, dx) L(\hat{\chi})}{h(f, \chi, \psi, dx) L(\chi)}$$

We are going to show that $\gamma(\chi, \psi, dx)$ is a meromorphic function of s and independent of f, and

$$\epsilon(\chi, \psi, dx) = \frac{h(\hat{f}, \hat{\chi}, \psi, dx)}{h(f, \chi, \psi, dx)}$$

Since every quasi-character χ is in the form of $\tilde{\chi}|\cdot|_F^s$, where $\tilde{\chi}$ is a unitary character, let p be the unique prime of F, U_n be the subgroup of the form $1 + p^n$, denote $\chi_{s,n}(x) = |x|_F^s \tilde{\chi}(\tilde{x})$, $\tilde{x} \in \mathfrak{o}_F^\times$ is defined as $x = \tilde{x}\pi_F^{\nu(x)}$, where p^n is the conductor of $\tilde{\chi}$. Define the standar non-trivial additive character on a local field to be $\psi = \psi_P(tr_{F/\mathbb{Q}_p}(\cdot))$, where p is the prime lying below $\mathfrak{p}$ and ψ_p is the standard non-trivial additive character on $\mathbb{Q}_p$.

Define $f_n(x) = \psi(x) char_{p^{-d-n}}(x)$.

If $n = 0$ (i.e.unramified case), since $\pi_F^{-d} o_F - \{0\} = \bigcup_{k=-d}^{\infty} \pi_F^k o_F^\times$, so

$$\begin{aligned}
Z(f_0, \chi_{s,0}) &= \int_{F^\times} f_0(x)\chi_{s,0}(x) d^*x \\
&= \int_{\pi_F^{-d} - \{0\}} |x|_F^s d^*x \\
&= \sum_{k=-d}^{\infty} \int_{\pi_F^k o_F^\times} |x|_F^s d^*x \\
&= \sum_{k=-d}^{\infty} q^{-ks} Vol(o_F^\times, d^*x) \\
&= Vol(o_F^\times, d^*x) \frac{q^{ds}}{1 - q^{-s}} \\
&= q^{ds} Vol(o_F^\times, d^*x)(1 - |\pi_F|_F^s)^{-1} \\
&= q^{ds} Vol(o_F^\times, d^*x) L(\chi_{s,0})
\end{aligned}$$

If $n > 0$ (i.e. the ramified case), since $\widetilde{\pi_F^k u} = \tilde{u}$, then

$$\begin{aligned}
Z(f_n, \chi_{s,n}) &= \int_{F^\times} f_n(x)\chi_{s,n}(x) d^*x \\
&= \int_{\pi_F^{-d-n} o_F - \{0\}} \psi(x)\tilde{\chi}\tilde{x}|x|_F^s d^*x \\
&= \sum_{k=-d-n}^{\infty} \int_{o_F^\times} \psi(\pi_F^k u)\tilde{\chi}(\widetilde{\pi_F^k u})|\pi_F^k u|_F^s d^*u \\
&= \sum_{k=-d-n} q^{-ks} \int_{o_F^\times} \int_{o_F^\times} \psi(\pi_F^k u)\tilde{\chi}(\tilde{x}) d^*u
\end{aligned}$$

For any multiplicative character $\omega : o_F^\times \to S^1$ and any additive character $\lambda : o_F \to S^1$, define the associated Gauss sum to be:

$$g(\omega, \psi) = \int_{o_F^\times} \omega(u)\lambda(u) d^*u$$

so we have

$$Z(f_n, \chi_{s,n}) = \sum_{k=-d-n} q^{-ks} g(\tilde{\chi}, \psi_{\pi_F^k})$$

where $\psi_{\pi_F^k}(x) = \psi(\pi_F^k x)$ We have the following lemma about Gauss sums:

Lemma 3.0.3. *Let ω and λ be taken as above with conductors p^n and p^r, respectively Let $c > 0$ be the number such that $d^*x = cdx$.. Then the following statements hold:*

(a) If $r < n$, then $g(\omega, \lambda) = 0$.

(b) If $r = n = 0$, then $|g(\omega, \lambda)|^2 = Vol(o_F^\times, d^*x)^2$.

(c) If $r = n$, *then* $|g(\omega,\lambda)|^2 = cVol(o_F, dx)Vol(U_r, d^*x)$.

(d) If $r > n$, *then* $|g(\omega,\lambda)|^2 = cVol(o_F, dx)(Vol(U_r, d^*x)) - q^{-1}Vol(U_{r-1}, d^*x)$.

Since the conductor of ψ is p^{-d}, then the conductor of ψ_{π_k} is (p^{-d-k}). By part (a), we have then since for $k > -d-n$, $g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) = 0$. So

$$Z(f_n, \chi_{s,n}) = q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})$$

Since $\tilde{\chi}$ and $\psi_{\pi_F^{-d-n}}$ have conductor p^n, then from part (b) and (c) in the lemma above, we have $g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) \neq 0$. As such, $X(f_n, \chi_{s,n})$ is essentially an exponential function with neither zeros nor poles. Recall that for $n > 0$, $L(\chi_{s,n}) = 1$ since $\chi_{s,n}$ is not ramified. Thus

$$Z(f_n, \chi_{s,n}) = q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) L(\chi_{s,n})$$

Now compute the Fourier transform of the test function f to determine $Z(\hat{f}, \hat{\chi_{s,n}}))$.

Lemma 3.0.4. *For* $n = 0$, *we have* $\hat{f}_0(y) = Vol(p^{-d}, dx)char_{o_F}(y)$, *where* $char_{o_F}(y)$ *is the characteristic function of* o_F. *For* $n > 0$, *we have* $\hat{f}_n(y) = Vol(p^{-d-n}, dx)1_{p^n-1}(y)$, *where* $char_{p^n-1}(y)$ *is the characteristic function of* $p^n - 1$.

We have computed $Z(f, \chi_{s,n})$ for the unramified and ramified cases, now compute $Z(\hat{f}, \hat{\chi_{s,n}})$ for both cases. If $n = 0$, using the lemma and $o_f - \{0\} = \bigcup_{k=0}^{\infty} \pi^k o_F^\times$, we have

$$\begin{aligned}
Z(\hat{f}, \hat{\chi_{s,n}}) &= \int_{F^\times} f_0(y)\hat{\chi_{s,0}}(y)d^*y \\
&= Vol(p^{-d}, dx)\int_{o_F - \{0\}} \hat{\chi_{s,0}}(y)d^*y \\
&= Vol(p^{-d}, dx)\sum_{k=0}^{\infty}\int_{\pi^k o_F^\times} |y|_F^{1-s} d^*y \\
&= Vol(p^{-d}, dx)Vol(o_F^\times, d^*x)\sum_{k=0}^{\infty} q^{-k(1-s)} \\
&= Vol(p^{-d}, dx)Vol(o_F^\times, d^*x)\frac{1}{1-q^{1-s}} \\
&= Vol(p^{-d}, dx)Vol(o_F^\times, d^*x)\frac{1}{1-\hat{\chi_{s,0}}(\pi_F)} \\
&= Vol(p^{-d}, dx)Vol(o_F^\times, d^*x)L(\hat{\chi_{s,0}}) \\
&= q^d Vol(o_F, dx)Vol(o_F^\times, d^*x)L(\hat{\chi_{s,0}})
\end{aligned}$$

Consequently, from equation about $Z(f_0, \chi_{s,0})$, we have

$$\gamma(\chi_{s,0}, \psi, dx) = q^{-d(s-1)} Vol(o_F, dx)\frac{L(\hat{\chi_{s,0}})}{L(\chi_{s,0})}$$

and

$$\epsilon(\chi_{s,0},\psi,dx)=q^{-d(s-1)}Vol(o_F,dx)$$

Now consider $n>0$. by definition $\hat{\chi_{s,n}}=\chi_{s,n}^{-1}|\cdot|_F=\tilde{\chi}^{-1}|\cdot|_F^{1-s}$. Since $\tilde{\chi}$ is unitary, then $\tilde{\chi}^{-1}=\bar{\tilde{\chi}}$. Note that the conductor of $\hat{\chi_{s,n}}$, which is just the conductor of $\tilde{\chi}^{-1}$, is also n. Using this fact and the lemma, and that $L(\chi_{s,n})=1$, we have

$$\begin{aligned}
Z(\hat{f}_n,\hat{\chi_{s,n}}) =&Vol(p^{-d-n},dx)\int_{p^n-1}\bar{\tilde{\chi}}(\tilde{y})|y|_F^{1-s}d^*y\\
=&Vol(p^{-d-n},dx)\int_{p^n-1}\bar{\tilde{\chi}}(y)d^*y\\
=&Vol(p^{-d-n},dx)\int_{1+p^n}\bar{\tilde{\chi}}(-y)d^*y\\
=&q^{d+n}Vol(o_F,dx)\bar{\tilde{\chi}}(-1)\int_{1+p^n}\bar{\tilde{\chi}}d^*y\\
=&q^{d+n}Vol(o_F,dx)Vol(1+p^n,d^*x)\tilde{\chi}(-1)L(\chi_{s,n})
\end{aligned}$$

If $n>0$, then by applying the translation invariance of the Haar measure, we have

$$Vol(U_n,d^*x)=\int_{U_n}d^*x=c\int_{(1+p^n)-\{0\}}|x|_F^{-1}dx=c\int_{p^n}dx=cVol(p^n,dx)=cq^{-n}Vol(o_F,dx).$$

As such,

$$Z(\hat{f}_n,\hat{\chi_{s,n}})=cq^dVol^2(o_F,dx)\tilde{\chi}(-1)L(\chi_{s,n})$$

Therefore, we have

$$\epsilon(\psi_{s,n},\psi,dx)=\gamma(\chi_{s,n},\psi,dx)=\frac{cq^dq^{-(d+n)s}Vol^2(o_F,dx)\tilde{\chi}(-1)}{g(\tilde{\chi},\psi_{\pi_F^{-d-n}})}$$

Applying the translation invariance of the Haar measure d^*u, we obtain

$$\begin{aligned}
\overline{g(\tilde{\chi},\psi_{\pi_F^{-d-n}})}=&\overline{\int_{o_F^\times}\tilde{\chi}(u)\psi(\pi_F^{-d-n}u)d^*u}\\
=&\int_{o_F^\times}\bar{\tilde{\chi}}(u)\psi(-\pi_F^{-d-n}u)d^*u\\
=&\tilde{\chi}(-1)\int_{o_F^\times}\bar{\tilde{\chi}}(u)\psi(\pi_F^{-d-n}u)d^*u\\
=&\tilde{\chi}(-1)g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}})
\end{aligned}$$

Since the conductor of $\tilde{\chi}$ and ψ_{-d-n} is p^n, then

$$g(\tilde{\chi},\psi_{\pi_F^{-d-n}})\overline{g(\tilde{\chi},\psi_{\pi_F^{-d-n}})}=cVol(o_F,dx)Vol(1+p^n,d^*x)=c^2q^{-n}Vol^2(o_F,dx)$$

Consequently, for $n > 0$ we have

$$\begin{aligned}\epsilon(\psi_{s,n},\psi,dx) =& \frac{cq^d q^{(-d-n)s} Vol^2(o_F,dx)\tilde{\chi}(-1)}{g(\tilde{\chi},\psi_{\pi_F^{-d-n}})} \frac{\tilde{\chi}(-1)g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}})}{\overline{g(\tilde{\chi},\psi_{\pi_F^{-d-n}})}} \\ =& \frac{cq^d q^{(-d-n)s} Vol^2(o_F,dx) g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}})}{c^2 q^{-n} Vol^2(o_F,dx)} \\ =& \frac{1}{c} q^{(-d-n)(s-1)} g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}})\end{aligned}$$

Consider the Gauss sum of $\tilde{\chi}$ and $\psi_{\pi_F^{-d-n}}$ defined by

$$g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}}) = \int_{o_F^\times} \bar{\tilde{\chi}}(u)\psi_{\pi_F^{-d-n}}(u)d^*u$$

Since ψ has conductor p^{-d}, then $\psi_{\pi_F^{-d-n}}$ defined by $\psi_{\pi_F^{-d-n}}(x) = \psi(\pi_F^{-d-n}x)$, has conductor p^n. For $a \in U/U_n$ and $1 + n\pi_F^n \in U_n$, we have

$$\psi_{\pi_F^{-d-n}}(a(1+\pi_F^n b)) = \psi_{\pi_F^{-d-n}}(a)\psi_{\pi_F^{-d-n}}(a\pi_F^n b) = \psi_{\pi_F^{-d-n}}(a)$$

becasue $p^n = \pi_F^n o_F$ is the conductor of $\psi_{\pi_F^{-d-n}}$. As such,

$$\begin{aligned}g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}}) =& \sum_{x\in U/U_n} \psi_{\pi_F^{-d-n}}(x)\bar{\tilde{\chi}} d^*u \\ =& Vol(U_n,d^*x) \sum_{x\in U/U_n} \bar{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x) \\ =& cq^{-n} Vol(o_F,dx) \sum_{x\in U/U_n} \bar{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x)\end{aligned}$$

Therefore,

$$\epsilon(\psi_{s,n},\psi,dx) = \frac{1}{c} q^{(-d-n)(s-1)} g(\bar{\tilde{\chi}},\psi_{\pi_F^{-d-n}}) = q^{(-d)(s-1)} q^{-ns} Vol(o_F,dx) \sum_{x\in U/U_n} \bar{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x)$$

For F non-Archimedean case, we have

$$h(f_n,\chi_{s,n},\psi,dx) = \begin{cases} q^{ds} Vol(o_F^\times,d^*x) & n=0 \\ q^{(d+n)s} g(\tilde{\chi},\psi_{\pi_F^{-d-n}}) & n>0 \end{cases}$$

and

$$h(\hat{f}_n,\hat{\chi_{s,n}},\psi,dx) = \begin{cases} q^d Vol(o_F,dx) Vol(o_F^\times,d^*x) & n=0 \\ cq^d Vol^2(o_F,dx)\tilde{\chi}(-1) & n>0 \end{cases}$$

where p is the unique prime ideal of F, q is the order of the residue field o_F/po_F, p^{-d} is the conductor of the additive character ψ, p^n is the conductor of $\chi_{s,n}$, and $\tilde{\chi}$ is the restriction of $\chi_{s,n}$ to o_F. Furthermore, note the dependence of h on dx and on d^*x.

Therefore, for p-adic characteristic zero local fields, we have

$$\gamma(\chi, \psi, dx) = \epsilon(\chi, \psi, dx)\frac{L(\hat{\chi})}{L(\chi)}$$

where $\epsilon(\chi, \psi, dx)$ is an entire function of s whose image lies in $\mathbb{C}^{\times}$. Applying (b), we have

$$L(\chi)Z(\hat{f}, \hat{\chi}) = \epsilon(\chi, \psi, dx)Z(f, \chi)$$

Since $L(\chi)$, $L(\hat{\chi})$ and $\epsilon(\chi, \psi, dx)$ do not have zeros, then the poles of $Z(f, \chi)$ are no worse than those of $L(\chi)$, which is independent of f. $\square$

Chapter 4

Fourier-Whittaker Expansions of Cusp Forms for GL_n

Let k denotes a global field, $\mathbb{A}$ its ring of adeles and ψ denotes a continuous additive character of $\mathbb{A}$ which is trivial on k. (π, V_π) is a cuspidal automorphic representation of $GL_n(\mathbb{A})$.

Let $\varphi(g) \in V_\pi$ be a cusp form in the space of π. If $f(\tau)$ is a holomorphic cusp form on the upper half plane $\mathfrak{h}$ with respect to $SL_2(\mathbb{Z})$, then f is invariant under integral translation, $f(\tau+1) = f(\tau)$ and thus has a Fourier expansion of the form

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$$

If $\varphi(g)$ is a smooth cusp form on $GL_2(\mathbb{A})$, then the translations correspond to the maximal unipotent subgroup $N_2 = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ and $\varphi(ng) = \varphi(g)$ for $n \in N_2(k)$. So, if ψ is any continuous character of $k\backslash\mathbb{A}$ we can define ψ-Fourier coefficient or ψ-Whittaker function by

$$W_{\varphi,\psi}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi^{-1}(x)dx$$

We have the corresponding Fourier expansion

$$\varphi(g) = \sum_{\psi} W_{\varphi,\psi}(g)$$

If we fix a single non-trivial character ψ of $k\backslash\mathbb{A}$, then by standard duality theory , the additive characters of the compact group $k\backslash\mathbb{A}$ are isomorphic to k via the map $\gamma \in k \mapsto \psi_\gamma$ where ψ_γ is the character $\psi_\gamma(x) = \psi(\gamma x)$. By changing variables, we have $W_{\varphi,\psi_\gamma}(g) = W_{\varphi,\psi}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$ if $\gamma \neq 0$. If we set $W_\varphi = W_{\varphi,\psi}$ for our fixed ψ, then the Fourier expansion of φ becomes

$$\varphi(g) = W_{\varphi,\psi_0}(g) + \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

Since φ is cuspidal

$$W_{\varphi,\psi_0}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx \equiv 0$$

so the Fourier expansion for φ is

$$\varphi(g) = \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

As for the Fourier expansion for a cusp form φ on $GL_n(\mathbb{A})$, the translation correspond to the maximal unipotent subgroup N_n. Fix the non-trivial continuous character ψ of $k\backslash\mathbb{A}$ as above. Extend it to a character of N_n by setting $\psi(n) = \psi(x_{1,2} + ... + x_{n-1,n})$ and define the associated Fourier coefficient or Whittaker function by

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{N_n(k)\backslash N_n(\mathbb{A})} \varphi(ng)\psi^{-1}(n)dn$$

Since φ is continuous and the integration is over a compact set, this integral is absolutely convergent, uniformly on compact sets.

Theorem 4.0.1. *Let $\varphi \in V_\pi$ be a cups form on $GL_n(\mathbb{A})$ and W_φ its associated ψ-Whittaker function. Then*

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g)$$

converges absolutely and uniformly on compact sets.

Proof. Denote P_n the mirabolic subgroup stabilizing $e_n = (0, ..., 0, 1) \in k^n$ as

$$P_n = \left\{ p = \begin{pmatrix} h & y \\ & 1 \end{pmatrix} | h \in GL_{n-1}, y \in k^{n-1} \right\} \simeq GL_{n-1} \ltimes Y_n$$

where

$$Y_n = \left\{ y = \begin{pmatrix} I_{n-1} & y \\ & 1 \end{pmatrix} | y \in k^{n-1} \right\} \simeq k^{n-1}$$

$\square$

Since $P_n \supset N_n$, we may define a Whittaker function attached to a cuspidal function φ on $P_n(\mathbb{A})$ by

$$W_\varphi(p) = \int_{N_n(k)\backslash N_n(\mathbb{A})} \varphi(np)\psi^{-1}(n)dn.$$

We will prove by induction that for a cuspidal function φ on $P_n(\mathbb{A})$, we have

$$\varphi(p) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p \right)$$

converges absolutely and uniformly on compact sets.

The function on $Y_n(\mathbb{A})$ given by $y \mapsto \varphi(yp)$ is invariant under $Y_n(k)$ since $Y_n(k) \subset P_n(k)$ and φ is automorphic on $P_n(\mathbb{A})$. Hence by abelian Fourier theory for $Y_n \simeq k^{n-1}$ we have

$$\varphi(p) = \sum_{\gamma \in \widehat{(k^{n-1}\backslash \mathbb{A}^{n-1})}} \varphi_\lambda(p)$$

where

$$\varphi_\lambda(p) = \int_{Y_n(k)\backslash Y_n(\mathbb{A})} \varphi(yp)\lambda^{-1}(y)dy$$

By duality theory, $\widehat{(k^{n-1}\backslash \mathbb{A}^{n-1})} \simeq k^{n-1}$. In fact, if we let $\langle , \rangle$ denote the pairing $k^{n-1} \times k^{n-1} \to k$ by $\langle x, y \rangle = \sum x_i y_i$ we have

$$\varphi(p) = \sum_{x \in k^{n-1}} \varphi_x(p)$$

so,

$$\varphi_x(p) = \int_{k^{n-1}\backslash \mathbb{A}^{n-1}} \varphi(yp)\psi^{-1}(\langle x, y \rangle)dy$$

$GL_{n-1}(k)$ acts on k^{n-1} with two orbits: $\{0\}$ and $k^{n-1} - \{0\} = SL_{n-1}(k) \cdot^t e_{n-1}$ where $e_{n-1} = (0, ..., 0, 1)$. The stabilizer of ${}^t e_{n-1}$ in $GL_{n-1}(k)$ is ${}^t P_{n-1}$, therefore $\varphi(p)$ can be written as

$$\varphi(p) = \varphi_0(p) + \sum_{\gamma \in GL_{n-1}(k)/{}^t P_{n-1}(k)} \varphi_{\gamma \cdot {}^t e_{n-1}}(p)$$

Since $\varphi(p)$ is cuspidal and Y_n is a standard unipotent subgroup of GL_n,

$$\varphi_0(p) = \int_{Y_n(k)\backslash Y_n(\mathbb{A})} \varphi(yp)dy \equiv 0$$

On the other hand,

$$\varphi_{\gamma \cdot {}^t e_{n-1}}(p) = \varphi_{{}^t e_{n-1}} \left(\begin{pmatrix} {}^t\gamma & 0 \\ 0 & 1 \end{pmatrix} p \right)$$

so

$$\varphi(p) = \sum_{\gamma \in P_{n-1}(k)\backslash GL_{n-1}(k)} \varphi_{{}^t e_{n-1}} \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p \right)$$

For $n = 2$, it is true for GL_2 as we showed ealier.

We have $P_n = GL_{n-1} \ltimes Y_n$ and $N_n = N_{n-1} \ltimes Y_n$, so $N_{n-1}\backslash GL_{n-1} \simeq N_n \backslash P_n$. Let $\tilde{P}_{n-1} = P_{n-1} \ltimes Y_n \subset P_n$, then $P_{n-1}\backslash GL_{n-1} \simeq \tilde{P}_{n-1}\backslash P_n$, so

$$\varphi_{{}^t e_{n-1}}(yp) = \psi(y_{n-1})\varphi_{{}^t e_{n-1}}(p), \quad y \in Y_n(\mathbb{A}) \simeq \mathbb{A}^{n-1}$$

so

$$\varphi(p) = \sum_{\gamma \in P_{n-1}(k)\backslash GL_{n-1}(k)} \varphi_{t_{e_{n-1}}}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) = \sum_{\delta \in \tilde{P}_{n-1}(k)\backslash P_n(k)} \varphi_{t_{e_{n-1}}}(\delta p)$$

Fix $p \in P_n(\mathbb{A})$, consider $\varphi'(p') = \varphi'_p(p')$ on $P_{n-1}(\mathbb{A})$ given by

$$\varphi'(p') = \varphi_{t_{e_{n-1}}}\left(\begin{pmatrix} p' & 0 \\ 0 & 1 \end{pmatrix} p\right)$$

φ' is left invariant by $P_{n-1}(k)$ and cuspidal on $P_{n-1}(\mathbb{A})$, then apply the inductive assumption:

$$\varphi'(p') = \sum_{\gamma' \in N_{n-2}(k)\backslash GL_{n-2}(k)} W_{\varphi'}\left(\begin{pmatrix} \gamma' & 0 \\ 0 & 1 \end{pmatrix} p'\right) = \sum_{\delta \in N_{n-1}(k)\backslash P_{n-1}(k)} W_{\varphi'}(\delta' p')$$

Therefore

$$\begin{aligned} \varphi(p) &= \sum_{\delta \in \tilde{P}_{n-1}(k)\backslash P_n(k)} \varphi_{t_{e_{n-1}}}(\delta p) \\ &= \sum_{\delta \in \tilde{P}_{n-1}(k)\backslash P_n(k)} \varphi'_{\delta_P}(1) \\ &= \sum_{\delta \in \tilde{P}_{n-1}(k)\backslash P_n(k)} \sum_{\delta' \in N_{n-1}\backslash P_{n-1}} W_{\varphi'_{\delta_P}}(\delta') \end{aligned}$$

Then

$$\begin{aligned} W_{\varphi'_{\delta_P}}(\delta') &= \int_{N_{n-1}(k)\backslash N_{n-1}(\mathbb{A})} \varphi'_{\delta_P}(n'\delta')\psi^{-1}(n')dn' \\ &= \int_{N_{n-1}(k)\backslash N_{n-1}(\mathbb{A})} \int_{Y_n(k)\backslash Y_n(\mathbb{A})} \varphi'(yn'\delta'\delta p)\psi^{-1}(y_{n-1})\psi^{-1}(n')dydn' \\ &= \int_{N_n(k)\backslash N_n(\mathbb{A})} \varphi(n\delta'\delta p)\psi^{-1}(n)dn \\ &= W_\varphi(\delta'\delta p) \end{aligned}$$

so

$$\begin{aligned} \varphi(p) &= \sum_{\delta \in \tilde{P}_{n-1}(k)\backslash P_n(k)} \sum_{\delta' \in N_{n-1}\backslash \tilde{P}_{n-1}} W_\varphi(\delta'\delta p) \\ &= \sum_{\delta \in N_n(k)\backslash P_n(k)} W_\varphi(\delta p) \\ &= \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) \end{aligned}$$

If φ is a cusp form on $GL_n(\mathbb{A})$, then for g in a compact subset Ω, $\varphi_g(p) = \varphi(pg)$ form a compact family of cuspidal functions on $P_n(\mathbb{A})$ under the weak operator topology, so

$$\varphi_g(1) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_{\varphi_g}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\right)$$

converges absolutely and uniformly, hence $\varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_{\varphi}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$ converges absolutely and uniformly for $g \in \Omega$.

Chapter 5

The Global Construction for GL_6

Let F be a global field, and $\mathbb{A}$ its ring of adeles. Let

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ -1 & & & & & \end{pmatrix}.$$

Define

$$GSp_6 = \{g \in Gl_6 :^t gJg = \mu(g)J, \mu(g) \text{ a scalar}\}$$

where ${}^tg$ denotes the transpose of g. Let Q denote the Siegel parabolic of GSp_6, i.e.,

$$Q = (GL_1 \times GL_3)R$$

We can identify $GL_1 \times GL_3$ with

$$m : (\alpha, g) \mapsto \begin{pmatrix} \alpha g & \\ & g^* \end{pmatrix}, \quad \alpha \in GL_1, g \in GL_3$$

where $g^* = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} {}^tg^{-1} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$.

R can be identified with

$$S = \{Y \in M_{3,3} :^t Y \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} Y\}$$

by the inverse of

$$Y \mapsto u(Y) = \begin{pmatrix} I & Y \\ & I \end{pmatrix}, \; Y \in S$$

Let φ be a nontrivial additive character of $F\backslash\mathbb{A}$, let N be the maximal standard unipotent subgroup of GL_6, thus N consists of all upper unipotent matrices. Given $n = (n_{ij}) \in N$, we define

$$\psi_N(n) = \psi(n_{12} + n_{23} + n_{34} - n_{45} + n_{56})$$

Let π be an irreducible cuspidal representation on $GL_6(\mathbb{A})$, with a central character ω_π, we shall denote its space as V_π in $L_0^2(Z(\mathbb{A})GL_6(F)\backslash GL_6(\mathbb{A}), \omega_\pi)$, where Z is the center of GL_6. We assume that π is generic, which means that the space of the functions generated by

$$W_\varphi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\psi_N(n)dn, \quad \varphi \in V_\pi, g \in GL_6(\mathbb{A})$$

is not identically zero. We call the space of functions above form the Whittaker model of π and denote it by $\mathcal{W}(\pi, \psi)$.

To construct the Eisenstein seires, we define a character χ_π of $Q(\mathbb{A})$ as follows. Let χ be a unitary character of $F^*\backslash\mathbb{A}^*$. Define χ_π on the embedding of $GL_1(\mathbb{A}) \times GL_3(\mathbb{A})$ in $GSp_6(\mathbb{A})$ as

$$\chi_\pi(m(\alpha, g)) = (\omega_\pi\chi^3)(\alpha)(\omega_\pi\chi^2)(\det g)$$

where $\alpha \in GL_1(\mathbb{A})$ and $g \in GL_3(\mathbb{A})$. We can extend χ_π to $Q(\mathbb{A})$ by letting it act trivially on $R(\mathbb{A})$.

Given $s \in \mathbb{C}$, set

$$I(s, \chi) = \mathrm{Ind}_{Q(\mathbb{A})}^{GSp_6(\mathbb{A})} \delta_Q^s \chi_\pi$$

where δ_Q is the modular function on Q. Given $f_s \in I(s, \chi_\pi)$, then it satisfies

$$f_s(m(\alpha, g)rh) = (\omega_\pi\chi^3)(\alpha)(\omega_\pi\chi^2)(\det g)$$

Define the Siegel Eisenstein series as (for $\Re(s)$ large)

$$E(g, f_s, \chi, s) = \sum_{\gamma \in Q(F)\backslash GSp_6(F)} f_s(\gamma g) \qquad g \in GSp_6(\mathbb{A})$$

The theory of Langlands on Eisenstein series shows that the above series converges absolutely for $\Re s$ large enough, has a meromorphic continuation to the whole complex plane, and satisfies a certain functional equation.

Let

$$w = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \in \mathrm{GL}_6$$

and set $j(g) = wgw^{-1}$ for $g \in GSp_6$. Our global integral is

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})GSp_6(F)\backslash GSp(\mathbb{A})} \varphi(j(g))E(g, f_s, \chi, s)dg$$

Here Z is the center of GSp_6, which is also the center of GL_6. We have

Theorem 5.0.1. *The integral $I(\varphi, f_s, \chi, s)$ converges absolutely for all $s \in \mathbb{C}$ except where the Eisenstsin series has a pole.*

Proof. Given an integer N, we have

$$|\varphi(j(g)\omega)| \leqslant C_{N,\varphi}|g|^N$$

where $C_{N,\varphi}$ is a constant depending on N and φ, and ω is in any given compact set. From reduction theory, we can write $GSp_6(\mathbb{A}) = Z(\mathbb{A})GSp_6(F)T_\epsilon\Omega$, where Ω is a compact set and

$$T_\epsilon = \left\{ t = \begin{pmatrix} abc & & & & & \\ & ab & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & b^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix} : |a|, |b|, |c| > \epsilon \right\}$$

We may assume that every $t \in T_\epsilon$ has entry 1 at all the finite places, and some fixed positive real number at the infinite places. Then we can write our integral as

$$I = \int_\Omega \int_{T_\epsilon} \varphi(j(t))E(t\omega, f_s, \chi, s)dtd\omega$$

Since $|\omega_\pi| = 1$, we may pull out the center, and using the fact that $E(tw, f_s, \chi, s)$ is a slowly increasing function, we finished the proof of the absolute convergence of I for $s \in \mathbb{C}$ which is not a pole of the Eisenstein series. $\square$

The next step is to prove I is Eulerian:

Theorem 5.0.2. *For $\Re(s)$ large we have*

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})V(\mathbb{A})\backslash GSp_6(\mathbb{A})} \int_{\mathbb{A}} W_\varphi(X(r)j(g)) f_s(g) dr dg$$

where V is the maximal unipotent of GSp_6 such that $V \subset N$.

Proof. We will use these notations for the rest of this proof:

$$\begin{aligned}
R^3 =& R \\
R^2 =& \{r_2(y_1, y_2) \in V : r_2(y_1, y_2) = I_6 + y_1(e_{12} - e_{56}) + y_2(e_{13} - e_{46})\} \\
R^1 =& \{r_1(y) \in V : r_1(y) = I_6 + y(e_{23} - e_{45})\} \\
R^3_c =& \{r^c_3(y_1, y_2, y_3) \in V : r^c_e(y_1, y_2, y_3) = I_6 + y_1 e_{26} + y_2 e_{35} + y_3 e_{46}\} \\
R^2_c =& \{r^c_2(y_1, y_2) \in V : r^c_e(y_1, y_2) = I_6 + y_1 e_{12} + y_2 e_{13}\} \\
R^1_c =& \{r^c_1(y_1) \in V : r^c_e(y_1) = I_6 + y_1 e_{23}\} \\
V^3 =& R^3 R^3_c \\
V^2 =& R^2 R^2_c V^3 \\
V^1 =& R^1 R^1_c V^2 = V
\end{aligned}$$

We first carry out the standard unfolding process of the Eisenstein series.

$$\begin{aligned}
I(\varphi, f_s, \chi, s) &= \int_{ZGSp_6(F)\backslash GSp_6(\mathbb{A})} \varphi(j(g)) E(g, f_s, \chi, s) dg \\
&= \int_{ZGSp_6(F)\backslash GSp_6(\mathbb{A})} \varphi(j(g)) \left(\sum_{\gamma \in Q(F)\backslash GSp_6(F)} f_s(\gamma g) \right) dg \\
&= \int_{Z(\mathbb{A})Q(F)\backslash GSp(\mathbb{A})} \varphi(j(g)) f_s(g) dg \\
&= \int_{Z(\mathbb{A})GL_3(F)R^3(\mathbb{A})\backslash GSp(\mathbb{A})} \int_{R^3(F)\backslash R^3(\mathbb{A})} \varphi(j(rg)) f_s(g) dr_3 dg
\end{aligned}$$

Consider the Fourier expansion with respect to $j(R^3_c)(F\backslash\mathbb{A})$, then

$$I(\varphi, f_s, \chi, s) = \int \sum_{\alpha,\beta,\gamma} \int \varphi\left[j\left(\left(r^c_3(y_1, y_2, y_3) \right) r_3 g \right) \right] \psi(\alpha y_1 + \beta y_2 + \gamma y_3) f_s(g) dy_i dr_3 dg$$

$$= \int \sum_{\alpha,\beta,\gamma} \int \varphi \left[j \left(\begin{pmatrix} 1 & & & 0 & 0 & 0 \\ & 1 & & 0 & 0 & y_1 \\ & & 1 & 0 & y_2 & y_3 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} r_3 g \right) \right] \psi(\alpha y_1 + \beta y_2 + \gamma y_3) f_s(g)$$

$$dy_i dr_3 dg$$

where y_i are integrated on $F\backslash\mathbb{A}$, r_3 is integrated on $R^3(F)\backslash R^3(\mathbb{A})$, and g is integrated on $Z(\mathbb{A})GSp_6(F)\backslash GSp_6(\mathbb{A})$.

Since φ is invariant under $GL_6(F)$, we have

$$\varphi(x_3(\alpha,\beta,\gamma)h) = \varphi(h), \quad h \in GL_6(\mathbb{A}), \alpha,\beta,\gamma \in F$$

Denote $v_3 = r_3^c r_3$, then

$$\begin{aligned} I(\varphi, f_s, \chi, s) &= \int \sum_{\alpha,\beta,\gamma} \int \varphi\left[j\left(\left(r_3^c(y_1,y_2,y_3)\right) r_3 g\right)\right] \psi(\alpha y_1 + \beta y_2 + \gamma y_3) f_s(g) dy_i dr_3 dg \\ &= \int \sum_\theta \int \varphi[j(v)rg]\theta(v) f_s(g) dv dg \\ &= \int \sum_\theta \int_{M_{3,3}(F)\backslash M_{3,3}(\mathbb{A})} \varphi\left(j\left(\left(\begin{matrix} I & X \\ & I \end{matrix}\right)\right) g\right) f_s(g) dX \end{aligned}$$

here θ is summed up for all characters on R_c^3, hence can be considered on all $X \in M_{3,3}$, g is integrated on $Z(\mathbb{A})GSp_6(F)\backslash GSp_6(\mathbb{A})$.

The group $GL_1 \times GL_3$, the Levi part of $Q(F)$, acts on the group of characters of R_c^3 modulo R^3 with two orbits, one is the trivial orbit, and we can choose a representative of the open orbit ψ_1 as $\psi_1(X) = \psi(x_4 - x_8)$.

For the orbit with respect to the trivial character, we get

$$\int \int_{M_{3,3}(F)\backslash M_{3,3}(\mathbb{A})} \varphi\left(j\left(\left(\begin{matrix} I & X \\ & I \end{matrix}\right)\right) g\right) f_s(g) dX dg$$

where the inner integral $\int_{M_{3,3}(F)\backslash M_{3,3}(\mathbb{A})} \varphi\left(j\left(\left(\begin{matrix} I & X \\ & I \end{matrix}\right)\right) g\right) dX = 0$ since φ is cuspidal.

As for the other orbit, the representative ψ_1 as $\psi_1(X) = \psi(x_4 - x_8)$ has stabilizer in $GL_1 \times GL_3$ as $GL_2(F)R^2(F)$ $GL_2(F)R^2(F)$, embedded in GSp_6 as

$$\begin{pmatrix} |g| & & & \\ & g & & \\ & & g^* & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & 0 & & & \\ & & 1 & & & \\ & & & 1 & 0 & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}$$

where $g \in GL_2$ and $y_1, y_2 \in F$.

$$\begin{aligned} I(\varphi, f_s, \chi, s) = &\int_{Z(\mathbb{A})GL_2(F)R^2(F)R^3(\mathbb{A})\backslash GSp_6(\mathbb{A})} \int_{M_{3,3}(F)\backslash M_{3,3}(\mathbb{A})} \varphi\left(j\left(\left(\begin{matrix} I & X \\ & I \end{matrix}\right) g\right)\right) \psi_1(X) \\ &f_s(g) dX dg \end{aligned}$$

Now consider the Fourier expansion with respect to $j(R_c^2)(F\backslash\mathbb{A})$. $GL_2(F)$ act on the character group of $j(R_c^2)(F)\backslash j(R_c^2)(\mathbb{A})$ with two orbits: one is the trivial orbit, and we can choose a representative of the open orbit as $\psi_2(n) = \psi(n_{12}+n_{24}-n_{35}+n_{56})$, since $GL_1 \cdot R_1$ is the stabilizer of ψ_2 in GL_2R^2, where $GL_1 \cdot R_1$ is embedded in GSp_6 as

$$\begin{pmatrix} \alpha &&&&& \\ & \alpha &&&& \\ && 1 &&& \\ &&& \alpha && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ & 1 & \beta &&& \\ && 1 &&& \\ &&& 1 & -\beta & \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \qquad \alpha \in GL_1,\ \beta \in F.$$

The trivial orbit contributes

$$\int\int \varphi(j(ng))f_s(g)dndg$$

which will be zero since φ is cuspidal.

So we have

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})GL_1(F)R^1(F)V^3(\mathbb{A})\backslash GSp_6(\mathbb{A})} \int_{V^2(F)\backslash V^2(\mathbb{A})} \varphi(j(ng))\psi_2(n)f_s(g)dndg.$$

where $V^2 = R^2R_c^2V^3$. Consider the Fourier expansion with respect to the unipotent group $R_c^1 = \{j(I + ye_{23})\}$, then

$$\int_{V^2(F)\backslash V^2(\mathbb{A})} \varphi(j(ng))\psi_{V^2}(n)dn = \int_{V^2(F)\backslash V^2(\mathbb{A})} \sum_{\alpha\in F} \int_{F\backslash\mathbb{A}} \varphi(j(X_1(m)ng))\psi_2(n)\psi(\alpha m)dmdn$$

where $X_1(m) = I + me_{23}$. Let N_3 be the unipotent subgroup of N_2 for which $n_{34} = 0$. Thus

$$N_2 = N_3 \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & r && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$$

Notice that

$$X(r) = j \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & r && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$$

Since φ is invariant under rational points, it equals to

$$\int_{N_3(F)\backslash N_3(\mathbb{A})}\sum_{\alpha\in F}\int_{(F\backslash\mathbb{A})^2}\varphi[X(\alpha)j(X_1(m)n)X(r)j(g)]\psi_{N_3}(n)\psi(\alpha m)drdmdn$$

then by conjugating $X(\alpha)j(X_1(m)n)X(r)$, it equals to

$$\int_{N_3(F)\backslash N_3(\mathbb{A})}\sum_{\alpha\in F}\int_{(F\backslash\mathbb{A})^2}\varphi[j(X_1(m)n)X(\alpha+r)X(\alpha+r)^{-1}j(X_1(m)n)^{-1}X(\alpha)j(X_1(m)n)X(r)j(g)]$$
$$\psi_{N_3}(n)drdmdn$$

and by changing the variavle for g, it is

$$\int_{N_3(F)\backslash N_3(\mathbb{A})}\sum_{\alpha\in F}\int_{(F\backslash\mathbb{A})^2}\varphi[j(X_1(m)n)X(\alpha+r)j(g)]\psi_{N_3}(n)drdmdn$$

Collapsing the summation and integration over α and r, we have

$$\int_{\mathbb{A}}\int_{N_4(F)\backslash N_4(\mathbb{A})}\varphi(j(n)X(r)j(g))\psi_{N_4}(n)dndr$$

where N_4 is the unipotent subgroup of N given by all matrices of the form

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & * & \\ & & 1 & 0 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

Also ψ_{N_4} is extended trivially from ψ_{N_3}. Consider the Fourier expansion with respect to $j(I+me_{43})$ with $m\in F\backslash\mathbb{A}$, the trivial orbit is 0 by cuspidality as before, and by taking the representative $\psi_4(I+ym_{34})=\psi(y)$, then

$$\begin{aligned}I(\varphi,f_s,\chi,s) &= \int\int_{\mathbb{A}}\sum_{\alpha\in F}\int_{F\backslash\mathbb{A}}\int_{N_4(F)\backslash N_4(\mathbb{A})}\varphi(j((I+me_{43})n)X(r)j(g))\times\psi_{N_4}(n)\\ &\quad\psi(\alpha m)f_s(g)dndmdrdg\\ &= \int_{Z(\mathbb{A})V(\mathbb{A})\backslash GSp_6(\mathbb{A})}\int_{\mathbb{A}}\int_{N(F)\backslash N(\mathbb{A})}\varphi(nX(r)j(g)\psi_N(n)f_s(g)dndg\\ &= \int_{Z(\mathbb{A})V(\mathbb{A})\backslash GSp_6(\mathbb{A})}\int_{\mathbb{A}}W_\varphi(X(r)j(g))f_s(g)drdg\end{aligned}$$

which is the identity we need to prove.

Now we need to justify the unfolding process by showing the absolute convergence of the above integrals for $\Re s$ large.

We started by showing the integral

$$\int|W_\varphi[j(X(r))g]f_s(g)|drdg$$

converges for $\Re s$ large. Here g is integrated over $Z(\mathbb{A})V(\mathbb{A})\backslash GSp_6(\mathbb{A})$, r over $\mathbb{A}$. From the Iwasawa decomposition of $GSp_6(\mathbb{A})$, it is enough to show the convergence of

$$\int |W_\varphi[j(X(r))t'k]||a^2b^4c^8|^s\delta_{B_0}^{-1}(t')|drdt'dk$$

Here r is integrated over $\mathbb{A}$, k over the maximal standard compact subgroup of $GSp_6(\mathbb{A})$, t' over the maximal torus of $GSp_6(\mathbb{A})$ modulo $Z(\mathbb{A})$ parametrized by

$$\begin{pmatrix} abc &&&&& \\ & ac &&&& \\ && a &&& \\ &&& 1 && \\ &&&& c^{-1} & \\ &&&&& b^{-1}c^{-1} \end{pmatrix} \in GSp_6$$

We denote by δ_{B_0} the modular function of the Borel subgroup of GSp_6 such that $Q \supset B_0$. Conjugate t' to the left, changing variables we obtain $j(\bar{X}(rt'k)) = j(t't'^{-1}\bar{X}(r)t'k) = j(t'\bar{X})(a^{-1}r)k)$, hence

$$\int |W_\varphi(tj(X(r))j(k))||t|^{\mu_s}|dtdrdk$$

where $t = j(t')$ and

$$|t|^{\mu_s} = |a|^{2s-4}|b|^{4s-6}|c|^{8s-10}$$

The integration in t is on $(\mathbb{A}^*)^3$. For this proof only, we denote $R = X(r)$. Write the Iwasawa decomposition of $r \in R$ as $r = u_Rt_Rk_R$, where u_R is the unipotent part, t_R the toral part, and k_R is in the maximal standard compact subgroup of $GSp_6(\mathbb{A})$. Thus we need to prove the convergence of

$$\int |W_\varphi(tt_Rk_Rj(k))||t|^{\mu_s}dtdrdk$$

where the integration over r is on $R(\mathbb{A})$. Let f be a smooth function with compact support on $GSp_6(\mathbb{A})$. It follows from [4] that every W_φ is a finite sum of convolutions of the type

$$\int W_0(gh)f(h)dh$$

the integration over $GSp_6(\mathbb{A})$ Thus, it is enough to prove the convergence for these functions. For such functions

$$|W_\varphi(tt_Rk_Rj(k))| = |\int W_0(tt_Rh)f(j(k)^{-1}k_R^{-1}h)dh|.$$

Write

$$\int_{GSp_6(\mathbb{A})} = \int_{V(\mathbb{A})\backslash GSp_6(\mathbb{A})}\int_{V(\mathbb{A})}$$

where V denotes the maximal unipotent of GSp_6. Thus, the right-hand side is equal to the absolute value of

$$\int W_0(tt_Rh)(\int f(j(k)^{-1}k_R^{-1}uh)\bar{\psi}(tt_Rut_R^{-1}t^{-1})du)dh$$

where u is integrated over $V(\mathbb{A})$ and h over $V(\mathbb{A})\backslash GSp_6(\mathbb{A})$. Since W_0 is a Fourier coefficient of a cusp form, it is bounded uniformly, and hence the above integral is bounded by the absolute value of

$$\int f(j(k)^{-1}k_R^{-1}uh)\bar{\psi}(tt_Rut_R^{-1}t^{-1})dudh$$

Following the same reasoning as in [8], we deduce that it is bounded by

$$\int_{R(\mathbb{A}}\int_{(\mathbb{A}*)^3}\Phi(\frac{bt_1}{t_2},\frac{ct_2}{t_3},\frac{at_3}{t_4},\frac{ct_4}{t_5},\frac{bt_5}{t_6})|t|^{\mu_s}d^*tdr \tag{5.1}$$

Here t_i are the torus elements of t_R, and Φ is a fixed positive-valued Schwartz Bruhat function. Thus we may assume that Φ is factorizable and, we are reduced to studying the local version. A change of variables in a, b, c shows that the local integrals equals

$$\int_{R(F)}\int_{(F*)^3}\varphi(\frac{t_1}{t_2},\frac{t_3}{t_4},\frac{t_5}{t_6},a,b,c)|a^2b^4c^8|^s|a^4b^6c^{10}|^{-1}\alpha_s(t_1,...,t_6)d^*ad^*bd^*cdr$$

where $\varphi \geqslant 0$ a local Schwartz-Bruhat function, and α_s depends on the absolute value of t_i of $1 \leqslant i \leqslant 6$. Here F stands for an arbitrary local field. This integral is convergent by Iwasawa-Tate thesis, and we will prove it later in details in local theory. $\square$

Chapter 6

Unramified Computation

Let F be a nonarchimedean field. We will assume all data to be unramified in this section. Let $\pi = \pi_v$ be an unramified admissible generic irreducible representation of GL_6, let $K(GL_6)$ be the standard maximal compact subgroup. By this assumption, there exists a vector $\xi \in V_\pi$, such that $\pi(k)\xi = \xi$. The vector ξ is unique up to a scalar. Since we assumed that π is generic, there exists a unique vector $W_v \in \mathcal{W}(\pi_v, \psi_v)$ such that $W_v(k) = W_v(e) = 1$ for all $k \in K(GL_6)$. Let f_s denote the unramified vector in $I(s, \chi_\pi)$ with $f_s(e) = 1$. Thus the central character ω_π and χ are unramified characters.

Since π is unramified, we may assume $\pi = Ind_B^{GL_6}\delta_B^{1/2}\mu$ where B is the standard Borel subgroup of GL_6, $\delta_B^{1/2}$ is the modular function of B and μ is an unramified character. Let T be the maximal torus of GL_6, and we may identify $t \in T$ as $t = diag(t_1, t_2, ..., t_6)$, so we can write μ as

$$\mu(diag(t_1, t_2, ..., t_6)n) = \prod_{i=1}^{6} \mu_i(t_i), \quad t_i \in F^*, \; n \in N$$

From the general theory, ${}^LGL_6 = \Lambda^3$, where Λ^3 is the exterior cube representation of $GL_6(\mathbb{C})$, and this representation has dimension 20. So to each π, we may attach a semi-simple conjugacy class t_π in $GL_6(\mathbb{C})$ whose representative is chosen to be $\mathrm{diag}(\mu_1(p), \mu_2(p), ..., \mu_6(p))$. Define the local twisted exterior cube L-function by

$$L(\pi \otimes \chi, \Lambda^3, s) = \det[I - \Lambda^3(t_\pi)\chi(p)q^{-s}]^{-1},$$

where I is the 20×20 identity matrix. In terms of coordinates, we have

$$L(\pi \otimes \chi, \Lambda^3, s) = \prod_{1 \leqslant i < j < k \leqslant 6} (1 - (\mu_i\mu_j\mu_k)(p)\chi(p)q^{-s})^{-1}$$

For a given character ω of F^*, denote

$$L(\omega, s) = (1 - \omega(p)q^{-s})^{-1}.$$

Then we have

Theorem 6.0.1. *For all unramified data and for $\Re s$ large,*

$$I(W, f_s, \chi, s) = \frac{L(\pi \otimes \chi, \Lambda^3, 2s - 1/2)}{L(\omega_\pi \chi^2, 4s) L(\omega_\pi^2 \chi^4, 8s - 2)}$$

Proof. The Iwasawa decomposition of GSp_6 is $GSp_6 = ZVT'K$, where K is the maximal compact subgroup of GSp_6, ZT' is the maximal torus of GSp_6. Let B' denote the Borel subgroup of GSp_6, so that $B' = ZT'V$. We can parametrize an element $t' \in T$ as

$$t' = diag(abc, ac, 1, a, c^{-1}, b^{-1}c^{-1}) \quad a, b, c \in F^*$$

We have

$$\delta_{B'}(t') = |a^4 b^6 c^{10}|, \quad \delta_P(t') = |a^2 b^4 c^8|$$

Thus

$$I(W, f_s, \chi, s) = \int_{(F^*)^3} \int_F W \left[\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & x & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} j(t') \right] |a^2 b^4 c^8|^s \chi(ab^2 c^4) \omega_\pi(bc^2) |a^4 b^6 c^{10}|^{-1}$$
$$dx d^*a d^*b d^*c$$

Here the measure on K is chosen so that $\int_K dk = 1$. Conjugating the torus to the left we obtain

$$I(W,f_s,\chi,s) = \int_{(F^*)^3}\int_F W\left[j(t')\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&ax&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right]|a^2b^4c^8|^s\chi(ab^2c^4)\omega_\pi(bc^2)|a^4b^6c^{10}|^{-1}$$

$$dxd^*ad^*bd^*c$$

$$= \int_{(F^*)^3}\int_F W\left[j(t')\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&y&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right]|a^2b^4c^8|^s\chi(ab^2c^4)\omega_\pi(bc^2)|a^4b^6c^{10}|^{-1}$$

$$d\frac{y}{a}d^*ad^*bd^*c$$

$$= \int_{(F^*)^3}\int_F W\left[j(t')\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&y&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right]|a^2b^4c^8|^s\chi(ab^2c^4)\omega_\pi(bc^2)|a^5b^6c^{10}|^{-1}$$

$$dyd^*ad^*bd^*c$$

We have, for $|x|>1$

$$\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&x&1&&\\&&&&1&\\&&&&&1\end{pmatrix} = \begin{pmatrix}1&&&&&\\&1&&&&\\&&x^{-1}&&&\\&&&x&&\\&&&&1&\\&&&&&1\end{pmatrix}\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&x&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}$$

$$\begin{pmatrix}1&&&&&\\&1&&&&\\&&0&-1&&\\&&1&x^{-1}&&\\&&&&1&\\&&&&&1\end{pmatrix}$$

with $\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 0 & -1 && \\ && 1 & x^{-1} && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \in K$. And since $\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \in K$ when $|x| \leqslant 1$, we have

$$I(W, f_s, \chi, s) = \int_{(F^*)^3} W(j(t'))|a^2b^4c^8|^s|a^5b^6c^{10}|^{-1}\chi(ab^2c^4)\omega_\pi(bc^2)d^*ad^*bd^*c +$$

$$+ \int_{(F^*)^3}\int_{|x|>1} W\left(j(t') \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && x^{-1} &&& \\ &&& x && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\right)$$

$$\times |a^2b^4c^8|^s|a^5b^6c^{10}|^{-1}\chi(ab^2c^4)\omega_\pi(bc^2)dxd^*ad^*bd^*c$$

Here the measure on F is chosen so that $\int_{|x|\leqslant 1} dx = 1$. Denote

$$t = j(t') = diag(abc, ac, a, 1, c^{-1}, b^{-1}c^{-1})$$

Changing variables $a \to ax^2$, $c \to cx^{-1}$ we obtain

$$\int_{(F^*)^3} W\left(\begin{pmatrix} abcx &&&&& \\ & acx &&&& \\ && ax & ax^2 && \\ &&& x && \\ &&&& c^{-1}x & \\ &&&&& b^{-1}c^{-1}x \end{pmatrix}\right)|a^2b^4c^8x^{-4}|^s|a^5b^6c^{10}|^{-1}\chi(ab^2c^4x^{-2})$$

$$\omega_\pi(bc^2x^{-4})d^*ad^*bd^*c$$

$$= \int_{(F^*)^3} W\left(\begin{pmatrix} x &&&&& \\ & x &&&& \\ && x & ax && \\ &&& x && \\ &&&& x & \\ &&&&& x \end{pmatrix} t\right)|a^2b^4c^8x^{-4}|^s|a^5b^6c^{10}|^{-1}\chi(ab^2c^4x^{-2})\omega_\pi(bc^2x^{-2})d^*ad^*bd^*c$$

$$= \int_{(F^*)^3} W(t)\,\omega_\pi(x)\psi(ax)|a^2b^4c^8|^s|x|^{-4s}|a^5b^6c^{10}|^{-1}\chi(ab^2c^4)\chi(x)^{-2}\omega_\pi(bc^2)$$

$$\omega_\pi(x)^{-2}d^*ad^*bd^*c$$

$$= |x|^{-4s}\chi(x)^{-2}\omega_\pi(x)^{-1}\int_{(F^*)^3} W(t)\,\psi(ax)|a^2b^4c^8|^s|a^5b^6c^{10}|^{-1}\chi(ab^2c^4)\chi(x)^{-2}\omega_\pi(bc^2)d^*ad^*bd^*c$$

Therefore,

$$I(W,f_s,\chi,s)=\int_{(F^*)^3} W(t)\omega_\pi(bc^2)\chi(ab^2c^4)|a^2b^4c^8|^s|a^5b^6c^{10}|^{-1}H(a)d^*ad^*bd^*c$$

where

$$H(a)=1+\int_{|x|>1}\omega_\pi^{-1}\chi^{-2}(x)|x|^{-4s}\psi(ax)dx$$

For $|\epsilon|=1$, we have $H(a\epsilon)=H(a)$, so we only need to compute $H(p^m)$, $m\geqslant 0$, where p is the generator of the maximal ideal in the ring of integer of F, $q^{-1}=|p|$, since

$$\int_{|x|=q^k}\psi(x)=\begin{cases}(1-q)q^k & k\leqslant 0\\ -1 & k=1\\ 0 & k\geqslant 2\end{cases}$$

we have

$$\begin{aligned}H(p^m)=&1+\int_{|x|>1}\omega_\pi^{-1}\chi^{-2}(x)|x|^{-4s}\psi(p^mx)dx\\ =&1+\sum_{k=1}^{\infty}q^{-4k}\int_{|x|=q^k}(\omega_\pi^{-1}\chi^{-2})(x)\psi(p^mx)dx\\ =&1+\sum_{k=1}^{\infty}q^{-4k+m}\int_{|x|=q^{k-m}}(\omega_\pi^{-1}\chi^{-2})(p^{-m}x)\psi(x)dx\\ =&\frac{1-\omega_\pi\chi^2(p)q^{-4s}}{1-\omega_\pi\chi^2(p)q^{-4s+1}}(1-\omega_\pi\chi^2(p)q^{(-4s+1)(m+1)})\end{aligned}$$

hence,

$$H(a)=\frac{1-\omega_\pi\chi^2(p)q^{-4s}}{1-\omega_\pi\chi^2(p)q^{-4s+1}}(1-\omega_\pi\chi^2(a)|a|^{4s-1}\omega_\pi\chi^2(p)q^{(-4s+1)})$$

and

$$\begin{aligned}I(W,f_s,\chi,s)=&\frac{L(\omega_\pi\chi^2,4s-1)}{L(\omega_\pi\chi^2,4s)}\int_{|a|,|b|,|c|\leqslant 1}W(t)\omega_\pi(bc^2)\chi(ab^2c^4)|a^2b^4c^8|^s|a^5b^6c^{10}|^{-1}\\ &(1-\omega_\pi\chi^2(a)|a|^{4s-1}\omega_\pi\chi^2(p)q^{(-4s+1)})d^*ad^*bd^*c\end{aligned}$$

Let $K(t)=\delta_B^{-1/2}W(t)$ where B is the standard Borel subgroup of GL_6 Thus $\delta_B(t)=|a^9b^{10}c^{16}|$. Write $a=p^{n_1}\epsilon_1$, $b=p^{n_2}\epsilon_2$, $c=p^{n_3}\epsilon_3$, where n_i are nonnegative integers and $|\epsilon_i|=1$, $1\leqslant i\leqslant 3$, write

$$d(n_1,n_2,n_3)=diag(p^{n_1+n_2+n_3},p^{n_1+n_3},p^{n_1},1,p^{-n_3},p^{-n_2-n_3}).$$

Since $W(t)=0$ if $|a|>1$ or $|b|>1$ or $|c|>1$ we obtain

$$\begin{aligned}I(W,f_s,\chi,s)=&\frac{L(\omega_\pi\chi^2,4s-1)}{L(\omega_\pi\chi^2,4s)}\sum_{n_1,n_2,n_3=0}^{\infty}K(d(n_1,n_2,n_3))\chi(p)^{n_1+2n_2+4n_3}\times\\ &\times\omega_\pi(p)^{n_2+2n_3}q^{(-2s+1/2)n_1+(-4s+1)n_2+(-8s+2)n_3}(1-(\omega_\pi\chi^2)(p)^{n_1+1}\\ &q^{(-4s+1)(n_1+1)})\end{aligned}$$

Here we choose the measure on a,b,c such that $\int_{|\epsilon|=1} d\epsilon = 1$. Let $x = \chi(p)q^{-2s+1/2})$. Thus

$$I(W,f_s,\chi,s) = \frac{L(\omega_\pi\chi^2,4s-1)}{L(\omega_\pi\chi^2,4s)} \sum_{n_1,n_2,n_3=0}^{\infty} K(d(n_1,n_2,n_3)) \times \\ \times \omega_\pi(p)^{n_2+2n_3} x^{n_1+2n_2+4n_3}(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)})$$

Now consider the right-hand-side. By the Poincare identity, we have

$$L(\pi\otimes\chi,\Lambda^3,2s-1/2) = \det(I-\Lambda^3(t_\pi)\chi(p)q^{2s-1/2} \\ = \sum_{n=0}^{\infty} trS^n(t_\pi)\chi(p)^n q^{(-2s+1/2)n}$$

where S^n denotes the symmetric nth power operation. Thus we need to prove the identity

$$(1-\omega_\pi(p)x^2)(1-\omega_\pi^2(p)x^4)\sum_{n=0}^{\infty} trS^n(t_\pi)x^n = \sum_{n_1,n_2,n_3=0}^{\infty} K(d(n_1,n_2,n_3))\omega_\pi(p)^{n_2+2n_3} \\ x^{n_1+2n_2+4n_3}(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)})$$

Let $\widetilde{\omega_i}$ $1\leqslant i\leqslant 5$ denote the ith fundamental representation of $GL_6(\mathbb{C})$. Let $(0,...,1,...,0)$, one in the ith position and zero elsewhere, denote the character of the representation $\widetilde{\omega_i}$ evaluated at t_π. By the Casselman-Shalika formula[9],

$$K(d(n_1,n_2,n_3)) = (n_2,n_3,n_1,n_3,n_2)$$

Thus we need to prove

$$\sum_{n=0}^{\infty} trS^n(t_\pi)x^n = (1-\omega_\pi(p)x^2)^{-1}(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_1,n_2,n_3=0}^{\infty} (n_2,n_3,n_1,n_3,n_2)\omega_\pi(p)^{n_2+2n_3} \\ x^{n_1+2n_2+4n_3}(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)})$$

Let $\overline{V}$ denote the 20-dimensional complex vector space that GSp_6 acts via the exterior cube representation. Let $\mathbb{C}[\overline{V}]$ denote the symmetric algebra, by Brion[10], $\mathbb{C}[\overline{V}]^{\overline{N}}$ is a polynomial algebra generated by 1, where $\overline{N}$ is the maximal unipotent of $GSp_6(\mathbb{C})$ It follows from Brion [10] that

$$trS^r(t_\pi) = \sum (n_2,n_3,n_1+n_4,n_3,n_2)\omega_\pi(p)^{n_2+2n_3+n_4+2n_5}$$

where the sum is over all $n_i \in N$, $1\leqslant i\leqslant 5$ satisfying $n_1+2n_2+4n_3+3n_4+4n_5 = r$. Thus

$$
\begin{aligned}
\sum_{r=0}^{\infty} trS^r(t_\pi)x^r = & \sum_{n_i=0,1\leqslant i\leqslant 5}^{\infty} (n_2, n_3, n_1+n_4, n_3, n_2)\omega_\pi(p)^{n_2+2n_3+n_4+2n_5} \\
& x^{n_1+2m_2+4n_3+3n_4++4n_5} \\
=&(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_i=0,1\leqslant i\leqslant 4}^{\infty} (n_2, n_3, n_1+n_4, n_3, n_2)\omega_\pi(p)^{n_2+2n_3+n_4} \\
& x^{n_1+2m_2+4n_3+3n_4} \\
=&(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_4=0}^{\infty}\sum_{r=n_4}^{\infty}\sum_{n_2=0,n_3=0}^{\infty} (n_2, n_3, r, n_3, n_2)\omega_\pi(p)^{n_2+2n_3+n_4} \\
& x^{r+2m_2+4n_3+2n_4}
\end{aligned}
\tag{6.1}
$$

The right-hand-side of (6.1) is

$$
\begin{aligned}
&(1-\omega_\pi(p)x^2)^{-1}(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_1,n_2,n_3=0}^{\infty} (n_2, n_3, n_1, n_3, n_2)\omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3} \\
&(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)}) \\
=&(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_1,n_2,n_3=0}^{\infty} (n_2, n_3, n_1, n_3, n_2)\omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3}\frac{1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)}}{1-\omega_\pi(p)x^2} \\
=&(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_1,n_2,n_3=0}^{\infty} (n_2, n_3, n_1, n_3, n_2)\omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3} \\
&\sum_{r=0}^{n_1}(1+\omega_\pi x^2+\ldots+\omega_\pi^{n_1}x^{2n_1}) \\
=&(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_1,n_2,n_3=0}^{\infty}\sum_{r=0}^{n_1} (n_2, n_3, n_1, n_3, n_2)\omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3}\omega_\pi^r x^{2r} \\
=&(1-\omega_\pi^2(p)x^4)^{-1} \sum_{n_1,n_2,n_3=0}^{\infty}\sum_{r=0}^{n_1} (n_2, n_3, n_1, n_3, n_2)\omega_\pi(p)^{n_2+2n_3+r}x^{n_1+2n_2+4n_3+2r}
\end{aligned}
$$

which is the exactly (2) after exchanging the position of r and x_1. Therefore, we proved (1) and hence the theorem. $\square$

Chapter 7

Local Theory

In this section, we will study the local integrals which come from our global constructions.

Let F be a nonarchimedean local field, $\mathcal{O}$ will denote the ring of integers in F, p a generator of the maximal ideal in $\mathcal{O}$, with $q = |p|^{-1}$.

Let π be an admissible generic representation of $GL_6(F)$ with central character ω_π. Let χ be a unitary character of F^*. Let $I(s,\chi_\pi) = \mathrm{Ind}_Q^{\mathrm{GSp}_6} \delta_Q^s \chi_\pi$. Thus $f_s \in I(s,\chi_\pi)$ is a smooth function which satisfies

$$f_s((\alpha,g)rh) = (\omega_\pi\chi^3)(\alpha)(\omega_\pi\chi^2)(\mathrm{det}g)\delta_Q^s((\alpha,g))f_s(h)$$

for all $(\alpha,g) \in \mathrm{GL}_1 \times \mathrm{GL}_3, r \in R, h \in \mathrm{GSp}_6$. Let K be the standard maximal compact subgroup of GSp_6. If μ is an unramified character of F^*, let $L(\mu,s) = (1-\mu(p)q^{-s})^{-1}$.

We are studying the local integral

$$I(W,f_s,\chi,s) = \int_{ZV\backslash \mathrm{GSp}_6} \int_F W(X(r)j(g))f_s(g)drdg$$

where $W \in \mathcal{W}(\pi,\psi)$,$f_s \in I(s,\chi)$, $X(r) = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && r & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$, Z is the center of GSp_6, V is the maximal unipotent of GSp_6 such that the maximal standard unipotent subgroup of GL_6 N(all upper unipotent matrices) and $V \subset N$, which is the center of GL_6, $j(g) = wgw^{-1}$, where $w = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \in \mathrm{GSp}_6$

7.1 Convergence, Meromorphic Continuation

Definition 7.1.1. *A gauge ξ on GL_6 is defined as a positive function, invariant on the left under N, on the right under K, given on A, which is the maximal torus, by*

$$\xi(a) = \phi(\alpha_1(a), \alpha_2(a), ..., \alpha_5(a))|\alpha_1(a)\alpha_2(a)...\alpha_5(a)|^{-k}.$$

With the Iwasawa decomposition of GSp_6, let T' be the maximal torus in GSp_6, the parametrization of $t' \in T'$ is $t' = \mathrm{diag}(abc, ac, 1, a, c^{-1}, b^{-1}c^{-1}), \quad a, b, c \in F^*$,

$$\mathrm{GSp}_6 = ZVT'K$$

Let B' denote the Borel subgroup of GSp_6, P be the parabolic group, then

$$\delta_{B'}(t') = |a^4b^6c^{10}|, \quad \delta_P(t') = |a^2b^4c^8|$$

Take K° as an open compact subgroup of K which stablizes W and f_s. Then we can write K as a finite union $\bigcup_i K^\circ k_i$, let $W_i = \rho(j(k_i))W$, $f_{s,i} = R(k_i)f_s$. We have

$$\begin{aligned} I(W, f_s, \chi, s) &= \int_{T'}\int_K\int_F W(X(r)j(t'k))f_s(t'k)drdkdt' \\ &= \sum_i \int_{T'}\int_{K^\circ}\int_F W(X(r)j(t'k^\circ)j(k_i))f_s(t'k^\circ k_i)drdk^\circ dt' \\ &= c \cdot \sum_i \int_{T'}\int_F W_i(X(r)j(t'))f_{s,i}(t')drdt' \end{aligned}$$

where c is the volume of K°. By proposition 2.2 in [11], for any generic representation of $G(F)$, there exists a finite set X of finite functions $\{\lambda\}$ defined on the torus and a corresponding set of Schwartz functions $\{\phi_\lambda\} \subset \mathscr{S}(F^{r-1})$ with the following property: for any W in $\mathcal{W}(\pi; \psi)$,

$$W(a) = \sum_{\lambda \in X} \lambda(a_1, a_2, ..., a_{r-1})\phi_\lambda(a_1, a_2, ..., a_{r-1}) \tag{7.1}$$

for

$$a = diag(a_1a_2...a_{r-1}, a_2...a_{r-1}, ..., a_{r-1}, 1),$$

which means W can be considered to be the finite combination of the gauges.

Theorem 7.1.1. *For ξ a gauge on GL_6,*

$$\xi\left[\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} j(t')\right] \neq 0, \quad x \in F, t' \in T'$$

implies that x belongs to a compact set independent of $t' \in T'$

Proof. Write $t' \in T'$ as

$$t' = \begin{pmatrix} abc &&&&& \\ & ac &&&& \\ && 1 &&& \\ &&& a && \\ &&&& c^{-1} & \\ &&&&& b^{-1}c^{-1} \end{pmatrix}$$

then,

$$\xi\left[\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} j(t')\right] = \xi\left[\begin{pmatrix} abc &&&&& \\ & ac &&&& \\ && a &&& \\ && ax & 1 && \\ &&&& c^{-1} & \\ &&&&& b^{-1}c^{-1} \end{pmatrix}\right]$$

If we rewrite ax as $\lambda x'$, which gives a change of variable $x' = ax/\lambda$, and the Iwasawa decomposition of $\begin{pmatrix} abc &&&&& \\ & ac &&&& \\ && a &&& \\ && \lambda x' & 1 && \\ &&&& c^{-1} & \\ &&&&& b^{-1}c^{-1} \end{pmatrix}$ is

$$\begin{pmatrix} abc &&&&& \\ & ac &&&& \\ && -\frac{a}{\lambda x'} & -a && \\ &&& -\lambda x' && \\ &&&& c^{-1} & \\ &&&&& b^{-1}c^{-1} \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 0 & 1 && \\ && 1 & 0 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & \frac{1}{\lambda x'} && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$$

Then $\xi\left[\begin{pmatrix} abc & & & & & \\ & ac & & & & \\ & & a & & & \\ & & \lambda x' & 1 & & \\ & & & & c^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix}\right]$ is nonzero implies

$\xi\begin{pmatrix} abc & & & & & \\ & ac & & & & \\ & & -\frac{a}{\lambda x'} & & & \\ & & & -\lambda x' & & \\ & & & & c^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix}$ is nonzero, hence $|\frac{ac}{-\frac{a}{\lambda x'}}| = |c\lambda x'|$ and $|\frac{-\frac{a}{\lambda x'}}{den}|$ is bounded. Since our choice for λ is any nonzero number, we can let $\lambda = 1/c \in F^{\times}$, we have $|x'|$ is bounded, so x' must land in a compact set independent of $t' \in T'$. □

Theorem 7.1.2. *Let π be a generic representation, let $W \in \mathcal{W}(\pi, \psi)$, then the integral $I(W, f_s, \chi, s)$ is absolutely convergent for $\Re s$ large, and it has a meromorphic continuation to the complex plane.*

Proof. Since

$$I(W, f_s, \chi, s) = vol(K^{\circ}) \cdot \sum_i \int_{T'} \int_F W_i(X(r)j(t'))f_{s,i}(t')drdt',$$

we only need to prove for each of the W_i and $f_{s,i}$ that $\int_{T'}\int_F W_i(X(r)j(t')f_{s,i}(t')drdt'$ is absolutely convergent for $\Re s$ large and admit a meromorphic continuation to the complex plane. By (3), we can reduce the problem to the case when W is a gauge function. By Now back to $I(W, f_s, \chi, s)$, By Theorem 1, $W(X(r)j(t') \neq 0$ implies that r belongs to a compact set independent of t'. Thus the integral $I(W, f_s, \chi, s)$ is a finite sum of integrals of the form

$$\int_{T'} W(j(t))f_s(t')dt'$$

with $W \in \mathcal{W}(\pi, \psi), f_s \in I(s, \chi_\pi)$. In fact, this integral is computed as

$$\int_{(F^*)^3} W\begin{pmatrix} abc & & & & & \\ & ac & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & c^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix} f_s\begin{pmatrix} abc & & & & & \\ & ac & & & & \\ & & 1 & & & \\ & & & a & & \\ & & & & c^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix} d^*ad^*bd^*c$$

Since $f_s \in I(s, \chi_\pi)$,

$$f_s \begin{pmatrix} abc & & & & & \\ & ac & & & & \\ & & 1 & & & \\ & & & a & & \\ & & & & c^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix}$$

$$=(\omega_\pi \chi^3)(\alpha)(\omega_\pi \chi^2)(\det g)\delta_Q^s((\alpha, g))f_s(I_6) = \mu_2(a,b,c)\nu(a,b,c)^s$$

where $\mu_2(a,b,c) = |a^5b^6c^{10}|^{-1}\chi(ab^2c^4)\omega_\pi(bc^2)$, $\nu(a,b,c) = |a^2b^4c^8|$. Since here W is a gauge, we have for $diag(abc, ac, a, 1, c^{-1}, b^{-1}c^{-1}) \in T'$

$$W \begin{pmatrix} abc & & & & & \\ & ac & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & c^{-1} & \\ & & & & & b^{-1}c^{-1} \end{pmatrix} = \phi(a,b,c)\mu_3(a,b,c)$$

where ϕ is a Schwartz function of a, b, c, $\mu_3(a,b,c)$ is a finite function. Let $\mu = \mu_2 \cdot \mu_3$, then the integral is in the form of

$$\int_{(F^*)^3} \phi(a,b,c)\mu(a,b,c)\nu(a,b,c)^s d^*ad^*bd^*c$$

which according to Tate's one dimensional case [as in the lemma below], the integral above converges and admits meromorphic continuation, and is a rational function of q^{-s}.

□

7.2 Nonvanishing result

Theorem 7.2.1. *Let f_s be a K standard section, i.e. its restriction to the standard maximal compact group K is independent of s Let W be a smooth vector in the Whittaker space of π. Then given $s_0 \in \mathbb{C}$, there is a choice of W and a K finite section f_s such that $I(W, f_s, \chi, s)$ is nonzero at $s = s_0$.*

Proof. Assume that $I(W, f_s, \chi, s)$ is zero at $s = s_0$ at all choice of data, then

$$I(W, f_{s_0}, \chi, s_0) = \int_{ZV\backslash GSp_6} \int_F W(X(r)j(g))f_{s_0}(g)drdg = 0$$

By Iwasawa Decomposition, $g = m(\alpha, g_1)k$ for $k \in K$, then

$$f_s(m(\alpha, g)k) = \omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}f_s(k).$$

We have

$$\begin{aligned} I(W, f_s, \chi, s) &= \int_{ZV\backslash GSp_6}\int_F W(X(r)j(g))f_s(g)drdg \\ &= \int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F\int_K W(X(r)j(m(\alpha, g)k)\omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6} \\ &\quad |\det g|^{4s-4}f_s(k)dkdrd^*\alpha dg \end{aligned}$$

Define for $\Re s$ large,

$$\begin{aligned} I_1(W, \chi, s, k) = \int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F & W(X(r)j((\alpha, g))k)\omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6} \\ & |\det g|^{4s-4}dtd^*\alpha dg \end{aligned}$$

Here $k \in K(G)$ and $(\alpha, g) \in GL_1 \times GL_3$. Then for $\Re s$ large,

$$I(W, f_s, \chi, s) = \int_{GL_3\cap K\backslash K} I_1(W, \chi, s, k)f_s(k)dk$$

So I_1 admits a continuation in s and such continuation in s as a function in k is locally constant.

Therefore, for all K-finite function σ, we have

$$\int I_1(W, \chi, s, k)\sigma(k)dk$$

is zero at $s = s_0$ for all smooth functions σ on $(GL_3 \cap K)\backslash K$ Thus $I_1(W, \chi, s, k)$ is zero at $s = s_0$ for all W. In particular, it is true for $k = e$, then

$$\begin{aligned} I_1(W, \chi, s, e) = \int_{ZV\cap GL_3\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F & W(X(r)j((\alpha, g))\omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}| \\ & \det g|^{4s-4}drd^*\alpha dg \end{aligned}$$

is zero at $s = s_0$ for all W. Define a Whittaker function W_1 by

$$W_1(g) = \int_{F^3} W\left(gj\begin{pmatrix} 1 & & & 0 & 0 & 0 \\ & 1 & & 0 & 0 & r_1 \\ & & 1 & 0 & r_2 & r_3 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right)\phi(r_1, r_2, r_3)dr_1dr_2dr_3$$

where ϕ is a smooth function with compact support on F^3. Then

$$I_1(W_1,\chi,s,e) = \int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times \mathrm{GL}_3}\int_F\int_{F^3} W(X(r)j\left((\alpha,g)\begin{pmatrix} 1 & & & 0 & 0 & 0 \\ & 1 & & 0 & 0 & r_1 \\ & & 1 & 0 & r_2 & r_3 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right)$$

$$\phi(r_1,r_2,r_3)\mu_1(\alpha,g,s)drf^*\alpha dg$$

where $\mu_1(\alpha,g,s)=\omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}drd^*\alpha dg$. Now conjugating $\begin{pmatrix} 1 & & & 0 & 0 & 0 \\ & 1 & & 0 & 0 & r_1 \\ & & 1 & 0 & r_2 & r_3 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$ to the left, we obtain for $\Re s$ large,

$$I_1(W_1,\chi,s,e)=\int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F\int_{F^3} W\left(X(r)j\left(m(\alpha,g)\begin{pmatrix}1&&&0&0&0\\&1&&0&0&r_1\\&&1&0&r_2&r_3\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right)\right)$$

$$\phi(r_1,r_2,r_3)\mu_1(\alpha,g,s)drf^*\alpha dg$$

$$=\int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F\int_{F^3} W(X(r)j((m(\alpha,g)\begin{pmatrix}1&&&0&0&0\\&1&&0&0&r_1\\&&1&0&r_2&r_3\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}$$

$$(X(r)j(m(\alpha,g))^{-1}(r)j(m(\alpha,g))\phi(r_1,r_2,r_3)\mu_1(\alpha,g,s)drf^*\alpha dg$$

$$=\int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F\int_{F^3} W(X(r)j(m(\alpha,g))$$

$$\psi\left(X(r)j(m(\alpha,g)\begin{pmatrix}1&&&0&0&0\\&1&&0&0&r_1\\&&1&0&r_2&r_3\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}(X(r)j(m(\alpha,g))^{-1}\right)\phi(r_1,r_2,r_3)$$

$$\mu_1(\alpha,g,s)drf^*\alpha dg$$

$$=\int_{Z(V\cap GL_3)\backslash \mathrm{GL}_1\times\mathrm{GL}_3}\int_F\int_{F^3} W(X(r)j(m(\alpha,g))\psi(\alpha g_{11}r_1+\alpha g_{12}r_2+\alpha g_{13}r_3)$$

$$\phi(r_1,r_2,r_3)\mu_1(\alpha,g,s)drf^*\alpha dg$$

Let μ_2 be the restriction of μ_1 to GL_2, since

$$\int_{F^3}\psi(\alpha g_{11}r_1+\alpha g_{12}r_2+\alpha g_{13}r_3)\phi(r_1,r_2,r_3)\mu_1(\alpha,g,s)dr_1dr_2dr_3$$

is an arbitrary smooth function on $GL_2R^2\backslash GL_1 \times GL_3$ (GL_2R^2 is embedded in GL_6 as

$$\begin{pmatrix} |g| & & & \\ & g & & \\ & & g^* & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}.$$

so let

$$I_2(W,\chi,s) = \int_{(V\cap GL_2)\backslash GL_2}\int_F (W(X(r)j(g))\mu_2(g,s)drdg$$

we have the meromorphic continuation of $I_2(W,\chi,s_0) = 0$ for all data.

Let

$$W_2(g) = \int_{F^2} W\left(gj\left(\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}\right)\right)\phi(y_1,y_2)dy_i$$

Then we have

$$I_2(W_2,\chi,s) = \int_{F^2}\int_{(V\cap GL_2)\backslash GL_2}\int_F W(X(r)j\left(g\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}\right)$$

$$\phi(y_1,y_2)\mu_2(g,s)drdgdy_1dy_2$$

We can write $g \in GL_2$ as $g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} g_0$ by conjugating to the left,

$$I_2(W_2,\chi,s) = \int_{F^2}\int_{(V\cap GL_2)\backslash GL_2}\int_F W(X(r)j\left(g\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}\right)\phi(y_1,y_2)$$

$$\mu_2(g,s)drdgdy_1dy_2$$

$$= \int_{F^2}\int_{(V\cap GL_2)\backslash GL_2}\int_F W(X(r)j\left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & 1 & & & \\ & & & a & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} g_0\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}\right)$$

$$\phi(y_1,y_2)\mu_2(g,s)drdgdy_1dy_2$$

$$= \int_{F^2}\int_{(V\cap GL_2)\backslash GL_2}\int_F W(X(r)j\left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & 1 & & & \\ & & & a & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} g_0\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}\right)$$

$$\left(X(r)\left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & 1 & & & \\ & & & a & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right)\right)^{-1} X(r)\left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & 1 & & & \\ & & & a & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right)$$

$$\phi(y_1,y_2)\mu_2(g,s)drdgdy_1dy_2$$

$$= \int_F^* \int_F W\left[X(r)\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right]\mu_3(a,s)drd^*a$$

where μ_3 is the restriction of μ_2 to $Z(GL_3)$. Therefore, the meromorphic continuation of

$$I_3(W,\chi,s) = \int_{F^*}\int_F W\left[X(r)\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right]\mu_3(a,s)drd^*a$$

is zero for all W, so $W(e) = 0$ for all W. This is a contradiction. $\square$

Chapter 8

The Analytic Properties of the Partial L-function

Let $\pi = \otimes_v \pi_v$ and $I(s, \chi_\pi) = \otimes_v I_v(s, \chi_\pi)$. Let S be a finite set including the Archimedean places such that outside of S all data is unramified. Given a character $\mu = \otimes_v \mu_v$ of $F^* \backslash \mathbb{A}^*$, we denote $L^S(\mu, s) = \prod_{v \notin S} L_v(\mu_v), s$ where $L_v(\mu_v, s)$ is the local degree one L-function of μ_v. Set

$$E^*(g, f_s, \chi, s) = L_S(\omega_\pi \chi^2, 4s) L_S(\omega_\pi^2 \chi^4, 8s - 2) E(g, f_s, \chi, s)$$

and

$$I^*(\varphi, f_s, \chi, s) = L_S(\omega_\pi \chi^2, 4s) L_S(\omega_\pi^2 \chi^4, 8s - 2) I(\varphi, f_s, \chi, s)$$

Lemma 8.0.1. *Let φ be a cusp form on $GSp_6(\mathbb{A})$, then*

$$\int_{GSp(F) \backslash GSp_6(\mathbb{A})} \varphi(j(g)) E(g, f, s) dg = 0$$

Proof. Let Q denote the parabolic subgroup of GSp_6 whose Levi part is $GL_1 \times GSp_4$. Let $I(s) = Ind_{Q(\mathbb{A})}^{GSp_6(\mathbb{A})} \delta_Q^s$, and for every $f \in I(s)$, define for $\Re(s)$ large

$$E(g, f, s) = \sum_{\gamma \in Q(F) \backslash GSp_6(F)} f(\gamma g, s)$$

For $\Re(s)$ large, by unfolding the Eisenstein series, we have

$$\int_{GL_1(F) GSp_4(F) R_3(F) \backslash GSp_6(\mathbb{A})} \varphi(g) f_s(g) dg$$

where R_3 is the unipotent radical subgroup of Q, and the embedding of $GL_1 \times GSp_4$ in GSp_6 is as

$$(\alpha, h) \to \begin{pmatrix} \alpha & & \\ & h & \\ & & \alpha^{-1} \end{pmatrix} \quad \alpha \in GL_1, h \in GSp_4.$$

Define

$$V^2 = \{v \in V : v = I_6 + r_1 e_{12} + r_2 e_{13} + r_3 e_{14} + r_4 e_{15}\}$$

Take the Fourier expansion with respect to $V^2(F)\backslash V^2(\mathbb{A})$, then the integral becomes (the last step is by cuspidality)

$$\int_{GL_1(F)GSp_4(F)R(F)\backslash GSp_6(\mathbb{A})} \int_{V^2(F)\backslash V^2(\mathbb{A})} \varphi(v_2 j(g))\varphi(\sum_{i=1}^{4} \alpha_i r_i) f_s(g) dv dg$$

$$\int_{GL_1(F)GSp_2(F)R_2(F)R_3(\mathbb{A})\backslash GSp_6(\mathbb{A})} \int_{V_6(F)\backslash V_6(\mathbb{A})} \varphi(vj(g)) f_s(g) dv dg = 0.$$

□

Theorem 8.0.2. *Let f_s be a standard K finite section which is an unramified outside of S, then*

(a) If $\omega_\pi \chi^2 = 1$ or $\omega_\pi^2 \chi^4 \neq 1$, then $I^*(\varphi, f_s, \chi, s)$ is entire.

(b) If $\omega_\pi^2 \chi^4 = 1$ but $\omega_\pi \chi^2 \neq 1$, then $I^*(\varphi, f_s, \chi, s)$ can have at most a simple pole at $s = 1/4$ or $s = 3/4$.

Proof. We first state the result from Ikeda [12]:

Lemma 8.0.3 (Ikeda)**.** *Suppose that k is a number field. If $f^{(s)}$ is a good section of $I(\omega, s)$, then the pole of $E(h, f^{(s)})$ are at most simple. The set of possible poles is as follows:*

(a) If ω is principal, say $\omega = 1$:

$$\{(n+1)/2 - m | m \in \mathbb{Z}, 0 \leqslant m \leqslant n+1, m \neq (n+1)/2\}$$

(b) If ω is not principal, but ω^2 is principal, say $\omega^2 = 1$:

$$\{(n-1)/2 - m | m \in \mathbb{Z}, 0 \leqslant m \leqslant n-1, m \neq (n-1)/2\}$$

(c) If ω^2 is not principal, then $E(h, f^{(s)})$ is entire.

We need to study the poles of E^* and its residue. By the functional equation of E^* and the fact that if $f^{(s)}$ is a good section of $I(\omega, s)$, then $M_{w_0} f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$ (where M_{w_0} is the intertwining operator), it is enough to prove for the case when $\Re(s) \geqslant 1/2$. By lemma above, the poles are as following: $E^*(g, f, \chi, s)$ has at most simple poles, and for $\Re(s) \geqslant 1/2$, we have

(a) If $\omega_\pi^2 \chi^4 \neq 1$, $E^*(g, f, \chi, s)$ is entire.

(b) If $\omega_\pi^2 \chi^4 = 1$, but $\omega_\pi \chi^2 \neq 1$, $E^*(g, f, \chi, s)$ has a simple pole at $s = 3/4$.

(c) If $\omega_\pi \chi^2 = 1$, $E^*(g, f, \chi, s)$ has simple poles at $s = 1$ and $s = 3/4$.

□

Theorem 8.0.4. *Let π be a cusp form on $GL_6(\mathbb{A})$. Let S be as above. Then*

$$L^S(\pi, \Lambda^3 \otimes \chi, s) = \prod_{v \notin S} L_v(\pi_v, \Lambda^3 \otimes \chi_v, s)$$

is entire unless $\omega_\pi^2 \chi^4 = 1$ and $\omega_\pi \chi^2 \neq 1$. In this case, the L-function can have at most a simple pole at $s = 0$ or $s = 1$.

Proof. By theorem 6.1, we can write

$$I^*(\varphi, f, \chi, s) = \prod_{v \in S} \Phi_v(W_v, f^{(v)}, \chi_v, s) L_S(\pi \times \chi, \Lambda^3, 2s - \frac{1}{2})$$

For $v \in S$, by theorem 7.4, we can choose data so that $\Phi_v(W_v, f^{(v)}, \chi_v, s)$ is nonzero, together with theorem 8.2, we proved this theorem. □

References

[1] David Ginzburg and Stephen Rallis. The exterior cube l-function for gl(6). *Compositio Mathematica*, 123(3):243–272, 2000.

[2] Stephen Gelbart and Freydoon Shahidi. *Analytic properties of automorphic L-functions*. Academic Press, 2014.

[3] André Weil. *Basic number theory.*, volume 144. Springer Science & Business Media, 2013.

[4] JW Cogdell. Notes on l-functions for gln. *School on Automorphic Forms on GL (n)*, 21:75–158, 2000.

[5] Hervé Jacquet, Ilya I Piatetskii-Shapiro, and Joseph A Shalika. Rankin-selberg convolutions. *American journal of mathematics*, 105(2):367–464, 1983.

[6] Robert P Langlands. *Euler products*, volume 1. Yale University Press New Haven, 1971.

[7] Robert P Langlands. *On the functional equations satisfied by Eisenstein series*, volume 544. Springer, 2006.

[8] Robert Phelan Langlands. Problems in the theory of automorphic forms. *Matematika*, 15(2):57–83, 1971.

[9] William Casselman and Joseph Shalika. The unramified principal series of p-adic groups. ii. the whittaker function. *Compositio Mathematica*, 41(2):207–231, 1980.

[10] Michel Brion. Invariants d'un sous-groupe unipotent maximal d'un groupe semi-simple. In *Annales de l'institut Fourier*, volume 33, pages 1–27, 1983.

[11] Hervé Jacquet, Ilja Iosifovitch Piatetski-Shapiro, and Joseph Shalika. Automorphic forms on gl (3) i. *Annals of Mathematics*, 109(1):169–212, 1979.

[12] Tamotsu Ikeda. On the location of poles of the triple l-functions. *Compositio Mathematica*, 83(2):187–237, 1992.