On the Computation of Suboptimal H^∞ Controllers for Unstable Infinite Dimensional Systems[1]

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Abstract

In this paper we show how to compute suboptimal H^∞ controllers, for a class of (possibly) unstable and (possibly) infinite dimensional plants, from a finite set of linear equations. A solution to the H^∞ suboptimal control problem for infinite dimensional *stable* plants was obtained in [6]. Also, in [13] and [14] the H^∞ *optimal* control problem was solved for unstable distributed plants. Our solution for the *suboptimal* control problem of *unstable* distributed plants is based on the techniques developed in [6] and [13]. We obtain a computable expression for the suboptimal H^∞ controllers and identify their finite and infinite dimensional parts.

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1 Introduction

The purpose of this paper is to give a solution to the two block H^∞ *suboptimal* control problem for a class of distributed plants with a finite number of *unstable poles*. The H^∞ suboptimal control problem for stable distributed plants was solved in [6] and the structure of all suboptimal controllers was determined in [12]. In [13] and [14], the classical skew Toeplitz approach developed in [2], [7], [8], [9], [11], was extended to solve the optimal control problem for unstable plants. In this paper, we combine certain ideas and observations from [6] and [13] to obtain an explicit formulae for suboptimal H^∞ controllers for a class of unstable distributed plants.

We consider SISO systems and use the frequency domain approach to H^∞ control. Following is an outline of the solution procedure given in this paper. First we use spectral factorization to reduce the 2-block problem to a 1-block problem. Then, Adamjan Arov Krein (AAK) formulae are used for parametrization of controllers. At this step, an extension of the techniques developed in [13] and [14] is used for reducing an infinite dimensional matrix problem, which appears in AAK formulae, to a finite dimensional matrix problem. The final formulae for the suboptimal H^∞ controllers, is a linear fractional transformation (LFT) on a free parameter in the unit ball of H^∞. The coefficients of this LFT are of the form rational plus inner times rational. If the plant is stable, this formula gives the structure previously observed in [12].

The rest of this paper is organized as follows. In Section 2, we formulate the 2-block problem and discuss its reduction to 1-block problem. In Section 3, the AAK formulae are used for the suboptimal 1-block H^∞ problem, and the associated infinite dimensional matrix problem is reduced to a finite dimensional one. The structure of suboptimal H^∞ controllers is given in Section 4, where we also apply our results to the stable plant case, and verify the controller structure given in [12]. Finally, in Section 5 we make some concluding remarks. See Appendix A for the notation used in the paper.

2 Suboptimal H^∞ control problem

In this section we will define the suboptimal two-block H^∞ control problem and reduce it to a one-block problem. We consider SISO, LTI, (possibly) infinite dimensional plants with finitely many unstable stable poles:

$$P = \frac{N_p}{D_p}$$

where $N_p \in H^\infty$ and $D_p \in \mathbb{R}H^\infty$. We assume that (i) $N_p = m_n N_o$, where $m_n \in H^\infty$ is inner (arbitrary) and $N_o \in H^\infty$ is outer, and (ii) N_p is analytic and non-zero at the zeros of D_p in the closed unit disc. We also write $D_p = m_d D_o$ where $m_d \in \mathbb{R}H^\infty$ is inner and D_o is outer. Under these assumptions there exist $X \in H^\infty$ and $Y \in H^\infty$ such that

$$XN_p + YD_p = 1. \tag{1}$$

Note that X must be chosen such that

$$Y = \frac{1 - N_p X}{D_p} \in H^\infty.$$

So, X must satisfy only finitely many interpolation conditions (i.e. $1 - N_p X$ must have zeros at the zeros of D_p in the closed unit disc, at least of the same multiplicity). Therefore, X can be chosen as rational. The set of all controllers which stabilize the plant $P = N_p/D_p$ can now be written in the from, [16],

$$C = \frac{X + QD_p}{Y - QN_p}$$

where $Q \in H^\infty$ is the free parameter. Let $S = (1 + PC)^{-1}$ and $T = 1 - S$ be the sensitivity and complementary sensitivity functions respectively.

We will consider the following mixed sensitivity reduction problem: we want to parametrize the set

$$\mathcal{C}_\rho = \left\{ C : C \text{ stabilizes } P, \quad \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \leq \rho \right\} \tag{2}$$

where $W_1, W_2 \in \mathbb{R}H^\infty$ are given weighting functions. As in [13] and [14], we define a function G from the conditions: (i) $W_1^* W_1 + W_2^* W_2 = G^* G$ and (ii) $G, G^{-1} \in \mathbb{R}H^\infty$. Since $G^{*-1} W_1^* W_1 \in \mathbb{R}L^\infty$, there exists a finite Blaschke product $b_1 \in \mathbb{R}H^\infty$ such that $w_0 = b_1 G^{*-1} W_1^* W_1 \in \mathbb{R}H^\infty$. Note that b_1 can be chosen as a finite Blaschke product with zeros at the poles of $G^{*-1} W_1^* W_1$ inside the unit disc, of the same multiplicity. We also define $Q_1 := G G_{no} Q$ and $g_0 := -W_1 W_2 G^{-1}$. Then, we choose $\widehat{w}_0 \in H^\infty$ such that $(G G_{no} X - \widehat{w}_0)/G_d \in H^\infty$. Since $\widehat{w}_0$ has to satisfy only finitely many interpolation conditions it can be chosen as rational. Now define

$$Q_2 := Q_1 + (G G_{no} X - \widehat{w}_0)/G_d,$$

$$\widehat{q} := G_{do}Q_2, \quad m := b_1 m_n m_d, \quad m_1 := b_1 m_n.$$

If

$$\sigma_1 = \max\left\{ \left\| \begin{array}{c} w_0(z_i) - m_1(z_i)\widehat{w}_0(z_i) \\ g_0(z_i) \end{array} \right\|_\infty , \left\| \begin{array}{c} w_0(z_j) \\ g_0(z_j) \end{array} \right\|_\infty \right.$$

$$\left. : \ z_j \ (z_i) \text{ is a zero of } G_{no} \ (G_{do}) \text{ on T} \right\} < \gamma_{opt}$$

then the problem reduces to the parametrization of

$$\mathcal{S}_\rho = \{\widehat{q} \in H^\infty \ : \ \left\| \left[\begin{array}{c} w_0 - \widehat{w}_0 m_1 - m\widehat{q} \\ g_0 \end{array} \right] \right\|_\infty \leq \rho\} \tag{3}$$

(see [13] and [3] for all details), where

$$\gamma_{opt} := \inf_{\widehat{q} \in H^\infty} \left\| \left[\begin{array}{c} w_0 - \widehat{w}_0 m_1 - m\widehat{q} \\ g_0 \end{array} \right] \right\|_\infty \tag{4}$$

which can be computed by the formulae given in [13]. For the stable plant case $\widehat{w}_0 = 0$, $m_2 = 1$, and the corresponding problem (3) has been solved in [6].

Now we assume that $\rho > \gamma_{opt}$, then $\|g_0\|_\infty \leq \gamma_{opt} < \rho$; so there exists an $f_\rho \in \mathbb{R}H^\infty$ with ${f_\rho}^{-1} \in \mathbb{R}H^\infty$ such that $|f_\rho|^2 + |g_0|^2 = \rho^2$. Hence (3) reduces to

$$\mathcal{S}_\rho = \{\widehat{q} \in H^\infty \ : \ \|\frac{w_0}{f_\rho} - \frac{\widehat{w}_0}{f_\rho} m_1 - m\frac{\widehat{q}}{f_\rho}\|_\infty \leq 1\}. \tag{5}$$

Let us define $q_1 := \widehat{q}/f_\rho$, $u_0 := w_0/f_\rho$, $\widehat{u}_0 := \widehat{w}_0/f_\rho$ and $u := u_0 - m_1\widehat{u}_0$. Then, we get

$$\mathcal{S}_\rho = \{f_\rho q_1 \ : \ \|u - mq_1\|_\infty \leq 1 \text{ and } q_1 \in H^\infty\}. \tag{6}$$

The problem defined by (6) is a one-block suboptimal H^∞ control problem. Since m is allowed to be an irrational inner function (or infinite Blaschke product) we cannot use the standard state space methods for solving this problem. Also, since u is possibly infinite dimensional, we cannot use the results of [6] directly, we need to use an observation similiar to the one given in [13].

3 A solution to the suboptimal H^∞ control problem

In this section, we parametrize all H^∞ functions $f_\rho q_1$ in the set $\mathcal{S}_\rho$, by using the AAK formulae (see [1]). We will need the following

Proposition 3.1: For u and m defined above, we have

$$\inf_{q_1 \in H^\infty} \|u - mq_1\|_\infty \; < 1 \; .$$

Proof: By the commutant lifting theorem, (see e.g. [4]) there exists a $\widehat{q} \in H^\infty$ such that

$$|w_0 - \widehat{w}_0 m_1 - m\widehat{q}|^2 + |g_0|^2 \;=\; \gamma_{opt}^2$$

$$|w_0 - \widehat{w}_0 m_1 - m\widehat{q}|^2 + |g_0|^2 + |f_\rho|^2 \;=\; \gamma_{opt}^2 + |f_\rho|^2$$

$$|w_0 - \widehat{w}_0 m_1 - m\widehat{q}|^2 \;=\; \gamma_{opt}^2 - \rho^2 + |f_\rho|^2$$

$$|u - mq_1| \;=\; 1 - \frac{\rho^2 - \gamma_{opt}^2}{|f_\rho|^2}.$$

Since f_ρ^{-1} is in H^∞ and $\rho > \gamma_{opt}$,

$$\|u - mq_1\|_\infty \;<\; 1 \; .$$

Hence

$$\inf_{q_1 \in H^\infty} \|u - mq_1\|_\infty \; < 1. \qquad \square$$

3.1 The AAK approach

In this section we will present a result from [1], and then apply it to our problem (6). For this purpose we will need some more notation. Let $H(m)$ be the orthogonal complement of mH_2 in H_2, $\mathbf{\Gamma}_{m^*u}$ be the Hankel operator defined from H_2 to $H_2^\perp$ as $\mathbf{\Gamma}_{m^*u} = \mathbf{P}_{H_2^\perp} m^* u$, where $\mathbf{P}_{H_2^\perp}$ denotes the orthogonal projection operator from L_2 to $H_2^\perp$. Now, define $\mathcal{R}$ as the reflection operator defined from $H_2^\perp$ to H_2, as $\mathcal{R}f(z) = z^{-1}f(z^{-1})$. Then, $\mathcal{R}^*$ is form H_2 to $H_2^\perp$ and $\mathcal{R}^* f(z) = f(z^{-1})z^{-1}$. Finally, let $\mathbf{S}$ be the shift operator defined on H_2 as $\mathbf{S}f(z) = zf(z)$ and $\mathbf{T} : H(m) \to H(m)$ be the compressed shift operator defined as $\mathbf{T} = \mathbf{P}_{H(m)}\mathbf{S}$, where $\mathbf{P}_{H(m)}$ is the orthogonal projection operator from H_2 to $H(m)$.

We will use $\mathbf{\Gamma}$ for the matrix representation of $\mathcal{R}\mathbf{\Gamma}_{m^*u}$. Following [1] we set $\mathbf{R}_\rho := (\mathbf{I} - \mathbf{\Gamma}\overline{\mathbf{\Gamma}})^{-1}$, $e(z) := 1$, $p := \mathbf{R}_\rho e$, and $q := \mathbf{S}\ \overline{\mathbf{\Gamma}}\ \overline{\mathbf{R}}_\rho e$.

Theorem 3.2 ([1]): Assume that $\|\mathbf{\Gamma}_{m^*u}\| < 1$; then the set of all $q_1 \in H^\infty$ satisfying

$$\|u - mq_1\| \ \leq \ 1$$

can be obtained from

$$f_{\varepsilon,\rho} := m^*u - q_1 = \frac{\varepsilon\ p^* + q^*}{p + \varepsilon\ q} \tag{7}$$

where ε is the free parameter in the unit ball of H^∞, i.e. in

$$\mathcal{B} = \{\varepsilon \in H^\infty \ : \ \|\varepsilon\|_\infty \leq 1\}. \ \ \square$$

Note that by Proposition 3.1 $\|\mathbf{\Gamma}_{m^*u}\| < 1$. Therefore, Theorem 3.2 is applicable to our problem. Hence, the solution of (6) amounts to finding p and q in (7). The rest of this section is devoted to computation of p and q.

3.2 Computation of p

Let us first define the 2-block operator $\mathbf{A} : H_2 \to H(m) \oplus H_2$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{P}_{H(m)}u(\mathbf{S}) \\ 0 \end{bmatrix}$$

as in [13]. Since $u(\mathbf{T})\mathbf{P}_{H(m)} = \mathbf{P}_{H(m)}u(\mathbf{S})$ (see [13]), we have

$$\mathbf{A}^*\mathbf{A} = [\mathbf{P}_{H(m)}u(\mathbf{T})^* \quad 0] \begin{bmatrix} u(\mathbf{T})\mathbf{P}_{H(m)} \\ 0 \end{bmatrix} = \mathbf{P}_{H(m)}u(\mathbf{T})^*u(\mathbf{T})\mathbf{P}_{H(m)},$$

$$\mathbf{A}^*\mathbf{A} = \mathbf{P}_{H(m)}u(\mathbf{T})^*mm^*u(\mathbf{T})\mathbf{P}_{H(m)}.$$

As shown in [5], $\mathbf{\Gamma}_{m^*u}|_{H(m)} = m^*u(\mathbf{T})\mathbf{P}_{H(m)}$, and

$$\mathbf{A}^*\mathbf{A}|_{H(m)} = \mathbf{\Gamma}^*_{m^*u}\mathbf{\Gamma}_{m^*u}|_{H(m)}.$$

But $\mathbf{\Gamma} = \mathcal{R}\mathbf{\Gamma}_{m^*u}$, and $\mathcal{R}^*\mathcal{R} = \mathbf{I}$, so

$$\mathbf{\Gamma}^*\mathbf{\Gamma} = \mathbf{\Gamma}^*_{m^*u}\mathbf{\Gamma}_{m^*u},$$

and

$$\mathbf{A}^*\mathbf{A}|_{H(m)} = \mathbf{\Gamma}^*\mathbf{\Gamma}|_{H(m)}. \tag{8}$$

Furthermore, $\mathbf{\Gamma}^* = \overline{\mathbf{\Gamma}}$ because $\mathbf{\Gamma}^T = \mathbf{\Gamma}$. By the AAK formulae, we have

$$(\mathbf{I} - \mathbf{\Gamma}^*\mathbf{\Gamma})p = e.$$

Also, $p = \mathbf{P}_{mH_2}p \;+\; \mathbf{P}_{H(m)}p$, so

$$(\mathbf{I} - \mathbf{\Gamma}^*\mathbf{\Gamma})\mathbf{P}_{mH_2}p + (\mathbf{I} - \mathbf{\Gamma}^*\mathbf{\Gamma})\mathbf{P}_{H(m)}p = 1$$

We now separate the above equation into its $H(m)$ and mH_2 components.
Since $\mathbf{\Gamma}|_{mH_2} = 0$, we have

$$(\mathbf{I} - \mathbf{\Gamma}^*\mathbf{\Gamma})\mathbf{P}_{mH_2}p = \mathbf{P}_{mH_2}p \in mH_2.$$

Also, by (8)

$$(\mathbf{I} - \mathbf{\Gamma}^*\mathbf{\Gamma})\mathbf{P}_{H(m)}p = (\mathbf{I} - \mathbf{A}^*\mathbf{A})\mathbf{P}_{H(m)}p \;\in H(m).$$

Therefore,

$$\mathbf{P}_{mH_2}p = \mathbf{P}_{mH_2}1 = \overline{m(0)}m(z). \tag{9}$$

Defining

$$y := \mathbf{P}_{H(m)}p \qquad \text{and} \qquad \mu(z) := -\mathbf{P}_{H(m)}1 = -1 + \overline{m(0)}m(z). \tag{10}$$

we obtain

$$(\mathbf{I} - \mathbf{A}^*\mathbf{A})y = \mathbf{P}_{H(m)}1 = 1 - \overline{m(0)}m(z),$$

which is equivalent to

$$(\mathbf{A}^*\mathbf{A} - \mathbf{I})y = \mu(z). \tag{11}$$

3.3 Solution of $(\mathbf{A}^*\mathbf{A} - \mathbf{I})y(z) = \mu(z)$

In this section, we will solve the equation (11) by using some of the formulae given in Appendix B and results of [13]. Let us re-write u_0 and $\widehat{u}_0$ as $u_0(z) =: B(z)/K(z)$, $\widehat{u}_0(z) =: C(z)/K(z)$, where $B(z), C(z), K(z) \in \mathbb{R}[z]$ and $1/K(z) \in H^\infty$, (see [13] and [14]). Let n denote the maximum of the degrees of the polynomials $B(z)$, $C(z)$ and $K(z)$. For simplicity we will assume that $n = \deg K$. By making a change of variable $x(z) = y(z)/K(z)$ we transform (11) into the following form

$$K(\mathbf{S})^*(\mathbf{A}^*\mathbf{A} - \mathbf{I})K(\mathbf{S})x(z) = K(\mathbf{S})^*\mu(z).$$

Let us define $\mathbf{R} := K(\mathbf{S})^*(\mathbf{A}^*\mathbf{A} - \mathbf{I})K(\mathbf{S})$, and $\lambda(z) := K(\mathbf{S})^*\mu(z)$. By [13], we know the effect of $\mathbf{R}$ on an arbitrary $x(z) \in H_2$, i.e. we have an explicit expression for $\mathbf{R}x(z)$. Furthermore by [15], $\lambda(z)$ can be computed easily, in fact

$$\lambda(z) = K(z^{-1})\mu(z) - V_-(z)\mathcal{K}\begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{n-1} \end{bmatrix}$$

where $K(z) = k_0 + k_1 z + \cdots + k_n z^n$,

$$V_-(z) = [\, z^{-n} \quad \cdots \quad z^{-1} \,],$$

$$\mathcal{K} = \begin{bmatrix} k_n & \cdots & \cdots & k_1 \\ 0 & k_n & \cdots & k_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix},$$

and $\mu_0, ..., \mu_{n-1}$ are the first n coefficients of the power series expansion of $\mu(z)$, i.e. $\mu(z) = \mu_0 + \mu_1 z + \cdots + \mu_{n-1}z^{n-1} + \cdots$. With this formula, we see that $\lambda(z)$ is in the form $\lambda(z) = t_1^\lambda + m t_2^\lambda$, where

$$t_1^\lambda(z) := K(z^{-1}) - V_-(z)\mathcal{K}[\mu_0 \ \mu_1 \ ... \ \mu_{n-1}]^T$$

$$t_2^\lambda(z) := -\overline{m(0)}K(z^{-1}).$$

Note that t_1^λ and t_2^λ are rational functions of z. We now have an equation of the form $\mathbf{R}x(z) = \lambda(z)$, and by [13] we know how to compute the left hand side:

$$\mathbf{R}x(z) = P(z, z^{-1})x_u + Q(z, z^{-1})mx_v + T(z)\Phi$$

where $x = x_u + mx_v$ with $x_u \in H(m)$, $x_v \in H_2$, and $P(z, z^{-1}) = B(z^{-1})B(z) - K(z^{-1})K(z)$, $Q(z, z^{-1}) = -K(z^{-1})K(z)$, $T(z)$ is a $3n + 2\ell$ row vector ($\ell :=$ dim m_2) with entries in $\mathbb{R}(z) + m(z)\mathbb{R}(z)$ (see Appendix D for explicit formula of $T(z)$), and Φ is an unknown $3n + 2\ell$ column vector depending on x (see [13]). Note that $P(z, z^{-1})$, $Q(z, z^{-1})$ and $T(z)$ are known and x_u, x_v and Φ are unknown.

Now consider the equation

$$P(z, z^{-1})x_u + Q(z, z^{-1})mx_v + T(z)\Phi = \lambda(z). \tag{12}$$

Taking the orthogonal projection onto mH_2, we obtain

$$m(z)[Q(z, z^{-1})x_v + T_v(z)\Phi] = \lambda_v(z) \tag{13}$$

where $T_v(z)$ is a row vector of rational functions (see Appendix D for an explicit formula of $T_v(z)$), and

$$\lambda_v(z) := \mathbf{P}_{mH_2}\lambda(z) = \mathbf{P}_{mH_2}(t_1^\lambda + mt_2^\lambda) = m\mathbf{P}_{H_2}m^*(t_1^\lambda + mt_2^\lambda).$$

So,

$$\lambda_v(z) = m\mathbf{P}_{H_2}(m^*t_1^\lambda + t_2^\lambda) = m\lambda_v^R$$

where $\lambda_v^R = \mathbf{P}_{H_2}(m^*t_1^\lambda + t_2^\lambda)$ is rational and can be computed by the projection formulae of Appendix B. From (13), we have

$$Q(z, z^{-1})x_v + T_v(z)\Phi = \lambda_v^R(z),$$

and hence

$$x_v(z) = \frac{\lambda_v^R(z) - T_v(z)\Phi}{Q(z, z^{-1})}. \tag{14}$$

Note that $x_v(z)$ is rational.

Assumption 1: The zeros of $K(z)$, denoted by $z_1, ..., z_n$, are distinct.

With this assumption, by (14) we get

$$T_v(z_i)\Phi = \lambda_v^R(z_i), \quad \text{for } i = 1, ..., n. \tag{15}$$

It is easy to see that (15) is a necessary and sufficient condition for x_v to be in H_2. Taking projection of (12) onto $H(m)$, we obtain

$$P(z, z^{-1})x_u + T_u(z)\Phi = \mathbf{P}_{H(m)}\lambda(z) =: \lambda_u(z), \tag{16}$$

where $T_u(z)$ is a row vector with entries in $\mathbb{R}(z) + m(z)\mathbb{R}(z)$ (see Appendix D for an explicit formula of $T_u(z)$) and $\lambda_u(z) = \lambda(z) - m(z)\lambda_v^R(z)$. Then,

$$x_u = \frac{\lambda_u(z) - T_u(z)\Phi}{P(z, z^{-1})}. \tag{17}$$

Let $z_{n+1}, ..., z_{3n}$ be the zeros of $P(z, z^{-1})$, where the first r zeros are those lying in the closed unit disc. As in [13], we make the following simplifying assumption.

Assumption 2: The zeros $z_{n+1}, ..., z_{3n}$ are distinct and nonzero.

Since $x_u \in H(m)$, by (17) we have

$$T_u(z_i)\Phi = \lambda_u(z_i) \quad \text{for } i = n + 1, ..., n + r. \tag{18}$$

But $m^* x_u \in H_2^\perp$, so

$$m^*(z_i)T_u(z_i)\Phi = m^*(z_i)\lambda_u(z_i) \quad \text{for } i = n + r + 1, ..., 3n. \tag{19}$$

Taking the orthogonal projection of (16) onto $m_1 H_2$, we obtain

$$\mathbf{P}_{m_1 H_2}(\lambda_u) = \mathbf{P}_{m_1 H_2}(P(z, z^{-1})x_u + T_u(z)\Phi). \tag{20}$$

As shown in [13], we have that

$$\mathbf{P}_{m_1 H_2}(P(z, z^{-1})x_u + T_u(z)\Phi) = m_1[P(z, z^{-1})x_q + T_q(z)\Phi]$$

where $x_u = x_p + m_1 x_q$, with $x_p \in H(m_1), x_q \in H(m_2)$, (see Appendix D for an explicit formula of $T_q(z)$). Since $\lambda_u = \mathbf{P}_{H(m)}\lambda = \lambda - m\lambda_v^R$, we have $\lambda_u = t_1^\lambda + m t_2^\lambda - m\lambda_v^R$. Now set $t_1^u = t_1^\lambda$ and $t_2^u = t_2^\lambda - \lambda_v^R$. Then, we have $\lambda_u = t_1^u + m t_2^u = t_1^u + m_1 t_3^u$, where $t_3^u = t_2^u m_2$. Define

$$\lambda_{u,1} := \mathbf{P}_{m_1 H_2}(\lambda_u) = \mathbf{P}_{m_1 H_2}(t_1^u + m_1 t_3^u),$$

then

$$\lambda_{u,1} = m_1 \mathbf{P}_{H_2} m_1^*(t_1^u + m_1 t_3^u) = m_1 \mathbf{P}_{H_2}(m_1^* t_1^u + t_3^u) = m_1 \lambda_{u,1}^R$$

where $\lambda_{u,1}^R = \mathbf{P}_{H_2}(m_1^* t_1^u + t_3^u)$ is rational and can be computed by the methods of Appendix B. From (20), we have

$$\lambda_{u,1}^R = P(z, z^{-1}) x_q + T_q(z)\Phi$$

so

$$\lambda_{u,1}^R(a_j) = P(a_j, a_j^{-1}) x_q(a_j) + T_q(a_j)\Phi, \qquad j = 1, ..., \ell,$$

where a_j's are the zeros of $m_2(z)$. We assume that a_j's are distinct. Following the ideas of [13] we obtain

$$\lambda_{u,1}^R(a_j) = P(a_j, a_j^{-1}) \sum_{i=0}^{\ell} \alpha_i f_i(a_j) + T_q(a_j)\Phi, \qquad j = 1, ..., \ell \text{ and} \tag{21}$$

$$\lambda_u(a_j) = P(a_j, a_j^{-1})\beta_j + T_u(a_j)\Phi, \qquad j = 1, ..., \ell. \tag{22}$$

where $\Phi = [\Phi_1, ..., \Phi_{3n}, \alpha_1, ..., \alpha_\ell, \beta_1, ..., \beta_\ell]^T$ and $f_i(z) := 1/(1 - \bar{a}_i z)$.

Combining (15), (18), (19), (21), (22) into a single matrix equation, we obtain the matrix formula

$$\Theta\Phi = C_\mu.$$

Again as in [13] we make the following assumption for simplicity.

Assumption 3: The zeros $z_{n+1}, ..., z_{n+r}$ of $P(z, z^{-1})$ inside the unit disc are disjoint from $a_1, ..., a_\ell$.

Suppose that Assumptions 1-3 hold, then by [13], $det(\Theta) \neq 0$, so $\Phi = \Theta^{-1} C_\mu$. Once Φ is found, using (14), $x_v(z)$ can be found. Note that $x_v(z)$ is rational. Define

$R_2^v(z) := x_v(z)$, then from (17), $x_u(z) = R_1^u(z) + m(z)R_2^u(z)$ with R_1^u and R_2^u rational. Recall that $x = x_u + mx_v$, so

$$x(z) = x_u(z) + m(z)x_v(z) = R_1^u(z) + m(z)(R_2^u(z) + R_2^v(z)),$$

and by $y(z) = K(z)x(z)$,

$$y(z) = K(z)R_1^u(z) + m(z)(K(z)R_2^u(z) + K(z)R_2^v(z)).$$

Recall that $p(z) = y(z) + m(z)\overline{m(0)}$, so

$$p(z) = K(z)R_1^u(z) + m(z)(K(z)R_2^u(z) + K(z)R_2^v(z) + \overline{m(0)}).$$

Defining

$$R_1(z) = K(z)R_1^u(z) \text{ and } R_2(z) = K(z)R_2^u(z) + K(z)R_2^v(z) + \overline{m(0)},$$

we have

$$p(z) = R_1(z) + m(z)R_2(z).$$

Note that R_1 and R_2 can be computed explicitly from the above arguments and the projection formulae given in Appendix B.

3.4 Computation of q

By the AAK formulae, we know that $q = \mathbf{S}\ \overline{\mathbf{\Gamma}}\ \overline{\mathbf{R}}_\rho e$, where $e(z) = 1$, and other symbols are as defined in Section 3.1. Note that

$$\overline{\mathbf{R}}_\rho e = \overline{p}(z), \quad \text{because} \quad \mathbf{R}_\rho e = p(z).$$

Since $\overline{\mathbf{\Gamma}} = \mathbf{\Gamma}^*_{m^*u}\mathcal{R}^*$, we have

$$q(z) = z\mathbf{\Gamma}^*_{m^*u}\mathcal{R}^*\overline{p}(z),$$

so

$$q(z) = z\mathbf{\Gamma}^*_{m^*u}z^{-1}p^*(z), \quad \text{because} \quad \mathcal{R}^*\overline{p}(z) = z^{-1}\overline{p}(z^{-1}) = z^{-1}p^*(z).$$

Hence

$$q(z) = z\mathbf{P}_{H_2} u^* m(\frac{R_1^*}{z} + m^* \frac{R_2^*}{z}),$$

$$q(z) = z\mathbf{P}_{H_2}(u_0^* - m^* m_2 \widehat{u}_0^*)(\frac{R_1^*}{z} m + \frac{R_2^*}{z}),$$

$$q(z) = z\left\{\mathbf{P}_{H_2}(u_0^* \frac{R_1^*}{z} m + u_0^* \frac{R_2^*}{z}) - \mathbf{P}_{H_2}(m_2 \widehat{u}_0^* \frac{R_1^*}{z} + \frac{R_2^*}{z} m_2 \widehat{u}_0^* m^*)\right\}. \tag{23}$$

The above projections can be computed by the methods of Appendix B. In summary we have

$$q(z) = R_3(z) + m(z) R_4(z)$$

where $R_3(z)$ and $R_4(z)$ are rational. In particular we can see from (23) that $R_4(z) = u_0^* R_1^*$. Hence,

$$q(z) = R_3(z) + m(z)(u_0^*(z) R_1^*(z)).$$

Remark 3.3: We can show that $p(z)$ is of the form $p = \widehat{R}_1 + (u_0^* R_3^*) m$ for some rational $\widehat{R}_1$, which can be computed explicitly. In order to see this, first note that from [1] we have $p - 1 = \mathbf{\Gamma}^* \mathbf{S}^* \overline{q}$. But $q(0) = 0$, so $\overline{q}(0) = 0$ and $\mathbf{S}^* \overline{q}(z) = z^{-1} \overline{q}(z)$. Therefore,

$$p - 1 = \mathbf{P}_{H_2} u^* m \mathcal{R}^* z^{-1} \overline{q}(z) = \mathbf{P}_{H_2} u^* m q^*(z),$$

$$p - 1 = \mathbf{P}_{H_2}(u_0^* - m^* m_2 \widehat{u}_0^*) m (R_3^* + m^* R_4^*),$$

$$p - 1 = \mathbf{P}_{H_2}(u_0^* - m^* m_2 \widehat{u}_0^*)(m R_3^* + R_4^*),$$

$$p = 1 + \mathbf{P}_{H_2}(u_0^* R_3^* m + u_0^* R_4^*) - \mathbf{P}_{H_2}(m_2 \widehat{u}_0^* R_3^* + R_4^* m_2 \widehat{u}_0^* m^*).$$

Hence,

$$p = \widehat{R}_1 + (u_0^* R_3^*) m,$$

where

$$\mathbf{P}_{H_2}(u_0^* R_3^* m + u_0^* R_4^*) = r_1 + u_0^* R_3^* m \mathbf{P}_{H_2}(m_2 \widehat{u}_0^* R_3^* + R_4^* m_2 \widehat{u}_0^* m^*) = r_2,$$

with r_1, r_2 rational (see Appendix B) and $\widehat{R}_1 = 1 + r_1 - r_2$.

Note that if m is infinite dimensional, $\widehat{R}_1 = R_1$, $R_2 = u_0^* R_3^*$, $R_4 = u_0^* R_1^*$.

4 Structure of suboptimal controllers

For implementation purposes it can important to identify the finite and infinite dimensional parts of a controller. This requires a careful study of the controller structure. In this section we will study the structure of all suboptimal H^∞ controllers which can be obtained from the formulae given in Section 3.

Recall that the set of all $q_1 \in H^\infty$ satisfying $\|m^* u - q_1\| \le 1$ is given by Theorem 3.2 as

$$m^* u - q_1 = f_{\varepsilon,\rho} = \frac{\varepsilon p^* + q^*}{p + \varepsilon q},$$

where $\varepsilon \in \mathcal{B}$ is the free parameter. That is

$$q_1 = \frac{m^* u (p + \varepsilon q) - \varepsilon p^* - q^*}{p + \varepsilon q},$$

$$q_1 = \frac{(m^* u_0 - \widehat{u}_0 m_2^*)(p + \varepsilon q) - \varepsilon p^* - q^*}{p + \varepsilon q}.$$

Since $p = R_1 + mR_2$ and $q = R_3 + m u_0^* R_1^*$, we have

$$\begin{aligned}(m^* u_0 - \widehat{u}_0 m_2^*)p - q^* &= (m^* u_0 - \widehat{u}_0 m_2^*)(R_1 + mR_2) - (R_3^* + m^* u_0 R_1) \\ &= (u_0 R_2 - R_3^* - \widehat{u}_0 R_1 m_2^*) - (\widehat{u}_0 m_2^* R_2) m,\end{aligned}$$

Defining $\eta_1 = u_0 R_2 - R_3^* - \widehat{u}_0 R_1 m_2^*$ and $\eta_2 = \widehat{u}_0 m_2^* R_2$, we obtain

$$(m^* u_0 - \widehat{u}_0 m_2^*)p - q^* = \eta_1 + m\eta_2$$

where η_1 and η_2 are rational. On the other hand, we can express p and q as $p = \widehat{R}_1 + m u_0^* R_3^*$ and $q = R_3 + mR_4$. So,

$$\begin{aligned}(u_0 m^* - \widehat{u}_0 m_2^*)q - p^* &= (m^* u_0 - \widehat{u}_0 m_2^*)(R_3 + mR_4) - (\widehat{R}_1^* + m^* \widehat{u}_0 R_3) \\ &= (-\widehat{R}_1^* - \widehat{u}_0 m_2^* R_3 + u_0 R_4) + (\widehat{u}_0 m_2^* R_4) m.\end{aligned}$$

Now defining $\eta_3 = -\widehat{R}_1^* - \widehat{u}_0 m_2^* R_3 + u_0 R_4$ and $\eta_4 = \widehat{u}_0 m_2^* R_4$, we have

$$(u_0 m^* - \widehat{u}_0 m_2^*)q - p^* = \eta_3 + m\eta_4.$$

Therefore,

$$q_1 = \frac{(\eta_1 + m\eta_2) + (\eta_3 + m\eta_4)\varepsilon}{(R_1 + mR_2) + (R_3 + mR_4)\varepsilon}. \tag{24}$$

Recall that $\widehat{q} = f_\rho q_1$, where f_ρ is invertible in $\mathbb{R}H^\infty$ and defined in Section 2. Hence

$$\widehat{q} = \frac{(\nu_1 + m\nu_2) + (\nu_3 + m\nu_4)\varepsilon}{(R_1 + mR_2) + (R_3 + mR_4)\varepsilon}, \tag{25}$$

where $\nu_i = f_\rho \eta_i$ for $i = 1, .., 4$. Note that ν_i's are rational.

Remark 4.1: If we define $F_\varepsilon = (\eta_1 + m\eta_2) + (\eta_3 + m\eta_4)\varepsilon$, from (24) we obtain

$$q_1 = \frac{F_\varepsilon}{p + \varepsilon q}$$

where F_ε has at most finitely many poles in D for all fixed $\varepsilon \in \mathcal{B}$ and the equality holds for $z \in$ T. We will prove that this holds for all $z \in D$. Since $F_\varepsilon = q_1(p + \varepsilon q)$, we have $F_\varepsilon \in L^2$. Let W be the (monic) polynomial of minimal order such that $WF_\varepsilon \in H_2$. Then, $W(p+\varepsilon q)q_1 = WF_\varepsilon$ for $z \in D$, because if two H_2 functions agree on the boundary of the unit disc, then they agree in the unit disc. Let w be a root of W, then by the choice of W, $w \in D$ and $W(w)F_\varepsilon(w) \neq 0$, but

$$W(w) \ \ (p + \varepsilon q)(w) \ \ q_1(w) = 0$$

and this contradicts with $W(p+\varepsilon q)q_1 = WF_\varepsilon$ for $z \in D$, so W has no root in D, $W = 1$. Hence,

$$(p + \varepsilon q)q_1 = F_\varepsilon \ \text{ for all } \ z \in D.$$

Furthermore $p + \varepsilon q$ is outer (see [1]), so

$$q_1 = \frac{F_\varepsilon}{p + \varepsilon q} \text{ for all } z \in D. \qquad \square$$

This completes the solution of the suboptimal H^∞ control problem. The relation between $\widehat{q}$ and the controller C is given in [14]. The set of all C_{subopt}'s can be written as a LFT on $\widehat{q}$, (see [14])

$$C_{subopt} = N_o^{-1} D_p \frac{A_1 + A_2\widehat{q}}{A_3 + A_4\widehat{q}} \tag{26}$$

where D_p, A_1, A_2 are rational and $A_3, A_4 \in \mathbb{R}(z) + m(z)\mathbb{R}(z)$, (see Appendix C for explicit formulae of A_i's). The suboptimal controller C_{subopt} is N_o^{-1} times a LFT on $\widehat{q}$, and $\widehat{q}$ is a LFT on ε, so

$$C_{subopt} = N_o^{-1} \frac{a_1(z) + a_2(z)\varepsilon}{a_3(z) + a_4(z)\varepsilon}$$

where and $a_i(z)$'s are in $\mathbb{R}(z) + m(z)\mathbb{R}(z)$ (see Appendix C for expilicit formulae).

Now, we will restrict our attention to infinite dimensional stable plants case, and show that the above structure agrees with the structure obtained in [12]. In this case, $\widehat{u}_0 = 0, m_2 = 1$ and m is infinite dimensional. From Section 3, we know that

$$p = R_1 + (u_0^* R_3^*)m, \quad q = R_3 + (u_0^* R_1^*)m$$

and

$$q_1 = \frac{(u_0 R_2 - R_3^*) + \varepsilon(-R_1^* + u_0 R_4)}{R_1 + (u_0^* R_3^*)m + \varepsilon(R_3 + u_0^* R_1^* m)}.$$

Now, defining

$$G_\varepsilon = \frac{u_0^* R_3^* + u_0^* R_1^* \varepsilon}{R_1 + R_3 \varepsilon}$$

and

$$G'_\varepsilon = \frac{(u_0 R_2 - R_3^*) + \varepsilon(-R_1^* + u_0 R_4)}{R_1 + R_3 \varepsilon},$$

we get

$$q_1 = \frac{G'_\varepsilon}{1 + m G_\varepsilon}.$$

But

$$\frac{(u_0 R_2 - R_3^*)}{u_0^* R_3^*} = \frac{(-R_1^* + u_0 R_4)}{u_0^* R_1^*} =: R_r,$$

where R_r is rational, so

$$G'_\varepsilon = R_r G_\varepsilon$$

and

$$q_1 = R_r \frac{G_\varepsilon}{1 + mG_\varepsilon}.$$

In [12], it was shown that for infinite dimensional stable plants the suboptimal H^∞ controllers have the following structure:

$$C_{subopt} = N_o^{-1} \frac{\widehat{q}}{h - m\widehat{q}}$$

where h is rational (see [14] for the definition of h). But since $\widehat{q} = f_\rho q_1$,

$$C_{subopt} = N_o^{-1} f_\rho R_r \frac{G_\varepsilon}{h(1 + mG_\varepsilon) - m(f_\rho R_r G_\varepsilon)},$$

$$C_{subopt} = N_o^{-1} h_c \frac{H_\varepsilon}{1 + mH_\varepsilon} \quad , \tag{27}$$

where $h_c = f_\rho R_r/(h - f_\rho R_r)$ is rational, $H_\varepsilon = G_\varepsilon(1 - f_\rho R_r/h)$. Note that H_ε is a LFT on ε with rational coefficients. Hence, (27) gives the structure of suboptimal controllers in the stable case. This structure is the same as the result obtained in [12]. Note that, in the stable case, m does not appear in the numerator.

5 Concluding Remarks

In this paper, we have combined the ideas and observations given in [6], [13], [14] and obtained a solution to the 2-block H^∞ suboptimal control problem for infinite dimensional systems with finitely many unstable poles. We have shown that the suboptimal controllers can be obtained by solving a set of finitely many linear equations. A computer program for solving these equations can be developed by combining the program of [17] (which constructs the matrix Θ of Appendix E, and the matrices of Appendix D) with a program which implements the projection formulae of Appendix B and the matrix C_μ of Appendix E.

In Section 4 we have studied the structure of all suboptimal H^∞ controllers. In the implementation of a controller it can be useful to identify its finite and infinite dimensional parts, [12]. We have showed that the set of all suboptimal H^∞ controllers are of the form N_o^{-1} times a LFT on a free parameter in the unit ball of H^∞, where the coefficients of this LFT are in $\mathbb{R}(z) + m\mathbb{R}(z)$, and N_o is the outer part of the numerator of the plant. Moreover, the rational coefficients in this LFT can be computed explicitly. For the 1-block suboptimal control problem of stable plants, we have verified the controller structure observed in [12].

Appendix A: Notation and Some Remarks

Here we introduce the basic notation and definitions which are used throughout the paper; and make some remarks regarding the transfer function models we consider.

$\mathbb{R}$: Real numbers,
$\mathbb{C}$: Complex numbers,
RHP: open right half plane in $\mathbb{C}$, $\{s \in \mathbb{C} \; : \; \text{Re } s > 0\}$,
D: open unit disc, $\{z \in \mathbb{C} \; : \; |z| < 1\}$,
$\mathbb{T}$: unit circle, $\{\zeta \in \mathbb{C} \; : \; |\zeta| = 1\}$,
$\mathbb{R}[z]$: polynomial functions of z with real coefficients,
$\mathbb{R}(z)$: rational functions of z with real coefficients,
L^∞: Banach space of essentially bounded functions on $\mathbb{T}$,
H^∞: L^∞ functions which admit bounded analytical extensions to D,
$H^\infty(RHP)$: Bounded analytic functions on RHP,
$\mathbb{R}H^\infty$: Real rational functions in H^∞,
L^2: Hilbert space of square integrable functions on $\mathbb{T}$,
H^2: L^2 functions which admit analytical extensions to D,
$\|G\|_n$: norm of G, when $G \in L^n$, H^n, $n = 2$, or ∞.

For a function $f \in L^2$, we define $f^*(z) := \overline{f(1/\overline{z})}$.

$$\text{If } f(z) = \sum_{k=0}^{\infty} a_k z^k \; , \text{ then } f^*(z) = \sum_{k=0}^{\infty} \overline{a_k} z^{-k}.$$

With this definition, if $f(z)$ is analytic in the unit disc D, then $f^*(z)$ is analytic in the complement of the unit disc, moreover $f^*(z) = \overline{f(z)}$ for $z \in \mathbb{T}$. We also define $\overline{f}(z) := f^*(z^{-1})$, in this case,

$$\overline{f}(z) = \sum_{k=0}^{\infty} \overline{a_k} z^k.$$

and if $f(z)$ is analytic in D, then $\overline{f}(z)$ is analytic in D too.

We would like to note that in this paper the systems are represented by their transfer functions, which are functions of the Laplace transform variable $s \in RHP$ (in the case of continuous time systems) or functions of the Z-transform variable $z \in D$ (for discrete time systems). Our solution to the suboptimal H^∞ control problem will be derived using functions defined on the unit disc ($z-$plane). This does not limit us to discrete time systems, since we can find a conformal map between the right half plane (RHP) and the unit disc (D). A simple example of such a map is

$$z = \frac{s-1}{s+1}\ , \qquad s = \frac{1+z}{1-z},$$

where $s \in RHP$ and $z \in D$. This conformal map transforms every point in RHP to a unique point in D and vice versa, the imaginary axis (boundary of RHP) is mapped to the unit circle (boundary of D). In particular, for the above example the points $j\infty$ and 0 in the $s-$plane are mapped to the points 1 and -1 in the $z-$plane.

Any function $F \in H^\infty(RHP)$ defined on RHP can be represented in terms of a function $f \in H^\infty$, and vice versa:

$$f(z) = F(\frac{1+z}{1-z}) \qquad \text{and} \qquad F(s) = f(\frac{s-1}{s+1}).$$

The conformal map between RHP and D preserves all the important properties of $F(s)$ as a bounded analytic function: e.g. $f(z)$ is a bounded analytic function on D and

$$\|F\|_\infty = \text{ess} \sup_{\omega \in \mathrm{R}} |F(j\omega)| = \text{ess} \sup_{\theta \in [0,2\pi]} |f(e^{j\theta})| = \|f\|_\infty.$$

In view of the above facts we can transform the problem data from RHP to D. For example if $P(s)$ represents the transfer function of the plant, it can also be represented by $p(z) = P(\frac{1+z}{1-z})$, as a function defined on the unit disc. Conversely, if the controller is given as a function of z, i.e. $c(z)$, then, its transfer function can be obtained from the inverse map, i.e. $C(s) = c(\frac{s-1}{s+1})$.

Appendix B: Some projection formulae

Now, we would like to present the formulae for computing projections of the form $\mathbf{P}_{H_2}(t_1 + mt_2)$ and $\mathbf{P}_{H_2}(t_1 + m^*t_2)$ where $t_1(z)$ and $t_2(z)$ are rational, $m(z)$ is inner and the arguments of $\mathbf{P}_{H_2}$ are in L^2, i.e. $t_1(z) + m(z)t_2(z) \in L^2$, and $t_1(z) + m^*(z)t_2(z) \in L^2$.

Note that both of these projections can be considered as a function of two variables where each variable is a rational function.

If $t_1(z) = A(z)/C(z)$, $t_2(z) = B(z)/C(z)$ with $A(z), B(z), C(z) \in \mathbb{R}[z]$, and $C(z) = \prod_{i=1}^{N}(z - z_i)$, then we define

$$C_+(z) = \prod_{|z_i|<1} (z - z_i) \quad \text{and} \quad C_-(z) = \prod_{|z_i|\geq 1} (z - z_i).$$

After reindexing, we may assume that $z_1, ..., z_M$ in D and $z_{M+1}, ..., z_N$ are not in D. Let $\widehat{D}$ be an unknown polynomial of degree at most $M - 1$. Then

$$t_1(z) + m(z)t_2(z) = \frac{A(z) + m(z)B(z)}{C(z)}$$

$$= \frac{A(z) + m(z)B(z) - \widehat{D}(z)C_-(z)}{C(z)} + \frac{\widehat{D}(z)}{C_+(z)}.$$

But $\widehat{D}(z)/C_+(z) \in H_2^{\perp}$ for all choices of $\widehat{D}$. Also

$$\frac{A(z) + m(z)B(z) - \widehat{D}(z)C_-(z)}{C(z)} = \frac{A(z) + m(z)B(z)}{C(z)} - \frac{\widehat{D}(z)}{C_+(z)} \in L^2.$$

If $\widehat{D}(z)$ satisfies $(\widehat{D}C_-)^{(l)}(z_j) = (A + mB)^{(l)}(z_j)$ for all $1 \leq j \leq M$ and $0 \leq l \leq L_j - 1$ where L_j is the multiplicity of the pole at z_j, then $(A - mB - \widehat{D}C_-)/C \in H_2$. By Leibnitz's formula, we obtain

$$\sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array} \right) \widehat{D}^{(k)}(z_j)C_-^{(l-k)}(z_j) = A^{(l)}(z_j) + \sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array} \right) m^{(k)}(z_j)B^{(l-k)}(z_j).$$

Since $C_-(z_j) \neq 0$, one can solve for $\widehat{D}^{(l)}(z_j)$ and given these values, one can construct a $\widehat{D}(z) \in \mathbb{R}[z]$ of degree at most $M - 1$, satisfying these interpolation conditions. If $C_-(z)$ has no multiple roots, then $\widehat{D}(z)$ can be constructed using Lagrange interpolation, i.e. construct a polynomial $\widehat{D}(z)$, of degree at most $M - 1$ such that

$$\widehat{D}(z_j) = \frac{A(z_j) + m(z_j)B(z_j)}{C_-(z_j)}.$$

It is clear that,

$$\mathbf{P}_{H_2}(t_1(z)+m(z)t_2(z)) = \frac{A(z)+m(z)B(z)-\widehat{D}(z)C_-(z)}{C(z)}$$

and

$$\mathbf{P}_{H_2^\perp}(t_1(z)+m(z)t_2(z)) = \frac{\widehat{D}(z)C_-(z)}{C(z)} = \frac{\widehat{D}(z)}{C_+(z)}.$$

For the computation of $\mathbf{P}_{H_2}(t_1+m^*t_2)$, we assume that $t_1(z)+m^*(z)t_2(z) \in L^2$. Let

$$\widehat{A}(z) = \mathbf{P}_{H_2^\perp}(t_1^*+mt_2^*) \in \mathrm{I\!R}(z).$$

which can be computed by the methods given above. Then,

$$t_1^*+mt_2^* = [(t_1^*-\widehat{A})+mt_2^*]+\widehat{A}$$

where $\widehat{A} \in H_2^\perp$ and $[(t_1^*-\widehat{A})+mt_2^*] \in H_2$. Define $c_0 := [(t_1^*-\widehat{A})+mt_2^*]|_{z=0}$, then

$$t_1^*+mt_2^* = (t_1^*-\widehat{A}+mt_2^*-c_0)+(c_0+\widehat{A})$$

where $(c_0+\widehat{A}) \in \mathrm{I\!R}+H_2^\perp$, and $(t_1^*-\widehat{A}+mt_2^*-c_0) \in H_2$ which vanishes at $z=0$. Therefore,

$$t_1+m^*t_2 = (t_1-\widehat{A}^*+m^*t_2-c_0^*)+(c_0^*+\widehat{A}^*)$$

where $(c_0^*+\widehat{A}^*) \in H_2$ and $(t_1-\widehat{A}^*+m^*t_2-c_0^*) \in H_2^\perp$. So

$$\mathbf{P}_{H_2}(t_1+m^*t_2) = c_0^*+\widehat{A}^* \quad \text{and} \quad \mathbf{P}_{H_2^\perp}(t_1+m^*t_2) = (t_1-\widehat{A}^*-c_0^*)+m^*t_2.$$

If $(t_1^*-\widehat{A})$ or t_2^* has a pole at $z=0$, then

$$(t_1^*-\widehat{A}) = z^{-k}N_1(z) \quad \text{and} \quad t_2^* = z^{-k}N_2(z)$$

with N_1 and N_2 are rational and analytic at $z=0$. In this case, $(t_1^*-\widehat{A})+mt_2^* = z^{-k}(N_1+mN_2)$, hence

$$c_0 = \frac{1}{k!}\{\frac{d^k}{dz^k}(N_1+mN_2)\}|_{z=0}$$

If both $(t_1^* - \widehat{A})|_{z=0}$ and $t_2^*|_{z=0}$ are finite, then $c_0 = (t_1^* - \widehat{A})|_{z=0} \ + \ m(0)\ t_2^*|_{z=0}$.

Note that both $\mathbf{P}_{H_2^\perp}(t_1 + mt_2)$ and $\mathbf{P}_{H_2}(t_1 + m^* t_2)$ are rational.

Appendix C: $a_1, a_2, a_3, a_4 \in \mathbb{R}(z) + m\mathbb{R}(z)$

If m is rational, there is nothing to prove. Now, assume that m is infinite dimensional. From [14] we have

$$\begin{aligned}
A_1 &= \widehat{w}_0 \\
A_2 &= m_d \\
A_3 &= G - m_n \widehat{w}_0 \\
A_4 &= m_n m_d.
\end{aligned}$$

Now, by substituting (25) in (26), we get

$$\begin{aligned}
a_1 &= D_p(A_1 \iota_3 + A_2 \iota_1) \\
a_2 &= D_p(A_1 \iota_4 + A_2 \iota_2) \\
a_3 &= A_3 \iota_3 + A_4 \iota_1 \\
a_4 &= A_3 \iota_4 + A_4 \iota_2
\end{aligned}$$

where

$$\begin{aligned}
\iota_1 &= \nu_1 + m\nu_2 \\
\iota_2 &= \nu_3 + m\nu_4 \\
\iota_3 &= R_1 + mR_2 \\
\iota_4 &= R_3 + mR_4.
\end{aligned}$$

Since $\nu_i = f_\rho \eta_i$, we have

$$\begin{aligned}
\iota_1 &= \nu_1 + m\nu_2 = f_\rho(\eta_1 + m\eta_2) \\
\iota_2 &= \nu_3 + m\nu_4 = f_\rho(\eta_3 + m\eta_4).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_1(z) &= D_p(A_1 \iota_3 + A_2 \iota_1) \\
&= D_p \widehat{w}_0 (R_1 + mR_2) + D_p m_d f_\rho (\eta_1 + m\eta_2)
\end{aligned}$$

$$
\begin{aligned}
&= D_p(m_d f_\rho \eta_1 + \widehat{w}_0 R_1) + m D_p(m_d f_\rho \eta_2 + \widehat{w}_0 R_2) \\
a_2(z) &= D_p(A_1 \iota_4 + A_2 \iota_2) \\
&= D_p \widehat{w}_0 (R_3 + m R_4) + D_p m_d f_\rho (\eta_3 + m \eta_4) \\
&= D_p(m_d f_\rho \eta_3 + \widehat{w}_0 R_3) + m D_p(m_d f_\rho \eta_4 + \widehat{w}_0 R_4) \\
a_3(z) &= A_3 \iota_3 + A_4 \iota_1 \\
&= (G - m_n \widehat{w}_0)(R_1 + m R_2) + m_n m_d f_\rho (\eta_1 + m \eta_2) \\
&= (G R_1 - m_n \widehat{w}_0 R_1) + m(G R_2 + m_d f_\rho \eta_1 m_n / m) \\
a_4(z) &= A_3 \iota_4 + A_4 \iota_2 \\
&= (G - m_n \widehat{w}_0)(R_3 + m R_4) + m_n m_d f_\rho (\eta_3 + m \eta_4) \\
&= (G R_3 - m_n \widehat{w}_0 R_3) + m(G R_4 + m_d f_\rho \eta_3 m_n / m)
\end{aligned}
$$

Note that $m_n/m = 1/m_2 = 1/m_d$ is rational. So, a_i's are in $\mathbb{R}(z) + m\mathbb{R}(z)$.

Appendix D: Explicit formulae of $T(z), T_v(z), T_u(z), T_q(z)$

In this appendix, we give explicit formulae of $T(z), T_v(z), T_u(z)$ and $T_q(z)$, (see [13] for proofs).

Note that $\Phi = [\Phi_1, ..., \Phi_{3n}, \alpha_1, ..., \alpha_\ell, \beta_1, ..., \beta_\ell]^T$, let $\gamma_- = [\gamma_{-n}, ..., \gamma_{-1}]^T = [\Phi_1, ..., \Phi_n]^T$, $\gamma_+ = [\gamma_0, ..., \gamma_{n-1}]^T = [\Phi_{n+1}, ..., \Phi_{2n}]^T$, $\delta = [\delta_0, ..., \delta_{n-1}]^T = [\Phi_{2n+1}, ..., \Phi_{3n}]^T$. Recall that $a_1, ..., a_\ell$ are zeros of $m_2(z)$ and we defined $f_i(z) = 1/(1 - \overline{a_i} z)$. We define the following matrices as in [13],

$$
\widehat{P} := \begin{bmatrix} P_{-n} & 0 & 0 \\ \vdots & \ddots & 0 \\ P_{-1} & \cdots & P_{-n} \end{bmatrix}, \widehat{Q} := \begin{bmatrix} Q_{-n} & 0 & 0 \\ \vdots & \ddots & 0 \\ Q_{-1} & \cdots & Q_{-n} \end{bmatrix},
$$

where $P(z, z^{-1}) = P_{-n} z^{-n} + \cdots + P_0 + \cdots + P_n z^n$ and $Q(z, z^{-1}) = Q_{-n} z^{-n} + \cdots + Q_0 + \cdots + Q_n z^n$,

$$
\widehat{B} := \begin{bmatrix} B_n^* & 0 & 0 \\ \vdots & \ddots & 0 \\ B_1^* & \cdots & B_n^* \end{bmatrix}, \widehat{C} := \begin{bmatrix} C_n^* & 0 & 0 \\ \vdots & \ddots & 0 \\ C_1^* & \cdots & C_n^* \end{bmatrix},
$$

where $B(z) = B_n z^n + \cdots + B_0$ and $C(z) = C_n z^n + \cdots + C_0$,

$$\tilde{B}^* := \begin{bmatrix} B_0^* & 0 & 0 \\ \vdots & \ddots & 0 \\ B_{n-1}^* & \cdots & B_0^* \end{bmatrix}, \tilde{C}^* := \begin{bmatrix} C_0^* & 0 & 0 \\ \vdots & \ddots & 0 \\ C_{n-1}^* & \cdots & C_0^* \end{bmatrix},$$

$$M_1 := \begin{bmatrix} (m_1)_0 & 0 & 0 \\ \vdots & \ddots & 0 \\ (m_1)_{n-1} & \cdots & (m_1)_0 \end{bmatrix}, M := \begin{bmatrix} (m)_0 & 0 & 0 \\ \vdots & \ddots & 0 \\ (m)_{n-1} & \cdots & (m)_0 \end{bmatrix},$$

where $m_1(z) = (m_1)_0 + (m_1)_1 z + \cdots + (m_1)_{n-1} z^{n-1} + \cdots$ and $m(z) = (m)_0 + (m)_1 z + \cdots + (m)_{n-1} z^{n-1} + \cdots$,

$$M_d := \begin{bmatrix} (m_d)_0 & 0 & 0 \\ \vdots & \ddots & 0 \\ (m_d)_{n-1} & \cdots & (m_d)_0 \end{bmatrix}, A_- := \begin{bmatrix} a_1^{-n} & \cdots & a_1^{-1} \\ \vdots & & \vdots \\ a_\ell^{-n} & \cdots & a_\ell^{-1} \end{bmatrix},$$

where $a_1, ..., a_\ell$ are zeros of $m_2(z)$,

$$A_+ := \begin{bmatrix} 1 & \cdots & a_1^{n-1} \\ \vdots & & \vdots \\ 1 & \cdots & a_\ell^{n-1} \end{bmatrix}, D_B := \begin{bmatrix} B(a_1) & 0 & 0 \\ 0 & \ddots & 0 \\ B(a_\ell) & \cdots & B(a_1) \end{bmatrix},$$

$$D_C := \begin{bmatrix} C(a_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C(a_\ell) \end{bmatrix}, D_P := \begin{bmatrix} P(a_1, a_1^{-1}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P(a_\ell, a_\ell^{-1}) \end{bmatrix},$$

$$D_m := \begin{bmatrix} m_1(a_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_1(a_\ell) \end{bmatrix}, \Lambda := \begin{bmatrix} f_1(a_1) & \cdots & f_\ell(a_1) \\ \vdots & & \vdots \\ f_1(a_\ell) & \cdots & f_\ell(a_\ell) \end{bmatrix}.$$

Now define

$$K_{C^*B}(z) := (C_0^* B_0 + \cdots + C_n^* B_n) + (C_0^* B_1 + \cdots + C_{n-1}^* B_n) z + \cdots$$

$$(C_0^* B_{n-1} + C_1^* B_n) z^{n-1} + (C_0^* B_n) z^n,$$

and

$$D_{C^*B} := \begin{bmatrix} K_{C^*B}(a_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_{C^*B}(a_\ell) \end{bmatrix}.$$

As in [13], we write

$$\begin{aligned} V_-(z) &:= [z^n, \cdots, z^{-1}], \\ V_+(z) &:= [1, \cdots, z^{n-1}], \\ F(z) &:= [f_1(z), \cdots, f_\ell(z)]. \end{aligned}$$

Then

$$T(z) = [\Psi_{\gamma_-}(z) \vdots \Psi_{\gamma_+}(z) \vdots \Psi_\delta(z) \vdots \Psi_\alpha(z) \vdots \Psi_\beta(z)]$$

where

$$\begin{aligned} \Psi_{\gamma_+}(z) &:= -V_-(z)\hat{P}, \\ \Psi_\delta(z) &:= -V_-(z)\hat{Q}M, \\ \Psi_\beta(z) &:= (F(z)D_C^* - m_1(z)F(z)D_B^* - m_1(z)V_-(z)\hat{B}A_+^* \\ &\qquad +V_-(z)M_1\hat{B}A_+^*)\Lambda^{-1}D_C, \\ \Psi_{\gamma_-}(z) &:= (V_+(z)(C^*(z^{-1})m_d(z) - B^*(z^{-1})m(z)) - V_-(z)(\hat{C}M_d - \hat{B}M))\hat{B}^*M_d^* \\ &\qquad -V_+(z)\tilde{C}\hat{B}^* \end{aligned}$$

and

$$\begin{aligned} \Psi_\alpha(z) &:= (-V_+(z)(C^*(z^{-1})m_d(z) + B^*(z^{-1})m(z)) + V_-(z)(\hat{C}M_d - \hat{B}M))\hat{B}^*M_d^*A_-^* \\ &\qquad -F(z)(K_{C^*B}(z)\mathrm{I}_{\ell\times\ell} - K_{B^*C}^*(0)\mathrm{I}_{\ell\times\ell} + D_{B^*C}^*). \end{aligned}$$

$$T_v(z) = [\Psi_{\gamma_-}^v(z) \vdots \Psi_{\gamma_+}^v(z) \vdots \Psi_\delta^v(z) \vdots \Psi_\alpha^v(z) \vdots \Psi_\beta^v(z)]$$

where

$$\begin{aligned} \Psi_{\gamma_-}^v(z) &:= V_+(z)(\hat{P}^* - \tilde{B}\hat{B}^*)M_d^*, \\ \Psi_{\gamma_+}^v(z) &:= [0, \ldots, 0]_{1\times n}, \\ \Psi_\delta^v(z) &:= -V_-(z)\hat{Q}, \\ \Psi_\alpha^v(z) &:= V_+(z)(\tilde{B}\hat{B}^* - \hat{P}^*)M_d^*A_-^*, \\ \Psi_\beta^v(z) &:= [0, \ldots, 0]_{1\times\ell}. \end{aligned}$$

$$T_u(z)\Phi = T(z)\Phi - m(z)T_v(z)\Phi$$

$$T_q(z) = \left[\Psi^q_{\gamma_-}(z) \vdots \Psi^q_{\gamma_+}(z) \vdots \Psi^q_{\delta}(z) \vdots \Psi^q_{\alpha}(z) \vdots \Psi^q_{\beta}(z)\right]$$

where

$$\begin{aligned}\Psi^q_{\gamma_-}(z) = \quad & V_+(z)\hat{P}^* - m_d(z)V_+(z)\hat{P}^* M_d^* + m_d(z)V_+(z)\tilde{B}\hat{B}^* M_d^* \\ & +(V_-(z)\hat{B}M_d - V_+(z)m_d(z)B^*(z^{-1}))\hat{B}^* M_d^* - F(z)D_m^* D_C^* \Lambda^{-1} A_+ \hat{B}^*,\end{aligned}$$

$$\Psi^q_{\gamma_+}(z) = [0, \ldots, 0]_{1\times n},$$

$$\Psi^q_{\delta}(z) = (V_-(z)m_d(z) - V_-(z)M_d)\hat{Q},$$

$$\begin{aligned}\Psi^q_{\alpha}(z) = \quad & -V_-(z)\hat{P}A_+^* + m_d(z)V_+(z)\hat{P}^* M_d^* A_-^* - m_d(z)V_+(z)\tilde{B}\hat{B}^* M_d^* A_-^* \\ & +(V_+(z)m_d(z)B^*(z^{-1}) - V_-(z)\hat{B}M_d)\hat{B}^* M_d^* A_-^* - F(z)D_m^* D_C^* D_B\end{aligned}$$

and

$$\Psi^q_{\beta}(z) = -F(z)D_B^*\Lambda^{-1}D_C + F(z)D_m^* D_C^* \Lambda^{-1} D_C.$$

Appendix E: Explicit formulae of Θ and C_μ

We can combine the equations (15), (18), (19), (21) and (22) in a single matrix form as $\Theta\Phi = C_\mu$, where Θ and C_μ are as follows. First set

$$\Theta_1 := \begin{bmatrix} T_q(a_1) \\ \vdots \\ T_q(a_\ell) \end{bmatrix} + \begin{bmatrix} 0 & \ldots & 0 & D_P\Lambda & 0 \end{bmatrix}_{\ell\times(3n+2\ell)},$$

$$\Theta_2 := \begin{bmatrix} T_u(a_1) \\ \vdots \\ T_u(a_\ell) \end{bmatrix} + \begin{bmatrix} 0 & \ldots & 0 & D_P \end{bmatrix}_{\ell\times(3n+2\ell)},$$

$$\Theta_3 := \begin{bmatrix} T_u(z_{n+1}) \\ \vdots \\ T_u(z_{n+r}) \\ \overline{m}_v(z_{n+r+1}^{-1})T_u(z_{n+r+1}) \\ \vdots \\ \overline{m}_v(z_{3n}^{-1})T_u(z_{3n}) \end{bmatrix},$$

$$\Theta_4 := \begin{bmatrix} T_v(z_1) \\ \vdots \\ T_v(z_n) \end{bmatrix}.$$

Then we have

$$\Theta := \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{bmatrix}.$$

Now for C_μ we define

$$C_{\mu 1} := \begin{bmatrix} \lambda_{u,1}^R(a_1) \\ \vdots \\ \lambda_{u,1}^R(a_1) \end{bmatrix},$$

$$C_{\mu 2} := \begin{bmatrix} \lambda_u^R(a_1) \\ \vdots \\ \lambda_u^R(a_1) \end{bmatrix},$$

$$C_{\mu 3} := \begin{bmatrix} \lambda_u(z_{n+1}) \\ \vdots \\ \lambda_u(z_{n+r}) \\ \overline{m}(z_{n+r+1})\lambda_u(z_{n+r+1}) \\ \vdots \\ \overline{m}(z_{3n})\lambda_u(z_{3n}) \end{bmatrix},$$

$$C_{\mu 4} := \begin{bmatrix} \lambda_u^R(z_1) \\ \vdots \\ \lambda_u^R(z_n) \end{bmatrix}.$$

Then,

$$C_\mu := \begin{bmatrix} C_{\mu 1} \\ C_{\mu 2} \\ C_{\mu 3} \\ C_{\mu 4} \end{bmatrix}.$$

References

[1] V. M. Adamjan, D. Z. Arov, and M. G. Krein, "Analytic properties of Schmidt pairs for a Hankel operator and generalized Shur–Takagi problem," *Math. USSR Sbornik* **15** (1971), pp. 31–73.

[2] H. Bercovici, C. Foias, and A. Tannenbaum, "On skew Toeplitz operators I," *Operator Theory: Advances and Applications* **32** (1988), pp. 21–43.

[3] Flamm, D.S., "Outer Factor 'Absorption' for H^∞ Control Problems, ISS Report No.55, July 31, 1990, Dept. of Elec. Eng., Princeton University, 1990.

[4] Foias, C. and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birkhäuser, Basel, 1990.

[5] C. Foias, H. Özbay and A. Tannenbaum, "Remarks on H^∞ optimization of multivariate distributed systems," IEEE Conference on Decision and Control, Austin, Texas, December 1988, pp. 985–986.

[6] C. Foias and A. Tannenbaum, "On the parametrization of the suboptimal solutions in generalized interpolation," *Linear Algebra and its Applications*, **122/123/124** (1989), pp. 145–164.

[7] C. Foias and A. Tannenbaum, "On the four block problem, I," *Operator Theory: Advances and Applications* **32** (1988), pp. 93–112.

[8] C. Foias and A. Tannenbaum, "On the four block problem, II : the singular system," *Operator Theory and Integral Equations* **11** (1988), pp. 726–767.

[9] C. Foias, A. Tannenbaum, and G. Zames, "Some explicit formulae for the singular values of a certain Hankel operators with factorizable symbol," *SIAM J. Math. Analysis* **19** (1988), pp. 1081–1091

[10] Doyle, J., K. Glover, P. P. Khargonekar, and B. Francis, "State space solutions to standard H^2 and H^∞ control problems," *IEEE Transactions on Automatic Control*, **26** (1981), pp. 4-16.

[11] H. Özbay and A. Tannenbaum, "A skew Toeplitz approach to the H^∞ optimal control of multivariable distributed systems," *SIAM J. Control and Optimization*, **28** (1990) pp. 653–670.

[12] H. Özbay and A. Tannenbaum, "On the structure of suboptimal H^∞ controllers in the sensitivity minimization problem for distributed stable plants," *Automatica*, March 1991, vol. 27, No. 2, pp. 293–305.

[13] H. Özbay, M. C. Smith and A. Tannenbaum, "Mixed sensitivity optimization for a class of unstable infinite dimensional systems," to appear in *Linear Algebra and its Applications*. A short version of the paper, under the title "Controller design for unstable distributed plants," appears in the Proc. of the American Control Conference, San Diego CA, May 1990, pp. 1583–1588.

[14] H. Özbay, M. C. Smith and A. Tannenbaum, "On the optimal two block H^∞ compensators for distributed unstable plants," Proceedings of the American Control Conference, Chicago IL, June 1992, pp. 1865–1869

[15] H. Özbay, "A simpler formula for the singular values of a certain Hankel operator, *Systems and Control Letters*, vol. 15, No. 5, (1990) pp. 381–390.

[16] M. C. Smith, "On stabilization and existence of coprime factorizations," *IEEE Transactions on Automatic Control*, 1989, pp. 1005–1007.

[17] H. Tu, *An H^∞ Optimization Method and Matlab Program for Linear Distributed Systems*, M.S. Thesis, University of Minnesota–Duluth, 1992.