ANALYTICITY OF SOLUTIONS OF THE GENERALIZED KORTEWEG-DE VRIES EQUATION WITH RESPECT TO THEIR INITIAL VALUES

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Abstract

It is shown that the initial value problem (IVP) of the generalized KdV equation

$$\partial_t u + \partial_x(a(u)) + \partial_x^3 u = 0, \qquad u(x,0) = \phi(x)$$

is well posed in the classical Sobolev space $H^s(R)$ with $s > 3/4$, which thus establishes a nonlinear map K from $H^s(R)$ to $C([-T,T]; H^s(R))$. Then it is proved that (i) if $a = a(x)$ is a C^∞ function on R to R, then K is infinitely many times Frechet differentiable; (ii) if $a = a(x)$ is a polynomial, then K is analytic, i.e. for any $\phi \in H^s(R)$, K has a Taylor series expansion

$$K(\phi + h) = \sum_{n=0}^{\infty} \frac{1}{n!} K^{(n)}(\phi)[h^n]$$

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Each term $y_n = K^{(n)}[h^n]$ in the series solves a linearized KdV equation. Thus any "small" perturbation $K(\phi+h)$ of $K(\phi)$ can be obtained by solving a series of linear problems.

The proof of these results relies on various smoothing properties of the associated linear KdV equation.

1 Introduction

This paper is mainly concerned with the initial value problem (IVP) for the generalized Korteweg-de Vries (KdV) equation:

$$\begin{cases} \partial_t u + \partial_x(a(u)) + \partial_x^3 u = 0, & t,\ x \in R \\ u(x,0) = \phi(x) \end{cases} \tag{1.1}$$

in which $a(x)$ is assumed to be a C^∞ function on R to R , though a weaker differentiablity suffices for most results below.

The KdV equation (i.e., $a(u) = u^2/2$ in (1.1)) and its generalized form (1.1) have been studied by many authors [1, 2, 4 - 6, 10 - 17, 19 - 23, 27 - 37, 42 - 44, 47]. For a complete list of references, see, for example, [13, 17, 22, 30]. In particular, it is well-known that the IVP (1.1) is locally well posed in the classical Sobolev space $H^s(R)$ with $s > 3/2$ (cf. [17]) and is globally well posed with some restrictions on $a(u)$ or the size of the initial data ϕ when $s \geq 2$ (see [5], [14], [17], [19] and [28] for the well-posedness of the IVP (1.1) in other function spaces). In the case that $a'(u) = u^k$ in (1.1) with k being a positive integer, Kenig, Ponce and Vega [23] recently proved that the IVP (1.1) is locally well posed in the space $H^s(R)$ with

$$\begin{cases} s > 3/4 & \text{if } k = 1 \\ s \geq 1/4 & \text{if } k = 2 \\ s \geq \frac{1}{12} & \text{if } k = 3 \\ s \geq \frac{k-4}{2k} & \text{if } k \geq 4 \end{cases} \tag{1.2}$$

and is global well-posedness when $1 \leq k \leq 3$ and $s \geq 1$. Their proof is based on careful analysis of various smooth properties of the associated linear problem together with the contraction mapping principle.

In this paper, we first continue to study the well-posedness of the IVP (1.1) in the space $H^s(R)$. While only assuming that $a(0) = a'(0) = 0$, we shall prove that the IVP (1.1) is locally well posed in the space $H^s(R)$ with $s > 3/4$ by using Kenig, Ponce and Vega's argument [22], [23] with a few modification. In order to state our result precisely, we introduce the following Banach spaces as Kenig, Ponce and Vega did in [23].

Let $s > 0$ and $T > 0$ be given. For

$$w: \quad R \times [-T, T] \to R,$$

define

$$\lambda_1(T, w) = \sup_{[-T,T]} \|w(., t)\|_s,$$

$$\lambda_2(T, w) = \left(\sup_x \int_{-T}^{T} |D^s \partial_x w(x, t)|^2 dt \right)^{1/2},$$

$$\lambda_3(T, w; l) = \left(\int_{-T}^{T} \|J^l \partial_x w(., t)\|_\infty^4 dt \right)^{1/4}$$

with $l \in [0, s - 3/4]$ where $J^s = (1 - \partial_x^2)^{s/2}$,

$$\lambda_4(T, w; r) = (1 + T)^{-\rho} \left(\int_R \sup_{[-T,T]} |J^r w(x, t)|^2 dx \right)^{1/2}$$

with $r \in [0, s - 3/4)$ and $\rho > 3/4$ being a fixed constant, and

$$\Lambda_{l,r}^s(T; w) = max \ \{\lambda_1(T, w), \ \lambda_2(T, w), \ \lambda_3(T, w; l), \ \lambda_4(T, w; r)\}. \quad (1.3)$$

Denote by

$$X_{l,r}^{T,s} = \left\{ w \in C([-T, T]; H^s(R)) \mid \Lambda_{l,r}^s(T; w) < \infty \right\} \quad (1.4)$$

with $(l, r) \in [0, s - \frac{3}{4}] \times [0, s - \frac{3}{4})$. This is a Banach space equipped with the norm

$$\|w\|_{X_{l,r}^{T,s}} := \Lambda_{l,r}^s(T; w).$$

Clearly, $X_{l,r}^{T,s}$ is a subspace of $C([-T,T];H^s(R))$ with stronger topology.

We shall prove that if $a(0)=a'(0)=0$, then for any $\phi\in H^s(R)$, there exists a $T>0$ depending only on $\|\phi\|_s$ such that he IVP (1.1) has a unique solution $u\in X_{l,r}^{T,s}$ where $s>3/4$ and $(l,r)\in[0,s-\frac{3}{4}]\times[0,s-\frac{3}{4})$. The global result is also obtained while the same restrictions are enforced on $a(u)$ in (1.1) or the size of the initial value ϕ as those in Kato [17] but with $s\geq 1$ instead of $s\geq 2$.

Thus the IVP (1.1) establishes a nonlinear map K from $H^s(R)$ to $X_{l,r}^{T,s}$ (or $C([-T,T];H^s(R))$). In the second part of this paper, we study differentiability of the map K. It has been known for many years that the map K is continuous from $H^s(R)$ to $C([-T,T];H^s(R))$ [2], [17] and is proved recently by Kenig, Ponce and Vega [23] being Lipschitz continuous in the case that $a'(u)=u^k$. In our early paper [47], we proved the map K corresponding to the classical KdV equation, i.e. $a'(u)=u$ in (1.1), is infinitely many times Frechet differentiable from $H^s(R)$ to $X_{l,r}^{T,s}$ and it has Taylor series expansion at any given $\phi\in H^s(R)$. That is to say, the map K is analytic from $H^s(R)$ to $X_{l,r}^{T,s}$.

In this paper we shall show that the nonlinear map K established by the IVP (1.1) is also infinitely many times Frechet differentiable from $H^s(R)$ to $X_{l,r}^{T,s}$. For any $n\geq 1$, its n-th derivative $K^{(n)}(\phi)$ at $\phi\in H^s(R)$, a n-linear map from the n-fold product space $(H^s(R))^n$ into $X_{l,r}^{T,s}$, can be constructed by solving a system of inhomogeneous linearized KdV equations. More precisely, for any $n\geq 1$ and $h_k\in H^s(R)$ $(k=1,2,...,n)$, denote by

$$w_{[1,...,n]}^{(n)}:=K^{(n)}(\phi)[h_1,...,h_n], \tag{1.5}$$

then it solves

$$\begin{cases} \partial_t w_{[1]}^{(1)}+\partial_x(a'(u)w_{[1]}^{(1)})+\partial_x^3 w_{[1]}^{(1)}=0 \\ w_{[1]}^{(1)}(x,0)=h(x) \end{cases} \tag{1.6}$$

for $n=1$ and

$$\begin{cases} \partial_t w_{[1,...,n]}^{(n)}+\partial_x(a'(u)w_{[1,...,n]}^{(n)})+\partial_x^3 w_{[1,...,n]}^{(n)}=-\partial_x(H_n) \\ w_{[1,...,n]}^{(n)}(x,0)=0 \end{cases} \tag{1.7}$$

for $n \geq 2$ where $u = K(\phi)$ and H_n is a polynomial of $w^{(j)}_{[i_1,...,i_j]}$ with $1 \leq i_1, ..., i_j \leq n$ and $1 \leq j \leq n-1$ (see section 3 for the structure of H_n).

If we choose $h_1 = h_2 = ... = h$ and denote by

$$y_n = K^{(n)}(\phi)[h^n],$$

which is a homogeneous polynomial of degree n from $H^s(R)$ to $X^{T,s}_{l,r}$, then

$$\begin{cases} \partial_t y_1 + \partial_x(a'(u)y_1) + \partial_x^3 y_1 = 0 \\ \\ y_1(x,0) = h(x) \end{cases} \tag{1.8}$$

for $n = 1$ and

$$\begin{cases} \partial_t y_n + \partial_x(a'(u)y_n) + \partial_x^3 y_n = -\partial_x \left\{M_n\right\} \\ \\ y_n(x,0) = 0 \end{cases} \tag{1.9}$$

for $n \geq 2$ where

$$M_n = \sum_{j=2}^{n} \frac{a^{(j)}(u)}{j!} \sum_{k_1+...+k_j=n} \frac{n!}{k_1!...k_j!} y_{k_1}...y_{k_j}$$

and the summation $\sum_{k_1+...+k_j=n}$ is over all possible $(k_1, ..., k_j)$ with $1 \leq k_1, ...k_j \leq n$ and $k_1 + ...k_j = n$.

Thus we can define the n-th Taylor polynomial P_n of K at $\phi \in H^s(R)$:

$$\begin{aligned} P_n(\phi)[h] \quad &:= \quad K(\phi) + \sum_1^n \frac{K^{(n)}(\phi)}{k!}[h^k] \\ &= \quad u + \sum_{k=1}^{n} \frac{y_k}{k!} \end{aligned} \tag{1.10}$$

for any $h \in H^s(R)$.

Let

$$z_n = K(\phi + h) - P_n(\phi)[h].$$

It is called the n-th Taylor remainder of K at ϕ and we shall see that z_n solves

$$\begin{cases} \partial_t z_0 + \partial_x(F_1(u,v)z_0) + \partial_x^3 z_0 = 0 \\ \\ z_0(x,0) = h(x) \end{cases} \tag{1.11}$$

for $n = 0$ and

$$\begin{cases} \partial_t z_n + \partial_x(F_1(u,v)z_n) + \partial_x^3 z_n = -\partial_x(G_n) \\ \\ z_n(x,0) = 0 \end{cases} \tag{1.12}$$

for $n \geq 1$ where

$$u = K(\phi), \qquad v = K(\phi + h),$$

$$G_n = \sum_{m=2}^{n+1} F_m(u,v) \sum_{k=0}^{n+1-m} z_k \sum_{k_1+...+k_{m-1}=n-k} q_{k_1}...q_{k_{m-1}} \tag{1.13}$$

with

$$q_m = \frac{y_m}{m!},$$

and

$$F_m(u,v) = \int_0^1 \lambda_1^{m-1}.... \int_0^1 \lambda_{m-1} \int_0^1 a^{(m)} \left(\prod_{j=1}^m \lambda_j v + (1 - \prod_{j=1}^m \lambda_j) u \right) d\lambda_m...d\lambda_1 \tag{1.14}$$

for $m = 1, ..., n+1$.

Letting $n \to \infty$ in (1.10), we obtain a formal Taylor series of K at $\phi \in H^s(R)$. Assuming that $a(u)$ is a polynomial, we shall prove that for any $\phi \in H^s(R)$, there exists a $\delta > 0$ such that if $h \in H^s(R)$ with $\|h\|_s \leq \delta$, then

$$K(\phi + h) = \sum_{k=0}^{\infty} \frac{K^{(n)}(\phi)}{n!}[h^n], \tag{1.15}$$

the series conversing uniformly about h with $\|h\|_s \leq \delta$ in the space $X_{l,r}^{T,s}$. In another word, the map K is analytic from $H^s(R)$ to $X_{l,r}^{T,s}$.

The paper is organized as follows.

– In section 2, we first list the estimates concerning the IVP

$$\partial_t u + \partial_x^3 u = f(x,t), \qquad u(x,0) = u_0(x) \tag{1.16}$$

which are needed to establish the nonlinear results. Then we consider the IVP for the following linear equation

$$\begin{cases} \partial_t u + \partial_x(a(v)u) + \partial_x^3 u = f(x,t), \quad x, \ t \in R \\ \\ u(x,0) = \phi(x) \end{cases} . \tag{1.17}$$

Let $s > 3/4$, $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ and $T > 0$ be given. We shall show that if $v \in X^{T,s}_{0,0}$, then for any $\phi \in H^s(R)$ and $f \in L^1([-T,T]; H^s(R))$, the IVP (1.17) has a unique solution $u \in X^{T,s}_{l,r}$ and

$$\|u\|_{X^{T,s}_{l,r}} \le \beta\left(\|v\|_{X^{T,s}_{0,0}}\right)\left(\|\phi\|_s + \int_{-T}^{T} \|f(.,t)\|_s dt\right) \tag{1.18}$$

This would be a key estimate to obtain differentiability of the map K.

– In section 3, we show that the map K defined by the IVP (1.1) is infinitely many times Frechet differentiable from $H^s(R)$ to $X^{T,s}_{l,r}$.

– In section 4, assuming that $a(u)$ in (1.1) is a polynomial, we show that the nonlinear map K is analytic from $H^s(R)$ to $X^{T,s}_{l,r}$.

Notations:

– The norm in $L^2(R)$ will be denoted by $\|.\|$ and the norm in $H^s(R)$ will be denoted by $\|.\|_s$. The notation $\|.\|_\infty$ is used to denote the norm in $L^\infty(R)$.

– $D^s = (-\partial_x^2)^{s/2}$ and $J^s = (1-\partial_x^2)^{s/2}$ denote the Riesz and the Bessell potential of order s respectively.

– $[A, B] = AB - BA$, where A, B are operators. Thus $[J^s; f]g = J^s(fg) - fJ^s g$ in which f is regarded as a multiplication operator.

– $H^\infty(R) := \cap_{s>0} H^s(R)$

– For $1 \le p,\ q \le \infty$ and $f:\ R \times [-T,T] \to R$,

$$\|f\|_{L^q_T L^p_x} = \left(\int_{-T}^{T}\left(\int_{-\infty}^{\infty} |f(x,t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}$$

and

$$\|f\|_{L^p_x L^q_T} = \left(\int_{-\infty}^{\infty}\left(\int_{-T}^{T} |f(x,t)|^q dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

2 Linear estimates

We use $\{W(t)\}_{-\infty}^{+\infty}$ to denote the unitary group which defines the solution of the IVP associated to

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, & \text{for } x,\ t \in R \\ \\ v(x,0) = v_0(x) \end{cases} \tag{2.1}$$

where

$$v(t) = W(t)v_0 = S_t * v_0$$

with $S_t(.)$ defined by the oscillatory integral

$$S_t(x) = c\int_{-\infty}^{+\infty} e^{ix\xi}e^{it\xi^3}d\xi.$$

Then the solution of the inhomogeneous equation

$$\begin{cases} \partial_t v + \partial_x^3 v = f(x,t), & x,\ t \in R \\ \\ v(x,0) = 0 \end{cases} \tag{2.2}$$

is expressed as

$$v(t) = \int_0^t W(t-\tau)f(.,\tau)d\tau.$$

Lemma 2.1 *For any $s \geq 0$,*

$$\left(\sup_x \int_{-\infty}^{\infty} |D^s\partial_x W(t)v_0|^2 dt\right)^{1/2} \leq c\|v_0\|_s, \tag{2.3}$$

and

$$\left(\int_{-\infty}^{\infty} \|D^{s+1/4}W(t)v_0\|_\infty^4 dt\right)^{1/4} \leq c\|v_0\|_s. \tag{2.4}$$

In addition, if $s > 3/4$, then

$$\left(\int_{-\infty}^{+\infty} \sup_{[-T,T]} |J^l W(t)v_0|^2(x)dx\right)^{1/2} \leq c(1+T)^\rho\|v_0\|_s \tag{2.5}$$

where $l \in [0, s-3/4)$ and ρ is a fixed constant larger than $3/4$.

Proof: see Kenig, Ponce and Vega ([22], Lemma 2.1, Theorem 2.4 and Corollary 2.9).

Remark 2.1 *(2.3) is a stronger version of local smoothing effect of Kato type and (2.4) is the global smoothing effect of Strichartz type [38] present in solutions of (2.1). The estimate (2.5), which gives a bound for the associated maximal function* $\sup_{[-T,T]}|W(t).|$*, is due to Vega [46].*

Lemma 2.2 *For any* $s \geq 0$ *and* $T > 0$*,*

$$\|W(t)v_0\|_s = \|v_0\|_s \tag{2.6}$$

and

$$\sup_{[-T,T]} \| \int_0^t W(t-\tau) f(.,\tau) d\tau \|_s \leq \int_{-T}^T \|f(.,\tau)\|_s d\tau. \tag{2.7}$$

Proof: (2.6) and (2.7) follow easily from Kato ([17], Lemma 3.1).

Lemma 2.3 *For any* $s \geq 0$ *and* $T > 0$*,*

$$\|D_x^s \partial_x \int_0^t W(t-\tau) f(.,\tau) d\tau\|_{L_x^\infty L_T^2} \leq c\|f\|_{L^1([-T,T];H^s(R))} \tag{2.8}$$

and

$$\|D^{s+\frac{1}{4}} \int_0^t W(t-\tau) f(.,\tau) d\tau\|_{L_x^\infty L_T^4} \leq c \int_{-T}^T \|f(.,\tau)\|_s d\tau. \tag{2.9}$$

If $s > 3/4$*, then*

$$\|J^l \int_0^t W(t-\tau) f(.,\tau) d\tau\|_{L_x^2 L_T^\infty} \leq c(1+T)^\rho \int_{-T}^T \|f(.,\tau)\|_s d\tau \tag{2.10}$$

where $l \in [0, s-3/4)$ *and* ρ *is a fixed constant larger than* $3/4$*.*

Proof: It follows from Lemma 2.1 by using Minkowski's integral inequality (see [47]).

Lemma 2.4 *Let* $s > 1/2$ *and* $T > 0$ *be given. Then there is a constant* $c > 0$ *such that*

$$\int_{-T}^T \|u\partial_x v\|_s dt \leq cT^{1/2}(1+T)^\rho \|u\|_{X_{0,0}^{T,s}} \|v\|_{X_{0,0}^{T,s}} \tag{2.11}$$

and

$$\int_{-T}^{T} \|\partial_x(uv)\|_s dt \le cT^{1/2}(1+T)^\rho \|u\|_{X_{0,0}^{T,s}} \|v\|_{X_{0,0}^{T,s}}. \tag{2.12}$$

for any $u,\ v \in X_{0,0}^{T,s}$.

Proof: see Lemma 2.4 in [47].

Lemma 2.5 *Let* $s > 0$ *be given. Then there is a constant* $c > 0$ *such that for any* $y_k \in H^s(R)$, $k = 1, 2, ..., m$,

$$\|\prod_{k=1}^{m} y_k\|_s \le c^m \left(\sum_{j=1}^{m} \|y_j\|_s \prod_{k=1,\, k\ne j}^{m} \|y_k\|_\infty \right) \tag{2.13}$$

and if $s > 1/2$,

$$\|\prod_{k=1}^{m} y_k\|_s \le c^m \prod_{k=1}^{m} \|y_k\|_s \tag{2.14}$$

for any $m \ge 2$ *where* c *in (2.14) may be different from* c *in (2.13).*

Proof: According to Kato and Ponce ([18], Lemma X4),

$$\|y_1 y_2\|_s \le c\{\|y_1\|_s \|y_2\|_\infty + \|y_2\|_s \|y_1\|_\infty\}$$

which is (2.13) with $m = 2$. Suppose that (2.13) is true for $m = N$. Then for $m = N + 1$,

$$\begin{aligned}
\|\prod_{k=1}^{N+1} y_k\|_s &= \|y_{N+1} \prod_{k=1}^{N} y_k\|_s \\
&\le c\left\{ \|y_{N+1}\|_s \|\prod_{k=1}^{N} y_k\|_\infty + \|y_{N+1}\|_\infty \|\prod_{k=1}^{N} y_k\|_s \right\} \\
&\le c\|y_{N+1}\|_s \prod_{k=1}^{N} \|y_k\|_\infty + c^{N+1} \sum_{j=1}^{N} \|y_j\|_s \prod_{k=1,\, k\ne j}^{N+1} \|y_k\|_\infty \\
&\le c^{N+1} \sum_{j=1}^{N+1} \|y_j\|_s \prod_{k=1,\, k\ne j}^{N+1} \|y_k\|_\infty
\end{aligned}$$

Thus (2.13) is proved by induction. As for (2.14), it follows directly from (2.13) since $\|y_k\|_\infty \le \|y_k\|_s$ if $s > 1/2$. The proof is completed. □

Lemma 2.6 *Let $b \in C^\infty(R;R)$ with $b(0)=0$. Then*

$$\|b(u)\|_s \leq \tilde{b}(\|u\|_s), \qquad s > 1/2$$

where $\tilde{b}(.)$ is a monotone increasing function depending only on b.

Proof: see Kato ([17], Lemma A.3).

Lemma 2.7 *Let $s > 1/2$ and $T > 0$ be given and assume $a \in C^\infty(R;R)$ with $a(0)=0$. Then there is a constant $c > 0$ such that for any $u \in X_{0,0}^{T,s}$ and $y \in X_{0,0}^{T,s}$,*

$$\int_{-T}^{T} \|\partial_x(a(u)y)\|_s dt \leq c\beta\left(\|u\|_{X_{0,0}^{T,s}}\right)\|y\|_{X_{0,0}^{T,s}} \tag{2.15}$$

where $\beta(.)$ is a continuous monotone increasing function only depending on a.

Proof: First of all,

$$\|\partial_x(a(u)y)\|_s \leq \|a'(u)y\partial_x u\|_s + \|a(u)\partial_x y\|_s$$

and it is easy to see by using Lemma 2.5 and Lemma 2.6 that

$$\begin{aligned}
\|a'(u)y\partial_x u\|_s &\leq \|(a'(u)-a'(0))y\partial_x u\|_s + |a'(0)|\|y\partial_x u\|_s \\
&\leq c\left\{\|a'(u)-a'(0)\|_s\|y\partial_x u\|_\infty + \|a'(u)-a'(0)\|_\infty\|y\partial_x u\|_s\right\} + \\
&\qquad +|a'(0)|\|y\partial_x u\|_s \\
&\leq c\left\{\|a'(u)-a'(0)\|_s + |a'(0)|\right\}\|y\partial_x u\|_s \\
&\leq c\beta_1(\|u\|_s)\|y\partial_x u\|_s
\end{aligned}$$

where $\beta_1(.) : R^+ \to R^+$ is a continuous monotone increasing function only depending on a. Using Lemma 2.4 yields

$$\begin{aligned}
\int_{-T}^{T}\|a'(u)y\partial_x u\|_s dt &\leq c\sup_{[-T,T]}\beta(\|u\|_s)\int_{-T}^{T}\|y\partial_x u\|_s dt \\
&\leq cT^{1/2}(1+T)^\rho\beta_1\left(\|u\|_{X_{0,0}^{T,s}}\right)\|u\|_{X_{0,0}^{T,s}}\|y\|_{X_{0,0}^{T,s}}.
\end{aligned}$$

In addition, applying Lemma 2.10 in [22], we obtain

$$
\begin{aligned}
&\|a(u)\partial_x y\|_s = \|J^s(a(u)\partial_x y)\| \\
&= \|a(u)D^s\partial_x y + a(u)(J^s - D^s)\partial_x y + [J^s; a(u)]\partial_x y\| \\
&\leq \|a(u)D^s\partial_x y\| + \|a(u)\|_\infty \|y\|_s + c\left\{\|\partial_x y\|_\infty \|a(u)\|_s + \right. \\
&\qquad \left. + \|a'(u)\partial_x u\|_\infty \|y\|_s\right\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{-T}^{T} \|a(u)\|_\infty \|y\|_s dt &\leq \sup_{[-T,T]} \|a(u)\|_s \int_{-T}^{T} \|y\|_s dt \\
&\leq cT \sup_{[-T,T]} \beta_2(\|u\|_s) \sup_{[-T,T]} \|y\|_s \qquad \text{(by Lemma 2.6)} \\
&\leq cT^{1/2}(1+T)^\rho \beta_2(\|u\|_{X_{0,0}^{T,s}})\|y\|_{X_{0,0}^{T,s}}
\end{aligned}
$$

where $\beta_2(.) : R^+ \to R^+$ is a continuous monotone increasing function depending only on a,

$$
\begin{aligned}
\int_{-T}^{T} \|a(u)\|_s \|\partial_x y\|_\infty dt &\leq \sup_{[-T,T]} \|a(u)\|_s \int_{-T}^{T} \|\partial_x y\|_\infty dt \\
&\leq \sup_{[-T,T]} \beta_2(\|u\|_s) \int_{-T}^{T} \|\partial_x y\|_\infty dt \\
&\leq \beta_2(\|u\|_{X_{0,0}^{T,s}})(2T)^{3/4} \left(\int_{-T}^{T} \|\partial_x y\|_\infty^4 dt\right)^{1/4} \\
&\leq cT^{1/2}(1+T)^\rho \beta_2(\|u\|_{X_{0,0}^{T,s}})\|y\|_{X_{0,0}^{T,s}},
\end{aligned}
$$

$$
\begin{aligned}
\int_{-T}^{T} \|a'(u)\partial_x u\|_\infty \|y\|_s dt &\leq \sup_{[-T,T]} (\|a'(u)\|_\infty \|y\|_s) \int_{-T}^{T} \|\partial_x u\|_\infty dt \\
&\leq cT^{1/2}(1+T)^\rho \beta_3(\|u\|_{X_{0,0}^{T,s}})\|u\|_{X_{0,0}^{T,s}}\|y\|_{X_{0,0}^{T,s}},
\end{aligned}
$$

where

$$\beta_3(r) = \sup_{|\lambda|\leq r} |a'(\lambda)|,$$

and

$$\int_{-T}^{T} \|a(u)D^s\partial_x y\| dt \leq T^{1/2} \int_{-T}^{T} \int_R |a(u)D^s\partial_x u|^2 dx dt$$

$$
\begin{aligned}
&\le \int_R \int_{-T}^{T} |u|^2 |D^s \partial_x u|^2 dx dt \sup_{[-T,T]} \|\frac{a(u)}{u}\|_\infty \\
&\le T^{1/2} \beta_4(\|u\|_{X_{0,0}^{T,s}}) \left(\int_R \sup_{[-T,T]} |u(x,t)|^2 dx \right)^{1/2} \left(\sup_x \int_{-T}^{T} |D^s \partial_x y|^2 dt \right)^{1/2} \\
&\le cT^{1/2}(1+T)^\rho \beta_4(\|u\|_{X_{0,0}^{T,s}}) \|u\|_{X_{0,0}^{T,s}} \|y\|_{X_{0,0}^{T,s}}
\end{aligned}
$$

where

$$
\beta_4(r) = \sup_{|\lambda| \le r} \left| \frac{a(\lambda)}{\lambda} \right| .
$$

Therefore,

$$
\int_{-T}^{T} \|\partial_x (a(u) y)\|_s dt \le cT^{1/2}(1+T)^\rho \beta(\|u\|_{X_{0,0}^{T,s}}) \|y\|_{X_{0,0}^{T,s}}
$$

for some constant $c > 0$ where

$$
\beta(r) = \max \{ r\beta_1(r),\ \beta_2(r),\ r\beta_3(r),\ r\beta_4(r) \} .
$$

The proof is completed. □

¿From the proof of the above lemma we may draw the following corollary.

Corollary 2.1 *Let $s > 1/2$ and $T > 0$ be given and assume $a \in C^\infty(R; R)$ with $a'(0) = 0$. Then for any $u \in X_{0,0}^{T,s}$,*

$$
\int_{-T}^{T} \|\partial_x (a(u))\|_s dt \le c\beta \left(\|u\|_{X_{0,0}^{T,s}} \right) T^{1/2} (1+T)^\rho \|u\|_{X_{0,0}^{T,s}} \tag{2.16}
$$

where $\beta(.)$ is a continuous monotone increasing function only depending on a.

Lemma 2.8 *Let $s > 1/2$, $T > 0$ and $a \in C^\infty(R; R)$ be given. Then there exists a constant $c > 0$ such that for any $u \in X_{0,0}^{T,s}$ and $y_k \in X_{0,0}^{T,s}$ with $k = 1, 2, ..., m$ and $m \ge 2$,*

$$
\int_{-T}^{T} \|\partial_x (a(u) \prod_{k=1}^{m} y_k)\|_s dt \le c^m \beta(\|u\|_{X_{0,0}^{T,s}}) \prod_{k=1}^{m} \|y_k\|_{X_{0,0}^{T,s}} \tag{2.17}
$$

where $\beta(.)$ is a continuous monotone increasing function only depending on a.

Proof: By applying (2.14) we have

$$\|\partial_x(a(u)\prod_{k=1}^{m} y_k)\|_s \le \|a'(u)\partial_x u\prod_{k=1}^{m} y_k\|_s + \sum_{k=1}^{m}\|\prod_{j=1,\, j\neq k}^{m} a(u)y_j\partial_x y_k\|_s$$

$$\le c^{m+1}\left(\|a'(u)-a'(0)\|_s + |a'(0)|\right)\|y_1\partial_x u\|_s\prod_{k=2}^{m}\|y_k\|_s +$$

$$+c^{m+1}\left(\|a(u)-a(0)\|_s+|a(0)|\right)\left(\sum_{k=2}^{m}\|y_1\partial_x y_k\|_s\prod_{j=2,\, j\neq k}^{m}\|y_j\|_s + \right.$$

$$\left. + \|y_2\partial_x y_1\|_s\prod_{j=3}^{m}\|y_j\|_s\right).$$

Thus, using Lemma 2.6 and Lemma 2,4, we have

$$\int_{-T}^{T}\|\partial_x(a(u)\prod_{k=1}^{m} y_k)\|_s dt \le c_1^{m+1}\sup_{[-T,T]}\left(\|a'(u)-a'(0)\|_s+|a'(0)|\right) *$$

$$*\prod_{k=2}^{m}\sup_{[-T,T]}\|y_k\|_s\int_{-T}^{T}\|y_1\partial_x u\|_s dt + c_1^{m+1}\sup_{[-T,T]}\left(\|a(u)-a(0)\|_s+|a(0)|\right) *$$

$$*\left(\sum_{k=2}^{m}\sup_{[-T,T]}\prod_{j=2,\, j\neq k}^{m}\|y_j\|_s\int_{-T}^{T}\|y_1\partial_x y_k\|_s dt + \prod_{j=3}^{m}\sup_{[-T,T]}\|y_j\|_s\sup_{[-T,T]}\|y_2\partial_x y_1\|_s\right)$$

$$\le c_1^{m+1}\beta_1(\|u\|_{X_{0,0}^{T,s}})T^{1/2}(1+T)^{\rho}\|u\|_{X_{0,0}^{T,s}}\prod_{k=1}^{m}\|y_k\|_{X_{0,0}^{T,s}} +$$

$$+mc_1^{m+1}\beta_1(\|u\|_{X_{0,0}^{T,s}})T^{1/2}(1+T)^{\rho}\prod_{k=1}^{m}\|y_k\|_{X_{0,0}^{T,s}}$$

$$\le c^m\beta(\|u\|_{X_{0,0}^{T,s}})\prod_{k=1}^{m}\|y_k\|_{X_{0,0}^{T,s}}$$

for some $c>0$. The proof is completd. $\Box$

To end this section, we consider the following linear problem:

$$\begin{cases} \partial_t u + \partial_x(a(v)u) + \partial_x^3 u = f(x,t), & x,\, t\in R \\ u(x,0)=\phi(x) \end{cases} \tag{2.18}$$

where $a(.)\in C^{\infty}(R;R)$ with $a(0)=0$.

Theorem 2.1 *Let $s > 3/4$, $(l, r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$, $T > 0$ and $v \in X_{0,0}^{T,s}$ be given. Then for any $f \in L^1([-T, T]; H^s(R))$ and $\phi \in H^s(R)$, there exists a unique solution $u \in X_{l,r}^{T,s}$ to (2.18) such that*

$$\|u\|_{X_{l,r}^{T,s}} \leq \beta(\|v\|_{X_{0,0}^{T,s}}) \left(\|\phi\|_s + \int_{-T}^{T} \|f(.,t)\|_s dt \right) \tag{2.19}$$

where β is a continuous monotone increasing function only depending on a.

Proof: We use the contraction principle argument that Kenig, Ponce and Vega used in [23].

For any given $\phi \in H^s(R)$ and $f \in L^1([-T, T]; H^s(R))$, denote by $u = \Phi(w)$, the solution of the following IVP

$$\begin{cases} \partial_t u + \partial_x^3 u = f - \partial_x(a(v)w) \\ \\ u(x, 0) = \phi(x) \end{cases} \tag{2.20}$$

where

$$w \in S_b^T = \left\{ w \in X_{0,0}^{T,s} | \quad \Lambda_{0,0}^s(T; w) \leq b \right\}$$

for some $b > 0$ to be determined.

We shall show that there exists a $b = b(\|\phi\|_s, \|f\|_{L^1([-T,T];H^s(R))}) > 0$ and a $T^* > 0$ such that $u = \Phi(w) \in S_b^{T^*}$ if $w \in S_b^{T^*}$ and

$$\Phi : S_b^{T^*} \to S_b^{T^*}$$

is a contraction map.

Consider the integral equation form of the IVP (2.20),

$$u(t) = W(t)\phi + \int_0^t W(t - \tau)\partial_x(a(v)w)(., \tau)d\tau. \tag{2.21}$$

Applying (2.3) - (2.10) and (2.15) to (2.21) leads to

$$\begin{aligned} \Lambda_{l,r}^s(t; u) &\leq c \left(\|\phi\|_s + \int_{-t}^{t} \|f(., \tau)\|_s d\tau \right) + c \int_{-t}^{t} \|\partial_x(a(v)w)\|_s d\tau \\ &\leq c \left(\|\phi\|_s + \int_{-T}^{T} \|f(., \tau)\|_s d\tau \right) + \\ &\quad ct^{1/2}(1 + t)^\rho \beta(\|v\|_{X_{0,0}^{T,s}})\Lambda_{0,0}^s(t; w) \end{aligned} \tag{2.22}$$

for $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$. In particular,

$$\Lambda^s_{0,0}(t;u) \le c\left(\|\phi\|_s + \int_{-T}^{T} \|f\|_s d\tau\right) + c\beta(\|v\|_{X^{T,s}_{0,0}}) t^{1/2}(1+t)^\rho \Lambda^s_{0,0}(t;w).$$

Choosing

$$b = 2\left(\|\phi\|_s + \int_{-T}^{T} \|f\|_s d\tau\right) \tag{2.23}$$

and $0 < T^* < T$ such that

$$cT^*(1+T^*)^\rho \beta(\|v\|_{X^{T,s}_{0,0}}) = \frac{1}{2}, \tag{2.24}$$

we obtain

$$\Lambda^s_{0,0}(T^*;u) \le b. \tag{2.25}$$

Thus, Φ is a map from $S^{T^*}_b$ to $S^{T^*}_b$.

For any $w_1,\ w_2 \in S^{T^*}_b$, let

$$z = \Phi(w_1) - \Phi(w_2).$$

Then

$$z(t) = \int_0^t W(t-\tau)\partial_x\left(a(v)(w_1-w_2)\right) d\tau$$

and

$$\begin{aligned}\Lambda^s_{0,0}(T^*;z) &\le c\int_{-T^*}^{T^*} \|\partial_x\left(a(v)(w_1-w_2)\right)\|_s d\tau \\ &\le c\beta(\|v\|_{X^{T,s}_{0,0}})\sqrt{T^*}(1+T^*)^\rho \Lambda^s_{0,0}(T^*;w_1-w_2)\end{aligned}$$

It follows from (2.24) that

$$\Lambda^s_{0,0}(T^*;z) \le \frac{1}{2}\Lambda^s_{0,0}(T^*, w_1-w_2). \tag{2.26}$$

Consequently, by the contraction principle, there exists a unique solution $u \in S^{T^*}_b$ such that

$$\Phi(u) = u$$

i.e.

$$u(t) = W(t)\phi - \int_0^t W(t-\tau)\left(\partial_x(a(v)u)\right)(\tau)d\tau + \int_0^t W(t-\tau)f(.,\tau)d\tau \tag{2.27}$$

for $-T^* < t < T^*$. In addition, it follows from (2.22), (2.23) and (2.25) that

$$\Lambda^s_{l,r}(T^*; u) \le c\left(\|\phi\|_s + \int_{-T}^{T} \|f\|_s d\tau\right) \tag{2.28}$$

for $(l, r) \in [0, s - \frac{3}{4}] \times [0, s - \frac{3}{4})$.

Finally, note that T^* determined by (2.24) only depends on $\beta(\|v\|_{X^{T,s}_{0,0}})$ and, in particular, it does not depend on f and ϕ. Thus, a standard argument shows that T^* can be extended to $T^* = T$ and (2.28) hold for $T^* = T$ with another c depending only on $\|v\|_{X^{T,s}_{0,0}}$ and a. The proof is completed. □

Remark 2.2 *Theorem 2.1 is still true if $a(v)$ in (2.18) is replaced by*

$$\int_0^1 a(\lambda v_1 + (1-\lambda)v_2)d\lambda$$

with $v_1,\ v_2 \in X^{T,s}_{0,0}$.

3 Well-posedness

In this section we consider the IVP of the generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x(a(u)) + \partial_x^3 u = 0, & x,\ t \in R \\ u(x,0) = \phi(x) \end{cases} \tag{3.1}$$

where $a = a(x)$ is assumed to be a C^∞ function from R to R and

$$a(0) = a'(0) = 0.$$

The following result is due to Kato [17]

Proposition 3.1 *Let $s > 3/2$ be given. Then*

(i) For each $\phi \in H^s(R)$, there exists a $T > 0$ depending only on $\|\phi\|_s$, and a unique solution u to (3.1) such that

$$u \in C([-T, T]; H^s(R)).$$

The map $\phi \to u$ is continuous from $H^s(R)$ to $C([-T,T];H^s(R))$.

(ii). If, in addition, $\phi \in H^{s'}(R)$ for some $s' > s$, then (i) is true with same T and with s replaced by s'.

(iii). There is a number $\gamma_a > 0$ depending only on a, which is called the ceiling for the given a, such that if $\phi \in H^s(R)$ with $s \geq 2$ and

$$\|\phi\|_1 \leq \gamma_a, \tag{3.2}$$

then there is a unique solution u to (3.1) such that

$$u \in C([0,\infty);H^s(R)).$$

Based on Kato's results, we shall prove the following thoerems by using Kenig, Ponce and Vega's argument in [22] with a few modification.

Theorem 3.1 *Let $s > 3/4$ be given. Then*

(i). For any $\phi \in H^s(R)$, there exists a $T > 0$ depending only on $\|\phi\|_s$, and a unique solution $u \in C([-T,T];H^s(R))$ to the IVP (1.1) satisfying

$$\left(\int_{-T}^{T} \|\partial_x u(.,t)\|_\infty^4 dt\right)^{1/4} < \infty$$

and moreover,

$$\|u\|_{X_{l,r}^{T,s}} \leq \eta(\|\phi\|_s)$$

where $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ and $\eta(.)$ is a continuous monotone increasing function with $\eta(0) = 0$.

(ii). For any $T' < T$, there exists a neighborhood U of ϕ in $H^s(R)$ such that the map

$$K : \phi \to u(.,t)$$

from U to $X_{l,r}^{T,s}$ is Lipschitz continuous.

Theorem 3.2 *Let $s \geq 1$ be given. Then Theorem 1.1 is true with T arbitrarily large provided that*

$$\|\phi\|_1 \leq \gamma_a$$

where γ_a is the ceiling of a defined in Theorem 3.1.

Remark 3.1 *According to Kato [17]*

$$\gamma_a = \infty$$

if

$$\lim_{|\lambda|\to\infty} \sup |\lambda|^{-6} a_2(\lambda) = 0 \tag{3.3}$$

where

$$a_2(\lambda) = 2\int_0^\lambda (\lambda - \mu) a'(\mu) d\mu.$$

Note that (3.3) is true if

$$\lim_{|\lambda|\to\infty} \sup |\lambda|^{-4} a'(\lambda) = 0.$$

or

$$a'(u) \le |u|^p$$

with $p \le 3$ as $|u| \to \infty$.

Remark 3.2 *Theorem 3.2 follows from Theorem 3.1 and the global a priori estimates for solutions of the IVP (3.1) due to Kato [17] by a standard argument.*

We start to prove Theorem 3.1 by establishing the following a priori estimate for solutions of (3.1).

Proposition 3.2 *Let $s > 3/4$, $T > 0$ and $(l, r) \in [0, s - \frac{3}{4}] \times [0, s - \frac{3}{4})$ be given. If $u \in X_{0,0}^{T,s}$ is a solution of (3.1), then there exists a $T_0 > 0$ depending only on $\|\phi\|_s$ such that*

$$\Lambda_{l,r}^s(T_0; u) < c\|\phi\|_s \tag{3.4}$$

where $c > 0$ is a constant independent of u.

Proof: Consider the integral equation form of (3.1),

$$u(t) = W(t)\phi - \int_0^t W(t-\tau)\partial_x(a(u))(\tau) d\tau. \tag{3.5}$$

Applying (2.3) - (2.10) to (3.1) yields

$$\Lambda^s_{l,r}(t;u) \le c\|\phi\|_s + c\int_{-t}^{t} \|\partial_x(a(u))(\tau)\|_s d\tau \tag{3.6}$$

for any $t \le T$. In particular, using (2.16), we have

$$\begin{aligned} \Lambda^s_{0,0}(t;u) &\le c\|\phi\|_s + c\int_{-t}^{t} \|\partial_x(a(u))(\tau)\|_s d\tau \\ &\le c\|\phi\|_s + c\beta(\Lambda^s_{0,0}(t;u))t^{1/2}(1+t)^\rho \Lambda^s_{0,0}(t;u). \end{aligned} \tag{3.7}$$

Note that $\beta(\Lambda^s_{0,0}(t;u))$ is a continuous increasing function of t. There exists a $t = T_0$ such that

$$c\beta(\Lambda^s_{0,0}(T_0;u))T_0^{1/2}(1+T_0)^\rho = 1/2 \tag{3.8}$$

and it follows from (3.7) that

$$\Lambda^s_{0,0}(T_0;u) \le 2c\|\phi\|_s, \tag{3.9}$$

Thus the following inequality must hold

$$c\beta(2c\|\phi\|_s)T_0^{1/2}(1+T_0)^\rho \ge 1/2,$$

which implies that $T_0 > M_1 > 0$ with $M_1 > 0$ depending only on $\|\phi\|_s$. (3.4) follows from (3.6), (2.16) and (3.9). The proof is completed. □

Proposition 3.3 *Let $s > 3/4$ and $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ be given. For any $\phi \in H^s(R)$, suppose $\phi_\epsilon \in H^\infty$, $\epsilon \in (0,1)$, with*

$$\lim_{\epsilon \to 0} \phi_\epsilon = \phi_0 \qquad \text{in } H^s(R).$$

Then there exist $T > 0$ such that for any $\epsilon \in (0,1)$, (3.1) has a unique solution $u_\epsilon \in C([-T,T]; H^\infty(R))$ with $u_\epsilon(x,0) = \phi_\epsilon(x)$ satisfying

$$\Lambda^s_{l,r}(T;u_\epsilon) < c\|\phi_\epsilon\|_s \tag{3.10}$$

for $0 < \epsilon < 1$ where c is the same constant as that in (3.4).

Proof: According to Theorem 3.1, $u_\epsilon \in C([-T_\epsilon, T_\epsilon]; H^\infty(R))$ where T_ϵ only depends on $\|\phi_\epsilon\|_s$. Since $\|\phi_\epsilon\|_s$ is uniformly bounded for $\epsilon \in (0,1)$, we may assume that

$$T_\epsilon > T_1 \qquad \text{for any } \epsilon \in (0,1)$$

for some $T_1 > 0$ and therefore

$$u_\epsilon \in X_{0.0}^{T_1,s}.$$

Thus it follows from Proposition 3.2 that

$$\Lambda_{l,r}^s(T; u_\epsilon) \le c\|\phi_\epsilon\|_s$$

where $T > 0$ is a constant independent of ϵ since $\|\phi_\epsilon\|_s$ is bounded. The proof is completed. $\square$

Proof of Theorem 3.1 For $\phi \in H^s(R)$, choose $\phi_\epsilon \in H^\infty(R)$ such that

$$\lim_{\epsilon \to 0} \phi_\epsilon = \phi \qquad \text{in } H^s(R).$$

By Proposition 3.3, there exists a $T > 0$ and $K > 0$ such that for any $\epsilon \in (0,1)$ the IVP

$$\begin{cases} \partial_t u_\epsilon + \partial_x(a(u_\epsilon)) + \partial_x^3 u_\epsilon = 0 \\ \\ u_\epsilon(x,0) = \phi_\epsilon(x) \end{cases}$$

has a unique solution $u_\epsilon \in C([-T,T]; H^\infty(R))$ satisfying

$$\Lambda_{l,r}^s(T; u_\epsilon) < K.$$

It suffices to show that u_ϵ is a Cauchy sequence in $X_{l,r}^{T,s}$ and then the limit u of u_ϵ as $\epsilon \to 0$ is the desired solution of the IVP (3.1) corresponding to the initial value ϕ.

Let $\epsilon' < \epsilon$ and

$$w = u_\epsilon - u_{\epsilon'}.$$

Then w solves

$$\begin{cases} \partial_t w + \partial_x(A(u_\epsilon, u_{\epsilon'})w) + \partial_x^3 w = 0 \\ \\ w(x,0) = \phi_\epsilon - \phi_{\epsilon'} \end{cases}$$

where

$$A(u,v)=\int_0^1 a'\left(\lambda u-(1-\lambda)(v)\right)d\lambda.$$

According to Theorem 2.1 and its remark,

$$\Lambda_{l,r}^s(T;w)\le c^*\|\phi_\epsilon-\phi_{\epsilon'}\|_s$$

for some $c^*>0$ independent of ϵ and therefore u_ϵ is a Cauchy sequence in $X_{l,r}^{T,s}$.

Finally, if $T'<T$, (3.1) defines a map k from a neighborhood U of ϕ in $H^s(R)$ to $X_{l,r}^{T,s}$. For any $\psi\in H^s(R)$ with $\|\psi-\phi\|_s\le\epsilon_0$ for some $\epsilon_0>0$, let u and v be the solutions of the IVP (3.1) with $u(x,0)=\phi(x)$ and $v(x,0)=\psi(x)$, respectively. Then, similarly, we have

$$\Lambda_{l,r}^s(T';u-v)\le c\|\phi-\psi\|_s$$

where c depends only on $\|\phi\|_s$. Therefore the map $\phi\to u$ is Lipschitz continuous. The proof is completed. □

4 Diffenrentiability

Let X, Y be two Banach spaces. An n-linear map from X to Y is a map from the n-fold product space X^n into Y such that

$$x_k\to f(x_1,\dots,x_n)$$

is linear for each fixed $(x_1,x_2,\dots,x_{k-1},x_{k+1},\dots,x_n)$, for $k=1,2,\dots,n$. A homogeneous polynomial of degree n from X to Y is a map of the form $x\to f(x,\dots,x)$, for some n-linea map f.

A map $f:\ X\to Y$ is Frechet differentiable at a point $x_0\in X$ if there exists a continuous linear map $f'(x_0):\ X\to Y$ so that

$$\|f(x)-f(x_0)-f'(x_0)(x-x_0)\|_Y=o(\|x-x_0\|_X)$$

as $x\to x_0$ where $f'(x_0)\in\mathcal{L}(X,Y)$ is called the Frechet derivative of f at x_0. f is said twice differentiable at x_0 if f is differentiable at each point in a neighborhood of x_0 and $x \to f'(x) \in \mathcal{L}(X,Y)$ is differentiable at x_0; and so on.

A map $f: U \subset X \to Y$, U open, is analytic in U if f is infinitely often differentiable at each point of U and if, for each $x \in U$, there exists $\delta = \delta(x) > 0$ so that whenever $\|h\|_X \le \delta$,

$$f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x)[h^k],$$

the series converging in Y-norm uniformly in $\|h\|_X \le \delta$.

Consider the IVP for the generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x(a(u)) + \partial_x^3 u = 0, & x,\ t \in R \\ \\ u(x,0) = \psi(x) \end{cases} \tag{4.1}$$

where $a \in C^\infty(R;R)$ with $a(0) = a'(0) = 0$.

According to Theorem 3.1, for any $\phi \in H^s(R)$, $s > 3/4$, there is a $T > 0$ and a neighborhood U of ϕ in $H^s(R)$ such that (4.1) defines a nonlinear map K from U to $X_{l,r}^{T,s}$

$$u := K(\psi)$$

for any $\psi \in U$ where u is the solution of (3.1). We shall show in this section that K is infinitely many times Frechet differentiable from U to $X_{l,r}^{T,s}$.

If we suppose that the map K is n times Frechet differentiable, then its n-th order derivative $K^{(n)}(\psi)$ at $\psi \in U$ is a symmetric n-linear map from $H^s(R)$ to $X_{l,r}^{T,s}$ and for any $h_1, ..., h_n \in H^s(R)$,

$$K^{(n)}(\phi)[h_1, ..., h_n] = \left\{ \frac{\partial^n}{\partial \xi_1 ... \partial \xi_n} K(\psi + \sum_{k=1}^{n} \xi_k h_k) \right\}_{0,...,0}.$$

As for the homogeneous polynomial $K^{(n)}(\psi)[h^n]$ of degree n induced by $K^{(n)}(\psi)$, it is given by

$$K^{(n)}(\psi)[h^n] = \left\{ \frac{d^n}{d\xi^n} K(\psi + \xi h) \right\}_{\xi=0}$$

for any $h \in H^s(R)$.

Denote by

$$w^{(n)}_{[1,...,n]} = K^{(n)}(\psi)[h_1,...,h_n] \quad \text{and } y_n = K^{(n)}(\psi)[h^n].$$

Then direct computation show that $w^{(n)}_{[1,...,n]}$ solves

$$\begin{cases} \partial_t w^{(1)}_{[1]} + \partial_x(a'(u)w^{(1)}_{[1]}) + \partial_x^3 w^{(1)}_{[1]} = 0 \\ \\ w^{(1)}_{[1]}(x,0) = h_1 \end{cases} \tag{4.2}$$

for $n = 1$ and

$$\begin{cases} \partial_t w^{(n)}_{[1,...,n]} + \partial_x(a'(u)w^{(n)}_{[1,...,n]}) + \partial_x^3 w^{(n)}_{[1,...,n]} = -\partial_x(H_n) \\ \\ w^{(n)}_{[1,...,n]}(x,0) = 0 \end{cases} \tag{4.3}$$

for $n \geq 2$ with $u = K(\psi)$ and

$$H_n = \sum_{j=2}^{n} \frac{a^{(j)}}{j!} \sum_{k_1+...+k_j=n} \sum_{\mho} w^{(k_1)}_{[i_1^1,...,i_{k_1}^1]} w^{(k_2)}_{[i_1^2,...,i_{k_2}^2]} ... w^{(k_j)}_{[i_1^j,...,i_{k_j}^j]} \tag{4.4}$$

where $\sum_{\mho}$ is the summation over all $(i_1^1,...,i_{k_1}^1,...i_1^j,...,i_{k_j}^j)$ satisfying

$$1 \leq i_1^m < i_2^m < ... < i_{k_m}^m \leq n$$

for $m = 1, 2, ..., j$ and

$$\bigcup_{m=1}^{j} \bigcup_{l=1}^{k_m} \{i_l^m\} = \{1, 2, ..., n\}.$$

As for y_n, it solves

$$\begin{cases} \partial_t y_1 + \partial_x(a'(u)y_1) + \partial_x^3 y_1 = 0 \\ \\ y_1(x,0) = h \end{cases} \tag{4.5}$$

for $n = 1$ and

$$\begin{cases} \partial_t y_n + \partial_x(a'(u)y_n) + \partial_x^3 y_n = -\partial_x(M_n) \\ \\ y_n(x,0) = 0 \end{cases} \tag{4.6}$$

for $n \geq 2$ where

$$M_n = \sum_{j=2}^{n} \frac{a^{(j)}(u)}{j!} \sum_{k_1+...+k_j=n} \frac{n!}{k_1!...k_j!} y_{k_1}..y_{k_j}.$$

On the other hand, according to Theorem 2.1, given $u \in X_{0,0}^{T,s}$ with $s > 3/4$, (4.2) defines a linear map $\mathcal{K}^{(1)}(u)$ from $H^s(R)$ to $X_{l,r}^{T,s}$ with $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$,

$$\mathcal{K}^{(1)}(u)[h_k] = w_{[k]}^{(1)}$$

where $w_{[k]}^{(1)}$ is the solution of (4.2) with the initial value $h_k \in H^s(R)$, $k = 1, 2....$ Inductively, (4.2)-(4.3) defines an n-linear map $\mathcal{K}^{(n)}(u)$ from the n-fold space $(H^s(R))^n$ to $X_{l,r}^{T,s}$ for any $n \geq 2$.

Proposition 4.1 *Let $s > 3/4$, $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$, $T > 0$ and $u \in X_{0,0}^{T,s}$ be given. Then for any $h_1, ..., h_n \in H^s(R)$, (4.2)-(4.3) has a unique solution $w_{[1,..,n]}^{(n)}$, which defines an n-linear map $\mathcal{K}^{(n)}(u)$ from the n-fold space $(H^s(R))^n$ to $X_{l,r}^{T,s}$. Moreover,*

$$\|\mathcal{K}^{(n)}(u)[h_1, ..., h_n]\|_{X_{l,r}^{T,s}} \leq c(n, \|u\|_{X_{0,0}^{T,s}}) \prod_{k=1}^{n} \|h_k\|_s \tag{4.7}$$

for any $n \geq 1$ and $h_1, ..., h_n \in H^s(R)$ where $c(n, .)$ is a continuous monotone increasing function from R^+ to R^+ with $c(n, 0) = 0$.

Proof: It is easy to see that (4.7) is true with $n = 1$ by applying (2.19) to (4.2). Similarly, for $n = 2$, applying (2.19) to (4.3) yields

$$\|w_{[1,2]}^{(2)}\|_{X_{l,r}^{T,s}} \leq \beta(\|u\|_{X_{0,0}^{T,s}}) \int_{-T}^{T} \|\partial_x H_2\|_s d\tau.$$

Then it follows from Lemma 2.8 that

$$\begin{aligned} \int_{-T}^{T} \|\partial_x H_2\|_s dt &= \int_{-T}^{T} \|\partial_x \left(\frac{a^{(2)}(u)}{2} w_{[1]}^{(1)} w_{[2]}^{(1)} \right) \|_s dt \\ &\leq c^2 \beta_2(\|u\|_{X_{0,0}^{T,s}}) \|w_{[1]}^{(1)}\|_{X_{0,0}^{T,s}} \|w_{[2]}^{(1)}\|_{X_{0,0}^{T,s}} \\ &\leq c^2 \beta_2(\|u\|_{X_{0,0}^{T,s}}) c^2(1, \|u\|_{X_{0,0}^{T,s}}) \|h_1\|_s \|h_2\|_s. \end{aligned}$$

Thus,

$$\begin{aligned}
\|w^{(2)}_{[1,2]}\|_{X^{T,s}_{l,r}} &\leq c^2\beta(\|u\|_{X^{T,s}_{0,0}})\beta_2(\|u\|_{X^{T,s}_{0,0}})c^2(1,\|u\|_{X^{T,s}_{0,0}})\|h_1\|_s\|h_2\|_s \\
&:= c(2,\|u\|_{X^{T,s}_{0,0}})\|h_1\|_s\|h_2\|_{X^{T,s}_{0,0}}.
\end{aligned}$$

Assume that (4.7) is true for $1 \leq k \leq N-1$. Applying (2.19) to (4.3) with $n = N$ yields

$$\begin{aligned}
&\|w^{(N)}_{[1,\ldots,N]}\|_{X^{T,s}_{l,r}} \leq \beta(\|u\|_{X^{T,s}_{0,0}}) \int_{-T}^{T} \|\partial_x H_N\|_s dt \\
&\leq \beta(\|u\|_{X^{T,s}_{0,0}}) \sum_{j=2}^{N} \frac{c^j}{j!}\beta_j(\|u\|_{X^{T,s}_{0,0}}) \sum_{k_1+\ldots+k_j=N} \prod_{l=1}^{j} \|w^{(k_l)}_{[i^l_1,\ldots,i^l_{k_l}]}\|_{X^{T,s}_{0,0}} \\
&\leq \beta(\|u\|_{X^{T,s}_{0,0}}) \sum_{j=2}^{N} \frac{c^j}{j!}\beta_j(\|u\|_{X^{T,s}_{0,0}}) \sum_{k_1+\ldots+k_j=N} \prod_{l=1}^{j} c(k_l,\|u\|_{X^{T,s}_{0,0}})\|h_{[i^l_1,\ldots,i^l_{k_l}]}\|_s \\
&:= c(N,\|u\|_{X^{T,s}_{0,0}}) \prod_{k=1}^{N} \|h_k\|_s.
\end{aligned}$$

The proof is completed by induction. $\square$

Corollary 4.1 *Let $s > 3/4$, $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$, $T > 0$ and $u \in X^{T,s}_{0,0}$ be given. Then (4.5)-(4.6) defines a homogeneous polynomial $\mathcal{K}^{(n)}(u)[h^n]$ of degree n from $H^s(R)$ to $X^{T,s}_{l,r}$ and*

$$\|\mathcal{K}^{(n)}(u)[h^n]\|_{X^{T,s}_{l,r}} \leq c(n,\|u\|_{X^{T,s}_{0,0}})\|h\|^n_s \tag{4.8}$$

for any $h \in H^s(R)$.

Now we may define formally the n-th Taylor polynomial $P_n(h)$ of the map K at some $\phi \in H^s(R)$ as

$$\begin{aligned}
P_n(h) &= K(\phi) + \sum_{k=1}^{n} \frac{1}{k!}\mathcal{K}^{(k)}(u)[h^k] \\
&:= u + \sum_{k=1}^{n} \frac{1}{k!} y_k
\end{aligned}$$

where $u = K(\phi)$.

Proposition 4.2 *Let z_n denote the n-th Taylor remainder of K at $\phi \in H^s(R)$, i.e.*

$$z_n = K((\phi + h) - P_n(h).$$

Then it solves

$$\begin{cases} \partial_t z_0 + \partial_x(F_1(u,v)z_0) + \partial_x^3 z_0 = 0 \\ \\ z_0(x,0) = h(x) \end{cases} \tag{4.9}$$

for $n = 0$ and

$$\begin{cases} \partial_t z_n + \partial_x(F_1(u,v)z_n) + \partial_x^3 z_n = -\partial_x(D_n) \\ \\ z_n(x,0) = 0 \end{cases} \tag{4.10}$$

for $n \geq 1$ where

$$u = K(\phi), \qquad v = K(\phi + h),$$

and

$$D_n = \sum_{m=2}^{n+1} F_m(u,v) \sum_{k=0}^{n+1-m} z_k \sum_{k_1+...+k_{m-1}=n-k} q_{k_1}...q_{k_{m-1}} \tag{4.11}$$

with $F_m(u.v)$ defined by (1.14) and

$$q_m = \frac{y_m}{m!}$$

for $m = 1, ..., n+1$.

Proof: Direct computation shows easily that (4.9) and (4.10) with $n = 1$ are true. Assume that (4.10) is true for $n = N$. Then, for $n = N + 1$, by definition,

$$z_{N+1} = z_N - \frac{1}{(N+1)!} y_{N+1} = z_N - q_{N+1}$$

where q_{N+1} solves

$$\begin{cases} \partial_t q_{N+1} + \partial_x(a'(u)q_{N+1}) + \partial_x^3 q_{N+1} = -\partial_x(E_{N+1}) \\ \\ q_{N+1} = 0 \end{cases} \tag{4.12}$$

with

$$E_{N+1}=\sum_{m=2}^{N+1}\frac{a^{(m)}(u)}{m!}\sum_{k_1+\ldots+k_m=N+1}q_{k_1}\ldots q_{k_m}.$$

Hence

$$\partial_t z_{N+1}+\partial_x^3 z_{N+1}=-\partial_x\left(F_1(u,v)z_N-a^{(1)}(u)q_{N+1}\right)-\partial_x(G_{N+1})$$

where

$$\begin{aligned}G_{N+1} &= \sum_{m=2}^{N+1}F_M(u,v)\sum_{k=0}^{N+1-m}z_k\sum_{k_1+\ldots.+k_{M_1}=N-k}q_{k_1}\ldots q_{k_{m-1}}- \\ &- \sum_{m=1}^{N+1}\frac{a^{(m)}(u)}{m!}\sum_{k_1+\ldots+k_m=N+1}q_{k_1}\ldots q_{k_m}.\end{aligned}$$

Note that

$$\begin{aligned}F_1(u,v)z_N-a'(u)q_{N+1} &= F_1(u,v)\left(z_N-q_{N+1}\right)+\left(F_1(u,v)-a'(u)\right)q_{N+1} \\ &= F_1(u,v)z_{N+1}+F_2(u,v)z_0q_{N+1},\end{aligned}$$

$$\begin{aligned}&\sum_{m=2}^{N+1}F_m(u,v)\sum_{k=0}^{N+1-m}z_k\sum_{k_1+\ldots+k_{m-1}=N-k}q_{k_1}\ldots q_{k_{m-1}} \\ &=\sum_{m=2}^{N+1}F_m(u,v)\sum_{k=0}^{N+1-m}\left(z_k-q_{k+1}\right)*\sum_{k_1+\ldots+k_{m-1}=N_k}q_{k_1}\ldots q_{k_{m-1}}+ \\ &\quad+\sum_{m=2}^{N+1}F_m(u,v)\sum_{k=0}^{N+1-m}q_{k+1}\sum_{k_1+\ldots+k_{m-1}=N-k}q_{k_1}\ldots q_{k_{m-1}} \\ &=\sum_{m=2}^{N+1}F_m(u,v)\sum_{k=1}^{N+2-m}z_k\sum_{k_1+\ldots+k_{m-1}=N+1-k}q_{k_1}\ldots q_{k_{m-1}}+ \\ &\quad+\sum_{m=2}^{N+1}F_m(u,v)\sum_{k_1+\ldots+k_m=N+1}q_{k_1}\ldots q_{k_m}\end{aligned}$$

and

$$\sum_{m=2}^{N+1}F_m(u,v)\sum_{k_1+\ldots+k_m=N+1}q_{k_1}\ldots q_{k_m}-\sum_{m=2}^{N+1}\frac{a^{(m)}(u)}{m!}\sum_{k_1+\ldots+k_m=N+1}q_{k_1}\ldots q_{k_m}$$

$$= \sum_{m=2}^{N+1} \left(F_m(u,v) - \frac{a^{(m)}(u)}{m!} \right) \sum_{k_1+...+k_m=N+1} q_{k_1}...q_{k_m}$$
$$= \sum_{m=2}^{N+1} F_{m+1}(u,v) z_0 \sum_{k_1+k_2+...+k_m=N+1} q_{k_1}...q_{k_m}$$
$$= \sum_{m=3}^{N+2} F_m(u,v) z_0 \sum_{k_1+...+k_{m-1}=N+1} q_{k_1}...q_{k_{m-1}}.$$

Thus

$$\begin{aligned} F_1(u,v)z_n - a'(u)q_{N+1} - G_{N+1} &= F_1(u,v)z_{N+1} + F_2(u,v)z_0 q_{N+1} + \\ &+ \sum_{m=2}^{N+1} F_m(u,v) \sum_{k=1}^{N+2-m} z_k \sum_{k_1+..+k_{m-1}=N+1-k} q_{k_1}...q_{k_{m-1}} + \\ &+ \sum_{m=3}^{N+2} F_m(u,v) z_0 \sum_{k_1+...+k_{m-1}=N+1} q_{k_1}...q_{k_{m-1}} \\ &= F_1(u,v)z_{N+1} + \sum_{m=2}^{N+2} F_m(u,v) \sum_{k=0}^{N+2-m} z_k \sum_{k_1+...+k_{m-1}=N+1-k} q_{k_1}...q_{k_{m-1}} \end{aligned}$$

and we conclude that

$$\begin{cases} \partial_t z_{N+1} + \partial_x (F_1(u,v) z_{N+1}) + \partial_x^3 z_{N+1} = -\partial_x (D_{N+1}) \\ \\ z_{N+1}(x,0) = 0 \end{cases}$$

which is (4.10) with $n = N+1$. The proof is completed by induction. $\square$

Theorem 4.1 *Let $s > \frac{3}{4}$ and $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ be given. Then, for any $\phi^* \in H^s(R)$, there exist a $T > 0$ and a neighborhood U of ϕ^* in $H^s(R)$ such that the nonlinear map K defined by the IVP (3.1) is infinitely many time Frechet differentiable in U from $H^s(R)$ to $X_{l,r}^{T,s}$. Its n-th derivative $K^{(n)}$ at $\psi \in U$ is given by*

$$K^{(n)}(\psi)[h_1,...,h_n] = \mathcal{K}^{(n)}(u)[h_1,...,h_n]$$

for any $h_1,...,h_n \in H^s(R)$ where $\mathcal{K}^{(n)}(u)$ is defined by (4.2) and (4.3) and $u = K(\psi)$.

Proof: We only need to prove that for any $\psi \in U$,

$$K(\phi + h) = \sum_{k=0}^{n} \frac{1}{k!} \mathcal{K}^{(k)}(\phi)[h^k] + o(\|h\|_s^n)$$

as $h \to 0$ in $H^s(R)$ uniformly for $\|\phi - \psi\|_s \leq \|h\|_s$ by the Converse Taylor Theorem (see [8]).

Let

$$v = K(\phi + h), \qquad u = K(\phi), \qquad y^{(k)} = \mathcal{K}^{(k)}(u)[h^k]$$

for $1 \leq k \leq n$ and

$$z_0 = v - u, \qquad z_1 = z_0 - y_0, \qquad z_n = z_{n-1} - \frac{1}{n!} y_n.$$

Then, by Proposition 4.2,

$$\begin{cases} \partial_t z_0 + \partial_x (F_1(u,v) z_0) + \partial_x^3 z_0 = 0 \\ \\ z_0(x,0) = h \end{cases} \tag{4.13}$$

and for $n \geq 1$,

$$\begin{cases} \partial_t z_n + \partial_x (F_1(u,v) z_n) + \partial_x^3 z_n F_1(u,v) = -\partial_x D_n \\ \\ z_n(x,0) = 0 \end{cases} \tag{4.14}$$

Choose $\delta_1 > 0$ such that

$$S_{\delta_1}(\psi) = \{\phi \in H^s(R),\ \|\phi - \psi\|_s \leq \delta_1\} \subset U.$$

Obviously, $K(\phi)$ is bounded on $S_{\delta_1}(\psi)$ since K is a continuous map.

By Corollary 4.1,

$$\|q_k\|_{X_{l,r}^{T,s}} \leq c(k, \|u\|_{X_{0,0}^{T,s}}) \|h\|_s^k, \qquad k = 1, 2, ..., n$$

where $u = K(\phi)$ and $c(k, \|u\|_{X_{0,0}^{T,s}})$ is uniformly bounded on $S_{\delta_1}(\psi)$. It suffices to prove that

$$\|z_n\|_{X_{l,r}^{T,s}} \leq \gamma_n \|h\|_s^{n+1} \tag{4.15}$$

for $n \geq 0$ where γ_n is uniformly bounded for $\phi \in S_{\delta_1}(\psi)$. Indeed, it is easy to obtain (4.15) for $n = 0$ by using Theorem 2.1 to (4.13) and if we suppose (4.15) is true for $n \leq N$, then applying Theorem 2.1 and Lemma 2.8 to (4.14) with $n = N+1$, we have that

$$\begin{aligned}
&\|z_{N+1}\|_{X_{l,r}^{T,s}} \\
&\leq c \sum_{m=2}^{N+2} \sum_{k=0}^{N+2-m} \sum_{k_1+\ldots+k_{m-1}=N+1-k} \int_{-T}^{T} \|\partial_x \left(F_m(u,v) z_k q_{k_1} \ldots q_{k_{m-1}} \right) \|_s dt \\
&\leq c \sum_{m=2}^{N+2} \sum_{k=0}^{N+2-m} \sum_{k_1+\ldots+k_{m-1}=N+1-k} c^m \beta_m \Lambda_{0,0}^s(T; z_k) \prod_{j=1}^{m-1} \Lambda_{0,0}^s(T; q_{k_j}) \\
&\leq c \sum_{m=2}^{N+1} \sum_{k=0}^{N+2-m} \sum_{k_1+\ldots+k_{m-1}=N+1-k} c^m \gamma_m \beta_m \prod_{j=1}^{m-1} c(k_j, \|u\|_{X_{0,0}^{T,s}}) \|h\|_s^{N+2} \\
&:= \gamma_{N+1} \|h\|_s^{N+2}
\end{aligned}$$

The proof is completed by induction. □

Corollary 4.2 (Taylor's Formula) *For any $\phi \in U$ and $h \in H^s(R)$ satisfying*

$$\phi + \xi h \in U, \qquad \textit{for any } \xi \in (0,1),$$

$$K(\phi + h) = \sum_{j=0}^{n-1} \frac{1}{j!} K^{(j)}(\phi)[h^j] + \int_0^1 \frac{(1-\xi)^{n-1}}{n!} K^{(n)}(\phi + \xi h)[h^n] d\xi$$

with any $n \geq 1$.

Proof: See [7, Theorem 8.14.3].

5 Analyticity

In the previous section we proved the map K is infinitely many times Frechet differentiable from $H^s(R)$ to $X_{l,r}^{T,s}$. Naturally the further question is whether the map K is an analytic map, i.e., whether it has Taylor series expansion at any $\phi \in H^s(R)$:

$$K(\phi + h) = \sum_{n=0}^{\infty} \frac{K^{(n)}(\phi)}{n!} [h^n], \tag{5.1}$$

where the series coverges in $X_{l,r}^{T,s}$ uniformly for $\|h\|_s \leq \delta$ with some $\delta > 0$ only depending on ϕ. For that purpose we need a better estimate of the n-th derivative $y_n = K^{(n)}(\phi)[h^n]$ of the map K at ϕ and the n-th Taylor remainder z_n.

Proposition 5.1 *Let $u \in X_{0,0}^{T,s}$ be given and y_n is the solution of (4.5)-(4.6). Then there exists a sequence $\alpha(n)$ given by*

$$\alpha(1) = 1, \tag{5.2}$$

and

$$\alpha(n) = \sum_{j=2}^{n} \frac{\beta_j}{j!} \sum_{k_1+...+k_j=n} \alpha(k_1)\alpha(k_2)...\alpha(k_j) \tag{5.3}$$

for $n \geq 2$ such that

$$\|y_n\|_{X_{l,r}^{T,s}} \leq c^n n! \alpha(n) \|h\|_s^n \tag{5.4}$$

for any $n \geq 1$ where $c > 0$ is a constant independt of n and h,

$$\beta_j = \beta_j(\|u\|_{X_{0,0}^{T,s}})$$

is a continuous monotone function depending only on $a^{(j)}$, and $\beta_j \equiv 0$ if $\alpha^{(j)} \equiv 0$.

Proof: Denote by $q_n = y_n/n!$. It solves

$$\begin{cases} \partial_t q_1 + \partial_x(a'(u)q_1) + \partial_x^3 q_1 = 0 \\ \\ q_1(x,0) = h(x) \end{cases} \tag{5.5}$$

for $n = 1$ and

$$\begin{cases} \partial_t q_n + \partial_x(a'(u)q_n) + \partial_x^3 q_n = -\partial_x\left(\sum_{j=2}^{n} \frac{a^{(j)}(u)}{j!} \sum_{k_1+...+k_j=n} q_{k_1}...q_{k_j}\right) \\ \\ q_n(x,0) = 0 \end{cases} \tag{5.6}$$

for $n \geq 2$.

Applying (2.19) to (5.5) yields

$$\|q_1\|_{X_{l,r}^{T,s}} \leq c\|h\|_s.$$

Assume that

$$\|q_m\|_{X^{T,s}_{l,r}} \le c^{2m-1}\alpha(m)\|h\|_s^m, \qquad \text{for } 1\le m\le N$$

Then applying (2.19) to (5.6) with $n = N+1$, we obtain

$$\begin{aligned}
\|q_{N+1}\|_{X^{T,s}_{l,r}} &\le c\sum_{j=2}^{N+1}\frac{1}{j!}\sum_{k_1+...+k_j=N+1}\int_{-T}^{T}\|\partial_x\left(a^{(j)}(u)q_{k_1}...q_{k_j}\right)\|_s dt\\
&\le c\sum_{j=2}^{N+1}\frac{\beta_j}{j!}\sum_{k_1+...+k_j=N+1}\prod_{l=1}^{j}\|q_{k_l}\|_{X^{T,s}_{0,0}}\\
&\le c\sum_{j=2}^{N+1}\frac{\beta_j}{j!}\sum_{k_1+...+k_j=N+1}\prod_{l=1}^{j}c^{2k_l-1}\|h\|_s^{k_l}\alpha(k_l)\\
&\le c^{2(N+1)-1}\alpha(N+1)\|h\|_s^{N+1}.
\end{aligned}$$

Thus we have proved by induction that

$$\|y_n\|_{X^{T,s}_{l,r}} \le c^{2n-1}n!\alpha(n)\|h\|_s^n$$

which is (5.4) with a different $c>0$. The proof is completed by induction. □

Proposition 5.2 *Let $s>3/4$, $T>0$, $(l,r)\in[0,s-\frac{3}{4}]\times[0,s-\frac{3}{4})$ and $u,\, v\in X^{T,s}_{0,0}$ be given. If z_n is the solution of (4.9)-(4.10) with y_n being the solution of (4.5)-(4.6), then there exists a sequence $\gamma(n)$ given by*

$$\gamma(0)=\eta_1 \tag{5.7}$$

$$\gamma(n)=\eta_1\sum_{m=2}^{n+1}\eta_m\sum_{k=0}^{n+1-m}\gamma(k)\sum_{k_1+...+k_{m-1}=n-k}\prod_{i=1}^{m-1}c^i\alpha(i) \tag{5.8}$$

for $n\ge1$ with $\alpha(n)$ given by (5.2)-(5.3) such that

$$\|z_n\|_{X^{T,s}_{l,r}} \le \gamma(n)\|h\|_s^{n+1}, \qquad n\ge 0 \tag{5.9}$$

where

$$\eta_j=\eta_j(\|u\|_{X^{T,s}_{0,0}},\|v\|_{X^{T,s}_{0,0}}),$$

depending only on $a^{(j)}$, is a continuous function of $\|u\|_{X^{T,s}_{0,0}}$ and $\|v\|_{X^{T,s}_{0,0}}$ with $\eta_j\equiv0$ if $\alpha^{(j)}\equiv0$, and $c>0$ is a constant independent of n and h.

Proof: It follows from (4.9) by applying (2.19) that

$$\|z_0\|_{X_{l,r}^{T,s}} \leq \eta_1 \|h\|_s.$$

Assume that (5.9) is true for all $0 \leq n \leq N-1$. Then applying (2.19) and (2.17) to (4.10) with $n = N$, we obtain that

$$\begin{aligned}
\|z_N\|_{X_{l,r}^{T,s}} &\leq \eta_1 \sum_{m=2}^{N+1} \sum_{k=0}^{N+1-m} \sum_{k_1+\ldots+k_{m_1}=N-k} \int_{-T}^{T} \|\partial_x \left(F_m z_k \prod_{i=1}^{m-1} q_{k_i} \right) \|_s dt \\
&\leq \eta_1 \sum_{m=2}^{N+1} \sum_{k=0}^{N+1-m} \sum_{k_1+\ldots+k_{m-1}=N-k} \eta_m \|z_k\|_{X_{0,0}^{T,s}} \prod_{i=1}^{m-1} \|q_{k_i}\|_s \\
&\leq \eta_1 \sum_{m=2}^{N+1} \sum_{k=0}^{N+1-m} \sum_{k_1+\ldots+k_{m-1}=N-k} \eta_m \gamma(k) \|h\|_s^{N+1} \prod_{i=1}^{m-1} c^i \alpha(i) \\
&\leq \gamma(N) \|h\|_s^{N+1}.
\end{aligned}$$

The proof is completed by induction. □

Now we consider the map K defined by (3.1) from an open set U in $H^s(R)$ to $X_{l,r}^{T,s}$ where $s > 3/4$ and $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$.

Proposition 5.3 *Let $\phi \in U$. If there is a $\delta_1 > 0$ and $c > 0$ such that*

$$S_{\delta_1}(\phi) = \{\psi \in H^s(R) | \quad \|\phi - \psi\|_s < \delta_1\} \subset U$$

and

$$\gamma(n) \leq c^n, \qquad n \geq 0 \tag{5.10}$$

uniformly for $\psi \in S_{\delta_1}(\phi)$ where $\gamma(n)$ is defined by (5.7)-(5.8) with $u = K(\phi)$ and $v = K(\psi)$, then there exists a $\delta > 0$ such that the series (5.1) coverges in $X_{l,r}^{T,s}$ uniformly for $\|h\|_s \leq \delta$.

Proof: Consider the n-th Taylor remainder

$$z_n = K(\phi + h) - \sum_{j=0}^{n} \frac{1}{j!} K^{(j)}(\phi)[h^j].$$

According to Proposition 5.2 and hypothesis (5.10),

$$\|z_n\|_{X_{l,r}^{T,s}} \leq \gamma(n) \|h\|_s^{n+1} \leq c^n \|h\|_s^{n+1}$$

where $c > 0$ is independent of n and $h \in H^s(R)$ with $\|h\|_s < \delta_1$. Thus if we choose $\delta > 0$ such that

$$\delta < \frac{1}{2c_1},$$

then

$$\|z_n\|_{X^{T,s}_{l,r}} \leq (\frac{1}{2})^n, \qquad n \geq 1$$

for any $h \in H^s(R)$ with $\|h\|_s \leq \delta$. The proof is completed. □

In the following we shall show that if $a(u)$ in (3.1) is a polynomial of degree N, i.e.

$$a(u) = \sum_{j=2}^{N} b_j u^j \tag{5.11}$$

with b_j, $j = 1, 2..., N$ being real constants, then hypothesis (5.10) is satisfied for any $\phi \in U$ and therefore the map K is an analytic map from $U \subset H^s(R)$ to $X^{T,s}_{l,r}$.

First we prove a technical lemma.

Lemma 5.1 *Let $N \geq 1$ be a given integer and α_n be a sequence given by*

$$\alpha_N(1) = 1, \tag{5.12}$$

$$\alpha_N(n) = \sum_{j=2}^{n} \frac{b_j}{j!} \sum_{k_1+...+k_j=n} \alpha_N(k_1)...\alpha_N(k_j), \quad \text{for } 2 \leq n \leq N-1, \tag{5.13}$$

and

$$\alpha_N(n) = \sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k_1+...+k_j=n} \alpha_N(k_1)....\alpha_N(k_j), \quad \text{for } n \geq N. \tag{5.14}$$

where b_j, $1 \leq j \leq N$ are given constants. Then there exists a constant $c > 0$ such that

$$\alpha_N(n) \leq c^n \tag{5.15}$$

for any $n \geq 1$.

Remark 5.1 *In the case $N = 2$ and $b_2 = 1$, the lemma is Proposition 3.4 in [47] where it is shown that*

$$\alpha_2(n) = \frac{2^{n-1}(2n-3)!!}{n!} \qquad \text{for any } n \geq 2.$$

Proof of Lemma 5.1: Note that that $\alpha_N(n)$, for any $n \geq 1$, is uniquelly determined by (5.12) and (5.13) inductively. In particular, we may obtain

$$\alpha_N(1),\ \alpha_N(2),\ ..., \alpha_N(N-1)$$

explicitly by computation.

Let $P_N(x)$ be a polynomial of degree N as follows

$$\begin{aligned} P_N(x,y) &= y - \sum_{j=2}^{N} \frac{b_j}{j!} \left(\sum_{k=1}^{N-1} \alpha_N(k) x^k + y \right)^j - \\ &- \sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k=j}^{N-1} x^k \sum_{k_1+...+k_j=k} \alpha_N(k_1)...\alpha_N(k_j). \end{aligned} \tag{5.16}$$

It is easy to check that

$$P_N(0,0) = 0, \qquad \frac{\partial}{\partial y} P_N(0,0) = 1.$$

Thus, according to the implicit function theorem,

$$P_N(x,y) = 0 \tag{5.17}$$

has a unique solution

$$y = f(x) \qquad \text{with } f(0) = 0$$

in the neighborhood of $x = 0$ such that

$$P_N(x, f(x)) = 0, \qquad \text{for any } |x| \leq \delta$$

where $\delta > 0$ is a constant. Besides, direct computation shows that $f^{(j)}(0) = 0$, for $j = 1, 2, ..., N-1$.

Moreover, $y = f(x)$ is a real analytic function in a neighborhood of $x = 0$, i.e. $f(x)$ has a Taylor series expansion at $x = 0$,

$$y = f(x) = \sum_{j=N}^{\infty} d_j x^j \tag{5.18}$$

which is uniformly convergent for $|x| < \delta$ with some $\delta > 0$.

To see this, let

$$g = \sum_{k=1}^{N-1} \alpha_N(k) x^k + y$$

and

$$z = \sum_{k=1}^{N-1} \alpha_N(k) x^n + \sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k=j}^{N-1} x^k \sum_{k_1+...+k_j=k} \alpha_N(k_1)...\alpha_N(k_j) := h(x).$$

Then, equation (5.17) may be written as

$$g - \sum_{j=2}^{N} \frac{b_j}{j!} g^j = z$$

which obviously has an analytic solution $g(z)$ in a neiborhood of $z = 0$ such that $g(0) = 0$. Thus $y = f(x) = g(h(x)) - \sum_{k=1}^{N-1} \alpha_N(k) x^k$ is an analytic function in a neighborhood of $x = 0$ sicne $h(x)$ is a polynomial of x.

Plugging (5.18) into (5.17) and denoting by

$$d_j = \alpha_N(j), \qquad \text{for } j = 1, 2, ..., N-1,$$

we have

$$\begin{aligned}
y &= \sum_{j=N}^{\infty} b_j x^j \\
&= \sum_{j=2}^{N} \frac{b_j}{j!} \left(\sum_{k=1}^{N-1} \alpha_N(k) x^k + \sum_{k=N}^{\infty} d_k x^k \right)^j - \\
&\qquad - \sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k=j}^{N-1} x^k \sum_{k_1+...+k_j=k} \alpha_N(k_1)...\alpha_N(k_j) \\
&= \sum_{j=2}^{N} \frac{b_j}{j!} \left(\sum_{k=1}^{\infty} d_k x^k \right)^j - \sum_{k=2}^{N} \frac{b_j}{j!} \sum_{k=j}^{N-1} x^k \sum_{k_1+...+k_j=k} d_{k_1}...d_{k_j} \\
&= \sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k=j}^{\infty} \sum_{k_1+...+k_j=k} d_{k_1} x^{k_1} d_{k_2} x^{k_2} ... d_{k_j} x^{k_j} -
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k=j}^{N-1} \sum_{k_1+...+k_j=k} d_{k_1}...d_{k_j} x^k \\
= & \sum_{j=2}^{N} \frac{b_j}{j!} \left(\sum_{k=N}^{\infty} \sum_{k_1+...+k_j=k} d_{k_1}...d_{k_j} \right) x^k \\
= & \sum_{k=N}^{\infty} \left(\sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k_1+...+k_j=k} d_{K_1}...d_{k_j} \right) x^k
\end{aligned}
$$

for any $|x| < \delta$. Thus we have

$$
d_j = \alpha_N(j), \qquad j = 1, 2, ..., N-1
$$

and

$$
d_k = \sum_{j=2}^{N} \frac{b_j}{j!} \sum_{k_1+....+k_j=k} d_{k_1}...d_{k_j}
$$

for any $k \geq N$. That is to say, d_j for $j \geq 1$ also satisfy the induction relation (5.12)-(5.13). By uniqueness we have

$$
\alpha_N(k) = d_k, \qquad \text{for any } k \geq 1.
$$

On the other hand, d_k, $k \geq 1$ are coefficients of Taylor series (5.18) and therefore we must have

$$
d_n \leq c^n, \qquad \text{for all } n \geq 1
$$

for some $c > 0$ independent of n. The proof is completed. □

Theorem 5.1 *Let $s > 3/4$ and $(l, r) \in [0, s - \frac{3}{4}] \times [0, s - \frac{3}{4})$ be given and suppose that $a(u)$ is (3.1) is a polynomial of degree N . Then for any $\phi \in H^s(R)$, there is a $T > 0$ and a neighborhood U of ϕ in $H^s(R)$ such that the IVP (3.1) define an analytic map K from U to $X^{T,s}_{l,r}$, i.e., for any $\psi \in U$, there is a $\delta > 0$ such that the Taylor series*

$$
K(\psi + h) = K(\psi) + \sum_{n=1}^{\infty} \frac{1}{n!} K^{(n)}(\psi)[h^n]
$$

uniformly converges for $\|h\|_s \leq \delta$ in the space $X^{T,s}_{l,r}$. Moreover, if denote by

$$
u = K(\psi), \qquad y_n = K^{(n)}(\psi)[h^n], \quad n \geq 1
$$

then

$$\begin{cases} \partial_t y_1 + \partial_x(a'(u)y_1) + \partial_x^3 y_1 = 0 \\ \\ y_1(x,0) = h \end{cases}$$

for $n = 1$ *and*

$$\begin{cases} \partial_t y_n + \partial_x(a'(u)y_n) + \partial_x^3 y_n = -\partial_x G_N(n) \\ \\ y_n(x,0) = 0 \end{cases}$$

for $n \geq 2$ *where*

$$G_N(n) = \begin{cases} \sum_{j=2}^{n} \frac{a^{(j)}(u)}{j!} \sum_{k_1+\ldots+k_j=n} \frac{n!}{k_1!\ldots k_j!} y_{k_1} \ldots y_{k_j} & \text{for } 2 \leq n \leq N-1 \\ \\ \sum_{j=2}^{N} \frac{a^{(j)}(u)}{j!} \sum_{k_1+\ldots k_j=n} \frac{n!}{k_1!\ldots k_j!} y_{k_1} \ldots y_{k_j} & \text{for } n \geq N. \end{cases}$$

Proof: Since U is an open subset in $H^s(R)$, there is a $\delta_1 > 0$ such that if $h \in H^s(R)$ with $\|h\|_s \leq \delta_1$, then

$$\psi + h \in U.$$

Denote by

$$v = K(\psi + h), \qquad u = K(\psi)$$

and

$$z_0 = v - u, \qquad z_n = z_{n-1} - \frac{1}{n!} y_n, \quad \text{for } n \geq 1.$$

Then, by proposition 5.1 and 5.2,

$$\|y_n\|_{X_{l,r}^{T,s}} \leq n! c^n \alpha(n) \|h\|_s^n$$

and

$$\|z_n\|_{X_{l,r}^{T,s}} \leq \gamma(n) \|h\|_s^{n+1}.$$

Note that $a^{(j)}(u) \equiv 0$ for $j \geq N+1$ since $a(u)$ is a polynomial of degree N by the assumption. Thus $\alpha(n)$ is given by

$$\alpha(1) = 1,$$

$$\alpha(n) = \sum_{j=2}^{n} \frac{\beta_j}{j!} \sum_{k_1+....+k_j=n} \alpha(k_1)...\alpha(k_j)$$

for $2 \le n \le N-1$ and

$$\alpha(n) = \sum_{j-2}^{N} \frac{\beta_j}{j!} \sum_{k_1+...+k_j=n} \alpha(k_1)...\alpha(k_j)$$

for $n \ge N$. Moreover,

$$\gamma(0) = \eta_1,$$

$$\gamma(n) = \eta_1 \sum_{j=2}^{n} \eta_j \sum_{k=0}^{n+1-j} \gamma(k) \sum_{k_1+...k_{j-1}=n-k} \prod_{i=1}^{j-1} c^{k_i} \alpha(k_i)$$

for $1 \le n \le N-1$ and

$$\gamma(n) = \eta_1 \sum_{j=2}^{N} \eta_j \sum_{k=0}^{n+1-j} \gamma(k) \sum_{k_1+...k_{j-1}=n-k} \prod_{i=1}^{j-1} c^{k_i} \alpha(k_i).$$

According to Proposition 5.3, we need to show that there is a $c_* > 0$ such that

$$\gamma(n) \le c_*^n$$

for any $n \ge N$ with c_* independent of n and $\|h\|_s \le \delta_1$. To this end, we first see from Lemma 5.1 that

$$\alpha(n) \le c_1^n, \qquad n \ge N$$

with some $c_1 > 0$ independent of n and h. Thus, for $0 \le j \le N$,

$$\begin{aligned} \sum_{k_1+...k_{j-1}=n-k} \prod_{i=1}^{j-1} c^{k_i} \alpha(k_i) &\le \sum_{k_1+...+k_{j-1}=n-k} (cc_1)^{n-k} \\ &\le (n-k-1)^N (cc_1)^{n-k} \\ &\le c_2^{n-k} \end{aligned}$$

for some $c_2 > 0$ independent of n and

$$\begin{aligned}
\gamma(n) &\leq c_3 \sum_{j=2}^{N} \sum_{k=0}^{n+1-j} \gamma(k) \sum_{k_1+\ldots+k_{j-1}=n-k} \prod_{i=1}^{j-1} c^{k_i}\alpha(k_i) \\
&\leq c_3 \sum_{j=2}^{N} \sum_{k=0}^{n+1-j} c_2^{n-k}\gamma(k) \\
&\leq c_3(N-2) \sum_{k=0}^{n-1} c_2^{n-k}\gamma(k) \\
&\leq \sum_{k=0}^{n-1} c_4^{n-k}\gamma(k)
\end{aligned}$$

for some $c_4 > 0$ independent of n where

$$c_3 = max\,\{\eta_1\eta_2, ..., \eta_1\eta_N\}$$

Assume that

$$\gamma(k) \leq c_4^k 2^{k-1}, \quad \text{for } 1 \leq k \leq n-1.$$

Then,

$$\begin{aligned}
\gamma(n) &\leq \sum_{k=0}^{n-1} c_4^{n-k}\gamma(k) \\
&\leq c_4^n + \sum_{k=1}^{n-1} c_4^{n-k} c^k 2^{k-1} \\
&\leq c_4^n 2^{n-1}
\end{aligned}$$

Thus we proved by induction that

$$\gamma(n) \leq 2^{n-1} c_4^n \leq c_*^n$$

for some $c_* > 0$ independent of n and h. The proof is completed. □

Corollary 5.1 *Assume that $a(u)$ in (3.1) is a polynomial. Then for any $T > 0$ and $s \geq 1$, the map K defined by the IVP (3.1) is analytic from $H^s(R)$ to $X_{l,r}^{T,s}$ with $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$.*

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