

**MONOTONICITY AND PARAMETRIZATION RESULTS  
FOR CONTINUOUS-TIME ALGEBRAIC RICCATI  
EQUATIONS AND RICCATI INEQUALITIES**

By

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Monotonicity and parametrization results  
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## Abstract.

The solution sets of algebraic Riccati equations  $\mathcal{R}_i(X) = A_i^*X + XA_i - XB_iB_i^*X + Q_i = 0 = (I, X_i)H_i(I, X_i)^* = 0, i = 1, 2$ , can be compared if  $H_1 \leq H_2$ . In this paper we obtain a comparison result for local frames of solutions of algebraic Riccati inequalities and prove parametrization results which involve different Riccati equations.

**Keywords:** continuous-time algebraic Riccati equation, algebraic Riccati inequalities, local frame, monotonicity, parametrization of solutions

**AMS Subject Classifications:** 15A24.

# 1 Introduction

Let

$$\mathcal{R}(X) = A^*X + XA - XBB^*X + Q = 0 \quad (1.1)$$

be a continuous-time algebraic Riccati equation (CARE) where  $A, BB^*, Q = Q^*$  and  $X = X^*$  are complex  $n \times n$  matrices. Put

$$H = \begin{pmatrix} Q & A^* \\ A & -BB^* \end{pmatrix}$$

such that  $\mathcal{R}(X) = (I, X)H(I, X)^*$ . Besides (1.1) we consider another CARE

$$\tilde{\mathcal{R}}(X) = \tilde{A}^*X + X\tilde{A} - X\tilde{B}\tilde{B}^*X + \tilde{Q} = 0 \quad (1.2)$$

with associated hermitian matrix

$$\tilde{H} = \begin{pmatrix} \tilde{Q} & \tilde{A}^* \\ \tilde{A} & -\tilde{B}\tilde{B}^* \end{pmatrix} \in \mathbb{C}^{2n \times 2n}.$$

It is well known that an inequality  $H \leq \tilde{H}$  induces relations between solutions of (1.1) and (1.2). For example, if  $Q \geq 0$  and if (1.1) has a positive-semidefinite solution then there exists a solution  $X \geq 0$  of (1.2), and if  $K$  and  $\tilde{K}$  are the least positive-semidefinite solutions of (1.1) and (1.2) respectively, then according to Wimmer (submitted) we have  $K \leq \tilde{K}$ . In this paper further results are presented which are implied by the inequality  $H \leq \tilde{H}$ . Section 2 deals with local frames of solutions of Riccati inequalities and Section 3 contains parametrization results.

## 2 The local frame

In their study of the noncausal estimation problem A. Lindquist and G. Picci (1991) considered the continuous-time algebraic Riccati inequality

$$\mathcal{R}(X) = A^*X + XA - XBB^*X + C^*C \geq 0$$

and introduced the following concepts. If  $Y$  is a solution of  $\mathcal{R}(X) \geq 0$  then a solution  $U$  (resp.  $W$ ) of the CARE  $\mathcal{R}(X) = 0$  is called an *internal lower*

(resp. upper) bound of  $Y$  if  $U \leq Y$  (resp.  $W \geq Y$ ) holds. A local frame of  $Y$  consists of a pair  $U, W$  of such internal bounds. A. Lindquist and G. Picci (1991, Section 6.4) used methods from the theory of stochastic linear systems to prove the subsequent result.

**Theorem 2.1** *Assume that  $(A, B)$  is controllable. Let  $Y$  be a solution of  $\mathcal{R}(X) \geq 0$ . Then there exists a tightest local frame of  $Y$ , i.e. there exist solutions  $U_-$  and  $W_+$  of  $\mathcal{R}(X) = 0$  such that*

$$U_- = \sup\{X \mid \mathcal{R}(X) = 0, X \leq Y\} \quad (2.1)$$

—and

$$W_+ = \inf\{X \mid \mathcal{R}(X) = 0, Y \leq X\}. \quad (2.2)$$

In this section we want to show how under the hypotheses  $0 \leq \mathcal{R}(Y) \leq \tilde{\mathcal{R}}(Y)$  and  $H \leq \tilde{H}$  the tightest local frames of  $Y$  with respect to  $\mathcal{R}(X) = 0$  and to  $\tilde{\mathcal{R}}(X) = 0$  can be compared. Let us note first that the preceding theorem can be related to results on semidefinite solutions of CAREs. If  $Y$  satisfies  $\mathcal{R}(Y) \geq 0$  then we have  $\mathcal{R}(Y) = P$  for some  $P \geq 0$ . Put

$$A_Y = A - BB^*Y.$$

Let  $X$  be a solution of  $\mathcal{R}(X) = 0$  and set  $\Delta = X - Y$ . Then it is not difficult to see that  $\Delta$  satisfies a CARE

$$\mathcal{A}(\Delta) = A_Y^* \Delta + \Delta A_Y - \Delta BB^* \Delta + P = 0. \quad (2.3)$$

Hence  $X = Y + \Delta$  is an internal upper (resp. lower) bound of  $Y$  if and only if  $\Delta \geq 0$  (resp.  $\Delta \leq 0$ ). Thus the existence of  $W_+$  and  $U_-$  in (2.1) and (2.2) is equivalent to the existence of a least positive-semidefinite and a greatest negative-semidefinite solution  $\Delta$  of (2.3). — The first part of the following lemma can be found in Geerts (1988), the second part which was mentioned already in the introduction is contained in Wimmer (submitted). We focus on positive-semidefinite solutions.

**Lemma 2.2** *Assume that  $(A, B)$  is stabilizable and  $Q \geq 0$ . (1) Then (1.1) has a solution  $X \geq 0$ , and the set of positive-semidefinite solutions of (1.1) has a least element. (2) If  $H \leq \tilde{H}$  then  $(\tilde{A}, \tilde{B})$  is also stabilizable. Let  $K$  and  $\tilde{K}$  be the least positive-semidefinite solutions of (1.1) and (1.2) respectively. Then we have  $K \leq \tilde{K}$ .*

Obviously, part (1) of the lemma yields a proof of Theorem 2.1. We shall see that part (2) leads to a comparison of tightest local frames.

**Theorem 2.3** *Assume that  $(A, B)$  is controllable and  $H \leq \check{H}$ . Let  $Y$  be a hermitian matrix such that  $0 \leq \check{\mathcal{R}}(Y) \leq \mathcal{R}(Y)$ . Then there exist  $\check{X}$  and  $\check{V}$  which satisfy  $\check{\mathcal{R}}(\check{X}) = \check{\mathcal{R}}(\check{V}) = 0$  and  $\check{V} \leq Y \leq \check{X}$ . Furthermore there exist*

$$\check{U}_- = \sup\{X \mid \check{\mathcal{R}}(X) = 0, X \leq Y\}$$

and

$$\check{W}_+ = \inf\{X \mid \check{\mathcal{R}}(X) = 0, Y \leq X\}.$$

Let  $\check{U}_-$  and  $\check{W}_+$  be given by (2.1) and (2.2). Then

$$\check{U}_- \leq U_- \leq Y \leq W_+ \leq \check{W}_+.$$

**Proof.** We are going to prove only those statements which are concerned with internal upper bounds of  $Y$ . Put  $\check{A}_Y = \check{A} - \check{B}\check{B}^*Y$  and  $\check{P} = \check{\mathcal{R}}(Y)$  and define

$$\check{A}(\check{\Delta}) = \check{A}_Y^* \check{\Delta} + \check{\Delta} \check{A}_Y - \check{\Delta} \check{B} \check{B}^* \check{\Delta} + \check{P} = 0 \quad (2.4)$$

as a counterpart to (2.3). Let

$$H_Y = \begin{pmatrix} \mathcal{R}(Y) & A_Y^* \\ A_Y & -BB^* \end{pmatrix}, \quad \check{H}_Y = \begin{pmatrix} \check{\mathcal{R}}(Y) & \check{A}_Y^* \\ \check{A}_Y & -\check{B}\check{B}^* \end{pmatrix}$$

be the hermitian matrices associated to (2.3) and (2.4). Put

$$S = \begin{pmatrix} I & O \\ Y & I \end{pmatrix}.$$

Then  $H_Y = S^* H S$  and  $\check{H}_Y = S^* \check{H} S$ . Hence  $H \leq \check{H}$  is equivalent to  $H_Y \leq \check{H}_Y$  and we can apply Lemma 2.2 to (2.3) and (2.4). To complete the proof recall that internal upper bounds of  $Y$  with respect to  $\mathcal{R}(X) = 0$  and  $\check{\mathcal{R}}(X) = 0$  are of the form  $Y + \Delta$  and  $Y + \check{\Delta}$ , where  $\Delta \geq 0$  and  $\check{\Delta} \geq 0$  are solutions of (2.3) and (2.4). □

### 3 Parametrization of solutions

Willems (1971) and Kučera (1972) use extremal solutions and projections to classify solutions of (1.1). With the monotonicity assumption  $H \leq \check{H}$  it is possible to obtain parametrization results which involve different CAREs.

Put  $E_\lambda(A) = \text{Ker}(A - \lambda I)^n$  and define  $E_{<}(A) = \oplus\{E_\lambda(A), \text{Re } \lambda < 0\}$  and  $E_{\geq}(A) = \oplus\{E_\lambda(A), \text{Re } \lambda \geq 0\}$  such that

$$\mathbf{C}^n = E_{<}(A) \oplus E_{\geq}(A). \quad (3.1)$$

Let  $P_{<}(A) : \mathbf{C}^n \rightarrow E_{<}(A)$  and  $P_{\geq}(A) : \mathbf{C}^n \rightarrow E_{\geq}(A)$  be the spectral projections induced by the decomposition (3.1). Similarly we define  $P_{\leq}(A)$  and  $P_{>}(A)$ .

In the theorem below we deal with three CAREs

$$\mathcal{R}_i(X) = A_i^* X + X A_i - X B_i B_i^* X + Q_i = 0$$

and their associated hermitian matrices

$$H_i = \begin{pmatrix} Q_i & A_i^* \\ A_i & -B_i B_i^* \end{pmatrix}.$$

$i = 1, 2, 3$ . Let  $X_i$  be a solution of  $\mathcal{R}_i(X) = 0$  and put

$$F_i = A_i - B_i B_i^* X_i,$$

$i = 1, 2, 3$ , and  $\Delta_j = X_j - X_{j-1}, j = 2, 3$ . It is not difficult to verify that

$$\begin{aligned} & \mathcal{R}_2(X_2) - \mathcal{R}_1(X_1) = \\ = & F_2^* \Delta_2 + \Delta_2 F_2 + \Delta_2 B_2 B_2^* \Delta_2 + (I, X_1)(H_2 - H_1)(I, X_1)^* = 0 \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} & \mathcal{R}_3(X_3) - \mathcal{R}_2(X_2) = \\ = & F_3^* \Delta_3 + \Delta_3 F_3 - \Delta_3 B_3 B_3^* \Delta_3 - (I, X_2)(H_3 - H_2)(I, X_2)^* = 0. \end{aligned} \quad (3.3)$$

The following auxiliary result will be needed.

**Lemma 3.1** *Let  $F, \Delta = \Delta^*, T = T^*, BB^*$  be complex  $n \times n$  matrices which satisfy*

$$F^* \Delta + \Delta F = \Delta BB^* \Delta + T \quad (3.4)$$

*Assume that*

$$\text{rank}(F - \lambda I, B) = n \text{ if } \text{Re } \lambda = 0. \quad (3.5)$$

*If  $\Delta \geq 0$  and  $T \geq 0$  then  $E_{<}(F) \subseteq \text{Ker } \Delta$ .*

**Proof.** It is well known that  $\Delta \geq 0, T \geq 0$  together with the Lyapunov-type equation (3.4) yield  $E_{<}(F) \subseteq \text{Ker } \Delta$ . Let us show that the additional assumption (3.5) implies  $E_{=}(F) = \oplus\{E_{\lambda}(F), \text{Re } \lambda = 0\} \subseteq \text{Ker } \Delta$ . We consider a Jordan chain  $y_1, y_2, \dots, y_k, y_1 \neq 0$ , of  $F$  where  $(F - \lambda I)y_i = y_{i-1}, i = 1, \dots, y_k, y_0 = 0$  and  $\text{Re } \lambda = 0$ . As induction hypotheses we assume  $y_{i-1} \in \text{Ker } \Delta$ . Then  $y_i^*(F^* \Delta + \Delta F)y_i = (\lambda + \bar{\lambda})y_i^* \Delta y_i = 0$  and (3.4) yields

$$y_i^* \Delta B = 0 \quad (3.6)$$

and  $y^* T = 0$ . Therefore

$$0 = y_i^* \Delta(\bar{\lambda} I + F) = y_i^* \Delta(-\lambda I + F) = 0. \quad (3.7)$$

From (3.5) - (3.7) follows  $y_i^* \Delta = 0$ , i.e.  $y_i \in \text{Ker } \Delta$ . Since  $y_0 \in \text{Ker } \Delta$  is trivially satisfied we have  $\langle y_1, \dots, y_k \rangle \subseteq \text{Ker } \Delta$ . □

**Theorem 3.2** *Assume  $H_1 \leq H_2 \leq H_2$  and*

$$\text{rank}(A_2 - \lambda I, B_2) = n \text{ if } \text{Re } \lambda = 0. \quad (3.8)$$

*Let  $X_i$  be a solution of  $\mathcal{R}_i(X) = 0, i = 1, 2, 3$ . If  $X_1 \leq X_2 \leq X_3$  then we have*

$$X_2 = X_1 P_{>}(F_2) + X_3 P_{<}(F_2) \quad (3.9)$$

*where  $F_2 = A_2 - B_2 B_2^* X_2$ .*

**Proof.** From (3.2) follows  $F_2^* \Delta_2 + \Delta_2 F_2 = -\Delta_2 B_2 B_2^* \Delta_2 - S$  and we have  $\Delta_2 \geq 0$  and  $S \geq 0$ . Therefore Lemma 3.1 yields  $E_{>}(F_2) \subseteq \text{Ker } \Delta_2$  or equivalently

$$\Delta_2 P_{>}(F_2) = 0. \quad (3.10)$$



Similarly (3.3) implies  $F_2^* \Delta_3 + \Delta_3 F_2 = \Delta_3 B_2 B_2^* \Delta_3 + T$ , and  $\Delta_2 \geq 0, T \geq 0$ . At this point we take (3.8) into account to conclude that  $E_{\leq}(F_2) \subseteq \text{Ker } \Delta_3$  or

$$\Delta_3 P_{\leq}(F_2) = 0. \quad (3.11)$$

Clearly (3.10) and (3.11) are equivalent to  $X_2 P_{>}(F_2) = X_1 P_{>}(F_2)$  and  $X_2 P_{\leq}(F_2) = X_3 P_{\leq}(F_2)$ . From  $P_{>}(F_2) + P_{\leq}(F_2) = I$  follows (3.9).  $\square$

Instead of (3.10) and (3.11) we could also work with  $\Delta_2 P_{\geq}(F_2) = 0$  and  $\Delta_3 P_{<}(F_2) = 0$  and derive a representation of the form

$$X_2 = X_1 P_{\geq}(F_2) + X_3 P_{<}(F_2). \quad (3.12)$$

In the special case where  $H_1 = H_2 = H_3$  the preceding theorem can be found in Ando (1988, p.50).

In the following we consider positive-semidefinite solutions and assume  $Q_i \geq 0$ .

**Corollary 3.3** *Let  $\mathcal{R}_i(X) = A_i^* X + X A_i - X B_i B_i^* X + C_i^* C_i$  and*

$$H_i = \begin{pmatrix} C_i^* C_i & A_i^* \\ A_i & -B_i B_i^* \end{pmatrix}$$

*$i = 2, 3$ , be given. Assume (3.8) and  $H_2 \leq H_3$ . If  $\mathcal{R}_i(X_i) = 0, i = 2, 3$ , and  $0 \leq X_2 \leq X_3$  then*

$$X_2 = X_3 P_{<}(A_2 - B_2 B_2^* X_2). \quad (3.13)$$

**Proof.** Define  $\mathcal{R}_1(X) = A_2^* X + X A_2 - X B_2 B_2^* X$  and

$$H_1 = \begin{pmatrix} 0 & A_2^* \\ A_2 & -B_2 B_2^* \end{pmatrix}.$$

Then  $X_1 = 0$  is a solution of  $\mathcal{R}_1(X) = 0$  and we have  $H_1 \leq H_2$ . Hence (3.12) yields (3.13).  $\square$

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