

**The Cohomology of Dihedral Nichols Algebras using
Koszul Complexes**

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Abstract

In [12], the authors equate the cohomology of Hurwitz spaces to the cohomology of a braided Hopf algebra called the quantum shuffle algebra. This quantum shuffle algebra has a subalgebra called a Nichols algebra, and in this document, we study this subalgebra's cohomology. In particular, we study the quadratic covers of Nichols algebras generated by the set of reflections in the dihedral group of order $2p$, where p is an odd prime. When $p = 3$, this algebra is isomorphic to the third Fomin-Kirillov algebra, whose cohomology was computed in [23]. We study the dual Koszul complex of this algebra and use its structure to provide an alternate proof of this result. We expand our study of the dual Koszul complex to the case where $p \geq 5$, and we prove that these algebras are infinite dimensional. We further prove preliminary results about their cohomology groups and state a conjecture equating their dimension in a fixed degree to a family of recursive sequences, $\mathcal{S}_p(n)$, related to the Lucas numbers.

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Chapter 1

Introduction

Given a braided vector space V , the Nichols algebra $\mathfrak{B}(V)$ is the subalgebra of the quantum shuffle algebra $\mathfrak{A}(V)$ generated by the degree 1 terms. Given a finite group G and a conjugacy closed subset \mathcal{C} , we can form a braided vector space V with basis \mathcal{C} . In this paper, we study the corresponding Nichols algebras. In Section 1.1, we explain the motivation for studying the cohomology of these algebras.

In particular, we analyze the case when $G = D_{2p}$, the dihedral group of order $2p$ where p is an odd prime and \mathcal{C} is the conjugacy class of reflections. In Chapter 3, we outline a process to construct the quadratic cover of this Nichols algebra, which we abbreviate as a QCN algebra and denote \mathcal{B} . Our main tool for studying this algebra is to write out the dual to its Koszul complex. The quadratic dual to \mathcal{B} , which we denote \mathcal{R} , has a well understood structure and has a stabilization element $S \in \mathcal{R}$ such that multiplication by S as a map $\mathcal{R}_n \rightarrow \mathcal{R}_{n+2}$ is eventually an isomorphism. Taking a quotient of the dual Koszul complex by the image of multiplication by S on the \mathcal{R} component of the complex yields a finite quotient complex. Explicitly, this complex of free \mathcal{B}^* modules has the form

$$0 \longleftarrow \mathcal{B}^{*1} \xleftarrow{\alpha_3} \mathcal{B}^{*p} \xleftarrow{\alpha_2} \mathcal{B}^{*2p-2} \xleftarrow{\alpha_1} \mathcal{B}^{*p} \xleftarrow{\alpha_0} \mathcal{B}^{*1} \longleftarrow 0.$$

Shockingly, the ranks of the free modules within this complex have symmetry about the central term. The main results in this document discuss when this symmetry extends

to the cohomology of this complex, and how this effects the cohomology of the algebra \mathcal{B} itself.

When $p = 3$, the algebra \mathcal{B} is isomorphic to the third Fomin-Kirillov algebra, which is 12 dimensional ([17]). In this case, the QCN algebra \mathcal{B} is equal to the Nichols algebra $\mathfrak{B}(V)$, and its cohomology has been computed in [23]. In this case, we can use the finiteness of the algebra and the symmetry in the complex to prove the following result, which follows immediately from Lemma 4.3.4.

Lemma 1.0.1. *Let $p = 3$ and let $K(\mathcal{B})^*/S$ denote the quotient dual Koszul complex. There is a vector space isomorphism $H^i(K(\mathcal{B})^*/S) \cong H^{4-i}(K(\mathcal{B})^*/S)$.*

Using this lemma, we can completely compute the cohomology of this quotient complex, which is shown in Theorem 4.3.8. Using this cohomology, we get Corollary 4.3.11, an alternate proof of Ştefan-Vay's cohomology result in [23].

However, when $p > 3$, this duality does not hold. If the duality did hold, then the degree 4 cohomology of the complex would be one-dimensional over k , but in the following Corollary, which appears in this paper as Corollary 5.2.4, we show that this is not the case.

Corollary 1.0.2. *Let $p > 3$. The degree 4 cohomology of the quotient dual Koszul complex $H^4(K(\mathcal{B})^*/S)$ is zero.*

Finally, we prove the following Theorem, which appears as Theorem 5.2.3, which says that the algebra \mathcal{B} is infinite dimensional for $p > 3$.

Theorem 1.0.3. *Let \mathcal{B} denote the QCN algebra corresponding to the dihedral group D_{2p} for an odd prime p greater than 3. Then \mathcal{B} is infinite dimensional.*

Moreover, we conjecture that the dimensions of the degree n part of these algebras are given by a family sequences called the \mathcal{S}_r , defined in Definition 5.0.4. This conjecture is named the Generalized Alternating Lucas Number conjecture due to a relationship between this family of sequences and an analogue of the Lucas numbers. Lemma 5.1.2 relates the cohomology of the Koszul complex of \mathcal{B} to this conjecture.

Theorem 1.0.4. *For an odd prime p , if the QCN algebra \mathcal{B} is Koszul, then the Generalized Alternating Lucas Number Conjecture holds, which is to say that the dimension of \mathcal{B}_n is $\mathcal{S}_p(n)$.*

1.1 Function Field Statistics

The goal of this section is to motivate for study of dihedral Nichols algebras by linking the cohomology of the D_6 Nichols algebra to an open question in arithmetic statistics and number theory. Number fields, which are finite algebraic extensions of \mathbb{Q} , are a fundamental object of study in number theory. A number field L/\mathbb{Q} has a basic invariant called its discriminant, denoted $\Delta_{L/\mathbb{Q}}$. In this introduction, we'll study the count of distinct number fields of a fixed degree, up to isomorphism, with a given discriminant. We denote this statistic as

$$N_n(a, b) = \#\{L/\mathbb{Q} \mid \deg(L/\mathbb{Q}) = n \text{ and } a \leq \Delta_{L/\mathbb{Q}} \leq b\}.$$

Note that setting $a = 0$ and $b = X$ yields the number of distinct number fields with discriminant bounded by X . A common conjecture, possibly first associated to Linnik, is that for a fixed n , the statistic $N_n(0, X)$ is asymptotically linear in X . This conjecture has been proven when $n = 3$ by Davenport and Hielbronn ([11]) and when $n = 4$ and $n = 5$ by Bhargava and his collaborators ([5], [6], [7], [9]), but it remains unknown for $n > 5$. Further, Taniguchi and Thorne ([24]) proved that the second order asymptotic growth of $N_3(0, X)$ is $X^{5/6}$. Independently, Bhargava, Shankar, and Tsimerman ([8]) proved the same result, but their error term was $O(X^{13/16+\epsilon})$.

Theorem 1.1.1 (Taniguchi-Thorne). *When $n = 3$,*

$$N_3(0, X) = \frac{1}{12\zeta(3)}X + \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{7/9+\epsilon}).$$

For the following paragraph, let $N_n(a, b)$ denote the analogue of the statistic described above which counts function fields rather than number fields. Recall that function fields are finite algebraic extensions of the field of rational functions $\mathbb{F}_q(t)$ over a finite field \mathbb{F}_q and are also a fundamental object of study in number theory. Let G be a finite group, m an integer, and $c \subseteq G$ a union of conjugacy classes. The space $\text{Hn}_{G,m}^c$ is a quotient of a union of components of the Hurwitz space $\text{Hur}_{G,m}^c$. Moreover, the k points of $\text{Hn}_{G,m}^c$, that is $\text{Hn}_{G,m}^c(k)$, are in bijection with function fields $L/k(t)$ with Galois group G and certain other characteristics such that, when added up over all transitive subgroups $G \leq S_n$, with $c = G \setminus \{1\}$, over the field \mathbb{F}_q , gives $N_n(0, X)$. Here,

X is a quantity that depends on the number of branch points in a logarithmic way. For instance, the contribution from the simple branching is q^n . We'll focus on the simply branched case, and for this reason we'll replace X with q^n .

In short, the function field analogue of $N_n(a, b)$ can be computed by enumerating $\text{Hn}_{G,m}^c$. Using the Grothendieck-Lefschetz fixed point theorem, we can enumerate these Hurwitz spaces by determining the trace of the Frobenius map on their homology. The homology of Hurwitz space was studied in [12], which gives the following formula, stated in this paper as Theorem 2.2.4

$$H_{c,et}^{2m-j}(\text{Hur}_{G,m}^c, k) \cong H_j(\text{Hur}_{G,m}^c, k) = \text{Ext}_{\mathfrak{A}(V)}^{m-j, -m}(k, k),$$

where $V = kc$ is the braided vector space generated by c and $\mathfrak{A}(V)$ denotes the quantum shuffle algebra it generates. In general, these quantum shuffle algebras are not fully understood, but they have a sub-algebra $\mathfrak{B}(V) \leq \mathfrak{A}(V)$ called the Nichols subalgebra that is more well-studied. As a heuristic, we make a number simplifying assumptions involving \mathfrak{B} and the Frobenius map. Namely, we replace $\text{Ext}_{\mathfrak{A}(V)}^{m-j, -m}$ in the theorem above with $\text{Ext}_{\mathfrak{B}(V)}^{m-j, -m}$ and assume the Frobenius map somehow extends naturally.

Consider the case that $G = S_3$, the set c denotes the set of transpositions, and k is the finite field \mathbb{F}_q . We denote $\text{Hur}_{S_3,m}^c$ as Hur_m . In this case, there is simple branching which further justifies replacing X with q^n . The algebra $\mathfrak{B}(V)$ is isomorphic to the third Fomin-Kirillov algebra, with Ext algebras computed by Ştefan and Vay in [23]. Their main result, which is stated in this paper as Theorem 2.4.3, is that

$$\text{Ext}_{\mathfrak{B}(V)}(k, k) = \mathcal{R}[X].$$

Here $\mathcal{R} = \mathfrak{B}(V)^\dagger$ is generated by three generators in bidegrees $1, -1$ and X is in bidegree $4, -6$. By the equality above,

$$H_0(\text{Hur}_m) = \text{Ext}_{\mathfrak{B}(V)}(k, k)^{m, -m} = \mathcal{R}_m,$$

and X corresponds to an element in $H_2(\text{Hur}_6) \cong H_{c,et}^{10}(\text{Hur}_6)$. Therefore, we can write

the homology of the Hurwitz space in degree m as

$$\mathcal{R}[X]_m = \mathcal{R}_m \oplus \mathcal{R}_{m-6}X \oplus \mathcal{R}_{m-12}X^2 \oplus \cdots \oplus \mathcal{R}_{(m \bmod 6)}X^{\lfloor m/6 \rfloor}.$$

We'll now compute the trace of our presumptive Frobenius map on this homology. By Lemma 3.2.1, the degree of \mathcal{R}_m stabilizes at 6. The action of the Frobenius map on this set of components is well understood. When m is odd it has no fixed points, and when m is even there is a single fixed component. So the contribution to the fixed point count from \mathcal{R}_m is 0 when m is odd and is q^m when m is even. Moreover, we can approximate the effect of Frob on x using Deligne's bounds for the eigenvalues of the Frobenius map over \mathbb{F}_q . On $H_{c,et}^{10}(\text{Hur}_6)$, the largest eigenvalues we can expect are $q^{10/2} = q^5$. We'll assume this is the eigenvalue so that $\text{Frob}(X) = q^5 X$. We'll further assume that multiplication is equivariant for the Galois action, which is to say that $\text{Tr}(\text{Frob} |_{X^m}) = \text{Tr}(\text{Frob} |_X)^m$. When m is even, the trace of Frob on $\mathcal{R}[X]_m$ is

$$q^m + q^{m-6}q^5 + q^{m-12}q^{10} + \cdots + q^{5(\lfloor m/6 \rfloor) + (m \bmod 6)} \approx \frac{1}{q-1} \left(q^{m+1} - q^{(5/6)m} \right).$$

Appropriately summing over all indices yields

$$N_3(0, q^m) \approx Cq^m + D(q^m)^{5/6}$$

for some constants C and D . In other words, up to various assumptions, the second order term in the function field count matches the known second order term in the number field count. This fascinating and surprising result motivates the further study of Nichols algebra cohomology.

Chapter 2

Background

2.1 Braided Vector Spaces

To start, we introduce the braid group, which can be defined in three equivalent ways. First, we can give an algebraic presentation of the braid group:

Definition 2.1.1. For any natural number $n > 0$, the Braid group B_n is given as

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \rangle.$$

The presentation of B_n is identical to a presentation of the symmetric group \mathcal{S}_n omitting the relations that $\sigma_i^2 = 1$. In the symmetric group, σ_i represents the inversion $(i, i + 1)$. This gives a natural map $B_n \rightarrow \mathcal{S}_n$. This natural map has a section $\mathcal{S}_n \rightarrow B_n$ known as the Matsumoto section, defined by representing an arbitrary permutation in \mathcal{S}_n as a minimum length word in the σ_i 's and lifting to the corresponding element in B_n with the same representation as a word in σ_i 's. Notably, this section is not a homomorphism.

We can alternatively define B_n in terms of configuration spaces.

Definition 2.1.2. Given a manifold M , the configuration space $\text{Conf}_n(M)$ is the topological space of all n element subsets of M .

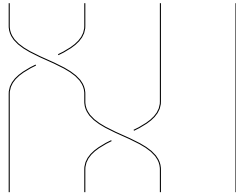
Taking $M = \mathbb{C}$, we can define the braid group as the fundamental group $\pi_1(\text{Conf}_n(\mathbb{C})) = B_n$. In fact, $\text{Conf}_n(\mathbb{C})$ is the Eilenberg-MacLane space $K(B_n, 1)$ ([3]).

Equivalently, an element in B_n can be thought of as a braid on n -strands. A braid consists of two identical sets of n points and n three-dimensional non-intersecting paths up to homotopy. In a braid, each path has a point from the first set as a starting point and a point in the second set as an ending point such that no point is used as a starting or ending point twice. These braids are realizations of elements in $\pi_1(\text{Conf}_n(\mathbb{C}))$. We can visualize braids by projecting them into two dimensions in the form of braid drawings.



Here are two simple examples of braid drawings on 4 strands

Multiplication of braids in B_n is given by gluing one braid on top of another and reparametrizing. The generators $\sigma_i \in B_n$ correspond to the braid that interchanges the i^{th} strand with the $(i+1)^{\text{st}}$ strand, passing the former in front of the latter, and leaves all other strands constant. With this interpretation, the braids in Figure 1 are σ_1 and σ_2 in B_4 and the braid below in Figure 2 is the product $\sigma_1\sigma_2$.



This is the result of multiplying the two braids shown in Figure 1.

Let k be a field and assume that all tensor products are taken over k . We retain this convention for the remainder of the paper. We will use often bar complex notation to represent elements in $V^{\otimes n}$, denoting $a_1 \otimes a_2 \otimes \dots \otimes a_n$ as $[a_1|a_2|\dots|a_n]$.

Definition 2.1.3. A **braided vector space** V is a finite dimensional vector space over a field k equipped with an invertible linear map $\sigma : V \otimes V \rightarrow V \otimes V$ that satisfies the following equation:

$$(\sigma \otimes \text{id}) \circ (\text{id} \otimes \sigma) \circ (\sigma \otimes \text{id}) = (\text{id} \otimes \sigma) \circ (\sigma \otimes \text{id}) \circ (\text{id} \otimes \sigma).$$

We refer to σ as a *braiding*. This equation is referred to as the braid equation and equates the quantities above as self maps of $V \otimes V \otimes V$.

If (V, σ) is a braided vector space, the braid group B_n acts on $V^{\otimes n}$ by equating the generator σ_i to the self map $\text{id}^{\otimes i-1} \otimes \sigma \otimes \text{id}^{\otimes n-i-1}$.

As we study various braided vector spaces and algebras defined over them in the coming sections, we will often look to decompose such algebras using a twisted tensor product, defined below.

Definition 2.1.4. If A and R are algebras over k whose multiplication maps are given by m_A and m_R respectively. A linear map $\sigma : R \otimes A \rightarrow A \otimes R$ is called a twisting map if it is compatible with m_A and m_R , which is to say that

$$\begin{aligned}\sigma \circ (\text{id}_R \otimes m_A) &= (m_A \otimes \text{id}_R) \circ (\text{id}_A \otimes \sigma) \circ (\sigma \otimes \text{id}_A) \\ \sigma \circ (m_R \otimes \text{id}_A) &= (\text{id}_A \otimes m_R) \circ (\sigma \otimes \text{id}_R) \circ (\text{id}_R \otimes \sigma),\end{aligned}$$

and σ is compatible with the units of A and R , which is to say that for all $a \in A$ and $r \in R$, $\sigma(r \otimes 1_A) = 1_A \otimes r$ and $\sigma(1_R \otimes a) = a \otimes 1_R$. Such a twisting map defines a multiplicative structure on $A \otimes R$, whose product is given by

$$m_{A \otimes R} = (m_A \otimes m_R) \circ (\text{id}_A \otimes \sigma \otimes \text{id}_R).$$

We call this multiplicative structure the **twisted tensor product**, and we denote it $A \otimes_\sigma R$.

2.2 The Quantum Shuffle Algebra

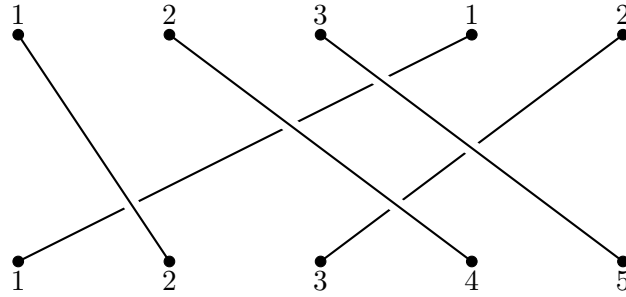
In this paper, we will turn our attention to understanding a specific algebra defined using a braided vector space known as the quantum shuffle algebra, and an important subalgebra called the Nichols subalgebra. To define this algebra, we first define a specific type of braid known as a shuffle.

Definition 2.2.1. An **m-n-shuffle** is a bijection $\tau : \{1, \dots, m\} \sqcup \{1, \dots, n\} \rightarrow \{1, \dots, m+n\}$ such that for all pairs $a, b \in \{1, \dots, m\}$ or $a, b \in \{1, \dots, n\}$, we have that $a < b$ if and

only if $\tau(a) < \tau(b)$. In other words, a shuffle does not change the order of $\{1, \dots, m\}$ or $\{1, \dots, n\}$.

A shuffle τ corresponds naturally to a permutation in \mathcal{S}_{m+n} and therefore naturally lifts to an element $\tilde{\tau} \in B_{m+n}$ according to the section described previously. If V is a braided vector space, τ acts on $V^{\otimes m+n}$ by first lifting to an element in B_{m+n} .

In this context of braid drawings, an m - n -shuffle τ corresponds to the braid drawing that labels the starting points of each strand according to the domain $\{1, \dots, m\} \sqcup \{1, \dots, n\}$, the ending points of each strand according to the codomain $\{1, \dots, m+n\}$, and connecting the points that are mapped according to τ , with the leftmost m strands passing in front of the rightmost n strands. For example, if τ is the 3-2-shuffle in which $(1, 2, 3) \mapsto (2, 4, 5)$ and $(1, 2) \mapsto (1, 3)$, then $\tilde{\tau}$ is the braid that is homotopic to the drawing below.



Definition 2.2.2. If (V, σ) is a braided vector space over k , the **quantum shuffle algebra** $\mathfrak{A}(V)$ is a braided graded bialgebra whose underlying coalgebra is the tensor coalgebra $T^{\text{co}}(V) = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$ with deconcatenation coproduct Δ defined as

$$\Delta([a_1|a_2|\dots|a_n]) = \sum_{i=1}^n [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_n],$$

and with multiplication \star defined as

$$[a_1|\dots|a_m] \star [b_1|\dots|b_n] = \sum_{\tau} \tilde{\tau} [a_1|\dots|a_m|b_1|\dots|b_n],$$

where the sum is over all m - n -shuffles τ .

A common construction we will use is, given a braided vector space (V, σ) , we can

construct another braided vector space V_ϵ by deforming σ by a sign. More formally, we construct the space $(V_\epsilon, \sigma_\epsilon)$ where $V_\epsilon = V$ and $\sigma_\epsilon = -\sigma$.

Example 2.2.3 ($V = k$). As an example, consider the trivial vector space $V = k$ with standard braiding $\sigma(a \otimes b) = b \otimes a$, so that V_ϵ is the braided vector space with braiding $\sigma(a \otimes b) = -b \otimes a$. Let $x_n = [1 \dots 1]$ be the degree n element in the tensor algebra with n instances of 1, the collection of which generate the tensor algebra. The action of an m - n -shuffle τ on x_n in V_ϵ can be easily shown to be $\tilde{\tau}x_n = \text{sgn}(\tau)x_n$ where $\text{sgn}(\tau) = (-1)^{|\tau|}$ where $|\tau|$ is the length of $\tilde{\tau}$ as a word in the σ_i 's, or equivalently where $|\tau|$ is the number of crossings in $\tilde{\tau}$ as a braid drawing.

In the quantum shuffle algebra $\mathfrak{A}(V_\epsilon)$, the product $x_m \star x_n = c_{m,n}x_{m+n}$ where

$$c_{m,n} = \sum_{\tau} \text{sgn}(\tau),$$

where this sum is over all m - n -shuffles τ . There are exactly $n + 1$ shuffles of 1 and n , whose signs alternate between 1 and -1 , so we can compute $c_{1,n} = c_{n,1} = 0$ if n is odd, and $c_{1,n} = c_{n,1} = 1$ if n is even. Therefore, $x_1 \star x_1 = 0$ and $x_1 \star x_{2n} = x_{2n+1}$. Moreover, one can compute by a straightforward recursion that $c_{2m,2n} = \binom{m+n}{n}$. These computations are symmetric, so we have $c_{n,m} = c_{m,n}$ for all m and n , so this algebra is commutative. Recall that the *exterior algebra* $\Lambda[x]$ is $k[x]/x^2$, and the *divided power algebra* $\Gamma[x]$ in degree d is the algebra generated additively by x_n in degree dn with multiplicative structure $x_n \star x_m = \binom{m+n}{n} x_{m+n}$.

Putting this all together, the quantum shuffle algebra $\mathfrak{A}(V_\epsilon)$ is generated multiplicatively by the degree 1 element x_1 and the degree 2 element x_2 and is isomorphic to $\Lambda[x_1] \otimes \Gamma[x_2]$.

◇

Our main reason for studying this quantum shuffle algebra is that it is the main object of study in [12]. The main result of this paper is included below. For the remainder of the paper, we will focus on the bigraded ring $\text{Ext}_R(k, k)$ corresponding to an algebra R . We refer to $\text{Ext}_R(k, k)$ as the cohomology of the algebra. Since we will exclusively be computing Ext with k as both inputs, we will occasionally refer to this algebra simply as Ext_R .

Theorem 2.2.4 (Ellenberg-Tran-Westerland). *If V is a braided vector space, braid group homology calculations with coefficients in V can be made by computing the bigraded Ext algebra of the quantum shuffle algebra $\mathfrak{A}(V_\epsilon)$. Formally,*

$$H_j(B_n; V^{\otimes n}) \cong \text{Ext}_{\mathfrak{A}(V_\epsilon)}^{n-j, -n}(k, k).$$

One should note that, in [12]’s presentation of this result, they use “ n ” as opposed to “ $-n$ ” in the bidegree of the Ext algebra. For the sake of this document, we’ll consider the degree in Hom to be the negative of what they use. In other words, if A and B are graded modules, we define the grading on a morphism $f \in \text{Hom}_R(A, B)$ to be the integer d such that f **increases** the degree of an element by d , or, in other words, f is degree preserving when viewed as a map $A \rightarrow \Sigma^d B$, the d^{th} suspension of B . In the construction of Ext, the bigrading on the algebra $\text{Ext}_R^i(k, k)$ is inherited from the grading on Hom. In their paper, this missing negative sign cascades into complications later in the paper, so we avoid this notational complexity and use $-n$ from the beginning.

Example 2.2.5 ($V = k$, continued). Let’s return to Example 2.2.3 where $V = k$ with the standard braiding. The vector space $V^{\otimes n}$ is generated by the class $x_n = [1] \dots [1]$, so $V^{\otimes n} \cong k$. Therefore, the term on the left in the theorem above becomes $H_j(B_n; k)$, and since the Eilenberg-MacLane space for B_n is $\text{Conf}_n(\mathbb{C})$, we can rewrite this as the topological homology $H_j(\text{Conf}_n(\mathbb{C}); k)$. We can recover the entirety of this Ext algebra by taking the sum over all n , which gives a union of configuration spaces on the left hand side,

$$H_* \left(\bigsqcup_n \text{Conf}_n(\mathbb{C}); k \right) = \text{Ext}_{\mathfrak{A}(V_\epsilon)}^*(k, k).$$

Turning to the right hand side, we computed in Example 2.2.3 that $\mathfrak{A}(V_\epsilon)$ is isomorphic to the tensor product of an exterior algebra with a divided power algebra, $\Lambda[x_1] \otimes \Gamma[x_2]$ where x_1 has degree 1 and x_2 had degree 2. We can compute Ext of these two factors independently. We will first further assume that k is a field of characteristic zero, in which case $\Gamma[x_2] = k[x_2]$. The following are standard facts about Ext of simple algebras which we use without proof.

- If R decomposes as a tensor product $R = R_1 \otimes R_2$, then the cohomology algebra similarly decomposes $\text{Ext}_R(k, k) \cong \text{Ext}_{R_1}(k, k) \otimes \text{Ext}_{R_2}(k, k)$

- If $R = k[x]$ is the single variable polynomial ring over k where x has degree d , then $\text{Ext}_{k[x]}(k, k) \cong k[y]/y^2 \cong \Lambda[y]$ where y has bidegree $(1, -d)$.
- If $R = \Lambda[x]$ is the exterior algebra where x has degree d , then $\text{Ext}_{\Lambda[x]}(k, k) \cong k[y]$ where y has bidegree $(1, -d)$.

Putting these facts together,

$$\text{Ext}_{\Lambda[x_1] \otimes \Gamma[x_2]}^*(k, k) \cong k[y_1] \otimes \Lambda[y_2],$$

where y_1 has bidegree $(1, -1)$ and y_2 has bidegree $(1, -2)$. This means that

$$H_* \left(\bigsqcup_n \text{Conf}_n(\mathbb{C}); k \right) = k[y_1] \otimes \Lambda[y_2].$$

We can therefore use Theorem 2.2.4 to compute the rational homology of the configuration space, so let $k = \mathbb{Q}$. The class $y_1 \in \text{Ext}^{1,-1}$ represents a 0 dimensional homology class in $\text{Conf}_1(\mathbb{C})$ and $y_2 \in \text{Ext}^{1,-2}$ represents a 1 dimensional homology class in $\text{Conf}_2(\mathbb{C})$. Moreover, all elements in the Ext algebra on the right are products y_1^n or $y_1^n y_2$. By the additivity of the bidegree, elements of the form y_1^n are in $\text{Ext}^{n,-n}$, so they represent the 0 dimensional homology class in $\text{Conf}_n(\mathbb{C})$, while elements of the form $y_1^k y_2$ are in $\text{Ext}^{k+1,-(k+2)}$ which represent 1 dimensional cohomology class in $\text{Conf}_{k+2}(\mathbb{C})$. Thus, the rational homology of $\text{Conf}_n(\mathbb{C})$ is as follows,

$$H_*(\text{Conf}_n(\mathbb{C}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0 \\ \mathbb{Q} & * = 1, n > 1 \\ 0 & \text{otherwise} \end{cases}$$

This is consistent with the original proof of this result, which was done by Arnol'd using a spectral sequence argument [2].

◇

Finally, an important object in the study of quantum shuffle algebras is the Nichols subalgebra $\mathfrak{B}(V) \leq \mathfrak{A}(V)$.

Definition 2.2.6. The **Nichols subalgebra** $\mathfrak{B}(V) \leq \mathfrak{A}(V)$ is the subalgebra generated (under the quantum shuffle product) by V , the degree 1 elements in $\mathfrak{A}(V)$.

Example 2.2.7 ($V = k$, continued). Concluding the previous example, when $V = k$ with the trivial braiding, we showed that $\mathfrak{A}(V_\epsilon) \cong \Lambda[x_1] \otimes \Gamma[x_2]$ where x_1 has degree 1 and x_2 has degree 2. In this case, $\mathfrak{B}(V) \cong \Lambda[x_1]$.

◇

2.3 Braided Hopf Algebras

The quantum shuffle algebra and its Nichols subalgebra as defined in the previous section are examples of braided Hopf algebras.

Definition 2.3.1. A **braided bialgebra** A is a braided vector space with a product $\mu : A \otimes A \rightarrow A$ and a coproduct $\Delta : A \rightarrow A \otimes A$ along with a unit $\nu : k \rightarrow A$ and counit $\epsilon : A \rightarrow k$. Here Δ is an algebra map where the algebra structure on $A \otimes A$ is given using the braiding on A , specifically

$$(a \otimes b) \times (c \otimes d) = [\mu \otimes \mu](a \otimes (\sigma(b \otimes c)) \otimes d).$$

Further, A is a **braided Hopf algebra** if there is a map $S : A \rightarrow A$ of braided vector spaces with

$$\mu \circ (S \otimes 1) \circ \Delta = \eta \circ \epsilon = \mu \circ (1 \otimes S) \circ \Delta.$$

In the case of the quantum shuffle algebra and its Nichols subalgebra, the product μ is the quantum shuffle product, the coproduct Δ is the standard coproduct on the tensor coalgebra, the unit ν is the inclusion map, and the counit ϵ is the section of ν which sends everything in $V^{\otimes n}$ to zero for $n > 0$. We will not make use of the map S , but since these algebras are connected bialgebras, there is a canonical choice of antipode as described in [Mil]. We can set $S(1) = 1$ and define S on positive degree elements as the unique solution to

$$\sum_{i=0}^n [a_0 | \dots | a_i] \star [a_{i+1} | \dots | a_n] = 0.$$

Definition 2.3.2. Given a braided Hopf algebra A , the **primitive** elements of A ,

denoted $P(A)$, is the set

$$P(A) = \{a \in A \mid \Delta(A) = a \otimes 1 + 1 \otimes a\},$$

and the **indecomposable** elements, denoted $Q(A)$, is the set

$$Q(A) = \ker(\epsilon) / \ker(\epsilon)^2.$$

In this context, there are a number of equivalent ways to more formally define the concept of a Nichols algebra. We'll use the following definition.

Definition 2.3.3. A braided Hopf algebra A is a **Nichols algebra** if the natural map $P(A) \rightarrow Q(A)$ is an isomorphism.

Given a braided vector space V , we can define $\mathfrak{B}(V)$ to be the unique Nichols algebra generated by V as its primitive elements. Note that in the construction of the quantum shuffle algebra $\mathfrak{A}(V)$, the coalgebra structure is isomorphic to the tensor coalgebra on V on which the only primitive elements are V . Since this algebra structure is unique, it follows that the Nichols subalgebra of the quantum shuffle algebra is in fact a Nichols algebra ([25], [1]).

2.4 Fomin-Kirillov Algebras

The Fomin-Kirillov algebra \mathcal{E}_n is an algebra which is known to be a Nichols algebra when n is 3, 4, or 5. It is conjectured to be true for larger n , but it is still unproven. For $n = 6$, it is not even known if the algebra \mathcal{E}_6 is finite dimensional.

Definition 2.4.1. For $n \geq 2$, the n^{th} **Fomin-Kirillov algebra** \mathcal{E}_n over k is the quadratic algebra with generators x_{ij} and $x_{ji} = -x_{ij}$ for $1 \leq i < j \leq n$ subject to the relations

- $x_{ij}^2 = 0$,
- $x_{ij}x_{kl} = x_{kl}x_{ij}$ when i, j, k, l are all distinct,
- $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$ when i, j, k are distinct.

Example 2.4.2. The third Fomin-Kirillov algebra \mathcal{E}_3 is generated by x_{12} , x_{23} , and x_{13} , which we'll denote as a , b , and $-c$ respectively. The relations described in the definition above tell us that $a^2 = b^2 = c^2 = 0$ and that $ab + bc + ca = ac + cb + ba = 0$. No generator commutes with another, but a basic algebraic manipulation of the relations above gives degree three relations called the braid relations: $aba = bab$, $aca = cac$, and $bc b = cbc$. This algebra can be decomposed as a twisted tensor product of the algebra generated by a and b , and the algebra generated by c . By studying these algebras, we can find that this algebra is generated additively by the generating set $\{1, a, b, ab, ba, aba, c, ac, bc, abc, bac, abac\}$. \diamond

The structure of higher Fomin-Kirillov algebras is more complicated ([10]). For example, in \mathcal{E}_4 , which is generated as an algebra by six elements, the braid relations are still satisfied, and there are similar cyclic relations called the claw relations, of the form $abca + bcab + cabc = 0$. This algebra can be shown to be generated additively by 48 elements with Hilbert series $(1+x)^2(1+x^2)^2(1+x+x^2)$. The braid and claw relations generalize to higher order cyclic relations, which are satisfied in larger Fomin-Kirillov algebras, but larger Fomin-Kirillov algebras have minimal relations that are not of this type. Larger Fomin-Kirillov algebras can similarly be decomposed as twisted tensor products, but the components become complicated and intractable as n grows.

Since we are interested in computing the cohomology of these algebras, we state the result of [23]. Ştefan and Vay use the fact that \mathcal{E}_3 can be decomposed as a twisted tensor product involving \mathcal{E}_2 and employ a spectral sequence argument to compute the cohomology of \mathcal{E}_3 . As a note, the algebra R in the theorem below is identical to the quadratic dual $\mathcal{E}_3^!$.

Theorem 2.4.3. (*Ştefan-Vay*) *Let R denote the symmetric braided algebra on three degree 1 generators, which is to say that R is generated by A , B , and C with relations $xy = zx$ for any permutation $\{x, y, z\}$ of $\{A, B, C\}$. Then,*

$$\text{Ext}_{\mathcal{E}_3}(k, k) \cong R[X],$$

where $R[X]$ is a polynomial ring over R and X has degree 4.

Chapter 3

The QCN Algebra for the Dihedral Group and its Quadratic Dual

3.1 The Quadratic Cover of the Nichols Algebra for the Dihedral Group

We denote the dihedral group of order $2p$ as $D_{2p} = \langle r, \omega \mid r^2, \omega^p, r\omega r\omega \rangle$ where r and ω represent a reflection and a rotation of a regular p -gon respectively. We can embed D_{2p} within the symmetric group S_p in the standard way, by labeling the vertices of a regular p -gon with the numbers 0 through $p - 1$ and viewing each symmetry in the dihedral group as the corresponding permutation of the vertices. Let \mathcal{C} denote the set of all reflections in D_{2p} . For odd p , let r_i denote the reflection about the straight line through the vertex of the regular p -gon labeled i . As an element in S_p , r_i is the product of $\frac{p-1}{2}$ two-cycles, specifically

$$r_i = \prod_{j=1}^{\frac{p-1}{2}} (i+j, i-j),$$

where all sums and differences are taken \pmod{p} .

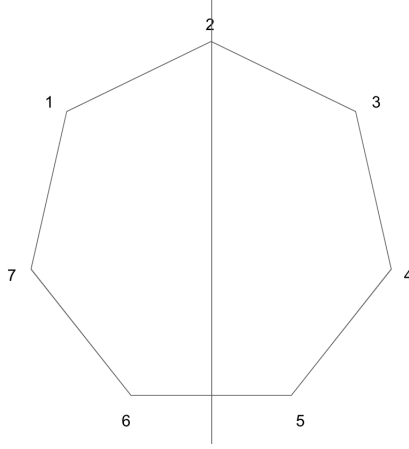


Figure 3.1: For example, in this regular 7-gon, the reflection r_2 is the product $(3, 1)(4, 7)(5, 6)$.

Take the generating rotation ω to be the permutation $(1, 2, \dots, p)$. If we conjugate r_i by ω , we get $r_i^\omega = \omega^{-1}r_i\omega = r_{i-1}$. It follows that for odd p , all reflections in D_{2p} are conjugate, and \mathcal{C} is a conjugacy class of D_{2p} . Specifically, $r_i = r_0^{\omega^{-i}}$ and $r_i^{r_k} = r_{2k-i}$. Note that if p is even, not all reflections in D_{2p} are conjugate. For example, in D_8 , the permutations $(2, 4)$ and $(1, 2)(3, 4)$ are reflections of a square. These reflections do not have the same cycle type, so they are not conjugate in S_4 , so they cannot be conjugate in the subgroup D_8 . For even values of p , the set \mathcal{C} is a union of conjugacy classes.

Consider the braided vector space $V = \mathbb{C}\{\mathcal{C}\}$ with braiding σ on $V \otimes V$ given by $v \otimes w \mapsto -w \otimes v^w$. Since the braiding was defined to be negative, this is actually the braided vector space V_ϵ following the notation in Section 2.2. We will omit the ϵ for the remainder of the paper. Further, we define the following operation on V .

Definition 3.1.1. Given v and w in V , define the operation

$$[v, w] = \sum_{j=0}^{d-1} (-1)^j \mu \sigma^j(v \otimes w),$$

where μ denotes multiplication and d is the order of $-\sigma$ on $v \otimes w$.

Using this operation, we define the algebra that will be the central object of this document.

Definition 3.1.2. Let \mathcal{B} be defined as the algebra

$$\mathcal{B} = T(V)/\{[r_i, r_j]\},$$

where i and j range over $0 \leq i, j \leq p-1$.

Note that if $i = j$, then $d = 1$ and $[r_i, r_i] = r_i \otimes r_i$. Given this, each generator r_i squares to zero in \mathcal{B} .

Example 3.1.3 ($p = 3$). We will work through the computation of the algebra \mathcal{B} when $p = 3$. The three reflections in D_6 can be labeled as r_1 , r_2 , and r_0 . Using the rule $r_i^{r_k} = r_{2k-i}$, the action of $-\sigma$ on $r_0 \otimes r_1$ is

$$r_0 \otimes r_1 \mapsto r_1 \otimes r_2 \mapsto r_2 \otimes r_0 \mapsto r_0 \otimes r_1,$$

with order $d = 3$. It follows that

$$[r_0, r_1] = r_0r_1 + r_1r_2 + r_2r_0$$

is a relation in \mathcal{B} . Similarly,

$$[r_0, r_2] = r_0r_2 + r_2r_1 + r_1r_0$$

is another relation. Finally, we note that $[r_0, r_1] = [r_1, r_2] = [r_2, r_0]$ and $[r_0, r_2] = [r_2, r_1] = [r_1, r_0]$. Since we know each generator always squares to zero, we get five relations: r_0^2 , r_1^2 , r_2^2 , $r_0r_2 + r_2r_1 + r_1r_0$, and $r_0r_1 + r_1r_2 + r_2r_0$. In this case, \mathcal{B} is isomorphic to the \mathcal{E}_3 , the third Fomin-Kirillov algebra as described in Example 2.4.2.

As a note, we can construct the algebra \mathcal{B} more generally using any group G rather than the dihedral group D_{2p} , where \mathcal{C} still denotes the conjugacy class of transpositions. When $G = S_k$, the algebra \mathcal{B} is isomorphic to \mathcal{E}_k . Since S_3 and D_6 are isomorphic, the result in this example follows immediately. \diamond

The following lemma more completely describes the structure of this algebra when p is an odd prime.

Lemma 3.1.4. *Let p denote an odd prime. The set of elements of the form $[r_i, r_j]$ has cardinality $2p - 1$, which is to say that there are $2p - 1$ independent degree 2 relations in \mathcal{B} .*

Proof. Recall that $r_i^{r_i^k} = r_{2k-1}$. Therefore $r_k^{r_k^{2k-i}} = r_{3k-2i}$ and $r_{2k-i}^{r_{2k-i}^{3k-2i}} = r_{4k-3i}$, and so on. By writing $i = 0k - (-1)i$ and $k = 1k - 0i$, we get an obvious pattern by applying the braiding $-\sigma$ repeatedly to $r_i \otimes r_k$.

$$r_{0k-(-1)i} \otimes r_{1k-0i} \mapsto r_{1k-0i} \otimes r_{2k-i} \mapsto r_{2k-i} \otimes r_{3k-2i} \mapsto r_{3k-2i} \otimes r_{4k-3i} \mapsto \dots$$

Each entry in this sequence of tensors is of the form $r_{mk-(m-1)i} \otimes r_{(m+1)k-mi}$. To find the degree of $-\sigma$, assume that two entries in this sequence of tensors are equal. Then, for some integers m and n ,

$$r_{mk-(m-1)i} \otimes r_{(m+1)k-mi} = r_{nk-(n-1)i} \otimes r_{(n+1)k-ni}.$$

Since the r_i 's are independent, this means that the indices are equal. In other words,

$$\begin{aligned} mk - (m-1)i &= nk - (n-1)i && \text{mod } p \\ (m-n)k - (m-1-n+1)i &= 0 && \text{mod } p \\ (m-n)(k-i) &= 0 && \text{mod } p. \end{aligned}$$

Since p is prime, this means that either $k - i = 0 \pmod{p}$ or $m - n = 0 \pmod{p}$. Since k and i are taken to be the indices on the labels of the reflection in D_{2p} , both k and i are between 0 and $p - 1$, so their difference divides p if and only if $i = k$. In this case, $-\sigma$ has order 1 as discussed previously. If k and i are distinct, then $m - n = 0 \pmod{p}$ must hold. Since m and n denote the index of the sequence of applying $-\sigma$ at which repetitions can happen, this means that the sequence has order p .

Taking orbits, $-\sigma$ partitions the set of tensors of the form $r_i \otimes r_k$ into $p - 1$ sets of size p and p sets of size 1. The sum of elements in each orbit gives an independent degree 2 relation in the construction of \mathcal{B} , so there are $2p - 1$ such relations. \square

As a result of this lemma, we will assume p to be an odd prime in for the remainder of this document, with the exception of the two examples at the end of this section. We

now relate the algebra \mathcal{B} to the Nichols algebra $\mathfrak{B}(V)$.

Lemma 3.1.5. *There is a well defined map $\mathcal{B} \rightarrow \mathfrak{B}(V)$ defined by sending elements of V to themselves and by sending tensor products to quantum shuffle products.*

Proof. To show that this map is well defined, we must show that $[r_i, r_j] = 0$ in $\mathfrak{B}(V)$. Since the only nonzero primitive elements in $\mathfrak{B}(V)$ are the generators in degree 1, it suffices to show that $[r_i, r_j]$, which has degree 2, is primitive. To do this, we compute $\Delta([r_i, r_j])$ explicitly. By the definition of Δ , for any a and b in $\mathfrak{B}(V)$

$$\Delta(a \otimes b) = ab \otimes 1 + 1 \otimes ab + a \otimes b + \sigma(a \otimes b) = ab \otimes 1 + 1 \otimes ab + (1 + \sigma)(a \otimes b).$$

Extending linearly, we can assert that

$$\Delta([r_i, r_j]) = \mu([r_i, r_j]) \otimes 1 + 1 \otimes \mu([r_i, r_j]) + (1 + \sigma)([r_i, r_j]).$$

Isolating the final term,

$$(1 + \sigma)([r_i, r_j]) = (1 + \sigma) \left(\sum_{j=0}^{d-1} (-1)^j \mu \sigma^j(r_i \otimes r_j) \right) = (1 + \sigma) \left(\sum_{j=0}^{d-1} (-1)^j \sigma^j \right) (r_i \otimes r_j).$$

But, $(1 + \sigma) \left(\sum_{j=0}^{d-1} (-1)^j \sigma^j \right)$ is 0 when viewed as an element of $\mathbb{Z}[\sigma]/\sigma^d$. Therefore,

$$\Delta([r_i, r_j]) = \mu([r_i, r_j]) \otimes 1 + 1 \otimes \mu([r_i, r_j]),$$

so $[r_i, r_j]$ is primitive in $\mathfrak{B}(V)$. □

Since the Nichols algebra is generated by V and the quantum shuffle product, the map $\mathcal{B} \rightarrow \mathfrak{B}(V)$ is surjective. By construction, it is an isomorphism on the degree 1 components. In degree 2, this map is an isomorphism if no elements in \mathcal{B}_2 are mapped to primitive elements. In the proof above, we showed that $a \otimes b$ is primitive in $\mathfrak{B}(V)$ if and only if $(1 + \sigma)(a \otimes b) = 0$. However, the annihilator of $(1 + \sigma)$, again as an element of $\mathbb{Z}[\sigma]/\sigma^d$, is exactly $\left(\sum_{j=0}^{d-1} (-1)^j \sigma^j \right)$. Therefore, the image of the relations in \mathcal{B} are the only primitive elements in $\mathfrak{B}(V)_2$. Therefore, this map is an isomorphism on degree 2 components. It follows that \mathcal{B} is isomorphic to the quadratic cover of the Nichols

subalgebra. For the remainder of the paper, we'll refer to the algebra \mathcal{B} corresponding to the dihedral group D_{2p} as its **QCN algebra**. Here QCN stands for quadratic cover of the Nichols algebra.

As mentioned previously, we will assume that p is an odd prime for the remainder of this paper. Before doing this, we briefly consider the structure of \mathcal{B} when p is not an odd prime.

Example 3.1.6 (p is not odd). When p is not odd, for example $p = 2$, the set \mathcal{C} is not a single conjugacy class but a union of conjugacy classes. As the smallest example, take $p = 2$ where $D_4 = \langle r, \omega \mid r^2, \omega^2, r\omega = \omega r \rangle$, which is isomorphic to the Klein Four group. This group is abelian, so the two reflections r and ωr are not conjugate. We'll denote ωr as s , and the algebra \mathcal{B} is generated by r and s with relations r^2, s^2 and $rs + sr$. This algebra is isomorphic to $\Lambda[r, s]$, the exterior algebra on two generators. \diamond

Example 3.1.7 (p is not prime). When p is odd but not prime, every reflection in D_{2p} is still conjugate. However, Lemma 3.1.4 fails as there are more than $2p - 1$ degree two relations of the form $[r_i, r_j]$. As an example, when $p = 9$, we can investigate the orbits of $-\sigma$ on the elements $r_i \otimes r_j$. There are elements on which $-\sigma$ has order 9. For example, the image of $-\sigma$ on $r_0 \otimes r_1$ is

$$r_0 \otimes r_1 \mapsto r_1 \otimes r_2 \mapsto \dots \mapsto r_8 \otimes r_0 \mapsto r_0 \otimes r_1.$$

However, some elements have a smaller orbit. For example, the image of $-\sigma$ on $r_0 \otimes r_3$ is

$$r_0 \otimes r_3 \mapsto r_3 \otimes r_6 \mapsto r_6 \otimes r_0 \mapsto r_0 \otimes r_3.$$

All together, there will be 9 relations of the generators squaring to zero, 6 relations which correspond to the orbits of $-\sigma$ of size 9, and 6 relations which correspond to orbits of $-\sigma$ of length 3.

Notably, the subalgebra generated by r_0, r_3 , and r_6 is isomorphic to the QCN algebra for D_6 , as are the subalgebra generated by r_1, r_4 , and r_7 and the subalgebra generated by r_2, r_5 , and r_8 . This has not been thoroughly studied, but is possible that the algebra \mathcal{B} constructed from D_{18} can be decomposed using the three copies of the D_6 QCN algebra as components, likely using a twisted tensor product. \diamond

3.2 The Quadratic Dual of the QCN Algebra

Since the relations in the definition of \mathcal{B} are generated in degree 2, \mathcal{B} is a quadratic algebra. We can therefore study its quadratic dual $\mathcal{R} = \mathcal{B}^\dagger$. Let $R_i \in V^*$ denote the dual element to $r_i \in V$. Since \mathcal{B} has $2p-1$ relations in degree 2, \mathcal{R} has the complementary $p^2 - (2p-1) = (p-1)^2$ independent degree 2 relations. These relations are of the form $R_a R_b = R_c R_d$ where $r_a \otimes r_b$ and $r_c \otimes r_d$ are in the same orbit of $-\sigma$ as described in the proof of Lemma 3.1.4. The orbits laid out in this proof identify these indices as $a = mk - (m-1)i$, $b = (m+1)k - mi$, $c = nk - (n-1)i$, and $d = (n+1)k - ni$ for some integers m, n, k , and i . For the remainder of this subsection, we will compute the Hilbert Series for this quadratic dual algebra. In other words, for any n , we will give the vector space dimension of the degree n part of the quadratic dual, denoted \mathcal{R}_n .

For preliminary results, note that \mathcal{R}_1 is spanned by R_0, R_1, \dots, R_{p-1} . Since \mathcal{R} is quadratic, there are no relations among these elements, so the dimension of \mathcal{R}_1 is p . Further, by construction \mathcal{R}_2 is spanned by the p^2 elements of the form $R_i R_j$ with $(p-1)^2$ independent degree 2 relations, so the dimension of \mathcal{R}_2 is $2p-1$. Finally, note that for all $n > 0$, there are p elements of the form R_i^n , and no relations in \mathcal{R} can be used to simplify them. Therefore the elements R_i^n are independent, so the dimension of \mathcal{R}_n is at least p for all $n > 0$. We now present the following lemma.

Lemma 3.2.1. *For all odd primes p , and for all $n > 2$, the vector space dimension of \mathcal{R}_n is $2p$.*

As a corollary of this lemma, we can state the Hilbert Series of \mathcal{R} .

Corollary 3.2.2. *The Hilbert Series of \mathcal{R} is*

$$H_{\mathcal{R}}(t) = 1 + pt + (2p-1)t^2 + \sum_{n=3}^{\infty} 2pt^n.$$

We can write this series in a closed form as

$$H_{\mathcal{R}}(t) = \frac{1 + pt + (2p-2)t^2 + pt^3 + t^4}{1 - t^2}.$$

To prove Lemma 3.2.1, we first define the following map.

Definition 3.2.3. Let φ denote the map from the set of monomials in \mathcal{R} to the dihedral group D_{2p} that is generated by $R_i \rightarrow r_i$ and by sending tensor products to group multiplication in the dihedral group.

Lemma 3.2.4. *The map φ is well-defined.*

Proof. Since the relations in \mathcal{R} are generated by relations of the form $R_a R_b = R_c R_d$, it suffices to show that in this case, $\varphi(R_a R_b) = \varphi(R_c R_d)$. We can rewrite a, b, c , and d in terms of integers i, k, m , and n as described at the start of this section. Note that both $R_a R_b$ and $R_c R_d$ are equal to $R_i R_k$ in \mathcal{B} , so it suffices to show that $\varphi(R_a R_b) = \varphi(R_i R_k) = \varphi(R_c R_d)$. Without loss of generality, we only show that $\varphi(R_a R_b) = \varphi(R_i R_k)$. The second equality can be shown with an identical argument by replacing m with n .

First, we compute $\varphi(R_i R_k)$. By definition, this is $r_i r_k$ as an element of the dihedral group. Using the earlier identity that $r_i = r_0 \omega^{-i}$, we get that

$$\varphi(R_i R_k) = \omega^i r_0 \omega^{-i} \omega^k r_0 \omega^{-k}.$$

Using the defining identity of the dihedral group that $r_0 \omega = \omega^{-1} r_0$, we can rewrite this as

$$\varphi(R_i R_k) = \omega^i \omega^i r_0 r_0 \omega^{-k} \omega^{-k} = \omega^{2i-2k}.$$

Using the exact same argument, replacing i and k with $a = mk - (m-1)i$ and $b = (m+1)k - mi$, we get

$$\varphi(R_a R_b) = \omega^{2(mk-(m-1)i)-2((m+1)k-mi)} = \omega^{2i-2k},$$

which completes the proof. □

The map φ allows us to prove the following lemma about about pairs of generators in \mathcal{R} .

Lemma 3.2.5. *Let R_{X_1}, R_{X_2} , and R_Y denote generators of \mathcal{R} where $X_1 \neq X_2$. There is a generator of \mathcal{R} , denoted R_Z , such that $R_{X_1} R_{X_2} = R_Z R_Y$.*

Proof. Let $\mathcal{O} \subseteq V \otimes V$ denote the orbit of $r_{X_1} \otimes r_{X_2}$ under $-\sigma$. By the construction of \mathcal{R} , $R_{X_1} R_{X_2} = R_Z R_Y$ if and only if $r_Z \otimes r_Y \in \mathcal{O}$. Assume to the contrary that no such

index Z exists. Then there p elements of \mathcal{O} each of the form $r_n \otimes r_m$ where $r_n \neq r_m$ and $r_m \neq r_Z$. There are p elements in this orbit and $p - 1$ choices for the second tensor term, so the pigeonhole principle gives us some index c such that $r_a \otimes r_c$ and $r_b \otimes r_c$ are both in \mathcal{O} where $r_a \neq r_b$. Since $r_a \otimes r_c$ and $r_b \otimes r_c$ are in the same orbit of $-\sigma$, $R_a R_c = R_b R_c$. Applying φ yields $r_a r_c = r_b r_c$ in D_{2p} . Multiplying on the right by r_c yields $r_a = r_b$, which is a contradiction, which completes the proof. \square

Now we prove a partial injectivity result about the map φ .

Lemma 3.2.6. *Assume X and Y are two monomials in \mathcal{R}_n for a fixed $n > 1$. Further assume that neither X nor Y is R_i^n for any i . Then, $\varphi(X) = \varphi(Y)$ if and only if $X = Y$ in \mathcal{R} .*

Proof. We proceed by induction. If $n = 2$, then $X = R_{X_1} R_{X_2}$ and $Y = R_{Y_1} R_{Y_2}$ where $X_1 \neq X_2$ and $Y_1 \neq Y_2$. Using Lemma 3.2.5, find an index Z such that $R_{Y_1} R_{Y_2} = R_Z R_{X_2}$. Then $\varphi(Y) = \varphi(X)$ implies that $r_{X_1} r_{X_2} = r_Z r_{X_2}$ in D_{2p} . Multiplying both sides by r_{X_2} on the right yields that $r_{X_1} = r_Z$. Since the r 's are independent, it follows that $X_1 = Z$, and therefore that $Y = R_{X_1} R_{X_2} = X$.

Therefore, we will assume $n > 2$ and that the result holds for $n - 1$. We write $X = R_{X_1} \dots R_{X_n}$ and $Y = R_{Y_1} \dots R_{Y_n}$. Without loss of generality, we'll assume that $Y_{n-1} \neq Y_n$. If this is not the case, let k be the largest index such that $Y_k \neq Y_{k+1}$. Since Y is not equal to a power of a generator, such a k must exist. Rewrite Y by using the relation which changes $R_{Y_k} R_{Y_{k+1}}$ to any other pair of generators. There are $p - 1$ such choices. Then relabel Y , and the largest index such that $Y_i \neq Y_{i+1}$ will be $k + 1$. Repeat this process until $Y_{n-1} \neq Y_n$. Similarly, we can assume that $R_{X_{n-1}} \neq R_{X_n}$. Further assume without loss of generality that X_1, \dots, X_{n-1} are not all the same. If this is not the case, we can rewrite X using the relation which changes the pair $R_{X_{n-2}} R_{X_{n-1}}$ to any other pair of generators. There are $p - 1$ such choices. One such choice changes $R_{X_{n-2}} R_{X_{n-1}}$ to a pair ending in the term R_{X_n} which ruins the previous assumption. Any other of the $p - 2$ choices gives the desired result.

Apply Lemma 3.2.5 to find an index Z such that $R_{Y_{n-1}} R_{Y_n} = R_Z R_{X_n}$. Note that $Z \neq X_n$. Using this relation, we can write $Y = R_{Y_1} \dots R_{Y_{n-2}} R_Z R_{X_n}$. As one last assumption without loss of generality, we can assume that $R_{Y_1}, \dots, R_{Y_{n-2}}, R_Z$ are not all identical. If they are, we can replace $R_{X_{n-1}} R_{X_n}$ with another pair using a relation

such that the first term is not $R_{X_{n-2}}$. There are $p - 2$ such choices. Repeating this process with the relabeled X will change the value of Z .

Now, $\varphi(Y) = \varphi(X)$ gives us that $r_{Y_1} \dots r_{Y_{n-2}} r_Z r_{X_n} = r_{X_1} \dots r_{X_{n-1}} r_{X_n}$. We assumed that $X_{n-1} \neq X_n$, and we know that $Z \neq X_n$. Therefore the last two digits of this word in the r_i 's are not the same and do not cancel. Therefore multiplying both by r_{X_n} on the right yields $r_{Y_1} \dots r_{Y_{n-2}} r_Z = r_{X_1} \dots r_{X_{n-1}}$. These are $\varphi(R_{Y_1} \dots R_{Y_{n-2}} R_Z)$ and $\varphi(R_{X_1} \dots R_{X_{n-1}})$ respectively, and by the assumptions in the previous paragraph, neither is a power of a single generator. Therefore, by the inductive hypothesis, $R_{X_1} \dots R_{X_{n-1}} = R_{Y_1} \dots R_{Y_{n-2}} R_Z$. Multiplying both sides by R_{X_n} gives $X = Y$ as desired. \square

Now we can use this map φ and the lemma above to complete the proof of Lemma 3.2.1.

Proof. (Lemma 3.2.1) Fix $n > 2$. Since all relations in \mathcal{R} are monomial relations, we can find a basis of \mathcal{R}_n by only considering monomials. Recall that each power R_i^n is independent in \mathcal{R}_n . There are p generators R_i , so this accounts for p independent elements in \mathcal{R}_n . Let $X \in \mathcal{R}_n$ be a degree n monomial that is not a power of a generator. Further, recall that, in D_{2p} , the product of two reflections is a rotation, and the product of a reflection and a rotation is reflection. We consider two cases.

If n is odd, then $\varphi(X)$ is a reflection since it is a product of n reflections. Using Lemma 3.2.6, if two such monomials get sent to the same reflection in D_{2p} , the two monomials are equal in \mathcal{R} . Since there are p reflections in D_{2p} , there are at most p independent monomials in \mathcal{R}_n which are not powers of generators. Consider the set $R_0^{n-1} R_1, R_0^{n-1} R_2, \dots, R_0^{n-1} R_{p-1}$, and $R_0^{n-2} R_1^2$. We can check that φ sends these monomials to r_1, r_2, \dots, r_{p-1} , and r_0 respectively. Since φ is well defined, none of these elements are equal to another in \mathcal{R}_n . It follows that there are exactly p independent monomials in \mathcal{R}_n which are not powers of generators.

Alternatively, if n is even, then $\varphi(X)$ is a rotation since it is a product of n reflections. The argument is identical as the odd case. There are p possible rotations in D_{2p} , and each rotation is mapped to by an element from the set $R_0^{n-1} R_1, R_0^{n-1} R_2, \dots, R_0^{n-1} R_{p-1}$, and $R_0^{n-2} R_1^2$. Note that this last element maps to the identity element in D_{2p} . As an aside, when n is exactly 2, the result does not hold since there is no element in \mathcal{R}_2 which

maps to the identity under φ other than the squares of the generators.

In both cases, there are exactly p independent monomials which are powers of generators and p independent monomials which are not powers of generators, so the vector space dimension of \mathcal{R}_n is $2p$. \square

In general, we'll refer to the basis $\{R_0^n, \dots, R_{p-1}^n, R_0^{n-1}R_1, \dots, R_0^{n-1}R_{p-1}, R_0^{n-2}R_1^2\}$ of \mathcal{R}_n for $n \geq 3$ as the *lexicographical* basis. Before leaving this section, we'll consider the example of the Fomin-Kirillov algebra when $p = 3$.

Example 3.2.7 (The Quadratic Dual when $p = 3$). Continuing Example 3.1.3, when $p = 3$, the algebra \mathcal{B} is isomorphic to the third Fomin-Kirillov algebra. In this case the quadratic dual algebra \mathcal{R} is generated by R_0 , R_1 , and R_2 with relations $R_0R_1 = R_1R_2 = R_2R_0$ and $R_0R_2 = R_2R_1 = R_1R_0$.

In degree 1, \mathcal{R}_1 is spanned by R_0 , R_1 , and R_2 . In degree 2, \mathcal{R}_2 is spanned by R_0^2 , R_1^2 , R_2^2 , R_0R_1 , and R_0R_2 . Note that $\varphi(R_0R_1) = (1, 2, 3)$ and $\varphi(R_0R_2) = (1, 3, 2)$ as elements in S_3 .

The proof of the Lemma above gives us a basis for \mathcal{R}_n in dimensions above 2. For example, in dimension 3, the basis is R_0^3 , R_1^3 , R_2^3 , $R_0^2R_1$, $R_0^2R_2$, and $R_0R_1^2$ with $\varphi(R_0^2R_1) = r_1$, $\varphi(R_0^2R_2) = r_2$, and $\varphi(R_0R_1^2) = r_0$.

This algebra is exactly the braided symmetric algebra on three generators mentioned in Theorem 2.4.3. \diamond

Remark 3.2.8. Before moving on, we note that, from Section 4.6 of [12], there is an isomorphism

$$\mathcal{R} \rightarrow \bigoplus_n H_0(\text{Hur}_{D_{2p},n}^{\mathcal{C},k})$$

where \mathcal{C} denotes the class of transpositions in D_{2p} .

3.3 The Stabilization Element of the Quadratic Dual

In Lemma 3.5 of [15], it is shown that this algebra \mathcal{R} has a stabilization element. In other words, there is an element $S \in \mathcal{R}$ of homogenous degree d such that right multiplication by S , viewed as a map $\mathcal{R}_n \rightarrow \mathcal{R}_{n+d}$ is an isomorphism for large enough n . They explicitly show that S can be written as a sum of the powers of the generators

of \mathcal{R} . In other words, for some n_0 and some d , multiplication by $R_0^d + \dots + R_{p-1}^d$ is an isomorphism $\mathcal{R}_n \rightarrow \mathcal{R}_{n+d}$ for $n \geq n_0$. In this section, we'll specifically find this element in the case that \mathcal{R} is the dual of the QCN algebra for D_{2p} . The proposition below asserts that $d = 2$ and $n_0 = 3$.

Proposition 3.3.1. *Let $S = R_0^2 + \dots + R_{p-1}^2$ in \mathcal{R} . Right multiplication by S is an isomorphism $\mathcal{R}_n \rightarrow \mathcal{R}_{n+2}$ for $n \geq 3$.*

Proof. We know that for $n \geq 3$, \mathcal{R}_n is spanned by the lexicographical basis

$$\{R_0^n, \dots, R_{p-1}^n, R_0^{n-1}R_1, \dots, R_0^{n-1}R_{p-1}, R_0^{n-2}R_1^2\}.$$

We multiply these elements by S , starting with those that are not powers of generators:

$$R_0^{n-1}R_1 \cdot S = R_0^{n-1}R_1R_0^2 + \dots R_0^{n-1}R_1R_{p-1}^2.$$

Since none of these monomials are powers of generators, we can use the map φ described in the previous section. Note that multiplying by R_i^2 does not affect the image of φ , so each monomial in the sum is mapped to the same thing by φ . The image of these monomials is the same as the image as $R_0^{(n+2)-1}R_1$. Using Lemma 3.2.6, it follows that $R_0^{n-1}R_1 \cdot S = pR_0^{n+1}R_1$. Similarly:

$$\begin{aligned} R_0^{n-1}R_1 \cdot S &= pR_0^{n+1}R_1 \\ &\vdots \\ R_0^{n-1}R_{p-1} \cdot S &= pR_0^{n+1}R_{p-1} \\ R_0^{n-2}R_1^2 \cdot S &= pR_0^nR_1^2 \\ R_0^n \cdot S &= R_0^{n+2} + (p-1)R_0^nR_1^2 \\ &\vdots \\ R_{p-1}^n \cdot S &= R_{p-1}^{n+2} + (p-1)R_0^nR_1^2. \end{aligned}$$

Since $\{R_0^{n+2}, \dots, R_{p-1}^{n+2}, R_0^{n+1}R_1, \dots, R_0^{n+1}R_{p-1}, R_0^nR_1^2\}$ is the lexicographical basis of \mathcal{R}_{n+2} , it can easily be seen that the images of multiplication by S in list above form a linearly independent set of $2p$ elements in \mathcal{R}_{n+2} . Therefore, multiplication by S is an isomorphism, as desired.

□

Before moving to the next section, we define another self map on \mathcal{R} .

Definition 3.3.2. Let $L : (\mathcal{R}_{>0})_n \rightarrow (\mathcal{R}_{>0})_{n+2}$ be defined by $L(R_i^n) = R_i^{n+2}$ and $L(R_0^{n-1}R_i) = R_0^{n+1}R_i$ and $L(R_0^{n-2}R_1^2) = R_0^nR_1^2$. In other words, L sends elements of the lexicographical basis of \mathcal{R}_n to their counterparts in \mathcal{R}_{n+2} . We therefore refer to L as the lexicographical map on \mathcal{R} .

Note that by construction, for monomials $X \in \mathcal{R}$, $\varphi(X) = \varphi(L(X))$. In general, this map is not as useful as the multiplication by S map since L is not defined on \mathcal{R}_0 , but we'll use it occasionally while studying \mathcal{R} in the coming sections. In particular, we will use the following lemma.

Lemma 3.3.3. *Let Y denote a basis element of \mathcal{R} and let R_i denote an arbitrary generator of \mathcal{R} . Then, $L(R_iY) = R_iL(Y)$.*

Proof. If $Y = R_i^k$ for some k , then both $L(R_iY)$ and $R_iL(Y)$ are R_i^{k+3} . Otherwise, neither $L(R_iY)$ nor $R_iL(Y)$ is a power of a generator of \mathcal{R} . Using that $\varphi(X) = \varphi(L(X))$,

$$\varphi(L(R_iY)) = \varphi(R_iY) = \varphi(R_i)\varphi(Y) = \varphi(R_i)\varphi(L(Y)) = \varphi(R_iL(Y)).$$

The result follows from Lemma 3.2.6. □

Chapter 4

Koszul Complexes

In this chapter, we'll describe the structure of Koszul complex of the the QCN algebra for D_{2p} . These Koszul complexes will be the main object of study that we will use to better understand these QCN algebras. Since we have a basis for the quadratic dual \mathcal{R} , we focus on the dual of the Koszul complex.

4.1 The Dual Koszul Complex

Let \mathcal{B} denote the QCN algebra for D_{2p} . In this section we study the Koszul complex and its dual for arbitrary odd primes p following Section 6.4 of [12]. To define the complex, we define the map ∂_x on \mathcal{B}^* to give this dual space the structure of a \mathcal{B} module.

Definition 4.1.1. The map ∂_x on $\phi \in \mathcal{B}^*$ is defined to be the map such that

$$\langle \partial_x \phi, v \rangle = \langle \phi, xv \rangle,$$

where $\langle \cdot, \cdot \rangle : \mathcal{B} \otimes \mathcal{B}^* \rightarrow k$ is the standard non-degenerate pairing, as outlined on the bottom of page 12 in [12].

The dual to the Koszul complex for \mathcal{B} takes the form $K(\mathcal{B})^* = \mathcal{B}^* \otimes \mathcal{R}$ with differential

$$\alpha(\psi \otimes x) = \sum_{i=0}^{p-1} \partial_{r_i} \psi \otimes R_i x.$$

Specifically, we can view the dual Koszul complex as a bigraded chain complex $K(\mathcal{B})_{m,n}^* = \mathcal{B}_m^* \otimes \mathcal{R}_n$ with differential

$$\alpha : K(\mathcal{B})_{m,n}^* \rightarrow K(\mathcal{B})_{m-1,n+1}^*.$$

Furthermore $K(\mathcal{B})^*$ is a right \mathcal{R} module, and that the action of \mathcal{R} commutes with the differential.

In degree d , the dual Koszul complex is given as $K(\mathcal{B})_d^* = \mathcal{B}^* \otimes \mathcal{R}_d = \mathcal{B}^{*\dim(\mathcal{R}_d)}$. Using the dimensions of \mathcal{R} given in Corollary 3.2.2, we can write out this complex explicitly.

$$\dots \longleftarrow \mathcal{B}^{*2p} \longleftarrow \mathcal{B}^{*2p} \longleftarrow \mathcal{B}^{*2p-1} \longleftarrow \mathcal{B}^{*p} \longleftarrow \mathcal{B}^{*1} \longleftarrow 0$$

Lemma 4.1.2. *For the dual Koszul complex $K(\mathcal{B})^*$ described above, $H^n(K(\mathcal{B})^*) = 0$ for all $n \geq 4$.*

Proof. We first appeal to Theorem 6.16 in [12]. As a dual statement to this result, the cohomology of this dual Koszul complex stabilizes as zero eventually. In other words, there is a constant q such that $H_n(K(\mathcal{B})^*) = 0$ for all $n \geq q$. We will show that $q = 4$ in this case.

Recall that \mathcal{R}_n is spanned as a vector space by the lexicographical basis

$$\{R_0^n, \dots, R_{p-1}^n, R_0^{n-1}R_1, \dots, R_0^{n-1}R_{p-1}, R_0^{n-2}R_1^2\}$$

for $n \geq 3$. From Definition 3.3.2, we have a map L which preserves this lexicographical basis. We can view L as a map on $K(\mathcal{B})^*$ by $L(x \otimes R) = x \otimes L(R)$. Note that the map $L : K(\mathcal{B})_n^* \rightarrow K(\mathcal{B})_{n+2}^*$ is invertible if $n \geq 3$.

We write the Koszul differential in degree d as $\alpha_d : K(\mathcal{B})_d^* \rightarrow K(\mathcal{B})_{d+1}^*$. We first show that L commutes with α . It suffices to show that these maps commute on elements of the form $x \otimes Y \in K(\mathcal{B})_d^*$ where Y is a basis element of \mathcal{R}_d . Explicitly, the compositions of these maps are

$$\alpha_{d+2}(L(x \otimes Y)) = \partial_{r_0}(x) \otimes R_0L(Y) + \partial_{r_1}(x) \otimes R_1L(Y) + \dots + \partial_{r_{p-1}}(x) \otimes R_{p-1}L(Y),$$

and

$$L(\alpha_d(x \otimes Y)) = \partial_{r_0}(x) \otimes L(R_0Y) + \partial_{r_1}(x) \otimes L(R_1Y) + \dots + \partial_{r_{p-1}}(x) \otimes R_{p-1}L(R_{p-1}Y).$$

The commutativity follows from Lemma 3.3.3.

To complete the proof, let q be the smallest integer such that $H^n(K(\mathcal{B})^*) = 0$ for all $n \geq q$. Assume $q > 4$. Therefore $\ker(\alpha_{n+1})/\text{im}(\alpha_n) = 0$. Since L commutes with the differential, $\alpha_n \circ L = L \circ \alpha_{n-2}$. Since $n > 4$, the maps $L : K(\mathcal{B})_{n-1}^* \rightarrow K(\mathcal{B})_{n+1}^*$ and $L : K(\mathcal{B})_{n-2}^* \rightarrow K(\mathcal{B})_n^*$ are invertible. It follows that

$$H^{n-2}(K(\mathcal{B})^*) = \ker(\alpha_{(n-2)+1})/\text{im}(\alpha_{n-2}) = 0.$$

An identical argument shows using $n + 1$ in place of n shows that

$$H^{n-1}(K(\mathcal{B})^*) = \ker(\alpha_{(n-1)+1})/\text{im}(\alpha_{n-1}) = 0.$$

This contradicts the minimality of q which gives the result. □

Further recall that right multiplication by the stabilization element $S = R_0^2 + \dots + R_{p-1}^2$ is an isomorphism as a map $\mathcal{R}_n \rightarrow \mathcal{R}_{n+2}$ for $n \geq 3$. Since $K(\mathcal{B})^*$ is a right \mathcal{R} module, we can view right multiplication as a map $S : K(\mathcal{B})_n^* \rightarrow K(\mathcal{B})_{n+2}^*$. Alternatively, we can view right multiplication by S as a map $S : K(\mathcal{B})^*[-2]_n \rightarrow K(\mathcal{B})_n^*$, from a shifted version of this dual Koszul complex to a non-shifted version. This map commutes with the differential, so we can compute its quotient and form a quotient complex. Since multiplication by S is eventually an isomorphism, this quotient complex has finite length.

Definition 4.1.3. The quotient complex $K(\mathcal{B})^*/S = \mathcal{B}^* \otimes \mathcal{R}/S$ is constructed in the diagram below.

$$\begin{array}{cccccccc}
\dots & \longleftarrow & \mathcal{B}^{*2p} & \longleftarrow & \mathcal{B}^{*2p-1} & \longleftarrow & \mathcal{B}^{*p} & \longleftarrow & \mathcal{B}^{*1} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
& & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S \\
\dots & \longleftarrow & \mathcal{B}^{*2p} & \longleftarrow & \mathcal{B}^{*2p} & \longleftarrow & \mathcal{B}^{*2p} & \longleftarrow & \mathcal{B}^{*2p-1} & \longleftarrow & \mathcal{B}^{*p} & \longleftarrow & \mathcal{B}^{*1} & \longleftarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \longleftarrow & 0 & \longleftarrow & \mathcal{B}^{*1} & \longleftarrow & \mathcal{B}^{*p} & \longleftarrow & \mathcal{B}^{*2p-2} & \longleftarrow & \mathcal{B}^{*p} & \longleftarrow & \mathcal{B}^{*1} & \longleftarrow & 0
\end{array}$$

From Corollary 3.2.2, the dimensions of \mathcal{R}_n were given as the coefficients in the Hilbert Series expansion of

$$\frac{1 + pt + (2p - 2)t^2 + pt^3 + t^4}{1 - t^2}.$$

The dimensions of $(\mathcal{R}/S)_n$, which appear as the B^* module ranks along the quotient complex we just constructed, are 1, p , $2p-2$, p , and 1. These are exactly the coefficients in the numerator above. Taking the quotient of \mathcal{R} by S has the effect of multiplying the Hilbert Series by $1 - t^2$.

We can explicitly construct a basis for \mathcal{R}/S in each dimension as a quotient of our lexicographical basis. Specifically, in $(\mathcal{R}/S)_2$, we have the relation $-R_{p-1}^2 = R_0^2 + \dots R_{p-2}^2$. In $(\mathcal{R}/S)_3$, we have relations $R_0^3 = -(p-1)R_0R_1^2$ and $R_i^3 = -(p-1)R_0^2R_i$ for $0 < i \leq p-1$. In $(\mathcal{R}/S)_4$, we have the relations $R_0^3R_i = 0$ for all $0 < i \leq p-1$, and $R_0^4 = \dots = R_{p-1}^4 = -(p-1)R_0^2R_1^2$. Therefore, the following is a basis for \mathcal{R}/S :

$(\mathcal{R}/S)_0$ has basis $\{1\}$

$(\mathcal{R}/S)_1$ has basis $\{R_0, \dots, R_{p-1}\}$

$(\mathcal{R}/S)_2$ has basis $\{R_0^2, \dots, R_{p-2}^2, R_0R_1, R_0R_2, \dots, R_0R_{p-1}\}$

$(\mathcal{R}/S)_3$ has basis $\{R_0^3, \dots, R_{p-1}^3\}$

$(\mathcal{R}/S)_4$ has basis $\{R_0^4\}$.

Since we have a short exact sequence of complexes

$$0 \rightarrow K(\mathcal{B})^*[-2] \rightarrow K(\mathcal{B})^* \rightarrow K(\mathcal{B})^*/S \rightarrow 0, \quad (4.1)$$

the following proposition is immediate.

Proposition 4.1.4. *There is a long exact sequence in cohomology:*

$$\begin{array}{ccccccccc}
 H^2(K^*) & & H^1(K^*) & & H^0(K^*) & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^4(K^*) & & H^3(K^*) & & H^2(K^*) & & H^1(K^*) & & H^0(K^*) \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 H^4(K^*/S) & & H^3(K^*/S) & & H^2(K^*/S) & & H^1(K^*/S) & & H^0(K^*/S)
 \end{array}$$

4.2 The Dual to the Quotient Complex

Reading off the bottom row of the diagram in the previous section, the quotient complex $K(\mathcal{B})^*/S$ has the form:

$$\cdots \longleftarrow 0 \longleftarrow \mathcal{B}^{*1} \xleftarrow{\alpha_3} \mathcal{B}^{*p} \xleftarrow{\alpha_2} \mathcal{B}^{*2p-2} \xleftarrow{\alpha_1} \mathcal{B}^{*p} \xleftarrow{\alpha_0} \mathcal{B}^{*1} \longleftarrow 0$$

Here α_i denotes the map $(K(\mathcal{B})^*/S)_i \rightarrow (K(\mathcal{B})^*/S)_{i+1}$. As clearly shown in this diagram, there is an apparent self-duality in the ranks of the free B modules which comprise this complex. In this section, we'll further expand upon this phenomenon. We first observe that using the basis of \mathcal{R}/S and the formulation of the dual Koszul differential in the previous paragraph, we can express α_0 and α_3 in terms of matrices with entries of the form ∂_{r_i} . Specifically,

$$\alpha_0 = \begin{bmatrix} \partial_{r_0} \\ \partial_{r_1} \\ \vdots \\ \partial_{r_{p-1}} \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} \partial_{r_0} & \partial_{r_1} & \cdots & \partial_{r_{p-1}} \end{bmatrix}.$$

These matrices are clearly transposes of each other. Computing the $(2p-2) \times p$ and $p \times (2p-2)$ matrices which represent the maps α_1 and α_2 is more complicated in general, but in the specific case when $p = 3$, we can determine these matrices by applying the Koszul differential to elements of the form $x \otimes Y$ where Y is a basis element of $(\mathcal{R}/S)_1$ or $(\mathcal{R}/S)_2$. As examples,

$$\begin{aligned} \alpha_1(x \otimes R_0) &= \partial_{r_0}(x) \otimes R_0^2 + \partial_{r_1}(x) \otimes R_1R_0 + \partial_{r_2}(x) \otimes R_2R_0 \\ &= \partial_{r_0}(x) \otimes R_0^2 + 0 \otimes R_1^2 + \partial_{r_2}(x) \otimes R_0R_1 + \partial_{r_1}(x) \otimes R_0R_2, \end{aligned}$$

and

$$\begin{aligned} \alpha_2(x \otimes R_0^2) &= \partial_{r_0}(x) \otimes R_0^3 + \partial_{r_1}(x) \otimes R_1R_0^2 + \partial_{r_2}(x) \otimes R_2R_0^2 \\ &= \partial_{r_0}(x) \otimes R_0^3 + \partial_{r_2}(x) \otimes \frac{-1}{2}R_1^3 + \partial_{r_1}(x) \otimes \frac{-1}{2}R_2^3. \end{aligned}$$

In this second computation, we use the identities of the form $R_1R_0^2 = R_0^2R_1 = \frac{-1}{2}R_1^3$ within \mathcal{R}/S . The full matrices for these maps are given below, with the example computations appearing in the first columns of each matrix.

$$\alpha_1 = \begin{bmatrix} \partial_{r_0} & 0 & -\partial_{r_2} \\ 0 & \partial_{r_1} & -\partial_{r_2} \\ \partial_{r_2} & \partial_{r_0} & \partial_{r_1} \\ \partial_{r_1} & \partial_{r_2} & \partial_{r_0} \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \partial_{r_0} & -\frac{1}{2}\partial_{r_0} & -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_1} \\ -\frac{1}{2}\partial_{r_1} & \partial_{r_1} & -\frac{1}{2}\partial_{r_0} & -\frac{1}{2}\partial_{r_2} \\ -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_1} & -\frac{1}{2}\partial_{r_0} \end{bmatrix}.$$

These matrices are not exactly transposes of each other. However, with some simple row operations we can convert the matrix for α_1 into the transpose of the matrix of α_2 . This suggests a change in basis will give us the self-duality within this quotient complex. We will expand upon this example in the next chapter. Explicitly,

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \partial_{r_0} & 0 & -\partial_{r_2} \\ 0 & \partial_{r_1} & -\partial_{r_2} \\ \partial_{r_2} & \partial_{r_0} & \partial_{r_1} \\ \partial_{r_1} & \partial_{r_2} & \partial_{r_0} \end{bmatrix} = \begin{bmatrix} \partial_{r_0} & -\frac{1}{2}\partial_{r_0} & -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_1} \\ -\frac{1}{2}\partial_{r_1} & \partial_{r_1} & -\frac{1}{2}\partial_{r_0} & -\frac{1}{2}\partial_{r_2} \\ -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_1} & -\frac{1}{2}\partial_{r_0} \end{bmatrix}^T.$$

Returning to a general p , we now turn our attention to the standard Koszul complex associated to \mathcal{B} , rather than its dual. The standard Koszul complex $K(\mathcal{B})_n = \mathcal{B} \otimes (\mathcal{R}^*)_n$ with differential $\beta : K(\mathcal{B})_n \rightarrow K(\mathcal{B})_{n-1}$ given by

$$d(x \otimes \psi) = \sum_{i=0}^{p-1} r_i x \otimes \partial_{R_i} \psi.$$

As with the dual Koszul complex, we can view the standard Koszul complex as a bigraded chain complex with differential

$$\beta : K(\mathcal{B})_{m,n} \rightarrow K(\mathcal{B})_{m+1,n-1}.$$

If we dualize the short exact sequence in Equation 4.1, we get the following short exact sequence involving the Koszul complex

$$0 \longleftarrow K(\mathcal{B})[-2] \xleftarrow{S^*} K(\mathcal{B}) \longleftarrow (K(\mathcal{B})^*/S)^* \longleftarrow 0.$$

This last term of the short exact sequence is the subcomplex of the Koszul complex which is the kernel of the map S^* . We can write it out explicitly as a complex of B modules whose ranks match the ranks in the quotient complex above.

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{B}^1 \xrightarrow{\beta_3} \mathcal{B}^p \xrightarrow{\beta_2} \mathcal{B}^{2p-2} \xrightarrow{\beta_1} \mathcal{B}^p \xrightarrow{\beta_0} \mathcal{B}^1 \longrightarrow 0$$

As with the quotient complex, we can express the maps β_i in this subcomplex in terms of matrices, this time with entries among the r_i 's. Specifically, we can write β_0 and β_3 as

$$\beta_0 = \begin{bmatrix} r_0 & r_1 & \dots & r_{p-1} \end{bmatrix} \quad \beta_3 = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{p-1} \end{bmatrix}.$$

As before, explicitly writing out the middle matrices in general is more complicated. As a note, if you take this subcomplex and write it in reverse, it has the form

$$0 \longleftarrow \mathcal{B}^1 \xleftarrow{\beta_0} \mathcal{B}^p \xleftarrow{\beta_1} \mathcal{B}^{2p-2} \xleftarrow{\beta_2} \mathcal{B}^p \xleftarrow{\beta_3} \mathcal{B}^1 \longleftarrow 0 \longleftarrow \dots$$

It is tempting to look at this reversed subcomplex of the Koszul complex and to look at the quotient Koszul complex $K(\mathcal{B})^*/S$ and say that they are, in some sense, the “same complex” since the ranks align and the matrices have the same structure, with B^* replaced with B and with each ∂_{r_i} in the matrix for the map α_j replaced with r_i in the matrix for the map β_{4-j} . However, the the structure of the two complexes as bigraded complexes does not match as the bigrading on the maps α_j and β_{4-j} are different. We’ll explore the relationship between these complexes in more depth in the next section.

4.3 The Cohomology of the Third Fomin-Kirillov Algebra

In this section, we’ll completely analyze the quotient dual Koszul complex described above in the case when $p = 3$. In doing so, we’ll compute its cohomology, which will provide an alternate proof Theorem 2.4.3, the main result of [SV].

4.3.1 Matrix Transposes

First, we’ll more explicitly study the matrices that make up this quotient dual Koszul complex. We can explicitly write out this complex as

$$\cdots \longleftarrow 0 \longleftarrow \mathcal{B}^{*1} \xleftarrow{\alpha_3} \mathcal{B}^{*3} \xleftarrow{\alpha_2} \mathcal{B}^{*4} \xleftarrow{\alpha_1} \mathcal{B}^{*3} \xleftarrow{\alpha_0} \mathcal{B}^{*1} \longleftarrow 0.$$

We label the matrix representations of the maps α_i as M_i .

Lemma 4.3.1. *The matrices in this complex satisfy the following conditions*

- $M_0 = M_3^T$
- *There exists an invertible matrix N such that $NM_1 = M_2^T$.*

Proof. This lemma follows immediately from the explicit representations of these matrices which we wrote in the previous section:

$$M_0 = \begin{bmatrix} \partial_{r_0} \\ \partial_{r_1} \\ \partial_{r_2} \end{bmatrix}, \quad M_3 = \begin{bmatrix} \partial_{r_0} & \partial_{r_1} & \partial_{r_2} \end{bmatrix},$$

and

$$M_1 = \begin{bmatrix} \partial_{r_0} & 0 & -\partial_{r_2} \\ 0 & \partial_{r_1} & -\partial_{r_2} \\ \partial_{r_2} & \partial_{r_0} & \partial_{r_1} \\ \partial_{r_1} & \partial_{r_2} & \partial_{r_0} \end{bmatrix}, \quad M_2 = \begin{bmatrix} \partial_{r_0} & -\frac{1}{2}\partial_{r_0} & -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_1} \\ -\frac{1}{2}\partial_{r_1} & \partial_{r_1} & -\frac{1}{2}\partial_{r_0} & -\frac{1}{2}\partial_{r_2} \\ -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_2} & -\frac{1}{2}\partial_{r_1} & -\frac{1}{2}\partial_{r_0} \end{bmatrix}.$$

The matrix N is

$$N = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

□

In this section, we'll transform this complex into an isomorphic complex such that the central matrices are exactly transposes of one another. To do this we need a change of basis matrix which we denote P .

Lemma 4.3.2. *There is a matrix P such that $P^T P = N$.*

Proof. We construct P explicitly. Since N has a block diagonal structure, we can immediately fill out most entries of such a matrix.

$$\begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

The defining equation for P is satisfied as long as $a^2 + c^2 = b^2 + d^2 = 1$ and $ab + cd = \frac{-1}{2}$. This is satisfied for any vectors (a, c) and (b, d) on the unit circle at an angle of $\frac{2\pi}{3}$. There are many such choices, for example $a = 1$, $c = 0$, $b = \frac{-1}{2}$, and $d = \frac{\sqrt{3}}{2}$. \square

Using this matrix P , we can create a map from our quotient complex to an isomorphic complex. This map uses P as a change of basis map in the degree 2 part of the complex.

Lemma 4.3.3. *The following complex is isomorphic to the quotient dual Koszul complex:*

$$0 \longleftarrow \mathcal{B}^{*1} \xleftarrow{M_3} \mathcal{B}^{*3} \xleftarrow{M_2 P^{-1}} \mathcal{B}^{*4} \xleftarrow{P M_1} \mathcal{B}^{*3} \xleftarrow{M_0} \mathcal{B}^{*1} \longleftarrow 0.$$

Further, the matrices which make up this complex are transposes, which is to say that $M_3 = (M_0)^T$ and $M_2 P^{-1} = (P M_1)^T$.

Proof. Consider this map between complexes. This map uses the matrix P as a change of basis on the central term of the complex.

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{B}^{*1} & \xleftarrow{M_3} & \mathcal{B}^{*3} & \xleftarrow{M_2} & \mathcal{B}^{*4} & \xleftarrow{M_1} & \mathcal{B}^{*3} & \xleftarrow{M_0} & \mathcal{B}^{*1} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow P & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{B}^{*1} & \xleftarrow{M_3} & \mathcal{B}^{*3} & \xleftarrow{M_2 P^{-1}} & \mathcal{B}^{*4} & \xleftarrow{P M_1} & \mathcal{B}^{*3} & \xleftarrow{M_0} & \mathcal{B}^{*1} & \longleftarrow & 0 \end{array}$$

In this diagram, all unlabeled vertical maps can be taken to be the identity. Since P is invertible, each individual vertical map in this map between chain complex is an isomorphism, so the chain map is an isomorphism. Moreover, in the second complex, the matrices realizing the maps are transposes of each other. By construction

$$\begin{aligned} M_2^T &= NM_1 \\ M_2^T &= P^T PM_1 \\ (P^{-1})^T M_2^T &= PM_1 \\ (M_2 P^{-1})^T &= PM_1. \end{aligned}$$

□

The resulting basis is much more difficult to work with than the lexicographical basis, so we won't write out the second complex explicitly. The duality in this transformed complex suggests that there may be an isomorphism $(K(\mathcal{B})^*/S)_n \rightarrow (K(\mathcal{B})^*/S)_{4-n}$. We will not construct such an isomorphism directly, but we will construct an isomorphism on cohomology. As a note, for higher values of p , we can construct a matrix P with the same properties. For $p > 3$, we can construct this matrix P by finding $p - 1$ vectors on the sphere S^{p-2} such that any two vectors have an angle of $\frac{2\pi}{3}$ between them. We will not use the matrix P in the general case, so we will not prove this remark.

4.3.2 Quotient Complex Self-Duality

We just showed that the quotient complex $K(\mathcal{B})^*/S$ can be written such that the matrices which describe the maps in this quotient complex are transposes. As a result, there is an apparent self-duality within this complex around the central term. In this section, we prove that this duality extends to the cohomology of the complex when $p = 3$.

Lemma 4.3.4. *Let \mathcal{B} denote a QCN that is finite dimensional. There is a vector space isomorphism $H^i(K(\mathcal{B})^*/S) \cong H^{4-i}(K(\mathcal{B})^*/S)$.*

To construct this isomorphism on cohomology, we'll study the dual to the quotient Koszul complex, $K(\mathcal{B})/S^*$. By the Universal Coefficient Theorem, the cohomology $H^i(K(\mathcal{B})^*/S)$ is linearly dual to the homology $H_i(K(\mathcal{B})/S^*)$. Before proceeding, we construct two crucial maps.

Lemma 4.3.5. *There is a map of vector spaces $\gamma : \mathcal{R}/S_n \rightarrow (\mathcal{R}^*/S^*)_{4-n}$ that satisfies $\gamma(R_i X) = \partial_{R_i}(\gamma(X))$ for all generators $R_i \in \mathcal{R}$.*

Proof. Recall that \mathcal{R}/S has a basis

$$\Lambda = \{1, R_0, \dots, R_{p-1}, R_0^2, \dots, R_{p-2}^2, R_0 R_1, \dots, R_0 R_{p-1}, R_0^3, \dots, R_{p-1}^3, R_0^4\}.$$

We construct a dual basis of $(\mathcal{R}/S)^*$ of elements $\{\delta_X\}_{X \in \Lambda}$ where δ_X is defined such that, given $Y \in \Lambda$, $\delta_X(Y) = 1$ if $X = Y$ and $\delta_X(Y) = 0$ otherwise.

Further recall that there are relations $R_0^3 = -(p-1)R_0 R_1^2$ and $R_i^3 = -(p-1)R_0^2 R_i$.

We define γ on this basis as:

- $\gamma(1) = \delta_{R_0^4}$,
- $\gamma(R_i) = \delta_{R_i^3}$ for $0 \leq i \leq p-1$,
- $\gamma(R_i^2) = \delta_{R_i^2} - \frac{1}{p-1} \sum_{j \neq i} \delta_{R_j^2}$, for $0 \leq i \leq p-2$,
- $\gamma(R_0 R_i) = -\frac{1}{p-1} \delta_{R_0 R_i}$, for $0 \leq i \leq p-1$,
- $\gamma(R_i^3) = \delta_{R_i}$ for $0 \leq i \leq p-1$,
- $\gamma(R_i^4) = \delta_1$.

Note that γ is defined such that, for X and Y in \mathcal{R}/S_n , we have $\gamma(X)(Y) = \delta_{R_0^4}(XY)$.

We will check that this map satisfies the desired commutativity relation explicitly by checking all possible cases. By symmetry, it suffices to check that $\gamma(R_0 X) = \partial_{R_0}(\gamma(X))$ for all basis elements X . By further symmetry, it suffices to check that the relation is satisfied for one basis element of each type on the list given above. For the second, third, and fifth type, we further consider the case where $R_i = R_0$ and where $R_i \neq R_0$. We illustrate by a series of commutative diagrams. Recall that by definition, $\partial_{R_0} \delta_X$ is the map $\delta_X(R_0 \cdot _)$.

- If $X = 1$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_0^4}) = \delta_{R_0^3}$ since $\delta_{R_0^4}(R_0 \cdot R_0^3) = 1$ and $\delta_{R_0^4}(R_0 \cdot _)$

$X) = 0$ for all other basis vectors X . The following diagram therefore commutes.

$$\begin{array}{ccc} 1 & \xrightarrow{\gamma} & \delta_{R_0^4} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0 & \xrightarrow{\gamma} & \delta_{R_0^3} \end{array}$$

- If $X = R_0$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_0^3})$. Since $\delta_{R_0^3}(R_0 \cdot R_i^2) = \delta_{R_0^3}(\frac{-1}{p-1}R_0^3) = \frac{-1}{p-1}$ for all $0 < i \leq p-1$ and $\delta_{R_0^3}(R_0 \cdot R_0^2) = 1$, it follows that

$$\partial_{R_0}(\delta_{R_0^3}) = \delta_{R_0^2} + \frac{-1}{p-1} \sum_{j \neq 0} \delta_{R_j^2}.$$

The following diagram therefore commutes.

$$\begin{array}{ccc} R_0 & \xrightarrow{\gamma} & \delta_{R_1^3} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0^3} \\ R_0^2 & \xrightarrow{\gamma} & \delta_{R_0^2} + \frac{-1}{p-1} \sum_{j \neq 0} \delta_{R_j^2} \end{array}$$

- If $X = R_1$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_1^3}) = -\frac{1}{p-1}R_0R_1$ since $\delta_{R_1^3}(R_0 \cdot R_0R_1) = \delta_{R_1^3}(\frac{-1}{p-1}R_1^3) = \frac{-1}{p-1}$ and $\delta_{R_1^3}(R_0 \cdot X) = 0$ for all other other basis vectors X . The following diagram therefore commutes.

$$\begin{array}{ccc} R_1 & \xrightarrow{\gamma} & \delta_{R_1^3} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0R_1 & \xrightarrow{\gamma} & \frac{-1}{p-1} \delta_{R_0R_1} \end{array}$$

- If $X = R_0^2$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_0^2} - \frac{1}{p-1} \sum_{j \neq 0} \delta_{R_j^2})$. Note that $\partial_{R_0} \delta_{R_i^2} = 1$ is $i = 0$ and 0 otherwise. Therefore,

$$\partial_{R_0}(\delta_{R_0^2} - \frac{1}{p-1} \sum_{j \neq 0} \delta_{R_j^2}) = \delta_{R_0}.$$

The following diagram therefore commutes.

$$\begin{array}{ccc} R_0^2 & \xrightarrow{\gamma} & \delta_{R_0^2} - \frac{1}{p-1} \sum_{j \neq 0} \delta_{R_j^2} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0^3 & \xrightarrow{\gamma} & \delta_{R_0} \end{array}$$

- If $X = R_1^2$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_1^2} - \frac{1}{\sum_{j \neq 1}} \delta_{R_j^2})$. Therefore,

$$\partial_{R_0}(\delta_{R_1^2} - \frac{1}{\sum_{j \neq 1}} \delta_{R_j^2}) = \frac{-1}{p-1} \delta_{R_0}.$$

The following diagram therefore commutes.

$$\begin{array}{ccc} R_1^2 & \xrightarrow{\gamma} & \delta_{R_1^2} - \frac{1}{\sum_{j \neq 1}} \delta_{R_j^2} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0 R_1^2 = \frac{-1}{p-1} R_0^3 & \xrightarrow{\gamma} & \frac{-1}{p-1} \delta_{R_0} \end{array}$$

- If $X = R_0 R_1$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\frac{-1}{p-1} \delta_{R_0 R_1})$. Note that $\partial_{R_0} \delta_{R_0 R_1} = \partial_{R_1}$. The following diagram therefore commutes.

$$\begin{array}{ccc} R_0 R_1 & \xrightarrow{\gamma} & \frac{-1}{p-1} \delta_{R_0 R_1} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0^2 R_1 = \frac{-1}{p-1} R_1^3 & \xrightarrow{\gamma} & \frac{-1}{p-1} \delta_{R_1} \end{array}$$

- If $X = R_0^3$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_0}) = \delta_1$. The following diagram therefore commutes.

$$\begin{array}{ccc} R_0^3 & \xrightarrow{\gamma} & \delta_{R_0} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0^4 & \xrightarrow{\gamma} & \delta_1 \end{array}$$

- If $X = R_1^3$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_{R_1}) = 0$. Moreover, $R_0 R_1^3 = 0$ in \mathcal{R}/S . The

following diagram therefore commutes.

$$\begin{array}{ccc} R_1^3 & \xrightarrow{\gamma} & \delta_{R_1} \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0 R_1^3 = 0 & \xrightarrow{\gamma} & 0 \end{array}$$

- If $X = R_0^4$, then $\partial_{R_0}(\gamma(X)) = \partial_{R_0}(\delta_1) = 0$. The following diagram therefore commutes.

$$\begin{array}{ccc} R_0^4 & \xrightarrow{\gamma} & \delta_1 \\ \downarrow R_0 \cdot & & \downarrow \partial_{R_0} \\ R_0^5 = 0 & \xrightarrow{\gamma} & 0 \end{array}$$

Up to symmetry, this is a complete list of all possible cases, which completes the proof. □

Lemma 4.3.6. *Assume \mathcal{B} is finite dimensional. There is an isomorphism $\int : \mathcal{B}^* \rightarrow \mathcal{B}$. Treating \mathcal{B}^* as a \mathcal{B} -module where x acts as ∂_x , the map \int is a \mathcal{B} -module isomorphism. In other words, $\int \partial_x \psi = x \int \psi$.*

Proof. Since \mathcal{B} is a finite dimensional Hopf algebra, this integral is canonically known as the Larson-Sweedler integral ([16]) which is a map $\int : \mathcal{B}^* \rightarrow \mathcal{B}$ and is a \mathcal{B} -module isomorphism as desired.

Since \mathcal{B} is a graded Hopf algebra, we can explicitly write out this integral as

$$\psi \mapsto \mu(\psi \otimes 1(\Delta(r_0 r_1 r_0 r_2))).$$

Note that under this map, δ_1 is mapped to $r_0 r_1 r_0 r_2$, the top dimensional class of \mathcal{B} , and that $\delta_{r_0 r_1 r_0 r_2}$ is mapped to 1. □

We can now prove the desired duality result.

Proof. (Lemma 4.3.4) To complete this proof, we need to establish an isomorphism $(K(\mathcal{B})^*/S)_n \rightarrow (K(\mathcal{B})/S^*)_{4-n}$. Recall that the domain is $\mathcal{B}^* \otimes \mathcal{R}/S_n$ and the codomain is $\mathcal{B} \otimes (\mathcal{R}^*/S^*)_{4-n}$. Using the maps γ and \int established in the previous two lemmas, we

form the isomorphism $x \otimes Y \mapsto \int x \otimes \gamma(Y)$ where $x \otimes Y \in (K(\mathcal{B})^*/S)_n$. Since \int and γ are each isomorphisms, this map on tensors is also an isomorphism. Moreover, this map commutes with the Koszul differentials. If we first apply the dual Koszul complex differential and then the isomorphism,

$$\begin{aligned} x \otimes Y &\mapsto \partial_{r_0}(x) \otimes R_0Y + \partial_{r_1}(x) \otimes R_1Y + \partial_{r_2}(x) \otimes R_2Y \\ &\mapsto \int \partial_{r_0}(x) \otimes \gamma(R_0Y) + \int \partial_{r_1}(x) \otimes \gamma(R_1Y) + \int \partial_{r_2}(x) \otimes \gamma(R_2Y). \end{aligned}$$

Alternatively, if we first apply the isomorphism and then apply the Koszul complex differential,

$$\begin{aligned} x \otimes Y &\mapsto \int x \otimes \gamma(Y) \\ &\mapsto r_0 \int x \otimes \partial_{R_0}\gamma(Y) + r_1 \int x \otimes \partial_{R_1}\gamma(Y) + r_2 \int x \otimes \partial_{R_2}\gamma(Y). \end{aligned}$$

We constructed these maps to satisfy $\int \partial_x \psi = x \int \psi$ and $\gamma(R_i X) = \partial_{R_i}(\gamma(X))$. Applying these equalities repeatedly gives the desired commutativity. Therefore, $(K(\mathcal{B})^*/S)_n \rightarrow (K(\mathcal{B})/S^*)_{4-n}$ is a chain isomorphism. Extending this to homology yields $H^{4-i}(K(\mathcal{B})^*/S) \cong H_i(K(\mathcal{B})/S^*)$ where $4-i$ was substituted for n . From the Universal Coefficient Theorem, $H^i(K(\mathcal{B})^*/S) \cong H_i(K(\mathcal{B})/S^*)$. Composing these isomorphisms yields

$$H^{4-i}(K(\mathcal{B})^*/S) \cong H^i(K(\mathcal{B})^*/S),$$

as desired. □

4.3.3 Koszul Complex Homology

In this section, we will completely calculate the homology of the Koszul complex of the third Fomin-Kirillov algebra. As described in Example 3.1.3, the third Fomin-Kirillov algebra is isomorphic to the QCN algebra when $p = 3$. This algebra is finite dimensional, so we can use the result of Lemma 4.3.4. Moreover, we'll use the long exact sequence of cohomology established in Proposition 4.1.4. First, we do two preliminary calculations using the fact that the Fomin-Kirillov algebra is a Nichols algebra.

Lemma 4.3.7. *Assume \mathcal{B} is a Nichols algebra. Then*

$$H^0(K(\mathcal{B})^*/S) \cong k \text{ and } H^1(K(\mathcal{B})^*/S) \cong k.$$

Proof. Let α_i denote the i^{th} map in the quotient Koszul complex, so that $H^0(K(\mathcal{B})^*/S) = \ker(\alpha_0)$ and $H^1(K(\mathcal{B})^*/S) = \ker(\alpha_1)/\text{im}(\alpha_0)$. Recall that $\alpha_0(\psi \otimes 1) = \partial_{r_0}\psi \otimes R_0 + \partial_{r_1}\psi \otimes R_1 + \dots + \partial_{r_{p-1}}\psi \otimes R_{p-1}$. The elements in the kernel of this map are the elements $\psi \in \mathcal{B}^*$ such that $\partial_{r_0}\psi = \partial_{r_1}\psi = \dots = \partial_{r_{p-1}}\psi = 0$. Since \mathcal{B} is a Nichols algebra, \mathcal{B}^* is also a Nichols algebra, and there is a unique element $\psi \in \mathcal{B}^*$ which satisfies this property (See Proposition 2.8 in [1]). This unique element is δ_1 , so the kernel of α_0 is generated over k by $\delta_1 \otimes 1$.

Now, we write the image of α_1 , leaving out the terms in the final $p-1$ terms since they do not contribute to this argument:

$$\begin{aligned} \alpha_1(x_0 \otimes R_0 + \dots + x_{p-1} \otimes R_{p-1}) &= \partial_{r_0}(x_0) - \partial_{r_{p-1}}(x_{p-1}) \otimes R_0^2 + \dots \\ &\quad + \partial_{r_{p-2}}(x_{p-2}) - \partial_{r_{p-1}}(x_{p-1}) \otimes R_{p-2}^2 + \\ &\quad + * \otimes R_0 R_1 + \dots + * \otimes R_0 R_{p-1}. \end{aligned}$$

Assume this is an element in $\ker(\alpha_1)$. Then, by independence of the basis vectors, each $\partial_{r_i}(x_i) - \partial_{r_{p-1}}(x_{p-1}) = 0$. Let $\psi = \partial_{r_{p-1}}(x_{p-1})$. If $\psi = 0$, it follows that $x_0 \otimes R_0 + \dots + x_{p-1} \otimes R_{p-1}$ is in the image of α_0 . If $\psi \neq 0$, then recall that $\partial_{r_i} \circ \partial_{r_i} = 0$ by construction. Therefore, $\partial_{r_{p-1}}\psi = 0$. Moreover, for $i \neq p-1$, we just showed that $\psi = \partial_{r_i}(x_i)$, so

$$\partial_{r_i}\psi = \partial_{r_i}(\partial_{r_i}(x_i)) = 0.$$

Therefore, ψ satisfies the condition that $\partial_{r_0}\psi = \partial_{r_1}\psi = \dots = \partial_{r_{p-1}}\psi = 0$. As described above, $\psi = c\delta_1$ for some constant c . Therefore, for each i , $\partial_{r_i}(x_i) = c\delta_1$, so $x_i = c\delta_{r_i}$. Therefore, $\ker(\alpha_1)$ is generated by the image of α_0 and the element $\delta_{r_0} \otimes R_0 + \dots + \delta_{r_{p-1}} \otimes R_{p-1}$. Further, recall that \mathcal{B}^* in degree 2 is generated by elements $\delta_{r_i r_j}$ where $i \neq j$. It follows that for all i , there is no $\psi \in \mathcal{B}^*$ such that $\partial_{r_i}\psi = \partial_{r_i}$. Therefore this element is not in the image of α_0 . Therefore, it generates the cohomology H^1 over k , which completes the proof. \square

We now prove the following theorem

Theorem 4.3.8. *Let \mathcal{B} denote a QCN algebra which is finite dimensional and is a Nichols algebra. The dual Koszul complex $K(\mathcal{B})^*$ has cohomology $H^0(K(\mathcal{B})^*) \cong k$ and $H^3(K(\mathcal{B})^*) \cong k$ and $H^i(K(\mathcal{B})^*) \cong 0$ for all i not equal to 0 or 3.*

Proof. Recall that in Lemma 4.1.2, we showed that $H^i(K(\mathcal{B})^*) = 0$ for all $i > 3$. We'll use the long exact sequence involving the quotient complex $K(\mathcal{B})^*/S$ to compute the remaining cohomology groups. We copy this long exact sequence below. Since \mathcal{B} is a Nichols algebra by assumption, we insert the result of Lemma 4.3.7 to simplify the complex.

$$\begin{array}{ccccccccc}
 H^3(K^*) & & H^2(K^*) & & H^1(K^*) & & H^0(K^*) & & 0 & & 0 \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 0 & & 0 & & H^3(K^*) & & H^2(K^*) & & H^1(K^*) & & H^0(K^*) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & H^4(K^*/S) & & H^3(K^*/S) & & H^2(K^*/S) & & k & & k
 \end{array}$$

From the segment $0 \rightarrow H^0(K^*) \rightarrow k \rightarrow 0$, we get that $H^0(K^*) = k$. We then look at the segment $0 \rightarrow H^1(K^*) \rightarrow k \rightarrow k \rightarrow H^2(K^*)$. Note that the first and last maps in this segment are the maps induced on cohomology by the multiplication by S map. Proposition 6.10 in [12] tells us that this is the zero map on cohomology. Since the last map is the zero map, the map $k \rightarrow k$ is an isomorphism of vector spaces. Therefore, it has trivial kernel. By the exactness of the sequence, the map $H^1(K^*) \rightarrow 0$ is the zero map. Again using the exactness of the sequence, $H^1(K^*)$ is then isomorphic to the image of the map coming into it. That map is $0 \rightarrow H^1(K^*)$, so $H^1(K^*) = 0$. We input these results where they appear in the long exact sequence.

$$\begin{array}{ccccccccc}
 H^3(K^*) & & H^2(K^*) & & 0 & & k & & 0 & & 0 \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 0 & & 0 & & H^3(K^*) & & H^2(K^*) & & 0 & & k \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & H^4(K^*/S) & & H^3(K^*/S) & & H^2(K^*/S) & & k & & k
 \end{array}$$

Using the result of Lemma 4.3.4, we can fill in more entries of the bottom row of this long exact sequence, namely that H^3 and H^4 of the quotient complex are also 1 dimensional over k .

$$\begin{array}{ccccccccc}
 H^3(K^*) & & H^2(K^*) & & 0 & & k & & 0 & & 0 \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 0 & & 0 & & H^3(K^*) & & H^2(K^*) & & H^1(K^*) & & k \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & k & & k & & H^2(K^*/S) & & k & & k
 \end{array}$$

Since this is an exact sequence, we can use the segment $0 \rightarrow k \rightarrow H^3(K^*) \rightarrow 0$ to conclude that $H^3(K^*) \cong k$. Filling this in in both location yields the following.

$$\begin{array}{ccccccccc}
 k & & H^2(K^*) & & 0 & & k & & 0 & & 0 \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 0 & & 0 & & k & & H^2(K^*) & & H^1(K^*) & & k \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & k & & k & & H^2(K^*/S) & & k & & k
 \end{array}$$

Now we look at the segment $0 \rightarrow k \rightarrow k \rightarrow H^2(K^*) \rightarrow 0$. This map $k \rightarrow k$

is injective, since its kernel is 0, as the image of the map $0 \rightarrow k$. Since this is an injective map between one dimensional vector spaces over k , it must be an isomorphism. Therefore, the next map in the exact sequence, the map $k \rightarrow H^2(K^*)$, must be the zero map. It follows that the next map in the exact sequence is an isomorphism, so $H^2(K^*) \cong 0$. This completes the proof. It is worth noting that a similar exact sequence computation yields that $H^2(K^*/S) = 0$, so the complete long exact sequence is included below.

$$\begin{array}{cccccc}
 k & 0 & 0 & k & 0 & 0 \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \downarrow \\
 0 & 0 & k & 0 & 0 & k \\
 \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
 0 & k & k & 0 & k & k
 \end{array}$$

□

We now prove the result for the third Fomin-Kirillov algebra as a corollary.

Corollary 4.3.9. *Let $K(\mathcal{B})^*$ denote the dual Koszul complex of the third Fomin-Kirillov algebra. This complex has cohomology $H^0(K(\mathcal{B})^*) \cong k$ and $H^3(K(\mathcal{B})^*) \cong k$ and $H^i(K(\mathcal{B})^*) \cong 0$ for all i not equal to 0 or 3.*

Proof. From Example 3.1.3, the third Fomin-Kirillov algebra is isomorphic to the QCN \mathcal{B} when $p = 3$, which is finite dimensional and a Nichols algebra. The result follows immediately from Theorem 4.3.8. □

To conclude this section, the following result allows us to translate information about the homology of the Koszul complex to information about the cohomology of the algebra \mathcal{B} itself. We thank Craig Westerland for the proof of the following result.

Theorem 4.3.10. *Let \mathcal{A} be a quadratic braided Hopf algebra with quadratic dual \mathcal{R} and Koszul complex K . If*

$$H^{p,q}(K^*) = \begin{cases} k & (p, q) = (0, 0) \\ V & (p, q) = (p_0, q_0) \\ 0 & \text{else,} \end{cases}$$

then $\text{Ext}_A(k, k) \cong T_{\mathcal{R}}(W)$, the tensor algebra over \mathcal{R} on W , where $W = R \otimes \Sigma^{(x,y)}V$ is the free R module on V for some bidegree (x, y) .

Proof. Let $\Omega(\mathcal{R}^*)$ denote the cobar complex on \mathcal{R}^* so that $H^*(\Omega(\mathcal{R}^*)) \cong \text{Ext}_{\mathcal{R}}(k, k)$. There is a map of differential graded algebras $\Omega(\mathcal{R}^*) \rightarrow A$ corresponding to the map

$$\text{Ext}_{\mathcal{R}}^{*,*}(k, k) \rightarrow A = \text{Ext}_{\mathcal{R}}^{**}(k, k)$$

defined by projecting onto the diagonal cohomology. This gives a change of rings spectral sequence

$$\text{Ext}_A(k, \text{Ext}_{\Omega(\mathcal{R}^*)}(A, k)) \Rightarrow \text{Ext}_{\Omega(\mathcal{R}^*)}(k, k).$$

By definition, $\text{Tor}_{\Omega(\mathcal{R}^*)}(k, k) \cong H_*B\Omega(\mathcal{R}^*)$, and since \mathcal{R}^* is a connected coalgebra, $H_*B\Omega(\mathcal{R}^*) \cong \mathcal{R}^*$. It follows that

$$\text{Ext}_{\Omega(\mathcal{R}^*)}(k, k) \cong \text{Tor}_{\Omega(\mathcal{R}^*)}(k, k)^* \cong \mathcal{R}.$$

Further, Proposition 6.5 in [12] shows that

$$\text{Tor}_{\Omega(\mathcal{R}^*)}(k, k) \cong H^*(K(\mathcal{R})^*),$$

and the Universal Coefficient Theorem shows that

$$\begin{aligned} \text{Ext}_{\Omega(\mathcal{R}^*)}(A, k) &\cong \text{Tor}_{\Omega(\mathcal{R}^*)}(k, k) \\ &= H_*(K(\mathcal{R})) \\ &= H^*(K(A)^*) \\ &= k \oplus V, \end{aligned}$$

where the last equality is the assumption in statement of this theorem.

Therefore, the change of rings spectral sequence has the form

$$\text{Ext}_A(k, k \oplus V) \Rightarrow \mathcal{R}.$$

The A -module structure on $k \oplus V$ is trivial, so the E_2 term of this spectral sequence is $\text{Ext}_A(k, k) \otimes (k \oplus V)$. Recall that \mathcal{R} is isomorphic to the diagonal component of

$\text{Ext}_A(k, k)$, so the differential in the change of rings spectral sequence must send the non-diagonal components to zero.

This is a spectral sequence of rings where $H^*(K(A)^*) = k \oplus V$ is the trivial ring, which cannot support a differential. Thus the only possible differential is

$$\text{Ext}_A(k, k) \otimes V \rightarrow \text{Ext}_A(k, k) \cdot 1.$$

We denote this differential as d . In order to get $E_\infty = \mathcal{R}$, this differential must be injective with cokernel \mathcal{R} . Therefore,

$$\text{Ext}_A(k, k) / \text{im}(d) \cong \mathcal{R}$$

and $\text{im}(d) \cong \text{Ext}_A(k, k) \otimes V$. This argument can be repeated to build up the tensor algebra inductively, which completes the proof that $\text{Ext}_A(k, k)$ is the tensor algebra over \mathcal{R} on $R \otimes V = W$. \square

The following corollary is immediate which is exactly the main result from [23] as stated in Theorem 2.4.3. The degree of the generator X can be extracted step-by-step in the proof above.

Corollary 4.3.11. $\text{Ext}_{\mathcal{E}_3}(k, k) \cong \mathcal{R}[X]$ where X is a degree 4 independent generator.

Finally, it is worth explicitly stating a generator for the non-trivial degree three cohomology. For simplicity, we instead look at the standard Koszul complex. The following element is in the non-trivial degree three homology of the Koszul complex:

$$2r_2r_0r_1 \otimes R_0^3 + 2r_2r_1r_0 \otimes R_1^3 + 2r_0r_2r_1 \otimes R_2^3 + r_0r_2r_0 \otimes R_0^2R_1 + r_0r_1r_0 \otimes R_0^2R_1 + r_1r_2r_2R_0R_1^2.$$

This element is in the kernel of the differential because of cancellation in \mathcal{B}_4 which comes from the uniqueness of the top class $r_0r_1r_0r_2$. In the next chapter, we'll show that when $p > 3$, no such top-dimensional class exists, and that the homology in degree three is trivial.

Chapter 5

Alternating Lucas Number Conjecture

In this chapter, we will study the QCN algebras \mathcal{B} for $p > 3$. Specifically, we will study their dimensions and the cohomology of the quotient dual Koszul complex as we did in the previous chapter for $p = 3$. Explicitly, we'll show that for $p > 3$, the QCN algebras are not finite dimensional and their quotient dual Koszul complexes do not have the same self-duality in cohomology as in Lemma 4.3.4.

As a preliminary calculation, we will explicitly compute the dimensions of \mathcal{B}_n when $p = 5$ for small values of n . In degree 1, \mathcal{B} is generated by the five generators which correspond to the reflections in D_{10} . In degree 2, there are 25 generators of degree 2, with 9 independent degree 2 relations, so \mathcal{B}_2 is 16 dimensional. Computing the dimensions in higher degrees is more difficult, but the next computation is included below.

Lemma 5.0.1. *When $p = 5$, the degree 3 component of \mathcal{B} has dimension 45.*

Proof. There are 125 degree three monomials in \mathcal{B}_3 . However, since $r_i^2 = 0$ in \mathcal{B} for all i , many of the degree three monomials are 0. Namely, there are 5 monomials of the form r_i^3 , there are 20 monomials of the form $r_j r_i^2$ with $r_i \neq r_j$, and there are 20 monomials of the form $r_i^2 r_j$ with $r_i \neq r_j$, all of which are zero in \mathcal{B}_3 . This leaves 80 degree three generators.

From the four relations of the form $\sum_{i=0}^4 r_i r_{i+k}$ where $k \in \mathbb{Z}/5\mathbb{Z}$ with $k \neq 0$, we can form 40 degree three relations by multiplying each relation by each r_i on the left or

the right. These relations are not independent. Notably, each relation includes exactly one symmetric monomial of the form $r_i r_j r_i$ and each symmetric monomial appears in exactly two degree three relations. Explicitly, to get the symmetric monomial $r_i r_j r_i$ within a relation, we can either left multiply the relation involving $r_j r_i$ by r_i , or we can right multiply the relation involving $r_i r_j$ by r_i . Therefore, for each symmetric monomial $r_i r_j r_i$, we can remove the monomial from the list of generators and form a relation that doesn't involve any symmetric monomials by subtracting the two relations which include $r_i r_j r_i$. This removes 20 monomials from the list of generators and cuts the relations in half, leaving 60 generators and 20 relations. These 20 relations are listed below, labelled by the symmetric monomial which was cancelled in their formation.

$$r_0 r_1 r_0 : r_0 r_2 r_1 + r_0 r_3 r_2 + r_0 r_4 r_3 - r_3 r_4 r_0 - r_2 r_3 r_0 - r_1 r_2 r_0 = 0$$

$$r_2 r_0 r_2 : r_2 r_4 r_1 + r_2 r_1 r_3 + r_2 r_3 r_0 - r_0 r_3 r_2 - r_3 r_1 r_2 - r_1 r_4 r_2 = 0$$

$$r_3 r_2 r_3 : r_3 r_0 r_1 + r_3 r_1 r_2 + r_3 r_4 r_0 - r_1 r_0 r_3 - r_2 r_1 r_3 - r_0 r_4 r_3 = 0$$

$$r_1 r_3 r_1 : r_1 r_2 r_0 + r_1 r_0 r_3 + r_1 r_4 r_2 - r_2 r_4 r_1 - r_3 r_0 r_1 - r_0 r_2 r_1 = 0$$

$$r_1 r_2 r_1 : r_1 r_3 r_2 + r_1 r_4 r_3 + r_1 r_0 r_4 - r_4 r_0 r_1 - r_3 r_4 r_1 - r_2 r_3 r_1 = 0$$

$$r_3 r_1 r_3 : r_3 r_0 r_2 + r_3 r_2 r_4 + r_3 r_4 r_1 - r_1 r_4 r_3 - r_4 r_2 r_3 - r_2 r_0 r_3 = 0$$

$$r_4 r_3 r_4 : r_4 r_1 r_2 + r_4 r_2 r_3 + r_4 r_0 r_1 - r_2 r_1 r_4 - r_3 r_2 r_4 - r_1 r_0 r_4 = 0$$

$$r_2 r_4 r_2 : r_2 r_3 r_1 + r_2 r_1 r_4 + r_2 r_0 r_3 - r_3 r_0 r_2 - r_4 r_1 r_2 - r_1 r_3 r_2 = 0$$

$$r_2 r_3 r_2 : r_2 r_4 r_3 + r_2 r_0 r_4 + r_2 r_1 r_0 - r_0 r_1 r_2 - r_4 r_0 r_2 - r_3 r_4 r_2 = 0$$

$$r_4 r_2 r_4 : r_4 r_1 r_3 + r_4 r_3 r_0 + r_4 r_0 r_2 - r_2 r_0 r_4 - r_0 r_3 r_4 - r_3 r_1 r_4 = 0$$

$$r_0 r_4 r_0 : r_0 r_2 r_3 + r_0 r_3 r_4 + r_0 r_1 r_2 - r_3 r_2 r_0 - r_4 r_3 r_0 - r_2 r_1 r_0 = 0$$

$$r_3 r_0 r_3 : r_3 r_4 r_2 + r_3 r_2 r_0 + r_3 r_1 r_4 - r_4 r_1 r_3 - r_0 r_2 r_3 - r_2 r_4 r_3 = 0$$

$$r_3r_4r_3 : r_3r_0r_4 + r_3r_1r_0 + r_3r_2r_1 - r_1r_2r_3 - r_0r_1r_3 - r_4r_0r_3 = 0$$

$$r_0r_3r_0 : r_0r_2r_4 + r_0r_4r_1 + r_0r_1r_3 - r_3r_1r_0 - r_1r_4r_0 - r_4r_2r_0 = 0$$

$$r_1r_0r_1 : r_1r_3r_4 + r_1r_4r_0 + r_1r_2r_3 - r_4r_3r_1 - r_0r_4r_1 - r_3r_2r_1 = 0$$

$$r_4r_1r_4 : r_4r_0r_3 + r_4r_3r_1 + r_4r_2r_0 - r_0r_2r_4 - r_1r_3r_4 - r_3r_0r_4 = 0$$

$$r_4r_0r_4 : r_4r_1r_0 + r_4r_2r_1 + r_4r_3r_2 - r_2r_3r_4 - r_1r_2r_4 - r_0r_1r_4 = 0$$

$$r_1r_4r_1 : r_1r_3r_0 + r_1r_0r_2 + r_1r_2r_4 - r_4r_2r_1 - r_2r_0r_1 - r_0r_3r_1 = 0$$

$$r_2r_1r_2 : r_2r_4r_0 + r_2r_0r_1 + r_2r_3r_4 - r_0r_4r_2 - r_1r_0r_2 - r_4r_3r_2 = 0$$

$$r_0r_2r_0 : r_0r_1r_4 + r_0r_4r_2 + r_0r_3r_1 - r_1r_3r_0 - r_2r_4r_0 - r_4r_1r_0 = 0$$

Clearly, these relations are still not independent. In this list above, every fourth relation is dependent on the three directly above it. Eliminating the last relation of each cluster gives a presentation of \mathcal{B}_3 with 60 generators and 15 relations. From analyzing the list above, we see that the remaining 15 relations are independent—The only relations which could have further dependences among them are those that share monomials. But, the relations that share monomials are clustered together and removing one relation from each cluster removed all dependence within that cluster. These relations are independent which completes the proof that \mathcal{B}_3 is 45 dimensional. \square

As a sequence the dimensions of the degree n part of the QCN algebra \mathcal{B} are

$$1, 5, 16, 45, \dots$$

Before stating the main conjecture of this chapter, we state the definition of a famous sequence, the Lucas numbers. The Lucas numbers are an analogue of the Fibonacci numbers using the same recurrence relation with different initial conditions.

Definition 5.0.2. The Lucas numbers $L(n)$ satisfy the recurrence relation $L(n + 2) = L(n + 1) + L(n)$ and the initial conditions $L(0) = 2$ and $L(1) = 1$.

The first few Lucas numbers can be written out as

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \dots$$

We now state the following conjecture about the dimension of \mathcal{B}_n when $p = 5$.

Conjecture 5.0.3 (Alternating Lucas Number Conjecture). The dimension of the degree n part of the QCN algebra \mathcal{B} for $p = 5$ is $L(2n + 2) - 2$.

Expanding this conjecture, we note that the sequence $L(2n + 2) - 2$ is a member of a family of recursive sequences $\mathcal{S}_r(n)$. We define this family of sequences below.

Definition 5.0.4. The \mathcal{S}_r family of sequences is a family of sequences that satisfy the additive recurrence relation

$$\mathcal{S}_r(n + 2) - (r - 2)\mathcal{S}_r(n + 1) + \mathcal{S}_r(n) - 2 = 0,$$

and the initial conditions $\mathcal{S}_r(0) = 1$ and $\mathcal{S}_r(1) = r$.

Note that $\mathcal{S}_r(2) = (r - 1)^2$ follows directly from this definition.

Lemma 5.0.5. *The sequence $L(2n + 2) - 2$ is identical to the sequence $\mathcal{S}_5(n)$.*

Proof. To show that these sequences are equivalent, first note that the first two terms of both sequences are 1 and 5. To show that the entire sequence are identical, it suffices to show that they satisfy the same recurrence. Starting with the relation

$$\mathcal{S}_5(n + 2) = 3\mathcal{S}_5(n + 1) - \mathcal{S}_5(n) + 2,$$

we will show that $L(2n + 2) - 2$ satisfies the same recurrence. This holds if and only if

$$L(2(n + 2) + 2) - 2 = 3[L(2(n + 1) + 2) - 2] - [L(2n + 2) - 2] + 2$$

We simplify using the definition of the Lucas numbers that $L(n + 2) = L(n + 1) + L(n)$.

$$\begin{aligned} L(2n + 6) - 2 &= 3L(2n + 4) - 6 - L(2n + 2) + 2 + 2 \\ L(2n + 5) + L(2n + 4) &= 2L(2n + 4) + L(2n + 4) - L(2n + 2) \\ L(2n + 4) + L(2n + 3) + L(2n + 4) &= 2L(2n + 4) + L(2n + 3) + L(2n + 2) - L(2n + 2). \end{aligned}$$

Both sides of this final line are equal to $2L(2n + 4) + L(2n + 3)$, which completes the proof. \square

The following lemma is not related to the study of QCN algebras since 4 is not prime, but it is an interesting fact about this family of sequences.

Lemma 5.0.6. *The sequence $\mathcal{S}_4(n) = n^2$.*

Proof. Again, to show these sequences are equivalent, we first note that both sequences begin with 1 and 4. It is straightforward to check that n^2 satisfies the same recurrence as $\mathcal{S}_4(n)$. A simple simplification yields

$$(n + 2)^2 = 2(n + 1)^2 - n^2 + 2.$$

□

As an additional preliminary fact about this sequence, we prove that $\mathcal{S}_r(n)$ is a family of increasing sequences, so long as $r > 3$.

Lemma 5.0.7. *Let $r > 3$. The sequences $\mathcal{S}_r(n)$ is a strictly increasing sequence, which is to say that $\mathcal{S}_r(n + 1) - \mathcal{S}_r(n) > 0$ for all $n \geq 0$.*

Proof. We proceed by induction. The first three terms of the sequence are 1, r , and $(r - 1)^2$, so the base case follows. Assume that $\mathcal{S}_r(n + 1) - \mathcal{S}_r(n) > 0$, and we'll show that $\mathcal{S}_r(n + 2) - \mathcal{S}_r(n + 1) > 0$. By definition,

$$\mathcal{S}_r(n + 2) - (r - 2)\mathcal{S}_r(n + 1) + \mathcal{S}_r(n) - 2 = 0.$$

We rearrange this equality as

$$\mathcal{S}_r(n + 2) - \mathcal{S}_r(n + 1) = (r - 3)\mathcal{S}_r(n + 1) - \mathcal{S}_r(n) + 2.$$

Since $r > 3$, we can look at the right side of this equality:

$$(r - 3)\mathcal{S}_r(n + 1) - \mathcal{S}_r(n) + 2 > \mathcal{S}_r(n + 1) - \mathcal{S}_r(n) + 2 > \mathcal{S}_r(n + 1) - \mathcal{S}_r(n).$$

This is greater than zero by assumption, so the result follows by induction. □

We can manipulate the recurrence relation in the definition of $\mathcal{S}_r(n)$ by plugging in

$n + 1$ in place of n and subtracting the original relation to get

$$\mathcal{S}_r(n + 3) - (r - 1)\mathcal{S}_r(n + 2) + (r - 1)\mathcal{S}_r(n + 1) - \mathcal{S}_r(n) = 0.$$

Repeating this process one more time yields the relation

$$\mathcal{S}_r(n + 4) - r\mathcal{S}_r(n + 3) + (2r - 2)\mathcal{S}_r(n + 2) - r\mathcal{S}_r(n + 1) + \mathcal{S}_r(n) = 0.$$

The coefficients in this last recurrence relation exactly match the ranks in the quotient dual Koszul complex from the previous chapter. If we assume the algebra \mathcal{B} to be Koszul, this suggests a much stronger conjecture, which has the Alternating Lucas Number conjecture as a corollary.

Conjecture 5.0.8 (Generalized Alternating Lucas Number Conjecture). Let \mathcal{B} denote the corresponding QCN algebra corresponding to the dihedral group D_{2p} for $p \geq 5$. The dimension of the degree n part of \mathcal{B} is $\mathcal{S}_p(n)$.

Finally, the following lemma gives us a closed form for the formal power series whose coefficients are given by the series $\mathcal{S}_p(n)$.

Lemma 5.0.9. *Let $r > 3$ and let*

$$A_r(t) = \sum_{n=0}^{\infty} \mathcal{S}_r(n)t^n.$$

Then

$$A_r(t) = \frac{1 + t}{1 - (r - 1)t + (r - 1)t^2 - t^3}.$$

Proof. We use a standard technique of [22]. For simplicity, in this proof we'll denote $\mathcal{S}_r(n) = s_n$. Starting with the recurrence relation that

$$s_n = (r - 1)s_{n-1} - (r - 1)s_{n-2} + s_{n-3},$$

we multiply each side by t^{n-3} and sum up over all $n \geq 3$. This gives

$$\frac{A_r(t) - s_0 - s_1t - s_2t^2}{t^3} = (r - 1)\frac{A_r(t) - s_0 - s_1t}{t^2} - (r - 1)\frac{A_r(t) - s_0}{t} + A_r(t).$$

Multiplying each side by t^3 and bringing all terms with $A_r(t)$ to one side yields

$$A_r(t)(1 - (r-1)t + (r-1)t^2 - t^3) = s_0 + s_1t + s_2t^2 - (r-1)ts_0 - (r-1)t^2s_1 + (r-1)t^2s_0.$$

Substituting in $s_0 = 1$, $s_1 = r$, and $s_2 = (r-1)^2$, the right hand side simplifies to $(1+t)$ and the result follows. \square

This yields the following corollary which demonstrates the relationship between the QCN algebras \mathcal{B} and the family of sequences $\mathcal{S}_r(n)$.

Corollary 5.0.10. *Let $H_{\mathcal{R}}(t)$ denote the Hilbert Series for the algebra \mathcal{R} , the quadratic dual to the QCN algebra for D_{2p} . Then,*

$$A_p(t) = \frac{1}{H_{\mathcal{R}}(-t)}.$$

Proof. We start by plugging $-t$ into the result from Corollary 3.2.2 to get

$$H_{\mathcal{R}}(-t) = \frac{1 - pt + (2p-2)t^2 - pt^3 + t^4}{1 - t^2}.$$

Note that $(1-t)(1 - (p-1)t + (p-1)t^2 - t^3) = 1 - pt + (2p-2)t^2 - pt^3 + t^4$, so the numerator simplifies to

$$H_{\mathcal{R}}(-t) = \frac{(1-t)(1 - (p-1)t + (p-1)t^2 - t^3)}{1 - t^2}.$$

Taking reciprocals and canceling $(1-t)$ yields

$$\frac{1}{H_{\mathcal{R}}(-t)} = \frac{1+t}{1 - (p-1)t + (p-1)t^2 - t^3} = A_p(t),$$

by Lemma 5.0.9. \square

5.1 The Euler Characteristic of the Quotient Koszul Complex

As remarked in the previous section, the ranks in the quotient dual Koszul complex for \mathcal{B} exactly match the coefficients in a recurrence relation for the sequence $\mathcal{S}_r(n)$. In this

section, we'll expand upon this relationship. Recall that this quotient complex is of the form

$$\cdots \longleftarrow 0 \longleftarrow \mathcal{B}^{*1} \xleftarrow{\alpha_3} \mathcal{B}^{*p} \xleftarrow{\alpha_2} \mathcal{B}^{*2p-2} \xleftarrow{\alpha_1} \mathcal{B}^{*p} \xleftarrow{\alpha_0} \mathcal{B}^{*1} \longleftarrow 0.$$

Further, recall that the dual Koszul complex, and therefore its quotient, can be regarded a bigraded chain complex where the differential $\alpha : \mathcal{B}_m^* \otimes \mathcal{R}_n \rightarrow \mathcal{B}_{m-1}^* \otimes \mathcal{R}_{n+1}$. Using this, we can decompose the dual quotient Koszul complex into a countable direct sum of diagonal sub-complexes indexed by integers k such that that bidegrees in each sub-complex $\mathcal{B}_m^* \otimes \mathcal{R}_n$ satisfy $m + n = k$. We can diagram this as

$$\begin{array}{cccccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
 0 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & 0 \\
 & \mathcal{B}_4^{*1} & \mathcal{B}_4^{*p} & \mathcal{B}_4^{*2p-2} & \mathcal{B}_4^{*p} & \mathcal{B}_4^{*1} & & \\
 0 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & 0 \\
 & \mathcal{B}_3^{*1} & \mathcal{B}_3^{*p} & \mathcal{B}_3^{*2p-2} & \mathcal{B}_3^{*p} & \mathcal{B}_3^{*1} & & \\
 0 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & 0 \\
 & \mathcal{B}_2^{*1} & \mathcal{B}_2^{*p} & \mathcal{B}_2^{*2p-2} & \mathcal{B}_2^{*p} & \mathcal{B}_2^{*1} & & \\
 0 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & 0 \\
 & \mathcal{B}_1^{*1} & \mathcal{B}_1^{*p} & \mathcal{B}_1^{*2p-2} & \mathcal{B}_1^{*p} & \mathcal{B}_1^{*1} & & \\
 0 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & 0 \\
 & \mathcal{B}_0^{*1} & \mathcal{B}_0^{*p} & \mathcal{B}_0^{*2p-2} & \mathcal{B}_0^{*p} & \mathcal{B}_0^{*1} & & \\
 0 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array}$$

First, we'll use this decomposition to further understand the $p = 3$ case outlined in the previous section.

Lemma 5.1.1. *When $p = 3$, if we divide the quotient dual Koszul complex into diagonal subcomplexes, there are exactly four subcomplexes with non-zero Euler characteristic.*

Proof. When $p = 3$, \mathcal{B} is 12 dimensional and $\mathcal{B}_n = 0$ for all $n > 4$. Therefore $\mathcal{B}_n^* = 0$ for all $n > 4$ as well. In this case, we can write out the entire quotient complex, subdivided into finitely many subcomplexes.

$$\begin{array}{cccccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & \mathcal{B}_4^{*1} & \mathcal{B}_4^{*3} & \mathcal{B}_4^{*4} & \mathcal{B}_4^{*3} & \mathcal{B}_4^{*1} & 0 & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & \mathcal{B}_3^{*1} & \mathcal{B}_3^{*3} & \mathcal{B}_3^{*4} & \mathcal{B}_3^{*3} & \mathcal{B}_3^{*1} & 0 & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & \mathcal{B}_2^{*1} & \mathcal{B}_2^{*3} & \mathcal{B}_2^{*4} & \mathcal{B}_2^{*3} & \mathcal{B}_2^{*1} & 0 & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & \mathcal{B}_1^{*1} & \mathcal{B}_1^{*3} & \mathcal{B}_1^{*4} & \mathcal{B}_1^{*3} & \mathcal{B}_1^{*1} & 0 & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & \mathcal{B}_0^{*1} & \mathcal{B}_0^{*3} & \mathcal{B}_0^{*4} & \mathcal{B}_0^{*3} & \mathcal{B}_0^{*1} & 0 & 0 \\
 \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.
 \end{array}$$

We can compute the Euler characteristic along these diagonals, remembering that \mathcal{B}_0^* , \mathcal{B}_1^* , \mathcal{B}_2^* , \mathcal{B}_3^* , and \mathcal{B}_4^* have dimensions 1, 3, 4, 3, and 1 respectively. For example, the $k = 2$ subcomplex is

$$0 \longleftarrow \mathcal{B}_0^{*4} \longleftarrow \mathcal{B}_1^{*3} \longleftarrow \mathcal{B}_2^{*1} \longleftarrow 0.$$

and has Euler characteristic $4 * 1 - 3 * 3 + 4 * 1 = -1$. Similarly, the $k=3$ subcomplex is

$$0 \longleftarrow \mathcal{B}_0^{*3} \longleftarrow \mathcal{B}_1^{*4} \longleftarrow \mathcal{B}_2^{*3} \longleftarrow \mathcal{B}_3^{*1} \longleftarrow 0.$$

and has Euler characteristic $3 * 1 - 4 * 3 + 3 * 4 - 1 * 3 = 0$. From similar computations, we can find exactly four subcomplexes with non-zero Euler characteristic, those indexed by $k = 0$, $k = 2$, $k = 6$, and $k = 8$. These subcomplexes each have Euler characteristic 1 or -1 . \square

Since Euler characteristic is cohomology invariant, the four subcomplexes with non-zero Euler characteristic correspond to the four non-trivial cohomology classes found in degrees 0, 1, 3, and 4 respectively of the quotient complex, as seen in the bottom row of last diagram in the proof of Theorem 4.3.8. Before moving on to $p > 3$, we note the relationship between the degrees of \mathcal{B}_n and $\mathcal{S}_3(n)$. Note that Lemma 5.0.7 does not hold for $r = 3$, and $\mathcal{S}_3(n)$ is not a strictly increasing sequence. Indeed, the sequence $\mathcal{S}_3(n)$ is a cyclic sequence:

$$1, 3, 4, 3, 1, 0, 1, 3, 4, 3, 1, 0, 1, 3, 4, 3, 1, 0, \dots$$

This gives the equality:

$$\dim(\mathcal{B}_n) = \mathcal{S}_3(n) - \mathcal{S}_3(n - 6),$$

where we take $\mathcal{S}_3(n) = 0$ for all $n < 0$. Here, we see that the Generalized Alternating Lucas Number conjecture fails when $p = 3$. Namely, the degrees of \mathcal{B}_n are given by a the relevant member of the \mathcal{S}_r family of sequence subtracted by a shifted version of itself.

Expanding upon this analysis, we have the following theorem.

Theorem 5.1.2. *For an odd prime p , if the QCN algebra \mathcal{B} is Koszul, then the Generalized Alternating Lucas Number Conjecture holds, which is to say that the dimension of \mathcal{B}_n is $\mathcal{S}_p(n)$.*

Proof. Consider the dual Koszul complex $K^* = K(\mathcal{B})^*$ as a bigraded complex with $K(\mathcal{B})_{m,n}^* = \mathcal{B}_m^* \otimes \mathcal{R}_n$. We can write the Hilbert series for this bigraded complex as

$$H_{K^*}(t, s) = \sum_{m,n} \text{rk}(K_{m,n}^*) t^m s^n = H_{\mathcal{B}}(t) H_{\mathcal{R}}(s).$$

Substituting $-t$ for s yields

$$H_{K^*}(t, -t) = \sum_{m,n} \text{rk}(K_{m,n}^*) (-1)^n t^{m+n}.$$

By viewing this sum as an alternating sum along the diagonals in the bigraded complex, this is equal to the sum of the Euler characteristics of the diagonal sub-complexes. Since the differential preserves $m + n$, this expression is homology invariant. It follows that

$$H_{H^*(K^*)}(t, -t) = H_{\mathcal{B}}(t)H_{\mathcal{R}}(-t).$$

If \mathcal{B} is Koszul, then its dual Koszul complex K^* is acyclic. Therefore,

$$H_{H^*(K^*)}(t, -t) = 1,$$

and it follows that

$$H_{\mathcal{B}}(t) = \frac{1}{H_{\mathcal{R}}(-t)}.$$

The result follows immediately from Corollary 5.0.10. \square

In this proof, we further see why the Generalized Alternating Lucas Number Conjecture fails when $p = 3$. When $p = 3$, \mathcal{B} is not Koszul and the dual Koszul complex has a single nontrivial cohomology class in bidegree $3, -3$. Therefore, $H_{H^*(K^*)}(t, -t) = 1 - t^6$. Dividing this by $H_{\mathcal{R}}(-t)$ as in the proof outlined above yields $H_{\mathcal{B}}(t) = A_p(t) - t^6 A_p(t)$, which exactly gives shift $\dim(\mathcal{B}_n) = \mathcal{S}_3(n) - \mathcal{S}_3(n - 6)$ computed previously.

As a final note in this section, since \mathcal{B} is a quadratic algebra, we can denote its set of generators V and its set of relations $R \subseteq V \otimes V$. As a vector space,

$$\mathcal{B}_n = V^{\otimes n} / \bigoplus_{i=1}^{n-1} V^{\otimes n-i-1} \otimes R \otimes V^{i-1}.$$

Theorem 5.1.2 establishes a link between \mathcal{B} being Koszul and the dimension of its n^{th} degree part. The following remark from page 484 of [4] also establishes such a connection using the modules $V^{\otimes n-i-1} \otimes R \otimes V^{i-1}$.

Remark 5.1.3 (Beilinson-Ginzburg-Soergel). The algebra \mathcal{B} is Koszul if and only if, for

all $n \geq 0$ the set $\{V^{\otimes n-i-1} \otimes R \otimes V^{i-1}\}_{i=1}^{n-1}$ generates a distributive lattice of submodules of $V^{\otimes n}$.

To unpack this statement, when $n = 3$, this is equivalent to saying that the following sequence is exact:

$$0 \rightarrow (V \otimes R) \cap (R \otimes V) \rightarrow (V \otimes R) + (R \otimes V) \rightarrow (V \otimes R) \oplus (R \otimes V) \rightarrow 0.$$

For larger n , a similar but longer sequence can be constructed involving pairwise intersections and further intersections. When that sequence is exact, we can compute the dimensions of $\bigoplus_{i=1}^{n-1} V^{\otimes n-i-1} \otimes R \otimes V^{i-1}$ to be an alternating sum of intersections of these modules,

$$\begin{aligned} \sum_{i=1}^{n-1} \dim(V^{\otimes n-i-1} \otimes R \otimes V^{i-1}) - \sum_{i \neq j}^{n-1} \dim(V^{\otimes n-i-1} \otimes R \otimes V^{i-1} \cap V^{\otimes n-j-1} \otimes R \otimes V^{j-1}) \\ + \cdots \pm \dim\left(\bigcap_{i=1}^{n-1} V^{\otimes n-i-1} \otimes R \otimes V^{i-1}\right). \end{aligned}$$

Assuming \mathcal{B} is Koszul, the dimension of \mathcal{B}_n is therefore p^n minus this alternating sum. Therefore p^n minus the alternating sum above is exactly $\mathcal{S}_p(n)$.

5.2 Asymmetrical Cohomology for $p > 3$

In the previous chapter, we showed that for QCN algebra for $p = 3$, which is in fact itself a Nichols algebra, the self-symmetry in ranks of the modules in the quotient dual Koszul complex extends to self-symmetry in the cohomology of this complex. This was formalized in the statement and proof of Lemma 4.3.4. The finite dimensionality of \mathcal{B} in this case was used, as the Hopf algebra Larson-Sweedler integral exists if and only if the algebra is finite dimensional ([16]). Explicitly, we wrote a formula for the integral using the top-dimensional class in \mathcal{B}_4 . In this final section we'll prove that for a generic $p > 3$, finding such a duality is equivalent to the finite dimensionality of \mathcal{B} . We then show that the algebra cannot be finite dimensional, which means that no such duality exists, and we explicitly prove that the degree 4 cohomology in the quotient dual Koszul complex is 0 for $p > 3$.

Theorem 5.2.1. *The following are equivalent:*

1. *The algebra \mathcal{B} is finite dimensional.*
2. *There is an element $x \in \mathcal{B}$ such that $r_i x = 0$ for all $1 \leq i \leq p - i$. This element is unique up to constant multiple.*
3. *There is a degree d such that \mathcal{B}_d is one dimensional as a vector space, and $\mathcal{B}_m = \{0\}$ for all $m > d$.*
4. *There is a chain complex isomorphism $(K(\mathcal{B})^*/S)_n \rightarrow (K(\mathcal{B})/S^*)_{4-n}$.*

Proof. Assume the first statement. The existence of the chain complex isomorphism in fourth statement was shown in the proof of Lemma 4.3.4.

Assume the fourth statement. Fix $n = 4$. By assumption, there is an isomorphism $(K(\mathcal{B})^*/S)_4 \rightarrow (K(\mathcal{B})/S^*)_0$. But $(K(\mathcal{B})^*/S)_4 \cong \mathcal{B}^* \otimes (\mathcal{R}/S)_4 \cong \mathcal{B}^*$ since $(\mathcal{R}/S)_4$ is one-dimensional. Similarly, $(K(\mathcal{B})/S^*)_0 \cong \mathcal{B} \otimes (\mathcal{R}^*/S^*)_0 \cong \mathcal{B}$ since $(\mathcal{R}^*/S^*)_0$ is one-dimensional. Since this map commutes with the Koszul differential, it is as \mathcal{B} module isomorphism. Therefore, this map satisfies the conditions of being a Larson-Sweedler integral. This integral exists only if \mathcal{B} is finite dimensional. Therefore the first and fourth statements are equivalent.

The first statement is equivalent to the existence of a Larson-Sweedler integral. Since \mathcal{B} is graded, by [16], this Larson-Sweedler integral is equivalent to finding an element x such that $r_i x = 0$ for all $1 \leq i \leq p - i$, and this integral is unique when it exists. Therefore the first and second statements are equivalent.

Assume the first statement. Since \mathcal{B} is finite dimensional, there must be a degree d such that $\mathcal{B}_m = \{0\}$ for all $m > d$. Given any $x \in \mathcal{B}_d$ and any generator r_i , the product $r_i x \in \mathcal{B}_{d+1} = \{0\}$. As stated in the previous paragraph, elements of this type are unique, up to constant multiple. It follows that \mathcal{B}_d is one dimensional as a vector space. Therefore the first statement implies the third.

Finally, assume the third statement. The dimension of \mathcal{B} as a vector space is $\sum_{i=0}^d \dim(\mathcal{B}_i)$, each of which is finite. Therefore \mathcal{B} is finite dimensional, and the third statement is equivalent to the first. \square

We now extend the implications of \mathcal{B} being finite.

Lemma 5.2.2. *If \mathcal{B} is finite dimensional, then*

- $H^4(K(\mathcal{B})^*/S) \cong k$, and
- \mathcal{B} is a Nichols algebra.

Proof. Recall that $K(\mathcal{B})^*/S$ is linearly dual to a subcomplex of the Koszul complex as described at the end of Section 4.2. This complex has the form

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{B}^1 \xrightarrow{\beta_3} \mathcal{B}^p \xrightarrow{\beta_2} \mathcal{B}^{2p-2} \xrightarrow{\beta_1} \mathcal{B}^p \xrightarrow{\beta_0} \mathcal{B}^1 \longrightarrow 0,$$

where the map β_3 is given by the matrix
$$\begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{p-1} \end{bmatrix}.$$

The degree 4 homology of this sequence is the kernel of this map β_3 . However, if $x \in \mathcal{B}^1$ is in the kernel of β_3 , then $r_0x = r_1x = \dots = r_{p-1}x = 0$. If \mathcal{B} is finite dimensional, then there is a unique element x satisfying this, by Theorem 5.2.1. Therefore, $H_4(K(\mathcal{B})^*/S) \cong k$. By the Universal Coefficient Theorem, $H^4(K(\mathcal{B})^*/S) \cong k$.

By Lemma 4.3.4, it follows that $H^0(K(\mathcal{B})^*/S) \cong k$. As described in the proof of Lemma 4.3.7, the degree 0 homology of this complex is exactly those elements $\psi \in \mathcal{B}^*$ such that $\partial_{r_0}\psi = \partial_{r_1}\psi = \dots = \partial_{r_{p-1}}\psi = 0$. Since there is a unique element satisfying this, it follows that \mathcal{B} is a Nichols algebra by Proposition 2.8 in [1]. □

We can now show that none of the equivalent conditions of Theorem 5.2.1 are satisfied for the QCN algebras when $p > 3$.

Theorem 5.2.3. *For $p > 3$, the algebra \mathcal{B} is infinite dimensional.*

Proof. Assume, to the contrary, that \mathcal{B} is finite dimensional. By Lemma 5.2.2, \mathcal{B} is a finite dimensional Nichols algebra. Therefore, we can apply Theorem 4.3.8, and the dual Koszul complex $K(\mathcal{B})^*$ has cohomology $H^0(K(\mathcal{B})^*) \cong k$ and $H^3(K(\mathcal{B})^*) \cong k$ and $H^i(K(\mathcal{B})^*) \cong 0$ for all i not equal to 0 or 3. Since there are is one non-trivial cohomology class of the dual Koszul complex, we can compute the Hilbert series of its cohomology,

as in the proof of Lemma 5.1.2. Labelling $(3, -m)$ as the bidegree of the non-trivial cohomology class, we can compute $H_{H^*(K^*)}(t, -t) = 1 \pm t^{3+m}$. Again in the proof of Lemma 5.1.2, we showed that $H_{H^*(K^*)}(t, -t) = H_{\mathcal{B}}(t)H_{\mathcal{R}}(-t)$. Therefore,

$$H_{\mathcal{B}}(t) = \frac{1}{H_{\mathcal{R}}(-t)} \pm \frac{t^{3+m}}{H_{\mathcal{R}}(-t)}.$$

By Corollary 5.0.10, it follows that

$$\dim(\mathcal{B}_n) = \mathcal{S}_p(n) \pm \mathcal{S}_p(n - 3 - m).$$

Since we assumed \mathcal{B} to be finite dimensional, there must be some d such that $\dim(\mathcal{B}_d) = 0$. Therefore $\mathcal{S}_p(d) \pm \mathcal{S}_p(d - 3 - m) = 0$. This contradicts Lemma 5.0.7 which affirmed that $\mathcal{S}_p(n)$ is a strictly increasing sequence, which completes the proof. \square

From this theorem, we get the following corollary about the homology of the quotient dual Koszul complex.

Corollary 5.2.4. *Let $p > 3$. The degree 4 cohomology of the quotient dual Koszul complex $H^4(K(\mathcal{B})^*/S)$ is zero.*

Proof. As described in the proof of 5.2.2, the degree 4 homology $H_4(K(\mathcal{B})/S^*)$ is generated by the elements $x \in \mathcal{B}$ such that $r_0x = r_1x = \dots = r_{p-1}x = 0$. Since \mathcal{B} is infinite dimensional, by Theorem 5.2.1, there are no non-zero elements that are torsion for all the generators r_i . Therefore, the degree 4 homology of this complex is zero. The result about the dual quotient complex follows by linear duality. \square

Finally, we will further study the cohomology of this dual Koszul complex in the case that \mathcal{B} is a Nichols algebra. In general, for $p > 3$, it is not know whether \mathcal{B} is itself a Nichols algebra.

Theorem 5.2.5. *If $p > 3$ and \mathcal{B} is a Nichols algebra, then $H^0(K(\mathcal{B}^*)) = k$ and $H^i(K(\mathcal{B}^*)) = 0$ for $i \notin \{0, 2\}$. Moreover, $\text{rk}(H^2(K(\mathcal{B}^*)))$ is either 0 or 2.*

Proof. We will plug our known cohomology computations into the long exact sequence from Proposition 4.1.4. By Lemma 4.3.7, we know that $H^0(K^*/S) = k$ and $H^1(K^*/S) =$

k . By Corollary 5.2.4, we know that $H^4(K^*/S) = 0$, which are filled in on the bottom row of this complex.

$$\begin{array}{ccccccccc}
 H^3(K^*) & & H^2(K^*) & & H^1(K^*) & & H^0(K^*) & & 0 & & 0 \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
 0 & & 0 & & H^3(K^*) & & H^2(K^*) & & H^1(K^*) & & H^0(K^*) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & H^3(K^*/S) & & H^2(K^*/S) & & k & & k
 \end{array}$$

Using the segment $0 \rightarrow 0 \rightarrow H^3(K^*) \rightarrow 0$, we get that $H^3(K^*) = 0$ by exactness. Similarly, the segment $0 \rightarrow H^0(K^*) \rightarrow k \rightarrow 0$ implies that $H^0(K^*) = k$. As in the proof of Theorem 4.3.8, we can use the fact that the multiplication by S map induces the zero map on homology to show that $H^1(K^*) = 0$.

$$\begin{array}{ccccccccc}
 0 & & H^2(K^*) & & 0 & & k & & 0 & & 0 \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
 0 & & 0 & & 0 & & H^2(K^*) & & 0 & & k \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & H^3(K^*/S) & & H^2(K^*/S) & & k & & k
 \end{array}$$

This completes the proof that $H^i(K(\mathcal{B}^*)) = 0$ for $i \notin \{0, 2\}$. We now show that the degree 2 cohomology is at most one dimensional. By Theorem 4.3.10, $\text{Ext}_{\mathcal{B}}(k, k)$ is isomorphic to $T_{\mathcal{R}}(H^2(K^*))$, the tensor algebra over \mathcal{R} freely generated by $H^2(K^*)$. However, by Theorem 3.12 in [20], $\text{Ext}_{\mathcal{B}}(k, k)$ is braided commutative. As a tensor algebra, $\text{Ext}_{\mathcal{B}}(k, k)$ can only be braided commutative if its space of generators has dimension at most 1. Therefore, $H^2(K^*)$ is at most one dimensional. \square

Therefore, under these assumptions, it follows that the dual Koszul complex can only have non-trivial cohomology in degree 2, and it has at most one cohomology class in degree 2. This is a partial result towards showing that the algebra \mathcal{B} is Koszul,

which would prove the Generalized Alternating Lucas Number conjecture. Finally, the following corollary uses this Koszul complex cohomology to compute the cohomology of the QCN algebra \mathcal{B} . It is immediate from Theorem 4.3.10.

Corollary 5.2.6. *If $p > 3$ and \mathcal{B} is a Nichols algebra, then either \mathcal{B} is Koszul, in which case $\text{Ext}_{\mathcal{B}}(k, k) \cong \mathcal{R}$, or $\text{Ext}_{\mathcal{B}}(k, k) \cong \mathcal{R}[X]$ where X corresponds to the generator of $H^2(K^*)$.*

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