FOURTH ORDER ACCURACY FOR A CUBIC SPLINE COLLOCATION METHOD

CHARLES A. McCARTHY*

This note is inspired by the paper of Bialecki, Fairweather, and Bennett [1] which studies a method for the numerical solution of $-u'' = f$ on an interval, and $-\Delta u = f$ on a rectangle, with zero boundary data; their scheme calculates that C^1 piecewise cubic U for which $-U''$ (or $-\Delta U$) agrees with f at the two (or four) Gauss quadrature points of each subdivision of the original domain. They observe that not only does their U approximate u to order h^4 at mesh nodes – h is the linear dimension of the subdivisions – but also U' agrees with u' to order h^4 in one dimension, and U_x, U_y, and U_{xy} agree to order h^4 with u_x, u_y, and u_{xy} in two dimensions. Agreement of U with u to fourth order is perhaps not so surprising – Gaussian quadrature of this order is, after all, fourth order accurate and one could well expect this order of accuracy to be reflected in a Gauss-inspired differential equation solver. But fourth order agreement also for the derivatives is a surprise, and is due to the particular nature of the collocation scheme. Indeed, U agrees with u to fourth order uniformly in Ω, but away from the nodal points their derivatives need not be so close. It is in this sense that the derivatives of U exhibit superconvergence at the nodes.

We refer to the survey article of Fairweather and Meade [3] for a history of the subject and an extensive bibliography.

This note consists of three sections. The first section treats $-u'' = f$ on $[0,1]$ by elementary methods, and admits a non-uniform mesh. At the mesh nodes, we show that $u - U$ and $u' - U'$ are dominated by a quantity on the order of $h^4(M_3 + M_4)$ (h is the maximum mesh size, M_p is a bound for $|f^{(p)}|$); this is a very special case of the general results of deBoor and Swartz [2]. A simple example shows that no derivative of lesser order than 4 can suffice to give a general result of fourth order accuracy.

The second section redoes the problem $-u'' = f$ on $[0,1]$, this time with only a uniform mesh, by Fourier series and the explicit calculations that are available; f is expressed as a Fourier sine series, $f(x) = \sum c_\nu \sin \pi\nu x$, and the necessary control of $f^{(iv)}$ is effected through the hypothesis that $\sum |c_\nu|\nu^4 < \infty$. Although the Fourier method is not well suited to this problem and the results are weaker than those of section 1, this method does generalize to give a treatment of the higher dimensional analogues $-\Delta u = f$, with little more trouble than the expected computational tedium and cumbersome notation. One may also carry through a similar treatment using a cosine series, $f = \sum_0^\infty c_\nu \cos \pi\nu x$, or the full exponential series $f = \sum_{-\infty}^\infty c_\nu \exp(2\pi i\nu x)$; the treatments and results with these

*Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, mccarthy@ math.umn.edu

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are conceptually equivalent but the details substantially more complex, and on balance it seems best to content ourselves with just the sine series.

Section 3 studies $-\Delta u = f$ on the unit square. With a partition of the square into congruent rectangles of size $h \times k$, and f given as $f(x,y) = \sum c_{\mu\nu} \sin \pi\mu x \sin \pi\nu y$ with $\sum |c_{\mu\nu}|(\mu^4 + \nu^4) = K < \infty$, we show that at the nodes (i.e., the vertices of the little rectangles), $u - V$, $u_x - V_x$, $u_y - U_y$, and $u_{xy} - U_{xy}$ are all dominated by a multiple of $K(h^4 + k^4)$. The aspect ratio $\kappa = \frac{k}{h}$ of the partitioning rectangles does not enter otherwise into these estimates. We believe these pointwise estimates to be new.

We find it both interesting and reassuring to have some idea of the magnitude of various constants that arise in our estimates, even though such detail is not strictly necessary for the proofs. In that spirit, we record here the exact error term for two point Gaussian quadrature on an interval of length a; verification consists simply of a four fold integration by parts.

LEMMA (Gaussian two-point quadrature). *Let $\varphi(t)$ be defined for $|t| \leq \frac{a}{2}$, with $\varphi^{(iv)}$ integrable. Then*

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \varphi(t)dt - \frac{a}{2}\left(\varphi\left(-\frac{a}{2\sqrt{3}}\right) + \varphi\left(\frac{a}{2\sqrt{3}}\right)\right) = \int_{-\frac{a}{2}}^{\frac{a}{2}} r(t)\varphi^{(iv)}(t)dt, \tag{0.1}$$

where

$$r(t) = \begin{cases} \frac{t^4}{24} + t^2a^2\left(\frac{1}{16} - \frac{1}{8\sqrt{3}}\right) + a^4\left(\frac{1}{384} - \frac{1}{288\sqrt{3}}\right), & |t| \leq \frac{a}{2\sqrt{3}}; \\ \frac{1}{24}\left(\frac{a}{2} - |t|\right)^4, & \frac{a}{2\sqrt{3}} \leq |t|. \end{cases}$$

This immediately gives a variety of one-panel error bounds such as

$$\frac{a^5}{4320}\|\varphi^{(iv)}\|_\infty \text{ or } a^4\left(\frac{1}{384} - \frac{1}{288\sqrt{3}}\right)\|\varphi^{(iv)}\|_1 \sim 6 \cdot 10^{-4} a^4 \|\varphi^{(iv)}\|_1.$$

1. An elementary method for $-u'' = f$ on $[0,1]$. On the interval $[0,1]$ we are given nodes $0 = x_0 < x_1 < \cdots < x_N = 1$. The interval I_n is $[x_{n-1}, x_n]$, its length $h_n = x_n - x_{n-1}$, and midpoint $\mu_n = (x_{n-1} + x_n)/2$; the points, two in each I_n, $\xi_n^{\pm} = \mu_n + \frac{h_n}{2\sqrt{3}}$ will be referred to as Gauss points. We set $h = \max h_n$, and measure the non-uniformity of the partition effected by these nodes with $\kappa = (\max h_n)/(\min h_n)$.

S is the linear space of C^1 functions on $[0,1]$ which vanish at the endpoints 0 and 1, and which are cubic polynomials on each I_n.

Given the problem $-u'' = f$ on $[0,1]$, $u(0) = u(1) = 0$, the scheme of Bialecki, Fairweather, and Bennett takes as approximate solution that element U of S which satisfies the $2N$ Gauss collocation conditions $-U''(\xi_n^{\pm}) = f(\xi_n^{\pm})$, $n = 1, \ldots, N$; it is known that this specifies U uniquely. To study the error $u - U$ we find it useful to also consider that element V of S which satisfies the $2N$ conditions $-\int_{I_n} V''(t)dt = \int_{I_n} f(t)dt$, $-\int_{I_n} tV''(t)dt = \int_{I_n} tf(t)dt$; V is uniquely determined because V'' is linear on each I_n and so is determined by its mean and first moment. For simplicity of exposition, we assume that derivatives of f through order 4 are bounded, $\|f^{(p)}\|_\infty = M_p < \infty$ $(p \le 4)$; even though this assumption could be weakened slightly – for instance, to f''' of finite variation – a simple example will show that fourth order accuracy cannot be achieved with less smoothness of f.

We may now state the principal theorem of this section.

THEOREM. *At the nodes x_n, $n = 0, \dots, N$, we have*

$$V(x_n) = u(x_n), \tag{1.1}$$

$$V'(x_n) = u'(x_n), \tag{1.2}$$

$$|u(x_n) - U(x_n)| \le C(M_3 + M_4)h^4, \tag{1.3}$$

$$|u'(x_n) - U'(x_n)| \le C(M_3 + M_4)h^4. \tag{1.4}$$

Uniformly on $[0,1]$, we have

$$|u(x) - U(x)| \le C(M_2 + M_3 + M_4)h^4 \tag{1.5}$$

$$|u'(x) - U'(x)| \le C\kappa(M_2 + M_3 + M_4)h^3. \tag{1.6}$$

C is an absolute constant (C about $2 \cdot 10^{-3}$ suffices for (1.3) and (1.4), and C about 1 suffices for (1.5) and (1.6)).

Proof. That V agrees with u, and V' with u', at the nodes x_n follows immediately from the formulas

$$\psi(x) = -(1-x)\int_0^x t\psi''(t)dt - x\int_x^1 (1-t)\psi''(t)dt, \text{ and} \tag{1.7}$$

$$\psi'(x) = \int_0^x t\psi''(t)dt - \int_x^1 (1-t)\psi''(t)dt, \tag{1.8}$$

valid when ψ is C^1, piecewise C^2 on $[0,1]$ and vanishes at 0 and 1. In particular, when x is the node x_k and $\psi = u - V$, $\int_0^{x_n} = \sum_{j \le n} \int_{I_j}$ and $\int_{x_n}^1 = \sum_{n<j} \int_{I_j}$; the definition of V is such that $\int_{I_j} \psi(t)dt$ and $\int_{I_j} t\psi(t)dt$ all vanish, whence $\psi(x_n) = 0$, $\psi'(x_n) = 0$ for all n. This gives (1.1) and (1.2).

To compare the two elements U and V of S, it is convenient to introduce the norm

$$Q^* = \max_n \left\{ \frac{1}{2}|Q''(\xi_n^+) + Q''(\xi_n^-)|, \quad \frac{h_n}{2\sqrt{3}}|Q''(\xi_n^+) - Q''(\xi_n^-)| \right\} \tag{1.9}$$

for elements $Q \in S$. Use of (1.7) and (1.8) yields $|Q(x_n)| \le Q^*$ and $|Q'(x_n)| \le Q^*$. To see this,

$$\begin{aligned}\int_{I_j} tQ''(t)dt &= \mu_j \int_{I_j} Q''(t)dt + \int_{I_j} (t-\mu_j)Q''(t)dt \\ &= \mu_j \frac{h_j}{2}(Q''(\xi_j^+) + Q''(\xi_j^-)) + \frac{h_j}{2}\frac{h_j}{2\sqrt{3}}(Q''(\xi_j^+) - Q''(\xi_j^-))\end{aligned}$$

because tQ'' is a quadratic polynomial on I_j, for which Gaussian quadrature is exact. Thus

$$\left|\int_{I_j} tQ''(t)dt\right| \le h_j\left(\mu_j Q^* + \frac{h_j}{2}Q^*\right) \le h_j x_j Q^* \le h_j Q^*.$$

Similarly,

$$\left|\int_{I_j} (1-t)Q''(t)dt\right| \le h_j\left((1-\mu_j)Q^* + \frac{h_j}{2}Q^*\right) \le h_j(1-x_{j-1})Q^* \le h_j Q^*$$

so that

$$|Q(x_n)| \le \sum_j ((1-x_{j-1}) \text{ or } x_j) h_j Q^* \le Q^* \tag{1.10}$$

and

$$|Q'(x_n)| \le \sum_j h_j Q^* = Q^*. \tag{1.11}$$

To apply (1.10) and (1.11) to the difference $E = V - U$, we estimate E^*:

$$\begin{aligned}\frac{1}{2}(E(\xi_n^+) + E(\xi_n^-)) &= \frac{1}{2}(V''(\xi_n^+) + V''(\xi_n^-)) - \frac{1}{2}(U''(\xi_n^+) + U''(\xi_n^-)) \\ &= \frac{1}{h_n}\int_{I_n} (-f(t))dt - \frac{1}{2}(-f(\xi_n^+) - f(\xi_n^-)) \\ &= -\frac{1}{h_n}\left\{\int_{I_n} f(t)dt - \frac{h_n}{2}(f(\xi_n^+) + f(\xi_n^-))\right\} \\ &= -\frac{1}{h_n}\int_{I_n} r(t)f^{(iv)}(t)dt,\end{aligned}$$

using the Gaussian quadrature formula (0.1). Thus

$$\left|\frac{1}{2}(E(\xi_n^+) + E(\xi_n^-))\right| \le \frac{h_n^4}{4320} M_4.$$

In the same way,

$$\begin{aligned}
&\frac{1}{2\sqrt{3}} h_n(E(\xi_n^+) - E(\xi_n^-)) \\
&= -\frac{2}{h_n}\left\{\int_{I_n} (t-\mu_n) f(t)dt - \frac{h_n}{2}((t-\xi_n^+) f(\xi_n) + (t-\xi_n^-) f(\xi_n^-))\right\} \\
&= -\frac{2}{h_n}\int_{I_n} r(t)((t-\mu_n)f(t))^{(iv)}\, dt,
\end{aligned}$$

so that

$$\begin{aligned}
\left|\frac{1}{2\sqrt{3}} h_n(E_n(\xi_+) - E_n(\xi_-))\right| &\le \frac{h_n^4}{2160}\|4f''' + (t-\mu_n) f^{iv}\|_\infty \\
&\le \frac{h_n^4}{540} M_3 + \frac{h_n^5}{4230} M_4.
\end{aligned}$$

Thus $E^* \le C(M_3 + M_4)h^4$, so that (1.1) implies (1.3) and (1.2) implies (1.4).

To obtain the estimates (1.5) and (1.6), we set $e = u - V$ and continue to set $E = V - U$ so that $u - U = e + E$. On I_n, write the cubic polynomial E as a linear combination of the Lagrange interpolating polynomials: L_1, L_2, M_1, M_2 defined by the conditions that both L's and M's and their derivatives vanish at the endpoints of F_n except for $L_1(x_{n-1}) = L_2(x_n) = M_1'(x_{n-1}) = M_2'(x_n) = 1$. Then on I_n, $E(x) = E(x_{n-1})L_1(x) + E(x_n)L_2(x) + E'(x_{n-1})M_1(x) + E'(x_n)M_2(x)$. $\|L_1\|_\infty$ and $\|L_2\|_\infty$ are bounded by 1 on I_n, while $\|M_1\|_\infty$ and $\|M_2\|_\infty$ are bounded by $\frac{4}{27}h_n$, so $\|E\|_\infty \le CKh^4$ follows from (1.1) - (1.4). Also, on I_n we have $\|L_1'\|_\infty$ and $\|L_2'\|_\infty$ bounded by $\frac{3}{8h_n} \le \frac{3}{8}\kappa h^{-1}$, while $\|M_1'\|_\infty$ and $\|M_2'\|_\infty$ are bounded by $\frac{5}{9}$; thus $\|E'\|_\infty \le CK\kappa h^3$. We estimate e by repeated use of the near value theorem, knowing that $e = e' = 0$ at the endpoints of I and $e^{(iv)} = -f''$ within I_n: e vanishes at x_{n-1} and x_n so $e'(\eta) = 0$ for some η interior to I_n; e' vanishes at x_{n-1}, η, and x_n, so $e''(\eta_1) = e''(\eta_2) = 0$ for some $\eta_1 \varepsilon (x_{n-1}, \eta)$ and $\eta_2 \in (\eta, x_n)$; thus $e'''(\zeta) = 0$ for some $\zeta \in (\eta_1, \eta_2)$. Integrating back to get e, we have $e'''(x) = \int_\zeta^x e^{(iv)}(t)dt$, $e''(x) = \int_{\eta_1}^x e'''(t)dt$, $e'(x) = \int_{x_{n-1}}^x e''(t)dt$, $e(x) = \int_{x_{n-1}}^x e'(t)dt$, which quickly yield the estimates $|e(x)| \le \frac{h_n^4}{24}\|f''\|_\infty$, $|e'(x)| \le \frac{h_n^3}{6}\|f''\|_\infty$. $\square$

As an example to show that some control of $f^{(iv)}$ is needed in order to obtain fourth order convergence at the nodes, consider the function $u = \sin^2 \pi N x$ on $[0,1]$, with nodes $x_n = \frac{n}{N}$; u and u' vanish at each node x_n. But at the Gauss points, $-u''(\xi_n^{\pm}) = 2(\pi N)^2 \cos \frac{\pi}{\sqrt{3}}$, non zero and independent of the particular Gauss points; thus $U(x) = (\pi N)^2 \cos \frac{\pi}{\sqrt{3}} x(1-x)$. The errors at the nodes of $u - U$ are of order N^2 (for x_n near $\frac{1}{2}$) and of $u' - U'$ also of order N^2 (for x_n near 0 or 1). These errors cannot be of the order $h^4 \|f^{(p)}\|_\infty = N^{-4}\|u^{(p+2)}\|_\infty$ unless p is at least 4.

2. A Fourier series approach to $-u'' = f$. Given the integer N, we impose a uniform mesh on $[0,1]$ with nodes at $x_n = \frac{n}{N}$, $n = 0, \ldots, N$; the interval $I_n = [x_{n-1}, x_n]$ has length $h = \frac{1}{N}$ and midpoint $\mu_n = \frac{n-\frac{1}{2}}{N}$. The Gauss points are $\xi_n^{\pm} = \mu_n \pm \frac{h}{2\sqrt{3}}$.

We assume that f is expressed as a Fourier sine series, $f(x) = \sum_{\nu=1}^{\infty} c_\nu \sin \pi\nu x$ for which $K = \sum_{\nu=1}^{\infty} |c_\nu|\nu^4 < \infty$. The exact solution u and its derivative are given by $u(x) = \sum_{\nu=1}^{\infty} \frac{c_\nu}{\pi^2\nu^2} \sin \pi\nu x$ and $u'(x) = \sum_{\nu=1}^{\infty} \frac{c_\nu}{\pi\nu} \cos \pi\nu x$. Our method is to determine that $U_\nu \in S$ for which $-U_\nu''(\xi_n^{\pm}) = \sin \pi\nu\xi_n^{\pm}$; then $U = \sum_{\nu=1}^{\infty} c_\nu U_\nu$ satisfies $-U''(\xi_n^{\pm}) = f(\xi_n^{\pm})$. For ν small, say $\nu < N$, U_ν contains useful information about u, and for these terms, it is important how well $U_\nu(x_n)$ and $U_\nu'(x_n)$ approximate $\sin \pi\nu x_n/(\pi\nu)^2$ and $\cos \pi\nu x_n/(\pi\nu)$. For large ν, U_ν essentially represents annoying distortion, alias effects, and random noise, so that only a rough qualitative estimate can be useful.

We begin with the formula that determines, for a cubic polynomial Q on the interval $|x| \le \frac{h}{2}$, the values of $Q''\left(\pm\frac{h}{2\sqrt{3}}\right)$ from $Q\left(\pm\frac{h}{2}\right)$ and $Q'\left(\pm\frac{h}{2}\right)$; we use the subscripts e and o for the even and odd parts of a function, so that for example $Q_o'(x) = \frac{1}{2}(Q'(x) - Q'(-x))$.

Lemma 2.1. *The cubic polynomial $Q(x)$ on $|x| \le \frac{h}{2}$ satisfies*

$$\frac{2}{h} Q_o'\left(\frac{h}{2}\right) = Q_e''\left(\frac{h}{2\sqrt{3}}\right) \tag{2.1}$$

$$-\frac{4}{h^2} Q_o\left(\frac{h}{2}\right) + \frac{2}{h} Q_e'\left(\frac{h}{2}\right) = \frac{1}{\sqrt{3}} Q_o''\left(\frac{h}{2\sqrt{3}}\right) \tag{2.2}$$

Proof. Verify (2.1) for $Q = 1$ and x^2, (2.2) for x, and x^3. □

Lemma 2.1 allows us to determine that function $U_\nu \in S$ for which $-U_\nu''(\xi_n^{\pm}) = \sin \pi\nu\xi_n^{\pm}$.

Lemma 2.2. *For ν not a multiple of $2N$, set*

$$A_\nu = \left(\frac{\frac{\pi\nu h}{2}}{\sin \frac{\pi\nu h}{2}}\right)^2 \left(\cos \frac{\pi\nu h}{2} \cos \frac{\pi\nu h}{2\sqrt{3}} + \frac{1}{\sqrt{3}} \sin \frac{\pi\nu h}{2} \sin \frac{\pi\nu h}{2\sqrt{3}}\right) \tag{2.3}$$

$$B_\nu = \frac{\frac{\pi\nu h}{2}}{\sin \frac{\pi\nu h}{2}} \cos \frac{\pi\nu h}{2\sqrt{3}} \tag{2.4}$$

Then the element $U_\nu \in S$ that at the nodes satisfies

$$U_\nu(x_n) = A_\nu \frac{\sin \pi\nu x_n}{(\pi\nu)^2}, \tag{2.5}$$

$$U'_\nu(x_n) = B_\nu \frac{\cos \pi\nu x_n}{\pi\nu}, \tag{2.6}$$

at the Gauss points satisfies

$$-U''_\nu(\xi_n^\pm) = \sin \pi\nu\xi_n^\pm.$$

In the exceptional cases $\nu = 2pN$, the function $U_\nu \in S$ for which $-U''_\nu(\xi_n^\pm) = \sin \pi\nu\xi_n^\pm$ satisfies

$$U_\nu(x_n) = 0, \quad U'_\nu(x_n) = (-1)^{p+1} \frac{h}{2\sqrt{3}} \sin \frac{\pi\nu h}{2\sqrt{3}}. \tag{2.7}$$

Proof. For $\nu \neq 2pN$, one verifies that

$$U_{\nu e} = A_\nu \frac{1}{(\pi\nu)^2} \sin \pi\nu\mu_n \cos \frac{\pi\nu h}{2}, \quad U'_{\nu o} = -B_k \frac{1}{\pi\nu} \sin \pi\nu\mu_n \sin \frac{\pi\nu h}{2},$$

and

$$-U''_{\nu e} = \sin \pi\nu\mu_n \cos \frac{\pi\nu h}{2\sqrt{3}}$$

satisfy $\frac{2}{h}U'_{\nu o} = U''_{\nu e}$ (cf. 2.1) for all n; the common factor $\sin \pi\nu\mu_n$ is the sole dependence on n. Also,

$$U_{\nu o} = A_\nu \frac{1}{(\pi\nu)^2} \cos \pi\nu\mu_n \sin \frac{\pi\nu h}{2}, \quad U'_{\nu e} = B_\nu \frac{1}{\pi\nu} \cos \pi\nu\mu_n \cos \frac{\pi\nu h}{2},$$

and

$$-U''_{\nu o} \cos \pi\nu\mu_n \sin \frac{\pi\nu h}{2\sqrt{3}}$$

satisfy $-\frac{4}{h^2}U_{\nu_o} + \frac{2}{h}U''_{\nu e} = \frac{1}{\sqrt{3}}U''_{\nu o}$ (cf. 2.2) for all n; the sole dependence on n is the common factor $\cos \pi\nu\mu_n$.

For $\nu = 2pN$, h is a period of $\sin \pi\nu x$, $\sin \pi\nu x$ is odd about each μ_n and thus on I_n the piecewise linear U''_ν must also be odd about μ_n. U'_ν is the integral of the h-periodic U''_ν, U''_ν has mean 0, so U'_ν is h-periodic. The continuity of U'_ν together with $0 = U_\nu(1) - U_\nu(0) = \int_0^1 U'_\nu(t)dt$ shows that U'_ν has mean value zero; periodicity of U'_ν gives $\int_{I_n} U'_\nu = 0$ for all n so $U_\nu(x_n) = 0$. (2.2) then gives $U'_\nu(x_n)$
$= -\frac{h}{2\sqrt{3}} \cos\left(\pi \cdot 2pN \left(n - \frac{1}{2}\right) h\right) \sin \frac{\pi\nu h}{2\sqrt{3}} = (-1)^{p+1} \frac{h}{2\sqrt{3}} \sin \frac{\pi\nu h}{2\sqrt{3}}$. $\square$

For small ν, we wish to know how close to 1 are the quantities A_ν and B_ν of (2.3) and (2.4). For large ν, we wish simply wish to estimate the magnitude of $a_\nu = A_\nu/(\pi\nu)^2$ and $b_\nu = B_\nu/(\pi\nu)$.

Lemma 2.3. *There exist universal constants C_A, C_B, C_a, C_b such that*

$$|A_\nu - 1| \le C_A(\nu h)^4 \quad (\nu < N), \tag{2.8}$$

$$|B_\nu - 1| \le C_B(\nu h)^4 \quad (\nu < N), \tag{2.9}$$

$$|a_\nu| \le C_a \quad (\textit{all } \nu), \tag{2.10}$$

$$|b_\nu| \le C_b \quad (\textit{all } \nu), \tag{2.11}$$

Proof. To show (2.8) and (2.9), consider the functions $A(t) = \left(\frac{t}{\sin t}\right)^2 \left(\cos t \cos \frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}} \sin t \sin \frac{t}{\sqrt{3}}\right)$ and $B(t) = \left(\frac{t}{\sin t}\right) \cos \frac{t}{\sqrt{3}}$. $A(t)$ and $B(t)$ are analytic for $|t| < \tau$, their first three derivatives vanish at $t = 0$, so $A(t) = 1 + t^4 q_1(t)$ and $B(t) = 1 + t^4 q_2(t)$ for t small; the analyticity of A and B for $|t| < \pi$ yields boundedness of q_1 and q_2 uniformly on $|t| \le \frac{\pi}{2}$. When $t = \frac{\pi\nu h}{2}$, $A_\nu = A(t)$ and $B_\nu = B(t)$, $|\nu| \le N$ corresponding to $|t| \le \frac{\pi}{2}$, which shows (2.8) and (2.9).

For large ν, but not an even multiple of N, we have $b_\nu = \frac{1}{\pi\nu} B_\nu = \frac{\frac{h}{2}}{\sin \frac{\pi\nu h}{2}} \cos \frac{t}{\sqrt{3}}$. Bound $\cos \frac{t}{\sqrt{3}}$ by 1, but use $\sin \frac{\pi\nu h}{2} \ne 0$ to see that $|\sin \frac{\pi\nu h}{2}| \ge \sin \frac{\pi h}{2}$, the worst case being when $\nu \equiv \pm 1 \pmod{2N}$. Then $|b_\nu| \le \frac{\frac{h}{2}}{\sin \frac{\pi h}{2}} \le \frac{1}{2}$ for all ν not of the form $2pN$. For the exceptional cases $\nu = 2pN$, (2.7) gives $|U_\nu'(x_n)| \le 1$ directly.

Similarly,

$$a_\nu = \frac{1}{(\pi\nu)^2} A_\nu = \left(\frac{\frac{h}{2}}{\sin \frac{\pi\nu h}{2}}\right)^2 \left(\cos t \cos \frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}} \sin t \sin \frac{t}{\sqrt{3}}\right)$$

the second factor is of the form $v \cdot Dw$ where $v = (\cos t, \sin t)$ and $w = \left(\cos \frac{t}{\sqrt{3}}, \sin \frac{t}{\sqrt{3}}\right)$ are unit vectors, $D = \operatorname{diag}\left(1, \frac{1}{\sqrt{3}}\right)$, and so is bounded by 1. As with b, $\left(\frac{\frac{h}{2}}{\sin \pi \frac{\nu h}{2}}\right)^2 \le \left(\frac{\frac{h}{2}}{\sin \frac{\pi h}{2}}\right)^2 \le \frac{1}{4}$ when $\nu \not\equiv 0 \pmod{2N}$, so $|a_\nu| \le \frac{1}{4}$. When $\nu = 2pN$, we have $U_\nu(x_k) = 0$ from (2.7). □

More elaborate calculations show that

$$\left(\frac{t}{\sin t}\right)^2 \left(\cos t \cos \frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}} \sin t \sin \frac{t}{\sqrt{3}}\right) = 1 + q_1(t) t^4$$

where $q_1(0) = \frac{1}{90}$, and q_1 increases monotonically to about .02 at $t = \frac{\pi}{2}$, so that C_A about .13 suffices. Also,

$$\frac{t}{\sin t} \cos \frac{t}{\sqrt{3}} = 1 + q_2(t) t^4$$

where $q_2(0) = -\frac{1}{270}$ and q_2 decreases monotonically to about $-.005$ at $t = \frac{\pi}{2}$, so that C_B about .03 suffices. We suspect that all terms in the Taylor series for q_1 and $-q_2$ are non-negative.

The fourth order accuracy of U follows immediately from Lemma 2.3:

THEOREM 2.4. *Let* $f(x) = \sum_{\nu=1}^{\infty} c_\nu \sin \pi\nu x$ *be given on* $[0,1]$, *with* $\sum_{\nu=1}^{\infty} |c_\nu|\nu^4 = K < \infty$. *Let* U *be that element of* S *for which* $-U''(\xi_n^{\pm}) = f(\xi_n^{\pm})$, *and let* $u(x) = \sum_{\nu=1}^{\infty} c_\nu(\pi\nu)^{-2} \sin \pi\nu x$. *Then at the nodes* $x_n = \frac{n}{N}$, *we have*

$$|u(x_n) - U(x_n)| \le Kh^4$$

and

$$|u'(x_n) - U'(x_n)| \le Kh^4$$

Proof. $U(x) = \sum_{\nu=1}^{\infty} c_\nu U_\nu(x)$, so $u - U$ may be written as

$$u(x) - U(x) = \sum_{\nu=1}^{N-1} c_\nu \left((\pi\nu)^{-2} \sin \pi\nu x - U_\nu(x)\right) + \sum_{\nu=N}^{\infty} c_\nu(\pi\nu)^{-2} \sin \pi\nu x - \sum_{\nu=N}^{\infty} c_\nu U_\nu(x).$$

At $x = x_n$, the first sum is $\sum_{\nu=1}^{N-1} c_\nu(\pi\nu)^{-2} \sin \pi\nu x_n(1 - A_\nu)$, with absolute value at most $\sum_{\nu=1}^{N-1} |c_\nu|(\pi\nu)^{-2} \cdot C_A(\nu h)^4 \le C_A h^4 \sum_{\nu=1}^{N-1} |c_\nu|(\nu/\pi)^2$; the second sum is at most

$$\sum_{\nu=N}^{\infty} |c_\nu|(\pi\nu)^{-2} \le N^{-4} \sum_{\nu=N}^{\infty} |c_\nu|N^4(\pi\nu)^{-2} \le h^4 \sum_{\nu=N}^{\infty} |c_\nu|\nu^4(\pi N)^{-2}.$$

The last sum contributes the principal error term with

$$\left|\sum_{\nu=N}^{\infty} c_\nu U_\nu(x_n)\right| \le \sum_{\nu=N}^{\infty} |c_\nu|\ |a_\nu| \le C_a N^{-4} \sum_{\nu=N}^{\infty} |c_\nu|\nu^4.$$

Thus $|u(x_n) - U(x_n)|$ is dominated by a quantity of the form CKh^4; chasing the constants shows that $\frac{1}{4}Kh^4$ suffices.

The derivative estimate is much the same:

$$u'(x) - U'(x) = \sum_{\nu=1}^{N-1} c_\nu((\pi\nu)^{-1} \cos \pi\nu x - U_\nu'(x)) + \sum_{\nu=N}^{\infty} c_\nu(\pi\nu)^{-1} \cos \pi\nu x - \sum_{\nu=N}^{\infty} c_\nu U_\nu'(x).$$

At $x = x_n$, the first sum is $\sum_{\nu=1}^{N-1} c_\nu(\pi\nu)^{-1} \cos \pi\nu x_n(1-B_\nu)$, bounded by $C_B h^4 \sum_{\nu=1}^{N-1} |c_\nu|\nu^3/\pi$; the second sum is bounded by $N^{-4} \sum_{\nu=N}^{\infty} |c_\nu|\nu^4(\pi N)^{-1}$; and the last sum is bounded by $C_b N^{-4} \sum_{\nu=N}^{\infty} |c_\nu|\nu^4$. Together, these three terms give at most $\frac{1}{2}Kh^4$. $\square$

3. $-\Delta u = f$ **on the unit square.** We follow the treatment of section 2 closely.

The unit square $\Omega = [0,1] \times [0,1]$ is partitioned into MN rectangles of size $h = \frac{1}{M}$ by $k = \frac{1}{N}$, with vertices at the nodes $\{(mh, nk) : m = 0, \ldots, M, n = 0, \ldots N\}$. The ratio k/h is denoted by κ. The Gauss points are the $4MN$ points $\Big\{(\xi,\eta) = \Big(\left(m - \frac{1}{2}\right) h \pm \frac{h}{2\sqrt{3}}, \left(n - \frac{1}{2}\right) k \pm \frac{k}{2\sqrt{3}}\Big) : m = 1, \ldots, M, n = 1 \ldots N\Big\}$. For the partitioning rectangles R, the Gaussian quadrature formula $\int\int_R \varphi(x,y)dxdy = |R| \left(\sum_{(\xi,\eta)\in R} \varphi(\xi,\eta)\right)$ is exact when φ is a polynomial of degree at most 3 in x and y separately; as in section 2, an analysis of the residual for general φ is not needed, its place taken instead by consideration of the individual terms of the Fourier series of φ.

We take f as given by a Fourier series $f(x,y) = \sum_{\mu\geq 1,\nu\geq 1} c_{\mu\nu} \sin \pi\mu x \sin \pi\nu y$, with $\sum |c_{\mu\nu}|(\mu^4 + \nu^4) = K < \infty$. The exact solution u to $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, is $u(x,y) = \sum c_{\mu\nu}(\pi^2\mu^2 + \pi^2\nu^2)^{-1} \sin \pi\mu x \sin \pi\nu y$, with first and second derivatives available through the obvious termwise differentiation.

The space S is the set of functions U which are C^1 on the unit square Ω, vanish identically on the boundary of Ω, and which are cubic polynomials within each partitioning rectangle. It is known that f has dimension $4MN$ and that an element $U \in S$ is uniquely determined by specifying the values of ΔU at the $4MN$ Gauss points. A basis for S, which we will not use further, but which serves well to introduce notation, is formed by the Lagrange interpolating polynomials. We first note that specification of the sixteen pieces of data U, $P = U_x$, $Q = U_y$, and $S = U_{xy}$ at each of the vertices of a rectangle specifies the cubic polynomial U uniquely: U and P at the vertices determine U along the horizontal edges, while Q and $S = Q_x$ at the vertices determine Q along the horizontal edges; then along any vertical line, U and Q at the endpoints determine U within the rectangle. The requirement that U vanish on the boundary means that also P vanishes on the horizontal sides of the rectangle and Q on the vertical sides, so that all told there are $(M-1)(N-1)$ degrees of freedom for U at interior nodes, $(M+1)(N-1)$ for P, $(M-1)(N+1)$ for Q, and $(M+1)(N+1)$ for S. An explicit basis may be formed with the one-variable Lagrange interpolating polynomials (cf. the proof of 1.5 and 1.6) L and M that respectively have value 1 and derivative 1 at a point, and have all other data zero: $L(x)L(y)$ gives a value of U, $M(x)L(y)$, $M(x)M(y)$, and $M(x)M(y)$ give P, Q, and S.

As in section 2, we find it convenient to work with the even and odd parts of a function, with respect to the center of a rectangle. The subscripts e and o denote the even and odd parts with respect to x and y, of the indicated function; as an example on a rectangle centered at $(0,0)$, $Q_{oe}(x,y) = \frac{1}{4}\left[Q(x,y) - Q(-x,y) + Q(x,-y) - Q(x,-y)\right]$ is that part of U_y which is odd in x and even in y.

Our first lemma gives the relationship between data at vertices and Laplacian at Gauss points that holds for a cubic polynomial.

LEMMA 3.1. *On the rectangle $|x| \le \frac{h}{2}$, $|y| \le \frac{k}{2}$, the values of U, P, Q, S at the vertices and the values of ΔU at the Gauss points $\left(\pm\frac{h}{2\sqrt{3}}, \pm\frac{k}{2\sqrt{3}}\right)$ satisfy*

$$\begin{aligned}
&(3.1) && && -\frac{2}{h}P_{oe} && -\frac{2}{k}Q_{eo} && +\frac{1}{3}\left(\frac{h}{k}+\frac{k}{h}\right)S_{oo} = -\Delta U_{ee}\\
&(3.2) && \frac{12}{h^2}U_{oe} && -\frac{6}{h}P_{ee} && -\left(\frac{8}{3k}+\frac{2k}{h^2}\right)Q_{oo} && +\left(\frac{h}{k}+\frac{1}{3}\frac{k}{h}\right)S_{eo} = -\sqrt{3}\Delta U_{oe}\\
&(3.3) && \frac{12}{k^2}U_{eo} && -\left(\frac{8}{3h}+\frac{2h}{k^2}\right)P_{oo} && -\frac{6}{k}Q_{ee} && +\left(\frac{k}{h}+\frac{1}{3}\frac{h}{k}\right)S_{oe} = -\sqrt{3}\Delta U_{eo}\\
&(3.4) && 16\left(\frac{1}{h^2}+\frac{1}{k^2}\right)U_{oo} && -\left(\frac{8}{h}+\frac{2h}{k^2}\right)P_{eo} && -\left(\frac{8}{k}+\frac{2k}{h^2}\right)Q_{oe} && +\left(\frac{k}{h}+\frac{h}{k}\right)S_{ee} = -3\Delta U_{oo}
\end{aligned}$$

(The even and odd parts of U, P, Q, S are to be evaluated at $\left(\frac{h}{2},\frac{k}{2}\right)$, and those of ΔU at $\left(\frac{h}{2\sqrt{3}},\frac{k}{2\sqrt{3}}\right)$).

Proof. Direct verification: (3.1) is satisfied when U is $1, x^2$, y^2, or x^2y^2, (3.2) by x, x^3, xy^2, x^3y^2, (3.3) by y, x^2y, y^3, x^2y^3, and (3.4) by xy, x^3y, xy^3, x^3y^3. $\square$

We are now ready to identify that element $U_{\mu\nu}$ of S for which $-\Delta U_{\mu\nu}(\xi,\eta) = \sin\pi\mu\xi \sin\pi\nu\eta$ at all Gauss points (ξ,η).

LEMMA 3.2. *There exist numbers a, b, c, d, depending on μ, ν, h, and k but independent of the partitioning rectangle $[(m-1)h, mh] \times [(n-1)k, nk]$, with the property that the cubic polynomial $U_{\mu\nu}$ defined through the sixteen conditions*

$$\begin{aligned}
&(3.5) && U_{\mu\nu}(x,y) = && a\sin\pi\mu x\sin\pi\nu y\\
&(3.6) && P_{\mu\nu}(x,y) = && b\cos\pi\mu x\sin\pi\nu y\\
&(3.7) && Q_{\mu\nu}(x,y) = && c\sin\pi\mu x\cos\pi\nu y\\
&(3.8) && S_{\mu\nu}(x,y) = && d\cos\pi\nu x\cos\pi\nu y
\end{aligned}$$

for each of the four vertices (x,y) of the partitioning rectangle, satisfies

$$(3.9) \qquad -\Delta U_{\mu\nu}(\xi,\eta) = \sin\pi\mu\xi\sin\pi\nu\eta$$

at the four Gauss points of the rectangle.

Proof. To make the formulas more manageable, we use the abbreviations $\sigma = \sin\frac{\pi\mu h}{2}$, $\gamma = \cos\frac{\pi\mu h}{2}$, $\sigma' = \sin\frac{\pi\nu k}{2}$, $\gamma' = \cos\frac{\pi\nu k}{2}$.

Translate lemma 3.1 to the center (x_0, y_0) of the partitioning rectangle, and substitute expressions (3.5) through (3.9), into (3.1)–(3.4). After removing common factors of $\sin\pi\mu x_0\sin\pi\nu y_0$, $\cos\pi\mu x_0\sin\pi\nu y_0$, $\sin\pi\mu x_0\cos\pi\nu y_0$, and $\cos\pi\mu x_0\cos\pi\nu y_0$ respectively from the four equations, we see that it is sufficient that a, b, c, d satisfy four simultaneous linear equations with augmented coefficient matrix

(3.10)
$$\left(\begin{array}{cccc|c} 0 & \frac{2}{h}\sigma\gamma' & \frac{2}{k}\gamma\sigma' & \frac{1}{3}\left(\frac{h}{k}+\frac{k}{h}\right)\sigma\sigma' & \cos\frac{\pi\mu h}{2\sqrt{3}}\cos\frac{\pi\nu k}{2\sqrt{3}} \\ \frac{12}{h^2}\sigma\gamma' & -\frac{6}{h}\gamma\gamma' & \left(\frac{8}{3k}+\frac{2k}{h^2}\right)\sigma\sigma' & -\left(\frac{h}{k}+\frac{1}{3}\frac{k}{h}\right)\gamma\sigma' & \sqrt{3}\sin\frac{\pi\mu h}{2\sqrt{3}}\cos\frac{\pi\nu k}{2\sqrt{3}} \\ \frac{12}{k^2}\gamma\sigma' & \left(\frac{8}{3h}+\frac{2h}{k^2}\right)\sigma\sigma' & -\frac{6}{k}\gamma\gamma' & -\left(\frac{k}{h}+\frac{1}{3}\frac{h}{k}\right)\sigma\gamma' & \sqrt{3}\cos\frac{\pi\mu h}{2\sqrt{3}}\sin\frac{\pi\nu k}{2\sqrt{3}} \\ 16\left(\frac{1}{h^2}+\frac{1}{k^2}\right)\sigma\sigma' & -\left(\frac{8}{h}+\frac{2h}{k^2}\right)\gamma\sigma' & -\left(\frac{8}{k}+\frac{2k}{h^2}\right)\sigma\gamma' & \left(\frac{k}{h}+\frac{h}{k}\right)\gamma\gamma' & 3\sin\frac{\pi\mu h}{2\sqrt{3}}\sin\frac{\pi\nu h}{2\sqrt{3}} \end{array}\right)$$

The determinant of the coefficient matrix is non-zero except when μ and ν are both even multiples of M and N. This fact about the coefficient matrix, and more, will be shown in the next lemma, and therewith the quantities a, b, c, d determined.

The exceptional case $\mu = 2pM$, $\nu = 2qN$ will be disposed of here: $-\Delta U$ is (h,k) translation invariant and odd in both x and y about the center (x_0, y_0) of the partitioning rectangle, so U, P, Q and S are (h,k) translation invariant, and thus U must vanish along the edges of every partitioning rectangle, giving

$$a = 0, b = 0, c = 0 \quad ((\mu,\nu) = (2pM, 2qN))\,; \tag{3.11}$$

(3.4) then gives

$$d = 3\frac{hk}{h^2+k^2}(-1)^{p+q}\cos\frac{\pi\mu h}{2\sqrt{3}}\cos\frac{\pi\nu k}{2\sqrt{3}} \quad ((\mu,\nu) = (2pM, 2qN))\,. \tag{3.12}$$

To complete the proof of Lemma 3.2 we yet need to show that the coefficient matrix of (3.10) is non-zero in the non-exceptional cases. But we will eventually need more, namely, the boundedness of a, b, c, d uniformly in μ, ν, h, and k. The next lemmas do this, without art. We simply calculate the determinant of the coefficient matrix, and all its 3×3 minors, to show that the inverse of the coefficient matrix has entries bounded uniformly in μ, ν, h, and k; since the right-hand side of (3.10) is bounded, this will show that a, b, c, d are likewise bounded uniformly.

Lemma 3.3. *The determinant D of the coefficient matrix of (3.10) is given by*

$$\begin{aligned} \frac{27}{16}h^2k^2D &= \sigma^2\gamma^2\gamma'^4(243+243\kappa^2) + \sigma'^2\gamma'^2\gamma^4(243+243\kappa^{-2}) \\ &\quad + \sigma^4\gamma'^4(324+189\kappa^2+27\kappa^4) + \sigma^2\sigma'^2\gamma^2\gamma'^2(702+513\kappa^2+513\kappa^{-2}) \\ &\qquad + \sigma'^4\gamma^4(324+189\kappa^{-2}+27\kappa^{-4}) \\ &\quad + \sigma^4\sigma'^2\gamma'^2(579+399\kappa^2+72\kappa^4+252\kappa^{-2}) \\ &\qquad + \sigma'^4\sigma^2\gamma^2(579+399\kappa^{-2}+72\kappa^{-4}+252\kappa^2) \\ &\quad + \sigma^4\sigma'^4(296+196\kappa^2+48\kappa^4+196\kappa^{-2}+48\kappa^{-4}), \end{aligned} \tag{3.13}$$

so that

$$\frac{27}{16}h^2k^2D \geq 148(\sigma^2+\sigma'^2)+189(\kappa^2\sigma^2+\kappa^{-2}\sigma'^2)+27(\kappa^4\sigma^4+\kappa^{-4}\sigma'^4) \tag{3.14}$$

Proof. (3.13) is simply a brute force calculation; recall that $\kappa = k/h$. To obtain (3.14), use $\sigma^2+\gamma^2 = 1 = \sigma'^2+\gamma'^2$ to obtain the expansions

$$\begin{aligned}\sigma^2+\sigma'^2 &= (\sigma^2\gamma^2\gamma'^4+\sigma'^2\gamma'^2\gamma^4)+(\sigma^4\gamma'^4+4\sigma^2\sigma'^2\gamma^2\gamma'^2+\sigma'^4\gamma^4)\\ &\quad +3(\sigma^4\sigma'^2\gamma'^2+\sigma'^4\sigma^2\gamma^2)+2\sigma^4\sigma'^4,\end{aligned} \tag{3.15}$$

$$\sigma^2 = \sigma^2\gamma^2\gamma'^4+\sigma^4\gamma'^4+2\sigma^2\sigma'^2\gamma^2\gamma'^2+2\sigma^4\sigma'^2\gamma'^2+\sigma^2\gamma'^4\gamma^2+\sigma^4\sigma'^4, \tag{3.16}$$

$$\sigma'^2 = \sigma'^2\gamma'^2\gamma^4+2\sigma^2\sigma'^2\gamma^2\gamma'^2+\sigma'^4\gamma^4+\sigma^4\sigma'^2\gamma'^2+2\sigma^2\sigma'^4\gamma'^2+\sigma^4\sigma'^4, \tag{3.17}$$

$$\sigma^4 = \sigma^4\gamma'^4+2\sigma^4\sigma'^2\gamma'^2+\sigma^4\sigma'^4, \tag{3.18}$$

$$\sigma'^4 = \sigma'^4\gamma^4+2\sigma^2\sigma'^4\gamma^2+\sigma^4\sigma'^4, \tag{3.19}$$

Next, collect like powers of κ in (3.13). The coefficients of κ^o is

$$\begin{aligned}&243(\sigma^2\gamma^2\gamma'^4+\sigma'^2\gamma'^2\gamma^4)+(324\sigma^4\gamma'^4+702\sigma^2\sigma'^2\gamma^2\gamma'^2+324\sigma'^4\gamma^4)\\ &\qquad +579(\sigma^4\sigma'^2\gamma'^2+\sigma'^4\sigma^2\gamma^2)+296\sigma^4\sigma'^4\end{aligned}$$

which, by (3.15), exceeds $\min\left(243, 324, \frac{702}{4}, \frac{579}{3}, \frac{296}{2}\right)(\sigma^2+\sigma'^2) = 148(\sigma^2+\sigma'^2)$. The coefficients of κ^2 and κ^{-2} in (3.13) similarly exceed $\min\left(243, 189, \frac{513}{2}, \frac{399}{2}, 252, 196\right)\sigma^2 = 189\sigma^2$ and $189\sigma'^2$ by (3.16) and (3.17) respectively, while the coefficients of κ^4 and κ^{-4} exceed $\min\left(27, \frac{72}{2}, 48\right)\sigma^2 = 27\sigma^2$ and $27\sigma'^2$ by (3.18) and (3.19). $\square$

We thus see from (3.14) that $D \neq 0$ unless $\sigma = \sigma' = 0$. But this is just the exceptional case in Lemma 3.2, for $0 = \sigma = \sin\frac{\pi\mu h}{2}$, $h = \frac{1}{M}$, happens only when μ is an even multiple of M, and $\sigma' = 0$ only when ν is an even multiple of N. The proof of Lemma 3.2 is thus complete.

We now calculate the 3×3 minors of the coefficient matrix of (3.10) and bound them by the determinant D.

LEMMA 3.4. *There are absolute constants $C_{ij} (i,j = 1,\ldots,4)$ such that the minors M_{ij} of the coefficient matrix of (3.10) satisfy $|M_{ij}| \leq C_{ij}D$ for $(\mu,\nu) \neq (2pM, 2qN)$.*

Proof. Note that the interchange $(\sigma,\gamma,h,\kappa) \leftrightarrow (\sigma',\gamma',k,\kappa^{-1})$ leaves D unchanged, but interchanges the pairs of minors $(M_{12}, M_{13}), (M_{31}, M_{21}), (M_{22}, M_{33}), (M_{23}, M_{32}), (M_{24}, M_{34})$, and (M_{42}, M_{43}), so it suffices to consider only one of each of these pairs.

For some of the minors we use elementary estimates such as

$$|\sigma h| \leq (\sigma^2+h^2)/2, \qquad |\sigma' k| \leq (\sigma'^2+k^2)/2 \tag{3.20}$$

$$|\kappa\sigma k| \leq (\kappa^2\sigma^2+k^2)/2, \qquad |\kappa^{-1}\sigma' h| \leq (\kappa^{-2}\sigma'^2+h^2)/2, \tag{3.21}$$

$$|\kappa\sigma\sigma'| \leq (\kappa^2\sigma^2+\sigma'^2)/2, \qquad |\kappa^{-1}\sigma\sigma'| \leq (\sigma^2+\kappa^{-2}\sigma'^2)/2, \tag{3.22}$$

$$\kappa\sigma^2 \leq (\kappa^2\sigma^2+\sigma^2)/2, \qquad \kappa^{-1}\sigma'^2 \leq (\kappa^{-2}\sigma'^2+\sigma'^2)/2, \tag{3.23}$$

$$|\kappa^3\sigma^3| \leq (\kappa^2\sigma^2+\kappa^4\sigma^4)/2, \qquad |\kappa^{-3}\sigma'^3| \leq (\kappa^{-2}\sigma'^2+\kappa^{-4}\sigma'^4)/2; \tag{3.24}$$

when $\mu \neq 2pM$ or $\nu \neq 2qN$, we also have respectively

$$h^2 = \frac{1}{M^2} \leq \sin^2 \frac{\pi\mu h}{2} = \sigma^2 \quad \text{ and } k^2 = \kappa^2 h^2 \leq \kappa^2\sigma^2, \text{ or}$$
$$k^2 \leq \sigma'^2 \text{ and } h^2 = \kappa^{-2}k^2 \leq \kappa^{-2}\sigma'^2$$

so in either case

$$h^2 + k^2 \leq (\sigma^2 + \sigma'^2) + (\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2); \tag{3.25}$$

finally, superfluous occurrences of $\sigma, \sigma', \gamma, \gamma', h$, and k are simply dominanted by 1; no attempt is made to combine terms to obtain better bounds. We now record each of the minors and indicate how they are dominated by the determinant D, with (3.14) as an intermediate step.

$$\begin{aligned}
M_{11} : \frac{9}{4}h^2k^2M_{11} &= h^2(81\gamma^3\gamma'^3 + 108\sigma^2\gamma\gamma'^3 + 63\sigma'^3\gamma^3\gamma' + 79\sigma^2\sigma'^2\gamma\gamma') \\
&\quad + k^2(81\gamma^3\gamma'^3 + 63\sigma^2\gamma\gamma'^3 + 108\sigma'^2\gamma\gamma'^3 + 79\sigma^2\sigma'^2\gamma\gamma') \\
&\quad + \kappa^2\sigma^2(9k^2\gamma\gamma'^3 + 12k^2\sigma^2\gamma\gamma') \\
&\quad + \kappa^{-2}\sigma'^2(9h^2\gamma^3\gamma' + 12h^2\sigma^2\gamma\gamma'), \text{ so} \\
\frac{9}{4}h^2k^2|M_{11}| &\leq 331((\sigma^2 + \sigma'^2) + (\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2)) + 21(\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2) \quad \text{(by 3.25)} \\
&\leq 3 \cdot \frac{27}{16}h^2k^2D \quad \text{(by 3.14);} \\
|M_{11}| &\leq \frac{9}{4}D.
\end{aligned}$$

$$\begin{aligned}
M_{21}(\text{and } M_{31}) : -\frac{9}{4}h^2k^2M_{21} &= h^2(27\sigma\gamma\gamma'^3 + 36\sigma^3\gamma'^3 + 27\sigma\sigma'^2\gamma^2\gamma' + 31\sigma^3\sigma'^2\gamma') \\
&\quad + k^2(27\sigma\gamma^2\gamma'^3 + 21\sigma^3\gamma'^3 + 36\sigma\sigma'^2\gamma\gamma' + 23\sigma^3\sigma'^2\gamma') \\
&\quad + \kappa^2\sigma^2(3k^2\sigma\gamma'^3 + 4k^2\sigma\sigma'^2\gamma') \\
&\quad + \kappa^{-2}\sigma'^2(9k^2\sigma\gamma^2\gamma' + 12h^2\sigma^3\gamma'), \text{ so} \\
\frac{9}{4}h^2k^2|M_{21}| &\leq 121((\sigma^2 + \sigma'^2) + (\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2)) + 21(\kappa^2\sigma^2 + \kappa^{-2}\sigma^{-2}) \quad \text{(by 3.25)} \\
&\leq \frac{27}{16}h^2k^2D \quad \text{(by 3.14);} \\
|M_{21}| &\leq \frac{3}{4}D, \quad |M_{31}| \leq \frac{3}{4}D.
\end{aligned}$$

$$
\begin{aligned}
M_{41}: -\frac{27}{4}h^2k^2M_{41} &= h^2(27\sigma\sigma'\gamma^2\gamma'^2 + 36\sigma^3\sigma'\gamma'^2 + 39\sigma\sigma'^3\gamma^2) \\
&\quad + k^2(27\sigma\sigma'\gamma^2\gamma'^2 + 39\sigma^3\sigma'\gamma'^2 + 36\sigma\sigma'^3\gamma^2) \\
&\quad + \kappa^2\sigma^2(9k^2\sigma\sigma'\gamma'^2 + 12k^2\sigma\sigma'^3) \\
&\quad + \kappa^{-2}\sigma'^2(9h^2\sigma\sigma'\gamma^2 + 12h^2\sigma^3\sigma'), \text{ so} \\
\frac{27}{4}h^2k^2|M_{41}| &\leq 102((\sigma^2+\sigma'^2) + (\kappa^2\sigma^2 + \kappa'^2\sigma'^2)) + 21(\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2) && \text{(by 3.25)} \\
&\leq \frac{27}{16}h^2k^2D && \text{(by 3.14);} \\
|M_{41}| &\leq \frac{1}{4}D.
\end{aligned}
$$

$$
\begin{aligned}
M_{12}(\text{and } M_{13}): -\frac{9}{8}h^2k^2M_{12} &= h\sigma(81\gamma^2\gamma'^3 + 108\sigma^2\gamma'^3 + 90\sigma'^2\gamma^2\gamma' + 100\sigma^2\sigma'^2\gamma) \\
&\quad + \kappa\sigma k(81\gamma^2\gamma'^3 + 63\sigma^2\gamma'^3 + 108\sigma'^2\gamma^2\gamma' + 64\sigma^2\sigma'^2\gamma) \\
&\quad + \kappa^3\sigma^3(9\gamma'^3 + 12k\sigma'^2\gamma) \\
&\quad + \kappa^{-2}\sigma'^2(36h\sigma\gamma^2\gamma' + 48h\sigma^3\gamma'), \text{ so} \\
\frac{9}{8}h^2k^2|M_{12}| &\leq |h\sigma|\cdot 371 + |\kappa\sigma k|\cdot 316 + 21\kappa^3\sigma^3 + 84\kappa^{-2}\sigma'^2 \\
&\leq 186(h^2+\sigma^2) && \text{(by 3.20)} \\
&\quad + 158(k^2+\kappa^2\sigma^2) && \text{(by 3.21)} \\
&\quad + 11(\kappa^2\sigma^2+\kappa^4\sigma^4) && \text{(by 3.24)} \\
&\quad + 84\kappa^{-2}\sigma'^2 \\
&\leq 186((\sigma^2+\sigma'^2) + (\kappa^2\sigma^2+\kappa^{-2}\sigma'^2)) + 186\sigma^2 \\
&\quad + 169(\kappa^2\sigma^2+\kappa^{-2}\sigma'^2) + 11\kappa^4\sigma^4 && \text{(by 3.25)} \\
&\leq 3\cdot\frac{27}{16}h^2k^2D && \text{(by 3.14);} \\
|M_{12}| &\leq \frac{9}{2}D, \quad |M_{13}| \leq \frac{9}{2}D.
\end{aligned}
$$

$$
\begin{aligned}
M_{22}(\text{and } M_{33}): -\frac{3}{8}h^2k^2M_{22} &= \sigma'^2(9h\gamma^3\gamma' + 7h\sigma^2\gamma\gamma') \\
&\quad + \kappa^{-2}\sigma'^2(9h\gamma^3\gamma' + 12h\sigma^2\gamma\gamma') \\
&\quad + \kappa\sigma^2(-5k\sigma'^2\gamma\gamma'), \text{ so} \\
\frac{3}{8}h^2k^2|M_{22}| &\leq 16\sigma'^2 + 21\kappa^{-2}\sigma'^2 + 5\kappa\sigma^2 \\
&\leq \frac{1}{9}\cdot\frac{27}{16}h^2k^2D && \text{(by 3.23 and 3.25);} \\
|M_{22}| &\leq \frac{1}{2}D, \quad |M_{33}| \leq \frac{1}{2}D.
\end{aligned}
$$

$$
\begin{aligned}
M_{32}(\text{and } M_{23}) : -\frac{9}{8}h^2k^2M_{32} &= h\sigma(27\sigma'\gamma^2\gamma' + 48\sigma'^3\gamma^2 + 80\sigma^2\sigma'^3) \\
&+ \kappa\sigma k(27\sigma'\gamma^2\gamma' + 9\sigma^2\sigma'\gamma'^2 + 36\sigma'^3\gamma^2 + 76\sigma^2\sigma'^3) \\
&+ \kappa^3\sigma^3(9k\sigma'\gamma'^2 + 12k\sigma'^3) \\
&+ \kappa^{-2}\sigma'^2(12h\sigma\sigma'\gamma^2 + 16h\sigma^3\sigma'), \text{ so} \\
\frac{9}{8}h^2k^2|M_{32}| &\le 73(h^2 + \sigma^2) && \text{(by 3.20)} \\
&+ 74(k^2 + \kappa^2\sigma^2) && \text{(by 3.21)} \\
&+ 11(\kappa^2\sigma^2 + \kappa^4\sigma^4) && \text{(by 3.24)} \\
&+ 28\kappa^{-2}\sigma'^2 \\
&\le 74((\sigma^2 + \sigma'^2) + (\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2)) + 73\sigma^2 \\
&+ 85(\kappa^2\sigma^2 + \kappa^{-2}\sigma'^2) + 11\kappa^4\sigma^4 && \text{(by 3.25)} \\
&\le \frac{27}{16}h^2k^2D && \text{(by 3.14);} \\
|M_{32}| \le D, \quad |M_{23}| \le D.
\end{aligned}
$$

$$
\begin{aligned}
M_{42}(\text{and } M_{43}) : -\frac{3}{8}h^2k^2M_{42} &= \kappa\sigma^2(6k\sigma'\gamma\gamma'^2 + 3k\sigma'^3\gamma) \\
&+ \sigma'^2(9h\sigma'\gamma^3 + 7h\sigma^2\sigma'\gamma) \\
&+ \kappa^{-2}\sigma'^2(3h\sigma'\gamma^3 + 4h\sigma^2\sigma'\gamma), \text{ so} \\
\frac{3}{8}h^2k^2|M_{42}| &\le 5(\sigma^2 + \kappa^2\sigma^2) + 16\sigma'^2 + 7\kappa^{-2}\sigma'^2 && \text{(by 3.22)} \\
&\le \frac{1}{9}\cdot\frac{27}{16}h^2k^2D && \text{(by 3.25);} \\
|M_{42}| \le \frac{1}{2}D, \quad |M_{43}| \le \frac{1}{2}D.
\end{aligned}
$$

$$
\begin{aligned}
M_{14} : -\frac{9}{16}h^2k^2M_{14} &= \kappa\sigma\sigma'(81\gamma^2\gamma'^2 + 171\sigma^2\gamma'^2 + 108\sigma'^2\gamma^2 + 148\sigma^2\sigma'^2) \\
&+ \kappa^{-1}\sigma\sigma'(81\gamma^2\gamma'^2 + 108\sigma^2\gamma'^2 + 171\sigma'^2\gamma^2 + 148\sigma^2\sigma'^2) \\
&+ \kappa^3\sigma^3(36\sigma'\gamma'^2 + 48\sigma'^3) \\
&+ \kappa^{-3}\sigma'^3(36\sigma\gamma^2 + 48\sigma^3), \text{ so} \\
\frac{9}{16}h^2k^2|M_{14}| &\le 508\kappa\sigma\sigma' + 508\kappa^{-1}\sigma\sigma' + 84\kappa^3\sigma^3 + 84\kappa^{-3}\sigma'^3 \\
&\le 259(\sigma'^2 + \kappa^2\sigma^2) + 259(\sigma^2 + \kappa^{-2}\sigma'^2) && \text{(by 3.22)} \\
&+ 42(\kappa^2\sigma^2 + \kappa^4\sigma^4) + 42(\kappa^{-2}\sigma'^2 + \kappa^{-4}\sigma'^4) && \text{(by 3.24)} \\
&\le 2\cdot\frac{27}{16}h^2k^2D && \text{(by 3.25);} \\
|M_{14}| \le 6D.
\end{aligned}
$$

$$
\begin{aligned}
M_{24}(\text{and } M_{34}) : -\frac{3}{16}h^2k^2M_{24} &= \kappa\sigma^2(27\sigma'\gamma\gamma'^2 + 16\sigma'^3\gamma) \\
&\quad + \kappa^{-1}\sigma'^2(36\sigma'\gamma^3 + 28\sigma^2\sigma'\gamma) \\
&\quad + \kappa^{-3}\sigma'^3(9\gamma^3 + 12\sigma^2\gamma), \text{ so} \\
\frac{3}{16}h^2k^2|M_{24}| &\le 22(\sigma^2 + \kappa^2\sigma^2) && \text{(by 3.23)} \\
&\quad + 32(\sigma'^2 + \kappa^{-2}\sigma'^2) && \text{(by 3.23)} \\
&\quad + 11(\kappa^{-2}\sigma'^2 + \kappa^{-4}\sigma'^4) && \text{(by 3.24)} \\
&\le \frac{1}{2}\cdot\frac{27}{16}h^2k^2D && \text{(by 3.25);} \\
|M_{24}| \le 5D, \quad |M_{34}| \le 5D.
\end{aligned}
$$

$$
\begin{aligned}
M_{44} : -\frac{1}{16}h^2k^2M_{44} &= \kappa\sigma^2(9\gamma\gamma'^3 + 7\sigma'^2\gamma\gamma') + \kappa^{-1}\sigma'^2(9\gamma^3\gamma' + 7\sigma^2\gamma\gamma'), \text{ so} \\
\frac{1}{16}h^2k^2|M_{44}| &\le 8(\sigma^2 + \kappa^2\sigma^2) + 8(\sigma'^2 + \kappa^{-2}\sigma'^2) && \text{(by 3.23)} \\
&\le \frac{1}{18}\cdot\frac{27}{16}h^2k^2D && \text{(by 3.25);} \\
|M_{44}| &\le \frac{3}{2}D
\end{aligned}
$$

One principal step in our analysis of $U_{\mu\nu}$, the boundedness uniform over all μ, ν of the values at the nodes, is complete. The other step, showing that $U_{\mu\nu}$ is a good approximant when both $\mu < M$ and $\nu < N$, will be done now. We set $A = a\frac{1}{\pi^2(\mu^2+\nu^2)}$, $B = b\frac{\pi\mu}{\pi^2(\mu^2+\nu^2)}$, $C = c\frac{\pi\nu}{\pi^2(\mu^2+\nu^2)}$, $D = d\frac{\pi^2\mu\nu}{\pi^2(\mu^2+\nu^2)}$, where a, b, c, d are the quantities defined in (3.5) – (3.8) of Lemma 3.2.

LEMMA 3.5. *There exist absolute constants C_A, C_B, C_C, C_D such that the quantities A, B, C, D defined above satisfy, when both $\mu < N$ and $\nu < N$,*

$$
\begin{aligned}
|1 - A| &\le C_A((\mu h)^4 + (\nu k)^4) \\
|1 - B| &\le C_B((\mu h)^4 + (\nu k)^4) \\
|1 - C| &\le C_C((\mu h)^4 + (\nu k)^4) \\
|1 - D| &\le C_D((\mu h)^4 + (\nu k)^4).
\end{aligned}
$$

Proof. The quantities A, B, C, D satisfy four simultaneous equations with augmented coefficient matrix similar to (3.10). If we set $s = \frac{\pi\mu h}{2}$, $t = \frac{\pi\nu k}{2}$, the set of equations which A, B, C, D satisfy has coefficient matrix

(3.26)

$$\begin{bmatrix} 0 & \mu^2\sigma t\gamma' & \nu^2 s\gamma\sigma' & \frac{1}{3}(\mu^2t^2+\nu^2s^2)\sigma\sigma' \\ 3\mu^2\sigma t\sigma' & -3\mu^2 s\gamma t^2\gamma' & (\mu^2t^2+\frac{4}{3}\nu^1 s^2)\sigma\sigma' & -(\mu^2t^2+\frac{1}{3}\nu^2s^2)s\gamma\sigma' \\ 3\nu^2 s\sigma\sigma' & \left(\frac{4}{3}\mu^2t^2+\nu^2s^2\right)\sigma\sigma' & -3\nu^2s^2\gamma t^2\gamma' & -\left(\frac{1}{3}\mu^2t^2+\nu^2s^2\right)\sigma t\gamma' \\ 4(\mu^2t^2+\nu^2s^2)\sigma\sigma' & -(4\mu^2t^2+\nu^2s^2)s\gamma\sigma' & -(\mu^2t^2+4\nu^2s^2)\sigma t\gamma' & (\mu^2t^2+\nu^2s^2)s\gamma t\gamma' \end{bmatrix}$$

and right-hand side

$$(\mu^2+\nu^2)\begin{bmatrix} s\cos\frac{s}{\sqrt{3}} & t\cos\frac{t}{\sqrt{3}} \\ \sqrt{3}s^2\sin\frac{s}{\sqrt{3}} & t\cos\frac{t}{\sqrt{3}} \\ \sqrt{3}s\cos\frac{s}{\sqrt{3}} & t^2\sin\frac{t}{\sqrt{3}} \\ 3s^2\sin\frac{s}{\sqrt{3}} & t^2\sin\frac{t}{\sqrt{3}} \end{bmatrix}. \tag{3.27}$$

(In addition to adjusting the columns of (3.10) to reflect the fact that A, B, C, D are the unknowns rather than a, b, c, d, we have, essentially for our own personal preference, multiplied through everything by $\mu^2+\nu^2$, and further, multiplied the individual equations by st, s^2t, st^2, s^2t^2 respectively so that all entries are analytic functions of s and t. $\sigma = \sin\frac{\pi\mu h}{2}$ becomes $\sigma = \sin s$, and $\gamma = \cos s$, $\sigma' = \sin t$, $\gamma' = \cos t$.)

Now solve these equations for A, B, C, D by Cramer's rule, and expand the determinants in power series in s and t about $(0,0)$. The determinant of the coefficient matrix (3.26) is

$$\begin{aligned} &9s^4t^4\mu^2\nu^2(\mu^2+\nu^2)(\nu^2s^2+\mu^2t^2) \\ &+s^4t^4(\mu^2+\nu^2)(\nu^6s^4-5\mu^2\nu^2(\mu^2+\nu^2)s^2t^2+\mu^6t^4) \\ &+O(s^{10}t^4+s^4t^{10}). \end{aligned} \tag{3.28}$$

When we evaluate the four determinants that form the numerators for A, B, C, D we obtain in each case the same expressions as (3.28); the only differences are in the $O(s^{10}t^4+s^4t^{10})$ terms. Thus when we form the quotients which are A, B, C, D, these are all of the form

$$1+O\left(\frac{s^6+t^6}{\nu^2s^2+\mu^2t^2}\right) = 1+O(s^4+t^4). \tag{3.29}$$

The extent to which this estimate is valid is given by (3.14): apart from $s=0$, $t=0$, the determinant of (3.28) differs from the D of (3.14) only by non-zero factors, and so the determinant of (3.28) is non-zero unless both $\sigma = \sin s = 0$ and $\sigma' = \sin t = 0$. The expressions for A, B, C, D are therefore analytic in the polydisc $(|s|<\pi)\times(|t|<\pi)$, and the estimates (3.29) are uniform in the compact sub-polydisc $\left(|s|\le\frac{\pi}{2}\right)\times\left(|t|\le\frac{\pi}{2}\right)$ which corresponds to $\mu \le M$, $\nu \le N$. $\square$

To get an idea of the magnitude of these quantities, we give the first non-constant term:

$$
\begin{aligned}
A &= 1 + s^4\left(\frac{1}{90}\frac{\mu^2}{\mu^2+\nu^2} + \frac{1}{54}\frac{\nu^2}{\mu^2+\nu^2}\right) + t^4\left(\frac{1}{54}\frac{\mu^2}{\mu^2+\nu^2} + \frac{1}{90}\frac{\nu^2}{\mu^2+\nu^2}\right) + \cdots \\
B &= 1 + s^4\left(\frac{1}{270}\frac{-\mu^2+\nu^2}{\mu^2+\nu^2}\right) + t^4\left(\frac{1}{54}\frac{\mu^2}{\mu^2+\nu^2} + \frac{1}{90}\frac{\nu^2}{\mu^2+\nu^2}\right) + \cdots \\
C &= 1 + s^4\left(\frac{1}{90}\frac{\mu^2}{\mu^2+\nu^2} + \frac{1}{54}\frac{\nu^2}{\mu^2+\nu^2}\right) + t^4\left(\frac{1}{270}\frac{\mu^2-\nu^2}{\mu^2+\nu^2}\right) + \cdots \\
D &= 1 + s^4\left(\frac{1}{270}\frac{-\mu^2+\nu^2}{\mu^2+\nu^2}\right) + t^4\left(\frac{1}{270}\frac{\mu^2-\nu^2}{\mu^2+\nu^2}\right) + \cdots
\end{aligned}
$$

The coefficients of higher order terms follow a pattern which, to us, is neither simple nor illuminating.

The conclusion that our Gauss collocation solution U to $-\Delta u = f$ is fourth order accurate at the nodes, as also are U_x, U_y, and U_{xy}, now follows immediately:

THEOREM 3.6. *Let f be given on the unit square Ω by the Fourier series $f(x,y) = \sum_{\mu,\nu} c_{\mu\nu} \sin \pi\mu x \sin \pi\nu y$ with $\sum |c_{\mu\nu}|(\mu^4+\nu^4) = K < \infty$. Let u be the solution to $-\Delta u = f$ on Ω, $u = 0$ on $\partial\Omega$, and let $U \in S$ be the C^1 piecewise cubic for which $-\Delta U = f$ at the Gauss points. Then there are absolute constants C, independent of the partition, such that $u - U$, $u_x - U_x$, $u_y - U_y$, and $u_{xy} - U_{xy}$ are all bounded by $CK(h^4 + k^4)$.*

Proof. Break the Fourier series indices into two sets

$$I = \{(\mu,\nu) : \mu < M, \nu < N\} \text{ and } J = \{(\mu,\nu) : \mu \geq M \text{ or } \nu \geq N\}.$$

The difference $u - U$ is then, at a node (x,y),

$$
\begin{aligned}
(u-U) = &\sum_I c_{\mu\nu} : \frac{\sin \pi\mu x \sin \pi\nu y}{\pi^2(\mu^2+\nu^2)}(1 - A_{\mu\nu}) \\
&+ \sum_J c_{\mu\nu} \frac{\sin \pi u x \sin \pi\nu y}{\pi^2(\mu^2+\nu^2)} \\
&- \sum_J c_{\mu\nu} \sin \pi\mu x \sin \pi\nu y \cdot a_{\mu\nu}
\end{aligned}
$$

The first sum is dominated by $\sum_I |c_{\mu\nu}| C_A \frac{\mu^4 h^4 + \nu^4 k^4}{\pi^2(\mu^2+\nu^2)}$ (lemma 3.5) and so by $C\left(\sum_I |c_{\mu\nu}|(\mu^2+\nu^2)\right)(h^4+k^4)$. The second sum is dominated by $\sum_J |c_{\mu\nu}| \leq$ $\left(\sum_J |c_{\mu\nu}|(\mu^4+\nu^4)\right)(h^4+k^4)$. The third sum is dominated by $\sum_J |c_{\mu\nu}| a_{\mu\nu} \leq C \sum_J |c_{\mu\nu}|$, the uniform estimate $a_{\mu\nu} \leq C$ being provided by lemma 3.4 for most μ, ν and by (3.11) for the exceptional cases $(\mu,\nu) = (2pM, 2qN)$; the third sum is thus dominated by

$C\left(\sum_J |c_{\mu\nu}|(\mu^4+\nu^4)\right)(h^4+k^4)$. Together, these terms provide the estimate for $u-U$ at the nodes. The estimates for the derivatives are the same. □

Estimates of $u-U$ and its derivatives may also be extended throughout Ω by examining how the Lagrange interpolating polynomials extend U from data at the nodes to the whole of the partitioning rectangles. Continuing the assumptions of theorem 3.6, we state without proof

COROLLARY 3.7.

(a) $|u-U| \le CK(h^4+k^4)$ *throughout* Ω

(b) $|u_x-U_x| \le CK\left(h^3+\frac{k^4}{h}\right)$ *throughout* Ω, *although* $|u_x-U_x| \le CK(h^4+k^4)$ *on the vertical sides of the partitioning rectangles*

(c) $|u_y-U_y| \le CK\left(\frac{h^4}{k}+k^3\right)$ *throughout* Ω, *although* $|u_y-U_y| \le CK(h^4+k^4)$ *on the horizontal sides of the partitioning rectangles*

(d) $|u_{xy}-U_{xy}| \le CK\left(\frac{h^3}{k}+\frac{k^3}{h}\right)$ *throughout* Ω, *although* $|u_{xy}-U_{xy}| \le CK\left(\frac{h^4}{k}+k^3\right)$ *on the vertical sides of the partitioning rectangles and* $|u_{xy}-U_{xy}| \le CK\left(h^3+\frac{k^4}{h}\right)$ *on the horizontal sides.*

Acknowledgements. It is a pleasure to thank Karin Bennett for introducing us to the subject matter of this note, and the Institute of Mathematics and its Applications for support. The more extensive calculations were either performed or checked with Maple V.

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